GARCH-Copula Approach to Estimation of Value at Risk for Portfolios

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Key words Stock indexes, Value at Risk (VaR), GARCH, GJR, Dependence structure, Copula, Backtesting.

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Contents

1 Introduction

In financial markets, one of the most asked questions is that what investors will lose on an investment or "how much" the risk is on a portfolio. All the investors want to minimize the risk and maximize the profit. With the rapid growth of the global financial markets, violations become much stronger than before. Some financial crisis happened during the last 20 years. Therefore, financial institutions and investors pay more and more attention to financial risk management nowadays. Then an important issue arises, that is how to measure risk properly in financial markets. The concept of value at risk (VaR) is designed to provide a persuasive answer that how to measure risk within a reasonable bound. Currently, VaR is considered as one of the most important risk measures in the world. It has been widely used for financial risk management. There is a survey by Wharton Wood Gundy that there are 30% of American non-financial groups using VaR to assess the risk of derivatives. There are 60% of some pension fund management companies using VaR as risk tools according to New York University. In mathematics and statistics, VaR of a portfolio is simply an estimate of a specified quantile of the probability distribution of the portfolio's value over a given period. In this paper, five major Asian stock markets are studied in a portfolio to estimate VaRs. In addition, GARCH-copula framework with Monte Carlo simulation is used to estimate VaRs of a portfolio. To be specific, GARCH models are used to model the margins, and copulas are used to combine them together into a multivariate distribution.

1.1 Literature Review

Several famous methods to measure VaR have been suggested during the last 20 years. One of them is the RiskMetrics model, which has been developed by the risk department of J.P. Morgan in 1994. This method has greatly promoted the popularity of VaR since it can summarize risk as a number. The common RiskMetrics model assumes the returns of a risky asset follow a conditional normal distribution with zero mean, and the variance being expressed as an exponential weighted moving average of historical squared returns. However, with the application of this model, some drawbacks have been revealed. The main problem is that the financial returns are usually heavy tailed which do not satisfy the assumption of the model well. Another issue is that, many financial returns have a strong dependence on market volatility. The volatility can have a large impact on pricing of the assets and forecasting the risk in financial markets. So other measurements have been developed, such as using GARCH model to measure VaR. Angelidis, Benos and Degiannakis (2003) proved that GARCH models have some advantages in estimating VaR, since GARCH models provide good simulation of the volatility. Moreover, the co-movements of financial returns are of great practical importance because of the complex correlation between them. It is, therefore, important to extend the univariate models to the multivariate GARCH (MGARCH) models. Goeij and Marquering (2004) modeled the conditional covariance between stock and bond markets returns by a multivariate GARCH approach. They showed strong evidence of heteroscedasticity and asymmetries in the covariance between stock and bond market returns. Morimoto and Kawasaki (2008) did some empirical experiments on different MGARCH model to test the estimation for VaR with high-frequency data. They showed that Dynamic Conditional Correlation model (DCC-GARCH) provided the best forecasting performance among the multivariate GARCH models tested, and the model was thus considered favorable for practical risk management. Other multivariate methods have also been used to estimate VaR, like switching regime model. Billio and Pelizzon (2000) introduced a multivariate switching regime model in order to calculate the VaR for ten Italian stocks and for several portfolios that had been generated by them. They tested the performance of their models using two backtesting measures and concluded that a switching regime specification was more accurate than other known methods, such as RiskMetrics or GARCH(1,1) under Normal and Student-t distribution.

Even though some models have been proved to have good performances in estimating VaR, some recent studies show that Multivariate models have some drawbacks, especially when the dimensions get higher and the returns are not well distributed. VaR is an estimate of the tails of an empirical distribution. Many applications assume that asset returns are normal distributed, while in practice they usually show skewness and excess kurtosis, resulting in an underestimation or overestimation of the true VaR. Another important issue is that how the dependency between assets can be measured when returns are non-normal. As a common sense, the risk between the assets are correlated and influenced by each other. Because of these drawbacks and restrictions, copula methods have been tried and used more and more in financial applications during the recent 15 years.

Copulas are widely used in modeling joint distributions because they do not require the assumption of joint normality and allow us to decompose a high dimensional joint distribution into its marginal distributions and use a copula function to link them together. Sklar (1959) introduced the concept of copula, and Embrechts et al. (1999) first used it in financial applications. Li (1999) studied the problem about the default correlation in credit risk models, and showed that the current CreditMetrics approach to default correlation through asset correlation was equivalent to using a normal copula function. Bouye (2000) showed that copulas can be extensively used to solve many financial problems. Cherubini and Luciano (2001) estimated the VaR using the Archimedean copula and the historical empirical distribution to estimate the marginal distributions. Clemente and Romano (2003) combined extreme value theory (EVT) and copula to study Italian capital market, and used Monte Carlo simulation to calculate VaR of different portfolios. The result showed that an EVT-copula model is much better than a traditional normal VaR model. Rosenberg and Schuermann (2004) used VaR as a risk measure and copula approach to study the integrated risk of market risk, credit risk and operation risk, and compared with other models to get the best VaR. To model the dependence structure between excess returns of "large cap" and "small cap" of stock indices, Patton (2004) used a group of frequently used copulas and focused on the dependence structure between two stock indices which are more correlated during the market downturn than they are in the upturn. Long Kang (2007) modeled the joint distribution of excess returns of four major assets by copula-GARCH approach (gauss, t, hierarchical and mixed copula) and showed student's t copula yields the highest log-likelihood. Indrajit Roy (2011) used copula-GARCH approach for the high-frequency multivariate data in four major foreign currencies and got best calculated VaR for seven multivariate models based on goodness of fit and backtesting.

