Constraints on planetary orbital evolution theories from detection of multiple transits

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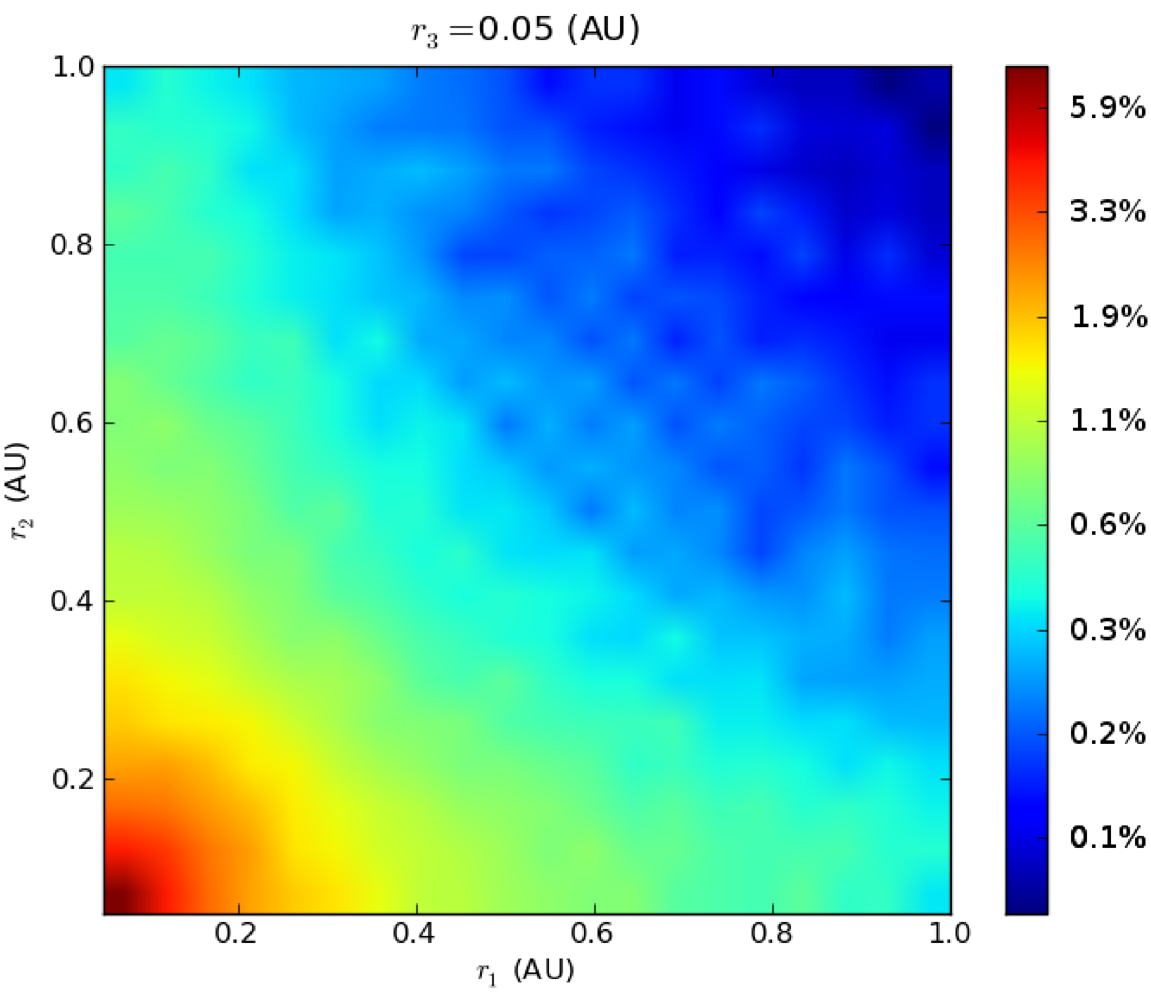

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Abstract

As observations of planetary systems are published from the Kepler mission more constraints can be put on the theory of planetary formation and evolution. This in turn can be used to more accurately tell what really has been observed, such as the frequency and orbital parameters with which a six planet system like the one around Kepler-11 will occur. Even if a system has been observed to have four transiting planets it is not necessarily true that it only have four planets.

Here a simple computer model is created for planetary systems consisting of a given number of planets with certain orbital parameters in order to calculate the frequency with which a certain number of transits will be seen. This is then tied to the data from the Kepler mission to make a rough prediction of the frequency for planetary systems with different number of planets

It is predicted that 3% of the observed systems have six planets, 13% have four planets and 14% have two planets. The other types of systems have very small and even negative prevalence.

The conclusion of this is that this *model is too simple* or the *assumptions are wrong*.

Introduktion

Målet för detta projekt var att *"beräkna sannolikheten för att se en multipel passage och kvantifiera hur denna sannolikhet beror på den ömsesidiga inklinationen mellan planeterna"*.

Vad innebär då detta? För att förklara det bör åtminstone *"multipel passage"* och *"ömsesidig inklination"* förklaras närmare.

Föreställ dig att du befinner dig utanför solsystemet, långt utanför Plutos bana, förbi Oorts moln och tittar in mot solen. Antar man att solen alltid har samma ljusflöde kan man ändå, om man har tur och befinner sig i rätt position, se hur solen periodiskt en kort stund bli svagare. Detta sker då en av planeterna blockerar lite av solljuset när den passerar mellan dig och solen. Perioden med vilken solen verkar svagare är densamma som planetens omloppstid och hur länge den blir svagare beror på hur länge planeten befinner sig framför solen. Har du riktig tur kan flera planeter passera mellan dig och solen, en *"multipel passage"*.

Om man gör detta flera gånger, tittar på solsystemet från en mängd olika håll, kommer man se varierande antal av planeter passera framför solen. Detta beror på en mängd olika faktorer däribland den *"ömsesidiga inklinationen"* mellan planeterna, dvs vinkeln mellan de plan i vilka planetbanorna ligger. Andra faktorer som även påverkar hur många passager man ser är: radien av planetbanorna, solens storlek och inte minst antalet planeter i systemet.

Dessa faktorer kan användas som begränsningar på hur solsystemet har utvecklats, t.ex hade planeterna skapats ur en tunn skiva av gas och stoft runt solen utan några interaktioner med andra planeter eller liknande så skulle de idag ha en väldigt liten ömsesidig inklination.

Föreställ dig nu att du gör samma experiment men tittar på andra stjärnor för att se hur många planeter du kan se passera framför stjärnorna de kretsar runt. Just detta har Nasas rymdfarkost *Kepler* gjort och har observerat 827 system där en planet passerar framför den stjärna vilken den kretsar runt, 115 med vilka två planeter passerar, 45 med tre, 8 med fyra, 1 med fem och 1 med sex.

Detta betyder däremot inte att t.ex alla system i vilka tre planeter gör en passage endast har tre planeter.

För att modelera detta har system med varierande antal planeter skapats.

Så givet antalet planeter i systemet, låt oss säga tre, måste systemet få ett utseende beroende på några parametrar. Antag också att planetbanorna är cirklar.

Den första uppsättningen parametrar är radien av deras planetbanor. Dessa kan inte alla vara lika stora då de skulle komma för nära och påverkar varandra så att systemet inte skulle vara stabilt över längre perioder.

Nästa uppsättning är banornas inklination gentemot ett gemensamt ursprungsplan. Denna inklination är en slumpmässig vinkel vars fördelning bestäms av en parameter. Som exempel kan tas en uniform fördelning mellan 0 och 5 grader där de tre planeterna får inklinationen 1, 3 och 3 grader respektive från den innersta till den yttersta.

Den tredje uppsättning parametrar för systemet är den vilken vid vilken planetbanorna skär det gemensamma ursprungsplanet. Ovan fick två planeter samma inklination, 3 grader, vilket betyder att de skulle haft en ömsesidig inklination om 0 grader. Om dessa två planeter får som tredje parameter vinklarna 0 och 180 grader betyder det däremot att de får en ömsesidig inklination om 6 grader.

Nu är systemet skapat och allt som behövs är två vinklar för att beskriva från vilken vinkel man tittar på systemet.

Om en planet nu har en sådan bana att det kommer att passera framför stjärnan sett från observatören så har den gjort en passage.

Detta repeteras för ett nytt system, antingen med samma antal planeter eller inte. I slutet står man med en lista över hur många av de system med tre planeter som genererar en singel, dubbel, trippel, eller ingen passage. Likaså för system med fler eller färre planeter. Man har även ett samband mellan den ömsesidiga inklinationen och antalet passager man ser.

Detta kan nu jämföras med vad rymdfarkosten Kepler har observerat och därigenom få en uppfattning om vad det egentligen är som har observerats, dvs hur den verkliga fördelning av antalet planeter per stjärna ser ut.

1 Introduction

As stated, the goal for this project was explicitly *"to calculate the probability to see multiple transits in a single system and to quantify how this probability depends on the mutual alignment of the planets"*, which seem rather straightforward. Just one free parameter (the mutual alignment) to vary for a good fit to data which in turn would put constraints on how planetary system forms. What could possibly go wrong?

Imagining the Solar system as the archetypal planetary system and disregarding some details, (such as alignment of the stellar equator to the planetary orbits etc) one of the next parameters needed is the longitude of the ascending node in relation to some reference direction, that is the angle at which the planet cross through the average plane of the planets. If all planets "cross" the average plane at the same longitude and observations are done in a direction such that this intersection is pointing towards the observer, all the planets in the system will be seen to transit. On the other hand, if they are not all aligned, a planet which have a lesser inclination relative to the observer than some other planet in the system might not transit. Looking at data for the planets in the Solar system there seem to be no preference for any value, so maybe it is not that unrealistic to assume it is uniformly distributed over 360 degrees, which is the same as to say that there is at least a rotational symmetry to the Solar system.

This longitude of ascending node is not the only extra parameter which has to be considered.

How should the mutual inclinations among the planets be described? Gillon et al. (2011) investigate this for the star GJ 1214 from the approach that; how does seeing one planet transit change the probability of seeing a second planet transit. They assume that the inclination of the planetary orbits are scattered with a normal distribution having a variance somewhere between 0 and 4 degrees. What they conclude is that this have a significant impact on seeing a second transit given that one has already been observed.

