LUND UNIVERSITY

Department of Statistics

Investigation of GARCH Models for the Estimation Power and Normality

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Abstract

The aims of the thesis are to investigate the estimation power and the normality of standardized residuals for Generalized autoregressive conditional heteroscedasticity models (GARCH). We facilitate the analysis by only dealing with GARCH(1, 1) models. We take use of MATLAB as the statistical programming tool for the simulation of the data and the estimation.

We define the meaning of estimation power in three ways. Firstly, how close estimated expectation of estimators is to the actual value given a value of biasness. Secondly, another way to define the estimation power is by calculating Root Mean Square Error (RMSE) of estimated values. Finally, we define it by how large proportion of significant models we get.

To analyze the estimation power, we perform three simulation tests to measure the biasness, the RMSE and the proportion of significant GARCH(1, 1) models. In addition, we analyze the normality by calculating the proportion of standardized residuals using the Jarque-Bera test.

Based on the results from those three simulation studies focused on the estimation power, we conclude that when the number of observations increases, it reduces the biasness of the estimated parameters. Secondly, the size of the GARCH and ARCH parameters plays a major role in determining the estimation power. The larger GARCH and ARCH effects are contained in the series, the better estimation power we get. Moreover, we conclude that for a given sum of GARCH and ARCH values being constant, the combination that has equal weight has the best estimation power.

Finally and most importantly, based on the result from the fourth simulation test, we understand that as long as we get an estimated model in which both estimated GARCH and ARCH parameters are significant, we have at least ninety percent of the standardized residuals that are normally distributed with the properties of zero mean and unit variance.

Contents

1. Introduction

Generalized autoregressive conditional heteroscedasticity models are often abbreviated as GARCH models. GARCH models, introduced by Bollerslev (1986), generalized Engle's (1982) earlier ARCH models to include the autoregressive (AR), as well as moving average (MA) terms. GARCH models can be more parsimonious (use fewer parameters), increasing computational efficiency.

The theory behind GARCH is that it includes past variances in the explanation of future variances. Due to this feature, GARCH has played a central role in analyzing financial time series for decades. We can apply GARCH models to such areas as risk management, option pricing, portfolio management and asset allocation and so forth.

The aims of the thesis are to investigate the estimation power and the normality of standardized residuals for GARCH models. We try to define the meaning of estimation power in three ways. Firstly, how close estimated expectation of estimators is to the actual value given a value of biasness. To get a good estimation power, the bias should be close to zero. Secondly, another way to define the estimation power is by calculating Root Mean Square Error (RMSE) of estimated values. If the RMSE is close to zero, it implies that the estimation power is good. Finally, we define the estimation power by how large proportion of significant models we get.

Based on the definitions of estimation power, we perform three simulation tests to investigate the estimation power in these three ways. In the first simulation study, we focus on measuring the biasness of estimated parameters when we increase the number of observations. In the second simulation test, we concentrate on how the actual GARCH and ARCH parameters affect the estimation power by comparing RMSE of estimated values. In the third simulation test, we focus on testing how the length of the series and the effect of using different combinations of actual GARCH and ARCH values influence the estimation power, especially getting estimated models for which both estimated GARCH and ARCH parameters are significant. Finally, we perform a simulation test to calculate the proportion of standardized residuals by increasing the length of series and using different combinations of actual GARCH and ARCH values. To facilitate the analysis, we only deal with GARCH(1, 1) models in the thesis.

2. Methodology

2.1. Generalized Autoregressive Conditional Heteroscedasticity model

Let $\{y_t$; t = 0,….,T$\}$ be an observed data series

$$y_t = C + \varepsilon_t \, , \tag{2.1}$$

where ε_t is the error term and C is a constant value and

$$\varepsilon_t = z_t \sigma_t \, . \tag{2.2}$$

z_t is the standardized residual that follows a normal distribution of zero mean and unit variance, while σ_t is a nonnegative process which is the square root of the conditional variance

$$\sigma_t^2 = K + \sum_{i=1}^{q} \alpha_i \, \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \, \sigma_{t-j}^2 \, , \tag{2.3}$$

where $K > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$. This is the generalized autoregressive heteroscedasticity model, denoted as GARCH(p,q) by Enders (1995).

To facilitate the analysis, we use GARCH(1, 1) models. The standardized residual is $z_t = \frac{\varepsilon_t}{\sigma_t}$ with the conditional variance of innovation

$$\sigma_t^2 = K + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2 \, . \tag{2.4}$$

In addition, ε_t is a stationary process if it is under the condition $\alpha + \beta < 1$. This condition is obtained from the unconditional variance of the error term ε_t

$$var(\varepsilon_t) = K/(1 - \alpha - \beta) \, . \tag{2.5}$$

For positive variance $1 - \alpha - \beta > 0 \;\; i.e. \;\; \alpha + \beta < 1$, where α is ARCH parameter and β is GARCH parameter.

2.2. Quasi-Maximum Likelihood Estimate

The quasi-maximum likelihood differs from maximum likelihood, and it deals with conditional variance of GARCH model that depends on the previous conditional variance and the error term. It was introduced by Robert Wedderburn in 1974.

Under the condition that the standardized residual z_t follows a normal distribution with zero mean and unit variance, we can maximize the quasi likelihood conditioned on σ_0^2 to obtain quasi-maximum likelihood that could be applied on series to estimate the model parameters. In our case we try to estimate GARCH(1, 1) parameters $\theta = (C, K, \alpha, \beta)$. We use the MATLAB build-in function (garchfit) and this function applies quasi-maximum likelihood to perform the estimation.

$$L_T\left(y_{0,\dots\dots,}y_T, \sigma_0^2; \theta\right) = L_T(\theta) = -(2T)^{-1} \sum_{t=1}^{T} (\ln \sigma_t{}^2(\theta) + \frac{\varepsilon_t{}^2}{\sigma_t{}^2(\theta)}) \,, \tag{2.6}$$

where

$$\sigma_t{}^2(\theta) = K + \alpha \sum_{k=0}^{\infty} \beta^k \varepsilon_{t-1-k}^2 \tag{2.7}$$

and

$$\varepsilon_t = y_t - C \,. \tag{2.8}$$

The quasi-maximum likelihood estimator (QMLE) maximizes the $L_T(\theta)$. The QMLE may be consistent, see e.g. Elie and Jeantheau (1995), and asymptotically normal, provided that the innovation has a finite fourth moment, even if it is far from Gaussian, see Hall and Yao (2003) as well as Berkes, Horváth and Kokoszka (2003). We show however that even though the estimators are asymptotically normal, they are often far from normal in finite sample sizes.

2.3. Biasness

Bias of an estimator (or mean-bias) in our case is the difference between the expectation of the estimated parameters and the actual value of the parameter. An estimator with zero bias is unbiased. Otherwise the estimator is said to be biased.

The smaller value of bias we get, the better estimation power we obtain.

$$Bias[\hat{\theta}] = E[\hat{\theta}] - \theta = E[\hat{\theta} - \theta]. \tag{2.9}$$

Under the condition $E(\hat{\theta}) \neq \theta$, where $\theta \in \{C, K, \alpha, \beta\}$, $\hat{\theta}$ is biased.

We estimate the bias of $\hat{\theta}$ by estimating the expectation of $\hat{\theta}$ using Monte Carlo method with N series. Letting $\hat{\theta}_i$ be the estimate from the ith simulated series, then

$$\widehat{E(\hat{\theta})} = N^{-1} \sum_{i=1}^{N} \hat{\theta}_i \tag{2.10}$$

is the estimated expected value of $\hat{\theta}$.

More of this will be discussed in the next section.

