Replication strategies of derivatives under proportional transaction costs
- An extension to the Boyle and Vorst model

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September 16, 2005

Abstract

When we introduce transaction costs the perfect Black and Scholes hedge, consisting of the underlying stock and a risk free asset, becomes infinitely expensive. By loosening the pure arbitrage argument and only considering the expected transaction costs, one can find an upper bound on the price of an option. In this essay this is done by using a framework presented by Leland (1985) and Boyle and Vorst (1992), which is based on rebalancing the hedge at predefined time-steps. However, their model is somewhat incomplete as they do not include the initial transaction cost of buying the hedge and the transaction cost of selling the hedge at maturity date. In this essay, an extension to their model is presented. This extension provides a framework that is consistent with their underlying model assumptions but incorporates the transaction costs mentioned above. In addition, we prove that these transaction costs have a significant effect on the price of an option.

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Chapter 1

Introduction

When Black and Scholes (1973) present their model for option pricing they assume a perfect frictionless market. This is generally not the case though, since most financial intermediaries are exposed to transaction costs when they want to trade a certain asset. What is first seen is that the perfect Black and Scholes hedge, consisting of the underlying stock and a risk free asset, will be infinitely expensive if we introduce transaction costs. There has been done a lot of research on how to find a solution to this problem. The main difficulties to overcome are how to hedge our derivatives in presence of transaction costs, and with respect to our hedge, how to price them. Many different hedging strategies has been presented during the last twenty years and they all have their advantages and disadvantages.

The main research on the subject starts with an article by Leland (1985), where he presents a discrete time approach to hedge derivatives in presence of transaction costs. Leland's model is further developed by Boyle and Vorst (1992), whose article has been one of the main sources for this essay. The strategies presented by these authors are built on rebalancing the hedge at predefined time-steps. The main advantage with this approach is that the transaction costs will be finite and that we can find an upper bound to our option price. On the other hand, we are forced to only rebalance our hedge at certain time-steps, which reduces the flexibility of how we want hedge our derivative. To make a continuous time approach to the problem, other move-based strategies has also been presented. Move-based strategies can be formed so that we only rebalance our hedging portfolio when the delta, gamma, or any other measure we want to use, reach a certain threshold. The main advantage with this approach is that we get smooth functions, so that we analytically can find an optimal solution to our trade-off between risk and cost, given a specific preference function. Although finding optimal solutions is very appealing, the model has many disadvantages. First, the model requires constant monitoring of the market. Secondly, the optimal solution postulate that it is optimal only to rebalance our portfolio so that it precisely get inside the threshold. This is all well of we only consider proportional transaction costs but if we include a fixed cost, ever so small, the strategy would be infinitely expensive. This follows since if an option increases or decreases in value we will get infinitesimal trades when we exceed our threshold. See Constantinides and Zariphopoulou (1999) for a good presentation on move-based strategies.

No matter which strategy we choose we are left with the fact that there is no strategy which perfectly replicates an option in presence of transaction costs. The question that remains is which strategy we want to use? In this essay we have chosen to study discrete time models, only rebalancing at predefined time-steps. Secondly, only proportional transaction costs are considered. This approach gives us a good basic understanding of what happens when we introduce transaction costs. Secondly, this model is fairly easy to implement practically. In chapter 2 we will define the basics of financial derivatives and arbitrage pricing models. We will also describe the dynamics of the Black and Scholes market. All the definitions needed to fully understand the scope of the next chapters will presented and explained in chapter 2. In chapter 3 we introduce transaction costs. First we describe the dynamics of the binomial market without transaction costs. After that we look on how the binomial model can be used to work with transaction costs by using the framework presented by Leland (1985) and Boyle and Vorst (1992). Later we will use the results from the binomial study to try to find an approximation on how to modify Black and Scholes formula to work with transaction costs. In the last section of chapter 3 we will try to make an extension to the model presented by Boyle and Vorst (1992). The purpose with this extension is to include transaction costs that are not calculated with in their model but which have great importance to the price of an option. These cost are the initial cost of buying the hedge and the cost of selling the hedge at maturity date. In chapter 4 we will simulate different scenarios and investigate on how the models defined in the previous chapters perform.

Chapter 2

Basic Theory

2.1 Financial Derivatives

The value of a derivative is derived from the underlying asset and is a contract between two parties. The payoff of a derivative is defined by its payoff function which will be denoted $\Phi(S)$. There are a wide variety of derivatives but the most common ones are the European call and put options and the American call and put options. These options gives the holder the right to buy or sell a stock for a certain price at a specific time or during a specific time interval. The price for which the holder is able to buy or sell a specific stock, denoted S, is called the strike price and will be denoted K. The difference between European options and American options are that European options may only be exercised at the maturity date, denoted T, while American options may be exercised at any time $t \leqslant T$. In this essay only European options will be discussed. (Hull, 2003, chapter 1)

2.2 Arbitrage Pricing

The difficulty with financial derivatives is to find the price at time t with respect to a certain payoff function. This may be done by arbitrage pricing. We will denote the price of a derivative at time t as $F_{t,s}$. To explain the basics we need a few definitions. (Björk, 2004, chapter 6 and 7)

Definition 2.2.1 *A portfolio, V^h with weights h_i, consisting of N assets is called self-financing if*

$$\frac{dV^h(t)}{dt} = \sum_{i=1}^{i=N} h_i(t) dS_i(t). \tag{2.1}$$

That is if no money is being added or withdrawn externally when the portfolio rebalances.

By looking at equation (2.1) we see that the change in value of a self-financing portfolio only depends on how much we have in our different assets, the h_i, and the change of value in the different assets, the dS_i. If we want to rebalance our portfolio, change the h_i, no money may be added our withdrawn to do so. In case of transaction costs we realize that if we want to rebalance a self-financing portfolio, the cost of doing so must be paid by the portfolio itself and not by any external aid.

Definition 2.2.2 *There exists an arbitrage* ***opportunity*** *if the following conditions hold:*

$i, \ V_t^h = 0$

$ii, \ P(V_T^h \geqslant 0) = 1$

$iii, \ P(V_T^h > 0) > 0$

We see that if the conditions in definition (2.2.2) are true we can buy an arbitrary amount of portfolio V^h at time t, since it will cost us nothing. At time T we know that the value of the portfolio for sure isn't worth less than zero, that is we take no risk. Furthermore we see that we have the possibility that the portfolio is worth more than zero. Given these conditions we realize that we can invest any amount at zero risk and eventually we will become tremendously rich, since the chance of profit is more than zero. In an arbitrage free market these possibilities never appear. In order to price derivatives we need a model for the stock price and a risk-neutral probability measure, such that there exists no arbitrage opportunities. We call this risk-neutral probability measure a martingale measure.

Definition 2.2.3 *A process X is called a martingale if*

$$E[X(s)] = X(t), t \leqslant s.$$

That is the process X has zero drift.

In other words the expected value of a process at a future time s, given what we know today, will be exactly the same as the value of the process at time t. (Björk, 2004, appendix C)

Another very important concept of arbitrage pricing is completeness. If a market is complete we mean that all derivatives can be replicated by a replicating portfolio. Since the replication criteria is very central in this essay, we want to show that the markets in which we act are complete. To show that a market is both free of arbitrage and complete, two very powerful theorems can be used. These theorems are called the fundamental theorems of arbitrage pricing.

Theorem 2.2.1 *The market model is free of arbitrage if and only if there exists a risk-neutral martingale measure $\mathbb{Q}$ that is equivalent with the observed price measure $\mathbb{P}$. That is a measure $\mathbb{Q} \sim \mathbb{P}$ such that the processes*

$$\frac{S_0(t)}{S_0(t)}, \frac{S_1(t)}{S_0(t)}, \dots, \frac{S_N(t)}{S_0(t)}$$

are martingales under $\mathbb{Q}$. S_i denotes the different assets in the current market.

Theorem 2.2.2 *Assuming absence of arbitrage, the market is complete if and only if the martingale measure $\mathbb{Q}$ is unique.*

Proof: See Björk (2004, chapter 10) for a heuristic proof.

These theorems may be a little bit hard to understand at first glance but remembering what we know about risk-neutrality from Hull (2003), a simple example can be made. Given a risky asset S_1 and a risk free asset S_0, we have the following conditions.

In order for the binomial market, see Cox et al. (1979), to be free of arbitrage we choose our probabilities so that the expected drift of a stock is equal to the risk free rate, that is

$$E^{\mathbb{Q}}\left[\frac{S_1(t+1)}{S_0(t+1)}\right] = \frac{S_1(t)}{S_0(t)}.$$

The probability measure which fulfill this condition is our $\mathbb{Q}$ measure. Furthermore we remember that the probabilities given are unique so the binomial market is also complete. With the price measure $\mathbb{P}$ we mean the actual price movement observed in the market. If our market is free of arbitrage the observed probability measure $\mathbb{P}$ is equivalent with the risk-neutral measure $\mathbb{Q}$, if not we have arbitrage opportunities.

In the Black and Scholes (1973) market, which is time continuous, the processes for the different assets must also follow the these conditions in order to be free of arbitrage and complete. These facts will be more thoroughly investigated later in this essay.

2.3 The Black and Scholes Market

When Black and Scholes (1973) derive their famous formula for pricing of european options they use an arbitrage argument. Furthermore they make a few assumptions about the market, which are necessary for the formula to hold. These ideal conditions are:

1. The short-term interest rate is known and is constant through time.
2. The stock price follows a random walk in continuous time with a variance rate proportional to the square of the stock price. Thus the distribution of possible stock prices at the end of any finite interval is lognormal. The variance rate of the return on the stock is constant.
3. The stock pays no dividends or other distributions.
4. The option is European, that is, it can only be exercised at maturity.
5. There are no transaction costs in buying or selling the stock or the option.
6. It is possible to borrow any fraction of the price of a security to buy it or to hold it, at the short-term interest rate.
7. There are no penalties to short selling. A seller who does not own a security will simply accept the price of the security from a buyer, and will agree to settle with the buyer on some future date by paying him an amount equal to the price of the security on that date.

