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Mechanism Design In Public Decisions And The Clarke-Groves Mechanism

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Abstract

This is a thesis in the field of mechanism design theory, a field of economic theory closely related to game theory. Instead of determine equilibriums of various games as in game theory, mechanism design tries to design a mechanism with respect to a desirable outcome of the game induced by the mechanism. The thesis can be viewed as an introduction to mechanism design theory in general, via a particular mechanism called the Clarke-Groves mechanism. Though the main aim with the thesis is to explain and discuss the Clarke-Groves mechanism. It is accomplished by using illustrative examples, definitions, theorems and proofs in relation to the Clarke-Groves mechanism.

The Clarke-Groves mechanism is a possible solution to the problem with decisions over public goods. For several settings and situations it is unlikely to achieve an efficient outcome of a decision over a public good. There are problems with externalities and free-riding in almost all contexts where the good decided over is public. Decisions over public goods concerns all agents in the society, since everybody is affected by the outcome of a decision over a public good. There are two delicate advantages with the Clarke-Groves mechanism. Firstly no agent can increase their own utility by misreporting their preferences, truthful report of the preferences is a dominant strategy, and secondly the efficient decision, the decision maximizing joint utility, is picked by the mechanism. Results for a general set of preferences regarding strategyproofness, efficiency and feasibility for the mechanism is presented in the thesis.

A generalization of the mechanism characterizing all strategyproof and efficient mechanisms is stated, these mechanisms are called pivotal mechanisms. What is more a theorem is established about the weigh between strategyproof and efficient mechanisms on the one hand and budget balanced mechanisms on the other hand. For a general set of preferences there exist no strategyproof, efficient and budget balanced mechanism. As a concluding remark of the thesis a discussion on the lack of applications of the Clarke-Groves mechanism is undertaken. While doing this the concept of secure implementation is examined. For a securely implementable mechanism all Nash Equilibriums and dominant strategy coincides. Thus the possibility of agents' ending up in other Nash equilibriums than the dominant strategy of truth telling is removed.

Keywords: Mechanism design, Clarke-Groves mechanism, Public decision, Strategyproof, Efficient, Feasibile.

1 Introduction

Goods can be divided into two classes: public and private goods. Private goods are what we in daily life think of when we talk about goods. In contrast to private goods we have the other type of goods, called public goods. Outside economic sciences they are often not even considered as a good. An example of a public good is the air quality in a city. The air quality affects everybody living in the city, and each level of the air quality leads to their consequences in terms of traffic, pollution from production of other goods, energy consumption etc. Analyzing and constructing economic models for public goods leads to other problems and solutions than with private goods.

Public goods have an interesting connection with externalities. An externality is a real effect in the economy that is not captured by price mechanisms in the market. Both positive and negative externalities exists. A classic example of a negative externality is pollution. Pollution or any type of environmental destruction that is not taken into account by the source of it, can be considered a negative externality. The externality leads in this example to a too high level of pollution. Efficiency gains can be made by lowering pollution. Externalities and public goods can be reasons for inefficient decisions, allocations, provision of goods and so forth. In a formal model these will all be treated as outcomes in the society, who could either be efficient or inefficient. The causes behind this inefficiency are similar for both public goods and externalities, as well as possible solution to it.[1]

For private goods without any significant externalities the market mechanisms are normally enough to achieve an efficient outcome, the outcome here is thought of as a general term for various contexts with private goods. In the case with externalities or public goods a market-failure could occur, resulting in an inefficient outcome. The theory of mechanism design can be used to deal with problems of inefficiency associated with public goods and externalities. Nowadays mechanism design is an important part of modern economic theory, 2007 years prize in economics in memory of Alfred Nobel was awarded to three economists for their contributions to mechanism design.

Mechanism design is closely related to game theory, as the fundamental tools are the same. Game theory has been developed to analyze situations with interactions between agents. The analysis in game theory is focused on finding equilibrium in the games

[1] Firstly a word about the concept efficiency in economics. The standard notion of efficiency in economics is Pareto optimality. A Pareto optimal allocation is one where no changes can be made to improve the welfare of one agent without decreasing it for another agent. Efficiency and Pareto optimality are often used as synonyms for each other.

constructed. An equilibrium in a game is the outcome that is logical consistent with the assumptions made in the model. Another interpretation of the equilibrium in a game is to view it as the predicted outcome for the current game. Mechanism design reverts the order. Instead of first describing the game and then analyzing it, the game is designed after a preferable outcome, hence the name mechanism design.

Let us first clarify what is the difference between a private and public good. Characterizing a good as public or private is a crucial step in the economic analysis, since depending on the characteristics of the good the analysis could end up in completely different conclusions. Which class a good belongs to is determined by the characteristics for the consumption of them. The separation is made upon:[2]
(i) How the consumption opportunity is affected by other agent's use of it.
(ii) The possibility of exclusion from consumption.
The consumption of a good is defined as either rival or non-rival. It is called non-rival when one agent's use of the commodity does not preclude another agent from using it, and rival when the use of it by one agent affects the consumption opportunity of it for another agent. The second aspect is whether it is possible to exclude agents from consuming the good. A public good is characterized as both non-rival and non-excludable in consumption. The consumption of these commodities does not preclude other agents from using it and it is not possible to exclude agents from using the commodity. The opposite is true for a private good, it is characterized as rival and excludable in consumption.

An apple is an example of a private good, when it is consumed no other agent can use it and the exclusion is also fulfilled. On the other hand Knowledge is a good example of public goods. Think of the algorithm for solving a second degree equation. One agent's use of this algorithm does not prevent the use of it from any other agent and there is no feasible way to exclude agents from using it. These characteristics are coherent with the definition of a public good. Before going into the details of mechanism design theory let me introduce what a mechanism is in an informal way. One interpretation of it is to think of a mechanism as the process in a group of people leading to the outcome, another way to see it is as the method used by the group choosing among alternative outcomes. To get better a grasp on what is meant by a mechanism let us look at a couple of examples.

Political leaders for the society can become the ruling regime in various ways. In

[2] The second criteria about exclusion is sometime left out. Exclusion is more a matter of technology and does not change the fundamental role for the goods characteristics in the analysis. Whether there exists a way to prevent agents from using the commodity changes with new technology. Later in this thesis the characterization of a good as non-rival in consumption is the important distinction between them.

democracies elections combined with variants of the majority rule is a common used method. With the majority rule the alternative preferred by the majority of all agents will be elected as the political leader. Here the majority rule is the method to determine the outcome, thus it is a mechanism. All other methods for determine the ruling regime in a society are also mechanisms. Social Choice theory, founded by Nobel Laureate Kenneth J. Arrow with his book Social Choice and Individual Values (1953), asks the fundamental question of how a society should aggregate individuals preferences to a common preference ordering? From a mechanism design point of view the question posed in *Social Choice theory* is a search for a mechanism with certain properties. In the book Arrow shows that a general rule for aggregating preferences in a group does not exist, satisfying some desirable properties. In the language of mechanism design Arrow proves the non-existence of such a mechanism. A related result is the *Gibbard-Satterthwaite theorem.* It states that the only existing non-manipulable voting rule for all possible preferences is dictatorial. These two theorems gives theoretical limits on voting procedures and collective decision in groups.

Until now all examples of mechanisms are from fields of economics not directly associated with markets and business, but mechanisms are also present in more typical business contexts. In every auction there exists rules for bidding and selling, these rules are examples of mechanisms. Two widely used rules for auctions are the *English* and the *Dutch* rules. For both types of auctions the whole procedure is public, i.e. unsealed messages from all agents in the auction. In an English auction the bidders raise each others' bids until only one person remains, who then pays the amount bid and receives the good. In a Dutch auction, the auctioneer calls out the prices in a descending order until somebody takes the offer. The person pays the last price called out by the seller.[3]

A third mechanism for auctions is the *Vickrey* auction named after its inventor the Nobel Laureate William Vickrey. It has the desirable property of being *strategyproof.* In the context of an auction this means that the dominant strategy for all agents is to reveal their true willingness to pay for the commodity sold at the auction.[4] In this thesis I will present the Clarke-Groves mechanism, which is a generalization of Vickrey's auction rules.

[3]Lucking-Reiley, David (2000).

[4]A more precise definition will appear later in the text.

1.1 Questions At The Issue

One of the central question mechanism design theory tries to answer is, if it is possible to create mechanisms that solves these inefficiencies due to externalities and public goods. In mechanism design and modern economic theory in general, the importance of information as a determinant of the outcome has been brought to light. When information is private and possibly valuable for the agents, incentives to keep such information to themselves could arise. An inefficient outcome of a public decision could then be made due to the unwillingness of the agents to reveal their private information. For almost all interesting public decisions an efficient outcome cannot be reached unless the agents true preferences are revealed. Mechanism design is an attempt of finding ways to extract true information from the agents in the society. This information about the preference for the public good is represented by the different *types* among the agents. The importance of correct information can be illustrated by examples. First let us take an illustrative but unrealistic example.

Imagine that there was a benevolent social planner ruling the country; her main objective is to do what is best for society. However what is best for a society is a debatable question. In this thesis as well as in the majority of work in economics, the criteria used is efficiency; the best allocation for the society is the efficient one. Though is still remains to specify what efficiency is, something I will get back to later on. With this aim for the social planner needs to take the right decisions regarding public goods. To make correct decisions', information about people's preferences is required. When she has certain information about their preferences she can weigh them and choose the efficient public decision. Removing the opportunity of having an all-knowing dictator, she has to acquire this knowledge about their preferences. However there are incentives for individuals to manipulate her by not telling the truth about their preferences for the public goods. Hence the information sent and received by her is not necessarily true, and she cannot be sure to make the correct decision if the information extracted from the agents could be false. Although the society is governed by a benevolent social planner, she cannot guarantee a Pareto optimal outcome without first finding a way of revealing the peoples preferences.

The importance of information for efficient decisions is not only an issue in economic theory, for the sake of theory. Here is a variant of a widespread example in societies regarding the crucial role of correct information. In a community voices been raised for building a statue at the main square. The statue is a public good, one agent's use of it does not preclude another agent from using it, and if it is situated in the main square the idea is not to exclude anybody from seeing it. There are two alternatives for the community; build or not build the statue. The Pareto optimal solution is to build the statue if and only if the joint valuation of the good is greater then the cost of it. The cost of the statue is known, but not the valuation of it by each agent. Therefore they decide to question all the agents on how much they would be willing to pay for undertaking the project. If the reported valuation is higher than the cost they will build the statue and finance it through taxes, otherwise the statue will not be built.

The procedure described above for deciding about the statue is a mechanism. If the project is undertaken then the sum of the reported willingness to pay for the statue is greater then its cost. But it is not certain that the decision made is Pareto optimal, because the answers from the agents are not likely to correspond with their true valuation, for the mechanism is easy to manipulate. If you are eager to have this statue built you are likely to exaggerate your willingness to pay for it, since your answer only determines if it will be built or not, and the burden in form of raised taxes is distributed among all agents. Somebody who does not want to build it can exaggerate its valuation in the opposite order, and report a lower willingness to pay for it. Thus, no matter what decision they make it is not at all sure that it is Pareto optimal. Without a mechanism for obtaining true information on the agents valuation of the good an efficient outcome is difficult to achieve. This is the main reason for the focus of strategyproof mechanisms.

A mechanism that induces all the agents to tell the truth is called strategyproof. One among several examples of strategyproof mechanisms is the *Clarke-Groves mechanism*. Another one is the Vickrey auction. The key point with strategyproof mechanisms is that from the agents perspective the best strategy for them is to reveal their true preferences. Instead of only relying on agents being honest when asked about their valuations, the truthfulness is guaranteed through the construction of the mechanism.

1.2 Purpose

The aim for this thesis is to explain thoroughly the Clarke-Groves mechanism by illustrative examples and a clear and comprehensive exposition of the theorems and relevant proofs. Furthermore an introduction to the theory of mechanism design and its central concepts will be presented to give a background for the mechanism in focus.