2.4. Root Mean Square Error

The Root Mean Square Error (RMSE)

$$RMSE(\hat{\theta}) = \sqrt{MSE(\hat{\theta})} = \sqrt{E\left[(\hat{\theta} - \theta)^2\right]}, \tag{2.11}$$

where $\theta \in \{C, K, \alpha, \beta\}$, is a frequently used measure of the difference between the estimated value and the actual value. RMSE is a good measure of precision, it serves to aggregate into a single measure of predictive power. The smaller RMSE we get, the better estimation power we obtain.

We estimate the RMSE of $\hat{\theta}$ by estimating the expectation of squared difference of estimator and the actual value using Monte Carlo method with N series.

The individual differences $\hat{\theta}_i - \theta$ are also called residuals where $\hat{\theta}_i$ be the estimate from the ith simulated series and the estimated $RMSE(\hat{\theta})$ is then

$$\widehat{RMSE}(\hat{\theta}) = \sqrt{N^{-1}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2}\,. \tag{2.12}$$

More of this will be discussed in the next section.

2.5. Wald test for significant GARCH(1, 1) models

We define a significant GARCH(1, 1) models to be a model where both estimated GARCH and ARCH parameters are significant from zero at the 5% level. The hypothesis test performed was the Wald test as follows:

$H_0: \theta = 0$

$H_1: \theta \neq 0.$

$$Wald\ statistic\ = \frac{(\hat{\theta} - \theta)^2}{\widehat{var}(\hat{\theta})}, \tag{2.13}$$

which is compared against a chi-square distribution with 1 degree of freedom and $\hat{\theta} \in \{\hat{C}, \hat{K}, \hat{\alpha}, \hat{\beta}\}$.

2.6. Jarque-Bera test for the normality

The Jarque–Bera test is a goodness-of-fit test for departure from normality, based on the sample kurtosis and skewness. The test is named after Carlos Jarque and Anil K. Bera (Jarque and Bera, 1987). The test statistic JB is defined as follows:

$$\widehat{JB} = \frac{T}{6}\left(\hat{s}^2 + \left(\hat{k} - 3\right)^2/4\right), \tag{2.14}$$

where the skewness $\hat{s}$ and the kurtosis $\hat{k}$ in our case are obtained by calculating the skewness and kurtosis of the estimated standardized residuals

$$\hat{z}_t = \frac{\hat{\varepsilon}_t}{\hat{\sigma}_t} \tag{2.15}$$

and the estimated conditional variance

$$\hat{\sigma}_t^2 = \widehat{K} + \sum_{i=1}^{q} \hat{\alpha}_i \, \hat{\varepsilon}_{t-i}^2 + \sum_{j=1}^{p} \hat{\beta}_j \, \hat{\sigma}_{t-j}^2 , \tag{2.16}$$

for t=1,…,T where T is the number of observations in the series.

The skewness is

$$\hat{s} = \frac{T^{-1}\sum_{t=1}^{T}(\hat{z}_t - \bar{z})^3}{(T^{-1}\sum_{t=1}^{T}(\hat{z}_t - \bar{z})^2)^{3/2}} \tag{2.17}$$

and the kurtosis is

$$\hat{k} = \frac{T^{-1}\sum_{t=1}^{T}(\hat{z}_t - \bar{z})^4}{(T^{-1}\sum_{t=1}^{T}(\hat{z}_t - \bar{z})^2)^2}, \tag{2.18}$$

for the mean

$$\bar{z} = T^{-1}\sum_{t=1}^{T} \hat{z}_t \, . \tag{2.19}$$

The null hypothesis and the alternative hypothesis of this test are

H_0: z_t's *are normally distributed*

H_1: $z_t^{'}s$ *are not normally distributed*.

3. Simulations

3.1. An overview of estimation power and properties of GARCH

The purpose of the simulation in this part is to investigate how estimated parameters C, K, GARCH and ARCH behave while number of observations increases. In addition, we want to investigate the effect of large and small actual GARCH and ARCH parameters.

First of all, we simulate 5000 series for GARCH(1, 1) with actual parameters C, K, GARCH and ARCH which are 0.2, 0.2, 0.2 and 0.1 respectively. Each series has 100 observations. Next, we estimate 5000 series using the GARCH(1, 1) model so that we can get 5000 estimated parameters for C, K, GARCH and ARCH respectively. Third, we calculate the averages for each of those 4 estimated parameters and subtract the actual parameters from the average estimated parameters. At the same time descriptive statistics of 4 parameters are calculated. Fourth, we repeat the procedure by increasing the number of observations in 200, 500, 1000, 2000 and 5000. The results are shown in Table 3.1. Finally, in order to investigate the effect with large and small values for GARCH and ARCH actual parameters, we repeat the procedure again by using actual parameters C, K, GARCH and ARCH being 0.2, 0.2, 0.5 and 0.4 respectively. The results are shown in Table 3.2.

Table 3.1 The biases of the estimated parameters for small actual parameters

	Actual value	T=100	T=200	T=500	T=1000	T=2000	T=5000
C	0.2	0.0003	-0.0006	-0.0003	-0.0004	0.0001	0.0000
K	0.2	-0.0564	-0.0406	-0.0225	-0.0136	-0.0045	0.0002
GARCH	0.2	0.1898	0.1414	0.0820	0.0503	0.0169	-0.0002
ARCH	0.1	0.0061	-0.0009	-0.0043	-0.0035	-0.0014	-0.0007

Table 3.2 The biases of the estimated parameters for large actual parameters

	Actual value	T=100	T=200	T=500	T=1000	T=2000	T=5000
C	0.2	0.0016	-0.0002	0.0002	0.0006	0.0003	0.0001
K	0.2	0.0887	0.0420	0.0155	0.0078	0.0036	0.0016
GARCH	0.5	-0.0701	-0.0388	-0.0136	-0.0067	-0.0032	-0.0015
ARCH	0.4	-0.0072	-0.0004	-0.0009	-0.0006	-0.0003	0.0001

According to the results in Table 3.1 and 3.2, there are two findings that we can obtain. Firstly, no matter what actual parameters of GARCH and ARCH are, when the number of observations increases, the biases are approaching zero either from positive or negative side. For instance, when the number of observations is 100 and the actual GARCH and ARCH are 0.2 and 0.1, the biases of GARCH and ARCH are 0.1898 and 0.0061 respectively. When the number of observations is 5000, biases of GARCH and ARCH are -0.0002 and -0.0007 respectively.

Secondly, with small actual GARCH and ARCH parameters, when number of observations increases, biases of GARCH and ARCH are from the positive side approaching to zero. It can be explained in other words that the average estimated parameters for GARCH and ARCH decrease to approach the actual values. On the contrary, the bias of K is from the negative side approaching to zero. It also can be explained that the average estimation of K is from small average estimated values approaching to the actual parameters. For example, bias of GARCH is from 0.1898 to -0.0002 while bias of K is from -0.0564 to 0.0002.

Relatively, with large actual GARCH and ARCH parameters, when number of observations increases, biases of GARCH and ARCH are from the negative side approaching to zero which can be interpreted as the average estimated GARCH and ARCH are from small estimated values approaching to the actual parameters . On the other hand, the bias of K is from the positive side approaching to zero. It also means that the average estimation of K is from large estimated values approaching to the actual value. For instance, bias of ARCH is from -0.0072 to 0.0001 while bias of K is from 0.0887 to 0.0016.