With these conditions Black and Scholes creates a portfolio consisting of a stock and a European call option,

$$V^{B\&S}(t) = h_1(t)S_t + h_2(t)F_{t,s}.$$

By choosing h_1 and h_2 such that the risk of the portfolio is equal to zero at all times, the no arbitrage argument stipulates that the drift of the portfolio must be equal to the risk free rate. With this in hand they derive the price for the European call option. See Björk (2004, chapter 7) for a comprehensive and good derivation.

2.3.1 The Dynamics of the Black and Scholes Market

In order to price an option under these market conditions we need a model for the stock price and the risk free asset. The stock price in Black and Scholes market is based on a Geometric Brownian Motion where the stochastic element derives from the Wiener process. (Björk, 2004, chapter 4)

Definition 2.3.1 *A stochastic process W is callad a* ***Wiener process*** *if the following conditions hold:*

$i,$ $W(0) = 0$

$ii,$ *The process W has independent increments, i.e. if $r < s \leqslant t < u$ then $W(u) - W(t)$ and $W(s) - W(r)$ are independent stochastic variables.*

$iii,$ *For $s < t$ the stochastic variable $W(t) - W(s)$ has the Gaussian distribution $\mathcal{N}[0, \sqrt{t-s}]$.*

$iv,$ *W has continous trajectories.*

To get some basic understanding of a Wiener process we will consider a simple example. Assume that we were to simulate a process with an equidistant time-grid so that $\Delta_t = 1$. The realization of this process would than be a set of independent increments so that $W(t+1) - W(t) \sim \mathcal{N}[0, 1]$. In figure 2.1 we see a realization of this Wiener process with 200 time-steps.

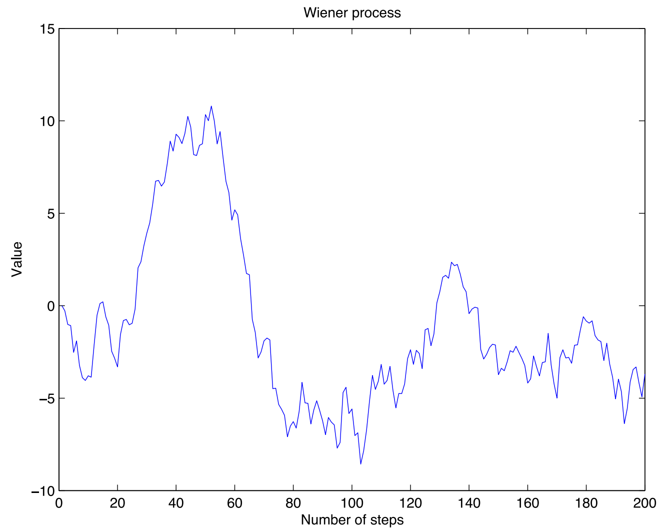

Figure 2.1: Simulation of a Wiener process with $\Delta_t = 1$.

With definition (2.3.1) in hand Black and Scholes discribes the dynamics of the stock price and the risk free asset as

$$dS_t = \mu S_t dt + \sigma S_t dW_t^{\mathbb{P}}, \tag{2.2}$$
$$dB_t = rB_t dt, \tag{2.3}$$

where μ, σ and r are deterministic constants and $W_t^{\mathbb{P}}$ is a Wiener process under the probability measure $\mathbb{P}$. We see that equation (2.2) is a stochastic differential equation, a SDE. If we want to solve a SDE, that is writing it in the form $S_t = \cdots$, we use Itô's formula.

Theorem 2.3.1 ***(Itô's formula)*** *Assume that the process X has a stochastic differential given by*

$$dX_t = \mu dt + \sigma dW_t,$$

where μ and σ are deterministic constants. Define the process Z by $Z_t = f(t, X_t)$. Then Z has a stochastic differential given by

$$df(t, X_t) = \left\{ \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} \right\} dt + \sigma \frac{\partial f}{\partial x} dW_t. \tag{2.4}$$

Proof: See Øksendal (2003, chapter 4)

By expressing a new variable $Z_t = ln(S_t)$ and applying Itô's formula we arrive at the following solutions for equations (2.2) and (2.3)

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t^{\mathbb{P}}}, \tag{2.5}$$

$$B_t = B_0 e^{rt}. \tag{2.6}$$

Remembering theorem (2.2.1) we want our $\mathbb{P}$ measure to be equivalent with our $\mathbb{Q}$ measure so that $\frac{S_t}{B_t}$ is a martingale. If we apply Itô's formula to $\frac{S_t}{B_t}$ we get the following dynamics under the probability measure $\mathbb{P}$

$$dS_t^B = S_t^B(\mu - r)dt + \sigma S_t^B dW_t^{\mathbb{P}}. \tag{2.7}$$

In order for this process to be a martingale we want it to have zero drift, which gives us that

$$\begin{aligned} E^{\mathbb{Q}}[dS_t^B] &= E^{\mathbb{Q}}[S_t^B(\mu - r)dt] + E^{\mathbb{Q}}[\sigma S_t^B dW_t^{\mathbb{P}}] \\ &= E^{\mathbb{Q}}[S_t^B(\mu - r)dt] + 0 \\ &= 0 \\ &\Rightarrow \mu = r. \end{aligned} \tag{2.8}$$

This leaves us with the fact that if our $\mathbb{P}$ measure is equivalent with our $\mathbb{Q}$ measure, the drift of the stock must be equal to the risk free rate. With this answer we see that equation (2.2) can be written as

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}, \tag{2.9}$$

which has the solution

$$S_t = S_0 e^{(r - \frac{\sigma^2}{2})t + \sigma dW_t^{\mathbb{Q}}}. \tag{2.10}$$

Furthermore we see that the measure $\mathbb{Q}$ is unique, which in accordance to theorem (2.2.2) gives us that the Black and Scholes market is complete.

□

2.4 The European Call Option

Having defined the Black and Scholes market we are ready to price our European call option under the risk-neutral measure $\mathbb{Q}$. The payoff function for the European call function is given below.

Definition 2.4.1 *A European call option have the following payoff function*

$$\Phi(S_T) = max[S_T - K, 0]. \tag{2.11}$$

To price this derivative we need the risk-neutral valuation formula.

Theorem 2.4.1 ***(Risk-neutral valuation formula)*** *The arbitrage free price at time t of the claim $\Phi(S)$ is given by*

$$F_{t,s} = e^{-r(T-t)} E^{\mathbb{Q}}_{t,s}[\Phi(S_T)]. \tag{2.12}$$

where $E^{\mathbb{Q}}$ is the expected value over probability measure $\mathbb{Q}$.

Proof: See Björk (2004, chapter 5)

If we use equation (2.12) on the European call option we arrive at the following result.

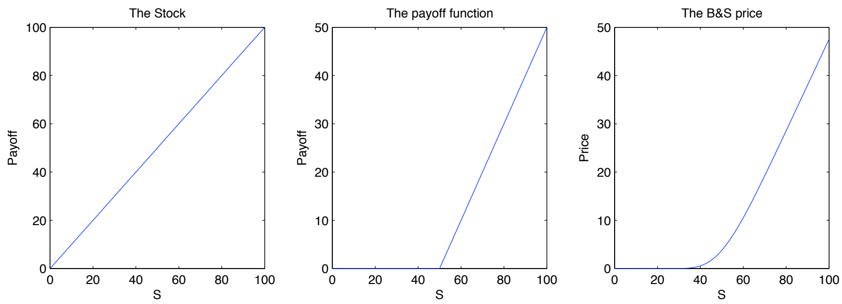

Figure 2.2: Payoff for a European call option with $K = 50, r = 0.05$ and $\sigma = 0.2$.

Theorem 2.4.2 ***(The Black and Scholes formula)*** *The price of a European call option at time t with the strike price K is given by*

$$F_{t,s} = S_t N(d_1) - e^{-r(T-t)} K N(d_2), \tag{2.13}$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left\{ ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T-t)\right\}, \tag{2.14a}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}. \tag{2.14b}$$

Proof: This derivation is taken from Rasmus (2005, chapter 6) and Björk (2004, chapter 7). The first thing we see is that we can write S_T as

$$S_T = S_t e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}z},$$

where $z \sim \mathcal{N}[0,1]$ and $\tau = T - t$. Given equation (2.12) we have that

$$\begin{aligned} e^{r\tau}F_{t,s} &= E^{\mathbb{Q}}_{t,s}[max(S_t - K, 0)] \\ &= E^{\mathbb{Q}}_{t,s}[S_T \mathbf{1}_{\{S_T > K\}}] - KE^{\mathbb{Q}}_{t,s}[\mathbf{1}_{\{S_T > K\}}] \\ &= E_1 + KE_2. \end{aligned}$$

By using the density function for $\mathcal{N}[0,1]$ we write E_1 as

$$\begin{aligned} E_1 &= \frac{1}{2\pi}\int_{z_0}^{\infty} S_t e^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}z} e^{-\frac{z^2}{2}} dz \\ &= \frac{se^{r\tau}}{2\pi}\int_{z_0}^{\infty} e^{-\frac{(z-\sigma\sqrt{\tau})^2}{2}} dz \\ &= S_t e^{r\tau} N[-z_0 + \sigma\sqrt{\tau}]. \end{aligned}$$

To get the value for z_0 we do the following

$$S_t e^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}z_0} = K \Leftrightarrow z_0 = \frac{ln(\frac{K}{S_t}) - (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}.$$

This gives us the desired result

$$E_1 = S_t e^{r\tau} N[d_1],$$

where

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}\left\{ ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(\tau) \right\}.$$

The term E_2 is much easier to solve

$$\begin{aligned} E_2 &= P^{\mathbb{Q}}[S_T > K] \\ &= P^{\mathbb{Q}}[S_t e^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}Z} > K] \\ &= N[d_2], \end{aligned}$$

where

$$d_2 = d_1 - \sigma\sqrt{\tau}.$$

□

Chapter 3

Transaction Costs

3.1 The Bank Point of View

To get a basic understanding of the effects of introducing transaction costs, we will start with a little example. Imagine that you were the CFO of large bank and that you would like to sell European call options. Now by selling these options the bank will get some initial income but it will at same time be exposed to a tremendous risk, since the underlying stock of the derivative can rise all time high. As a CFO you would like to reduce this risk by hedging the derivative. In the Black and Scholes market the optimal thing to do would be to create a self-financing replicating portfolio consisting of the underlying stock and the risk free asset. This portfolio would then be rebalanced every time the stock price changed and at maturity date the portfolio would give exactly the same payoff as the derivative. (Leland, 1985)

