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Comparing Mean-Variance and CVaR
optimal portfolios, assuming bivariate
skew-t distributed returns

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Abstract

In this paper we are building portfolios consisting of the S&P 500 index and a T-bond index. The portfolio weights are chosen in such a way that the risk for the portfolio is minimized. To be able to minimize the risk for a portfolio, we first have to specify how to measure the portfolios risk. There are several ways of measuring the risk for a portfolio. In this paper we are investigating how the portfolio weights differ whether we measure the portfolios risk by the variance or by the Conditional Value-at-Risk (CVaR). To measure the risk for the portfolios we first estimated a two-dimensional density function for the returns of the assets, using a skew student-t distribution. The time horizon for each portfolio was one week. The result shows that the weights in the S&P 500 index always were lower for the portfolios constructed by minimizing CVaR. The reason for this is that the distribution for the returns of the S&P 500 index exhibits a negative skewness and has fatter tails than the returns of the T-bond index. This fact isn't taken care of when choosing weights according to the variance criteria, which leads to an underestimation of the risk associated with the S&P 500 index. The underestimation of the risk leads to an overestimation of the optimal weights in the S&P 500 index.

Contents

1	Introduction	2
2	Description of the CVaR approach	7
3	The probability density function of the assets returns	10
3.1	The properties of asset returns	10
3.2	Description of the standard DCC model	12
4	Estimating the density function of the assets returns	15
5	Results and analysis	18
6	Conclusions	21
7	References	23
A	Tables	25
B	Figures	27

1 Introduction

Extreme price movements in financial markets are rare, but are still an important issue in risk management. After the market crash on Wall Street in 1987 and in the Asian market in 1998, extreme price movements have attracted a greater deal of attention. Risk management can be divided into several subareas; for example credit risk, liquidity risk and market risk.¹ In this paper we will only consider the market risk. The market risk for a financial security is often measured by its variance² (or the closely relative measure; volatility)³. The variance of a security's price tells us how large price fluctuations that can be expected. During the last years there are tendencies indicating that the volatility of the financial market has increased (see figure (1)), which makes it even more important to consider the market risk when investing in financial securities.

In economics, expected utility maximization is the most often used preference representation⁴. Therefore we want a risk measure to be consistent with expected utility maximization, i.e. the risk measure should rank the investment opportunities the same way as expected utility maximization criteria does. When using expected utility maximization you have to specify a utility function. The utility function must be non-decreasing and concave to satisfy the most often used preferences nonsatiety and risk aversion⁵. If we assume these properties of the utility function, the stochastic dominance criteria is consistent with expected utility maximization and the n -th order stochastic dominance is defined inductively as⁶

$$F^{(n)}(x) \equiv \int_{-\infty}^x F^{(n-1)}(u)du, \quad (1)$$

¹See Tsay [2002].

²The variance of X is defined as $E[(X - E[X])^2]$.

³See Yoshida and Yamai [2001].

⁴ibid.

⁵See Danthine and Donaldsson [2001].

⁶See Yoshida and Yamai [2001].

where

$$F^{(1)}(x) \equiv F(x), \quad (2)$$

and $F(x)$ is the distribution function. If X and Y are two investment opportunities, X dominates Y in the sense of n -th order stochastic dominance if $F_X^{(n)}(u) \leq F_Y^{(n)}(u)$ for $\forall u \in \mathbb{R}$. If X dominates Y in the sense of n -th order stochastic dominance, X also dominates Y in the sense of all higher order stochastic dominance⁷. Further, a risk measure consistent with n -th order stochastic dominance is also consistent with the lower orders⁸. Using stochastic dominance we do not have to specify the utility function, and therefore we can be more general. The problem with stochastic dominance is that it is restrictive. The n -th order is more restrictive than the $(n + 1)$ -th order⁹ and the first order is mostly too restrictive to be of any practical use¹⁰.

There are several ways of measuring the market risk. The variance, which we mentioned above is one of the oldest and most popular risk measures¹¹. It is a simple and efficient measure for the market risk, but it also has its drawbacks. If we want to know as much as possible about the properties of a stochastic variable, we prefer to know the whole probability distribution function. The variance is only considering the two first moments¹² of the probability distribution function. If we do not have a clue about the higher order moments this is the optimal risk measure. But, empirical data is strongly indicating that the distribution function of stocks etc. often shows both skewness and fat tails¹³ (which are functions of the three and four first moments). Therefore, in risk management we should, if possible, consider also these higher order moments. Also, the variance is only consistent with second order stochastic dominance if the mean of profit and loss are equal

⁷See Yoshida and Yamai [2001].

⁸ibid.

⁹ibid.

¹⁰See Danthine and Donaldsson [2001].

¹¹See Markowitz [1952].

¹²The i :th moment of X is defined as $E[X^i]$.

¹³See Tsay [2002].

between the investment opportunities¹⁴. These well-known properties has brought other risk measures up to date. An alternative risk measure is Value at Risk (VaR). Today VaR has become a widely used risk measure in portfolio and risk management¹⁵.

Value-at-Risk

VaR can be defined as the maximal loss during a given time period for a given probability, or more loosely speaking, as the minimal loss under extraordinary market conditions¹⁶. VaR is an attractive measure of risk because it is relative easy to understand, and VaR has become the standard extreme risk measure in the financial industry¹⁷. Despite its popularity, VaR has the following undesirable properties.

1. Intuitively, a weighted portfolio with several securities is less riskier than the individual risks for the securities, due to diversification. But VaR for a portfolio can sometimes be greater than the sum of the VaRs for the individual instruments in the portfolio. This undesirable property is called lack of sub-additivity and tells us that maybe VaR is not the very best risk measure.¹⁸
2. VaR as a function of the portfolio weights is not convex and can therefore exhibit multiple local extremes. This is a problem when trying to determine the weights that minimize VaR.¹⁹
3. VaR is only consistent with second order stochastic dominance when investment opportunities returns have the same type of elliptical distribution with finite variance and the same mean.²⁰

¹⁴See Yoshihara and Yamai [2001].

¹⁵See Hull [2003].

¹⁶See Rockafellar and Uryasev [1999].

¹⁷See Tsay [2002].

¹⁸See Rockafellar and Uryasev [1999].

¹⁹ibid.

²⁰See Yoshihara and Yamai [2001].

4. VaR only gives us the maximal loss given a certain level of confidence²¹. For example, VaR can tell us that with a certain probability we will not lose more than a specific amount of money in a certain period of time. But what if the loss exceeds this amount? Then VaR is the minimum amount we will lose, i.e. the loss will be greater than VaR. Sometimes it is more interesting to know how much we can expect to lose, given that the loss is greater than some threshold. A measure that tells us this is called Conditional VaR (CVaR).