1.3 Delimitation

This is a theoretical thesis, the goal is to present the theory *per se*. Examples are used to explain concepts and theoretical issues, not for illustrating applications of the theory in the society, although I do want to emphasize that real-world applications exist and many troublesome issues in the society can be analyzed with mechanism design. This thesis focus, however is theoretical and will not go into details of applications of the theory.

1.4 Outline Of The Thesis

This thesis is built up around the Clarke-Groves mechanism, as mentioned the main purpose is to present and discuss the mechanism. In order to understand the mechanism and understand the reasons for taking interest in it a general setting for a public decision process is presented, firstly with an example and secondly a generalization of it given in form of a formal model. This is the content of Chapter Two. Chapter Three contains fundamental parts of mechanism design theory necessary in order to present and discuss the Clarke-Groves mechanism. It also connects mechanism design theory with public decisions processes. In the last section of this chapter the three criterion of efficiency, strategyproofness and feasibility for evaluating a mechanism are defined. In Chapter Four the Clarke-Groves mechanism is presented and it is proved that it meets the three criterion defined in the preceding section. There is also an example illustrating how the mechanism works for a decision over a public good with a cost of c to produce.

Chapter Five is the last chapter before the conclusion. It contains a generalization of the Clarke-Groves mechanism to a broader class of mechanisms called pivotal mechanisms, all mechanisms satisfying the three criterion belongs to the pivotal class. A negative result is then presented on the trade-off between a budget balanced mechanism on the one hand and a strategyproof and efficient mechanism on the other hand. It is stated in the form of a theorem and a complete proof for it is given. As a concluding part of the thesis the important question, on why applications of the Clarke-Groves mechanism or other versions of pivotal mechanisms are almost non-existence in real life situations, is addressed

1.5 Literature And References

For a theoretical oriented thesis the question about footnotes and references for the text is a bit complicated. Since with all already established and proved economic theory there exists several available sources for it, thus it is not important whether person A or person

B has formulated the version employed by me as the author of this thesis. There may exist smaller differences in how definitions, theorem and proofs are stated but the meaning is the same in all of them. Though in order to avoid misunderstanding on what is my words and what is taken from the literature and facilitate further research one should point out from what source the theory presented is built on. In this thesis the main source is the book *Axioms of Cooperative Decision Making* (1988) by Hervé Moulin. It is a graduate textbook using the methods of axiomatic decision theory for various fields of economics. I have foremost used chapter eight, which treats strategyproof mechanisms.

Although my presentation builds heavily on Moulin's, the vast majority of the text is my own words and explanations. As an illustration of the proportions between my words and his let us use the proof of the Clarke-Groves theorem found in Moulin's book. He completes the proof in nine rows, while for me in this thesis it is explained in four pages. Considering the aim of this thesis is to explain the Clarke-Groves mechanism, the focus on explaining the proof is justified.

2 General Setting For A Public Decision Process

Economic theory develops models and then the analysis is done out of these objects. Assumptions are stated in the beginning of the work, when the model is presented. This is also the procedure for mechanism design. But before going into the theory of mechanism design we need a frame and a background for the public decision process. A basic model for the public decision process is presented. All the following discussion and modeling start out from this model. Instead of presenting the model straight away, an example is now introduced. It is constructed to illustrate a typical public decision process. The model and its assumptions are stated a couple of paragraphs down, after the example is presented.

2.1 Basic Example

The following example, later on referred to as the basic example, has a similar role in the thesis as the basic model will have. It is thought of as a general situation for the process of a certain type of public decisions, and it will follow us throughout the whole thesis. The purpose with the basic example is two-folded, first as a motivation to the use of mechanism design in economics and second as a tool to better understand concepts and definitions introduced in the theory. Considering the aim of this thesis is to explain the Clarke-Groves mechanism the latter part is particularly important. Furthermore a reader who is not used with formal mathematical definitions and notations this example is supposed to facilitate the understanding, and it also serves as an argument in favor of this formal style of doing economic theory.

It is now time to introduce the basic example.[5] In order to have a public decision the number of people involved has to be at least two, in this example there are five persons. These five persons constitutes a group. Here there are five members in the group engaged in the decision, but any other number of persons larger then two also constitutes a group. By characterizing this group as a collection of n elements, elementary set theory from mathematics can be used. A collection of n elements is in the language of set theory called a finite set. Sometimes the elements comprising the group concerned with the public decision are not persons. An example are households in a neighborhood considering building a new park, then the elements in the group are households and not primarily persons. Therefore agents are used as a general term for the elements in the group, *i.e.* the finite set is a collection of n agents. This set of n agents is called N. Two different

[5] This example is a modified version of the example found in Moulin (1988) chapter 8 page 202.

sets N_1 and N_2 with n_1 and n_2 agents are only separated by the number of elements they contain.

Likewise a group comprise of at least two persons, a decision requires at least two alternatives to choose between. Thus using the concepts introduced above the collection of all alternatives is a set, and it is named A. This set A is also finite, and the elements are called decisions instead of alternatives. Since it is a decision over a public good the elements of the set are all alternatives for the public good. There are two elements in the set of decisions for the basic example, they are called b and c. One interpretation is to think of them as two alternative locations for a public good, such as a bridge or a public daycare center.

In a context with identical preferences for the public good, the decision process is simple, choose one agent and she can make the decision for the group. However identical preferences is not the general case, it is rather the opposite that is true. When there exist different opinions among the agents on a public good, a decision has to be taken in the group. All agents are affected by the decision, due to the publicness of the good the decision concerns. As a reminder, public goods are characterized by the fact that the use of it by one agent does not preclude another agent from using it. Whether or not all agents participates in the decision process does not matter, the decision taken has an impact on everybody. This is why the determination of the good as private or public is of importance.

All of the five agents in the basic example has different preferences over b and c. Now a decision has to be taken in the society, either b or c. Before proceeding with the example let us think of the more general situation with n agents and their preferences for the various public decisions available in the set A. In a situation were all the agents preferences are known a reasonable thing to do is to sort all agents after their preferences, agents with identical preferences are grouped together and agents with different preferences are separated. These classes are called *types*, where the agents are sorted into. Returning to the example, the types of the five agents in the basic example determines their utility associated with the two outcomes b and c. With other words the utility function has type as one of the two arguments, the second argument will be the chosen public decision. Each agent are of different types in the example, resulting in different utility levels for every entry in Table 1 below. Table 1 is a summary over the basic example, and later when the basic example is used to illustrate or explain a concept I will use the information found in Table 1.

Table 1 states the utilities associated to the two decision b and c for agent 1 to agent 5. Decision b is the efficient decision in our example, because it maximizes joint utility. Joint utility for the two alternatives b and c are found in the last column. Computing joint utility is done through adding together all agents utility numbers for each separate public decision, for every decision joint utility can be calculated. When computing joint utility in the example it is only the difference in utility between b and c that matters. Therefore, without loss of generality, a normalization of the numbers has been done. Later on in the thesis when the Clarke-Groves mechanism is applied to the example it is convenient to have it in the normalized form.

Table 1 Utilities for the two decisions in the basic example.

Decision \ *Agent*	1	2	3	4	5	$\sum_{i=1}^{5} \hat{u}_i(a, v_i)$
b	$+6$	$+9$	-10	-11	$+13$	$+7$
c	-6	-9	$+10$	$+11$	-13	-7

2.2 Basic Model: Assumptions And Notations

It is now time to introduce the basic model. The model is from here on used for analyzing public decisions. When going into the details of Clarke-Groves mechanism, and further refinements of it the basic model maintains as our starting point. An overview and removing possible ambiguities are two of the main advantage with collecting the assumptions, notations and the overall content of the model in one part of the thesis. In section 2.1 I introduced and touched upon all concepts excepts number five and six below. Further discussion on them together with some of the other are hold after this list is presented. Here is the basic model:

1. The society consist of n agents given by the set N, indexed $i = 1, \ldots, n$
2. There exist a finite set A with k elements containing all possible public decisions.
3. There exist a set $V = \langle V_1, V_2, \ldots, V_n, \rangle$, where V_i are all possible types for agent i. Each agent $i \in N$ is characterized by a type $v_i \in V_i$.
4. Each agent has a utility function $\hat{u}_i(a, v_i)$, for every public decision $a \in A$.[6]

[6]For not mixing together the notations used here and later in the thesis the utility function for one decision $a \in A$ is denoted $\hat{u}_i(a, v_i)$ and not only $u_i(a, v_i)$.

5. The net utility for each agent is $\hat{u}_i(a, v_i) + t_i$, where t_i is a monetary transfer.
6. Knowledge about the types are private information, and only the agents themselves knows their own type v_i.

The monetary transfer, presented in number five above, can be used for influencing the agents reported preferences regarding the public decisions. In a general setting the transfer could be of both of positive and negative value. Net utility is a common used concept in economics, here it is defined as the agents utility for a certain public decision a and a monetary transfer of t. More crucial and possible disputable is how the form of preferences are set up through net utility. Assumption five claims that the form of the agents' preferences can be separated between the monetary transfer and the public decision. Agents' preferences are said to be additively separable in the public decision and in money. Furthermore the preferences are linear in money, which is the same as describing them as quasi-linear. This assumption implies a separation between the transfer of money t and the public decision a. Thus the utility function is not affected by the transfer. The transfer only appears as a shift in net utility for the agent, and the utility function remains unaffected.[7]

Number four above states that each agent has its own utility function, this means that the utility from a decision $a \in A$ may differ among the agents. This difference between the agents in utility for the same decision, could be represented, as mentioned earlier, through characterizing the agents by their type v_i. The types $v = (v_1, v_2, \ldots, v_n)$ of the agents determines which outcome is desirable, *i.e.* which outcome maximizes joint utility. If all types were known, the Pareto optimal solution could be reached by maximizing joint utility, conditioning on the types. But the types are private information, it is impossible to condition on the types if we do not have the information about them.

When only the agents know which type they are, the information about the types has to come from the agents themselves. Though there is nothing who guarantees truthful reports from the agents on their type. A rational agent reveals its true type v_i if and only if it is in its own interest, or in game theory terms truthful reporting has to be at least a weakly dominant strategy for guaranteeing truth telling. Without the information about the types the Pareto optimal decision is difficult to reach. One way around this

[7]Although the quasi-linear setting is a common used model for public decisions processes, there are examples of models where general preferences are allowed. Papers using models without an assumption of quasi-linear utility functions have still found interesting results on strategyproofness and other criterion for mechanisms. Examples of such research is found in the papers by: Demange and Gale (1985), Sun and Yang (2003) and Andersson and Svensson (2008).

problem is to construct a mechanism that reveals the agents types'. This is a general setting for topics discussed in mechanism design. First, there exist a set A of possible decisions and the desirable one depends on their types v, but these are private information. Thus a mechanism is designed to reveal the types of the agents, based on these messages $m = (m_1, m_2, \ldots, m_n)$ the decision is made.

3 Mechanism Design In A Public Decision Context

This chapter of the thesis aims for describing the fundamental parts of mechanism design, necessary for analyzing public decisions. There are several definitions stated in this part, where the conceptual understanding of them is the focus. In the first section 3.1 some of the elementary parts of mechanism design theory is presented. After that in 3.2 the ideas presented so far are connected to the public decision and more specifically to the basic model. Finally three desirable criterion for evaluating a mechanism are defined in section 3.3.

3.1 Fundamental Concepts In Mechanism Design Theory

Hitherto I have given several examples of what a mechanism could be, the idea has been to introduce the concept of a mechanism through well-known examples of them. Examples of mechanisms will also appear in the remaining part of this thesis. Though for doing a rigorous analysis it is necessary to define formally what is meant by a mechanism, examples and explaining words are not enough.