Even though in multi planet systems two planets have been seen to transit and they have a small mutual inclination - 88.55 and 88.12 degrees[1] - it is not necessary that a third planet in that system will also transit[2]. This is the case for Kepler-9. Holman et al. (2010) use transit timing variations to conclude that there is a third planet in the system which has not been seen to transit. Further discussion of the mutual inclination in relation to observing multiple transit from the Kepler data is in Steffen et al. (2010). Simulations (Juric & Tremaine 2008) of dynamical relaxation of planets in planet-planet interactions produce many systems with high inclinations but it also result in deficiencies of circular orbits.

What is the radial distribution of planets? Somewhat of an inclination to what these values might be are found in Lissauer et al. (2011) in which they provide details for a system, Kepler-11, with 6 planets all within 0.5AU of its host star.

One caveat that it might not be all that simple is that in Malmberg & Davies (2008) they look at what would happen to a planetary system which become disturbed by a second star. The end result is that two distributions of planetary systems are produced, one with high and one with low eccentricities. Which the outcome will be, depends on whether the system is initially dominated by a single massive planet or not. Anyhow, none of these systems alone can reproduce observations but a combination of them can. More possibilities for creating systems with small semi major axis and large planets are by disc migration discussed in Lin et al. (1996). While Weidenschilling &

[1]The Extrasolar Planets Encyclopaedia **http://exoplanet.eu/**

[2]Third planet, Kepler-9d, do transit (Torres et al. 2011)

Marzari (1996) on the other hand do the same thing but for gravitational scattering. In Winn et al. (2010) it is concluded that hot Jupiters have high obliquities and that they are not created by disc migration. Is it possible then to compare hot Jupiters which have high obliquities to hot Jupiters which have undergone disc migration?

Another interesting approach is from Savransky et al. (2011) in which they derive the distribution of keplerian orbital elements from initial assumptions on the parameters. If this problem is possible to invert, data - if not from Kepler then maybe from PLATO - can be used to understand the parameter distributions. Still, they are not sure on the radial distribution of planets.

Could this then be modeled?

The goal will not be that ambitious, but instead it will be more along the line of trying out a simple idea and see if it could be viable. If viable, it could be extended to maybe separate the observations into categories depending on the formation scenario of planetary systems.

In order to keep it as simple as possible the simulations were done as a exploration in the parameter space due to the lack of distributions for the parameters describing the system.

In section 2 a description of how the systems are created in relation to the parameters and what will constitute a transit, the result is validated against theoretical models and finally an example of simulated data is shown. In section 3 the simulation is done and presented together with the result for varying the parameters. Finally in section 4 a prediction of the number of planetary systems is made and a comparison is made to Lissauer et al. (2011).

2 Method

2.1 Coordinate system

Two coordinate systems are used; (X, Y, Z) which is tied to the center of the star in relation to the observer and (x, y, z) which describe the planets "average orbit" from which the individual planets are then further transformed by additional rotations.

2.1.1 For the star (X, Y, Z)

The coordinate system (X, Y, Z) is in its general form centered at the star; the Z-axis is taken parallel to the rotational axis of the star, the Y-axis is in the same plane as the Z-axis and the vector pointing to the observer, finally the X-axis is taken so to form a right-handed system. Note that the vector pointing to the observer will make an angle (inclination of the system) with respect the Y-axis. However, this will be simplified in this specific case by not considering the star anything other then a [stellar] disc with its normal pointing towards the observer. That is, take the Y-axis to point directly towards the observer, Z-axis to be parallel to what is considered "up" for the observer and finally the X-axis to make the system right handed(there is no practical distinction between right- and left-handed as this will only affect what is considered a "positive" angle in the rotations).

If needed - for considering the rotation of the star - this simplification can be left out, more specific, if the stellar equator is close to coplanar with the orbital plane of the planets constraining the inclination of the star can change the probability for observing a transit (Beatty & Seager 2010).

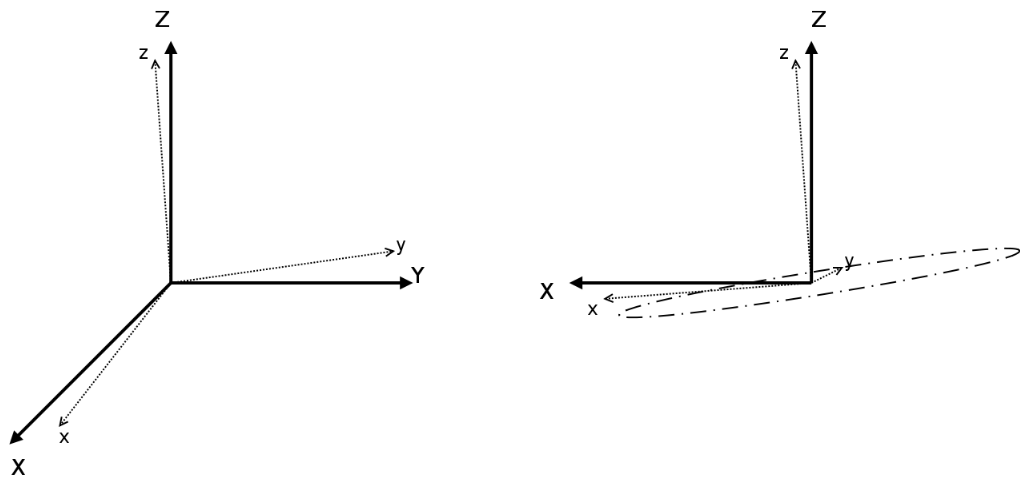

Figure 1: *Left:* The two coordinate systems (X, Y, Z) and (x, y, z). The first, (X, Y, Z) is centered at the star with the Y-direction pointing directly towards the observer and the Z-direction parallel to what the observer consider "up". The second system (x, y, z) describe the "average plane" of the planetary system and can be oriented in any way with respect to the observer, in this case it is only slightly rotated.
Right: The flat projection, or what the observer sees, onto the XZ-plane. Also included is a depiction of a planetary orbit by the dash-dotted line. Originally this planetary orbit is a circle in the XY-plane then rotated, first corresponding to alignment with respect to the "average plane" or xy-plane then the whole planetary system, (x, y, z), is rotated with respect to the (X, Y, Z) system. If a point on the dash-dotted ellipse, or the planetary orbit, is close enough to the origin of the projection it will be considered a transit.

2.1.2 For the planet (x, y, z)

Associated with the planets is another system (x, y, z) basically just for describing how the average plane of the planets is rotated with respect to the (X, Y, Z)-system. With *average* is here meant just some arbitrary plane - the xy-plane in (x, y, z) - which makes is simpler to describe the individual planetary orbits. With respect to this plane each planet can be described by a ascending node and inclination. The ascending node is the intersection of the planets orbital plane and the xy-plane and the inclination is the maximum angle between the planets orbital plane and the xy-plane.

Both these coordinate systems, $(X, Y, Z$ and (x, y, z), are drawn in figure 1 to the left slightly rotated with respect to one an another. The observer will see what is projected onto the XZ-plane and this is drawn to the right in the same figure, in this part a planetary orbit is also included.

This is, to reiterate, not a plane necessary connected to the Laplace plane of the planets at some moment or to the invariable plane. It is just a plane which makes the description when doing the simulations simpler without any specific physical meaning.

2.2 Rotational matrices

Rotations of an object can be described as extrinsic rotations in a fixed coordinate system. In this specific case, the rotations are made with respect to the coordinate system of the star (X, Y, Z) and described by three matrices. It could be thought of as one initial rotation which rotate the obit of a planet in the (x, y, z) system, or the average plane, creating the planetary system and then one further rotation for how the the observer sees the system of planets

Each rotation is made around one of the axes X, Y or Z and can be described as a 3x3 matrix.

$$R_X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix} \tag{1}$$

$$R_Y = \begin{pmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{pmatrix} \tag{2}$$

$$R_Z = \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

The three rotation matrices are then combined into a the final rotation matrix.

2.3 Description of planetary system

2.3.1 Planetary orbits

The orbit of a planet is initially described as a finite set of points in the (X, Y, Z)-system. For simplicity, the orbits used in these simulations are circular but can be exchanged for any other shape in order to more truly reflect a planetary system eg. elliptical orbits with the star at one focus or similar.

2.3.2 Active transformations

For multi planet systems the orbital planes of the individual planets are not all necessary coplanar but for that matter not totally independent. So, the transformation angles for the orbital planes are decomposed into two; one describing the rotation of the “average” plane with angles independent and uniformly distributed between 0 and 2π (the system can be orientated in any direction with respect to the observer but if necessary this could be restricted) and a second describing the deviation from the average plane, i.e. for the angles above $\alpha = \alpha_{\text{mean}} + \alpha_{\text{deviation}}$ etc for the other angles.

Note again, that the average plane is just used to distribute the planetary orbits around some common reference, this reference is not necessarily the average plane in the sense of sum of rotational vectors or similar but just used to describe how the system is rotated with respect to the observer.

The planetary orbit is then actively transformed by applying the rotational matrix to the orbit, one rotational matrix for each planetary orbit.

Important to note here is that this involves two set of random variables, one describing how the system is rotated and then a second set describing how the individual planets are rotated with respect to the system. This means that the rotation angles cannot simply be added and then used in the rotation but instead the two rotations have to be made in succession with separate matrices describing each rotation. Also, the order in which the rotations are done is important. The rotation describing, so to say, the individual planets in the system are to be performed first and then the whole system has to be rotated.

A point $(x, y, z)_{\text{initial}}$ on the initial circular orbit in the XY-plane is then transformed according to

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\text{final}} = R_Z R_Y R_X R_z R_y R_x \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\text{initial}} \tag{4}$$

2.3.3 Geometric transits

In this document with a *transit* is meant if any point on the orbit of the planet passes between the observer and stellar disc, unless it is in connection with observations. A transit by the planet does not necessarily mean that it will be observed for any number of reasons (too small planet, too shallow transit, etc.).