Table 3.3 Descriptive Statistics for estimated parameters with small actual parameters

	Actual value	T=100	T=1000	T=5000
C	0.2			
Mean		0.200	0.200	0.200
S.D.		0.054	0.017	0.007
Skewness		0.022	-0.024	-0.029
Kurtosis		2.998	2.886	2.883
K	0.2			
Mean		0.144	0.188	0.201
S.D.		0.094	0.068	0.037
Skewness		0.052	0.713	-0.352
Kurtosis		1.887	2.687	2.804
GARCH	0.2			
Mean		0.390	0.245	0.195
S.D.		0.362	0.250	0.132
Skewness		0.345	0.908	0.472
Kurtosis		1.622	2.963	2.983
ARCH	0.1			
Mean		0.106	0.098	0.100
S.D.		0.113	0.042	0.018
Skewness		1.259	0.242	0.026
Kurtosis		4.664	2.864	3.005

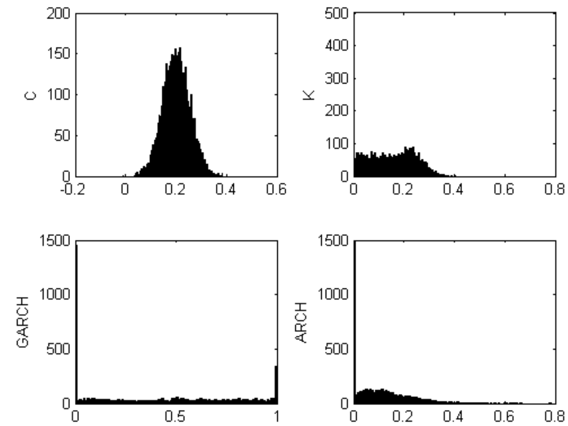

Figure 3.1 Distribution of parameters for T = 100

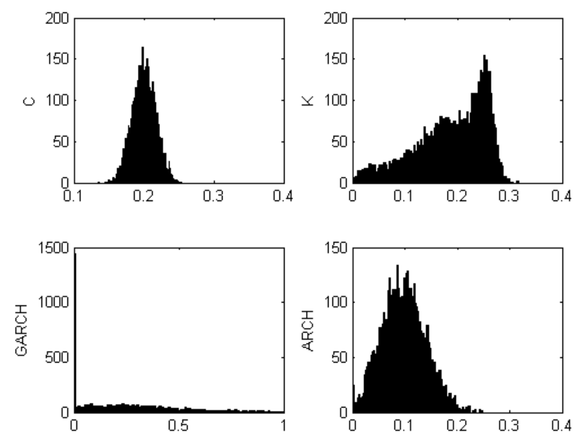

Figure 3.2 Distribution of parameters for T = 1000

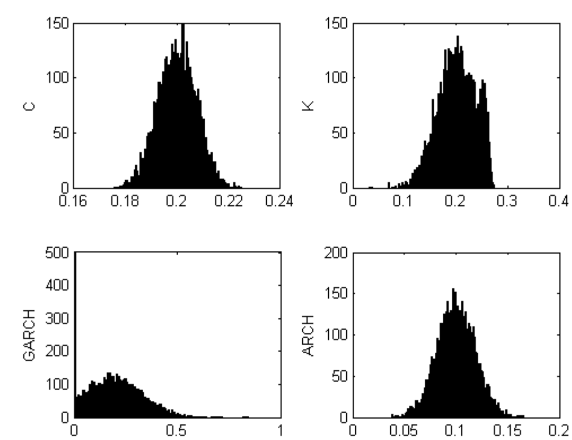

Figure 3.3 Distribution of parameters for T = 5000

According to the descriptive statistics in Table 3.3 for the small actual values of the parameters, when number of observations increases, some findings can be drawn. First of all, the properties of estimated parameter C are not affected by increasing the number of observations except for the sample standard deviation. When number of observations increases, sample standard deviation decreases from 0.054 to 0.007. Secondly, estimated parameter K is affected by the increase of the number of observations. The distribution of K shows that when the number of observations increases, the distribution is tending to be normal with negative skewness. There is an interesting thing occurred when T = 1000 and 5000, K has a high peak between 0.2 and 0.3. Looking at the GARCH parameter when T = 100, we find that the distribution is approximately uniform with two spikes close to 0 and 1. When T = 1000, the left spike remains unchanged while the right spike disappears. When T = 5000, the distribution is getting to be normal with the positive skewness which is affected by the spike at the value 0 and the frequency at zero starts decreasing from 1500 to 500. Finally, when T = 100, there is a left spike with skewness = 1.259 and kurtosis = 4.664 for ARCH parameter. When T increases, the spike is disappearing and the distribution tends to be normal.

According to the correlation coefficient in Tables 3.4-3.6 for actual values of C=0.2, K=0.2, GARCH=0.2 and ARCH=0.1, some findings can be drawn. Firstly, parameter C does not correlate to the others. In addition, there is an extremely strong negative correlation between estimated parameter K and GARCH, this negative correlation is getting slightly stronger as T increases. There is a weak positive correlation between estimated parameter K and ARCH. Moreover, there is a negative correlation between estimated parameters GARCH and ARCH. When the number of observations increases, the negative correlation is getting weaker from -0.50 to -0.32.

The correlations between estimated parameters with actual values C=0.2, K=0.2, GARCH=0.2 and ARCH=0.1 are given in Table 3.4-3.6. In figures 3.4-3.6, the estimated parameters are shown pairwise for the 5000 series for different lengths of series.

Table 3.4 Correlation matrix for T = 100

	C	K	GARCH	ARCH
C	1	-0.01	0.00	0.00
K		1	-0.93	0.26
GARCH			1	-0.50
ARCH				1

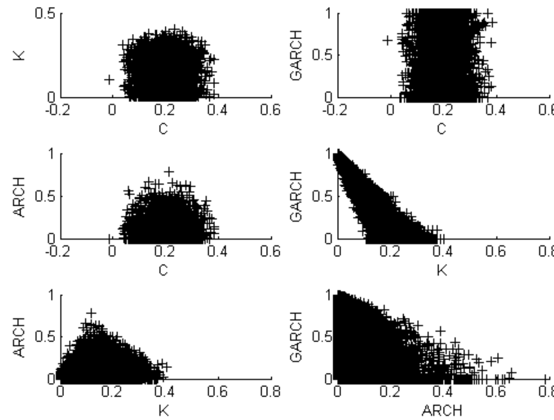

Figure 3.4 Pairwise comparisons for T = 100

Table 3.5 Correlation matrix for T = 1000

	C	K	GARCH	ARCH
C	1	0.00	0.01	-0.02
K		1	-0.98	0.29
GARCH			1	-0.41
ARCH				1

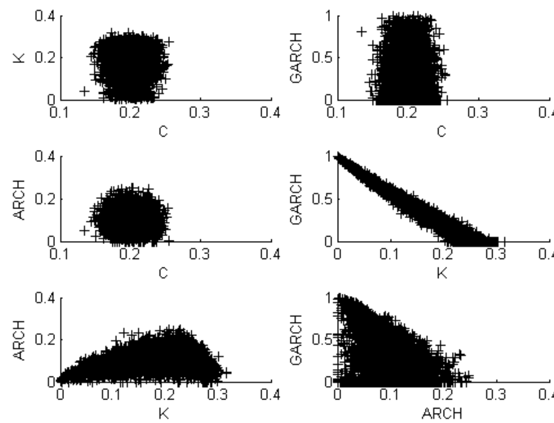

Figure 3.5 Pairwise comparisons for T = 1000

Table 3.6 Correlation matrix for T = 5000

	C	K	GARCH	ARCH
C	1	-0.01	0.02	-0.02
K		1	-0.99	0.22
GARCH			1	-0.32
ARCH				1

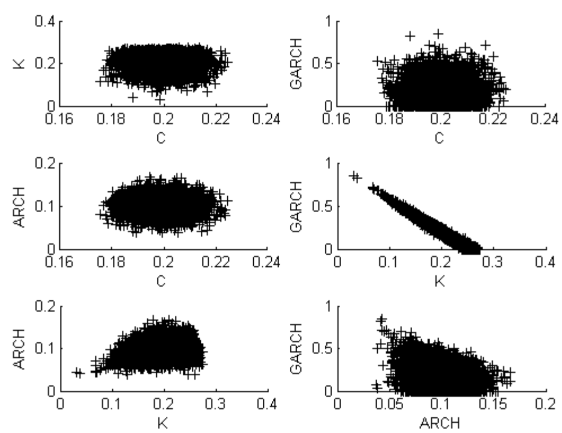

Figure 3.6 Pairwise comparisons for T = 5000

Table 3.7 Descriptive Statistics for estimated parameters with large actual parameters