Definition 3.1.1 *A mechanism is an object $\langle M_1, M_2, \ldots, M_n, f(m)\rangle$ where, M_i is the set of all possible messages from agent i. The object also contains a rule that assigns a decision to each possible configuration of reported messages, this is the function $f(m)$.*

This formal definition translates the concept of a mechanism into mathematical language, and models can be built around it. It also clarifies what is meant by a mechanism. Several examples of well-known mechanisms were given before, one of them are the rules for an auction. Another example of a mechanism, not mentioned before, are the various systems on how the rents are set for apartments and on what criteria tenants are chosen. The two extremes are pure market rents and a total rent-control. Depending on the mechanism, here constituted as the rules for determine rents, the outcome will change.

Together with the definitions above of the model the mechanism “induces” a game for the public decision process. Analyzing the constructed mechanisms can be performed with the aid of game theoretical tools. As with all games in game theory the players has well defined strategies and payoff functions. For the current game they are:

1. A strategy for agent i is to pick a message $m_i(v_i) \in M_i$ for every $i \in N$.
2. The payoff function is given by the players utility functions $\hat{u}_i(f(m), v_i)$, where $v_i \in V_i$ is agent $i's$ type and $f(m)$ produces a decision $a \in A$.

Although the model can be interpreted in a game theory environment, one big obstacle remains regarding the set of mechanisms. In the definition of a mechanism there is no restriction on neither the function nor the set of messages, thus all complete and non-contradictory rules for deciding among several alternatives in a group are covered by this definition of a mechanism. Using this definition in a model in order to find the best possible mechanism leads to complications, the first problem is that the set of all possible mechanisms is unbounded. Finding the best mechanism from an unbounded set is an overwhelming task. Fortunately, there exist a smaller class of mechanisms that are sufficient for most purposes. They are called *direct mechanisms.*

Definition 3.1.2 *A direct mechanism is a mechanism* $\langle M_1, M_2, \ldots, M_n, f(m)\rangle$*, where* $M_i = V_i$ *for all* $i \in N$

In the basic example a direct mechanism implies that the agents report their utility for the two scenarios b and c, a restriction on the domain of answers for the agents is the consequence imposed by using direct mechanisms. All other conceivable mechanism used for acquiring knowledge about the agents preferences for the two possible decision are taken out of the picture. Though a restriction is done in the example, the agents are still able to misreport their utility level for b and c. Worth pointing out is that direct mechanism does not mean that all agents reports their type truthfully, it only restricts the form of the mechanism.

A direct mechanism is one where all agents sends their type (or preferences) as the message. This is an obvious restriction of the mechanism. It is done because we want to reduce the set of all conceivable mechanisms, to the smaller set of direct mechanisms. In a general setting we would like to think of all possible mechanisms, who are normally more complicated then the direct mechanisms are. In order to analyze these we would have to specify their characteristics and understand how they work. As pointed out above, only to specify how they work and construct a mathematical model for each distinct mechanism takes a lot of effort and cannot be done in any straight forward way.

The restriction to direct mechanisms is an example of how good modeling works in economic theory. A complex reality needs to be simplified to make a model that is workable and logical consistent. Assumptions are done through balancing the models correspondence to reality and the possibility to use it for an analysis. There is no doubt on that direct mechanisms makes the models and the solution concepts easier. Regarding the first point on whether the assumptions in the model significantly limits the correspondence to the reality, one could raise the objection that models for public decision based on direct mechanisms are not of much value, since almost all mechanisms for public decisions are not at all direct mechanisms. It looks like we solved one problem and created another one.

But this is not the case, direct mechanisms incorporates almost all mechanisms that are of our interest. Hence the balance between reality and the assumptions in the model is maintained. The reason for this is that no matter how complicated the mechanisms are in real life situations we can reduce it to the direct mechanism scenario. And the same equilibrium with the more complicated mechanism can be achieved as with the direct mechanisms. This conclusion follows from the *Revelation Principle.* The Revelation Principle tells us that, if an allocation can be supported as an equilibrium in an arbitrary game form, then a direct-revelation game can be constructed and the same allocation will be an equilibrium under the new form. Thus, limiting the analysis to direct mechanism does for most situations not lead to a loss of generality in the conclusions drawn from the model. [8]

3.2 Specific Definitions Related To The Public Decision

Mechanism design theory aims for constructing mechanisms such that a given outcome can be supported as an equilibrium. In order to apply mechanism design theory to a public decision process a definition of what is meant by an outcome is needed.[9]

Definition 3.2.1 *An outcome is a pair (a,t) where $a \in A$ is a public and costless decision, and $t = (t_1, t_2, \ldots, t_n)$ is a vector of monetary transfers. Agents i's preferences are described by a vector $u_i \in \mathbb{R}^A$ such that his net utility for an outcome (a,t) is $\hat{u}_i(a, v_i) + t_i$.*

For the basic example found in section 2.2 on page 12 an outcome (a, t) could be any of the two public decisions b and c together with an arbitrary vector $t = (t_1, t_2, t_3, t_4, t_5)$ of monetary transfers. The vector t has five entries because there are five agents in the basic example, thus the length of the vector t is given by n the number of agents.

Since we already assumed the existence of a utility function for each agent, her preferences for every public decision a can be represented by a numerical value $\hat{u}_i(a, v_i) \in \mathbb{R}$.

[8]This exposition of the principle is kept informal, I will use the principle but not go into the details of it. The details and subtleties can be found in the work of last years Nobel Laureate Roger Myerson. He was primarily awarded for his work with the revelation principle, an important part of mechanism design theory.

[9]The remaining definitions in this chapter concerns mechanisms in a public decision and follows the set up used in Moulin (1988), chapter 8 page 203-205.

A vector $u_i = (\hat{u}_i(a_1, v_i), \hat{u}_i(a_2, v_i), \ldots, \hat{u}_i(a_k, v_i))$ for agent i is then constructed out of all the utility values for each public decision.[10] In the basic example agent $1's$ preferences u_1 is a vector with dimension two, because there are two public decisions, and $u_1 = (+8, -8)$. The length of the vector u_i is determined by k, the number of possible decisions, which is equal to the number of elements in set A containing all possible public decisions. Describing agent $i's$ preferences with $u_i \in \mathbb{R}^A$ is a more concise way to write this. $\mathbb{R}^A$ is the Cartesian product of $\mathbb{R}$ k times, hence $\mathbb{R}^A = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$. As in the previous definition of the utility function, it has two arguments: one is the public decision and the second is the type for agent i. Similarly is the net utility for agent i defined as before.

In the discussion before on public versus private goods, the distinction between them was made on how one agents use of the commodity affected another agents possibility to use it. For a public good one agents use of it does not preclude another agent to use it. A decision is public when the decision concerns alternatives for public goods, whether all agents are involved in the decision procedure is irrelevant for the sake of its publicness. Therefore all agents in the group are affected by the decision, whether or not they were part of the decision process. This is also the rationale for the definition of an efficient decision as the decision where joint utility is maximized. I will get back to this in section 3.3.

A mechanism was defined in Definition 3.1.1 on page 15 for the general case, with almost no restrictions at all on the form of it. Then direct mechanisms were introduced in Definition 3.1.2 on page 16. Limiting the analysis to the smaller class of direct mechanisms is not a problem for most situations of our interest, due to the Revelation Principle explained above. Interpreting all these limitations of the mechanism in the context of a public decision process and adding the requirement of injectivity to the function gives us the following definition.

Definition 3.2.2 *Given (N, A) both finite sets, there exist a function f(u) in the mechanism that associates each utility profile $u = (u_1, u_2, \ldots, u_n)$ to an outcome $(a(u), t(u))$, where $u_i \in \mathbb{R}^A$*

For the general mechanism defined in Definition 3.1.1 the function $f(\cdot)$ has m as its argument, where $m = (m_1, m_2, \ldots, m_n)$ is the reported messages from all agents. Because

[10] The utility function u_i could also be seen as a vector valued function, with m arguments one for each possible decision. This function returns m utility values, one for each decision. The important part to understand is that u_i contains a utility value for each possible decision. Whether the function is defined as vector valued or not does not make any significant difference. And to avoid unnecessary confusion the vector valued function is not introduced.

of the Revelation Principle the utility profile u can be used as the argument for the function $f(\cdot)$, without loss of generality. The function $f(u)$ is nothing more then a rule that connects a utility profile u with an decision a and a transfer t. This is the same as saying the mapping from the utility profile u to the outcome $(a(u), t(u))$ is called function $f(u)$.

Let us look closer at what is meant by a utility profile u. A utility profile u contains all the information needed about the agents preferences to make the efficient decision, given that the profile is true. In our setting the utility profile can be thought of as a matrix with n rows and k columns. Where n is the number of agents and k is the number of available decisions. Row i is the vector u_i, the same u_i as in Definition 3.2.1 on page 17, describing agent $i's$ utility for each of the k possible public decisions. The matrix is constituted by n such rows, where each row is the utility vector u_i for agent i. Using the same notation as when discussing u_i after Definition 3.2.1, the utility profile u can be expressed as:

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} \hat{u}_1(a_1, v_1) & \hat{u}_1(a_2, v_1) & \dots & \hat{u}_1(a_k, v_1) \\ \hat{u}_2(a_1, v_2) & \hat{u}_2(a_2, v_2) & \dots & \hat{u}_2(a_k, v_2) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{u}_n(a_1, v_n) & \hat{u}_n(a_2, v_n) & \dots & \hat{u}_n(a_k, v_n) \end{pmatrix}$$

Using the basic example we obtain an illustration of a utility profile. There are five agents and two public decisions in the basic example. Hence the matrix u, the utility profile for the basic example, is a 5×2 matrix. The first column in u are the utilities for decision b for the five agents, and similarly is the second column the utilities for decision c.

$$u = \begin{pmatrix} +8 & -8 \\ +9 & -9 \\ -10 & +10 \\ -12 & +12 \\ +11 & -11 \end{pmatrix}$$

The next part of the preparation before presenting the Clarke-Groves Mechanism is to formalize, define and explain concepts such as *efficiency*, *feasibility* and *strategyproofness.*

3.3 Three Criterion For Mechanism Design In A Public Decision

Efficiency is a benchmark used in almost all economic models, and it is mainly interpreted in terms of Pareto optimality. The central role for efficiency in economic science may be explained by the close relation between the essence of economics and the idea of efficiency. A widely used definition of economics is: economics studies the allocation of scarce resources among people. When studying allocations the concept of efficiency is useful as a tool for the analysis, everything equal an efficient allocation is better then an inefficient allocation. Though in a proper analysis it is necessary to define what is meant with efficiency, and this is generally done through defining efficiency in the accurate model used for the issue. Thus, this definition of efficiency is chosen to be accurate with respect to the basic model of a public decision process.

Defintion 3.3.1 *The mechanism is **efficient** if it selects an efficient public decision at all utility profiles. This means that a(u) is an efficient public decision if* [11]*:*

$$\sum_{i=1}^{n} \hat{u}_i(a) = \max_{b \in A} \left\{ \sum_{i=1}^{n} \hat{u}_i(b) \right\} \tag{1}$$

With this definition of efficiency decision b is the efficient one among b and c. Because joint utility is $+7$ for b and -7 for c, the numbers are found in the last column of Table 1 on page 12. From here on when a mechanism is characterized as efficient or not, the meaning of it is in Definition 3.3.1 above. Even though there exist an efficient decision, another decision might as well be chosen. A group of rational agents acts on their own behalf, all of them together are not likely to act such that they maximize joint utility, which would be the efficient strategy for the group as a whole. Therefore two more definitions characterizing the mechanism are introduced, the first one treats the problem of agents acting on their own behalf and thereby misreporting their preferences, which results in an inefficient decision. To guarantee an efficient decision the mechanism must have truth telling as a weakly dominant strategy. In other words telling the truth has to be the best strategy for the agents. This property of the mechanism is defined below.

Definition 3.3.2 *The mechanism is **strategyproof** if for each utility profile u, each agent $i \in N$ and each utility function $w_i \in \mathbb{R}^A$, agent i cannot benefit from reporting w_i whenever her true utility function is u_i:*

$$\hat{u}_i(a(u)) + t(u) \geq \hat{u}_i(a(w_i, u_{-i})) + t(w_i, u_{-i}) \tag{2}$$

[11] In order to keep the functions readable and not too messy, form here on the argument of the types v_i is omitted from the utility functions. The type v_i still determines the utility function for agent i, it is only left our for convenience reasons and to simplify the notations.