Figure 2 shows how the orbit is connected to the transit probability. The orbit of a planet can be described by a vector parallel to the normal of its orbital plane. The magnitude of the vector could be used as measure of the semi major axis and its inclination can be expressed in regular polar coordinates. If the planet is to transit, the angle between its orbital plane and the line of sight has to be less then $p = \arcsin(R_{\text{star}}/r_{\text{planet}})$, in other words, the projected distance of the planet has to be less the the radius of the star. The phase space of the orbit is then a shell (r_{planet} is fixed), and a fraction of this surface represent a transit. In figure 2 this is drawn as a grey symmetric band on a shell of radius r_{planet}, in the same figure is the planet (red dot) and its orbit (dashed ellipse). The outline of the shell describing the whole phase space is dotted. p is the maximum angle the orbital plane can make with the line of sight for there to still be a transit.

The probability for a transit is then calculated as the integration of the density over this surface. The density is uniformly distributed over the whole surface. The integration can be arranged into a more convenient form by measuring the inclination from the line of sight

$$p = \arcsin\left(\frac{R_{\text{star}}}{r_{\text{planet}}}\right) \tag{5}$$

$$P_{\text{transit}} = \frac{1}{4\pi} \int_{\theta=0}^{2\pi} \int_{\phi=\pi/2-p}^{\pi/2+p} \sin(\phi)\, d\phi d\theta \tag{6}$$

$$= \frac{1}{4\pi} 2\pi \left[-\cos(\phi)\right]_{\phi=\pi/2-p}^{\pi/2+p} = \sin(p) = \frac{R_{\text{star}}}{r_{\text{planet}}} \tag{7}$$

2.3.4 Transits

As mentioned above the planetary orbit is described by a finite number of points which mean that even a planetary orbit (still points) not yet rotated when looked at from the side might not present any points on the stellar disc.

Instead of increasing the number of points describing the orbit the point which is closest to make a transit could be used together with the point next to it in order to make an approximation of the circular orbit by a straight line. The minimum distance from the this line to the center of the star can then be used to determine if there would be a transit, ie. some point on the orbit would come close enough if the number of points only was increased enough.

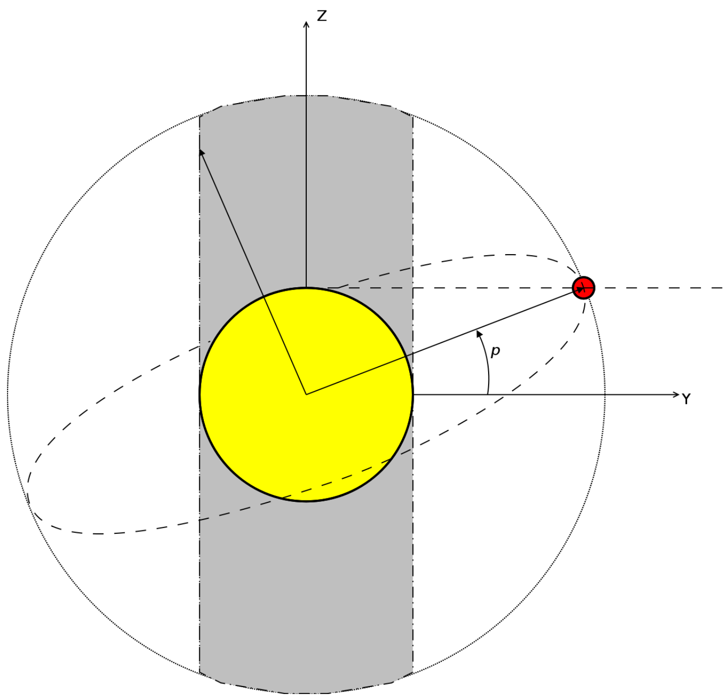

Figure 2: A transit will occur if the least distance between the orbit and the line of sight is less then the radius of the star, which in this case is $r_{\text{planet}}\sin(p) \leq R_{\text{star}}$. The space of possible transits will create a symmetric band(grey area) on the sphere constituting the phase space of orbits. The probability of transit, or the area of the symmetric band to the area of the whole sphere, is then $P_{\text{transit}} = R_{\text{star}}/r_{\text{planet}}$

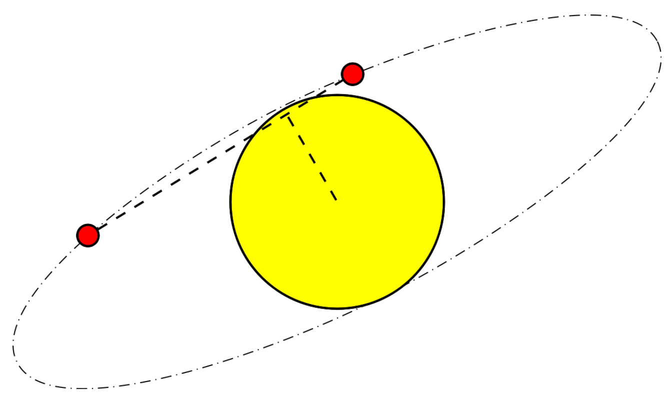

Figure 3: A pictorial of a transit. This is basically what the observer would see to the right in figure 1; The yellow circle is the stellar disc and the red circles the discrete points used to describe the planetary orbit. Since the number of points are finite (~ 50 points in the simulation) it could be that none of them would be close enough for it to be considered a transit. The true orbit(dash-dotted) of the planet is then approximated with a straight line(dashed) through the points and the least distance to the center of the star in the projection is calculated. If the distance is less then the radius of the star it is counted as a transit.

Also mentioned above, was the simplification to take the Y-direction to point directly towards the observer. As the star is so far away from the observer in comparison to the planets the orbital points could be projected flatly on the XZ-plane which in this case amounts to just look at the coordinates of the orbital points in the (X, Y, Z)-system. A transit is then said to have occurred if some point of the planetary orbit (which are just a set of coordinates) have coordinates such that $x_{\text{final}}^2 + z_{\text{final}}^2 < r_{\text{star}}^2$ (note that this does not say anything about the physical properties of the planet). And, as the number of points are finite, instead of increasing the number of points, the point of the orbit with smallest $x_{\text{final}}^2 + z_{\text{final}}^2$ is taken together with the point next to it in order to calculate approximately the closest a point could get to the center of the star.

In figure 3 the discrete points used for the orbits are drawn as red circles. The arc connecting the two points is a small section of the planetary orbit from figure 1 which is approximated by the straight line connecting the two dots. The smallest distance from this line is compared to the radius of the star to conclude if there has been a transit or not.

Doubling the size of the semi major axis, doubles the distance between the points. To keep the distance between the points require that the number of points have to go up as the semi major axis increase. As an example; let the semi major axis be 1 AU and the number of points describing the orbit be 50, then the projected distance between two points is roughly 0.1 AU which is to be compared to the radius of the star of roughly 0.005 AU, or it would be roughly 5% chance for a point to actually be in front of the star even if the orbit was observed straight from the side.

2.3.5 Example of simulation

An example of this whole process is to imagine a system with two planets. Let the two planets have a semi major axis of one and two astronomical units respectively. If this was a real system the two planets would probably not be totally coplanar, but their orbital planes would make some angle to each other. Still, both planetary orbits could be described in relation to another plane, the “average plane”, with a ascending node and inclination. To begin, two set of points would be created describing two circles with correct radius in the (X, Y, Z)-system representing the orbits of the two planets. Each planet need an inclination, which in this case was drawn from a Rayleigh distribution with a mean suitable for the system under investigation (it is kept as a free parameter) and each planet also need a ascending node which was drawn from a uniform distribution over 2π radians. These angles are used to create the rotational matrices for the individual planets.

The whole planetary system could be oriented with respect to the observer in any way so a new set of rotational matrices is need for the whole system.

Multiplying the set of coordinates for the planetary orbits with the respective rotation for the planets and then with the matrices describing the rotation of the system gives the two set of final coordinates describing the orbits after they have been rotated.

The projected point closes to the origin in the XZ-plane is then used to make an approximation of the planetary orbit by a straight line. The least distance of this line and the origin in the XZ-plane is then used to determine if there would be a transit or not.

2.4 Validation

In order to ensure that the the result is indeed a randomly rotated system, a number of rotated system were generated.

As a first step in validating the program is to take a single planet around a star which has an isotropic distribution of its rotational vector. For this kind of system, with a specific sized star the probability for a transit is just the fraction of the star radius to the semi major axis of the planet orbit $R_{\text{star}}/r_{\text{planet}}$.

This can easily be extended to an arbitrary number of planets with varying semi major axes, as long as they are isotropically distributed. In this case the probability become the product of probability for transit for each planet respectively, $\prod_i \frac{R_{\text{star}}}{r_{\text{planet},i}}$.

Doing the simulation for a two planet system, the probability for a double transit at different combinations of semi major axes are shown in figure 4 to the left. To the right are cuts in the data compared to analytical solutions. The two curves *“close”* and *“far”* correspond the the case of a single planet but with an additional factor since one planet is held fixed at certain distance from the star. The middle curve, *“equal”* is the probability to have a double transit with both the planets at exactly the same distance from the star, this would be the diagonal cut.

Next simple step is to validate it for a flat system. If the orbits are circular and all coplanar, the the probability for a multi transit should be equal to the probability for the outermost planet to transit. Eg. for a four planet system, the probability to see a quadruple transit is equal to the probability for the outermost of the four planets to transit.

This validation is only for very simple and unrealistic cases. If, for example, the rotation above were done in the reverse order - first multiply with the rotational matrix for the system then with the rotational matrix for the planet - it would not be caught.

2.5 Example of three planet system

The data retained from the simulations is for a given set of planetary radii the number of no, single, double,... up to “number of planets in the system”-multiple transits.

