LUND UNIVERSITY
School of Economics and Management

Option Hedging with Transaction Costs

Sonja Luoma

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Supervisor: Erik Norrman

Abstract

This thesis explores how transaction costs affect the optimality of hedging when using Black-Scholes option pricing model. Further, a number of models developed to improve the hedging results of Black-Scholes, when accounting for transaction costs, are analysed and compared. To numerically evaluate these strategies, extensive Monte-Carlo simulations are generated and the results of the strategies are computed in risk-return frameworks.

The general finding is that the variable bandwidth delta and fixed bandwidth delta strategy showed the best results, whereas Black-Scholes model expectedly generated poor results. However, slightly unpredictably the asset tolerance strategy did not outperform the Black-Scholes strategy. While an overall ranking between the hedging approaches could be defined, the optimal strategy and the relative difference between the strategies varied with the level of risk aversion.

Keywords: Geometric Brownian Motion, Black-Scholes, Transaction Costs, Option Hedging, Monte-Carlo Simulations.

List of Notations

Δ – Delta.

ε_t – Brownian motion in discrete time.

δt – Time interval between hedging in discrete time.

Γ – Gamma.

k – Proportional transaction costs.

K - Strike price.

μ - Drift.

N(0, 1) – Normal distribution with mean 0 and variance 1.

r - Interest rate.

σ – Volatility.

S_t - Stock price at time t.

T – Time to maturity.

W – Brownian motion.

CONTENTS

1 Introduction

1.1 Background

For most people hedging practice is tangible in some form, e.g. buying insurance in order to reduce the risk of facing high medical expenses. In financial terms, hedging relates to the practice of reducing risk of unfavourable movements of the assets in a portfolio. While derivative markets have grown extensively over time, many private investors still tend to leave out derivatives when investing, possibly due to a perception of derivatives market being too complex. Hedging is more commonly a tool for corporations and firms. Hedging using derivatives (futures, options and swaps) can be crucial for corporations with a heavy exposure to some specific risk. For instance an airline is highly dependent on fuel which is known for very volatile price movements. Thus by hedging crude oil which is highly correlated with fuel, the airline can stabilise the future expenses of fuel. Trading in derivatives is evidently also largely traded by portfolio managers of hedge funds, taking on both long and short positions depending on the strategy of the fund. Sophisticated programs for calculating complex strategies are continuously developed by firms to further increase the margins of hedge funds. While the initial idea of hedging is to reduce risk, many hedge funds have reported large profits but with a leveraged position in derivatives markets increasing the risk, thus rather speculating than hedging.

Although Alfred Jones has been mentioned as the creator of the first hedge fund in 1949, he did not at that point trade options. In fact options were not widely traded before 1973 when the first options exchange, Chicago Board Option Exchange was founded (CBOE). Black-Scholes presented their famous formula for option pricing the same year which would significantly develop trading in options and become one of the main formulas in modern portfolio theory (Hull, 2003, p. 234). While it gives good indications for analyses of option prices, the formula has some major drawbacks; one of which is it assumes continuous trading in the underlying asset without transaction costs. Black-Scholes is an appealing formula in theory but will in reality make perfect hedging impossible as continuous hedging would generate unbounded trading costs. Consequently, different developments have been suggested to make hedging estimates more realistic. It is thus of interest to compare and analyse different hedging approaches to see in what way they adjust for transaction costs and how well they perform.

1.2 Objectives

The focus in this study will be on analysing theoretically how transaction costs affect the hedging approach of Black-Scholes and how this has led to the development of hedging strategies adapted for transaction costs. The main purpose is to examine different basic approaches suggested in previous studies to find an optimal hedging strategy. Further, to deepen the analysis and compare the different approaches, extensive Monte-Carlo simulations are generated.

1.3 Delimitations

The study is limited to analysing approaches of hedging a European option. As Black-Scholes model is the basis of this paper, the option hedging is restricted to follow the assumptions of Black-Scholes theory (see section 2.4.1).

While the area of research is vast with some strategies more complex than others, the focus in this thesis lies on strategies that are simple enough to realistically be computed in an everyday life. More complex strategies which are significantly more time-consuming and not widely used in practice will not be covered in depth.

For the simulations, the stock price will be restricted to a discrete evolution with a time interval of 1 day which will also be the shortest hedging interval for the different strategies.

1.4 Outlining the Thesis

The next section will cover assumptions underlying the Black-Scholes option pricing model and the option theory itself with related mathematical theories. This is followed by section three which describes the theoretical framework of hedging with transaction costs, presenting hedging strategies suggested in other studies as well as theoretical analysis of these approaches. The fourth section consists of the empirical part, presenting the Monte-Carlo simulations and the related results. Lastly, conclusions and suggested future research in section five concludes the study.

2 Mathematical Frameworks

2.1 Brownian motion

A Brownian motion is an essential part in the theory of finance in continuous time and a main assumption in Black-Scholes option pricing model as it explains the random movement of the stock price. For a stochastic process to be a Brownian motion, the following conditions need be fulfilled. (Nielsen, 1999, p. 5)

- $W_0 = 0$, the random movement is at t_0 equal to zero.
- The processes of W_t are independent in such way that when referring to times units as $r < s \leq t < u$, W_u-W_t and W_s-W_r are independent stochastic variables.
- When $0 \leq s < t$, the stochastic variable W_t –W_s has constant variance per unit time $N(0, t-s)$.
- W_t is continuous.

2.2 Geometric Brownian Motion

A geometric Brownian motion is a stochastic process where the logarithm of the random movement is following a Brownian motion. The stock price is assumed to have a log-normal distribution and can be defined by the stochastic process which follows a geometric Brownian motion and fulfills the following stochastic differential equation (Hull, 2003, p. 223):

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1}$$

here μ, and σ, are positive constants. W_t is the Brownian motion and dt the deterministic component.

Introducing a function $V_t = ln\ S_t$ and using the Itô formula (see Appendix A for details) we get the following process:

$$dV = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t \tag{2}$$

Thus,

$$V_t - V_0 = lnS_t - lnS_0 = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t$$

$$V_t = lnS_t = lnS_0 + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t$$

From this we get the evolution of the stock price to be:

$$S_t = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\} \quad (3)$$ [1]

The figure below presents five simulated paths of the evolution of a stock price (blue lines) according to the equation for the stock price evolution. The red line simply represents the expected stock price according to the formula $E(S_T) = S_0 exp^{\mu T}$ (ibid. pp. 235) in a risk neutral world.