Now assume that there would be a transaction cost involved every time you needed to rebalance your portfolio. Then the dynamic replication strategy described would result in infinitely high transaction cost no matter how small the transaction costs would be or how short the time to maturity would be. This follows since the Brownian motion which drives the stock price has infinite variation. We conclude that there doesn't exist a continuous-time strategy that perfectly hedges the European call option. Clearly we realize that an other strategy must be engaged. Furthermore we realize that although the payoff of an option doesn't change when we introduce transaction costs, the price will. This naturally follows since the bank has to increase the price to cover for the transaction cost of the replicating portfolio. Furthermore we can no longer use Black and Scholes formula to calculate the price of an option since this formula doesn't include the expected transaction costs. (Leland, 1985)

Theoretically the transaction costs can be modulated in numerous different ways but mostly the transaction costs may be assumed to be fixed, proportional or a combination of both fixed and proportional. In this essay only proportional transaction costs will be used since this, in the bank point of view, is most realistic. As a private person we are used to pay a fixed cost when buying or selling a share but the banks which trade in much larger volumes are mainly exposed to proportional transaction costs. If a fixed cost is also included it would be so small in comparison to the proportional costs that we say it is insignificant. Furthermore almost every research that is done on the subject uses proportional transaction costs. This may also be explained by with using proportional transaction costs, we get continuous expressions, which are much easier to optimize analytically. (Constantinides and Zariphopoulou, 1999)

In this essay only the stock is exposed to proportional transaction costs. Including transaction costs on the risk free asset as well makes the model much more complicated, which is noted by Boyle and Vorst (1992). Not including transaction costs on the risk free asset doesn't have to make the model more unrealistic, since the risk free asset is usually interpreted as the bank account. Further, we assume that banks don't pay any transaction costs when moving money to their own accounts. Moreover, we will pay the same proportional transaction cost rate when we buy a stock as when we sell one.

3.2 A Discrete Time Approach

We have realized that we cannot rebalance our portfolio every time the stock price moves so we lessen our demand on creating a perfect replicating portfolio at all times by rebalancing only at certain time-steps. This leaves us with the choice of how often we want to rebalance. If we only rebalance our portfolio a few times our transaction costs will be low but our risk exposure will be high. Clearly we have to make a trade-of between risk and cost. Furthermore if we use a true arbitrage argument to bound our option prices it will be necessary to consider the maximum possible transaction cost rather than the average transaction costs. This follows since if we were to use an arbitrage argument, we would have to create a portfolio consisting of the stock and the derivative and eliminate all risk. Since the transaction costs also will be stochastic we have that eliminating all risk in this case includes eliminating the possibility of maximum transaction costs. This will in term give us that the upper bound on the option price would be extremely high. To get around this problem we will only consider the expected transaction costs of our replicating portfolio. An argument to support this approach of only calculating the expected transaction costs rather than the maximum transaction costs can be made by a simple example. (Leland, 1985)

Imagine that we would rebalance our portfolio twice every week for a derivative with one year to maturity. The maximum transaction cost would occur if the rebalancing between the stock and the risk free asset would be 100% at every time step. This would result in a turnover of 5200%[1], which is highly unlikely.

Even though both Leland and Boyle and Vorst only calculate with expected transaction cost, we still have the fact that if we let our time-step go to zero, the cost of creating a replicating portfolio will be infinite. Once again this leaves us with the result that there doesn't exist a strategy that perfectly replicates an option at all times. If we were let our time step go to zero we would soon see that the best strategy would be to dominate our call option by having a long position in the underlying stock at all times. The fact that this really is the optimal strategy when we let our time-step go to zero is proven by Soner et al. (1995). Now having the actual stock price as an upper bound for our option price isn't very interesting in the economic point of view. One can note that there are other theories which provide tighter upper bounds than the actual stock price when the time-step goes to zero. These theories include finding a martingale measure for a market with transaction costs and with this measure one can use a true arbitrage argument to price options in a viable price system. If the reader is interested, see Reisman (2001).

[1] $0.5 * 2 * 52 * 100\% = 5200\%$

3.3 A Modified Binomial Model

First we will create a model for proportional transaction costs within the binomial model by using the framework presented by Leland (1985) and Boyle and Vorst (1992). We will later use the results found in the binomial model to try creating a modified Black and Scholes formula, which include the expected transaction costs, by modifying the input volatility. We start by examining the dynamics of the binomial market without transaction costs.

3.3.1 The Dynamics of the Binomial Market

In the binomial model we have a time grid, $t_0, t_1, t_2, \cdots, t_n = T$, with equivalent time distances, $\Delta = t_i - t_{i-1}$. For every time step the stock may either go up, with a factor u and probability p_u, or down, with a factor d and probability p_d.

Definition 3.3.1 *The stock dynamics in an n period binomial model under the probability measure $\mathbb{P}$ is given by*

$$S_{t_i} = S_{t_{i-n}} Z_{\mathbb{P}}^n = \begin{cases} S_{t_{i-n}} u^n & \textit{with probability} \quad p_u^n \\ \vdots & \\ S_{t_{i-n}} u^{n-k} d^k & \textit{with probability} \quad \binom{n}{k} p_u^{n-k} p_d^k \\ \vdots & \\ S_{t_{i-n}} d^n & \textit{with probability} \quad p_d^n \end{cases}, \tag{3.1}$$

where $Z_{\mathbb{P}}$ is a stochastic variable with the distribution

$$Z_{\mathbb{P}} = \begin{cases} u & \textit{with probability} \quad p_u \\ d & \textit{with probability} \quad p_d \end{cases}. \tag{3.2}$$

$$\begin{array}{ccccc} & & & & Su^2 \\ & & & \nearrow & \\ & & Su & & \\ & \nearrow & & \searrow & \\ S & & & & Sud \\ & \searrow & & \nearrow & \\ & & Sd & & \\ & & & \searrow & \\ & & & & Sd^2 \end{array}$$

Remembering once again theorem (2.2.1), we want to find the probability distribution under the risk-neutral measure $\mathbb{Q}$ so that

$$E^{\mathbb{Q}}\left[\frac{S_{t_{i+1}}}{B_{t_{i+i}}}\right] = \frac{S_{t_i}}{B_{t_i}}.$$

Given the risk free asset B, which has the dynamics $B_{t_{i+1}} = B_{t_i} e^{r\Delta}$, we have the following distribution under the measure $\mathbb{Q}$.

Theorem 3.3.1 *Under the risk neutral measure $\mathbb{Q}$, the stochastic variable Z has the following distribution*

$$Z_{\mathbb{Q}} = \begin{cases} u & \textit{with probability} \quad q_u \\ d & \textit{with probability} \quad q_d \end{cases}, \tag{3.3}$$

where

$$q_u = \frac{e^{r\Delta} - d}{u - d}, \tag{3.4a}$$

$$q_d = \frac{u - e^{r\Delta}}{u - d}. \tag{3.4b}$$

Proof: See Cox, Ross, and Rubinstein (1979)

We also realize that $d \leqslant e^{r\Delta} \leqslant u$, to avoid arbitrage. Given this and the fact that our measure $\mathbb{Q}$ is unique, we have, in accordance to theorem (2.2.2), that the binomial model is both complete and free of arbitrage.

□

In our case we are going from a continuous time model to a discrete time model. To express the stock S in an n-period model with grid spacing Δ, we need a definition for u and d.

Definition 3.3.2 *Going from a continuous time model to a n-period discrete time model, the following values are used for u and d*

$$u_n = e^{\sigma\sqrt{\Delta}}, \tag{3.5}$$

$$d_n = e^{-\sigma\sqrt{\Delta}}, \tag{3.6}$$

where σ is the volatility of the stock.

Given the fact that the binomial market is complete, we know that we can create a self-financing replicating portfolio for all derivatives in this market. We let the pair (t_i, k) denote each node in the binomial tree, where k is the number of up-steps. The stock price at time t_i can then be written as

$$S_{t_i} = S_{t_0} u^k d^{i-k}, \tag{3.7}$$

and the replicating portfolio can be written as

$$V^h_{t_i}(k) = x_{t_i}(k) + y_{t_i}(k) S_{t_i}, \tag{3.8}$$

where x_{t_i} is the number of money units in the risk free asset and y_{t_i} is the position in the stock. With this we are ready to formulate a binomial algorithm that gives us the x_{t_i} and y_{t_i} of our replicating portfolio at each node (t_i, k) in the binomial tree.

Theorem 3.3.2 *A self-financing replicating portfolio, V^h, in the binomial market under the probability measure $\mathbb{Q}$ can be computed recursively using the scheme*

$$\begin{aligned} V^h_{t_i}(k) &= \frac{q_u V^h_{t_{i+1}}(k+1) + q_d V_{t_{i+1}}(k)}{e^{r\Delta}}, \\ V^h_T(k) &= \Phi(S_{t_o} u^k d^{i-k}). \end{aligned} \tag{3.9}$$

The weights in the replicating portfolio is then given by

$$x_{t_i}(k) = \frac{u V^h_{t_i}(k) - d V^h_{t_i}(k+1)}{(u-d) e^{r\Delta}}, \tag{3.10a}$$

$$y_{t_i}(k) = \frac{V^h_{t_i}(k+1) - V^h_{t_i}(k)}{(u-d) S_{t_i}}. \tag{3.10b}$$

$$
\begin{array}{ccccc}
 & & & & x_{t_2}(2), y_{t_2}(2) \\
 & & & \nearrow & \\
 & & x_{t_1}(1), y_{t_1}(1) & & \\
 & \nearrow & & \searrow & \\
x_{t_0}(0), y_{t_0}(0) & & & & x_{t_2}(1), y_{t_2}(1) \\
 & \searrow & & \nearrow & \\
 & & x_{t_1}(0), y_{t_1}(0) & & \\
 & & & \searrow & \\
 & & & & x_{t_2}(0), y_{t_2}(0)
\end{array}
$$

Proof: See Cox, Ross, and Rubinstein (1979)

If we want to price European options in the binomial market, we use the risk-neutral valuation formula (2.12). Applying this formula we end up with the following expression.