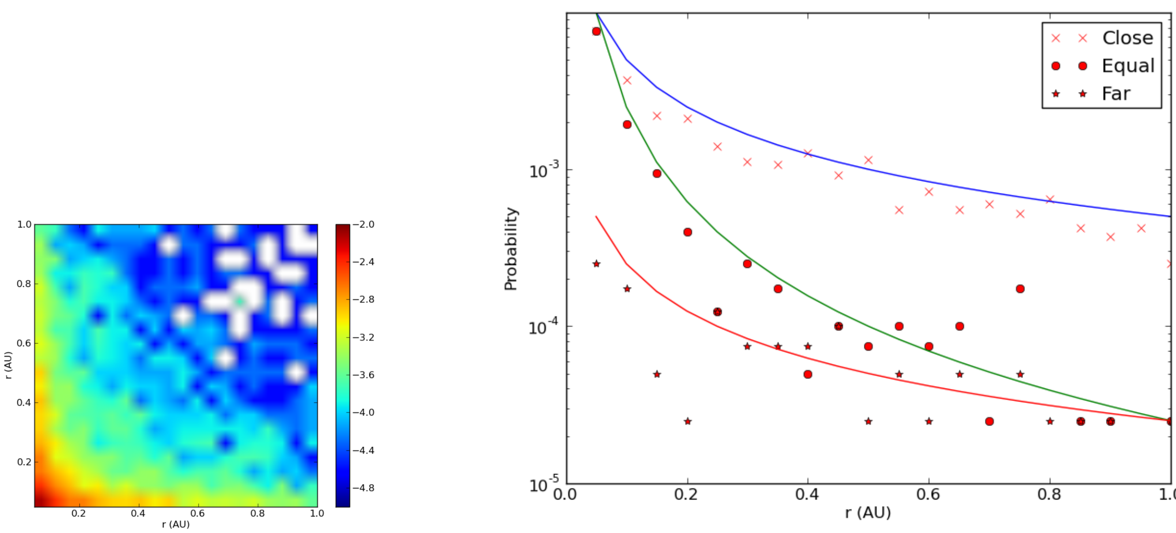

Figure 4: *Left:* Probabilities in log_{10} of a double transit in a two planet system with isotropic orbits, that is they are totally unrelated to each other in sense of inclination, with semi major axis in the range of 0.05-1.0 AU
Right: Cuts made in the plot to the left. Top curve is made with one planet fixed at 0.05 AU and correspond to the bottom row in the left picture. Bottom curve is made with one planet fixed at 1.0 AU and likewise correspond to the top row in the left picture. The middle curve is with both planets at the same location, or the diagonal of the left picture. Discrete markers are values taken from the plot and the curves are what would be expected from theory.

This count is strongly dependent on how far from the star the planets are. In the case of a three planet system with isotropic orbits, doubling the semi major axis decreases the probability eight times. $\prod_{i=1}^{3} \frac{R_{\text{star}}}{2r_{\text{planet},i}} = \frac{1}{8} \prod_{i=1}^{3} \frac{R_{\text{star}}}{r_{\text{planet},i}}$.

That a system has isotropic orbits is maybe a bit of a stretch, instead the probability of triple transits in a rather flat three planet system is shown in figure 5. Each semi major axis goes from 0.05AU to 1AU in 20 steps with 30000 systems simulated for each point. Two different mutual inclinations are considered, 1 and 2 degree to the left and right, respectively. The plots show the semi major axis for the first and second planet with three selections of semi major axis for the third planet.

As all planets are close to the star the number of counts give a probability for a transit of roughly 6% for the system with one degree of mutual inclination. Looking at the same plot but for the system with two degree of mutual inclination this probability has dropped significantly. In the extreme case. in which all three planets have isotropic orbits, the transit probability would be down to $\prod_{i=1}^{3} \frac{R_{\text{star}}}{r_{\text{planet},i}} = \left(\frac{0.005AU}{0.05AU}\right)^3 = 0.1\%$.

Moving the third planet out to ~0.5AU but keeping the two inner planets close to the star there is yet another significant drop in transit probabilities. This time the probability is roughly down to 0.6% for the system with one degree of mutual inclination and even lower for the system with two degree. Still, it is very much larger then what would be expected for a system with isotropic orbits(0.01%).

When the third planet is as far out as it gets the probability keep on dropping. The probability for a transit with two planets far in and one far out in the system with one degree inclination is maybe not as big as for the system with two degree of mutual inclination but with the third planet at 0.5AU, but still the probability is much more extended, i.e. the probability to have a triple transit with planets at 0.5AU, 0.5AU and 1.0AU with a mutual inclination of one degree is larger then the probability to have a triple transit with planets at 0.5AU, 0.5AU and 0.5AU but with a mutual inclination of two degree.

Even changing the mutual inclination slightly will have a big impact on the probabilities, as can be seen from comparing the left to the right column of plots in the figure.

Also, in the last plot there are some white points. These are points with no transits and they are symmetric around the diagonal since the problem was assumed symmetric among the planets thereby reducing the total computational load.

3 Simulation and Result

3.1 Parameters

The simulated systems had quite a few restrictions, consisting both in practical considerations of the shear computational burden and the memory footprint to more fundamental restrictions such as what kind of systems were really simulated. Even though this is a very serious restriction it was made more in the spirit of trying not to make too many assumptions which might result in, albeit correct, explorations for specific cases.

First out is the limited computational capacity. Options were offered to run it on more able equipment which would made it possible to both increase the resolution of the parameter space and

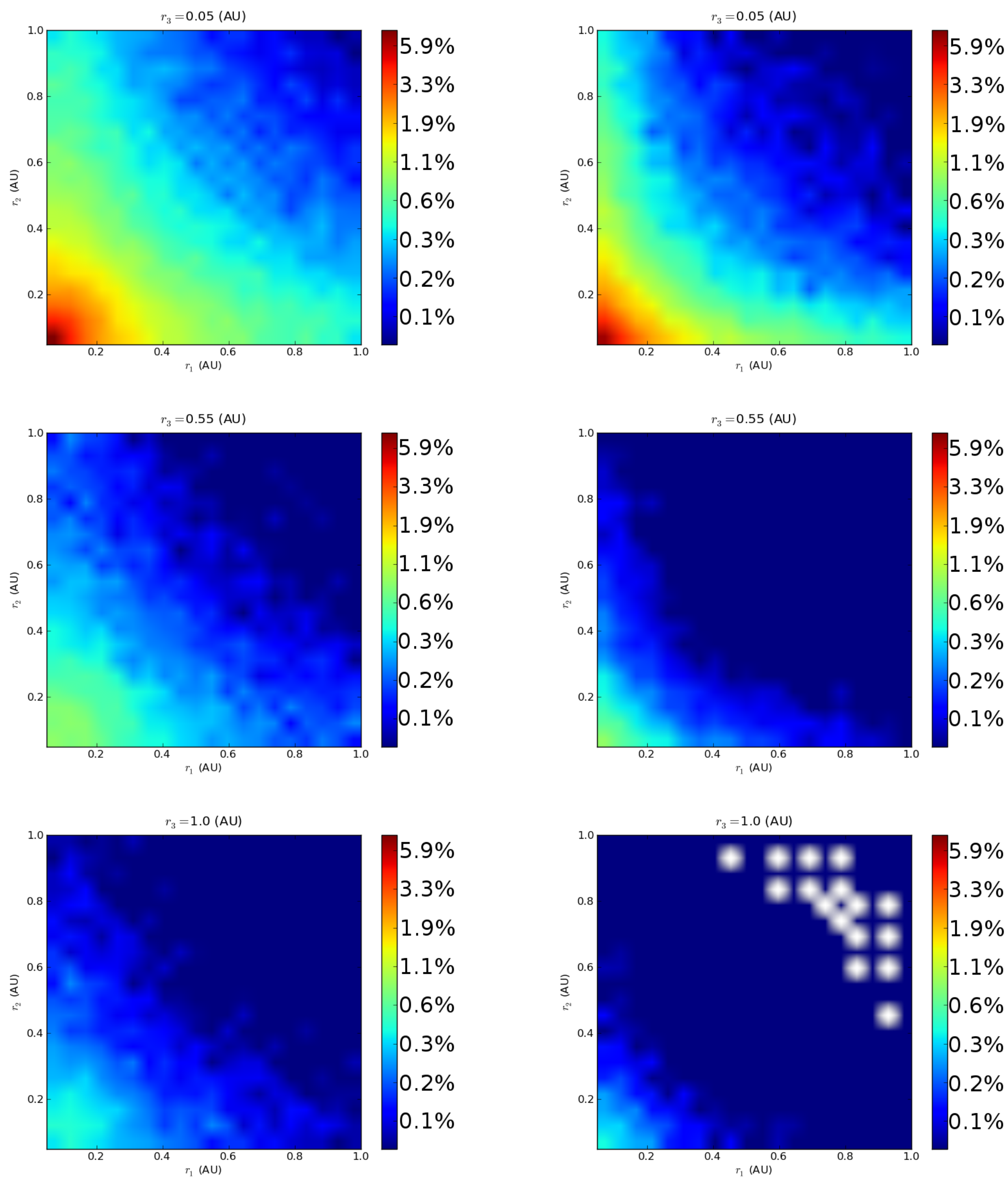

Figure 5: Probabilities of triple-transits in in a three planet system. Each point is sampled 30000 times, the semi major axis of the orbits goes from 0.05 to 1.0 AU in 20 steps. Left and right column is for one an two degree of mutual inclination respectively. Top row is for the third planet with semi major axis at 0.05 AU, middle row is for third planet at 0.55 AU and bottom row at 1.0 AU. The probability of seeing a triple transit is higher in the flatter system (left) and the closer the planets are to the star the more do the probability increase. The white dots in the lover right plot are an artifact from the restricted number of samples and the symmetry is due to another computational artifact.

how thorough each point in the parameter space could be explored. However, this would probably not have more effect on the result than just changing the assumptions a bit.

Next the planetary system simulated had to be considered. The stated goal was to make comparison to/predict the result from the Kepler and PLATO mission. This give an indication of how large the extent of the planetary systems could be, namely in the order of a astronomical unit for observing a few transits during a few year. Also, for a single planet with isotropic orbit doubling the semi major axis halves the transit probability. In general for the simulations it was mostly the inner orbits which made the large contributions to transit counts. So in the end the semi major axis was kept below a single astronomical unit. For inner limit 0.05 astronomical units were used. This value was chosen rather arbitrary and might actually not be such a sensible choice as there are quite a few transiting extra solar planets with semi major axis well below this value.

Once the extent of the system has been considered a natural question to ask is how many planets are there to be in the system and how closely can they be packed. Leaving the second question aside for the moment and only considering the first; how many planets are there in a system. This would introduce more unknown parameters and would be more inclined to answer question about the ensemble of planetary systems. So, once again, to keep it simple this is just left as a fixed parameter. Instead of populating the systems with random number of planets, specific systems with a given number of planets are only considered.