In Definition 3.3.2 the crucial part is to understand the meaning of equation (2) with the inequality. This inequality is a concise expression for the preceding discussion before the definition. The correct interpretation of the definition is: For every agent i in the group there exist no misrepresentation w_i of agent $i's$ utility for the public decisions, where the agent is made better of then when reporting her true utility u_i. To clarify the notations, a word about u_{-i} should be said. This u_{-i} is a utility profile, with no values on row i for agent i. A complete utility profile u needs to incorporate each agents preferences, thus for completing u_{-i} to a proper utility profile u agent $i's$ preferences u_i is added. Throughout the remaining part of this thesis when this notation with u_{-i} is used it has this interpretation. As an illustration of this notation u_{-i} the basic example is used. Here is the utility profile for the basic example without agent $3's$ utility:

$$u_{-3} = \begin{pmatrix} +8 & -8 \\ +9 & -9 \\ -12 & +12 \\ +11 & -11 \end{pmatrix}$$

Why does this concept of strategyproofness play a central role in Mechanism Design? Say that the mechanism not is strategyproof, then the agents might as well still tell the truth for other reasons, such as honesty. This does not diminish the importance of having a strategyproof mechanism. First of all relying on the agents telling the truth, because they are honest is not a solid ground for achieving overall efficient public decisions. And the personal incentives for misrepresent your preferences increases when the possible benefits for it grows. Secondly, those agents telling the truth even without strategyproof mechanisms continues to tell the truth under the strategyproof mechanism. [12]

Although the mechanism is strategyproof and efficient, if it leads to a budget deficit then a source to finance it has to be found before the mechanism can be implemented.

[12] There exist a refinement of the criterion of strategyproofness. For a mechanism with truth telling as a dominant strategy nobody can improve their utility by misreporting their preferences. Though if a group of agents cooperates and misreports their preferences in a coordinated manner, it might be possible to manipulate the mechanism and increase your own utility. This fact has led to an extension of the criteria strategyproofness for mechanisms. When no group of agents can improve their utility by misreporting their preferences than the mechanism is coalitionally strategyproof, see Demange and Gale (1985) and Andersson and Svensson (2008). I will not discuss this extension of the criterion further, the idea was to point out that it exist and it is one part of a growing research in mechanism design.

With a deficit imposed by the mechanism, the mechanism has to be financed in one or another way, using existing taxes or increasing them are the two most natural solutions. But none of them is a solution without drawbacks. Taking money from existing tax revenues, means less expenditures for the rest of the public sector. Decreasing the funds for schools, social security, research and so forth because of a deficit due to the mechanism is a policy tough to motivate. Increasing the taxes for funding a deficit of a mechanism is also solution with its own problems to argue for. The best way out of this problem would be to avoid the deficit in the first round through a clever construction of the mechanism, and then there is no deficit at all. When a mechanism impose no deficit it is called *feasible*. This is the last definition out of three for characterizing a mechanism.

Definition 3.3.3 *The mechanism is* ***feasible*** *if t(u) exhibits no deficit at any utility profile u. Thus, for all u:*

$$\sum_{i=0}^{n} t_i(u) \leq 0 \tag{3}$$

The sum $\sum_{i=0}^{n} t_i(u)$ is a summation of all the transfers from the agents. For a mechanism with a positive sum of transfer, the amount of negative transfers is smaller then the positive ones. In this scenario the mechanism leads to a budget deficit, since the money paid from the agents with negative transfer does not cover the money receive to those agents with positive transfers. Before the mechanism can be used, the deficit needs a solution. Money from some other source has to be taken to cover up for the deficit the mechanism imposed, and the same critique used earlier in the discussion can be raised. A better scenario is when the sum of transfer is smaller then or equal to zero, then no deficit occur and the mechanism is ready to be implemented.

4 Clark-Groves Mechanism

In this chapter the *Clarke-Groves* mechanism is presented and discussed thoroughly. First of all there is a presentation of the mechanism, then a proof of the Clarke-Groves theorem is given. After that a longer example of the mechanism with an indivisible public good is found. The purpose with this example is to further develop the understanding and provide an possible application of the mechanism.

4.1 Clarke-Groves Theorem

The Clarke-Groves mechanism is sometimes referred to as the pivotal mechanism. This name for it points out the key idea behind the mechanism. Recall the goal is to construct a mechanism that fulfills each of the three characteristics: efficiency, strategyproofness and feasibility as they were defined in the previous section 3.3. A monetary transfer imposed on each agent is the tool used by the mechanism to achieve this. In fact the Clarke-Groves mechanism manage to meet all the three desirable criterion for a mechanism in a public decision. It does this by taxing some of the winners, namely those agents who prefer the efficient decision to all the other decision so much that if they were removed from the group another decision would be the efficient one. Though, the revenues from the winners are not redistributed to the losers, and this is the mechanisms principal weakness that will be discussed extensively in the last chapter of this thesis.

Let us return to the basic example from page 12 to illustrate how the transfer can induce the agents to pick the efficient decision. As established earlier b is the efficient public decision in the basic example, because joint utility is maximized with b. In order to influence the agents to reveal their true preferences, only the agents who's existence in the group changes the efficient decision from b to c are taxed. All the other agents are not taxed at all and get a zero transfer. Those agents who prefer b over c to such an extent that if they were removed from the group the decision would change are called *pivotal*, since they are decisive in the public decision.[13]

Definition 4.1 *An agent i is pivotal if the efficient decision for the group of agents $N \setminus \{i\}$ differs from the efficient decision in N, the group of all agents.*

In the Definition above I used the new notation $N \setminus \{i\}$, this produces a new set consisting of all agents in N except agent i. The operation $\setminus$ is the counterpart of subtraction with

[13] My presentation of the Clarke-Groves mechanism is built on Moulin (1988), chapter 8 pages 201-205.

sets instead of numbers, with one natural difference. Subtracting the elements in the set Δ from the set Γ requires that all elements in Δ also exist in Γ, otherwise it is not a well defined operation.

There are two pivotal agents in the basic example, agent 2 and 5. Given the definition for a pivotal agent a quick check in Table 1 on page 12 and comparing the joint utility for b and c in the two sets $\{1,3,4,5\}$ and $\{1,2,3,4\}$ one finds that the efficient decision is c. In all of the three other coalitions of agents $N \setminus \{i\}$ for $i = 1,3,4$, decision b remains the efficient decision. Hence, agent 2 and 5 are the only two pivotal agents in the basic example. All non-pivotal agents has a zero transfer in the Clarke-Groves mechanism, in the basic example agent 1, 3 and 4 are non-pivotal and are therefore not taxed.

The tax paid by a pivotal agent equals the loss in utility incurred by the other agents $N \setminus \{i\}$ due to picking the efficient decision b in favor of c, where c is the most preferred decision in the group $N \setminus \{i\}$. Let us calculate the tax t_i' for the two pivotal agents 2 and 5. For agent 5 the tax t_5' is the loss in joint utility for the group {1,2,3,4} when switching from decision c to b:[14]

$$t_5' = (-6 - 9 + 10 + 11) - (6 + 9 - 10 - 11) = 12$$

Similarly for agent 2, her tax t_2' is calculated as the loss in joint utility for the group $\{1,3,4,5\}$ when switching from decision c to b:

$$t_2' = (-6 + 10 + 11 - 13) - (6 - 10 - 11 + 13) = 4$$

If the Clarke-Groves mechanism is strategyproof, then truth telling is a dominant strategy for all agents. To see this consider agent 2, one of the two pivotal agents, and her alternatives. Exaggerating her valuation for the efficient decision b does not change anything, her tax stays at 4 and the decision remains at b. Her other possibility is to lower her report for decision b such that the efficient decision changes from b to c. This strategy would cancel her tax, but at the same time imply a utility loss of 18 when changing from decision b to c. A canceled tax of 4 is eaten up by the utility loss of 18. For the pivotal agent 2 her dominant strategy is to report her true preferences for the two public decisions b and c. Reasoning in the same manner regarding agent 5, the other pivotal agent, she can only change the decision and cancel her tax if she decreases her reported valuation for decision b. Such a strategy would cancel her tax of 12 and lead to a loss of utility of 26 when going

[14] I use the notation t_i' for pointing out that this is a tax, and not the transfer t_i. When the transfer t_i is defined for the Clarke-Groves mechanism, the pivotal agents tax is then treated as a transfer and is therefore of negative value.

from decision b to c. For both of the tax paying pivotal agents, any strategy where the efficient decision remains at b will lead to no change at all. It is only a strategy making decision c instead of b look like the efficient one, that changes the outcome to decision c and cancels their tax. Though, the loss of utility for the pivotal agents rules out the gain in income from the canceled tax.

Agents 1, 3 and 4 are the losers in the public decision process, their preferred alternative c is not the efficient one. Any strategy for one of the non-pivotal agents where it looks like c instead of b is the efficient decision changes the outcome. Suppose it is agent 3 who considers this strategy where she exaggerates her dislike for decision b. Undertaking this strategy means an increased utility level of 20 when the decision goes from b to c. Then at the same agent 3 is pivotal and therefore has to pay a tax equal to the exact loss of utility for the agents $\{1, 2, 4, 5\}$ when switching from decision b to c:

$$t'_3 = (6 + 9 - 11 + 13) - (-6 - 9 + 11 - 13) = 34$$

With a tax at 34 and the gain of utility amounting to 20 is not enough to motivate this strategy for agent 3. Doing the same calculations for agent 1 and 4 gives the same conclusion. The benefits for switching from decision b to c are offset by the tax imposed on the new pivotal agent. Hence, truth-telling is also a dominant strategy for agent 1, 3 and 4 and we already showed that for agent 2 and 5. Since all the agents has truth telling as a dominant strategy the Clarke-Groves mechanism is strategyproof in the basic example. Furthermore it picks the efficient decision b and it is feasible, with a sum of the transfers from the pivotal agents 2 and 5 of -16. This surplus of 16 is not redistributed to the losing agents 1, 3 and 4.

For establishing a general result referring to one example where the Clarke-Groves mechanism satisfies the three criterion of strategyproofness, efficiency and feasibility is not enough. It is necessary to formalize the content of the mechanism and prove that it satisfies the three criterion for a general setting.

theorem 1 ***(Clark-Groves Theorem)***
A Clark-Groves mechanism, also called a pivotal mechanism, is a mechanism such that for every possible utility profile it produces a public decision a(u). This decision is efficient, feasible and strategyproof. The tax imposed on the pivotal agents is given by[15]*:*

$$t_i(u) = \sum_{j \neq i}^{n} \hat{u}_j(a(u)) - \max_{b \in A} \left\{ \sum_{j \neq i}^{n} \hat{u}_j(b) \right\} = \hat{u}_{N \setminus i}(a(u)) - \max_{b \in A} \hat{u}_{N \setminus i}(b) \tag{4}$$

[15] As a concise way of writing $\sum_i^n u(\cdot)$ I sometimes use the notation $u_N(\cdot)$, similarly $u_{N\setminus i}(\cdot)$ can be written instead of $\sum_{j \neq i}^n u(\cdot)$.