Conditional Value-at-Risk

CVaR, also called expected tail loss (ETL) or expected shortfall, is measuring the expected loss given that the loss is greater than VaR, i.e the average of the worst α percent of losses (for some specified significance level α). Therefore CVaR is always greater than or equal to VaR. CVaR is known to have better properties than VaR. It is a measure that is both convex and sub-additive. Due to the convexity it is easier to minimize CVaR (than VaR) with respect to the portfolio weights.²² Optimizing CVaR and VaR most often gives similar portfolio weights. This is not surprising since CVaR and VaR are closely related to each other. If the simultaneous distribution for the securities is symmetric, then the optimization of CVaR and VaR gives the same portfolio mixture.²³ CVaR is consistent with second order stochastic dominance. Hence, CVaR is consistent with expected utility maximization under weaker conditions than the variance and VaR are. But, if we are able to rank the investments by second order stochastic dominance, CVaR is not consistent with expected utility maximization. Hence, CVaR is not consistent with stochastic dominance of orders higher than two. A drawback for CVaR is that we need a larger sample size than for VaR to have the same level of accuracy.²⁴

²¹See Rockafellar and Uryasev [1999].

²²ibit.

²³ibit.

²⁴See Yoshihara and Yamai [2001].

Purpose and problem discussion

The task for every portfolio and risk manager is to minimize the risk and maximize the expected return. A high expected return is often connected with a high level of risk, which is a result from peoples risk aversion. Therefore, in practise, the managers task is to minimize the risk level given a specific expected return or to maximize the expected return given a certain level of risk. The mostly used risk measures are VaR and the variance criteria²⁵. As mentioned earlier, these measures have some undesirably properties and are not measuring the total risk exposure and are not always consistent with expected utility maximization²⁶. Therefore, when you are optimizing a portfolio, with respect to its weights, from these two criterions, maybe you get stuck with a portfolio which does not coincide the best with your preferences (about risk and expected return). To handle this optimization problem we will optimize CVaR of the portfolio instead of optimizing the portfolios VaR. CVaR is considered to be a better risk measure²⁷, although it is not always consistent with expected utility maximization²⁸. To have a more realistic model, we will allow the returns from the securities to follow a more general distribution than the normal distribution, allowing fat tails and skewness.

The aim of this paper is to find the optimal portfolio mixture of the Standard&Poor 500 index and an index of seven to ten years T-bonds, using CVaR as preference. The optimal portfolios when considering CVaR will be compared to the optimal portfolios optimized by the variance criteria. The assets are supposed to follow a bivariate skew-t distribution, hence the optimal portfolios with regards to the variance criteria will not be consistent with maximum expected utility, although it is possible that the CVaR and the variance portfolios does not differ in any practical meaning.

²⁵See Hull [2003].

²⁶See Rockafellar and Uryasev [1999] and Yoshiba and Yamai [2001]

²⁷See Rockafellar and Uryasev [1999].

²⁸See Yoshiba and Yamai [2001].

2 Description of the CVaR approach

This section is mostly collected from Rockafellar and Uryasev [1999] and Uryasev [2000]. Let the vector \mathbf{x} represent a portfolio of financial instruments. $\mathbf{x} = (x_1, \dots, x_n)^T$, where x_j is the portfolio weight of instrument j and

$$x_j \geq 0 \quad j = 1, \dots, n \quad \text{and} \quad \sum_{j=1}^n x_j = 1. \quad (3)$$

This restriction means that we are not allowing short sales.

Further, the vector \mathbf{y} is a random variable generating the returns of the securities. $\mathbf{y} = (y_1, \dots, y_n)^T$ where y_j is the return of security j . $L(\mathbf{x}, \mathbf{y})$ stands for the loss of the portfolio associated with weight vector \mathbf{x} . The loss of portfolio \mathbf{x} is given by

$$L(\mathbf{x}, \mathbf{y}) = -[x_1 y_1 + \dots + x_n y_n] = -\mathbf{x}^T \mathbf{y}. \quad (4)$$

Notice, the definition in (4) means that if the return of the portfolio \mathbf{x} is positive, the loss function $L(\mathbf{x}, \mathbf{y})$ will have a negative sign. Also, if we assume that the underlying probability distribution of \mathbf{y} can be represented by a density function $p(\mathbf{y})$, then the probability of the loss $L(\mathbf{x}, \mathbf{y})$, not exceeding a threshold α , is given by

$$\Psi(\mathbf{x}, \alpha) = \int_{L(\mathbf{x}, \mathbf{y}) \leq \alpha} p(\mathbf{y}) d\mathbf{y} \quad (5)$$

As a function of α , $\Psi(\mathbf{x}, \alpha)$ is the cumulative distribution function for the loss associated with portfolio \mathbf{x} . $\Psi(\mathbf{x}, \alpha)$ is non-decreasing with respect to α . If $p(\mathbf{y})$ is continuous, it follows that $\Psi(\mathbf{x}, \alpha)$ is also continuous, and therefore the derivative of $\Psi(\mathbf{x}, \alpha)$ will exist.

The β -VaR of a portfolio is the lowest amount α such that, with probability β , the loss will not exceed α . The β -CVaR is the expectation of the loss conditioning on that the loss is greater than α . The value of β -VaR is defined by

$$\alpha_\beta(\mathbf{x}) = \min\{\alpha \in \mathbb{R} : \Psi(\mathbf{x}, \alpha) \geq \beta\} \quad (6)$$

and the value of β -CVaR is defined by

$$\begin{aligned}\phi_\beta(\mathbf{x}) &= (1 - \beta)^{-1} \int_{L(\mathbf{x}, \mathbf{y}) \geq \alpha_\beta(\mathbf{x})} L(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) d\mathbf{y} \\ &= \alpha_\beta(\mathbf{x}) + (1 - \beta)^{-1} \int_{\mathbf{y} \in \mathbb{R}^n} [L(\mathbf{x}, \mathbf{y}) - \alpha_\beta(\mathbf{x})]^+ p(\mathbf{y}) d\mathbf{y},\end{aligned}\quad (7)$$

where $[t]^+ = \max(t, 0)$. In the definition of $\phi_\beta(\mathbf{x})$ we use Baye's rule, i.e. $f(x | y) = \frac{f(x, y)}{f(y)}$, where $f(\cdot)$ is a probability density function.