	Actual value	T=100	T=1000	T=5000
C	0.2			
Mean		0.202	0.201	0.200
S.D.		0.096	0.029	0.013
Skewness		0.057	-0.018	0.032
Kurtosis		3.257	3.061	3.389
K	0.2			
Mean		0.289	0.208	0.201
S.D.		0.206	0.042	0.018
Skewness		1.710	0.584	0.049
Kurtosis		7.518	3.941	4.797
GARCH	0.5			
Mean		0.430	0.493	0.499
S.D.		0.211	0.054	0.024
Skewness		-0.244	-0.127	0.210
Kurtosis		2.784	3.616	6.571
ARCH	0.4			
Mean		0.393	0.400	0.399
S.D.		0.174	0.054	0.025
Skewness		0.227	0.025	0.158
Kurtosis		2.906	3.037	6.433

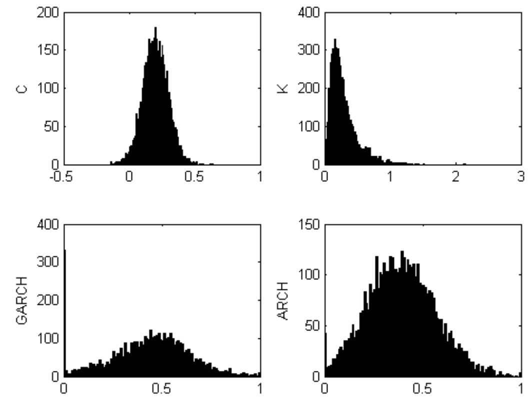

Figure 3.7 Distribution of parameters for T = 100

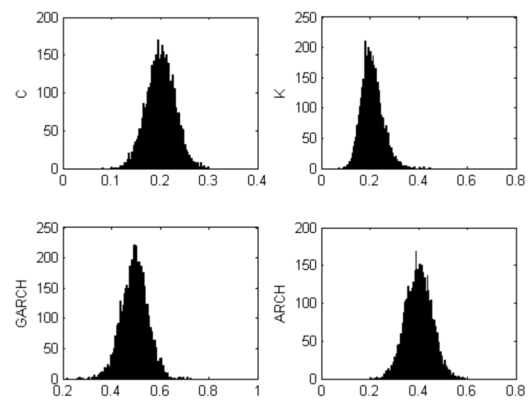

Figure 3.8 Distribution of parameters for T = 1000

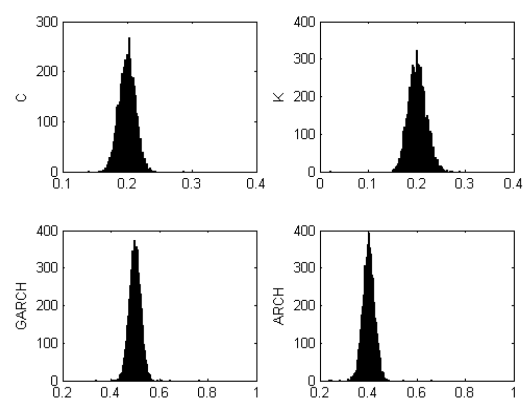

Figure 3.9 Distribution of parameters for T = 5000

According to the descriptive statistics for the large actual values of parameters shown in Table 3.7, when number of observations increases, some findings can be drawn. Firstly, when the number of observations has an increase from 100 to 5000, the sample standard deviation for estimated parameter C has a decrease from 0.096 to 0.013. The other properties of estimated parameter C are not affected by changing the number of observations. Thus, we get a similar result while we compare it with small actual parameter values. Secondly, estimated parameter K is affected by the increase in the number of observations. At the beginning the distribution has a positive skewness with a heavy tail. When number of observations increases, the distribution is tending to be in belt shape with high kurtosis. The skewness decreases to 0.049 when T = 5000. The distribution of estimated parameters GARCH starts following a normal distribution with a left spike when T = 100. When T = 1000, the left spike disappears. When T = 5000, the shape seems to be normal but the kurtosis is equal to 6.571 which is higher than expected for the normal distribution. Finally, when T = 100, there is a small left spike with frequency near to 50 for ARCH parameter. When T increases, the estimated parameters of ARCH behave similar to the estimated GARCH parameters.

Basically, we find that when we compare the estimation power, we can get a better estimation when the actual values of GARCH and ARCH parameters are large.

According to the correlation coefficient for C=0.2, K=0.2, GARCH=0.5 and ARCH=0.4 given in Table 3.8-3.10 and figures 3.10-3.12, some findings can be drawn. First of all, parameter C does not correlate to the others. Second, there is a strong negative correlation between estimated parameter K and GARCH, this correlation is not affected by T. There is a weak positive correlation between estimated parameter K and ARCH. When T increases, it is slightly increasing from 0.15 to 0.25. Moreover, there is a negative correlation between estimated parameter GARCH and ARCH. When the number of observations increases, the negative correlation is getting stronger from -0.57 to -0.72.

The correlations between estimated parameters with actual values C=0.2, K=0.2, GARCH=0.5 and ARCH=0.4 are given in Table 3.8-3.10. In figures 3.10-3.12, the estimated parameters are shown pairwise for the 5000 series for different lengths of each series.