Theorem 3.3.3 *The arbitrage free price at time t_i of a European option with payoff function $\Phi(S)$ is given by*

$$F_{t_i,s} = \frac{1}{e^{r\Delta(n-i)}} \sum_{k=0}^{n-i} \binom{n-i}{k} q_u^k q_d^{(n-i)-k} \Phi(S_{t_i} u^k d^{(n-i)-k}). \tag{3.11}$$

Proof: See Cox, Ross, and Rubinstein (1979)

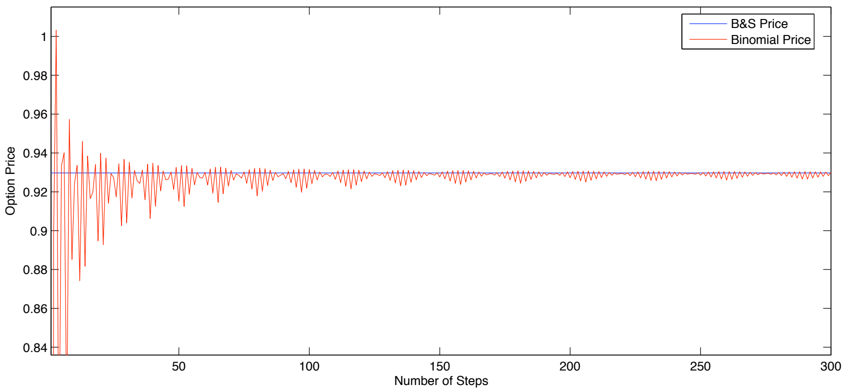

Figure 3.1: Price of a European call option with $S_{t_0} = 40$, $K = 50$, $r = 0.05$ and $\sigma = 0.2$.

As seen in figure (3.1), the binomial option price converge to the Black and Scholes price when the number of steps get large. This is not a surprising fact, given the law of large numbers and the central limit theorem, which yields that the stock in the binomial model has a log-normal distribution just like in the Black and Scholes model. However it is an important fact in our model, since we will use the binomial model to try to find an alternative Black and Scholes formula. (Cox et al., 1979)

3.3.2 Dynamics with Transaction Costs

We start by introducing the proportional transaction cost rate for the stock, which will be denoted λ. In order to keep our replicating portfolio self-financing, we must at time

t_i calculate our x_{t_i} and y_{t_i} so that the cost of rebalancing at time t_{i+1} is included in the portfolio.

Definition 3.3.3 *In order for our replicating portfolio to be self-financing when proportional transaction costs are present, the following conditions must hold:*

$$x_{t_i}(k)e^{r\Delta} + y_{t_i}(k)S_{t_i}u = x_{t_{i+1}}(k+1) + y_{t_{i+1}}(k+1)S_{t_i}u + \lambda|y_{t_i}(k) - y_{t_{i+1}}(k+1)|S_{t_i}u \quad (3.12)$$

$$x_{t_i}(k)e^{r\Delta} + y_{t_i}(k)S_{t_i}d = x_{t_{i+1}}(k) + y_{t_{i+1}}(k)S_{t_i}d + \lambda|y_{t_i}(k) - y_{t_{i+1}}(k)|S_{t_i}d \quad (3.13)$$

We see that equation (3.12) expresses that the value of the portfolio if the stock goes up is exactly enough to buy the replicating portfolio and cover the transaction costs, whereas equation (3.13) has similar interpretation if the stock goes down. (Boyle and Vorst, 1992)

In case of the European call option we realize that

$$y_{t_{i+1}}(k) \leqslant y_{t_i}(k) \leqslant y_{t_{i+1}}(k+1). \quad (3.14)$$

This follows since if our stock price goes up, our call option will be worth more because the chance that it will be in the money at time T will increase. To match this situation with our replicating portfolio we take a larger position in the underlying stock. For the European put option the equalities are reversed so that

$$y_{t_{i+1}}(k) \geqslant y_{t_i}(k) \geqslant y_{t_{i+1}}(k+1). \quad (3.15)$$

We will continue our calculations on the European call option and we see that we can rewrite equations (3.12) and (3.13) as

$$x_{t_i}(k)e^{r\Delta} + y_{t_i}(k)S_{t_i}\overline{u} = x_{t_{i+1}}(k+1) + y_{t_{i+1}}(k+1)S_{t_i}\overline{u}, \quad (3.16)$$
$$x_{t_i}(k)e^{r\Delta} + y_{t_i}(k)S_{t_i}\overline{d} = x_{t_{i+1}}(k) + y_{t_{i+1}}(k)S_{t_i}\overline{d}, \quad (3.17)$$

where

$$\overline{u} = u(1+\lambda),$$
$$\overline{d} = d(1-\lambda).$$

Equations (3.16) and (3.17) are both linear and can easily be solved. Another very important fact in the model presented by Leland (1985) and Boyle and Vorst (1992) is that they assume that buying the initial replicating portfolio as well as selling the replicating portfolio at maturity date is free in terms of transaction costs. That is we only calculate the transaction costs necessary to maintain the replicating portfolio until maturity date. In section 3.5 a framework to include these costs will be presented but until then we follow the model presented by Leland and Boyle and Vorst.

We are now ready to formulate a simple algorithm for calculating the weights of the replicating portfolio throughout our binomial tree.

1. Look at all the nodes at time t_n. For every node in which $\Phi(S_{t_n}) > 0$ we take a long position in the underlying stock and a short position in the risk free asset equal to the strike price. That is $y_{t_n} = 1$ and $x_{t_n} = -K$ if $\Phi(S_{t_n}) > 0$, else $y_{t_n} = 0$ and $x_{t_n} = 0$.

2. Go one time-step back. Use equations (3.16) and (3.17) to calculate y_{t_i} and x_{t_i} for every node in this time step.
3. If $i > 0$, goto 2, else quit.

After finishing the algorithm, we can easily calculate the value of a self-financing replicating portfolio at any node (t_i, k) by the following formula

$$V_{t_i}^h(k) = x_{t_i}(k) + y_{t_i}(k) S_{t_i}. \tag{3.18}$$

Given equation (3.1), we see that the price of at European call option without transaction costs is given by the discounted expectation of the maturity value of the option. We will now use this approach in the case of proportional transaction costs. We have that

$$\begin{aligned} V_{t_0}^h(0) &= x_{t_0}(0) + y_{t_0}(0) S_{t_0} \\ &= \big[p_u\{x_{t_1}(1) + y_{t_1}(1) S_{t_0} u(1+\lambda)\} \\ &+ p_d\{x_{t_1}(0) + y_{t_1}(0) S_{t_0} d(1-\lambda)\}\big]/e^{r\Delta}. \end{aligned} \tag{3.19}$$

If we extend to two steps we get

$$\begin{aligned} V_{t_0}^h(0) &= x_{t_0}(0) + y_{t_0}(0) S_{t_0} \\ &= \big[p_u p_{uu}\{x_{t_2}(2) + y_{t_2}(2) S_{t_0} u^2(1+\lambda)\} \\ &+ p_u p_{ud}\{x_{t_2}(1) + y_{t_2}(1) S_{t_0} ud(1-\lambda)\} \\ &+ p_d p_{du}\{x_{t_2}(1) + y_{t_2}(1) S_{t_0} ud(1+\lambda)\} \\ &+ p_d p_{dd}\{x_{t_2}(0) + y_{t_2}(0) S_{t_0} d^2(1-\lambda)\big]/e^{r2\Delta}. \end{aligned} \tag{3.20}$$

In particular we have that

$$\begin{aligned} x_{t_1}(1) + y_{t_1}(1) S_{t_0}\overline{u} = \big[&p_{uu}\{x_{t_2}(2) + y_{t_2}(2) S_{t_0}\overline{u}u\} \\ &+ p_{ud}\{x_{t_2}(1) + y_{t_2}(1) S_{t_0} u\overline{d}\}\big]/e^{r\Delta}, \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} x_{t_1}(0) + y_{t_1}(0) S_{t_0}\overline{d} = \big[&p_{du}\{x_{t_2}(1) + y_{t_2}(1) S_{t_0}\overline{u}d\} \\ &+ p_{dd}\{x_{t_2}(0) + y_{t_2}(0) S_{t_0}\overline{d}d\}\big]/e^{r\Delta}, \end{aligned} \tag{3.22}$$

which in after some reshuffling yields that

$$p_{uu} = \frac{e^{r\Delta}(1+\lambda) - \overline{d}}{(\overline{u} - \overline{d})} \qquad \text{and} \qquad p_{ud} = (1 - p_{uu}), \tag{3.23}$$

$$p_{du} = \frac{e^{r\Delta}(1-\lambda) - \overline{d}}{(\overline{u} - \overline{d})} \qquad \text{and} \qquad p_{dd} = (1 - p_{du}). \tag{3.24}$$

With p_{uu}, we mean the probability that the stock should go up if the last step also was an up-step, and with p_{du}, we mean the probability that the stock should go up if the last step was a down-step. By looking at equations (3.23) and (3.24), we immediately realize that $0 < p_{du} < p_{uu} < 1$ if $\lambda > 0$. In other words we say that our process has a "memory" of one step. This is an interesting fact since it results in a totally new process that differs from the binomial process both under the probability measure $\mathbb{P}$ and under the probability measure $\mathbb{Q}$. We will call this new probability measure $\mathbb{O}$ and denote the initial probabilities with (o_u, o_d), and the probabilities with "memory" with (o_{uu}, o_{du}). Another way of interpreting the process $\mathbb{O}$ is by saying that it has two states, e_1 and e_2. We do the following definition. (Boyle and Vorst, 1992)

Definition 3.3.4 *Our stochastic variable Z under the probability measure $\mathbb{O}$ is given by*

$$\begin{aligned} Z_{\mathbb{O}}(e_1) &= \begin{cases} u & \textit{with probability} \quad o_{uu} \\ d & \textit{with probability} \quad o_{ud} \end{cases}, \\ Z_{\mathbb{O}}(e_2) &= \begin{cases} u & \textit{with probability} \quad o_{du} \\ d & \textit{with probability} \quad o_{dd} \end{cases}. \end{aligned} \tag{3.25}$$

With e_i we define the state in a Markov chain given by

$$\overline{\mathbb{O}} = \begin{pmatrix} o_{uu} & o_{ud} \\ o_{du} & o_{dd} \end{pmatrix}, \tag{3.26}$$

where the initial distribution is given by

$$\begin{pmatrix} o_u & o_d \end{pmatrix}$$

Note that theorems (2.2.1) and (2.2.2) are not applicable to our new process $Z_{\mathbb{O}}$. This follows since this process only describes the movement of our replicating portfolio with transaction costs and not the actual process of any market asset.