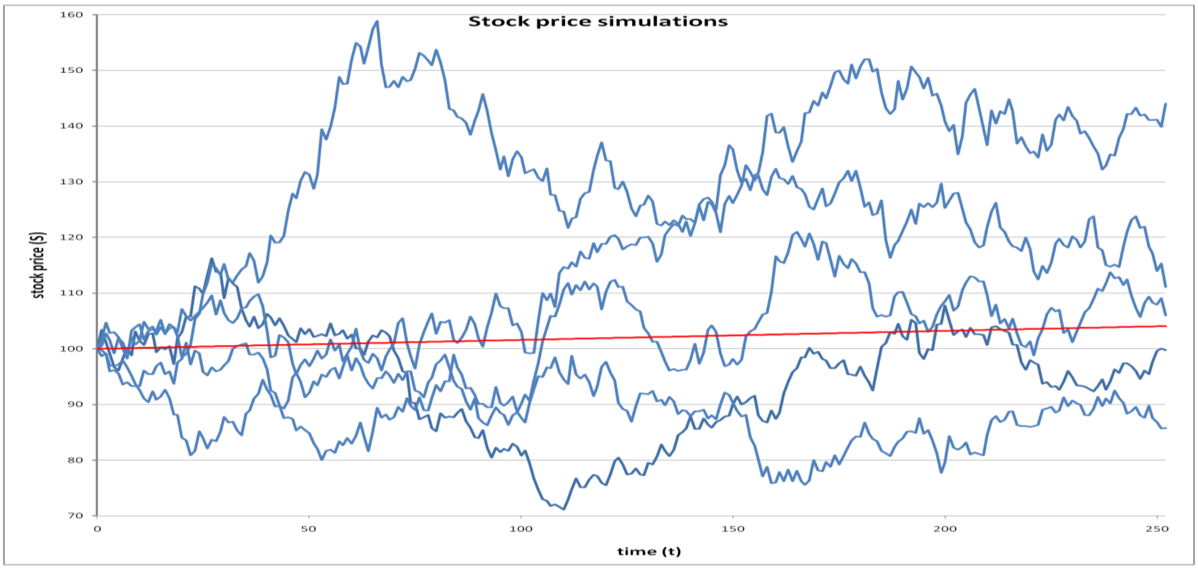

Figure 1:
Stock price simulations over one year (252 trading days) when assuming S = 100, r = 0,04, σ = 0,3, $\delta t = \frac{1}{252}$.

[1] The discrete time-version of the model is defined by (Hull, 2003, p.411):

$$S_{(t+\delta t)} = S_t exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)\delta t + \sigma\sqrt{\delta t}\varepsilon_t\right\} \quad \varepsilon_t \sim N(0,1)$$

2.3 Vanilla options

A holder of a European call option has the right but not the obligation to buy the underlying stock, S, to strike price, K, when the option expires. On the contrary, the option writer is obliged to sell the underlying stock to the strike price if the option is exercised. The payoff for the call option at maturity for a holder of a European call option is:

$$V(S(T),T) = \max(S(T) - K, 0) \tag{4}$$

Conversely, the payoff for a writer of a European call option is:

$$V(S(T),T) = -\max(S(T) - K, 0) \tag{5}$$

A holder of a European put option has the right but not the obligation to sell the underlying asset to the strike price at maturity. Thus the payoff for a holder of a European put option is:

$$V(S(T),T) = \max(K - S(T), 0) \tag{6}$$

and the payoff for a writer of a European put option:

$$V(S(T),T) = -\max(K - S(T), 0) \tag{7}$$

(James, 2003, pp. 15-16)

2.4 Black-Scholes

2.4.1 Black-Scholes assumptions

There exist some necessary assumptions made by Black-Scholes for the Black-Scholes option pricing:

1. The stock price follows a geometric Brownian Motion where μ, and σ, are constant.
2. No restrictions on short selling or the size of fractions of the underlying asset available for trading.
3. No derivatives pay dividends.
4. No taxes or transaction costs.
5. No arbitrage opportunities exist that are risk-free.
6. The risk-free rate is constant with borrowing and lending made to the same risk-free rate.
7. Continuous trading.

(Hull, 2003, p. 242)

All these assumptions simplify the calculations but do not necessarily hold in reality. In this thesis the fourth assumption will be violated to examine how to account for transaction costs to reach better hedging results. Indirectly the seventh assumption will additionally be violated when analyzing the best hedging strategy.

2.4.2 Black-Scholes model in complete markets

To hedge an option position the investor can invest in a risk-free asset (e.g. bank account) and a risky asset (e.g. stock). The investment in the risk-free asset *x(t)* follows an evolution where r is a constant.

$$dx(t) = rx(t)dt \tag{8}$$

(Björk, 1999, p. 76)

The value of the underlying risky asset is considered to follow a geometric Brownian motion as described already earlier.

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{9}$$

By applying (9) on Itô’s lemma (A4) (See Appendix A for derivation of Itô’s lemma), we get the following equation defining the dynamics of the option price, V:

$$dV_t = \left\{\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial s^2}\right\}dt + \sigma S\frac{\partial V}{\partial S}dW \tag{10}$$

As both the option and the stock are driven by the same uncertainty, dW, it is possible to eliminate the uncertainty by forming a portfolio of opposite positions in the stock and the option. Thus in the Black-Scholes model, the risk an option writer faces can be hedged away completely by holding Δ-shares equal to $\frac{\partial V}{\partial S}$ in the stock.

By cancelling out the uncertainty, equation (10) can in a risk neutral world intuitively be written as:

$$\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} = rV \tag{11}$$

This is the well-known Black-Scholes partial differential equation for option pricing (Hull, 2003, p. 243).

The expected value of a European call option is the solution to the partial equation above with the boundary condition $V(S,T) = \max(S-K,0)$ and defined by:

$$V(S,t) = SN[d_1(S,t)] - Ke^{-r(T-t)}N[d_2(S,t)] \tag{12}$$

where N is the cumulative normal distribution function ϕ(0, 1)

.

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + r + \left(\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

(Björk, 1999, p. 90[2])

By using put-call-parity, $V(S,t)_{put} + S = V(S,t)_{call} + Ke^{-r(T-t)}$, the price for the corresponding put option can be found (ibid. pp. 109-110).

2.5 Greeks

The Greeks define the sensitivities of derivatives, hence a way of managing and measuring the risk-exposure. Looking at options, the Greeks measure how the option value changes with respect to changes in some parameter, e.g. the underlying asset. While several risk measures of Greeks exist and they all are to some extent important for risk evaluations, only two Greeks will be discussed in this thesis. These two are delta and gamma which are important for some of the strategies analysed.

2.5.1 Delta

The delta of an option measures the sensitivity of changes in the underlying asset to the option value, defined as the first derivative of the option value, V, on the underlying asset, S (Hull, 2003, p.302). Thus for an investor hedging a short position in an option, the delta hedge defines how large fraction of the underlying asset needs to be held in the portfolio for every short call option in order to hedge the risk.

$$\Delta(t)_{call} = \frac{\partial V(t,S)}{\partial S} = N(d_1) \qquad (13)$$

(James, 2003, p. 54)

[2] see pp. 84-90 for details on the derivation of equation (12).

2.5.2 Gamma

Gamma is the second derivative of the option value on the underlying asset. It measures the sensitivity of delta on the price movements of the underlying asset, thus showing how fast the delta position changes when the underlying asset price changes.

$$\Gamma(t) = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V(t,S)}{\partial S^2} = \frac{N'(d_1)}{S\sigma\sqrt{T-t}} \tag{14}$$

$$N'(d_1) = \exp\left(\frac{-(d_1)^2}{2}\right) * \frac{1}{\sqrt{2\pi}}$$

(ibid. pp. 54-55)

2.6 Direct application of hedging with transaction costs on Black-Scholes model

A common approach in studies of hedging with transaction costs has been to assume proportional transaction costs (see e.g. Mohamed, 1994, Clewlow and Hodges, 1997, and Zakamouline, 2006). Violating the Black-Scholes assumption of no transaction costs and applying proportional transaction costs on the Black-Scholes model would imply the need of hedging in discrete time to avoid infinitely large trading costs. The option writer would invest delta shares $\Delta(t)$ in the stock and $V(t, S(t)) - \Delta(t)(1+k)S(t)$ in the bank account, where $V(t, S(t))$ is the amount received for writing an option. Rebalancing the portfolio at discrete time intervals, will reduce the transaction costs in total, while the risk will increase leading to a hedging error[3] as the hedge will no longer be perfect.