Table 3.8 Correlation matrix for T = 100

	C	K	GARCH	ARCH
C	1	0.01	-0.01	0.01
K		1	-0.75	0.15
GARCH			1	-0.57
ARCH				1

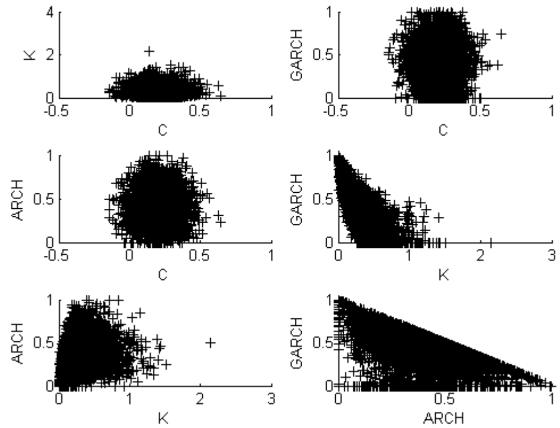

Figure 3.10 Pairwise comparison for T = 100

Table 3.9 Correlation matrix for T = 1000

	C	K	GARCH	ARCH
C	1	-0.02	0.02	-0.02
K		1	-0.77	0.24
GARCH			1	-0.70
ARCH				1

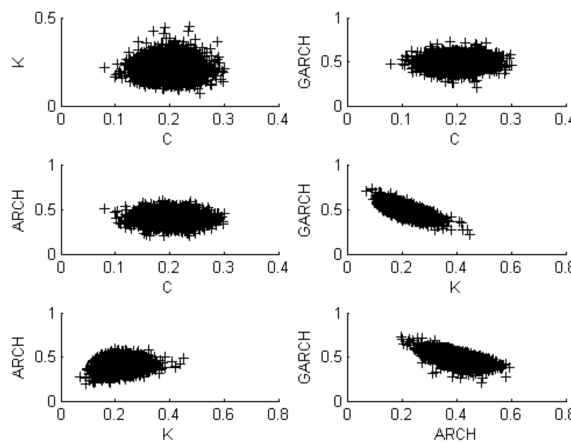

Figure 3.11 Pairwise comparison for T = 1000

Table 3.10 Correlation matrix for T = 5000

	C	K	GARCH	ARCH
C	1	0.01	-0.01	0.01
K		1	-0.76	0.25
GARCH			1	-0.72
ARCH				1

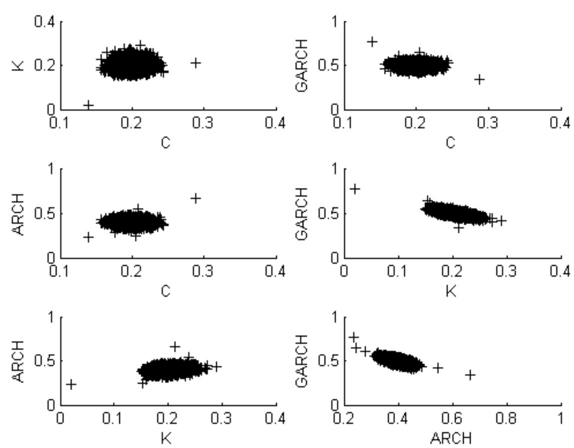

Figure 3.12 Pairwise comparison for T = 5000

3.2. Estimation power by varying actual parameters of GARCH and ARCH

In the previous simulation, we showed how the estimation power behaves when the number of observations increases. In this part, the purpose of the simulation is to investigate to what extent the actual GARCH and ARCH values affect the power of estimation. We use RMSE to measure the estimation power using different combinations of actual GARCH and ARCH parameters. Before we start the simulation, we need to know how many repetitions are needed to get sufficient accuracy.

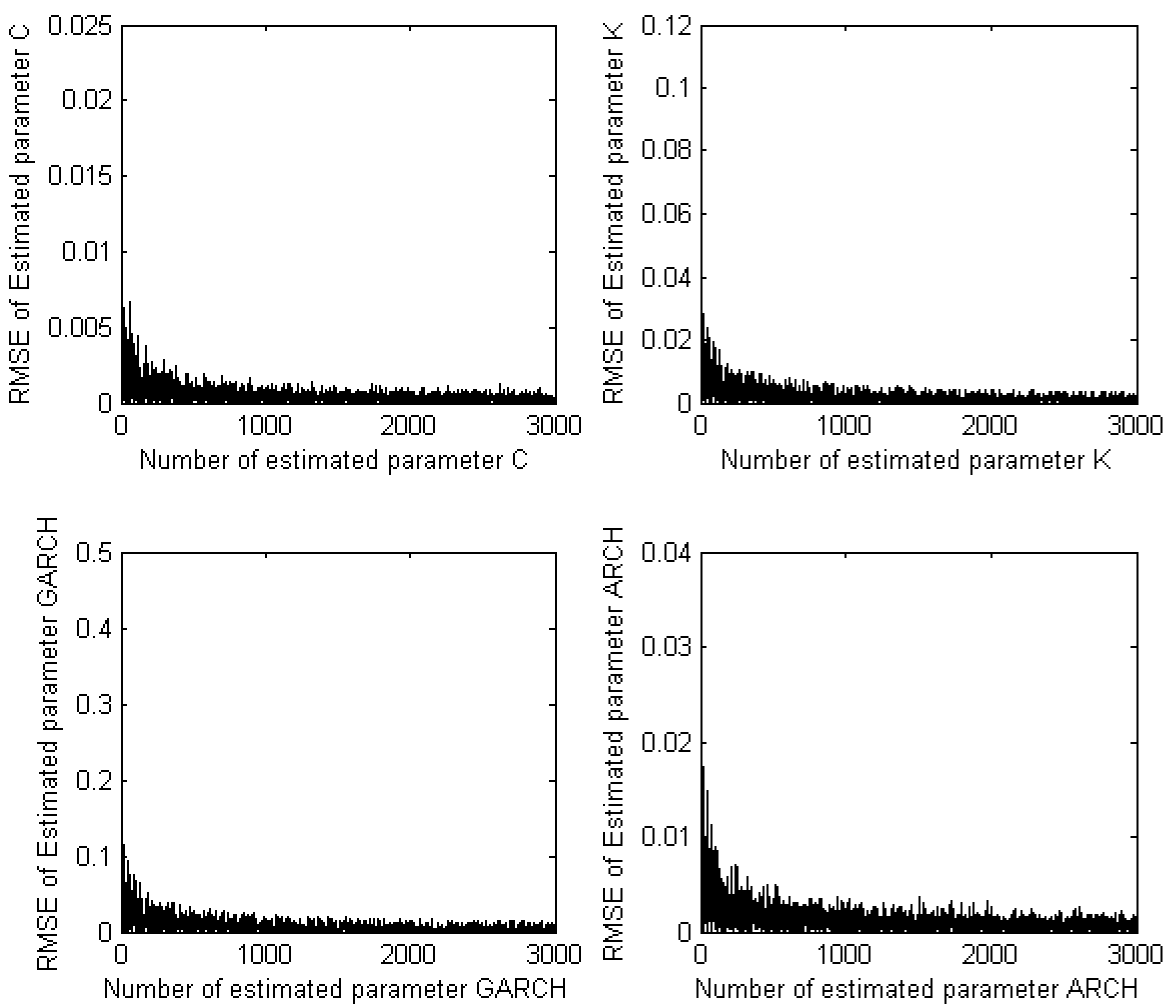

Figure 3.13 Checking steady state of RMSE for estimated parameters

According to the Figure 3.13, the estimated RMSE of estimated C, K, GARCH and ARCH parameters are stable when the number of series (i.e. N in (2.12)) is larger than 1000 approximately. Thus, to be sure that the simulation is accurate enough, we set the number of simulated series to be 5000.

The procedure of simulation is performed in the following way. Firstly, we simulate 5000 series and the length of each series is 100. The initial values of C and K are 0.2. Next, we do the estimation for those 5000 series and get 5000 estimated parameters of C, K, GARCH and ARCH. Then, we calculate the RMSE using those 5000 estimated parameters. Third, we repeat the procedure above by using different combinations of actual GARCH and ARCH values. The minimum of actual GARCH and ARCH values are 0.1 and the maximum of actual GARCH and ARCH values are 0.8. The actual GARCH and ARCH values are increased by 0.1 at a time subject to that the sum of the GARCH and ARCH values are less than 1. Finally, this procedure is repeated by using 1000 and 5000 as the length of each series.

The results for the length of 5000 observations are shown in Tables 3.11-3.14. Since the results for the length of 100 and 1000 observations are similar to the ones in these tables for the length of 5000 observations, these results deferred to the appendix in tables A.1-A.8.