Consider the function $F_\beta(\mathbf{x}, \alpha)$

$$F_\beta(\mathbf{x}, \alpha) = \alpha + (1 - \beta)^{-1} \int_{\mathbf{y} \in \mathbb{R}^n} [L(\mathbf{x}, \mathbf{y}) - \alpha]^+ p(\mathbf{y}) d\mathbf{y}.\quad (8)$$

$F_\beta(\mathbf{x}, \alpha)$ is convex and continuously differentiable with respect to α . The partial derivative of $F_\beta(\mathbf{x}, \alpha)$ with respect to α is given by

$$\begin{aligned}\frac{\partial F_\beta(\mathbf{x}, \alpha)}{\partial \alpha} &= 1 + (1 - \beta)^{-1} \int_{\mathbf{y} \in \mathbb{R}^n} \frac{\partial}{\partial \alpha} [L(\mathbf{x}, \mathbf{y}) - \alpha]^+ p(\mathbf{y}) d\mathbf{y} \\ &= 1 + (1 - \beta)^{-1} \int_{\mathbf{y} \in \mathbb{R}^n} \frac{\partial}{\partial \alpha} ([L(\mathbf{x}, \mathbf{y}) - \alpha] 1_{L(\mathbf{x}, \mathbf{y}) \geq \alpha}) p(\mathbf{y}) d\mathbf{y} \\ &= 1 + (1 - \beta)^{-1} \int_{\mathbf{y} \in \mathbb{R}^n} (-1_{L(\mathbf{x}, \mathbf{y}) \geq \alpha} + [L(\mathbf{x}, \mathbf{y}) - \alpha] \delta_{L(\mathbf{x}, \mathbf{y}) - \alpha}) p(\mathbf{y}) d\mathbf{y} \\ &= 1 - (1 - \beta)^{-1} \int_{\mathbf{y} \in \mathbb{R}^n} 1_{L(\mathbf{x}, \mathbf{y}) \geq \alpha} p(\mathbf{y}) d\mathbf{y} \\ &= 1 - (1 - \beta)^{-1} \underbrace{\int_{L(\mathbf{x}, \mathbf{y}) \geq \alpha} p(\mathbf{y}) d\mathbf{y}}_{=1 - \Psi(\mathbf{x}, \alpha)} = (1 - \beta)^{-1} (\Psi(\mathbf{x}, \alpha) - \beta),\end{aligned}\quad (9)$$

where 1_z is a indicator function taking the value one for $z \geq 0$ and zero elsewhere and δ is a dirac spike with all mass at δ_0 . Because of the convexity of $F_\beta(\mathbf{x}, \alpha)$, the local minimum of the function with respect to α is equal to the global minimum. The α which minimize $F_\beta(\mathbf{x}, \alpha)$ is the value setting (9) to zero, i.e. the α making $\Psi(\mathbf{x}, \alpha) = \beta$. Hence,

$$\min_{\alpha} F_\beta(\mathbf{x}, \alpha) = \phi_\beta(\mathbf{x})\quad (10)$$

and minimizing the β -CVaR of the loss associated with \mathbf{x} over all $\mathbf{x} \in X$ is therefore equivalent to minimizing $F_\beta(\mathbf{x}, \alpha)$ over all $(\mathbf{x}, \alpha) \in X \times \mathbb{R}$. Furthermore, if the loss function $L(\mathbf{x}, \mathbf{y})$ is convex, which indeed it is, $F_\beta(\mathbf{x}, \alpha)$ is convex with regards to (\mathbf{x}, α) . Therefore, the minimization of CVaR is an instance of convex programming.

3 The probability density function of the assets returns

3.1 The properties of asset returns

When studying the returns from financial securities, there are some properties that are often recurring. In this section we are going to discuss some of these properties.

Fat tails

The returns of financial assets are usually assumed to follow a normal distribution. An advantage with this assumption is that the normal distribution has nice properties and is easy to work with. But empirical data indicates that the returns usually are not normal. The empirically observed returns often exhibit a greater number of extreme values than what is generated from the normal distribution, i.e. the distributions of the returns of the assets have fatter tails than the normal distribution. This feature is also called leptokurtosis. Therefore, assuming the returns of the assets are normal distributed will result in an underestimation of the risk.²⁹

Volatility clustering

It is often observed that the absolute values of the returns in financial time-series are correlated. This means that if a high absolute value is observed, then the absolute value of the next observation is likely to be high as well. This phenomena is called volatility clustering. The effect of the volatility clustering will become more clear as the frequency of the data sampling

²⁹See Wong and Vlaar [2003].

increases. One possible economical explanation of this feature is that the arrival of new information is serially correlated.³⁰

Asymmetric volatility phenomenon

Empirical data is indicating that the volatility of assets returns often react differently to a big price increase than to a big price drop. A negative shock often has a larger impact on the volatility than a positive shock. This property is called the asymmetric volatility phenomenon. A possible explanation for this phenomena is that when a company's equity declines in value, the leverage of the company increases. In turn, the higher leverage makes that the risk associated with the company increases, which makes the volatility rise even more. Another reason to the asymmetric volatility phenomena is called the volatility feedback effect. If the volatility is expected to increase, the returns required by the investors increases, which lowers the price of the equity.³¹

Correlation of asset returns

When going from investing in a single asset to forming a portfolio of several assets, it is not enough to just look at the expected returns and the risks of the different individual securities. You also have to take into account the covariances between the returns of the assets. It is not unusual that there is a strong positive correlation between the returns. This means that if there is a loss in one asset, there will probably be losses in the other assets as well. This results in that the diversification effects are greater when the correlation between the assets are low, and this is an important fact when

³⁰See Wong and Vlaar [2003].

³¹ibit

trying to minimize the risk of a portfolio. Therefore, for being successful in portfolio management, it is very important to have accurate estimates for the volatilities and the correlations for the returns of the financial assets. Studies have shown that not only the volatility is time varying, the correlation is varying over time as well. This fact is important, though it makes the modeling a bit more complicated. But ignoring this fact could result in an underestimation of the risk associated with a portfolio. Empirical data is also indicating that correlation between assets returns are much higher in bear markets than in bull markets.³² A model that is able to take care of the time varying variance and correlation is the Dynamic Conditional Correlation (DCC) model, which is a multivariate GARCH model proposed by Engle (2002).

3.2 Description of the standard DCC model

The standard DCC model is a multivariate GARCH model that allows time-varying correlation. An advantage of the model is that it is relative parsimonious. Most GARCH models are not parsimonious and the number of parameters grows fast with the number of assets³³. In the DCC model the number of parameters grows only linearly with the number of assets. The DCC model is also flexible since it allows the volatility of different assets to follow different GARCH models.

Under the assumption of normal innovations, i.e. the sequence $\{z_t\}$ in (11) is iid normal distributed, the model can be estimated consistent in two steps. In the first step, an univariate GARCH model for each asset return is estimated. In the second step the standardized residuals (from step one) are used to estimate the time varying correlation matrix.³⁴ The two steps procedure results in that the number of parameters to be estimated simultaneously are

³²See Wong and Vlaar [2003].

³³See Tsay [2002].

³⁴See Wong and Vlaar [2003].

relatively small. Although the two steps procedure is consistent, it is not efficient, due to that the parameters of the variance processes and the correlation processes are not simultaneously estimated.

The standard DCC model is defined as follows

$$y_t = \mu_t + \Sigma_t^{1/2} z_t \quad (11)$$

$$\Sigma_t = D_t R_t D_t \quad (12)$$

$$D_t = \text{diag}(\sigma_{11,t}, \dots, \sigma_{kk,t}) \quad (13)$$

$$R_t = (\text{diag } Q_t)^{-1/2} Q_t (\text{diag } Q_t)^{-1/2} \quad (14)$$

$$Q_t = (1 - a - b)\bar{Q} + a\varepsilon_{t-1}\varepsilon_{t-1}^T + bQ_{t-1}, \quad (15)$$

$y_t = (y_{1t}, \dots, y_{kt})^T$ are the returns at time t for the different assets, where y_{it} is the return for asset i at time t .