3.4 A Modified Black and Scholes formula

We will now use the results from the previous section to develop a modified Black and Scholes formula, which take in count the expected transaction costs. To start with we introduce a new stochastic variable X, which has the same probability distribution as $Z_{\mathbb{O}}$ but with the following corresponding values (Boyle and Vorst, 1992)

Definition 3.4.1 *Our stochastic variable X under the probability measure $\mathbb{O}$ is given by*

$$\begin{aligned} X(e_1) &= \begin{cases} ln(u) & \textit{with probability} \quad o_{uu} \\ ln(d) & \textit{with probability} \quad o_{ud} \end{cases}, \\ X(e_2) &= \begin{cases} ln(u) & \textit{with probability} \quad o_{du} \\ ln(d) & \textit{with probability} \quad o_{dd} \end{cases}. \end{aligned} \tag{3.27}$$

Our initial distribution is still given by $\begin{pmatrix} o_u & o_d \end{pmatrix}$. By following equations (3.19) and (3.20), we receive the following result if we let the number of periods increase.

Theorem 3.4.1 *The value of a replicating portfolio at time t_0 is given by*

$$V^h_{t_0} = \frac{E\left[\{(1+\widehat{X_n}\lambda)S_{t_0}e^Y - K\}I_{\{S_{t_0}e^Y \geqslant K\}}\right]}{e^{r\Delta n}}, \tag{3.28}$$

where

$$Y = \sum_{k=1}^{n} X_k, \tag{3.29a}$$

$$\widehat{X_n} = 1 \quad \textit{if} \quad X_n = ln(u), \tag{3.29b}$$

$$\widehat{X_n} = -1 \quad \textit{if} \quad X_n = ln(d). \tag{3.29c}$$

Proof: Look at equations (3.19) and (3.20). Extend the number of periods and use an induction argument.

It is easily seen that if we set $\lambda = 0$ the above theorem will transform into the risk-neutral valuation formula, (2.12), for the European call option. This follows since if we set our transaction costs to zero we get that

$$\begin{aligned} o_{uu} &= o_{du} = q_u, \\ o_{ud} &= o_{dd} = q_d, \end{aligned}$$

and we're back to the probability measure $\mathbb{Q}$.

We will continue our calculations by examining the variable Y. The values used for u and d will be given by definition (3.3.2). By simple reasoning we immediately realize that the higher the transaction costs are the higher the volatility will be. This follows since if our first step is an up-step the probability for another up-step and a high value of Y will be higher. The contrary will occur if our first step is a down-step. The result of this leads to a higher volatility for Y. We continue be giving two helpful results.

Lemma 1 *The variance and expected value of Y has the following behavior for large n and small λ*

$$V(Y) = \sigma^2\left[1 + O(\lambda^2) + \left\{\frac{2\lambda}{\sigma} + O(\lambda^3)\right\}\sqrt{\frac{n}{T}}\right] + O\left(\sqrt{\frac{T}{n}}\right), \tag{3.30}$$

$$E(Y) = r - \frac{1}{2}\{V(Y)\} + O\left(\sqrt{\frac{T}{n}}\right) + O(\lambda^2). \tag{3.31}$$

Lemma 2 *The covariance of $(\widehat{X_n}\lambda, Y)$ and the expected value of $\widehat{X_n}\lambda$ has the following behavior for large n and small λ*

$$C(\widehat{X_n}\lambda, Y) = 4\lambda^2 + O\left(\sqrt{\frac{T}{n}}\right), \tag{3.32}$$

$$E(\widehat{X_n}\lambda) = -\lambda\left\{\lambda + O\left(\sqrt{\frac{T}{n}}\right)\right\}. \tag{3.33}$$

Proof: See Boyle and Vorst (1992)[2]

The function $O(\cdot)$ is called ordo and can be treated as zero when its arguments get small. Given this we do the following approximation for large n and small λ

$$\begin{aligned} V(Y) &\approx \sigma^2\left(1 + \frac{2\lambda}{\sigma}\sqrt{\frac{n}{T}}\right) = \widehat{\sigma}^2, \\ E(Y) &\approx r - \frac{1}{2}\widehat{\sigma}^2, \\ C(\widehat{X_n}\lambda, Y) &\approx 0, \\ E(\widehat{X_n}\lambda) &\approx 0. \end{aligned} \tag{3.34}$$

Finally we are ready to give the modified Black and Scholes formula.

[2]Note that Boyle and Vorst (1992) has set $T = 1$ in their derivation.

Theorem 3.4.2 *For large n and small λ the initial value, $t = 0$, of a self-financing replicating portfolio is approximately equal to the Black and Scholes formula, (2.13), with a modified variance given by*

$$\sigma^2 \left(1 + \frac{2\lambda}{\sigma} \sqrt{\frac{n}{T}}\right). \tag{3.35}$$

Proof: First we remind us that given the law of large numbers and the central limit theorem, the process e^Y has a lognormal distribution for large n. This gives us that S_T can be written as

$$S_T = S_{t_0} e^{(r - \frac{1}{2}\widehat{\sigma}^2)T + \widehat{\sigma}\sqrt{T}z},$$

where $z \sim \mathcal{N}[0, 1]$. With this in hand the proof is analogues with that of the Black and Scholes formula on page 8 with $t = 0$.

□

3.5 An Extension to the Boyle and Vorst Model

In the models presented by Leland (1985) and Boyle and Vorst (1992) no transaction costs are calculated with for buying the initial hedge at time t_0, as well as selling it at time t_n. However an extension to their model that includes these transaction costs will be presented in this section. The reason for including these transaction costs is that they do have great importance for the price of an option as well as they make the model more realistic. If we want to price an option it naturally follows that we have to include all transaction costs, not only the inter-temporal transaction costs. All the theory in this section will be my own derivations.

We will start by looking on how we could modify the binomial model to include the initial transaction cost at time t_0 and transaction cost of selling the hedge at time t_n. To include the initial transaction cost is easy and it is given by $\lambda y_{t_0} S_{t_0}$. Including the transaction cost of selling the hedge is a little bit more tricky. If we were to use a pure arbitrage argument we would need to calculate with the maximum possible transaction cost at maturity date, but we want our extension to be consistent with the models presented by Leland and Boyle and Vorst. Therefor only the expected transaction cost of selling will be considered. To get an adequate pricing formula we need to add the discounted expected transaction cost of selling to the value of the replicating portfolio at time t_0. The discounted expected transaction cost of selling is given by

$$\frac{\sum_{k \in I} \binom{n}{k} q_u^k q_n^{n-k} S_{t_0} u^k d^{n-k} \lambda}{e^{rT}}, \tag{3.36}$$

where I denotes the set of k so that $\Phi(S_{t_0} u^k d^{n-k}) > 0$. That is if our European call option is in the money. Having included these transaction costs an upper bound for the price of the European call option can be given.

Theorem 3.5.1 *The price of a European call option in a binomial market with proportional transaction costs can be given by*

$$F_{t_0,s} = x_{t_0} + y_{t_0} S_{t_0} + \lambda y_{t_0} S_{t_0} + \frac{\sum_{k \in I} \binom{n}{k} q_u^k q_n^{n-k} S_{t_0} u^k d^{n-k} \lambda}{e^{rT}}, \tag{3.37}$$

where I is the set of k so that $\Phi(S_{t_0} u^k d^{n-k}) > 0$.

Proof: Add the initial transaction cost and the discounted expected transaction cost of selling to the value of the replicating portfolio provided by the Boyle and Vorst (1992) framework.

□

To provide a method to include the initial transaction cost and the transaction of selling in the Black and Scholes approximation we will look at the delta at time t_0 and the the discounted expected cost of selling in accordance to the Black and Scholes market.

Theorem 3.5.2 *The delta of a European call option is given by*

$$\Delta = \frac{\partial F_{t,s}}{\partial s} = N[d_1] \tag{3.38}$$

Proof: See Björk (2004, chapter 9).

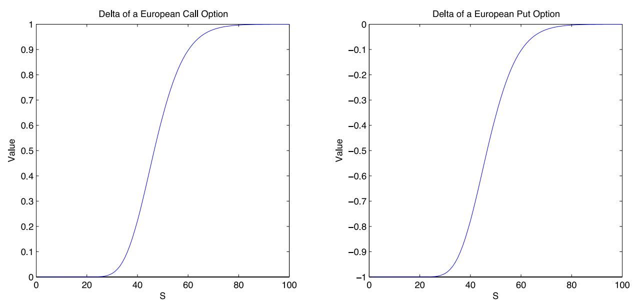

Figure 3.2: Delta of european options. $K = 100$, $r = 0.05$, $\sigma = 0.2$ and $T = 1$.

The expected cost of selling in the Black and Scholes market is given by (compare with the derivation on page 8)

$$\begin{aligned} E[\lambda S_T \mathbf{1}_{\{S_T > K\}}] &= \frac{\lambda}{2\pi} \int_{z_0}^{\infty} S_{t_0} e^{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}z} e^{-\frac{z^2}{2}} dz \\ &= \frac{\lambda S_{t_0} e^{rT}}{2\pi} \int_{z_0}^{\infty} e^{-\frac{(z-\sigma\sqrt{T})^2}{2}} dz \\ &= \lambda S_{t_0} e^{rT} N[-z_0 + \sigma\sqrt{T}] \\ &= \lambda S_{t_0} e^{rT} N[d_1]. \end{aligned} \tag{3.39}$$

With this we have a method to extend the modified Black and Scholes model to work with the initial transaction cost and the transaction cost of selling.