[3] The difference between the value of the replicating portfolio and the option payoff.

3. Theoretical Approach to Hedging With Transaction Costs

Previous studies on hedging with transaction costs can be divided into time-based and move-based strategies. In a time-based strategy, the rebalancing occurs at regular time intervals without looking separately at the optimality of each rehedging, it is rather comparable with the discretisation approach of the Black-Scholes model. While in a move-based strategy, the portfolio rebalance is dependent on movements of the underlying asset.

In move-based strategies, a bandwidth around the Black-Scholes delta is computed, forming a so called no transaction region. Rehedging only occurs when the hedge ratio lies outside the boundaries of this region. This has been a widely used approach of finding the optimal hedging strategy with transaction costs, both for theories based on the utility maximization approach but also more simple strategies (see e.g. Mohamed, 1994 or Martellini and Priaulet, 2002).

3.1 Previous research

Leland (1985) appears to be one of the first studies suggesting a way of including transaction costs in option hedging. Based on the Black-Scholes model Leland presented an option replication model where proportional transaction costs are accumulated in the Black-Scholes volatility. Bensaid et al. (1992) applied the strategy of "super-replicating" where the portfolio is dominating rather than replicating the derivative asset. This study was followed by Soner et al. (1995) and Cvitanic et al. (1999) proving that the cheapest strategy for the writer of a European call option in presence of transaction costs gives the optimal "super-replication" of simply buying one share of the underlying asset and keeping it until maturity. However, looking at the economic value of this strategy, it is of little interest for the option writer.

The utility maximization strategy maximizing the utility of the investor to form the optimal strategy was first presented by Neuberger and Hodges in 1989 (Zakamouline, 2006, p. 435). A no transaction region was formed where the boundaries defined where the investor is indifferent between the utility of rehedging and the utility of not making any changes to the portfolio. It has been seen as one of the most effectual developments for optimal hedging with transaction costs and has become a common approach developed further in subsequent studies. Several studies based on this approach have reached good empirical results (e.g. Mohamed, 1994, Clewlow & Hodges, 1997, and Zakamouline, 2006). While all utility-based strategies seem to outperform other strategies they have restricted usability as they are computationally time-consuming and need to be monitored and adjusted continuously. A common use of these strategies by portfolio managers or corporations would not be realistic due to trading being characterized by a fast paced environment where decisions have to be made instantly.

To deal with the complexity of the utility-maximisation based strategies, Whalley and Wilmot (1997) presented an asymptotic approximation of the hedging strategies assuming small transaction costs. Simpler approaches to rehedging with a no-transaction region has been developed by Martellini and Priaulet (2002), Henrotte (1993) and Whalley and Wilmott (1993). In these strategies the boundaries are simply computed depending on the risk aversion with respect to the movements of the underlying asset.

3.2 Hedging Approaches

Hedging approaches of interest for the analysis of this study will be presented below.

3.2.1 Leland

Leland (1985) assumed proportional transaction costs and developed a time-based strategy based on the Black-Scholes model in discrete time steps but with adjusted volatility. The strategy is optimized with respect to transaction costs and the length of the time interval. The modified volatility input of Leland’s strategy is solely for a short position in a call option (Leland, 1985, p. 1289):

$$\sigma_m^2 = \sigma^2 \left(1 + \frac{\sqrt{\frac{2}{\pi}}k}{\sigma\sqrt{\delta t}} \right) \tag{15}$$

Thus a risk averse hedger chooses a small δt to hedge more often than a more risk tolerant hedger.

3.2.2 Delta Tolerance

Whalley and Wilmott (1993) introduced a hedging strategy dependent on the movements of the option delta and where the market is continuously monitored. They proposed a constant tolerance level around Black-Scholes delta, rebalancing the hedging portfolio back to the Black-Scholes delta whenever the hedging ratio moved outside the defined tolerance level. The boundaries of the no transaction region in Whalley and Wilmott (1993) can be defined as:

$$\Delta = \frac{\partial V}{\partial S} \pm H \tag{16}$$

The constant tolerance level is denoted H, while $\frac{\partial V}{\partial S}$ denotes the normal Black-Scholes delta. The predefined tolerance level is derived from the investor's level of risk aversion where a higher risk aversion yields a smaller tolerance level leading to larger transaction costs.

3.2.3 Asset Tolerance

A similar study to the delta tolerance strategy is the asset tolerance strategy developed by Henrotte 1993 and commonly a part of other analyses of hedging strategies (e.g. Martellini and Priaulet, 2002, and Zakamouline, 2006). Instead of hedging being dependent on the delta hedge, the percentage change of the underlying asset is the control variable in the asset tolerance strategy. The market is monitored continuously and the portfolio is rebalanced to its Black-Scholes delta when the percentage change of the underlying stock price goes beyond a predefined level. Again this amount is dependent of the investors risk aversion. The point of rebalancing in Henrotte's strategy can be defined as (Martellini and Priaulet, 2002, p. 28):

$$i=0,1,2,.. \quad t_i < T \qquad \Delta = \frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}} \pm h \tag{17}$$

where the tolerance level measured as the percentage change of the underlying asset, is denoted h.

3.2.4 Fixed Bandwidth Delta

Martellini and Priaulet (2002) used a move-based strategy comparable with the delta tolerance strategy in their analyses of hedging strategies. Similarly to the delta tolerance strategy the rebalancing boundaries are given by (Martellini and Priaulet, 2002, p. 29):

$$\Delta = \frac{\partial V}{\partial S} \pm H \tag{18}$$

The difference in strategy compared to the delta tolerance strategy is that the rehedging is only performed to the nearest hedging boundary rather than to the Black-Scholes delta. (Priaulet 2002).

3.2.5 Variable Bandwidth Delta

Whalley and Wilmott (1997) presented an asymptotic solution to the computationally time consuming utility maximization strategy. By assuming small transaction costs they derived the following formula for the boundaries of the no transaction region (Whalley and Wilmott, 1997, p. 319).

$$\Delta = \frac{\partial \mathrm{V}}{\partial \mathrm{S}} \pm \left(\frac{3}{2}\frac{\mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}\mathrm{k}\mathrm{S}\,\Gamma^2}{\gamma}\right)^{\frac{1}{3}} \tag{19}$$

γ is the level of risk aversion of the hedger. When transaction costs go to zero ($k \to 0$), the formula approaches the delta hedge of Black-Scholes. When the absolute risk aversion increases, the no transaction region logically decreases.

The investor will similarly to the fixed bandwidth strategy only rebalance to the nearest boundary. However, in the strategy of Whalley and Wilmott (1997) the bandwidth is a function of variables which are not constant over time. The no transaction region widens the closer the option is to maturity, the higher the price of the underlying asset and the higher the option gamma.