In the preceding example the transfer for the pivotal agents was perceived as a tax and the order were opposite for the terms in equation (4). By that trick the transfer turned into a tax and was always of positive value. The order of the terms in (4) used in theorem 1 is preferable for a general definition of a transfer. Defined in this manner we are not obliged to treat the transfer as a tax, and do not have to keep in mind whether it is the tax or the transfer we are speaking of. With the definition of the transfer $t_i(u)$ found in (4) it is also easier to express the utility for each agent when several efficient decisions coexists in the same set A, of available public decisions. A multiple of efficient decisions is not unlikely to occur. When several efficient public decisions coexists at a given utility profile then all the efficient outcomes can be chosen. All of them will yield the same final utility for every agent:

$$\begin{aligned} S_i^*(u) &= \hat{u}_i(a(u)) + t_i(u) = \hat{u}_i(a(u)) + \sum_{j\neq i}^{n} \hat{u}_j(a(u)) - \max_{b\in A}\left\{\sum_{j\neq i}^{n} \hat{u}_j(b)\right\} \\ &= \max_{a\in A} \hat{u}_N(a) - \max_{b\in A} \hat{u}_{N\setminus i}(b) \end{aligned}$$

Proof of theorem 1:

This proof aims to establish that the Clarke-Groves mechanism satisfies the three criterion of efficiency, strategyproofness and feasibility. The first part of the proof concerns the efficiency criteria. In definition 3.1.1 of efficiency on page 15 an efficient decision a maximizes joint utility, this condition is expressed mathematically in equation (1). Thus, our goal is to demonstrate that the mechanism always picks the efficient decision and thereby meets equation (1).

Let us start with the trivial case where there are no pivotal agents at all in N over the public decision A. If there are no pivotal agents then according to the definition of a pivotal agent, the efficient decision a remains no matter which one of the agents are removed. Moreover there are no incentives for the agents to deviate, since nobody of the agents pays any tax. For the scenario with not one single pivotal agent the Clarke-Groves mechanism meets the efficiency criterion, stated in equation (1). Returning now to the more realistic case with one or several pivotal agents in N, and picking one of these pivotal agents i. Her transfer is given by equation (4) in theorem 1:

$$t_i(u) = \sum_{j\neq i}^{n} \hat{u}_j(a(u)) - \max_{b\in A}\left\{\sum_{j\neq i}^{n} \hat{u}_j(b)\right\} = \hat{u}_{N\setminus i}(a(u)) - \max_{b\in A} \hat{u}_{N\setminus i}(b)$$

Note that $a(u)$ maps a public decision for every argument u, given an argument u the output of the function $a(u) \in A$. Consider again the pivotal agent i (it is nothing particular with agent i the argument holds for any of the pivotal agents j) and let us add her utility $\hat{u}_i(a(u))$ for the decision $a(u) \in A$ with the transfer $t_i(u)$:

$$\begin{aligned} \hat{u}_i(a(u)) + t_i(u) &= \hat{u}_i(a(u)) + u_{N\setminus i}(a(u)) - \max_{b \in A\setminus\{a(u)\}} \hat{u}_N(b) \\ &= \hat{u}_N(a(u)) - \max_{b \in A\setminus\{a(u)\}} \hat{u}_N(b) \geq 0 \end{aligned} \tag{$\star$}$$

Why the inequality in equation ($\star$) is true follows from the definition of a pivotal agent. Because the Definition 4.1 on page 23 of a pivotal agent implies that the decision $a(u)$ maximizes joint utility $\hat{u}_N(\cdot)$ for the N agents over the set A of all decisions. What is more for all public decisions $a \in A$ and $b \in A$, and as remarked earlier $a(u) \in A$, we have:

$$\max_{b \in A} \hat{u}_N(b) \geq \hat{u}_N(a(u)) \iff \max_{b \in A} \hat{u}_N(b) - \hat{u}_N(a(u)) \geq 0 \tag{$\star\star$}$$

Putting the two conditions together, given by the two equations ($\star$) and ($\star\star$):

$$\begin{aligned} &\max_{b \in A} \hat{u}_N(b) \geq \hat{u}_N(a(u)) \geq \max_{c \in A\setminus\{a(u)\}} \hat{u}_N(c) \\ &\Longrightarrow b = a(u) \quad \Longrightarrow \max_{b \in A} \hat{u}_N(b) = \hat{u}_N(a(u)) \end{aligned}$$

Hereby the first part of the proof is completed. Thus the Clarke-Groves mechanism satisfies the efficiency criteria. The next part of the proof of theorem 1 demonstrates why the strategyproof criteria holds for the Clarke-Groves mechanism. Speaking of truth telling as a dominant strategy is only another way of expressing the definition 3.3.2 of strategyproofness with the language of game theory. A dominant strategy for agent i in game theory is the best possible strategy for agent i, no matter what actions the other agents undertake. Assuming a utility function $v_i(s)$ exist for the accurate situation, where s is a vector with all agents strategies. Using the utility function $v_i(s)$, the condition for a dominant strategy s_i^* for agent i is:[16]

$$v_i(s_{-i}, s_i^*) \geq v_i(s_{-i}, s_i) \qquad \text{, where } s_i \text{ is any other strategy for agent } i$$

[16]Often a distinction between weak and strict dominant strategies are made, with strict meaning strict inequality and weak also allowing equality between the utilities for two different strategies. In section 5.3 there is a shorter discussion on the difference between weak- and strict- dominant strategies and its relation to mechanism design.

In other words any other strategy s_i different from the dominant one s_i^* gives her a lower utility pay-off, regardless of the other players strategies. Interpreting this in our setting of theorem 1, it is enough to show that given an arbitrary utility profile u agent i obtains the highest utility by reporting $u_i \in \mathbb{R}^A$, any other reported utility vector $w_i \in \mathbb{R}^A$ leads to a lower utility for agent i.

As the first step a utility profile u is fixed, secondly pick an agent i and a message $w_i \in \mathbb{R}^A$ where w_i is the reported utility vector. The vector w_i composed of k entries, where each entry has the utility associated with each decision $a \in A$ for agent i. To avoid confusion about notations and keep the proof readable the following notations are determined: the efficient decision $a(u) = a$ and $a(u_{-i}, w_i) = b$. The function $a(\cdot)$ has a utility profile as its argument. Decision b arise when agent i deviates from using truth-telling as her strategy and reports w_i. By showing that agent i cannot improve her utility by sending w_i as her message, it is demonstrated that the Clarke-Groves mechanism indeed is strategyproof. Through comparing the net utility for agent i in the two scenarios, the first with truth telling u_i as her strategy and the second with misreported message w_i, a conclusion on the question of strategyproofness is determined. When u_i is the message sent the net utility is composed of the utility $\hat{u}_i(a)$ for decision a and the transfer $t_i(u)$, the latter defined by equation (4) in theorem 1. Likewise is agent $i's$ net utility for the message w_i given by the utility $\hat{u}_i(b)$ for decision b and the transfer $t_i(u_{-i}, w_i)$:

$$
\begin{aligned}
\hat{u}_i(a) + t_i(u) &= \hat{u}_i(a) + \hat{u}_{N\setminus i}(a) - \max_{c\in A} \hat{u}_{N\setminus i}(c) = \hat{u}_N(a) - \max_{c\in A} \hat{u}_{N\setminus i}(c) && (*)\\
&\geq \hat{u}_N(b) - \max_{c\in A} \hat{u}_{N\setminus i}(c) = \hat{u}_i(b) + \hat{u}_{N\setminus i}(b) - \max_{c\in A} \hat{u}_{N\setminus i}(c) && (**)\\
&= \hat{u}_i(b) + t_i(u_{-i}, w_i)\\
&\implies \qquad \hat{u}_i(a) + t_i(u) \geq \hat{u}_i(b) + t_i(u_{-i}, w_i)
\end{aligned}
$$

In the equations above demonstrating why the Clarke-Groves mechanism meets the criteria of strategyproofness, defined in Definition 3.3.2 on page 20 the crucial step of the derivation lays in the inequality between the equations $(*)$ and $(**)$. What happens is that agent i sends away w_i as her message, resulting in a new decision b. This gives a new joint utility $\hat{u}_N(b)$ and the the other term $\max\limits_{c\in A} \hat{u}_{N\setminus i}(c)$ remains as before. Decision a is a efficient decision, this implies that it maximizes joint utility, thus the joint utility for decision b is lower than or equal to a. Therefore any other strategy w_i different from u_i is weakly dominated by the truth-telling report of u_i, and the conclusion in the last equation is correct. This concludes the part of proving strategyproofness for the Clarke-Groves mechanism.

As the last part of establishing theorem 1 the criteria of feasibility, formalized in Definition 3.3.3 on page 22, is demonstrated. The proof of it is straight forward. Recall that for a feasible mechanism the sum of transfers is less than or equal to zero:

$\sum_{1=i}^{n} t_i(u) \leq 0$, and the transfer $t_i(u)$ for the Clarke-Groves mechanism is given by:

$$t_i(u) = \hat{u}_{N\setminus i}(a(u)) - \max_{b \in A} \hat{u}_{N\setminus i}(b)$$

A sum consisting of only negative transfers $t_i(u)$ is always negative. In other words to obtain a positive sum of transfers at least one of the transfers has to be positive. Therefore, assume there exist an agent i with a positive transfer $t_i(u)$:

$$\begin{aligned} t_i(u) > 0 \quad &\iff \quad \hat{u}_{N\setminus i}(a(u)) - \max_{b \in A} \hat{u}_{N\setminus i}(b) > 0 \\ &\iff \quad \hat{u}_{N\setminus i}(a(u)) > \max_{b \in A} \hat{u}_{N\setminus i}(b) \qquad (\diamond) \end{aligned}$$

In the expression ($\diamond$) above a contradiction is derived. At most there could be equality between them, but not an inequality. It is impossible with $\hat{u}_{N\setminus i}(a(u)) > \max_{b \in A} \hat{u}_{N\setminus i}(b)$. Maximizing a function is a search for the highest value of the function, given the domain it is defined on. Expression ($\diamond$) implies that the argument $a(u) \in A$ used in the function $\hat{u}_{N\setminus i}(\cdot)$ gives a higher utility than all of the other conceivable arguments in the domain A. Though $a(u) \in A$ is part of the domain for the function $\hat{u}_{N\setminus i}(\cdot)$ that is being maximized and is available as an argument for it. Thus, let $a(u) = b$ in the function $\max_{b \in A} \hat{u}_{N\setminus i}(b)$ and the inequality in ($\diamond$) is hereby impossible:

$$\hat{u}_{N\setminus i}(a(u)) > \max_{b \in A} \hat{u}_{N\setminus i}(b) = \hat{u}_{N\setminus i}(a(u))$$

From this contradiction one can draw the conclusion that there does not exist any agent $i \in N$ such that her transfer $t_i(u) > 0$. A sum without positive components can never be of positive value. It is at most eqaul to zero, if all the terms are equal to zero. Hence the sum of transfer $\sum_{1=i}^{n} t_i(u)$ for the Clarke-Groves mechanism is smaller than or equal to zero. This completes the proof for theorem 1. It is now established that the Clarke-Groves mechanism is strategyproof, efficient and feasible.

4.2 Clarke-Groves Mechanism And An Indivisible Public Good

In all public decision processes similar in setting to the basic model described on page 12 and the definitions in section 3, the Clarke-Groves mechanism provides a strategyproof, efficient and feasible mechanism. Some of the assumptions in the basic model and in the definitions are essential for the general result established by the proof of theorem 1, and some are not. One of these assumption possible to loosen up without any impact on the general result of theorem 1 is the assumption of a costless public decision. Adding a cost of c if the public good is produced does not impose a constraint on the Clarke-Groves mechanism. Though, the cost of the project demands funding for it, and the conditions are changed in the settings for the basic model. In the remaining part of this section I will provide an example of a public decision process, where the cost component is added. It is constructed for illustrating the Clarke-Groves mechanism in a different setting than in the basic example used until now.