The RMSE of estimated parameters C, K, GARCH and ARCH with the 5000 series and 5000 observations in each series are as follows:

Table 3.11 RMSE of estimated parameter C

		Actual ARCH parameter value							
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Actual GARCH parameter value	0.1	0.007	0.007	0.007	0.008	0.007	0.008	0.012	0.008
	0.2	0.008	0.008	0.008	0.008	0.008	0.009	0.009	
	0.3	0.008	0.009	0.008	0.009	0.010	0.010		
	0.4	0.009	0.010	0.010	0.011	0.012			
	0.5	0.010	0.011	0.012	0.015				
	0.6	0.012	0.012	0.015					
	0.7	0.013	0.017						
	0.8	0.019							

min	0.007
max	0.019

Table 3.12 RMSE of estimated parameter K

		Actual ARCH parameter value							
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Actual GARCH parameter value	0.1	0.029	0.018	0.015	0.013	0.012	0.011	0.013	0.010
	0.2	0.036	0.022	0.017	0.014	0.014	0.013	0.012	
	0.3	0.039	0.024	0.019	0.015	0.015	0.014		
	0.4	0.045	0.026	0.019	0.016	0.016			
	0.5	0.044	0.026	0.020	0.018				
	0.6	0.047	0.025	0.021					
	0.7	0.044	0.026						
	0.8	0.039							

min	0.010
max	0.047

Table 3.13 RMSE of estimated parameter GARCH

		Actual ARCH parameter value							
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Actual GARCH parameter value	0.1	0.120	0.066	0.045	0.033	0.028	0.022	0.020	0.016
	0.2	0.132	0.072	0.049	0.036	0.029	0.025	0.020	
	0.3	0.124	0.067	0.046	0.032	0.028	0.022		
	0.4	0.118	0.061	0.038	0.029	0.023			
	0.5	0.096	0.048	0.032	0.022				
	0.6	0.079	0.035	0.023					
	0.7	0.053	0.023						
	0.8	0.027							

min 0.016
max 0.132

Table 3.14 RMSE of estimated parameter ARCH

		Actual ARCH parameter value							
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Actual GARCH parameter value	0.1	0.018	0.022	0.025	0.026	0.029	0.032	0.041	0.034
	0.2	0.019	0.023	0.024	0.028	0.029	0.030	0.032	
	0.3	0.019	0.022	0.024	0.026	0.029	0.031		
	0.4	0.018	0.021	0.023	0.027	0.026			
	0.5	0.017	0.020	0.023	0.024				
	0.6	0.017	0.018	0.021					
	0.7	0.015	0.017						
	0.8	0.012							

min 0.012
max 0.041

According to the results in table 3.11-3.14 and in table A.1-A.8, we observe some findings. Firstly, when the number of observations increases, RMSE for 4 parameters decreases in whole but the distribution of minimum and maximum values for those 4 parameters remain the same. It shows that the effect of number of observations is only on the power of getting estimated parameters but it does not affect the effect of using different combination of GARCH and ARCH parameters. For instance, the minimum and maximum RMSE of estimated parameter C is min

= 0.049, 0.015, 0.007 and max = 0.133, 0.04, 0.019 with the same combinations which are (GARCH = 0.1, ARCH = 0.1) for minimum RMSE and (GARCH = 0.8, ARCH = 0.1) for maximum RMSE respectively.

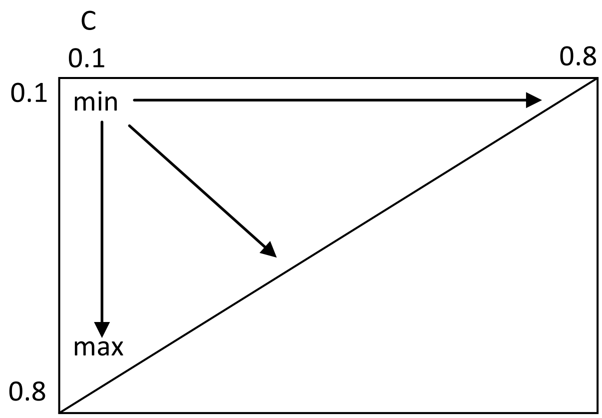

Figure 3.14 Size direction of RMSE for C

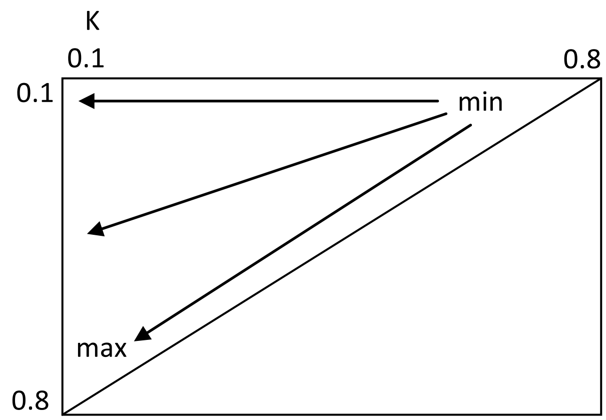

Figure 3.15 Size direction of RMSE for K

Based on the results in Table 3.11, 3.12, A.1, A.2, A.5 and A.6, we find that there is a similarity between parameter C and K. To get a minimum RMSE for C and K, the actual values of GARCH and ARCH should be close to zero. In addition, the RMSE of C and K are getting larger when the sum of actual GARCH and ARCH values is approaching to one. Moreover, we can get the maximum RMSE for C and K by using the same combination of actual GARCH and ARCH values which are near to GARCH = 0.8 and ARCH = 0.1.

Apart from the similarity, parameter C and K also have a difference. We obtain the minimum and maximum of RMSE of C when the actual GARCH and ARCH parameters are close to (GARCH = 0.1, ARCH = 0.1) and (GARCH = 0.8, ARCH = 0.1) respectively. As shown in figure 3.14, RMSE of C is getting larger from (GARCH = 0.1, ARCH = 0.1) to (GARCH = 0.8, ARCH = 0.1). On the other hand, we obtain the minimum and maximum RMSE of K when the actual GARCH and ARCH parameters are near to (GARCH = 0.1, ARCH = 0.8) and (GARCH = 0.8, ARCH = 0.1) respectively. According to the figure 3.15, we find that RMSE of K starts getting larger from (GARCH = 0.1, ARCH = 0.8) to (GARCH = 0.8, ARCH = 0.1).

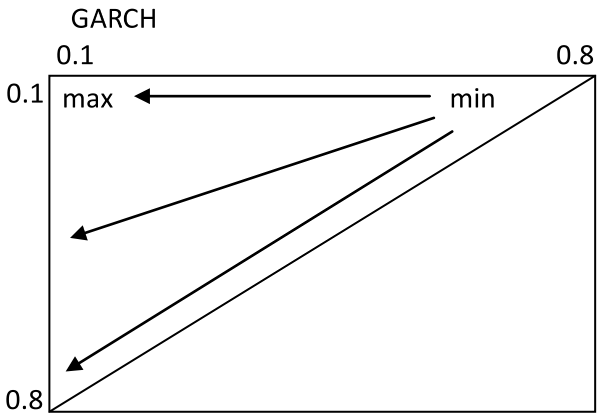

Figure 3.16 Size direction of RMSE for GARCH

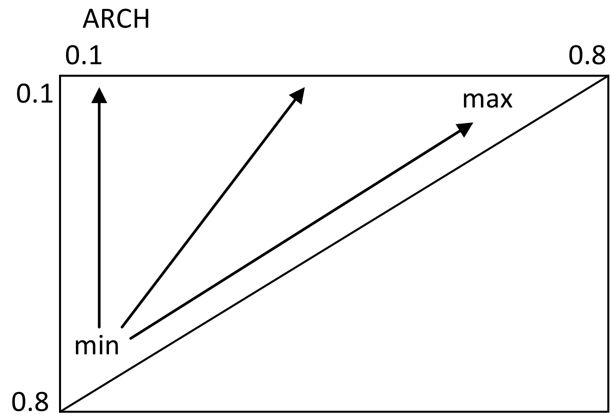

Figure 3.17 Size direction of RMSE for ARCH

The GARCH and ARCH parameters also have a similarity based on the RMSE and the similarity behaves opposite compared to the parameters C and K. According to the tables 3.13, 3.14, A.3, A.4, A.7 and A.8, we find that the minimum RMSE for GARCH and ARCH are on the right diagonal which means that we can get the minimum RMSE for GARCH and ARCH when the sum of actual GARCH and ARCH values is close to one. In addition, the RMSE of GARCH and ARCH are getting larger when the actual GARCH and ARCH values are approaching to zero.