3.3 Transaction costs

While transaction costs interferes with the Black-Scholes model to find a perfect hedging strategy, the complexity of transaction costs make it impossible to exactly account for them when hedging. All strategies presented in this thesis, assume proportional transaction costs. Yet in reality transaction costs are made up of a fixed part and a proportional part which are affected by several factors. Main factors affecting the size of transaction costs are the bid-ask spread and brokerage fees (Gemmill, 1993, p. 12). As is known, large and small investors tend to have different conditions when trading. While a large trader can often make agreements with a brokerage house to pay a flat commission rate independent of the trade size, a smaller trader is faced with a brokerage fee consisting of a minimum fee as well as a fee decreasing with the size of transaction. Several previous studies (Mohamed, 1994, Clewlow and Hodges, 1997 and Zakamouline, 2006) have used proportional transactions costs equal to 1%. Hull (2003, pp. 158) suggests a representative amount of commission costs to be 1-2 % of the stock value which would generate clearly higher result than 1% when accounting for other costs involved in transaction costs of trading. Thus transaction costs are complex and one unified price will never represent all strategies or hedgers. For simplicity we will account for only proportional transaction costs and assume they are 1% which is in line with previous studies.

3.4 Theoretical Analyses of the Hedging Strategies

As have been concluded by several other studies, no strategy perfectly replicates the option when accounting for transaction costs. The problem consists of the conflicting issue of rehedging as often as possible and simultaneously minimizing negative hedging errors.

3.4.1 Analysis of Time-Based Strategies

Black-Scholes model was developed for complete markets. As the risk can completely be eliminated by rehedging continuously to the delta-hedge, the derivatives bear no risks. But when accounting for transaction costs the hedging costs will become infinite as the stock price is assumed to evolve with a Brownian motion which shows infinite variation over time. Consequently, hedging in discrete time will reduce costs but increase hedging errors. Yet, theoretically the Black -Scholes is reasonably simple to compute with only one unobservable parameter, namely the volatility.

Looking at Leland's strategy, it should be an improvement of the Black-Scholes formula by the introduction of adjusted volatility. When incorporating transaction costs of trading in the price movements of the underlying asset, the volatility will increase. For a short call option, the increased hedging volatility will lead to a decreased value of gamma when option gamma is high (due to gamma being negatively related to volatility). It makes sense to adjust for volatility and in that way reduce the transaction costs for the option writer. For an option buyer the transaction costs would react in the opposite way and the model would need some modification to be optimal. While the Leland strategy has been developed to improve the Black-Scholes model, the Leland portfolio is also hedged at fixed intervals and is thus rehedged without considering if it is optimal to hedge at every time interval.

Based on an economic theory point of view it does not seem sensible to reach an optimal strategy by hedging based on time. Yet this does not directly mean it would not be possible to mathematically manage to find the optimal time-interval for hedging the portfolio based on Leland that would outperform other models. While time-based hedging does not seem to be optimal for the result, it simplifies the work of hedging as the movements of the underlying stock are not necessarily continuously monitored.

3.4.2 Analysis of Move-Based Strategies

The fixed bandwidth delta, the asset tolerance and the delta tolerance strategy generate fixed boundaries for a no transaction region around the Black-Scholes delta. While these strategies evidently reduce transaction costs as they are rehedged depending on the movement of the underlying asset, the strategies are nevertheless not exceptionally dynamic. The drawback with these three strategies is that they cannot account for certain risks that evolve with the value of the underlying asset. The asset tolerance strategy simply uses the price of the underlying asset to rehedge without using any sort of risk measure, whereas the delta tolerance and the fixed bandwidth delta strategy take in consideration the delta measure of the hedged option. Other sensitivity measures (Greeks) such as gamma are not a part of the strategies. In a high gamma area a small change in the underlying asset will largely affect the delta. Thus fixed bandwidths cannot adjust to this and reduce the rebalancing.

Davis and Norman (1990) show it to be more optimal to only hedge to the nearest boundary in order to save in transaction costs. This would thus logically imply that it is more optimal to hedge using the fixed bandwidth strategy than the delta tolerance and supposedly also the asset tolerance strategy.

As has been mentioned before, the utility-based strategy has clearly showed best performance in previous studies and is undoubtedly the strategy that generates best results. While it is too complex and time-consuming to use in everyday trading, the asymptotic approach of Whalley and Wilmott's variable bandwidth strategy is faster and simpler to compute. As the no transaction region is not fixed, as in the three above analysed move-based strategies, it better adjusts for changes affecting the strategy in reality. Looking at the asymptotic model of Whalley and Wilmott's model, the no transaction region is adjusted by both delta and gamma. A high gamma would generally imply more frequent rehedging, the no transaction region is positively related to gamma and thus by increasing the bandwidth of the no-transaction region when gamma is high, the transaction costs can be reduced.

All strategies seem to have improved in some way the basic Black-Scholes approach to be implemented in the real world. Looking from an economic point of view, the move-based strategies should outperform the time based strategies. Whereas the asymptotic solution of Whalley and Wilmott theoretically sounds most appealing, the model has been developed assuming small transaction costs. Increasing the transaction costs could make this model less reliable.

4 Empirical Results

4.1 Simulations

To examine the performance of the different hedging strategies, the price progress of the underlying asset (assumed to be a stock) has been generated by Monte Carlo simulations. Each simulation generates a probable path for the stock price and is repeated several times to get an expected average path for the stock price and an expected payoff. While the results of a large number of simulations are realistic, the results are not based on real market data and are thus only theoretically valid. Further the shortest time interval between rehedging is one day. As a result the move-based strategies which are theoretically based on continuous monitoring will be restricted to discrete monitoring with time intervals of one day.

For the analysis, the stock evolution is simulated by the following model (see section 2.2 for derivation of the model)

$$S_{(t+\delta t)} = S_t exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)\delta t + \sigma\sqrt{\delta t}\varepsilon_t\right\} \quad \varepsilon_t \sim N(0,1) \tag{20}$$

Daily updates for the stock price is generated for a six-month period (126 trading days) to hedge a short call option for an equally long time period. Thus, $T = \frac{126}{252}$ and $\delta t = \frac{1}{252}$ (δt is varying later on for the time-based strategies). $\mu = 0{,}04$, $\sigma = 0{,}3$ and the option is written at-the money with $S_0 = 100$ and K=100.

To begin with, the price of a six-month call option using Black-Scholes model (equation (12)) is calculated. This is the value in cash the option writer will initially receive at t_0. Further the evolution of the stock price was simulated 1000 times to look at the Black-Scholes strategy without transaction costs (assuming the same values on the parameters as previously). This was solely done to get some perspective on how exact results 1000 simulations would generate.
The delta of the option is calculated at t_0 and the proportion of it is invested in the stock to keep the portfolio delta neutral. The invested amount for the hedging position, together with the related transaction costs are drawn from the bank account. For the time-based strategies, the hedging portfolio is rebalanced to be delta neutral at every time interval $t \in [0; T]$, while the move-based hedging portfolios are monitored at every time interval but only hedged if needed.

Every time a rehedging occurs the transaction costs are also drawn from the bank account. At maturity, the value of the strategy is calculated by subtracting a possible exercise pay out from the then received strike price added to the value of the hedging portfolio. Further the value is compounded back to time t_0. Any value above or below zero will thus be a hedging error for the strategy. As the calculated option price in the beginning is assumed to be the Black-Scholes value for an option, all strategies should show negative values at maturity when accounting for transaction costs. The question is rather which strategy shows the least negative value to a certain level of risk.