Consider a society thinking on the question of whether they should undertake a specific public good project, for example building a tunnel for the railway through their city.[17] As mentioned above the cost for the project is c, and if the project is realized then a lump sum tax to finance the project of c/n is imposed on every agent. Similarly to the basic model the society consists of n agents and each of them has a utility function $\hat{u}_i(a, v_i)$. In the set A of public decisions there are two elements: undertake the project denoted by $a = 1$ and not undertake the project denoted by $a = 0$. The utility function is defined as:

$$\hat{u}_i(0, v_i) = 0 \qquad \text{and} \qquad \hat{u}_i(1, v_i) = b_i - \frac{c}{n} \qquad \text{, for every agent } i \in N \qquad (\bullet)$$

Where b_i is the benefit obtained by agent i when using the public good. We are only interested in direct mechanisms, see the discussion on the revelation principle why focusing on direct mechanisms impose no loss of generality for the argument, hence the agents report their benefit b_i truthful or not. A sensible rule for when the project should be realized is if the sum of all reported benefits b_i for all agents $i \in N$ is larger than or equal to the cost of the project c, otherwise it should not be undertaken:

$$\sum_{i=1}^{n} b_i > c \quad \Longrightarrow a = 1 \qquad \text{and} \qquad \sum_{i=1}^{n} b_i \leq c \quad \Longrightarrow a = 0$$

It arise two scenarios when the Clarke-Groves mechanism is applied to the current example. First (i) where $\sum_{i=1}^{n} b_i \leq c$ and thus the good is not produced and secondly (ii) the good is produced due to $\sum_{i=1}^{n} b_i > c$. In scenario (ii) all agents pay the lump sum tax of $\frac{c}{n}$ for financing the project and then a transfer of t_i, determined by the mechanism. The lump sum tax disappears in the other case (i), since there is no cost for producing the public good and only the transfer of t_i remains. Recall how the transfer $t_i(u)$ was defined

[17] This example is taken from Moulin (1988), chapter 8 page 206.

in equation (4) in theorem 1 on page 25:

$$t_i(u) = \hat{u}_{N\setminus i}(a(u)) - \max_{b\in A} \hat{u}_{N\setminus i}(b)$$

Using the utility function defined in (•) gives us:

$$(i) : t_i(u) = 0 \qquad , \text{if} \sum_{i\neq j}^{n} b_i \leq c\frac{n}{n-1} \text{ otherwise:} \quad t_i(u) = -\Big(\sum_{i\neq j}^{n} b_i - c\frac{n}{n-1}\Big)$$

$$(ii) : t_i(u) = -\frac{c}{n} \quad , \text{if} \sum_{i\neq j}^{n} b_i \geq c\frac{n}{n-1} \text{ otherwise:} \quad t_i(u) = -\Big(\frac{c}{n} + c\frac{n}{n-1} - \sum_{i\neq j}^{n} b_i\Big)$$

These are the transfer $t_i(u)$ for the Clarke-Groves mechanism in the current example plus the additional lump sum tax of c/n. Dividing the transfer component wise into the lump sum tax and the transfer imposed due to the mechanism one can see that it is only the pivotal agents who are exposed to a transfer from the mechanism. All the other non-pivotal agents incurs a zero transfer. In scenario (i) the efficient decision is to not undertake the project and the condition for a non-pivotal agent is $\sum_{i\neq j}^{n} b_i \leq c$, then no transfer is imposed on agent i. Though, when $\sum_{i\neq j}^{n} b_i > c$ agent i is pivotal and pays the transfer of $\sum_{i\neq j}^{n} b_i - c\frac{n}{n-1}$. Similarly in (ii) a non-pivotal agent is characterized by $\sum_{i\neq j}^{n} b_i \geq c$ and thus only pays the lump sum tax. For a pivotal agent i in scenario (ii) the condition is $\sum_{i\neq j}^{n} b_i < c$ and then she pays the lump sum tax of $\frac{c}{n}$ plus the additional transfer of $c\frac{n}{n-1} - \sum_{i\neq j}^{n} b_i$.

For demonstrating that the Clarke-Groves mechanism indeed is strategyproof, efficient and feasible an argument similar to the one used for the basic example could be used. I will not go through this argument once more for this example. In this example as well in many other the main drawback with the Clarke-Groves mechanism is the budget surplus it generates. The size of this budget surplus depends primarily on how many pivotal agents there are. If the efficient decision is to not produce the good, we are in scenario (i), then no lump sum tax is needed for financing the production. Every transfer comes from a pivotal agents and it has negative value, since the difference of $\sum_{i\neq j}^{n} b_i - c\frac{n}{n-1}$ always is negative for a pivotal agent in scenario (i). This leads to a revenue of all payed transfers $t_i(u)$ for the mechanism, and the money is not redistributed to the loser of the public decision. A larger number of pivotal agents, more transfers are payed.

Likewise in scenario (ii) a revenue is the result of the public decision process. First the cost for production of the good c is covered by the lump sum tax of $\frac{c}{n}$, additionally are there the transfers from the pivotal agents. All of these transfers are negative, the condition for being a pivotal agent implies that the transfer is negative. A negative transfer means a revenue for the mechanism designer. Though the revenue can never be equal to or larger than the cost of the project. Hence, the lump sum tax is necessary for financing the production of the public good. It is worth wile to investigate what happens in this example if there is no lump sum tax imposed on the agents to finance the good. Doing the same calculation as above for the transfers $t_i(u)$ gives us:

$$(i) : t_i(u) = 0 \quad , \text{if} \sum_{i \neq j}^{n} b_i \leq c \text{ otherwise:} \quad t_i(u) = -\Big(\sum_{i \neq j}^{n} b_i - c\Big)$$

$$(ii) : t_i(u) = 0 \quad , \text{if} \sum_{i \neq j}^{n} b_i \geq c \text{ otherwise:} \quad t_i(u) = -\Big(c - \sum_{i \neq j}^{n} b_i\Big)$$

Nothing of importance changes in scenario (i), it is only a matter of proportions for the magnitude of the transfers. The interesting scenario is (ii), here a surplus is generated and the question is whether the surplus is big enough to cover the cost c of production. Let us look closer at this, the maximum surplus occurs when there are n pivotal agents and all of them pays the transfer of $t_i(u)$:[18]

$$\sum_{i=1}^{n} t_i(u) = \sum_{i=1}^{n} \Big(c - \sum_{i \neq j}^{n} b_i\Big) = nc - (n-1)\sum_{i=1}^{n} b_i$$

This is the highest achievable income from the mechanism without imposing a lump sum tax. By subtracting the cost c of production we obtain a benchmark on whether the revenues from the transfers cover the cost of producing the good. It turns out that there is alway a budget deficit, when there is no lump sum tax.

$$\sum_{i=1}^{n} t_i(u) - c = (n-1)\Big(c - \sum_{i=1}^{n} b_i\Big) < 0$$

The last inequality follows from the condition for being in scenario (ii) and taking the efficient decision of producing the good, which is when $\sum_{i=1}^{n} b_i > c$. Thus, the funding for the project has to come from another source than the Clarke-Groves mechanism. When a lump sum tax of $\frac{c}{n}$ is used there will be some agents with a low valuation for the public good that would prefer to not participate at all in the project. This is not possible since the good is public and therefore no agent can be excluded from using it, when it is once produced.

[18]Since we are interested in whether the revenues from the mechanism covers the cost c of production the transfers are seen from the collectors perspective and a negative transfer for the agent is an income from the collectors point of view.

5 Strategyproofness, Efficiency And Budget Balance

The last chapter before the conclusion generalize the results established in the previous chapter 4. Starting in section 5.1 a result is presented on that all mechanisms satisfying the two criterion of efficiency and strategyproofness belongs to the class of pivotal mechanisms, and the Clarke-Groves mechanism is a particular type of these pivotal mechanisms. In section 5.2 a theorem on the impossibility for a mechanism to be budget balanced and at the same time being efficient and strategyproof is given and the proof of it is provided. The last part 5.3 treats briefly the problems with implementation of the mechanism.

5.1 Generalization Of The Clark-Groves Mechanism

Theorem 1 on page 25 established that the Clark-Groves mechanism meets the three criterion defined in section 3.3. A follow up question is if there exist other mechanisms who also satisfies these desirable criterion? As a matter of fact all efficient and strategyproof mechanisms are of a specific form. The Clark-Groves mechanism is one example of these mechanisms meeting the two criterion, for all conceivable utility profiles. Those belonging to the class of mechanisms satisfying the two criterion are called pivotal mechanisms. What is more theorem 1 demonstrated that the Clark-Groves mechanism is feasible, although this is not a general result for all pivotal mechanisms. In Definition 3.3.3 on page 22 of a feasible mechanism the sum of transfers has to be equal to or smaller than zero. Our last theorem in the next section demonstrates that there does not exist a mechanism meeting both strategyproofness and efficiency, and at the same time having the sum of transfer always equal to zero, for all types of preferences. Before going into further details on the relation between strategyproof and efficient mechanisms with the criteria of feasibility let us look at the theorem characterizing every efficient and strategyproof mechanism.[19]

theorem 2 ***The Theorem Of Pivotal Mechanisms***
For all $i \in N$, denote by $h_i(u_{-i})$ an arbitrary numerical function for all utility profiles u_{-i}, where u_{-i} is a $(n-1) \times m$ matrix. Then consider a mechanism $(a(\cdot), t(\cdot))$ such that:

$$a(u) \text{ is an efficient decision for each utility profile } u$$

$$t_i(u) = \hat{u}_{N\setminus i}(a(u)) - h_i(u_{-i}) \text{ for each utility profile } u \text{ and all } i \in N \tag{5}$$

This mechanism is strategyproof. Conversely any strategyproof and efficient mechanism is of this form.

[19] This version of the theorem is found in Moulin (1988), page 209.

Looking back at how the transfer $t_i(u)$ was defined for the Clark-Groves mechanism in theorem 1 by equation (4) on page 25 and comparing with theorem 2 above gives us:

$$\begin{aligned} t_i(u) &= \hat{u}_{N\setminus i}(a(u)) - h_i(u_{-i}) && \text{, the general form of the transfer from theorem 2} \\ &= \hat{u}_{N\setminus i}(a(u)) - \max_{b\in A} \hat{u}_{N\setminus i}(b) && \text{, the transfer for the Clark-Groves mechanism} \end{aligned}$$

Theorem 2 determines the structure of every strategyproof and efficient mechanism, though the exact construction of the function $h_i(u_{-i})$ is up to the mechanism designer to find out. The Clark-Groves mechanism provides an concrete example of how this function could look, given the frames set by theorem 2. Although the function $h_i(u_{-i})$ can have several forms, in order to use it the construction of the function should make economic sense. Which is exactly what the choice of the function $h_i(u_{-i})$ to $\max_{b\in A} \hat{u}_{N\setminus i}(b)$ in the Clarke-Groves mechanism does. With this choice of $h_i(u_{-i})$ the transfer $t_i(u)$ paid by the pivotal agents equals the loss in utility incurred by the other agents when switching from their preferred decision to the efficient decision. I will not go through the proof for theorem 2, a selection is necessary when the space is limited.[20]

5.2 Impossibility Result

Using the Clarke-Groves mechanism to solve the public decision in the basic example was close to a perfect solution. All the three desirable criterion for a mechanism are fulfilled, the Clarke-Groves mechanism is efficient, strategyproof and feasible. A last obstacle to overcome is the budget surplus generated by the mechanism. A surplus that is not redistributed to the losers in the public decision. This is a drawback for the mechanism and the last obstacle to overcome. Another version of this imbalance of the budget is found in section 4.2 with the example of an indivisible public good produced for a cost of c. If the efficient decision is to undertake the project the production cost of c needs funding. A lump sum tax of c/n is imposed on every agent to finance the project. Without a lump sum tax a deficit occurs, and having a lump sum tax there is no use for the surplus generated by the mechanism. It seems like there is a relation between efficiency and strategyproofness on the one hand and budget balance on the other hand. In the following theorem the general result for efficient and strategyproof mechanism contra budget balance is demonstrated. Since theorem 2 already specified the form of all strategyproof and efficient mechanisms, theorem 3 focus on all mechanisms in the pivotal class and their relation to budget balance.[21]

[20]A proof of theorem 2 is found in Moulin(1988), chapter 8 pages 209-210.