Considering now the mutual relations of planets in the system one of the first question to ask; how are the planetary orbits distributed around the average plane, as described above? Again, this is kept as a fixed parameter used to explore different kinds of systems thus some are kept totally flat and then gradually the mutual inclination between the planets are increased. A Rayleigh distribution was used with mean $\sigma\sqrt{\pi/2}$ describing the average mutual inclination between a planetary orbit and the average plane. This for two reasons, one is that the mutual inclination of the Solar system can be described by this type of distribution and secondly that Lissauer et al. (2011) used this which makes it easier to compare and validate the program to their values. To complete the rotation of the individual planets the longitude of the ascending node has to be given. If all planets had the same longitude of ascending node it would mean that if aligned up with the direction to the observer all planets would be seen to transit, no matter how large their mutual inclination would be. It could also imagined that the ascending nodes were all bunched up together. Looking at the data for the Solar system, the ascending nodes of the planets[1] and larger asteroids[2] seem not to have any particular distribution. The choice was then made to just let the ascending node be uniformly distributed between 0 and 2π radians. This will, however, have a slight side effect; the mutual inclination between the orbital planes in the direction if the observer will no longer be given by the specified angles, but instead be slightly less. Re-expressing everything in the angles in the direction of the observer would be an alternative but also take away some of the simplicity of how the system is constructed.

The simulations were done by varying the radial distribution for each planet from the minimum value of semi major axis to the maximum value allowed. This means that some of the points in the parameter space will all have the same semi major axis. Not very physically realistic if in addition the planets are distributed in a flat disc. But instead of excluding them from the simulation they were included and sub selections of the simulated points were then made, such as considering how close two planets could come to each other. This will be referred to as the packing number which is how

[1]NASA National Space Science Data Center **http://nssdc.gsfc.nasa.gov/**

[2]JPL Small-Body Database Browser **http://ssd.jpl.nasa.gov/**

close in numbers of Hill radii the next planet could be. The Hill radius is $r_{\mathrm{H}} \simeq a\,(m_p/(3M_{\mathrm{star}}))^{1/3}$ which for a system consisting of a star similar in mass to the Sun and a planet similar to the Earth gives $r_{\mathrm{H}} \simeq 0.01a$. For the system to be *"Hill stable"*, at least in the case of a two planet system, the initial fractional orbital separation has to be at least $\Delta \simeq 2.4(\mu_1 + \mu_2)^{1/3}$ (Gladman 1993) which work out to an initial separation of roughly 3 Hill radii, or a packing number of 3.

3.2 Different Packing

One thing that could be prudent to point out is what effect the packing has on the result. In figure 6 the probability to seeing different number of transits for a system with four planets are given for different packing, i.e. how close two planets can be put to each other. What is striking is that as the system is more sparsely packed the probability goes up. This is probably not entirely true.

To have a better understanding of what might be going on a small explanation of how they were created might be in place. Each curve is created from the same simulation. That is specifically; one simulation was done for a system with one degree of mutual inclination containing four planets. Starting with the first planet a specific value semi major axis was chosen, this value will later change as this planet can be placed anywhere in the range of possible semi major axis. Since the packing ratio was given it could be calculated which other values of semi major axis were available to the second planet and one of them were chosen. This was then repeated for the third and fourth planet resulting in nested loops placing the planets at different semi major axis and counting the number of transits. The normalization will then be the product of how many simulations were done for each specific combination of semi major axis times the number of possible ways to create a system with four planets adhering to the rules of how tightly the planets could be packed. Changing the rule of how tightly the planets could be packed resulted in fewer possible combinations for semi major axes in the system. This number of combinations decreases faster then the number of observed transits due to fewer summations which have the result that the probability for a transit goes up as the system has to be more sparsely packed.

For this specific case 10000 simulations were used stepping the semi major axis in 10 steps from 0.05 AU to 1.0 AU. The number of combinations for a packing of 0, 1, 15, 20, 30, 40 and 50 were 10000, 5040, 3360, 2424, 984, 528 and 96 respectively.

So for this way of doing it, it is not possible to compare the probabilities for different packing ratios.

3.3 Mutual Inclination

Leaving aside the problem mentioned before, that the number distribution of planets in a system is unknown, and instead just looking at how many transits there could be expected in a system with a given number of planets and with given parameter for the mutual inclination.

A packing number of one Hill radius was chosen in the first simulation which is shown in figure 7. Probably too optimistic assumption as to how tight a six planet could be packed, but the two innermost planets in Kepler-11 are roughly 0.01 AU or 10 Hill radii apart. This is on the other hand about what the resolution in semi major axis is in the simulation.

Also only four planets were used in the simulation, but looking at the figure further down (figure 9) with increasing number of planets, the shape of the curve is basically the same just either stretched or compressed laterally depending on if there were many or few planets in the system simulated.

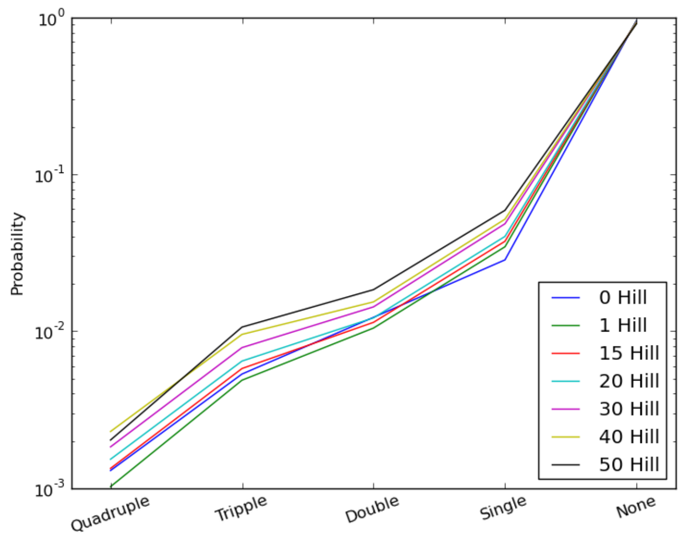

Figure 6: Simulation showing the probability for none to quadruple transits in a four planet system with one degree of mutual inclination between the planets for different limits on how close the planets can be packed, or put to each other. The packing is expressed as number of Hill radii the planets has to be separated by, the Hill radius is calculated for earth mass body. The normalization in the probabilities depend on the number of steps for the semi major axis used in the simulation and this normalization factor changes as the packing is changed even though the number of steps in the simulation stays the same. The curves do not change shape as the system is either more densely or sparsely packed, with the exception of zero Hill radii.

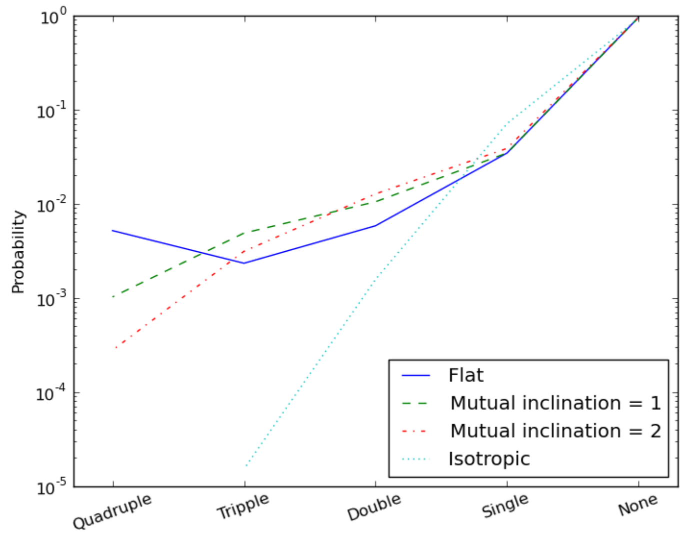

Figure 7: Probability of seeing none to quadruple transit in a four planet system as the mutual inclination changes. The packing is set to 1 Hill radius. The probability of seeing a double transit is actually higher in the system with mutual inclination of two degree compared to the two flatter systems. For a single transit the probability is, however, greatest for the system with isotropic orbits.

The two extremes are for either a completely flat or completely isotropic system. For the isotropic system there are no quadruple transits. As the simulation was for 10000 systems, in the most fortunate case the probability for a quadruple transit is roughly $(R_{\text{star}}/r_{\text{planet}})^4$ which for the simulated system comes out to $1/10000$ if all planets could be in the innermost position simulated. The normalization is not only number of simulations done with all the planets as close to the star as possible but also taken into account that there is a packing criteria and the radial distribution of semi major axis come into play. So maybe not that much can be said about the absolute numbers but the result turn more qualitatively.

The middle of the curve, double and triple transits, show some flattening out. One thing with might contribute to this is that there is a factor describing in how many ways three or two planets could be chosen from the four.

Still, the chances of seeing multiple transits are good as long as the system contain many planets and is relatively flat. Doubling the mutual inclination more then halves the probability for seeing a quadruple transit but for a triple or double transit there is a a much smaller relative change and even an increase respectively.

Comparing this to the situation in which the packing number is 40 Hill radii (figure 8) the change seem to follow what is shown for the packing. Also in this case the probability for a double transit actually increase as the mutual inclination is increased.

The general trend then seem to be that the probability for low order of transits increase as the mutual inclination increase. In a four planet system the probability to see a double transit improves with slightly increased mutual inclination but then again drops off as it increases even further. For

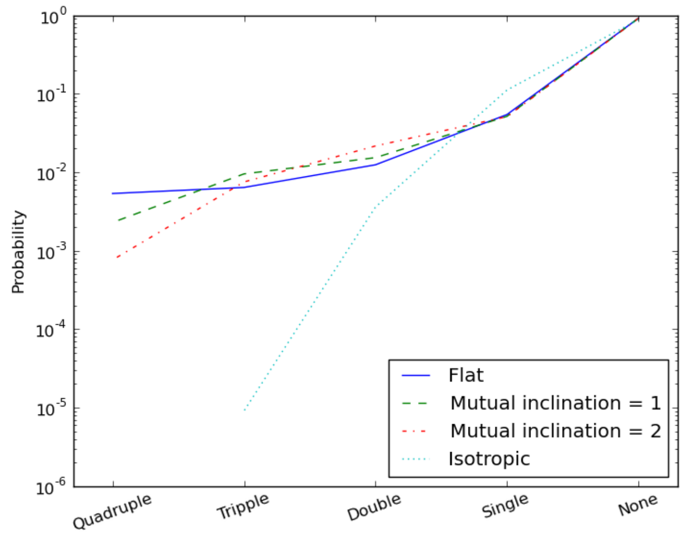

Figure 8: Same as in figure 6 but with changed packing to a more sparse system. The statements for double and single transits are still valid. Note the change in probabilities between this and the more dense system in relation to figure 5, the absolute value changes but this is probably due to computational effects.

totally isotropic systems, the probability of seeing a single transit seem to be the best. This might be a problem though, that there is a degeneracy in the mutual inclination.