As it is difficult to directly compare the level of risk aversion of different strategies by simply looking at on one or a few levels of the parameter, the risk aversion parameter is spanned over a larger variety of risk aversions. For every level of risk aversion, the mean hedging error is computed against the mean standard deviation of the hedging error.

For the time-based strategies, the time span parameter for the hedging intervals was between 1 to 50 days. Thus 50 different hedging intervals were simulated 1000 times each. While for move-based strategies fixed bandwidth delta and delta tolerance the risk tolerance applied varied from 0,01 to 0,5 and for asset tolerance 0,005 to 0,1. The risk aversion for the variable bandwidth strategy was spanned over 0,005 to 20. Also these strategies were simulated for 50 different hedging intervals, 1000 times for every level of risk aversion. To speed up the simulation process in Excel, the add-in Risk Solver Platform was used to manage the computation. Even though all strategies used 50 different values of the hedging frequency, they were not strictly equal, e.g. a time interval of 1 or 2 days does not compare directly with a risk tolerance of 0,01 or 0,02. Similarly even between the move based strategies the risk tolerance is not directly comparable. However, the whole spans for all strategies illustrate quite well the span where even a more extreme risk averse and risk tolerant hedger would lie within.

4.2 Results

The Black-Scholes option price when using Black-Scholes model (equation (12)) was equal to 9,39044048. For a hedging time interval of one day the hedging error turned out to be 0,000135. When no transaction costs are involved and continuous hedging is taken place the result of the simulation should be zero. Looking at the result, 1000 simulations gave a relatively accurate result. A lot more simulations could possibly have generated an even more exact result; however the difference is also due to the fact that the portfolio was discretely hedged.

Figure 2 displays the results of the different hedging strategies in a risk-return framework. A high standard deviation is related to the risk parameter (tolerance level) being high (low risk aversion in the variable bandwidth strategy). Thus looking at figure 2, in overall the more risk tolerant a hedger is, the smaller the hedging errors. This is consistent with less hedging reducing transaction costs but also increasing the standard deviation. For a given level of risk, a rational hedger always chooses to maximize the return. Or likewise, for a given level of return, the hedger would choose to minimize the risk. Analysing the results, no strategy is optimal for all levels of risk. The variable bandwidth strategy outperforms the other strategies when assuming moderate risk aversion. Yet at moderate level it is only slightly better than the fixed bandwidth strategy. Both these strategies clearly outperform the other strategies by being 25% less expensive than the third best strategy at moderate level of risk tolerance (around Std Dev 1,5). Further the fixed bandwidth strategy is clearly better for more risk tolerant hedgers. As assumed the fixed bandwidth strategy outperforms both the delta tolerance and asset tolerance strategy. While the variable bandwidth strategy was only slightly better than the fixed bandwidth strategy, looking at figure 2 one can see that the variable bandwidth strategy did not generate as large hedging errors as the other strategies for highly risk averse hedgers. This could be due to the fact that the bandwidth is more dynamic and increases in areas where fixed bandwidths would generate large transaction costs. Thus increasing the risk aversion a lot will still not make the mean hedging error a lot larger (which was also the case when trying to increase the risk aversion significantly). The asset tolerance strategy surprisingly shows worse results for most risk levels and with poorer results than the Black-Scholes model. For a very risk averse hedgers Leland’s strategy shows the smallest standard deviation. Overall Leland shows as expected better results than Black-Scholes.

The ranking between the strategies were in large extent in line with previous studies (e.g. Mohamed, 1994, Clewlow and Hodges, 1997 and Zakamouline, 2006). Zakamouline (2006) presented similar results except for the asset tolerance strategy which in his research performed clearly better than the time-based strategies, but also outperformed the delta tolerance strategy.

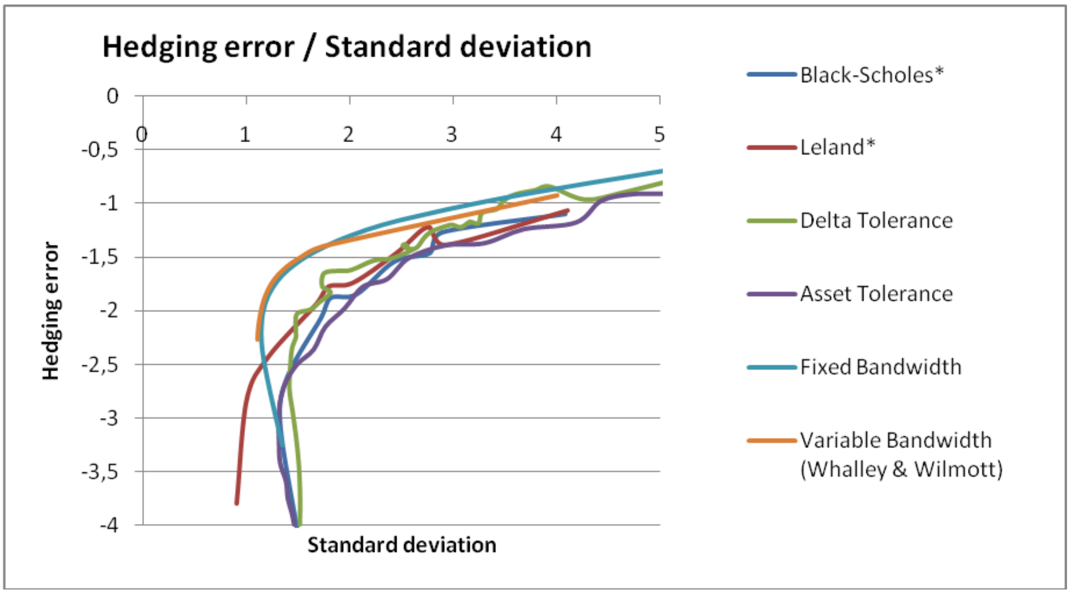

Figure 2

The result is based on 1000 simulations for each 50 risk parameter (risk aversions/time interval). An option writer hedges a six month at-the-money option with strike price 100. k = 1%, v = 30%, and r = 4%.
*only simulation values for time intervals hedged evenly up to maturity have been presented in the graph. This due to time intervals not evenly divisible generated in general poorer and more unstable results and making the graph unclear. All values can however be found in Appendix B (also including the values of some strategies that generated values outside the values of the axes).

While the risk-return framework is common for analysing different results in finance it might not in this case be optimal to base the analysis solely on the result using standard deviation as risk measure. When counting the mean standard deviation, both positive and negative deviations are computed. Thus a positive standard deviation, which in fact is favourable for the hedger, is increasing the mean standard deviation and thus shows a poorer result for the hedging strategy. Therefore the mean results of the hedging errors are also computed against Value at Risk (VaR) at a 95% confidence interval. Thus the risk measure shows the potential loss by looking at the maximum loss with a 5% likelihood for a hedging period of 6 months.