$\mu_t = (\mu_{1t}, \dots, \mu_{kt})^T$ is the conditional mean vector at time t , where μ_{it} is the conditional mean for asset i at time t .

Σ_t is the conditional covariance matrix at time t .

z_t is a $k \times 1$ vector of iid random variables with mean zero and variance one.

D_t is a diagonal matrix with the conditional standard deviations from the univariate GARCH models ($\sigma_{ii,t}$) at the diagonal.

R_t is the conditional correlation matrix at time t .

Both Σ_t , D_t and R_t are $k \times k$ matrices.

$\varepsilon_t = (\varepsilon_{1t}, \dots, \varepsilon_{kt})^T$ with $\varepsilon_{it} = (y_{it} - \mu_{it})/\sigma_{ii,t}$ are the standardized residuals received from the univariate processes for the assets returns.

Q_t is a $k \times k$ symmetric positive definite matrix and \bar{Q} is the $k \times k$ unconditional correlation matrix of ε_t .

a and b are scalar parameters which are going to be estimated. If we assume that the residuals from the different assets impact the correlations differently, then a and b are replaced by two diagonal matrixes. The model is then called the generalized DCC model³⁵. For (11) to be meaningful Σ_t must be positive definite, which means that a, b have to satisfy $a, b \geq 0$, and $a + b < 1$, and the univariate GARCH processes must satisfy equation (17). All conditioning are based upon the information available at time $t-1$.

³⁵See Wong and Vlaar [2003].

4 Estimating the density function of the assets returns

We will make the assumption that the assets returns are following a multivariate GARCH model, or to be more specific, a standard DCC model. Our financial time-series is the S&P 500 index and an index of seven to ten years T-bonds. The data observations are from 1986-01-02 to 2005-04-26. We will consider the weekly log returns of the indexes. For computing the weekly returns we have used the Wednesday's price for each week. In case there was no trade at a specific Wednesday, we have considered the Tuesday's price for that week instead. By choosing week data we only have to make one period forecasts of the density function when forecasting the CVaR for the following week. Considering GARCH-models, one period forecasting is much easier to work with than multi period forecasting³⁶. When estimating the density function, we first look at the univariate time-series to identify two appropriate univariate models for the mean and the volatility. All programs for the estimation procedure is implemented in C++ .

Choosing the univariate models for the mean and the volatility

An Autocorrelation Function (ACF) of the S&P 500 and the T-bond index shows some serial autocorrelation at five percent significance level (see figures (2) and (3)). A plot of the indexes indicates volatility clustering (see figures (4) and (5)). Usually, a GARCH(1,1) model is successful in capturing this property of the volatility³⁷. An AR(1) (Autoregressive model with one lag) can be considered as an MA(∞) (Moving average model)³⁸, therefore we assume that both the indexes are following an AR(1)-GARCH(1,1). A

³⁶See Tsay [2002].

³⁷See for instance Engle [2001].

³⁸See for instance Tsay

time-serie r_t is following an AR(p)-GARCH(m,s) if

$$\begin{aligned} r_t &= \phi_0 + \phi_1 r_{t-1} + \dots + \phi_p r_{t-p} + \varepsilon_t & \varepsilon_t &= \sigma_t z_t \\ \sigma_t^2 &= \alpha_0 + \sum_{i=1}^m \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^s \beta_j \sigma_{t-j}^2, \end{aligned} \quad (16)$$

where $\{z_t\}$ is a sequence of iid random variables with mean 0 and variance 1. If the univariate volatility processes (i.e. the univariate GARCH processes) are mean reverting, which is a very reasonably economic property, (16) must satisfy

$$1 - \sum_{i=1}^m \alpha_i - \sum_{j=1}^s \beta_j > 0. \quad (17)$$

Both indexes indicate fat tails and left skewness (see figures (6) and (7)), which further indicates that it is not probable that the returns follow a normal distribution. Assuming the innovations $\{z_t\}$ in (16) are $N(0, 1)$ distributed will result in that the process r_t in (16) will produce fatter tails than the normal distribution, but no skewness, since the normal distribution is symmetric. Therefore, as the data is indicating, we will assume that $\{z_t\}$ is skew Student-t distributed³⁹. This assumption allows the indexes to feature both skewness and heavy fat tails. Unfortunately, we are then not able to use the two steps procedure described for the standard DCC model when estimating the assets covariance function. Remember, the consistency in the two steps procedure only holds if normal innovations are assumed. However, we can use quasi maximum likelihood (QML) to estimate ε_t in (15) and then estimate the correlation part together with the parameters ξ and ν . In this way we will still have consistent estimators of the parameters although not efficient ones. In our case we will estimate the parameters in one step because we are only considering two assets. But if we should have considered more than two assets we should probably have used the two steps procedure with QML proposed by Bauwens and Laurent (2004). The reason is that the parameters to be estimated simultaneous in the one step procedure grows

³⁹See Bauwens and Laurent [2004].

with seven parameters for every additional asset in our case. In the one step procedure we only estimate seven parameters simultaneous. Estimating 21 or more parameters simultaneous can be a hard task. The standard DCC model disregards asymmetric correlation and volatility, but since we assume skewed innovations $\{z_t\}$ our model allows those features.

Estimating the density function

The parameters are estimated by the conditional Maximum Likelihood method (see table (1),(2) and (3)). For each index there is a total of seven parameters to be estimated, and together with the two parameters for the correlation, we will totally estimate 16 parameters simultaneous. At five percent confidence level all parameters are significant except from three of the parameters for the T-bond process. The lag parameter (ϕ_1) in the AR part, the constant parameter (α_0) in the univariate GARCH part and the skewness parameter (ξ) are not significant at five percent confidence level (see table (2)). The skewness parameter is negative for both indexes. The degrees of freedom parameters are both low, which means that the tails of the indexes are fatter than if we had assumed normal distributed innovations. The standardized residuals \hat{z}_t and the standardized residuals squared \hat{z}_t^2 have no significant serial correlation at five percent confidence level. When comparing the time varying correlation model (the standard DCC model) to the model with constant correlation (i.e. $a = b = 0$), the former model is highly significant at five percent confidence level, i.e. the correlation between S&P 500 and the T-bond index is time varying (see tables (4) and (5)).