[21]Similarly to theorem 2 this version of theorem 3 is taken from Moulin (1988), page 210.

theorem 3 ***(Green-Laffont's Impossibility Theorem)***
For general preferences there exist no ***strategyproof***, ***efficient*** *and* ***budget balanced*** *mechanism.*

Efficiency is defined as in Definition 3.3.1 on page 20 and strategyproofness is defined as in Definition 3.3.2 on page 20. The definition of a budget balanced mechanism is identical to the Definition 3.3.3 on page 22 for a feasible mechanism with the only difference that the sum of transfers has to be equal to zero and not smaller than or equal to zero as for a feasible mechanism. Theorem 3 is a negative result, meaning that no matter how much effort is laid down in the search of a strategyproof, efficient and budget balanced mechanism we will never find one for a general set of preferences. Though, if the set of preferences is restricted then there might exist a strategyproof, efficient and budget balanced mechanism. How this statement about all possible mechanisms could be made and why it is true is derived in the following proof for theorem 3.

Proof of theorem 3:[22]
First the set of mechanisms under our light can be reduced to the class of pivotal mechanisms. Since, without loss of generality it is enough to consider only pivotal mechanisms. Because from theorem 2 it follows that every strategyproof and efficient mechanism belongs to the pivotal class of mechanisms. Thus if it is demonstrated that no pivotal mechanism is budget balanced then there is no efficient, strategyproof and budget balanced mechanism at all. Using theorem 2 and more specifically equation (5) on page 33 we know that the transfer for a strategyproof and efficient mechanism is:

$$t_i(u) = \hat{u}_{N\setminus i}(a(u)) - h_i(u_{-i}) \quad \text{for each utility profile } u \text{ and all } i \in N \tag{5}$$

Since all strategyproof and efficient mechanisms has a transfer of the form in equation (5) there cannot exist a budget balanced mechanism with a structure of the transfer included in equation (5), if theorem 3 is correct. This proof relies on the principle of proof by contradiction, it is a common used tool for establishing theorems in mathematics. The idea is to use the parts in the theorem that are already established and the new statements are proven by assuming the opposite of what is claimed in the theorem and derive a contradiction. A conclusion is drawn that the assumption must be wrong, and thereby the opposite is true. Which is what was claimed in the theorem. In this proof the established facts are that a strategyproof and efficient mechanism (belonging to the pivotal class) has a transfer corresponding to equation (5) and the assumption is that the mechanism

[22] This proof builds on notes from Tommy Andersson and chapter 8.4 in Moulin (1988).

also is budget balanced. By deriving a contradiction, the assumption of the existence for a budget balanced pivotal mechanism is false and the conclusion of the non-existence of such a mechanism is established and theorem 3 is correct. Assuming that a pivotal mechanism is budget balanced is equivalent to assuming that the sum of transfers, where the transfers t_i are given by equation (5), is equal to zero:

$$\sum_{i=1}^{n} t_i(u) = 0 \iff \sum_{i=1}^{n} \hat{u}_{N\setminus i}(a(u)) = \sum_{i=1}^{n} h_i(u_{-i}) \iff$$
$$\sum_{i=1}^{n} h_i(u_{-i}) = (n-1)\max_{b\in A} \hat{u}_N(b) \implies h(u) = \sum_{i=1}^{n} h_i(u_{-i}) = (n-1)\max_{b\in A} \hat{u}_N(b)$$

From this assumption of budget balance a function $g(u)$ is constructed as:

$$g(u) = \max_{b\in A} \hat{u}_N(b) \tag{5}$$

This gives us the relation of $h(u) = (n-1)g(u)$, which will be used in order to derive the contradiction. To prove theorem 3 it is enough to find one type of preferences, where it does not exist a strategyproof, efficient and budget balanced mechanism. Theorem 3 only claims the non-existence for all types of preferences. [23] Let us construct the following utility functions:

$$\begin{cases} \hat{u}_i(a) & = 1 \\ \hat{u}_i(b) & = 1 + \frac{1}{n} \\ \hat{u}_i(c) & = 0 \qquad \text{, for any } c \in A \setminus \{a, b\} \end{cases} \qquad \begin{cases} \hat{v}_i(a) & = 1 \\ \hat{v}_i(b) & = 0 \qquad \text{, for any } b \in A \setminus \{a\} \end{cases}$$

Our next step is to introduce the new set S, it is a subset of N. In general a subset Γ of the set Δ consists of elements from the set Δ, the empty set is often included and is denoted $\emptyset$. Using notations from elementary set theory the new subset S of N is expressed as $S \subseteq N$. A new utility profile $w^\Lambda = (w_1^\Lambda, w_2^\Lambda, \dots, w_n^\Lambda)$, where Λ can be the subset S, the empty set $\emptyset$ or the whole society N, w^Λ is constructed with the aid of the utility functions introduced above. As mentioned before all utility profiles can be described by a matrix, the utility profile w^M is $(n \times m)$ matrix. An entry for agent i and decision $d \in A$ in the matrix w^Λ is determined by:

[23] Earlier mentioned Nobel Laureate Eric Maskin et al characterized in what settings there can exist a strategyproof, efficient and at same time budget balanced mechanism.

$$\begin{cases} w_i^S(d) &= \hat{u}_i(d) \qquad \text{for all } i \in S \\ w_i^S(d) &= \hat{v}_i(d) \qquad \text{for all } i \in N \setminus S \\ w_i^N(d) &= \hat{u}_i(d) \qquad \text{for all } i \in N \\ w_i^{\emptyset}(d) &= \hat{v}_i(d) \qquad \text{for all } i \in N \end{cases} \tag{6}$$

Recall how the function $g(u)$ is defined in equation (5), it is a function maximizing joint utility for all agents $i \in N$ over every public decisions $b \in A$. Let us use the utility profile w^{Λ} as an argument for the function $g(\cdot)$. Depending on the choice of Λ different decisions are picked as the maximizer of joint utility.

$$g(w^N) = g(u) = \max_{d \in A} \hat{u}_N(d) = \sum_{i=1}^{n} \hat{u}_i(b) = \sum_{i=1}^{n} \left(1 + \frac{1}{n}\right) = n + 1 \tag{7}$$

$$g(w^S) = g(v) = \max_{d \in A} \hat{v}_N(d) = \sum_{i=1}^{n} \hat{v}_i(a) = \sum_{i=1}^{n} (1) = n \quad \text{, for all } S \neq N \tag{8}$$

$$g(w^{\emptyset}) = g(v) = \max_{d \in A} \hat{v}_N(d) = \sum_{i=1}^{n} \hat{v}_i(a) = \sum_{i=1}^{n} (1) = n \tag{9}$$

Following the notations used before $u = (\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n)$ and likewise $v = (\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n)$, where u and v are utility profiles. Hence, the first equality in equation (7), (8) and (9) tells which utility function $\hat{u}_i(\cdot)$ or $\hat{v}_i(\cdot)$ is the correct one, for the three different choices of Λ when w^{Λ} is used as an argument for the function $g(\cdot)$. In the third equality the decisions maximizing joint utility are stated, for all three alternatives of the set Λ. Anyone with doubts regarding the choices of public decisions for maximizing joint utility in equations (7) – (9) will be convinced after looking back at how the utility functions were defined on the previous page.

Until now all I said about the subset S is that it is a subset of N. An example of a subset S to the set N, using the basic example on page 12, is the group S consisting of agent 1 and 2. Furthermore of importance is that in general it exist 2^N different subsets S of N, having the empty set and the whole society of N agents included in the subsets S. As a last new notation denote the complete collection of these different subsets S with δ. The total collection δ is a new set comprised of all subsets S of N, and as mentioned the total number of distinct subsets is 2^N. Thus δ has 2^N elements and each element in δ is a subset S of N. Every element S in δ has either an even or odd number of agents in it. A separtion of δ into two new subsets is done upon this characteristic. The first subset δ_1 consists only of even number of agents' in every element S, and the second subset δ_2 consists only of odd number of agents' in every element S in it. There are 2^{N-1} elements in the two subsets δ_1 and δ_2. Hence adding δ_1 and δ_2 the set δ is obtained with its 2^N elements.

Let us look back at how the function $g(u)$, defined in equation (5), is related to $h(u)$. There is only a constant term of $(n-1)$ separating them apart in the relation $h(u) = (n-1)g(u)$. The constant term is a scalar and can be viewed over without loss of generality for the coming argument. There are two crucial steps left of this proof. Firstly recall how $h(u)$ is set up. It is a sum of n functions $h_i(u_{-i})$ given by $h(u) = \sum_{i=1}^{n} h_i(u_{-i})$, where agent $i's$ preferences plays no role for each function $h_i(u_{-i})$. Expressing this mathematically means that the argument for every function $h_i(u_{-i})$ are all preferences except agent $i's$, u_{-i} is a reduced utility profile since agents $i's$ preferences are not included. Having this said about $h(u)$ the following conclusion can be drawn:

$$\sum_{i=1}^{n} h_i(w_{-i}^{S}) = \sum_{i=1}^{n} h_i(w_{-i}^{S'}) \quad \text{if } w_{-i}^{S} = w_{-i}^{S'} \text{ for every } i \in N \quad \Longleftrightarrow \quad h(w^S) = h(w^{S'})$$

Moreover for every $S \in \delta_1$ there exist a $S' \in \delta_2$ such that $S \setminus \{i\} = S' \setminus \{i\}$ and thus $w_{-i}^{S} = w_{-i}^{S'}$, considering the result in the preceding equation we have:

$$\sum_{S \in \delta_1} h(w^S) = \sum_{S \in \delta_2} h(w^S) \quad \Longleftrightarrow \quad \sum_{S \in \delta_1} g(w^S) = \sum_{S \in \delta_2} g(w^S) \tag{10}$$

The society consists of N agents, where N is either an even or odd number. Let us assume that N is an even number. In the subset δ_1 consisting of only even groups of agents, and one element is the whole society N. For the set δ_2 there exist no element S where all agents in N is included. This leads to different values of the highest achievable joint utility when the function $g(w^S)$ is evaluated.

$$\sum_{S \in \delta_1} g(w^S) = (n+1) + n(2^{N-1} - 1) \tag{11}$$

$$\sum_{S \in \delta_2} g(w^S) = n(2^{N-1}) \tag{12}$$

When S is equal to N we are in the third row of equation (6), which determines the correct utility function to use, and the maximum joint utility is obtained by picking decision a and it is equal to $n+1$. For all the remaining elements S in δ_1 we are in row 2 and thus the highest achievable joint utility is n with decision b. Since N is an even number there does not exist any element S in δ_2 where all agents in N are included. Hence, the highest achievable joint utility is n for all elements S. A symmetrical argument can be made if N is odd. In both scenarios the contradiction arise of that the two sums in equation

(11) and (12) are not equal, contradicting to equation (10). Reasoning by the principle of contradiction the assumption of the existence of a budget balanced pivotal mechanism is false, and thus theorem 3 is established. Hereby the proof of theorem 3 is completed.

5.3 Discussion On The Absence Of Applications For Pivotal Mechanisms

A natural question is why the Clarke-Groves mechanism with the three desirable characteristics is not widely used for public decisions in real life. It was invented in the early seventies and has been around ever since that. There are of course many possible explanations for it. One fundamental objection to the Clarke-Groves mechanism and similarly for other invented mechanisms, is that they are too complex. Unless the agents understands the mechanism and foresee what happens if a specific strategy is chosen then it does not make any difference whether truth telling is a dominant or dominated strategy. Another explanation for the lack of applications is more general, it attacks the question of rationality. It argues that agents outside economic theory and models, *i.e* in reality may not act according to computations of equilibriums, as in game theory. This hypothesis about the lack of rationality in agents actions has found support in laboratory experiments. In studies by Attiyeh et al (2000), Kawagoe and Mori (2001) results are given on that agents use their dominant strategy less than half of the time. Agents low use of dominant strategies could be explained by the hypothesis about lack of rationality. Irrational agents have no reason for adopting dominant strategies. [24]

There are experimentalists, for example Attiyeh et al (2000), who have out of these results drawn the conclusion that many of these mechanisms have no practical applications, they may though be interesting for other reasons. In Rothkopf (2007) 13 reasons to why the Clarke-Groves mechanism and the related Vickrey auction is not used more is given. The author perceives both as theoretical elegant mechanisms, though this does not help when they lack other qualities in terms of practical use. Among these 13 arguments there are theoretical as well as practical reasons to why the mechanisms are not seen more often in real life applications. One of the arguments is that the dominant strategy is a weak equilibrium and thereby other equilibriums may arise. If weakly dominated strategies are used instead of the dominant strategy truth telling, then the fundamental idea with the Clarke-Groves and other strategyproof mechanisms can be questioned. As a counterpart

[24] Additional references to experimental studies with similar result of that the agents adopt dominant strategies to a rather modest degree are found in Cason et al (2006).

to this view I will present some of the ideas in a article of Cason et al (2006). In this article the authors investigates if they found an answer and solution to why many laboratory experiments shows that agents does not act according to the dominant strategy of truth telling, and thereby the main idea with the Clarke-Groves mechanism and many other mechanisms is undermined. A concept called *secure implementation* is their answer for the poor performance by pivotal mechanisms to reveal the agents true valuations in real world situations, it was partly developed by one of the authors to the article Saijo et al. (2003). A mechanism is securely implementable if all Nash equilibriums and dominant strategies coincides in the game constructed by the mechanism. Given that the mechanism is efficient this criteria of secured implementation says: all Nash equilibriums must be social optimums in a securely implementable mechanism.