By measuring the RMSE of GARCH and ARCH, we obtain the minimum and maximum RMSE of GARCH when the actual GARCH and ARCH parameters are close to (GARCH = 0.1, ARCH = 0.8) and (GARCH = 0.1, ARCH = 0.1) respectively. According to figure 3.16, RMSE of GARCH starts getting larger from (GARCH = 0.1, ARCH = 0.8) to (GARCH = 0.1, ARCH = 0.1). On the other hand, we obtain the minimum and maximum of RMSE of ARCH when the actual GARCH and ARCH parameters are close to (GARCH = 0.8, ARCH = 0.1) and (GARCH = 0.1, ARCH = 0.1) respectively. Figure 3.17 shows that RMSE of ARCH starts getting larger from (GARCH = 0.8, ARCH = 0.1) to (GARCH = 0.1, ARCH = 0.1).

3.3. Proportion of significant GARCH(1, 1) by varying number of observations

In this part we are interested in whether the length of series and the combination of actual GARCH and ARCH parameters have effects on the proportion of getting significant GARCH(1, 1). The meaning of getting significant GARCH models is that both estimated GARCH and ARCH parameters are significant at 0.05 level of significance.

We study the proportion of getting significant GARCH(1, 1) by increasing the number of observation and using different combinations of actual ARCH and GARCH values. This proportion is calculated by dividing the number of estimated models that have significant ARCH and GARCH parameters by the total number of estimated models. The level of significance is 0.05.

As we know that all simulated series contain both GARCH and ARCH effects, a certain proportion of significant GARCH(1, 1) models also at the same time gives a complementary proportion of wrongly rejected GARCH(1, 1) models. If the estimated parameters are not significant, it implies that the estimation method does not perform well. Those insignificant models are the models where at least one of the estimated GARCH or ARCH parameter is not significant.

Table 3.15 The proportion of getting significant GARCH(1, 1) models with N = 1000 series

T	C = 0.2					
	K = 0.2					
	ARCH = 0.2			ARCH = 0.4		ARCH = 0.6
	GARCH = 0.2	GARCH = 0.4	GARCH = 0.6	GARCH = 0.2	GARCH = 0.4	GARCH = 0.2
100	0.011	0.046	0.126	0.060	0.266	0.134
200	0.074	0.199	0.461	0.183	0.653	0.364
500	0.191	0.542	0.936	0.471	0.954	0.802
1000	0.329	0.828	0.996	0.765	1.000	0.986
2000	0.489	0.977	1.000	0.973	1.000	1.000
5000	0.865	1.000	1.000	1.000	1.000	1.000

Based on the results in Table 3.15, we can conclude the following findings.

First of all, the proportion increases when the number of observations increases. For instance, under (GARCH = 0.2, ARCH = 0.2), the proportion is 0.011 when T = 100. When the number of observations increases to 2000, the proportion is 0.489.

In addition, the larger the sum of actual GARCH and ARCH values is, the higher proportion of significant GARCH(1, 1) we get. For example, when we compare the same number of observations such as T = 2000, under (GARCH = 0.2, ARCH = 0.2), the proportion is only 0.489. However, under (GARCH = 0.4, ARCH = 0.4), the proportion of getting significant GARCH(1, 1) is 1. Even though we increase the number of observations to T = 5000 with actual values (GARCH = 0.2, ARCH = 0.2), the proportion is only 0.865. It proves that to get significant GARCH and ARCH estimated parameters, the actual values of GARCH and ARCH play an important role.

Besides, there is an interesting finding that even though the sum of different combinations is the same, the effect of different combinations can be very different. Let's take the sum of GARCH and ARCH is 0.8 as an example. As we can see there are 3 combinations which are (GARCH = 0.2, ARCH = 0.6), (GARCH = 0.4, ARCH = 0.4) and (GARCH = 0.2, ARCH = 0.6), the sum of actual GARCH and ARCH values is equal to 0.8. Based on comparing the proportion, the best combination to get higher proportion is (GARCH = 0.4, ARCH = 0.4). As the Table 3.15 shows that no matter how long the series we use, the combination of (GARCH = 0.4, ARCH = 0.4) always gets the highest proportion. For example, when T = 500, the proportion of (GARCH = 0.4, ARCH = 0.4) is 0.954 but the proportions for combinations of (GARCH = 0.6, ARCH = 0.2) and (GARCH = 0.2, ARCH = 0.6) are only 0.936 and 0.802 respectively. It shows that when actual GARCH and ARCH values are high and equal, we get highest proportion.

In addition, when we do a comparison with the other two combinations, we find that when the length of series is short such as T = 100, the combination of (GARCH = 0.6, ARCH = 0.2) has a lower proportion than (GARCH = 0.2, ARCH = 0.6). However, when the length of series increases such as T = 200, 500, …etc, the chance to get a higher proportion significant

GARCH(1, 1) for (GARCH = 0.6, ARCH = 0.2) is getting larger. It shows that the effect of actual GARCH value has a higher relation to the length of series than the actual ARCH value.

3.4. Investigating the proportion of normality for standardized residuals

The findings that we got in the previous simulations show 2 important results. Firstly, the length of series affects the estimation power. Secondly, when the sum of actual GARCH and ARCH is close to one, we can get good estimation results. On the other hand, if the sum of actual GARCH and ARCH is far from one, for instance GARCH = 0.05 and ARCH = 0.05, the estimation power is bad.

Based on the findings above, we are interested to investigate whether the length of series and the actual values of GARCH and ARCH affect the normality of standardized residuals. In addition, we evaluate the normality by measuring the proportion of getting normally distributed standardized residuals and Jarque-Bera test is used to perform the hypothesis to test the normality.

We obtain the standardized residuals only from the significant GARCH(1, 1) models. Those insignificant GARCH(1, 1) models are disregarded. The reason why we do not collect standardized residuals from insiginificant GARCH(1, 1) models is that in the normal procedure when we perform an estimation of a time series, if the estimated parameters are not significant, we may have doubt that the series fit GARCH(1, 1) models. Therefore, we discard the estimated GARCH(1, 1) models and try to perform other estimated models instead.

In order to take the effect of the length of series and actual values of GARCH and ARCH into consideration, we use four pairs of actual GARCH and ARCH values which are (GARCH = 0.05, ARCH = 0.05), (GARCH = 0.05, ARCH = 0.45), (GARCH = 0.45, ARCH = 0.05) and (GARCH = 0.45, ARCH = 0.45) to simulate the series with the length of each series = 50, 500 and 5000 respectively. The actual values of C and K are equal to 0.2. Moreover, to make sure that all standardized residuals are from the GARCH(1, 1) model with both estimated GARCH and ARCH parameters significant, we discard, during the simulation process, the estimated results if at most one of GARCH and ARCH are significant until we collect 500 significant estimated models.

Table 3.16 The proportion of getting normal standardized residuals

500 series	Model 1		Model 2	
T	ARCH = 0.05		ARCH = 0.45	
	GARCH = 0.05	GARCH = 0.45	GARCH = 0.05	GARCH = 0.45
50	0.95	0.96	0.94	0.96
500	0.93	0.93	0.96	0.95
5000	0.95	0.95	0.95	0.95

An important finding is that as long as the estimated GARCH and ARCH parameters are both significant, the length of series and actual values of GARCH and ARCH do not affect the possibility of getting normally distributed standardized residuals. As you can see the proportion of getting standardized residuals using T = 50 with the combination of (GARCH = 0.05, ARCH = 0.05) is as high as the one using T = 5000 with the combination of (GARCH = 0.45, ARCH = 0.45).

However, when we tried to simulate the process, especially using short length of series such as T = 50 with small actual GARCH and ARCH values for 500 series, it took us almost 24 hours to get the result. It shows that with short length of series and small actual GARCH and ARCH values, it is rarely possible to get a GARCH(1, 1) model that estimated GARCH and ARCH parameters are significant.