5 Results and analysis

When the parameters in the DCC model are estimated, the next task is to estimate the weights for the different portfolios. Two types of portfolios are constructed, where both the portfolio types are combinations of two assets; the S&P 500 and the T-bond indexes, with no short sales allowed. The portfolios are rebalanced once a week (every Wednesday). The weights of the portfolios are estimated during a time period of one year, from 2004-04-21 to 2005-04-20. The first type of portfolios are constructed by optimizing the CVaR. The portfolio weights are chosen as those weights which minimize the CVaR for a time period of one week and with a specific level of confidence. We are considering portfolios with three different levels of confidence; $\beta = 0.90$, $\beta = 0.95$ and $\beta = 0.99$. The second type of portfolios are chosen with respect to the variance criteria, i.e. the portfolio weights are those weights minimizing the variance for the portfolio. From now on we name the different portfolio types as CVaR and variance portfolios. Totally we will estimate $4 \cdot 52 = 208$ portfolios (156 CVaR portfolios (52 for each level of confidence) and 52 variance portfolios). The optimal portfolio weights in the S&P 500 index for the portfolios are shown in the figures (8), (9) and (10). In the figures we see that the optimal portfolio weights in the S&P 500 index are always lower for the CVaR portfolios than for the variance portfolios. This is true for all three levels of confidence. The reason for this is that the variance criteria is not considering the extra risk arisen from the negative skewness and the kurtosis. This is because the variance criteria is only considering the first two moments for the distribution of the returns. The S&P 500 index is more skewed and has fatter tails than the T-bond index (see the parameters in table (2)), and can therefore be regarded as more risky. When choosing weights according to the variance criteria this fact is not taken care of, which leads to an underestimation of the risk associated with the S&P 500 index. This underestimation of the risk leads to an overestimation of the optimal portfolio weights in the S&P 500 index, which explains the different optimal portfolio weights for the two portfolio types. In figure (11) the optimal

portfolio weights for the CVaR portfolios for all three levels of confidence are shown together with the optimal weights for the variance portfolios (note that the variance portfolios are independent of the level of confidence). The figure shows that the difference, in optimal portfolio weights, between the two portfolio types is greater for higher levels of confidence. The reason is that the effect from the negative skewness and the kurtosis from the S&P 500 index becomes more important when looking at the distribution (for the returns) further out in the tails.

Another aspect that can be interesting to investigate is whether the CVaR and VaR differ much between the two portfolio types. VaR for the variance portfolios are given by the α which minimizes (8), and CVaR for these portfolios are given by (10), where \mathbf{x} are the portfolio weights. Since the CVaR portfolios are constructed by minimizing CVaR, it is obvious that CVaR for these portfolios always are going to be less than or equal to CVaR from the variance portfolios. However, when comparing VaR between the two portfolio types, it is not for sure that VaR should be lower for the CVaR portfolios than for the variance portfolios. Nevertheless, there is a strong connection between CVaR and VaR, so one can expect the result to be similar. It is also clear that, as we have mentioned earlier, VaR always are lower than CVaR. Figures (12), (13) and (14) show CVaR and VaR for both portfolio types for the different levels of confidence. The figures confirm that CVaR always are lower for the CVaR portfolios than for the variance portfolios. This observation is more obvious for the higher levels of confidence, even if the difference is not very big anywhere (The curves are almost identical). In average the variance portfolios has 0.023 percent higher CVaR than the 99-CVaR portfolios. The average difference in CVaR between the variance portfolios and the other two CVaR portfolios ($\beta = 0.95$ and $\beta = 0.99$) is slightly less than 1/2 respective 1/4 of this difference. Hence, there seems to be a linear relationship in CVaR for the variance and the β -CVaR portfolios with respect to the confidence level β , at least a local one. When comparing VaR for the two portfolio types, the result is similar. The figures show that

VaR are almost identical for the two portfolio types, but when studying the figures very carefully, you can see that VaR are in general a little bit lower for the CVaR portfolios than for the variance portfolios, especially for higher levels of confidence. But, this is not true everywhere; in some places at the lowest confidence level, you can see that VaR are lower for the variance portfolios than for the CVaR portfolios.

6 Conclusions

Due to risk aversion, risk management is a highlighting issue in finance. The most popular measures for the market risk is the variance and VaR, although they both have some undesirable properties. The variance is only considering the second order moment of a stochastic variable, and not the risk associated with skewness and kurtosis. Therefore, when considering the variance of a security we may underestimate the risk. VaR of a portfolio can sometimes be greater than the sum of the VaRs of the individual instruments in the portfolio. When considering a portfolio with many instruments it can be hard to find the optimal portfolio mixture, which minimizes VaR, since VaR is not a convex function with respect to the portfolio weights. Also, VaR only tells us the minimum loss given an extreme (negative) price movement. These are all undesirable properties belonging to VaR. An alternative and better risk measure than VaR is CVaR. CVaR, which is the expected loss given extraordinary market conditions (or the expected loss given that the loss exceeds some threshold), does not exhibit VaR's undesirable properties.

In this paper we have constructed portfolios, consisting of the S&P 500 index and a seven to ten years T-bond index, by minimizing CVaR and the variance for the portfolios, without allowing short sales. The distribution of the returns is allowed to exhibit the most common properties observed in empirical data, such as fat tails and skewness (see section 3 and 4). Minimizing these two risk measures gives similar portfolio mixtures, but the weight in the S&P 500 index is always lower for the CVaR portfolios than for the variance portfolios, (see figures (8) to (10)). This leads to that the CVaR and VaR for the two portfolio types does not differ that much either, even if the CVaR always are a little bit lower for the CVaR portfolios, (see figures (12) to (14)). These results indicate that, in practise, for those assets, it does not matter that much whether we optimize our portfolios by minimizing CVaR or using the variance criteria when building our portfolios. In either case we will be stuck with a similar risk for the portfolio. A reason for this result is

that both the securities we are considering exhibit both left skewness and fat tails. It is important to be aware of that it is the relative risk between the securities that matters. For example, if we have two instruments with exactly the same skewness and kurtosis, it does not matter if we only consider the mean and the variance, because we are not underestimating the relative risk. In this case, the skewness and kurtosis of our two indexes do not differ that much. A good guess might be that if we add options, which exhibit much more skewness and fatter tails, into the portfolio, the results should differ in favor for the CVaR approach.

Even if two different approaches of minimizing the risk of a portfolio give the same portfolio weights, there are other things you should consider when choosing an approach. CVaR is a convex function of the portfolio weights and is therefore easier to optimize, especially when we are considering many instruments. Another reason for choosing the CVaR approach is that it gives a concrete value of the risk, i.e. the expected value of the loss given an extreme price fall. This is not the case for the variance approach, where we instead might use confidence interval to receive a concrete value of the risk. When assuming that the returns are following an ordinary distribution, like the normal and Student-t distribution, this is often not a problem however, since every statistical software program have built-in functions for this type of calculating. But, when assuming more uncommon (and more complicated) distributions for the returns this may not be the fact. In these cases you have to, by yourself, for example by Monte Carlo simulation, create these confidence intervals, and this might give you some extra problem.