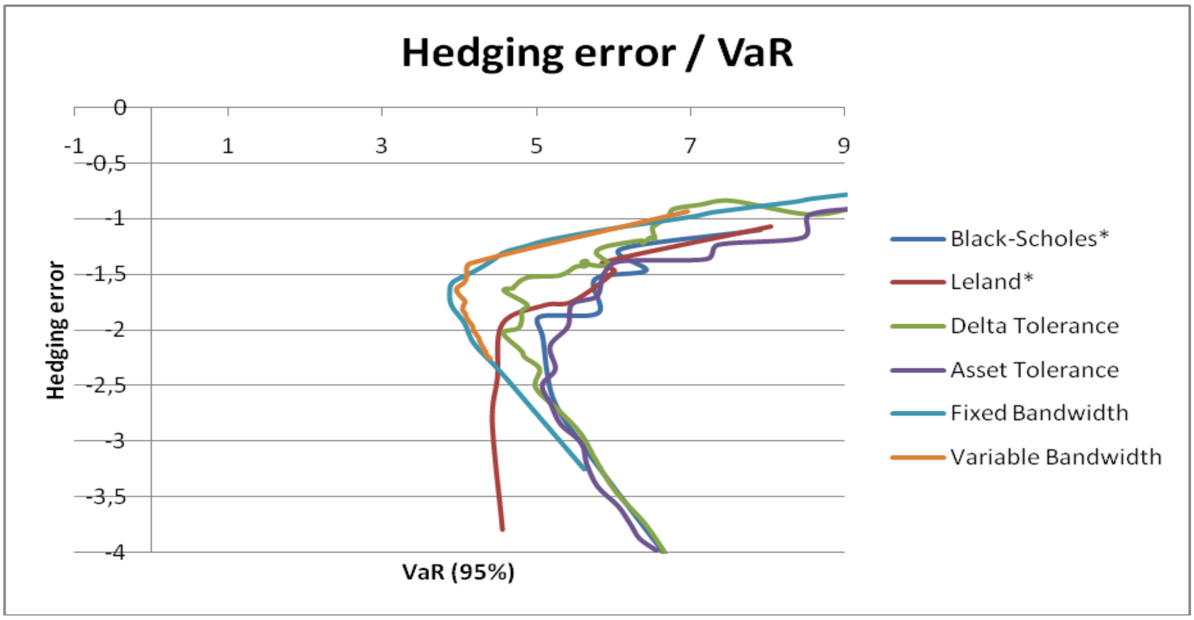

Figure 3

See figure 2 for more details on parameters and *.

The results of the two risk frameworks are generally fairly similar for the ranking between the strategies; they do however show some variations. While the variable bandwidth strategy was the best strategy for moderate risk aversion in Figure 2, this is less clear when using VaR at a 95% confidence interval as risk measure. However, on the whole the fixed and variable bandwidth delta strategy outperforms the other strategies, e.g. for a moderate hedging level their VaR was 4,3 which is approximately 20% less than the third best strategy.

Leland's strategy does not show the best result for the very lowest level of risk tolerance as when standard deviation was used as risk measure, possibly indicating that other strategies accounted for more positive variation in the results for levels when the risk tolerance was low.

5 Conclusions

5.1 Main Findings

The results of the simulations show that no hedging strategy is optimal for all levels of risk aversions. Thus the optimal strategy depends on the specific situation and level of risk aversion defining the investor. The variable bandwidth delta strategy is theoretically the most favourable of the strategies studied. Together with the fixed bandwidth strategy it also generated largely better results than the other strategies. An aspect talking further in favour of these strategies is the fact that the stock price was limited to being monitored discretely rather than continuously. Consequently move-based strategies were made less powerful as they should theoretically be monitored continuously.

While the asset tolerance strategy was expected to be weaker than the other move-based strategies due to a sensitivity measure where only the stock price affected the hedging , it was not expected perform of bottom rank. The discretisation of the monitoring could possibly partly be the reason why the asset tolerance strategy performed worse than the Black-Scholes strategy. While Black-Scholes model was expected to perform worse than the other strategies (which it nearly did), it is nevertheless widely used in practice. One explanation could be its simplicity and easiness to grasp. Although the move-based strategies here are simple to compute, the difficulty still lies in deciding exactly the level of risk aversion.

The results also verify the previously mentioned fact that it will never be profitable (return wise) to hedge an option when assuming transaction cost. With the strategies analysed here the hedging error is below -1 for most hedgers, unless the risk of the hedging strategy is high. This does not change the interest for derivatives. As mentioned in the introduction the market for derivatives including option markets have been expanding greatly over time. It is evident that hedge funds use a lot more complex strategies mixing several different derivatives but possibly also taking on more risk to have the potential to be lucrative and generate large profits. Some corporations hedging to reduce their exposure to a certain risk might still use a simple strategy as it is worth hedging with a slight loss in portfolio value, in favour of stable cash flows.

5.2 Future research

There exists a vast area to explore full of opportunities for future research. This thesis has only covered a very small part of the field of option hedging. The aim with option hedging is to provide a way to reduce risk for the investor. However, the results presented here also show that transaction costs tend to generate negative returns when hedging. It could thus be of relevance to compare more complex options or option hedging strategies and relate the results with plain vanilla options hedging strategies.

Another way of following up this thesis could be to use real data for the study. In a stock market, jumps usually occur from the closing of one day to the opening the following day. This would favour time-based strategies which can easier be adjusted to jumps and the relative results between the different hedging approaches could thus show some interesting results. In addition analysing how well or badly the option hedging strategies performs during a longer time period covering both a financial crisis and more stable time periods, could be subject for a study. As the assumption of log-normal distribution of stocks tends to be questioned especially during financial crisis in favour of more skewed probability distributions with fat tails, this type of study could give some intuition on how accurate the strategies are over time.

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Appendices

Appendix A

An Itô process is used to represent the process of a price or portfolio value and is defined by the integral equation:

$$S(t) = S_0 + \int_0^t \mu \, ds + \int_0^t \sigma \, dW \qquad \text{(A1)}$$

which in short form can be defined as the stochastic differential equation (SDE):

$$dS = \mu_t dt + \sigma_t dW_t \qquad \text{(A2)}$$

(Nielsen, 1999, pp. 52-53)

When assuming a process V as a function of S (stock price) and t (time), the diffusion dV can be found by using Itô's lemma. An intuition to the derivation can be made by taking the Taylor expansion of V(S,t) around S(t) up to the second order terms (Björk, 1998, p. 38), which gives:

$$V(s(t) + \Delta S, t + \Delta t) \approx V(s(t), t) + \frac{\partial V}{\partial S}\Delta S + \frac{\partial V}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 V}{(\partial S^2)}(\Delta S)^2 + \frac{1}{2}\frac{\partial^2 V}{(\partial t)^2}(\Delta t)^2 + \frac{\partial^2 V}{\partial t \partial S}\Delta t \Delta S$$

When Δt and Δs are very small it can be written as:

$$dV = \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{(\partial S^2)}(dS)^2 + \frac{1}{2}\frac{\partial^2 V}{(\partial t^2)}(dt)^2 + \frac{\partial^2 V}{\partial t \partial S}dtdS \qquad \text{(A3)}$$

Using the formal multiplication rules (see Björk, (1998, p.38) for more details on the rules for stochastic calculus) $(dt)^2 = 0$, $dt*dS=0$ and $(dS)^2 = dt$ on the equation (A3) we get Itô's lemma:

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S}(dS)^2 \qquad \text{(A4)}$$

See Björk, (1999, pp.39-40) for a framework of the full formal proof.