Theorem 3.5.3 *The price of a European call option in a Black and Scholes market with proportional transaction costs can be given by*

$$F_{t_0,s} = S_{t_0} N[d_1(\hat{\sigma})] - e^{-rT} K N[d_2(\hat{\sigma})] + 2\lambda S_{t_0} N[d_1(\sigma)], \tag{3.40}$$

where $\hat{\sigma}$ is the modified input volatility given by equation (3.35).

Proof: Add the initial transaction cost and the discounted expected transaction cost to the modified Black and Scholes formula.

□

Chapter 4

Simulation and Results

4.1 Method

All the simulations are programmed and executed with Matlab. Furthermore all the graphic is created with Matlab. With the finished code we simulated and compared the process with a number of different input variables. The result and conclusions of these simulations will be covered in the next two sections.

4.2 Results

To get a good view of how the binomial algorithm works we look at table 4.1. We see, if we look at the last column of the last two matrixes, that the values are equal and that we exactly replicate the payoff of a European call option. When we start moving backwards through the matrixes though, the value of the replicating portfolio with transaction costs gets higher and higher in comparison to the one without. This is well expected and shows us that the algorithm presented on page 16 does indeed give us the value for a self-financing replicating portfolio.

We will now test the modified binomial presented by Leland (1985) and Boyle and Vorst (1992) with a number of different input values. The result is shown in table 4.2. We have chosen to rebalance our portfolio once a month, once a week or once every trading day[1]. We notice that the number of times we rebalance our portfolio has great impact on the transaction cost. The result of hedging every day instead of every month is a 4-5 times higher transaction costs. Furthermore we see that the further out of the money we write our call option the higher the relative transaction costs will be. The difference between an option that is written 20% under the current stock price to one that is written 20% over the current stock price is a 18-20 times higher cost increase due to transaction costs. This follows since an option that is written far out of the money starts with a small y_{t_0} compared to one that is written in the money. And if our option that is written out of the money gets in the money we will have to buy much larger proportions of the stock at a higher stock price. These cases must be taken into account in the model and results in a higher difference of price on options written out of the money.

[1]The number of trading days of OMX is equal to 253 during 2005.

t_0	t_1	t_2	t_3	t_4	t_5
		S_{t_i}			
100.0000	109.3565	119.5884	130.7776	143.0138	156.3948
0	91.4441	100.0000	109.3565	119.5884	130.7776
0	0	83.6202	91.4441	100.0000	109.3565
0	0	0	76.4657	83.6202	91.4441
0	0	0	0	69.9233	76.4657
0	0	0	0	0	63.9407
		x_{t_i}			
-50.3645	-66.2051	-83.0079	-98.0199	-99.0050	-100.0000
0	-33.7235	-49.8053	-71.0066	-99.0050	-100.0000
0	0	-15.3208	-26.8395	-47.0183	-100.0000
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
		y_{t_i}			
0.6202	0.7591	0.8927	1.0000	1.0000	1.0000
0	0.4308	0.5867	0.7741	1.0000	1.0000
0	0	0.2044	0.3275	0.5246	1.0000
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
		$V^h_{t_i}(\lambda = 0)$			
10.8059	16.1248	23.3853	32.7578	44.0088	56.3948
0	4.9497	8.1603	13.1596	20.5834	30.7776
0	0	1.3808	2.6129	4.9444	9.3565
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
		$V^h_{t_i}(\lambda \neq 0)$			
11.6576	16.8027	23.7441	32.7578	44.0088	56.3948
0	5.6717	8.8629	13.6466	20.5834	30.7776
0	0	1.7728	3.1057	5.4406	9.3565
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

Table 4.1: Output result with $S_{t_0} = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.2$, $\lambda = 0.01$, $\Delta = 0.2$ and $T = 1$

To price a European call option we have to include the transaction cost of buying the initial hedge and the transaction cost of selling the hedge at maturity date. In table 4.3 we see a simulation on the modified Boyle and Vorst binomial model that includes these cost. Not surprisingly including these cost makes the difference to the no transaction cost case even higher. What is interesting is that the discrepancy between an option that is written in the money to one that is written out of the money decreases drastically. The difference between an option that is written 20% under the current stock price to one that is written 20% over the current stock price is now only a 9-11 times higher relative cost increase due to transaction costs. This follows since if an option that is written in the money it starts with a larger proportion in y_{t_0} and has a higher probability to be in the money at maturity date. When we include transaction costs for buying and selling the hedge, the options that are written in the money will naturally be effected the most. Finally in table 4.4 we see a comparison between Boyle and Vorst's binomial model and the modified Boyle and Vorst binomial model. We see that the difference between rebalancing once a week to once a day is very small, which is well expected. The only reason for why we see a small difference is that the expected cost of selling changes (and converges due to the Law of Large Numbers) when the number rebalancing times increases. Either way we see that including the transaction cost of buying the hedge and the transaction cost of selling the hedge has a significant importance.

We will now take a look on the performance of our Black and Scholes approximation presented at page 20. In table 4.5 we see a comparison between Boyle and Vorst's binomial model and Boyle and Vorst's Black and Scholes approximation. We notice that the difference between these models is very small and gets even smaller as the number of time-steps increase, which is well expected given the law of large numbers and the central limit theorem. Generally we can conclude that Boyle and Vorst's binomial model gives us slightly larger values on our replicating portfolio. We also notice that the difference between these models increase greatly when our λ increases. Still the maximum difference between the models in our simulation isn't more than about 0.5%. The reason for these differences can be explained by that in Boyle and Vorst's approximation they set a few terms to zero that might have had some small effect on results. Especially those terms that included λ would have had a larger impact when we increase the transaction cost. In figures 4.1 and 4.2, we see a graphical comparison between the original Black and Scholes formula and the Boyle and Vorst's Black and Scholes approximation.

To price an option with the Black and Scholes approximation we need to add the transaction costs of buying the hedge and selling the hedge at maturity date. This is done by using our extension to the Boyle and Vorst approximation, see equation (3.40) on page 21. In table 4.6 we see a comparison between the modified Boyle and Vorts binomial model and the modified Boyle and Vorst Black and Scholes approximation. We see that the difference between these two models decreases when the number of time-steps increases, which is well expected. The difference between our two models increases a little bit more when we increase the transaction cost than it does with Boyle and Vorst's models. Still the differences between our modified Boyle and Vorst models are very small and are seldom over 0.5%. In figure 4.3 we see a plot on our Boyle and Vorst extension together with Boyle and Vorst's Black and Scholes approximation. We see that the spread between these to models only depends on the transaction cost, which is well expected. The only thing that happens when we increase the number of rebalancing times is that the relative price increase, for including the transaction costs of buying the hedge and selling the hedge, gets smaller. Clearly figure 4.3 gives us another proof of that that including these transaction costs has significant importance.

	T/Δ					
	12	52	253	12	52	253
K	$V^h_{t_0}$			$\Delta[V^h_{t_0}(\lambda \neq 0), V^h_{t_0}(\lambda = 0)]$ in %		
			$\lambda = 0\%$			
80	24.5507	24.5739	24.5903	0	0	0
90	16.6601	16.6915	16.7030	0	0	0
100	10.2858	10.4122	10.4575	0	0	0
110	6.0333	6.0558	6.0434	0	0	0
120	3.3200	3.2508	3.2477	0	0	0
			$\lambda = 0.125\%$			
80	24.6120	24.6988	24.8714	0.2497	0.5082	1.1429
90	16.7832	16.9368	17.2330	0.7388	1.4694	3.1732
100	10.4565	10.7483	11.1723	1.6591	3.2275	6.8348
110	6.2090	6.4071	6.7967	2.9129	5.8003	12.4657
120	3.4671	3.5543	3.9069	4.4317	9.3375	20.2993
			$\lambda = 0.25\%$			
80	24.6745	24.8277	25.1658	0.5044	1.0327	2.3405
90	16.9053	17.1776	17.7414	1.4718	2.9121	6.2167
100	10.6238	11.0713	11.8323	3.2860	6.3303	13.1462
110	6.3815	6.7449	7.4918	5.7713	11.3779	23.9680
120	3.6128	3.8506	4.5320	8.8198	18.4526	39.5472
			$\lambda = 0.5\%$			
80	24.8027	25.0940	25.7721	1.0266	2.1166	4.8058
90	17.1466	17.6461	18.6984	2.9202	5.7188	11.9461
100	10.9494	11.6839	13.0282	6.4507	12.2137	24.5819
110	6.7173	7.3855	8.7504	11.3373	21.9569	44.7928
120	3.8999	4.4230	5.6952	17.4677	36.0597	75.3631
			$\lambda = 1\%$			
80	25.0690	25.6454	26.9821	2.1111	4.3602	9.7264
90	17.6177	18.5337	20.4164	5.7480	11.0370	22.2319
100	11.5683	12.8020	15.0801	12.4683	22.9514	44.2032
110	7.3568	8.5551	10.9077	21.9371	41.2712	80.4902
120	4.4579	5.4964	7.7548	34.2746	69.0783	138.7831
			$\lambda = 2\%$			
80	25.6253	26.7590	29.2396	4.3773	8.8917	18.9070
90	18.5168	20.1444	23.3149	11.1446	20.6865	39.5847
100	12.7046	14.7399	18.3955	23.5155	41.5638	75.9072
110	8.5331	10.5832	14.3904	41.4342	74.7611	138.1183
120	5.5157	7.4195	11.1842	66.1352	128.2358	244.3791

Table 4.2: Simulation with the Boyle and Vorst's binomial model. $S_{t_0} = 100$, $r = 0.05$ and $T = 1$.