3.4 Number of planets

Increasing the number of planets from four up to six and performing similar calculations it does show (figure 9) that the curve levels of for intermediate number of transits and then decrease further as the relative number of transits for the system goes up.

Also shown in figure 9 is a dotted curve which is for a four planet system but with higher resolution in radial distance, it follows the ordinary four planet curve except for the case of single transit in which case it is lower.

Overall it is what maybe could be expected. As the number of planets in a system grows, the chances of seeing an intermediate number of transits stay relatively high since there are a number of choices of which planets in the system should transit. For seeing all planets in a system transit it means that all the orbits have to be almost coplanar.

4 Discussion

4.1 Predicting the frequency of planetary systems

One tempting thing to do is to use the data presented from Kepler (Borucki et al. 2011) for the number of transiting planets together with the probability for observing a given number of transits

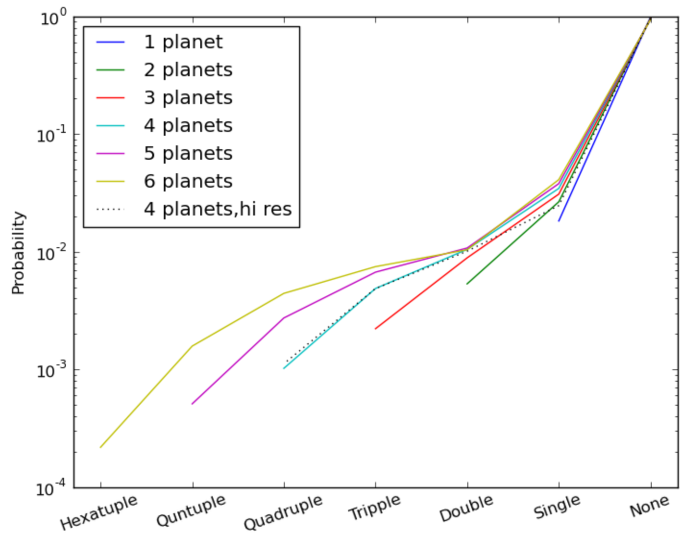

Figure 9: Probability of seeing a given number of transit in a system consisting of varying number of planets. The systems all have the same packing (least separation between planets) of 1 Hill radius, same mutual inclination of 1 degree and with a semi major axis varying from 0.05 to 0.5 AU. The line corresponding to four planets can be compared to the green line in figure 6 or the green dashed line in figure 7.

calculated above to figure out the frequency at which a system with a given number of planets occur.

The only simulation available for six planet was one in which the mutual inclination was one degree. Keeping things simple, it was assumed that this would also be a good description of what a system with less then six planets would look like.

How close the system could be packed do have significance but in the figure with different packing the overall shape do not change that much with packing so, quite arbitrary, the packing was chosen to be no higher then one Hill radius. Even thouh if it would be changed to fulfill the Hill stability criterion or three Hill radii it would not matter as the resolution in the simulation was around 10 Hill radii.

Taking the data from Borucki et al. (2011) that out of 156435 stars 809 have a single transiting planet, 115 have two, 45 have three, eight have four and then a single system as five and six transiting planets respectively. This gives the observed relative frequencies, which then together with the calculated probabilities is used to solve for the relative frequencies.

That is solve

$$\begin{pmatrix} o_0 \\ o_1 \\ o_2 \\ o_3 \\ o_4 \\ o_5 \\ o_6 \end{pmatrix} = \begin{pmatrix} p_{0o} & p_{10} & p_{20} & p_{30} & p_{40} & p_{50} & p_{60} \\ 0 & p_{11} & p_{21} & p_{31} & p_{41} & p_{51} & p_{61} \\ 0 & 0 & p_{22} & p_{32} & p_{42} & p_{52} & p_{62} \\ 0 & 0 & 0 & p_{33} & p_{43} & p_{53} & p_{63} \\ 0 & 0 & 0 & 0 & p_{44} & p_{54} & p_{64} \\ 0 & 0 & 0 & 0 & 0 & p_{55} & p_{65} \\ 0 & 0 & 0 & 0 & 0 & 0 & p_{66} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} \tag{8}$$

for the a_j:s where a_j is the frequency with which j-multiple planet system occur, o_i is the observed frequency of i-multiple transits and p_{ij} is the probabilify of observing a i-multiple transit in a j-multiple planet system. The o_is are from Kepler data and the p_{ij}s are from the simulation.

As could be expected this produces some unrealistic results such that the relative frequencies for three and five planet systems are negative (-2% and -7%). They are relatively small, but negative non the less. Also, the relative frequency for two planet system is small (1%). For six planet systems the relative frequency is rather big in comparison (3%). For one and four planet systems the relative frequency is 14% and 13% respectively.

If simplifying it by assuming that three and five planet systems do not contribute at all, the probability of seeing transits can be calculated. In figure 10 the probabilities for transits are drawn as red circles using this assumption, as comparison the data from Kepler is drawn as red 'x':s. The overall behavior does not seem too bad but is is to be remembered that there is only a single transit of six and five planet respectively.

Another limitation is that this assumes that all systems which do transit are also detected.

Now, this can not be taken too serious but it shows a different approach to the one taken in Lissauer et al. (2011). They use that planets are uniformly distributed in $\log(P)$ where P is the orbital period varying between 3 to 125 days, which would mean that it is exponential in semi major axis, to poulate their systems. When a first planet has been created a second is created observing the Hill stabiliy criteria of how close they could put the second planet. The number of planets per system is calculated from three different distributions. Moreoever, they take into consideration the physical aspect of the planets as well as the observability of a transit. They also find that in general the mutual inclinations of the system is higher then the one degree used in the simulation.

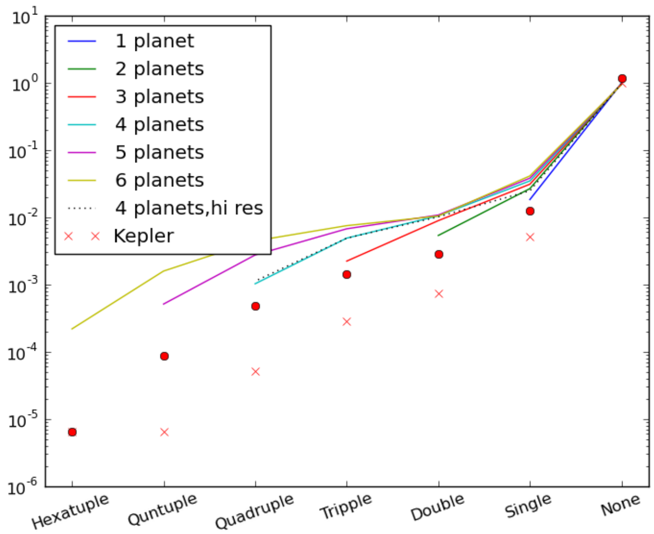

Figure 10: Same as figure 8 but with data from Kepler(red x's) added normalized to the total number of stars under observation. Red circles are predictions made from the simulations but with the coefficients for the different kinds of system censored to only include positive terms.

Also, this does not take into account any observational data. As an example, Kepler object of interest 500 KOI-500, have five planetary candidates. The one furthest out is 29 "stellar radii" out, which when compared to this simulation would be 0.15 AU. This is far in compared to the cut off of the simulation at 1AU. If planets are this far in, considerations should be given to how they have gotten that far in, i.e. have they undergone disc migration or not? If they have undergone disc migration they will probably have migrated through mean motion resonances. If it has not undergone disc migration it might have a mutual inclination which is much larger then the one simulated.

None of these very important details have been considered but instead it was simplified to just use the probability of multiple transits of flat multi planet systems.

Predicting the number of multiple transit which will be seen by Kepler or PLATO become very difficult very fast without any good assumption as to what the distributions of the parameters are. One such thing is that Kepler have found (Lissauer et al. 2011) that around 16% of all multi transit system are in low order resonances. This is a consideration which has been ignored in the above discussion as to how the planets are distributed radially.

5 Conclusion

As could have been expected the mutual inclination of the planets in a system have a impact on the probability of observing transits. As the number of *transits to planets* in a system increase the more important the mutual inclination gets. For example, in the four planet case (figure 7) increasing the mutual inclination decreases the probability of quadruple and triple transits but increase it for double. Still, for this to be quantified, knowledge of the the unerlying distribution of the inclination is needed.

Important also is the number of planets in the system and a how frequent such a system is. Even if a system with say six planets is rare, it could have impact on the number of systems observed with four transiting planets. Unless these systems are looked more carefully on, with for example transit timing variations, it become more important to have a good model for the systems in order to connect number of observed transits to the number of planets in the system.

Kepler have provided a wealth of information which is being put to good use but one argument for PLATO has been that it will, in contrast to Kepler, observe brighter stars. Since the brighter stars also are involved in the formation and evolution of planetary systems Kepler will not provide a compete sample which in turn will lead to assumptions for the brighter stars. Of course, not each and every one of the stars can be measured but a bigger and more representative sample makes for better understanding.

Now this is not to say that the reason for the lack of quantitative result herein is due to lack of data. Many significant improvements could be made; The radial distribution of planets is one obvious thing and a uniform distribution is probably not realistic. Another is that Kepler has observed that many planets are in/close to low order resonances, which has not been considered here at all. This might be a better way to create a system - place a initial planet and then use resonances to place the other.

Also the resolution in the simulations is a bit low but with more planets in the system the computational load goes up. If systems with different number of planets are to be compared without regard to distribution of parameters it was thought that the resolution had to be set the same for all simulations and that also the systems with six planets should have simulation times on a reasonable scale(even though it took a few hours). Data from Kepler should have been used to create empirical distributions in order to not waste too much computational time.