The most widely used SDE for defining the stock price movement, which Black-Scholes option model is based on, is the SDE already defined by (1) in section 2.2:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \qquad \text{(A5)}$$

Assuming the stock follows a log-normal distribution, by introducing $V_t = ln\ S_t$ and applying it to Itô's lemma (A4) we get:

$$dV = \frac{1}{S}dS + \frac{1}{2}\left(-\frac{1}{S^2}\right)(dS)^2$$

$$= \frac{1}{S}(\mu S dt + \sigma S dW) + \frac{1}{2}\left(-\frac{1}{S^2}\right)\sigma^2 S^2 dt$$

$$= (\mu dt + \sigma dW) - \frac{1}{2}\sigma^2 dt$$

$$dV = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t \qquad \text{(A6)}$$

Appendix B

Results from the Monte-Carlo simulations.

Black-Scholes			
Simulation	Mean	Std Dev	VaR 95%
1	-4,12348	1,506168	6,780485
2	-3,05169	1,327719	5,611577
3	-2,61076	1,400136	5,212228
4	-2,31918	1,455939	5,027396
5	-2,2167	1,546005	5,084653
6	-2,06873	1,720914	5,081481
7	-1,88017	1,814138	5,040744
8	-1,83407	1,92072	5,321915
9	-1,84682	2,065458	5,799438
10	-1,72446	2,020054	5,347169
11	-1,6995	2,135803	5,454838
12	-1,6772	2,195487	5,592378
13	-1,64275	2,336563	5,920884
14	-1,54083	2,431003	5,760726
15	-1,58889	2,3734	5,606601
16	-1,47599	2,634631	6,026128
17	-1,5032	2,673585	5,908166
18	-1,46121	2,775549	6,425671
19	-1,44954	2,756127	6,292558
20	-1,39833	2,699372	5,982897
21	-1,26998	2,888566	6,100167
22	-1,35885	2,930583	6,264708
23	-1,46708	3,000235	6,672855
24	-1,435	3,065548	6,759245
25	-1,4437	3,185941	7,087602
26	-1,36101	3,295384	7,151425
27	-1,3203	3,193772	6,826416
28	-1,37848	3,174842	7,030974
29	-1,36937	3,291894	7,372114
30	-1,44989	3,315572	7,034142
31	-1,33987	3,474874	7,25368
32	-1,19833	3,575715	7,118013
33	-1,21323	3,556302	7,470091
34	-1,23426	3,620036	7,195371
35	-1,2303	3,536242	7,275984
36	-1,24134	3,486874	7,025794
37	-1,28723	3,572863	7,341037
38	-1,36751	3,616532	7,36096
39	-1,3818	3,715342	7,549106
40	-1,36127	3,823169	7,879872
41	-1,3371	3,928805	8,243362
42	-1,09798	4,081698	7,911925
43	-1,12643	4,073713	7,959753
44	-1,12766	4,064086	8,191729
45	-1,15432	4,053619	8,570419
46	-1,12135	4,038181	8,12245
47	-1,10765	4,045197	8,304137
48	-1,06187	4,017367	7,865272
49	-1,13455	4,06866	8,153501
50	-1,0741	4,093348	8,141684

Leland			
Simulation	Mean	Std Dev	VaR 95%
1	-3,79862	0,907714	5,444931
2	-2,83085	1,004423	4,555037
3	-2,42942	1,213537	4,422595
4	-2,12574	2,002289	5,105433
5	-2,08614	1,934636	5,013228
6	-1,94067	1,672599	4,488847
7	-1,77544	1,791533	4,553234
8	-1,66398	2,710197	5,682656
9	-1,73819	2,030803	5,095521
10	-1,56176	2,80725	5,737948
11	-1,56822	2,804507	5,615505
12	-1,51574	2,973903	5,862254
13	-1,49615	3,152357	6,265364
14	-1,46416	2,446737	5,468626
15	-1,44123	3,094587	5,770245
16	-1,30225	3,566359	6,941262
17	-1,36654	3,288043	6,196482
18	-1,38643	2,758094	6,007163
19	-1,31089	3,601887	6,584538
20	-1,27988	3,301396	6,15676
21	-1,21658	2,924704	5,864084
22	-1,17866	3,851652	6,761803
23	-1,35689	3,661892	6,784749
24	-1,30576	3,586923	6,71268
25	-1,38884	3,354187	6,768533
26	-1,20855	4,267092	7,503423
27	-1,14642	4,103077	7,195202
28	-1,21412	3,974751	7,353767
29	-1,26174	3,908049	7,422845
30	-1,32617	3,794142	6,942478
31	-1,24437	3,811816	7,424798
32	-1,02758	4,533076	7,503723
33	-1,01272	4,485321	7,699211
34	-1,11834	4,522886	7,727193
35	-1,06664	4,461821	7,481508
36	-1,0751	4,331717	7,677992
37	-1,12694	4,3579	7,536819
38	-1,24134	4,311642	7,637717
39	-1,26751	4,305724	7,741849
40	-1,24916	4,291773	7,763358
41	-1,22868	4,296926	8,136782
42	-1,06441	4,104091	8,036926
43	-0,99938	4,970124	8,714722
44	-0,97339	4,958533	8,492859
45	-0,99847	4,950982	8,819043
46	-0,9645	4,9063	8,220007
47	-0,95283	4,918333	8,619708
48	-0,90928	4,896406	8,238528
49	-0,94485	4,96627	8,220551
50	-0,94463	4,976449	8,322033

Delta Tolerance			
Simulation	Mean	Std Dev	VaR 95%
1	-3,99437	1,522087	6,665986
2	-3,74028	1,525486	6,41104
3	-3,44536	1,512952	6,023742
4	-3,18161	1,48637	5,785049
5	-2,93402	1,451474	5,581583
6	-2,72757	1,421325	5,299294
7	-2,5207	1,429769	4,993949
8	-2,35101	1,446957	5,044923
9	-2,24614	1,484421	4,852834
10	-2,18637	1,484035	4,80555
11	-2,03015	1,493788	4,572777
12	-1,98878	1,623073	4,760771
13	-1,92246	1,707053	4,81206
14	-1,82817	1,818646	4,816265
15	-1,77174	1,737887	4,88174
16	-1,64855	1,756103	4,584362
17	-1,62624	1,892801	4,701293
18	-1,61898	2,016722	4,704645
19	-1,52933	2,23677	4,882682
20	-1,5106	2,404027	5,307764
21	-1,43114	2,608968	5,515993
22	-1,42696	2,579515	5,644122
23	-1,38114	2,541789	5,658534
24	-1,39372	2,516563	5,596871
25	-1,42314	2,584709	5,787073
26	-1,40656	2,652833	5,946834
27	-1,30645	2,731794	5,777525
28	-1,25037	2,813215	5,906197
29	-1,19766	2,974763	6,349757
30	-1,20691	3,022443	6,409667
31	-1,21891	3,080442	6,383094
32	-1,16907	3,158402	6,484348
33	-1,17036	3,176377	6,485899
34	-1,18267	3,247255	6,544363
35	-1,08546	3,280726	6,508609
36	-1,04912	3,422891	6,569203
37	-0,97646	3,482413	6,71847
38	-0,91357	3,596157	6,786299
39	-0,87475	3,789613	7,127664
40	-0,83961	3,929966	7,539292
41	-0,96148	4,273179	8,536226
42	-0,89151	4,656849	9,19993
43	-0,74076	5,296408	10,07064
44	-0,76331	5,340682	10,07064
45	-0,79661	5,35225	10,19568
46	-0,74873	5,3729	10,07064
47	-0,75103	5,404511	10,19568
48	-0,77392	5,445102	10,19568
49	-0,72136	5,465766	10,07064
50	-0,70556	5,516831	10,07064