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A Tables

Table 1

95-percent C.I for the estimated parameters for the S&P 500 index

Parameter	Lower interval limit	Estimated value	Upper interval limit
ϕ_0	0.1174	0.2248	0.3322
ϕ_1	-0.1306	-0.0740	-0.0175
α_0	0.0014	0.0677	0.1340
α_1	0.0492	0.0859	0.1226
β	0.8615	0.9021	0.9428
ξ	0.7223	0.7991	0.8758
ν	3.7679	6.6220	9.4761

Table 2

95-percent C.I for the estimated parameters for the T-bond index

Parameter	Lower interval limit	Estimated value	Upper interval limit
ϕ_0	0.1012	0.1537	0.2062
ϕ_1	-0.0823	-0.0231	0.0361
α_0	0.0000	0.0590	0.1246
α_1	0.0185	0.0649	0.1112
β	0.7551	0.8644	0.9738
ξ	0.8891	0.9854	1.0817
ν	5.0066	9.1197	13.2328

Table 3
95-percent C.I for the estimated parameters in the DCC model

Parameter	Lower interval limit	Estimated value	Upper interval limit
<i>a</i>	0.0268	0.0517	0.0765
<i>b</i>	0.8988	0.9332	0.9676

Estimated parameters

Model I: constant correlation i.e. $a = b = 0$.

Model II: the standard DCC model.

$$a = 0.0517, b = 0.9332$$

Table 4

The loglikelihood values

Model	loglikelihood	# parameters
Model I	3384.2	0
Model II	3325.1	2

Table 5

The LR test statistics

TEST	LR	critical χ^2
Model I vs II	118.2	5.9915

B Figures

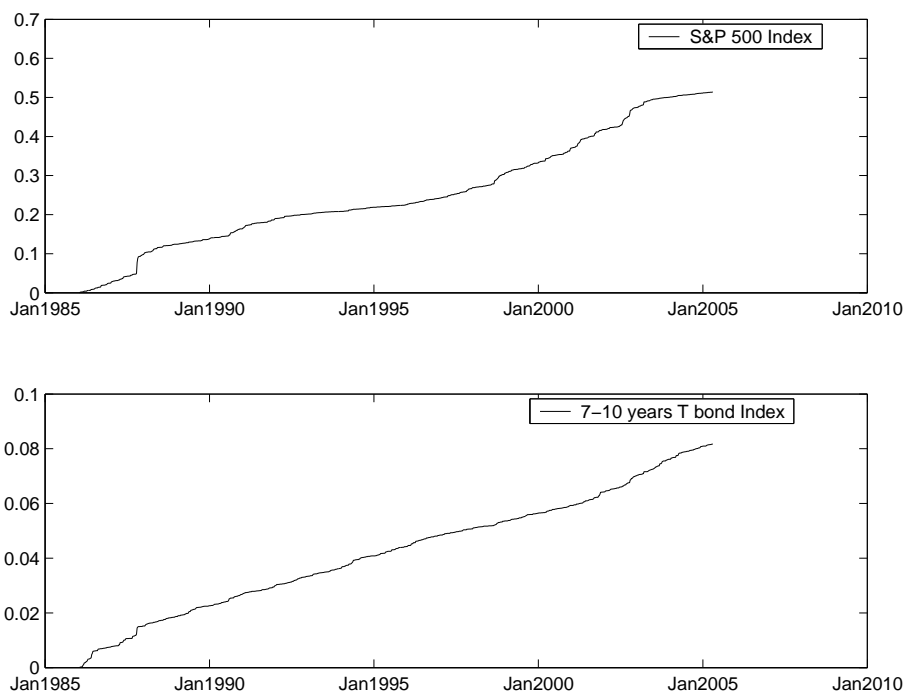


Figure 1: The figure shows how the volatility has developed. A straight line indicates constant volatility. When the slope of the line increases the volatility increases as well. The figure shows some tendencies of an increased volatility during the last years. This tendency is more obvious for the S&P 500 index than for the T-bond index

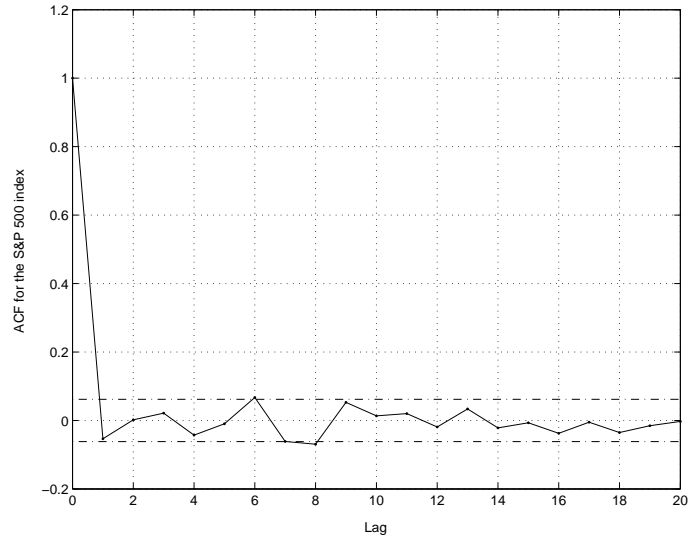


Figure 2: The figure shows the ACF of the S&P 500 index. The horizontal lines is the five percent confidence level.

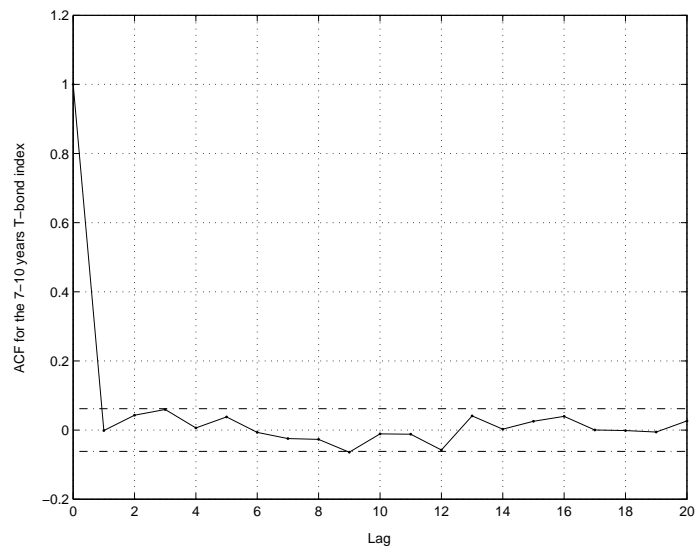


Figure 3: The figure shows the ACF of the T-bond index. The horizontal lines is the five percent confidence level.