	T/Δ					
	12	52	253	12	52	253
K		$F_{t_0,s}$		$\Delta[F_{t_0,s}(\lambda \neq 0), F_{t_0,s}(\lambda = 0)]$ in %		
			$\lambda = 0\%$			
80	24.5507	24.5739	24.5903	0	0	0
90	16.6601	16.6915	16.7030	0	0	0
100	10.2858	10.4122	10.4575	0	0	0
110	6.0333	6.0558	6.0434	0	0	0
120	3.3200	3.2508	3.2477	0	0	0
			$\lambda = 0.125\%$			
80	24.8391	24.9322	25.1018	1.1749	1.4581	2.0799
90	16.9766	17.1343	17.4326	1.8996	2.6529	4.3680
100	10.6017	10.9006	11.3309	3.0704	4.6901	8.3516
110	6.3306	6.5231	6.9093	4.9279	7.7166	14.3281
120	3.5412	3.6249	3.9829	6.6638	11.5086	22.6401
			$\lambda = 0.25\%$			
80	25.1278	25.2927	25.6232	2.3509	2.9250	4.2002
90	17.2911	17.5709	18.1374	3.7875	5.2687	8.5875
100	10.9139	11.3755	12.1488	6.1062	9.2510	16.1726
110	6.6253	6.9782	7.7193	9.8132	15.2318	27.7323
120	3.7624	3.9944	4.6888	13.3245	22.8734	44.3760
			$\lambda = 0.5\%$			
80	25.7057	26.0175	26.6752	4.7046	5.8743	8.4786
90	17.9147	18.4267	19.4810	7.5302	10.3958	16.6318
100	11.5287	12.2907	13.6594	12.0831	18.0411	30.6183
110	7.2077	7.8568	9.2130	19.4652	29.7392	52.4481
120	4.2041	4.7194	6.0239	26.6293	45.1775	85.4844
			$\lambda = 1\%$			
80	26.8616	27.4702	28.7563	9.4127	11.7859	16.9415
90	19.1416	20.0770	21.9595	14.8948	20.2828	31.4701
100	12.7245	14.0120	16.3410	23.7088	34.5731	56.2605
110	8.3469	9.5123	11.8551	38.3474	57.0766	96.1665
120	5.0841	6.1179	8.4541	53.1352	88.1975	160.3140
			$\lambda = 2\%$			
80	29.1655	30.3452	32.7174	18.7972	23.4853	33.0501
90	21.5279	23.1863	26.3616	29.2181	38.9106	57.8257
100	15.0112	17.1560	20.9252	45.9404	64.7679	100.0969
110	10.5431	12.5401	16.3437	74.7494	107.0745	170.4398
120	6.8255	8.7438	12.6856	105.5888	168.9736	290.6075

Table 4.3: Simulation with the modified Boyle and Vorst binomial model. $S_{t_0} = 100$, $r = 0.05$ and $T = 1$.

	T/Δ					
	52	52	52	253	253	253
K	M Bin	MM Bin	Δ in %	M Bin	MM Bin	Δ in %
			$\lambda = 0.125\%$			
80	24.6988	24.9322	0.9450	24.8714	25.1018	0.9264
90	16.9368	17.1343	1.1664	17.2330	17.4326	1.1581
100	10.7483	10.9006	1.4169	11.1723	11.3309	1.4198
110	6.4071	6.5231	1.8113	6.7967	6.9093	1.6559
120	3.5543	3.6249	1.9857	3.9069	3.9829	1.9458
			$\lambda = 0.25\%$			
80	24.8277	25.2927	1.8730	25.1658	25.6232	1.8172
90	17.1776	17.5709	2.2899	17.7414	18.1374	2.2320
100	11.0713	11.3755	2.7468	11.8323	12.1488	2.6748
110	6.7449	6.9782	3.4602	7.4918	7.7193	3.0365
120	3.8506	3.9944	3.7322	4.5320	4.6888	3.4603
			$\lambda = 0.5\%$			
80	25.0940	26.0175	3.6799	25.7721	26.6752	3.5043
90	17.6461	18.4267	4.4240	18.6984	19.4810	4.1857
100	11.6839	12.2907	5.1931	13.0282	13.6594	4.8453
110	7.3855	7.8568	6.3812	8.7504	9.2130	5.2870
120	4.4230	4.7194	6.7014	5.6952	6.0239	5.7716
			$\lambda = 1\%$			
80	25.6454	27.4702	7.1155	26.9821	28.7563	6.5755
90	18.5337	20.0770	8.3268	20.4164	21.9595	7.5579
100	12.8020	14.0120	9.4523	15.0801	16.3410	8.3613
110	8.5551	9.5123	11.1880	10.9077	11.8551	8.6854
120	5.4964	6.1179	11.3079	7.7548	8.4541	9.0169
			$\lambda = 2\%$			
80	26.7590	30.3452	13.4019	29.2396	32.7174	11.8943
90	20.1444	23.1863	15.1004	23.3149	26.3616	13.0680
100	14.7399	17.1560	16.3912	18.3955	20.9252	13.7514
110	10.5832	12.5401	18.4900	14.3904	16.3437	13.5737
120	7.4195	8.7438	17.8490	11.1842	12.6856	13.4237

Table 4.4: Simulation with Boyle and Vorst's binomial model, M Bin, and our modified Boyle and Vorst binomial model, MM Bin. $S_{t_0} = 100$, $r = 0.05$ and $T = 1$.

	T/Δ					
	52	52	52	253	253	253
K	M Bin	M B&S	Δ in %	M Bin	M B&S	Δ in %
$\lambda = 0\%$						
80	24.5739	24.5888	-0.0606	24.5903	24.5888	0.0060
90	16.6915	16.6994	-0.0475	16.7030	16.6994	0.0213
100	10.4122	10.4506	-0.3671	10.4575	10.4506	0.0663
110	6.0558	6.0401	0.2606	6.0434	6.0401	0.0544
120	3.2508	3.2475	0.1019	3.2477	3.2475	0.0054
$\lambda = 0.125\%$						
80	24.6988	24.7139	-0.0608	24.8714	24.8694	0.0078
90	16.9368	16.9420	-0.0310	17.2330	17.2285	0.0263
100	10.7483	10.7819	-0.3118	11.1723	11.1644	0.0701
110	6.4071	6.3894	0.2764	6.7967	6.7926	0.0605
120	3.5543	3.5507	0.1017	3.9069	3.9064	0.0140
$\lambda = 0.25\%$						
80	24.8277	24.8426	-0.0598	25.1658	25.1633	0.0100
90	17.1776	17.1801	-0.0145	17.7414	17.7358	0.0315
100	11.0713	11.1004	-0.2614	11.8323	11.8234	0.0751
110	6.7449	6.7251	0.2937	7.4918	7.4867	0.0683
120	3.8506	3.8464	0.1099	4.5320	4.5308	0.0259
$\lambda = 0.5\%$						
80	25.0940	25.1078	-0.0548	25.7721	25.7681	0.0154
90	17.6461	17.6428	0.0188	18.6984	18.6904	0.0429
100	11.6839	11.7041	-0.1720	13.0282	13.0168	0.0872
110	7.3855	7.3611	0.3318	8.7504	8.7428	0.0863
120	4.4230	4.4168	0.1417	5.6952	5.6922	0.0525
$\lambda = 1\%$						
80	25.6454	25.6546	-0.0358	26.9821	26.9740	0.0299
90	18.5337	18.5179	0.0855	20.4164	20.4025	0.0681
100	12.8020	12.8048	-0.0223	15.0801	15.0626	0.1163
110	8.5551	8.5198	0.4152	10.9077	10.8940	0.1260
120	5.4964	5.4835	0.2354	7.7548	7.7466	0.1070
$\lambda = 2\%$						
80	26.7590	26.7521	0.0256	29.2396	29.2196	0.0684
90	20.1444	20.1001	0.2205	23.3149	23.2858	0.1249
100	14.7399	14.7076	0.2201	18.3955	18.3619	0.1831
110	10.5832	10.5208	0.5932	14.3904	14.3602	0.2099
120	7.4195	7.3856	0.4580	11.1842	11.1606	0.2116

Table 4.5: Simulation with Boyle and Vorst's binomial model, M Bin, and Boyle and Vorst's Black and Scholes approximation, M B&S. $S_{t_0} = 100$, $r = 0.05$ and $T = 1$.

	T/Δ					
	52	52	52	253	253	253
K	MM Bin	MM B&S	Δ in %	MM Bin	MM B&S	Δ in %
			$\lambda = 0\%$			
80	24.5739	24.5888	-0.0606	24.5903	24.5888	0.0060
90	16.6915	16.6994	-0.0475	16.7030	16.6994	0.0213
100	10.4122	10.4506	-0.3671	10.4575	10.4506	0.0663
110	6.0558	6.0401	0.2606	6.0434	6.0401	0.0544
120	3.2508	3.2475	0.1019	3.2477	3.2475	0.0054
			$\lambda = 0.125\%$			
80	24.9322	24.9460	-0.0552	25.1018	25.1016	0.0008
90	17.1343	17.1445	-0.0591	17.4326	17.4309	0.0096
100	10.9006	10.9411	-0.3704	11.3309	11.3236	0.0640
110	6.5231	6.5018	0.3275	6.9093	6.9050	0.0615
120	3.6249	3.6225	0.0661	3.9829	3.9782	0.1199
			$\lambda = 0.25\%$			
80	25.2927	25.3069	-0.0560	25.6232	25.6276	-0.0175
90	17.5709	17.5849	-0.0796	18.1374	18.1406	-0.0180
100	11.3755	11.4188	-0.3794	12.1488	12.1418	0.0572
110	6.9782	6.9499	0.4074	7.7193	7.7116	0.1009
120	3.9944	3.9900	0.1089	4.6888	4.6744	0.3080
			$\lambda = 0.5\%$			
80	26.0175	26.0365	-0.0729	26.6752	26.6968	-0.0806
90	18.4267	18.4525	-0.1394	19.4810	19.5001	-0.0976
100	12.2907	12.3409	-0.4068	13.6594	13.6537	0.0423
110	7.8568	7.8107	0.5897	9.2130	9.1925	0.2233
120	4.7194	4.7039	0.3289	6.0239	5.9794	0.7442
			$\lambda = 1\%$			
80	27.4702	27.5118	-0.1514	28.7563	28.8313	-0.2602
90	20.0770	20.1373	-0.2994	21.9595	22.0219	-0.2836
100	14.0120	14.0785	-0.4719	16.3410	16.3362	0.0291
110	9.5123	9.4191	0.9897	11.8551	11.7933	0.5241
120	6.1179	6.0579	0.9912	8.4541	8.3209	1.6002
			$\lambda = 2\%$			
80	30.3452	30.4667	-0.3987	32.7174	32.9342	-0.6580
90	23.1863	23.3389	-0.6539	26.3616	26.5246	-0.6143
100	17.1560	17.2549	-0.5731	20.9252	20.9092	0.0763
110	12.5401	12.3194	1.7912	16.3437	16.1588	1.1440
120	8.7438	8.5344	2.4531	12.6856	12.3094	3.0561

Table 4.6: Simulation with the modified Boyle and Vorst binomial model, MM Bin, and the modified Boyle and Vorst Black and Scholes approximation, MM B&S. $S_{t_0} = 100$, $r = 0.05$ and $T = 1$.