	Asset Tolerance		
Simulation	Mean	Std Dev	VaR 95%
1	-3,99008	1,465169	6,562999
2	-3,88118	1,441651	6,341203
3	-3,74769	1,4033	6,219437
4	-3,58852	1,385706	6,061642
5	-3,41478	1,329094	5,798995
6	-3,21548	1,318372	5,657903
7	-3,02167	1,315537	5,583986
8	-2,85056	1,327694	5,314136
9	-2,67913	1,376469	5,193609
10	-2,5013	1,48968	5,078561
11	-2,35961	1,652445	5,247508
12	-2,1494	1,764593	5,18016
13	-1,97557	1,95166	5,410189
14	-1,77303	2,129461	5,465201
15	-1,70218	2,362244	5,787897
16	-1,50599	2,571542	5,871955
17	-1,38867	2,924645	6,081842
18	-1,36747	3,30239	7,209647
19	-1,23744	3,68013	7,36871
20	-1,17441	4,187083	8,459465
21	-0,97167	4,430688	8,541286
22	-0,91002	4,742736	9,220152
23	-0,90889	5,043674	9,681401
24	-0,89141	5,39956	10,10272
25	-0,79369	5,646703	11,09489
26	-0,72742	5,839715	11,80805
27	-0,59701	5,81041	11,60095
28	-0,59617	5,970543	11,90606
29	-0,56485	6,067707	12,29982
30	-0,53952	6,167384	12,50226
31	-0,57672	6,263018	12,74774
32	-0,54052	6,264332	12,74774
33	-0,52152	6,293982	12,92992
34	-0,52101	6,307178	12,92992
35	-0,50943	6,304398	12,92992
36	-0,50018	6,304946	12,92992
37	-0,51149	6,342441	12,95314
38	-0,5013	6,347003	12,95314
39	-0,50271	6,352455	12,95314
40	-0,50546	6,355382	12,95314
41	-0,52792	6,42773	13,16111
42	-0,51804	6,415797	13,16111
43	-0,51804	6,415797	13,16111
44	-0,51804	6,415797	13,16111
45	-0,51343	6,394057	13,16111
46	-0,51343	6,394057	13,16111
47	-0,51263	6,394599	13,16111
48	-0,51263	6,394599	13,16111
49	-0,51263	6,394599	13,16111
50	-0,51263	6,394599	13,16111

	Fixed Bandwidth Delta		
Simulation	Mean	Std Dev	VaR 95%
1	-3,25535	1,351708	5,622137
2	-2,72395	1,226157	4,962419
3	-2,37459	1,159135	4,527002
4	-2,12749	1,151758	4,190851
5	-1,93929	1,18527	4,062511
6	-1,78997	1,256922	3,900905
7	-1,67148	1,352577	3,872949
8	-1,57451	1,469028	3,912278
9	-1,49337	1,593721	4,142183
10	-1,42177	1,731926	4,331347
11	-1,35626	1,874652	4,471532
12	-1,30144	2,0184	4,594294
13	-1,24993	2,167378	4,862911
14	-1,20391	2,321198	5,095442
15	-1,16372	2,475371	5,386839
16	-1,12977	2,628933	5,6547
17	-1,09614	2,781083	5,940173
18	-1,06336	2,93282	6,240838
19	-1,03254	3,082469	6,555672
20	-1,00318	3,231404	6,848951
21	-0,97504	3,379186	7,102098
22	-0,9474	3,527269	7,277475
23	-0,92164	3,674162	7,554305
24	-0,89719	3,822667	7,830958
25	-0,87267	3,97048	8,106658
26	-0,84926	4,114967	8,378981
27	-0,82547	4,257742	8,563958
28	-0,80301	4,397087	8,823457
29	-0,7821	4,534216	9,096716
30	-0,76113	4,668454	9,37074
31	-0,74035	4,800917	9,645914
32	-0,71883	4,931429	9,920535
33	-0,69929	5,058228	10,16947
34	-0,68058	5,179817	10,40854
35	-0,66364	5,297854	10,68789
36	-0,65005	5,40944	10,70561
37	-0,63769	5,516527	10,90945
38	-0,6266	5,618891	11,11298
39	-0,61725	5,714886	11,31143
40	-0,6088	5,804836	11,50727
41	-0,5983	5,889243	11,64187
42	-0,58676	5,963216	11,66612
43	-0,58047	6,00101	11,82637
44	-0,57446	6,036507	11,83642
45	-0,5679	6,071683	11,84219
46	-0,56191	6,105641	11,99225
47	-0,55649	6,138365	12,15881
48	-0,55076	6,170451	12,27721
49	-0,54536	6,201396	12,42342
50	-0,54109	6,230524	12,44636

	Variable Bandwidth Delta		
Simulation	Mean	Std Dev	VaR 95%
1	-0,92888	4,004878	6,955338
2	-1,39954	1,743909	4,133659
3	-1,50306	1,523647	4,091317
4	-1,57148	1,414264	4,057579
5	-1,62447	1,348088	3,957551
6	-1,66943	1,301046	3,97108
7	-1,70797	1,26809	4,020976
8	-1,74277	1,244045	4,063651
9	-1,77457	1,22405	4,04912
10	-1,80331	1,208555	4,035303
11	-1,82899	1,195847	4,035999
12	-1,85273	1,185261	4,066288
13	-1,8748	1,176403	4,075578
14	-1,89538	1,168704	4,086238
15	-1,91459	1,162329	4,106154
16	-1,93249	1,156811	4,122681
17	-1,94939	1,151875	4,144143
18	-1,96544	1,147459	4,160293
19	-1,98086	1,143786	4,168082
20	-1,9958	1,140398	4,171312
21	-2,01008	1,137394	4,169741
22	-2,02362	1,134789	4,196539
23	-2,03651	1,132308	4,206976
24	-2,04888	1,130057	4,21578
25	-2,06085	1,127993	4,227317
26	-2,07233	1,126071	4,243911
27	-2,08328	1,124224	4,247517
28	-2,09386	1,12256	4,24888
29	-2,1041	1,121019	4,256369
30	-2,11405	1,119552	4,26251
31	-2,12366	1,118186	4,268376
32	-2,13295	1,116903	4,273986
33	-2,14197	1,115734	4,279135
34	-2,15076	1,114682	4,288797
35	-2,15932	1,113736	4,298077
36	-2,16766	1,112851	4,306955
37	-2,1758	1,112046	4,307253
38	-2,18377	1,111314	4,307349
39	-2,19155	1,110661	4,321213
40	-2,19913	1,110118	4,334342
41	-2,20652	1,109637	4,332882
42	-2,2137	1,109179	4,330335
43	-2,22074	1,108785	4,336122
44	-2,22767	1,108448	4,345086
45	-2,23442	1,108142	4,353966
46	-2,24099	1,107859	4,361638
47	-2,24742	1,10761	4,370639
48	-2,2537	1,10743	4,380331
49	-2,25987	1,10729	4,388262
50	-2,26589	1,107202	4,394093