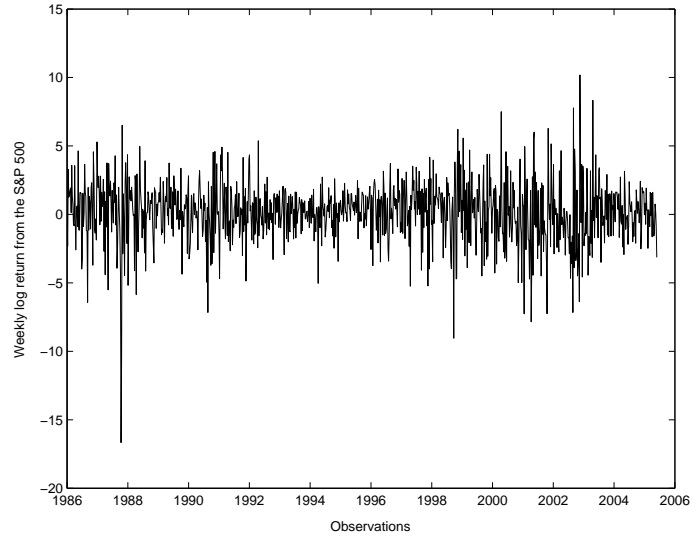


Figure 4: The figure shows the log returns of the S&P 500 index. The figure indicates volatility clustering.

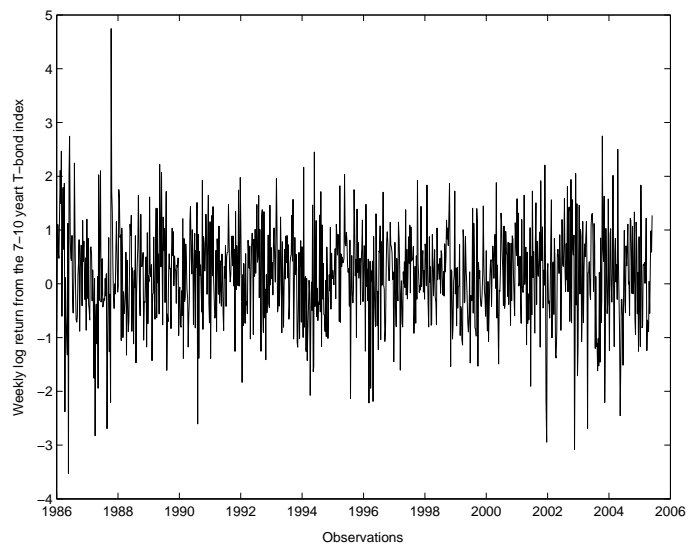


Figure 5: The figure shows the log returns of the T-bond index. The figure indicates volatility clustering.

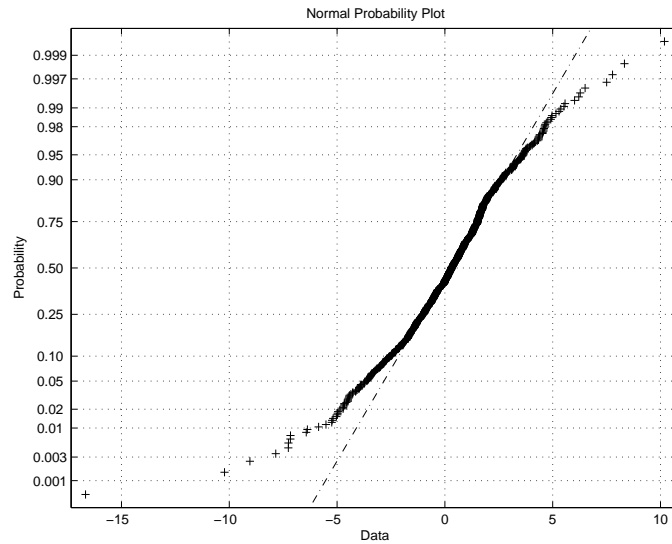


Figure 6: The normplot shows that the S&P 500 index is not normally distributed

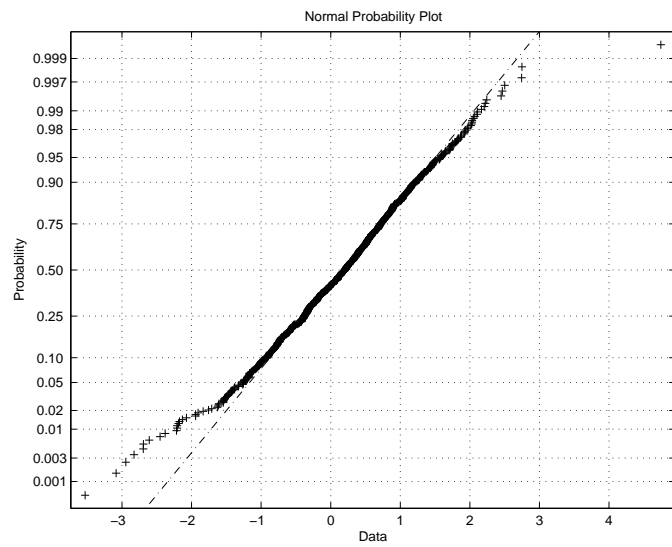


Figure 7: The normplot shows that the T-bond index is not normally distributed

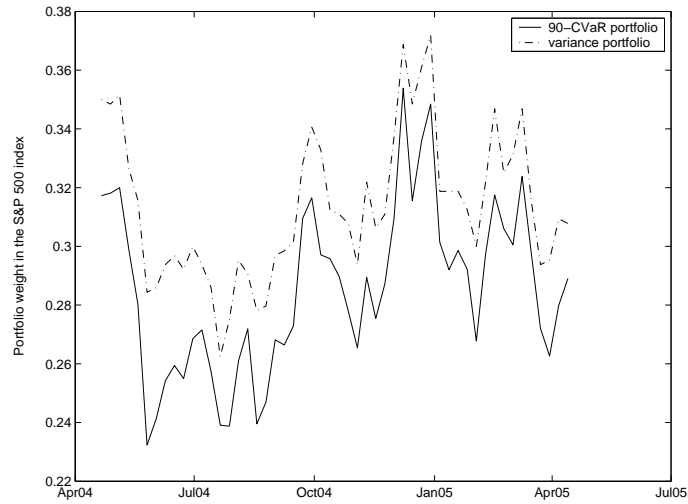


Figure 8: The figure shows the optimal weights in the S&P 500 index for the CVaR portfolios and the variance portfolios for $\beta = 0.9$.

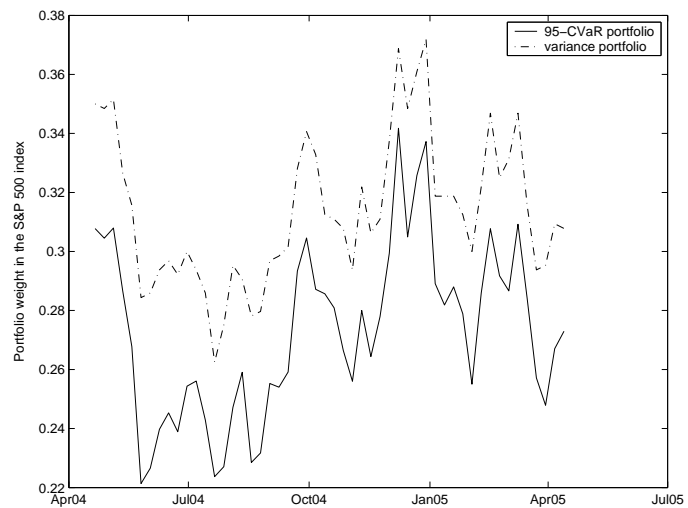


Figure 9: The figure shows the optimal weights in the S&P 500 index for the CVaR portfolios and the variance portfolios for $\beta = 0.95$.