4. Conclusion

Based on the results from the previous simulation tests, we can obtain some interesting conclusions for the estimation power and properties of GARCH model and the normality of standardized residuals as well.

First of all, based on measuring the biasness for the estimated parameters, we can conclude that when number of observations increases, it reduces the bias of the estimated parameters. At the same time we can conclude that with actual GARCH values equal to or larger than 0.2 and ARCH values equal to or larger than 0.1, we get a bias which is less than or equal to absolute 0.05. This occurs when the length of the series is larger than or equal to 2000.

In addition, the size of the GARCH and ARCH effect that are contained in the series also plays a major role in determining the estimation power. The larger the GARCH and ARCH effects are, the better estimation power we get. Based on measuring the RMSE, we can conclude that when the sum of actual values of GARCH and ARCH is close but less than one, we get the best estimation power.

Moreover, according to the third simulation measuring the estimation power by proportion of getting GARCH(1, 1) model with both GARCH and ARCH parameters significant, we conclude that if the sum of the actual GARCH and ARCH values is the same, the combination which has the equal GARCH and ARCH values gives the best estimation power. On the other hand, we find that GARCH effect has a higher relation with the number of observations than the ARCH effect.

Finally and most importantly, based on the result from the fourth simulation test, we understand that as long as we get an estimated model in which both estimated GARCH and ARCH parameters are significant, we have at least 90 percent chance to obtain standardized residuals with the properties of zero mean and one variance.

References

Berkes, I., Horváth, L., Kokoszka, P. (2003). Garch processes: Structure and estimation. *Bernoulli*, Vol. 9, 201–227.

Bollerslev, T. (1986). Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics*, Vol. 31, 307-327.

Elie, L., Jeantheau, T. (1995). Consistency in heteroskedastic models. *Comptes Rendus de l 'Académie des Sciences*, Vol. 320, 1255–1258.

Enders, W. (1995). *Applied Econometric Time Series*. John Wiley & Sons, Inc., New York.

Engle, R. F. (1982). Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of United Kingdom Inflation. *Econometrica*, Vol. 50, 987-1007.

Hall, P., Yao, Q. (2003). Inference in arch and garch models with heavy-tailed errors. *Econometrica*, Vol. 71, 285–317.

Jarque, C. M., Bera, A. K. (1987). A test for normality of observations and regression residuals. *International Statistical Review*, Vol. 55, 163-172.

Wedderburn, R. W. M. (1974). Quasi-likelihood functions, generalized linear models, and the Gauss-Newton method. *Biometrika*, Vol. 61, 439-447.

Appendix. Root Mean Square Error of section 3.2 for series of length 100 and 1000

Number of observations = 100
Number of series = 5000
C = 0.2
K = 0.2

Table A.1 RMSE of estimated parameter C

		Actual ARCH parameter value							
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Actual GARCH parameter value	0.1	0.049	0.054	0.052	0.053	0.053	0.056	0.057	0.060
	0.2	0.055	0.055	0.061	0.058	0.061	0.066	0.069	
	0.3	0.056	0.059	0.066	0.065	0.069	0.076		
	0.4	0.062	0.071	0.073	0.077	0.086			
	0.5	0.072	0.076	0.087	0.097				
	0.6	0.085	0.090	0.107					
	0.7	0.096	0.131						
	0.8	0.133							

Min = 0.049
Max = 0.133

Table A.2 RMSE of estimated parameter K

		Actual ARCH parameter value							
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Actual GARCH parameter value	0.1	0.113	0.091	0.085	0.076	0.073	0.075	0.069	0.074
	0.2	0.109	0.100	0.092	0.089	0.094	0.099	0.109	
	0.3	0.114	0.107	0.107	0.109	0.123	0.111		
	0.4	0.126	0.129	0.140	0.145	0.149			
	0.5	0.146	0.176	0.187	0.215				
	0.6	0.202	0.236	0.293					
	0.7	0.323	0.430						
	0.8	0.659							

Min = 0.069
Max = 0.659

Table A.3 RMSE of estimated parameter GARCH

		Actual ARCH parameter value							
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Actual GARCH parameter value	0.1	0.475	0.359	0.314	0.237	0.212	0.186	0.156	0.139
	0.2	0.401	0.359	0.285	0.253	0.216	0.179	0.158	
	0.3	0.369	0.315	0.286	0.236	0.208	0.185		
	0.4	0.347	0.309	0.278	0.246	0.191			
	0.5	0.343	0.319	0.276	0.227				
	0.6	0.354	0.302	0.244					
	0.7	0.368	0.278						
	0.8	0.369							

Min = 0.139
Max = 0.475

Table A.4 RMSE of estimated parameter ARCH

		Actual ARCH parameter value							
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Actual GARCH parameter value	0.1	0.109	0.152	0.175	0.195	0.207	0.227	0.225	0.221
	0.2	0.113	0.144	0.171	0.193	0.220	0.212	0.197	
	0.3	0.118	0.148	0.181	0.193	0.202	0.207		
	0.4	0.115	0.145	0.172	0.187	0.182			
	0.5	0.113	0.147	0.169	0.166				
	0.6	0.114	0.148	0.151					
	0.7	0.110	0.132						
	0.8	0.104							

Min = 0.104
Max = 0.227

Number of observations = 1000
Number of series = 5000
C = 0.2
K = 0.2

Table A.5 RMSE of estimated parameter C

		Actual ARCH parameter value							
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Actual GARCH parameter value	0.1	0.015	0.016	0.017	0.017	0.017	0.018	0.018	0.018
	0.2	0.017	0.017	0.017	0.019	0.019	0.020	0.021	
	0.3	0.019	0.018	0.020	0.020	0.021	0.022		
	0.4	0.020	0.022	0.022	0.024	0.026			
	0.5	0.021	0.024	0.025	0.030				
	0.6	0.024	0.029	0.033					
	0.7	0.032	0.038						
	0.8	0.040							

Min = 0.015
Max = 0.040

Table A.6 RMSE of estimated parameter K

		Actual ARCH parameter value							
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Actual GARCH parameter value	0.1	0.062	0.040	0.031	0.028	0.024	0.024	0.030	0.025
	0.2	0.070	0.048	0.039	0.032	0.027	0.028	0.024	
	0.3	0.079	0.054	0.041	0.038	0.033	0.031		
	0.4	0.089	0.061	0.049	0.040	0.038			
	0.5	0.103	0.065	0.047	0.043				
	0.6	0.121	0.068	0.049					
	0.7	0.133	0.070						
	0.8	0.129							

Min = 0.024
Max = 0.133

Table A.7 RMSE of estimated parameter GARCH

		Actual ARCH parameter value							
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Actual GARCH parameter value	0.1	0.254	0.151	0.101	0.079	0.059	0.052	0.042	0.037
	0.2	0.260	0.160	0.107	0.077	0.062	0.056	0.043	
	0.3	0.251	0.151	0.101	0.082	0.062	0.049		
	0.4	0.238	0.147	0.101	0.071	0.054			
	0.5	0.224	0.120	0.076	0.055				
	0.6	0.200	0.095	0.052					
	0.7	0.150	0.060						
	0.8	0.082							

Min = 0.037
Max = 0.260

Table A.8 RMSE of estimated parameter ARCH

		Actual ARCH parameter value							
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Actual GARCH parameter value	0.1	0.042	0.052	0.063	0.074	0.077	0.070	0.074	0.075
	0.2	0.041	0.048	0.053	0.060	0.062	0.071	0.070	
	0.3	0.040	0.045	0.051	0.060	0.065	0.066		
	0.4	0.040	0.048	0.057	0.059	0.061			
	0.5	0.038	0.046	0.050	0.054				
	0.6	0.037	0.044	0.046					
	0.7	0.035	0.039						
	0.8	0.029							

Min = 0.029
Max = 0.077