Another way to do this type of investigation could have been to just manipulate statistical distributions. The big upside with this would be that it is much more computationally efficient. The large majority of simulations here resulted in no transit and only a small fraction of the computation was used if there was a transit or not. This could have been improved by a iterative solution for calculating if there was a transit or not. On the other hand a non circular orbit would make it harder to determine if there would be a transit or not but by manipulating the full orbit this was not really a concern as it just were points being rotated.

To sum up the result for this simulation, this simulation

- does not realistically mimic planetary systems
- indicate that there is a ambiguity in transit probability as the mutual inclination is changed
- indicate that the number of transits seen depend on how many planets there are in the system
- the mutual inclination have a significant impact on the transit probability, especially for high order of transits

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7 Description of program

7.1 First step

After implementing the ideas of creating the planetary orbit and projection to determine if a transit has occurred or not, the first check comes; to see if the result is the same as predicted by theory for a single planet on orbital radius r_{planet} with isotropic inclination transiting in front of a star with radius R_{star}. That is, $P_{\text{transit}} = \frac{R_{\text{star}}}{r_{\text{planet}}}$. A caveat is that, the theoretical transit is for a point-like planet transiting in front of a circular stellar disc whereas the program uses a dirty little trick in creating the planetary orbit as a finite number (50-100 really) of points, rotating the orbit and calculating the least distance of the line interpolating the clostest point and one of its neighbors to the center of the star This result in that the simulated probability for transit should be ever so slightly higher then the theoretical value, independent of the number of points used in the creation of the orbit. However, this is not such a catastrophic failure as it could either be compensated by decreasing the size of the stellar disc or arguing that between two points on the orbit a straight line is a good approximation to the more correct arc.

7.2 Second step

As a natural extension to the first step is to apply it for two planets with independent isotropic inclination. From a theoretical standpoint this will just amount to the product of respective probabilities for given orbital radii. Simulation wise it is also very simple, just add another planet.

The only thing which started to be a problem was the construction of the program. Up to this point it had been constructed as a very function based program with a small, separate function for each small task to be done. However, as the choice was done to implement it in Python and function calls are not very fast in interpreted language, creating a bottleneck if they are called over and over again. If the number of function calls in the simulation was seriously reduced, i.e. just for specific set of parameters and not over whole ranges or parameters it might have been ok. But as the number of calls grow with the power of number of planets making it all but grinding to a complete stop for multi-planet systems with finer parameter grids. So the program was changed/re-written to make use of LAPACK through Numpy and trying to solve it with matrix computations as much as possible reducing the number and extent of loops.

7.3 Third step

As the program was still very much sequential even though it is *embarrassingly parallel* next up was to try to gain some more speed by utilizing the capabilities of the computer some more.

7.3.1 Threads

First thing that come to mind was to use multiple threads since Python has a nice framework for worker-queues and the problem can easily be chopped up into small work-lumps to feed the threads.

Dividing the simulation into chunks deleniated by the set of parameters. For a given set of radii for the planets it is assumed that all combinations have the same probabilities since there is no considerations given to particular physical properties for the planets. That is, it is equally probable that there will be a transit independent of which particular planet is where eg. for a two planet system it does not matter if planet 1 is at 0.01AU and planet 2 at 0.02AU or the reverse.

A queue of limited length is set up and populated from a loop. If the queue is full the loop waits for a slot to become free until feeding in more work to the queue. This is done until all work has been pushed into the queue. In the same loop a reciprocating queue containing the done work and result is polled and taken care of. This to ensure that the threads doing the work always will have something to do but at the same time all the work which have been done is accounted for, that is keep the work queue fully populated and the result queue empty so the thread can run un-interrupted.

The threads work by polling a the work queue, doing the simulation and returning the result.

The problem with this approach/implementation was that it did not utilize the option of multi-processors. So the next improvement was to use multiple processes doing the work to see if it could utilize more of the available processing power.

7.3.2 Processes

By trying the above with processes instead of threads there might be the possibility to have the workload fully distributed over multiple processors thereby reducing the time for doing a simulation run.

The implementation was very similar between the two and not too much rework of the code was needed. One thing that changed, as mentioned before, was that instead of all possible combinations of parameters it was reduced such that for radius(the only parameter which ran freely) $r_1 \leq r_2 \leq r_3$ instead of the running independent and over all possible values.

7.3.3 Result

Running the threaded-code on a multi core computer did not have as much improvement as hoped, not all the processors/cores were fully utilized. If this is the result from memory sharing between the threads or just a bad implementation was never investigated.

The code with multiprocess did however show the desired improvement, fully utilizing the cores of two and four processors.

It could have been interesting to just try it out on a multi core/multiprocessor to see if it would scale nicely with the number of available . Also, it could have been a better approach to just writing it in a distributed fashion since there really is no need for synchronization or heavy communication involved. But as the main objective not was a digression into programming but having a working program with reasonable efficiency to get some result it was not further evolved.

7.4 Fourth step

As the number of planets grow more data can be retained from the simulations, that is for a given number of planets what are the probabilities for seeing a given number of transits for the parameters.