Through conducting experiments with two different mechanisms, where one satisfies the condition of secure implementation and the other does not, a benchmark on the influence of secure implementation is supposed to arise. My plan is to briefly discuss their theoretical tools and then concentrate on the results from the experiments they report in their article. Undertaking a thorough treatment of implementation theory demands an additional thesis. Instead I will use some of the concepts developed until now in this field of mechanism design and focus on this article. This section 5.3 should be seen as an attempt to answer the question of the applicability of mechanism design, given the limited amount of space available in a bachelor thesis.

For social choice and mechanism design theory the criteria of strateygyproofness plays a central role. If a mechanism meets this criteria the agents' true preferences are supposed to be revealed, and the necessary information for a efficient decision is obtained. Although a mechanism is strategyproof and thereby truth telling is a dominant strategy, it may exist Nash equilibriums different from the dominant strategy. When the agents end up in of these bad Nash equilibriums, different from the dominant strategy outcome and the social optimum, then they might stay there. Because truth telling is a weak and not a strict dominant strategy and thus alternative strategies may give the same outcome for certain set of actions. This is a possible explanation for the low use of dominant strategies when the mechanisms are tested in laboratory experiments. The undesirable situation where Nash equilibriums exists outside the dominant strategy disappears if the only Nash equilibriums are the ones in the dominant strategy. When this is the conditions for a mechanism it satisfies secure implementation.

Recall the difference between a dominant strategy and a Nash equilibrium. A dominant strategy s_i^* for an agent i yields the highest utility, no matter what strategy s_{-i} the other agents use. In a Nash equilibrium (s'_{-i}, s'_i) every agent i cannot receive a higher utility by deviating from $s_i^{'}$ given that the other agents use strategy $s_{-i}^{'}$. Each dominant strategy is also a Nash equilibrium, but the contrary is not true a Nash equilibrium is not by necessity a dominant strategy. Below are two examples of games in matrix form to illustrate the difference between dominant strategies and Nash equilibriums. Two separate mechanisms, both satisfying the three criterion defined in section 3.3, induced these two games. There are two agents in both games, agent 1 and 2. They have two options for their strategies: either playing True or playing False. For a game in matrix form the payoff for a given strategy pair is found in the corresponding box of the matrix describing the game. Here are the two games induced by the two different mechanisms.

Game 1; not secured implementation

1 \ 2	*False*	*True*
False	$(2, 1)$	$(1, 1)$
True	$(2, 2)$	$(3, 3)$

Game 2; secured implementation

1 \ 2	*False*	*True*
False	$(1, 1)$	$(2, 3)$
True	$(2, 3)$	$(2, 4)$

Think of the two games as simplified examples of the game induced by a mechanism. The purpose is to explain the difference between mechanisms satisfying secured implementation and those who does not meet it. In Game 1 there exist two Nash equilibriums *(False, False)* and *(True, True)*, $(\cdot, \cdot)$ stands for a strategy combination for agent 1 and agent 2. The latter equilibrium *(True, True)* is part of both agents dominant strategy of playing *True*, in contrast to the first bad equilibrium *(False, False)* where the agents use their dominated strategy *False*. Hence the mechanism behind game 1 is not securely implementable. Similarly in game 2 both agents has truth-telling as their dominant strategy, though the only existing Nash equilibrium is *(True, True)* and therefore the mechanism meets the criteria of secured implementation.

The hypothesis in the article by Cason et al (2006) is that Nash equilibriums outside the dominant strategy influence the agents behavior in a negative way and decreases the use of dominant strategies. This is tested in a laboratory experiment through constructing two different mechanisms; one is securely implementable and the other is not, and provide them to a group of agents. Instead of presenting thoroughly the two mechanisms with belonging utility functions and transfers, it is enough to point out that one meets the requirement of being securely implementable and the other does not. We are more interested in the results of the experiment on whether secure implementation makes any difference in the rate agents adopt their dominant strategy, than the theoretical subtleties and details in the two mechanisms.

Before discussing the results I will briefly explain the structure and background to the experiment. It was conducted at two universities, Tokyo Metropolitan University in Japan 1998 and at Purdue University in USA 2003. In total there were four session with 20 participants for every session. In each session the game was repeated 8-10 times, and before each round the 20 participants were paired together two and two. During the whole experiment no participants were ever with the same person twice, it was a new constitution of pairs for every period. Two of the sessions were with the mechanism satisfying the criteria of secured implementation, here called treatment S as in the article, and two with the mechanism who had Nash equilibriums outside the dominant strategy, here called treatment P as in the article.

Common for both mechanisms was that the society comprised of two persons and they had to decide over a public good. For the mechanism in treatment P not satisfying secured implementation the public decision regarded a binary public good, similar in the general setting to the example with an indivisible public good in the preceding section 4.2. It was only a question about whether the good should be produced or not. For treatment S where the mechanism was securely implementable the decision concerned the level of a public good. Both of the mechanisms had truth telling as a dominant strategy and thereby it was also a Nash equilibrium, though one of them (treatment P) had other Nash equilibriums as well except truth telling. Payoff tables were employed in all sessions, and the participants picked a strategy by deciding on a row or column in the table. There were 25 available strategies for both mechanisms, depending on their report the mechanism assigned a decision for the public good. The reason for the use of tables instead of explaining the mechanism for the participants was to remove complexity and ambiguities that could hide the difference between the two mechanisms in terms of secure implementation. Remember that the aim with the experiment was the search of a benchmark for the difference in the use of dominant strategy between secured and not secured implementation.

From the experiment they obtained a data set of 180 observations for each mechanism, each observation consisted of a pair of valuations from the two agents paired together before each period in the four sessions. As mentioned above, the payoff tables had 25 alternatives for the agents to choose between depending on which entry they report the mechanism maps a public decision. When they chose an entry on the table it is indirectly a valuation of the public good. In order to obtain a benchmark for the difference in the use of dominant strategies a characterization of the experiment is made upon the outcome for the two mechanisms, the material is separated into three classes: dominant strategy equilibrium, Nash equilibrium distinct from the dominant strategy and last all other outcomes. As a summary of the experiment conducted the results are listed in Table 2 below.

Table 2 Results of the experiment

Outcome \ Treatment	P	S
Dominant Strategy Equilibrium	90	146
Nash Equilibrium	61	0
Other	29	34
Total	180	180

As Table 2 shows the frequency of agents adopting to the dominant strategy is significantly higher for the secured implementation mechanism than the other with Nash equilibriums distinct from the dominant strategy. Several statistical tests are done with the data in the article. For both mechanisms the data cannot reject the hypothesis that the participants median choice was dominant strategy, in any of the periods. This confirms the importance of dominant strategy being a criteria for a successful mechanism. Two separate statistical tests analyzing the data showed that the degree to which dominant strategies was adopted by the participants was significantly higher with treatment S than treatment P in 8 out 10 for one of the tests and in 7 out of 10 for the other test. Agents more frequent use of dominant strategy in treatment S than in treatment P supports the argument for securely implementable mechanisms.

Another hypothesis was that deviations from dominant strategies is not random. Interpreting this in the experiment the deviations for treatment P would primarily be to other Nash equilibriums. A look in Table 2 for other outcomes than dominant strategies and Nash equilibriums with treatment P gives us the number 29. The proportion of other outcomes to non dominant strategies outcomes is 29/90, at a first glance this might seem as a large proportion of all the deviations from the dominant strategy. Though in total there are 298/621 outcomes that are not Nash equilibriums, a significantly higher proportion than 29/90. If we compare the proportions of Nash equilibriums to the total number of outcomes for the two mechanisms it is almost the same. For treatment P 151/180 and for treatment S 146/180, this indicates that Nash equilibriums are a better predictor of agents behavior than the dominant strategy are.

Further research on this topic of implementation is needed, even if the authors hypothesis on the importance of secure implementation was supported by this experiment other research might show the opposite. Advocating Nash equilibriums in favor of dominant strategies as an important criteria for implementation is probably the right thing to do. For dynamic processes this argument for Nash equilibriums is even more valid, a Nash equilibrium is a resting point. If the agents are in a Nash Equilibrium they are not likely to move to another Nash equilibrium. Though the fundamental objection to mechanisms against the complexity and the assumptions of rationality in agents behavior are harder to overcome. A realistic environment for a mechanism is vast more complex than payoff tables. First of all agents needs to understand the mechanism, a goal tough enough on its own. When this step is taken then the crucial question remains on whether the agents behave rational or not.

Regarding a concluding remark on the applicability of mechanism design one should note that it is a big and costly project to implement a mechanism in a real world scenario. I perceive the possible application for mechanism design on large project and questions of considerable importance for the society as a whole. Examples of these possible applications are: major environmental issues, substantial public good project and other large investments the society has to decide on.

6 Conclusion

In the beginning of this thesis it was pointed out that truthful information about the agents preferences are necessary to obtain, in order to achieve an efficient allocation of public goods in a society. The idea behind mechanism design theory is to design mechanisms in accordance to outcomes. A game is induced by the mechanism and the equilibrium of the game is the desirable outcome, after which the mechanism was designed. After specifying a general model for a public decision process and defining three criterion: efficiency, strategyproofness and feasibility, by which the mechanism is evaluated. I demonstrated that the Clarke-Groves mechanism satisfies each of the three criterion. It is due to a clever construction of a tax imposed on everybody that the Clarke-Groves mechanism manages to meet each of the three criterion.

The Clarke-Groves mechanism and all other mechanisms satisfying the criterion of efficiency and strategyproofness belongs to the pivotal class of mechanisms. Every mechanism in the pivotal class has a transfer of a specific type, and thus all mechanisms with these types of transfers are strategyproof and efficient, and some of them are also feasible as the Clarke-Groves mechanism is. There is one major drawback with all pivotal mechanisms, they are not budget balanced. This general result for all mechanisms is formulated in theorem 3 and a complete proof of it is also given. Except the budget imbalance associated with all pivotal mechanism they are all theoretical elegant mechanisms, with truth telling as a dominant strategy and all satisfies the efficiency criteria. However applications of them to real public decisions processes are few so far.

These problems and similar questions are all issues concerning implementation of mechanism design, on its own a complete field of research. Since my focus in the thesis is theoretical rather than empirical I discussed the problems of implementation by questioning the role of dominant strategy as a determinant of equilibriums in games, and argued for the hypothesis driven in Cason et al (2006) that Nash equilibrium and not dominant strategy is the important concept when agents chose their strategy. Every dominant strategy is also a Nash equilibrium, but the contrary is not true. It is possible to have Nash equilibriums different from the dominant strategy. In the experiment conducted it turned out that a securely implementable mechanism, meaning that all Nash equilibriums is part of the dominant strategy, leads to a higher adoption of the dominant strategy than the scenario when the mechanism has Nash equilibriums distinct from the dominant strategy.

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