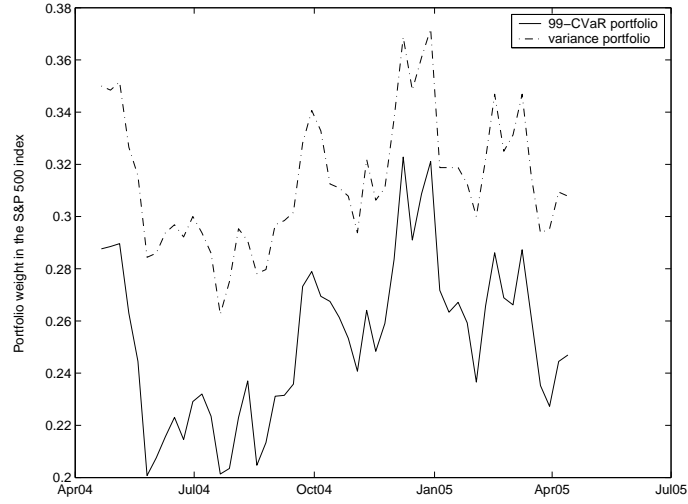


Figure 10: The figure shows the optimal weights in the S&P 500 index for the CVaR portfolios and the variance portfolios for $\beta = 0.99$.

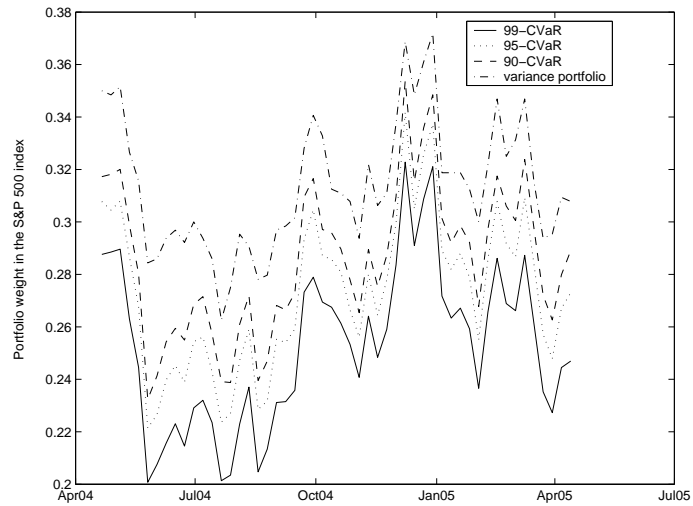


Figure 11: The figure shows the optimal weights in the S&P 500 index for the three different CVaR portfolios and for the variance portfolios.

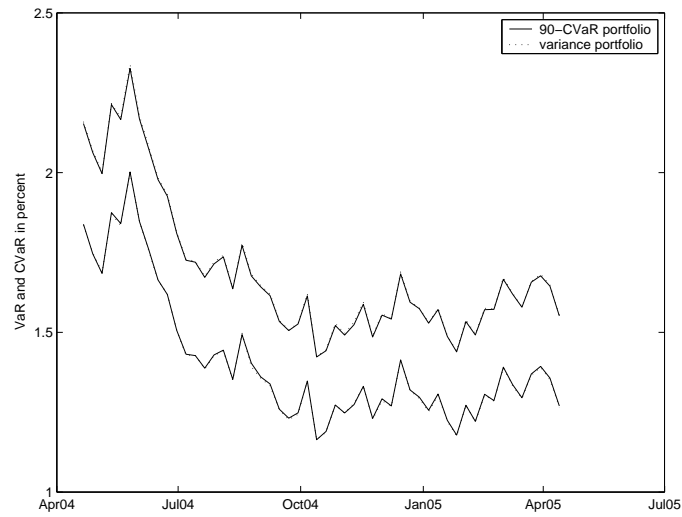


Figure 12: The figure illustrates the differences between CVaR and VaR for the two portfolio types for $\beta = 0.90$. The upper two lines show CVaR for the two portfolio types, and the lower two lines show VaR.

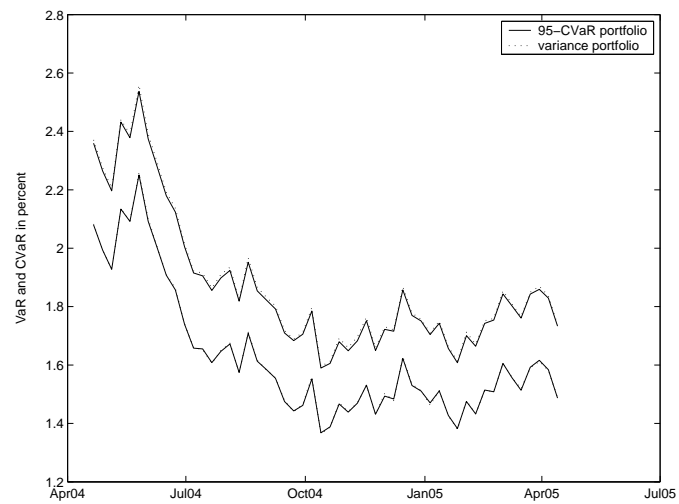


Figure 13: The figure illustrates the differences between CVaR and VaR for the two portfolio types for $\beta = 0.95$. The upper two lines show CVaR for the two portfolio types, and the lower two lines show VaR.

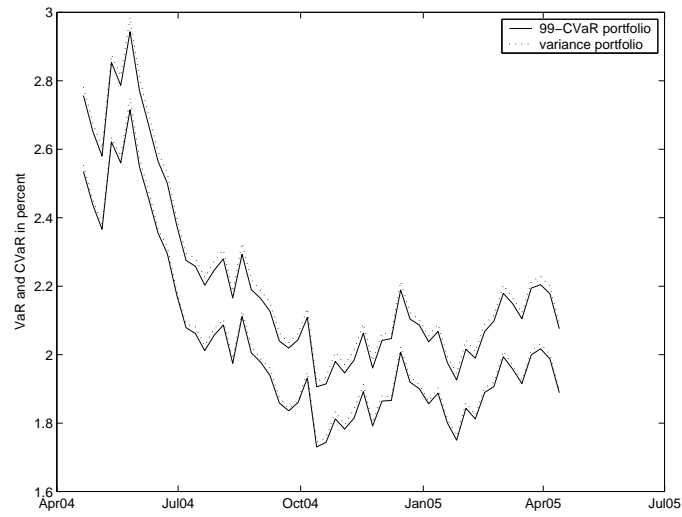


Figure 14: The figure illustrates the differences between CVaR and VaR for the two portfolio types for $\beta = 0.99$. The upper two lines show CVaR for the two portfolio types, and the lower two lines show VaR.