Appendix - Program code

Below is shown the code used to do the calculations for the six-planet system, some of the parts consist of badly written nested loops but I could not figure out a way to write it in a shorter and more flexible way.

```
from time import time
t_start = time()
import Queue
import multiprocessing as mp
import numpy as np
import matplotlib.pyplot as plt
import itertools as it
import cPickle as cP

## Copied from Python.org
def combinations_with_replacement(iterable, r):
    # combinations_with_replacement('ABC', 2) --> AA AB AC BB BC CC
    pool = tuple(iterable)
    n = len(pool)
    if not n and r:
        return
    indices = [0] * r
    yield tuple(pool[i] for i in indices)
    while True:
        for i in reversed(range(r)):
            if indices[i] != n - 1:
                break
        else:
            return
        indices[i:] = [indices[i] + 1] * (r - i)
        yield tuple(pool[i] for i in indices)


## Create the work and result queue
```

```
work_queue = mp.JoinableQueue(4)
result_queue = mp.JoinableQueue()

## The workers, ie. process doing a job
## a job consist of one set of parameters
## the job is staken from the work queue
## simulation is done
## and result put into result queue
class WorkerProcess(mp.Process):
    def __init__(self,work_queue,result_queue,r_seed,parameters):
        # Initialize mp.process
        mp.Process.__init__(self)
        # ???
        self.work_queue = work_queue
        self.result_queue = result_queue
        self.r_seed = r_seed
        self.parameters = parameters

    def run(self):

        print self.r_seed
        np.random.seed(self.r_seed)
        print np.random.uniform()

        [system_rotation,n_samples,n,r_star,planet_rotation,n_planets,r_planet] = self.parameters

        t_x = 0
        t_y = 0

        R_xp = np.zeros([n_samples,3,3])
        R_yp = np.zeros([n_samples,3,3])
        R_zp = np.zeros([n_samples,3,3])

        R_xs = np.zeros([n_samples,3,3])
```

```
R_ys = np.zeros([n_samples,3,3])
R_zs = np.zeros([n_samples,3,3])

theta = np.linspace(0,2*np.pi,n+1)[0:-1]

R0 = np.zeros([n_samples,3,n])

R = np.zeros([n_samples,3,3])
Rp = np.zeros([n_samples,3,3])
Rs = np.zeros([n_samples,3,3])

zeros = np.zeros(n_samples)
ones = np.ones(n_samples)

transit = np.zeros([n_planets,n_samples])

n_transits = np.zeros(n_planets+1)

x_min = np.zeros(n_samples)
z_min = np.zeros(n_samples)

while True:
    planet = self.work_queue.get()

    transit[:,:] = 0

    if system_rotation[0] != 0:
        alpha_system = np.random.uniform(0,system_rotation[0],n_samples)
    else:
        alpha_system = np.zeros(n_samples)
    if system_rotation[1] != 0:
        beta_system = np.random.uniform(0,system_rotation[1],n_samples)
    else:
        beta_system = np.zeros(n_samples)
```

```
if system_rotation[2] != 0:
    gamma_system = np.random.uniform(0,system_rotation[2],n_samples)
else:
    gamma_system = np.zeros(n_samples)


for k in xrange(n_planets):

    r_x = r_planet[planet[k]]

    r  = np.array([r_x*np.cos(theta)-t_x,r_x*np.sin(theta)-t_y,0*theta])

    if planet_rotation[0] != 0:
        alpha_planet = np.random.rayleigh(planet_rotation[0],n_samples)
    else:
        alpha_planet = np.zeros(n_samples)
    if planet_rotation[1] != 0:
        beta_planet = np.random.rayleigh(planet_rotation[1],n_samples)
    else:
        beta_planet = np.zeros(n_samples)
    if planet_rotation[2] != 0:
        gamma_planet = np.random.uniform(0,planet_rotation[2],n_samples)
    else:
        gamma_planet = np.zeros(n_samples)

    cap = np.cos(alpha_planet)
    sap = np.sin(alpha_planet)
    cbp = np.cos(beta_planet)
    sbp = np.sin(beta_planet)
    cgp = np.cos(gamma_planet)
    sgp = np.sin(gamma_planet)

    cas = np.cos(alpha_system)
    sas = np.sin(alpha_system)
```

```
cbs = np.cos(beta_system)
sbs = np.sin(beta_system)
cgs = np.cos(gamma_system)
sgs = np.sin(gamma_system)

R_xp[:,0,0] = ones
R_xp[:,1,1] = cap
R_xp[:,1,2] = -sap
R_xp[:,2,1] = sap
R_xp[:,2,2] = cap

R_yp[:,0,0] = cbp
R_yp[:,0,2] = sbp
R_yp[:,1,1] = ones
R_yp[:,2,0] = -sbp
R_yp[:,2,2] = cbp

R_zp[:,0,0] = cgp
R_zp[:,0,1] = -sgp
R_zp[:,1,0] = sgp
R_zp[:,1,1] = cgp
R_zp[:,2,2] = ones

R_xs[:,0,0] = ones
R_xs[:,1,1] = cas
R_xs[:,1,2] = -sas
R_xs[:,2,1] = sas
R_xs[:,2,2] = cas

R_ys[:,0,0] = cbs
R_ys[:,0,2] = sbs
R_ys[:,1,1] = ones
R_ys[:,2,0] = -sbs
R_ys[:,2,2] = cbs
```

```
        R_zs[:,0,0] = cgs
        R_zs[:,0,1] = -sgs
        R_zs[:,1,0] = sgs
        R_zs[:,1,1] = cgs
        R_zs[:,2,2] = ones

        for l in xrange(n_samples):
            Rp[l] = np.dot(R_zp[l],np.dot(R_yp[l],R_xp[l]))
            Rs[l] = np.dot(R_zs[l],np.dot(R_ys[l],R_xs[l]))
            R[l] = np.dot(Rs[l],Rp[l])

        R0 = np.dot(R,r)

#        index = np.unique(np.nonzero(R0[:,0,:]**2+R0[:,2,:]**2 < np.ones([n_samples,n])*r_star**2)[0])

        closest_to_transit = np.argsort(R0[:,0,:]**2+R0[:,2,:]**2,axis=1)
#        a = closest_to_transit[:,0]
#        b = closest_to_transit[:,2]
        for l in xrange(n_samples):
            a = closest_to_transit[l,0];
            cntr = 1
            while True:
                b = closest_to_transit[l,cntr]
                if abs(a-b) == 1 :
                    break
                cntr = cntr+1
#            m = (R0[l,2,b[l]]-R0[l,2,a[l]])/(R0[l,0,b[l]]-R0[l,0,a[l]])
#            x_min[l] = (R0[l,0,a[l]]*m**2-R0[l,2,a[l]]*m)/(1+m**2)
#            z_min[l] = m*(x_min[l]-R0[l,0,a[l]]) + R0[l,2,a[l]]
            m = (R0[l,2,b]-R0[l,2,a])/(R0[l,0,b]-R0[l,0,a])
            x_min[l] = (R0[l,0,a]*m**2-R0[l,2,a]*m)/(1+m**2)
            z_min[l] = m*(x_min[l]-R0[l,0,a]) + R0[l,2,a]
```

```
                index = np.nonzero(x_min**2+z_min**2 < r_star**2)[0]

                transit[k,index] = 1

            for k in xrange(n_planets+1):
                n_transits[k] = sum(sum(transit,0)==k)

            self.result_queue.put([planet,n_transits])
            self.work_queue.task_done()

n_samples = 10000
n_orbit_points = 50
r_star = 0.005
r_min = 0.05
r_max = 1
n_step = 10
n_planets = 6
save_step = 1000
system_rotation = [2*np.pi,2*np.pi,2*np.pi]
planet_rotation = [1.0/180*np.pi*np.sqrt(2/np.pi),0,2*np.pi]
random_seed = [53936480,894014178,105939643,103517350,718503960]

#P = np.zeros([n_step**n_planets,n_planets+1])
r = np.linspace(r_min,r_max,n_step)
#P = np.zeros([n_planets+1,n_step,n_step])
P = []

for i in range(mp.cpu_count()):
    print i
    p = WorkerProcess(work_queue,result_queue,random_seed[i],
[system_rotation,n_samples,n_orbit_points,r_star,planet_rotation,n_planets,r])
    p.daemon = True
    p.start()
```

```
#plt.plot(r,(M[:,0,2]/n_samples),r,(r_star/r)*(r_star/r_min))
#plt.imshow(np.log10(M[:,:,2]),origin='lower',extent=(r_min,r_max)*2),plt.colorbar()


#combinations = it.combinations(range(n_step),n_planets)
#diagonal = np.tile(range(n_step),[n_planets,1]).transpose()
#planet_configurations = it.chain(diagonal,combinations)

planet_configurations = combinations_with_replacement(range(n_step),n_planets)
#planet_configurations = []
#ratio_n = 5
#for x in range(n_step):
#    planet_configurations.append((x,np.floor(x/ratio_n**(2.0/3))))

iteration = 0
for planet in planet_configurations:
    iteration += 1
    if (iteration%save_step == 0) :
        cP.dump(P,open('P_saved.p','wb'))
    work_queue.put(planet)
    print 'Put work for ', planet
    try:
        r_planet,n_transits = result_queue.get_nowait()
        result_queue.task_done()
        print 'Got back for ', r_planet
#        P[:,r_planet[n_planets-2],r_planet[n_planets-1]] = n_transits
        P.append((tuple(r_planet),tuple(n_transits)))
    except Queue.Empty:
        pass

work_queue.join()
work_queue.close()
```

```
while True:
    try:
        r_planet,n_transits = result_queue.get_nowait()
        result_queue.task_done()
        print 'Got back for ', r_planet
#        P[:,r_planet[n_planets-2],r_planet[n_planets-1]] = n_transits
        P.append((tuple(r_planet),tuple(n_transits)))
    except Queue.Empty:
        cP.dump(P,open('P_saved.p','wb'))
        break
print 'Done with it'
result_queue.join()
result_queue.close()

size_of_matrix = np.concatenate((np.ones(n_planets)*n_step,[n_planets+1]))
M = np.zeros(size_of_matrix)
planet_configuration = it.chain(P)
for planet in planet_configuration:
    for coord in it.permutations(planet[0],n_planets):
        M[coord] = planet[1]

#V = map(lambda n: 1.0/1.5**n, range(30))
#R_x1,R_x2 = np.meshgrid(r_y,r_x)
#CR = plt.contour(R_x1,R_x2,P/n_samples,V,colors='k')
#plt.clabel(CR,inline=1,fontsize=10)
#plt.imshow(P,origin='lower')
#plt.colorbar()
#plt.plot(r_x,P[0,:]*1.0/n_samples,label='Simulated')
#xx = np.linspace(r_x_min,r_x_max,n_x*10)
#plt.plot(xx,r_star/xx,'r',label='theoretical')
#plt.legend()
#plt.show()
```

```
t_stop = time()
print 'Consumed time: ', t_stop-t_start

#plt.figure(1)
#plt.imshow(np.log(P),origin='lower')
#plt.colorbar()
#plt.title('Log count double')

#plt.figure(2)
#plt.imshow(np.log(P_double),origin='lower')
#plt.colorbar()
#plt.title('Log count double')

#plt.show()

#plt_number = 0
#for k in [0,2,19]:
#    plt_number += 1
#    plt.figure(plt_number)
#    for l in [0,1,2,3]:
#        bah = 220+l+1
#        plt.subplot(bah)
#        plt.title('$r_3=$'+str(round(r[k],3))+' $n_t=$'+str(l) )
#        plt.imshow(M[k,:,:,l]*1.0/n_samples,origin='lower',extent=(r_min,r_max)*2,vmin=0,vmax=0.25)
#        plt.colorbar()

#N1 = []
#N2 = []
#for x in P:
#    N1.append(x[1][1])
#    N2.append(x[1][2])

#plt.plot(N1,N2,'rx')
#
```

```
##P = cp.load(open('P_saved.p','rb'))
#plt.xlabel('$N_1$')
#plt.ylabel('$N_2$')
#plt.title('1:'+str(ratio_n))
#plt.show()


#prob = np.zeros(n_step)
#for k in range(n_step):
##    prob[k] = np.sum(M[k,(np.abs(r-r[k]) > r[k]*2E-2*10),2])/np.sum(M[k,:,1]+M[k,:,2])
#    prob[k] = np.sum(M[k,(np.abs(r-r[k]) > r[k]*2E-2*10),2])/(0.5*(M[k,k,1]+2*M[k,k,2]))


#plt.plot(r,prob)
#plt.show()


n_Hill = 40
c_Hill = 0.01
cntr = 0
prob = np.zeros(n_planets+1)
for first_planet in range(n_step):
    r_1 = r[first_planet]
    for second_planet in range(n_step):
        r_2 = r[second_planet]
        if (np.abs(r_1 - r_2) < max(r_1,r_2) * c_Hill*n_Hill):
            # The points are too closed packed
            continue
        for third_planet in range(n_step):
            r_3 = r[third_planet]
            if ((np.abs(r_1 - r_3) < max(r_1,r_3) * c_Hill*n_Hill) or
                (np.abs(r_2 - r_3) < max(r_2,r_3) * c_Hill*n_Hill) ):
                continue
            for fourth_planet in range(n_step):
```

```
        r_4 = r[fourth_planet]
        if ((np.abs(r_1 - r_4) < max(r_1,r_4) * c_Hill*n_Hill) or
            (np.abs(r_2 - r_4) < max(r_2,r_4) * c_Hill*n_Hill) or
            (np.abs(r_3 - r_4) < max(r_3,r_4) * c_Hill*n_Hill) ):
            continue
        for fifth_planet in range(n_step):
            r_5 = r[fifth_planet]
            if ((np.abs(r_1 - r_5) < max(r_1,r_5) * c_Hill*n_Hill) or
                (np.abs(r_2 - r_5) < max(r_2,r_5) * c_Hill*n_Hill) or
                (np.abs(r_3 - r_5) < max(r_3,r_5) * c_Hill*n_Hill) or
                (np.abs(r_4 - r_5) < max(r_4,r_5) * c_Hill*n_Hill) ):
                continue
            for sixth_planet in range(n_step):
                r_6 = r[sixth_planet]
                if ((np.abs(r_1 - r_6) < max(r_1,r_6) * c_Hill*n_Hill) or
                    (np.abs(r_2 - r_6) < max(r_2,r_6) * c_Hill*n_Hill) or
                    (np.abs(r_3 - r_6) < max(r_3,r_6) * c_Hill*n_Hill) or
                    (np.abs(r_4 - r_6) < max(r_4,r_6) * c_Hill*n_Hill) or
                    (np.abs(r_5 - r_6) < max(r_5,r_6) * c_Hill*n_Hill) ):
                    continue
                prob = prob + M[first_planet,second_planet,
third_planet,fourth_planet,fifth_planet,sixth_planet,:]
                cntr = cntr + 1


print prob,cntr,n_samples

print prob/(cntr*n_samples)

plt.semilogy(range(n_planets+1),prob,label='Packing='+str(n_Hill))

ax = plt.gca()
#plt.axis(max(ax.get_xlim()+0.5)+ax.get_ylim())
plt.xticks(range(n_planets+1),('None','Single','Double','Triple',
```

```
'Quadruple','Quintuple','Hexatuple'),rotation=20)
plt.axis((max(ax.get_xlim())+0.3,min(ax.get_xlim())-0.3) + ax.get_ylim())

plt.legend()

print n_Hill,prob,cntr,n_samples


plt.show()
```