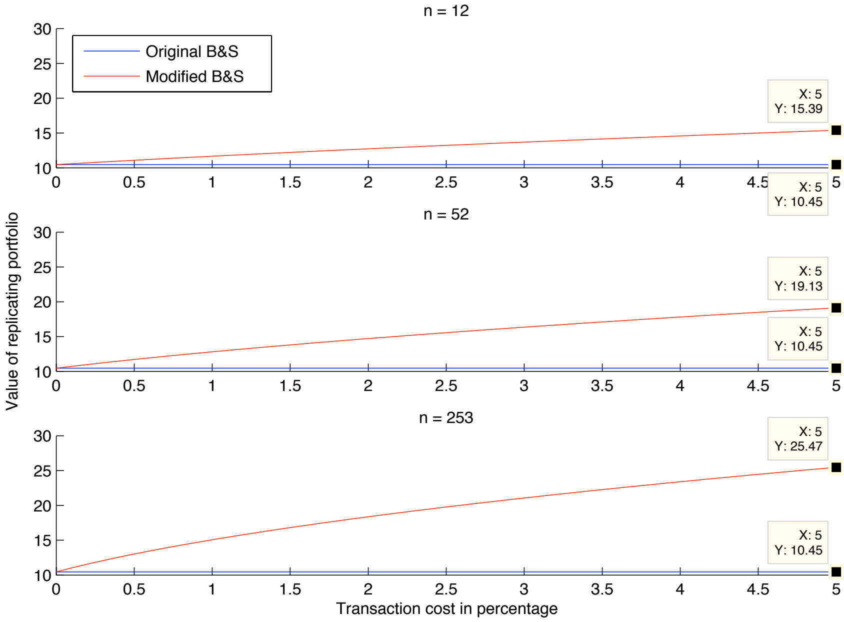

Figure 4.1: Difference between the original B&S formula and Boyle and Vorst's B&S approximation. $S_{t_0} = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.2$ and $T = 1$.

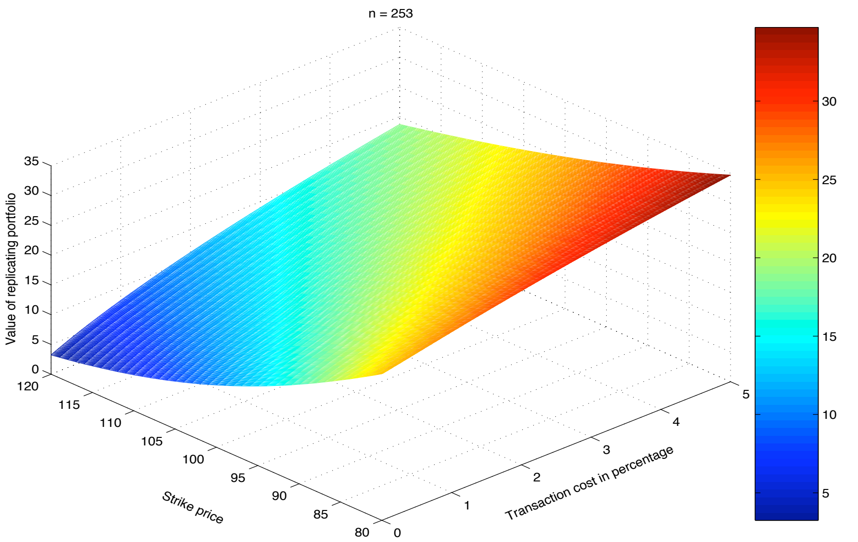

Figure 4.2: The Boyle and Vorst B&S approximation. $S_{t_0} = 100$, $r = 0.05$, $\sigma = 0.2$ and $T = 1$.

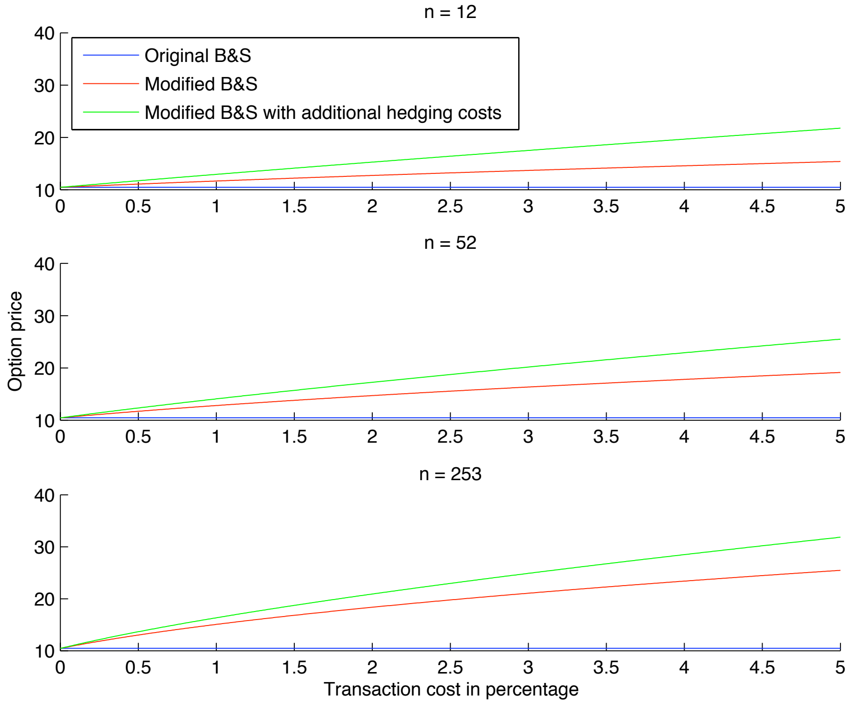

Figure 4.3: Boyle and Vorst's Black and Scholes approximation and the modified Boyle and Vorst Black and Scholes approximation. $S_{t_0} = 100$, $r = 0.05$, $\sigma = 0.2$ and $T = 1$.

4.3 Conclusions

The effect of introducing transaction costs is that the perfect Black and Scholes hedge becomes infinitely expensive. The problem is than to find a model that entails an upper bound on the price of an option. By using the framework presented in the articles by Leland (1985) and Boyle and Vorst (1992) an upper bound for the price of an option can be found. However, as we have seen, their models are somewhat incomplete as they do not include the initial transaction cost of buying the hedge and the transaction cost of selling the hedge at maturity date. Further, a motivation for not including these costs is not mentioned in any of the articles. Modifications to make their model more complete are however easily done as we have shown in section 3.5. The main conclusions from the results of our simulations are:

- When we introduce transaction costs options that are written out of the money will face the greatest relative price increase.
- Including transaction costs for buying the hedge and selling the hedge at maturity date has great importance if we want to price an option.
- Including transaction costs for buying the initial hedge and selling the hedge at maturity date decreases the spread in the relative price increase by a factor ≈ 2.

Even with our modification at hand, it is still important that we realize that several simplifications have been made. The most important ones are:

- Only proportional transaction costs are considered.
- The transaction cost factor of the risk free asset is zero.
- The transaction cost factor for buying an asset is the same as the transaction cost factor for selling an asset.
- We only rebalance our portfolio at predefined time-steps.
- We only rebalance our portfolio with respect to the delta factor.
- We do not use a pure arbitrage argument to price our options.
- The model does not include any aspects of optimality.

Not including any fixed transaction cost is motivated by the assumptions that the fixed cost is insignificant in comparison to the proportional transaction cost. In real life, however, there is almost always a fixed cost involved even though it is small. Therefore including a fixed cost as well could make the model a little bit more realistic. By letting the transaction cost of the risk free asset equal zero, we assume that intermediaries can transfer funds between accounts for free. Even though this transaction cost is likely to be very small, including it could help to improve the model. Assuming that we are exposed to the same proportional transaction cost when we want to sell an asset as when we want to buy an asset is also very unlikely since there is almost always a spread between the bid price and the ask price. Incorporating this fact in the model could also make it more realistic.

Our hedge is only rebalanced at predefined time-steps and with respect to the delta factor. This is a fact that drastically reduces the flexibility of the model since it could be more effective to hedge against an other factor, i.e. gamma. Furthermore, it could be better to use a move-based strategy by only rebalancing when our hedging factor reaches a certain threshold. However testing these other strategies requires the presentation of a totally new framework and is therefore beyond the scope of this essay.

Not using a pure arbitrage argument is a very important simplification, which is needed to get lower upper bounds on our option prices. However, it is important to realize that only calculating with expected transaction costs results in a risk exposure. The smaller we let our time-step be, the smaller the risk of the replicating portfolio will be. On the other hand, we are still constrained by the fact that If we want to use a pure arbitrage argument in continuous time the upper bound would be equal to the underlying stock. Regardless we are stuck with the fact that we have to make a trade-off between cost and risk.

Finally, it is important to remember that the model presented in this essay is not tested against any preference function of any kind. We do not investigate what would be the optimal trade-off between cost and risk for a certain individual or for some financial intermediary. In this essay we simply investigate some of the effects when transaction costs are involved. As we mentioned in chapter 1 there are examples of move-based strategies that provide a framework for finding optimal solutions for hedging derivatives under proportional transaction costs. Still, finding these optimal solutions also requires a lot of simplifications and only work for a specific set of functions.

Even though we are aware of the weaknesses of the types of models presented in this essay, they provide a framework that works fairly well and is popular in industry, see Martellini (2000). The problem of pricing options under transaction costs is still an active field of research in the literature but as this study shows there is no nontrivial hedging portfolio for option pricing under transaction costs.

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