1.2 Structure of the Paper

This paper is organized as follows. Chapter 1 gives an introduction and a literature review of VaR. Chapter 2 introduces the basic mathematical concepts of VaR, and gives three traditional estimation methods of VaR. Chapter 3 briefly introduces basic volatility models - GARCH type models and their advantages and improvements. The GARCH models are used to model the margins later in empirical work. Chapter 4 introduces the dependence structure modeling, and the copula theory will be along with the five copulas that will be used in empirical work. Chapter 5 carries out empirical work with the corresponding results in the form of tables and figures. Chapter 6 gives summary of this paper, and conclusion based on the empirical results.

2 Value at risk

Value at risk (VaR) is widely used by financial institutions as a measure of risk on a portfolio of risky assets in financial world. It is the most well-known risk measure, which is simply defined as the maximum loss at a given time horizon with a given confidence level on the portfolio. The given time horizon is usually one day, one week or one month, and the confidence level is usually 90%, 95%, and 99%. For example, if the VaR on a portfolio is $ 10 million at a one day with 95% confidence level, the biggest loss of the portfolio is $ 10 million tomorrow under the 95% confidence level. In other words, there is only 5% chance that the value of the portfolio will drop more than $ 10 million over a day.

2.1 A Short History of Value at Risk

Value at risk became popular in banks during the 1990s. It was during this period that the name "Value at Risk" entered the financial lexicon. But VaR measure had been used for a long time before this time. It first came from Harry Markowitz's portfolio theory based on the mean-variance method to study portfolio optimization. During late 1980s and early 1990s, a few number of institutions started to implement VaR measure to study market risk and capital allocation. In 1989, the CEO of J.P. Morgan asked for a daily report that fully described the market risk of portfolios in the bank. After that, J.P. Morgan subsequently set up a system to measure risk. In 1994 the department known as RiskMetrics built a method to calculate VaR. It largely promoted the use of VaR among the company's clients. In 1995, the Basel Committee on Banking Supervision utilized market risk capital requirements for banks. These were based on a crude VaR measure, but the committee also approved the banks' own proprietary VaR measures. It included a set of qualitative and quantitative standards for the process of risk management that are applied to banks. Thus, models of risk evaluation entered into a common framework. The 1996 Amendment to the Basel II has brought an added capital margin for the market risk, including for the bank's trading portfolio and for other portfolios consisting of financial instruments. To estimate the market risk, banks may use the standard method, together with the internal models for determining VaR. Internal models are more advantageous for large banks, as they take into account the correlations between assets, and require a much lower capital cost. In the last decade, VaR has become the most popular measure of risk in financial companies and has even begun to find acceptance outside financial firms.

2.2 Measuring Value at Risk

Even though VaR has so many advantages in measuring risk, one of the most important and challenging issues is how to measure VaR. It is quite a mathematical and statistical question. Let us check its mathematical expression in this section.

Definition 2.1 (Value at risk) Value at risk of a long position over time horizon l with probability $p=1-\alpha$ and the real $VaR_t(p,l)$ is such that

$$p=\Pr\left[\Delta V(l)\le \mathrm{VaR}_t(p,l)\right]=F_l(\mathrm{VaR}_t(p,l)) \ , \qquad (1.1)$$

where VaR is defined as a negative value (loss), $\Delta V(l)$ is the change of portfolio value from time t to time $t+l$, and $F_l(x)$ is the cumulative distribution function of $\Delta V(l)$.

Remarks

(1) VaR is a single, summary and statistical measure of possible loss of the portfolio.
(2) VaR should contain a given time horizon, usually it takes over one day, one week, and so on.
(3) VaR is based on a confidence level α, which represents the possible maximum loss over a given period at this probability. In other words, loss greater than VaR is suffered only with a specified small probability p. The confidence levels usually take 90%, 95% and 99% based on the choice of different firms.
(4) This definition can also written as $VaR_\alpha=\inf\{x\mid F_l(x)>\alpha\}$. It means the p-th quantile of $F_l(x)$, or the smallest real number x that satisfies $F_l(x)\ge\alpha$.
(5) $F_l(x)$ is usually unknown in practice, and need to be estimated based on a specified method.
(6) VaR is not subadditive, which means the VaR of a portfolio can be larger than the sum of the VaRs of its components.

2.3 Estimating Value at Risk

Considering the mathematical definition above, there are several methods to estimate VaR. In this section, we mainly introduce three widely used methods to estimate VaR, namely Historical Simulation, Variance-Covariance (Parametric) method and Monte Carlo Simulation.

2.3.1 Historical Simulation

Historical simulation is the simplest method of estimating VaR. It is a non-parametric model that does not require any assumptions of the returns. In this approach, VaR is estimated by generating a time series of returns of the portfolio, then it is obtained by running the portfolio through actual historical observations and computing the changes that would occur in each period. The basic idea of that method is that the history will appear again in the future. Since it assumes the distribution of the portfolio returns does not change, VaR is just the 100p% quantile of historical series with α ($\alpha=1-p$) confidence level.

Since the historical simulation approach does not need any assumptions on the distribution of returns, it is very easy to compute. Another issue is that since the distribution of the returns does not rely on any parameters or any distributed forms, there is no need to discuss the "bad" part of returns, like heavy tails. Besides, the approach does not take the volatility of the returns into account, which helps decrease the risk of the model. Because of these advantages of historical simulation, it became the basic method of estimating VaR in Basel Accord in 1993.

However, the shortcomings were revealed with the increasing usages of the approach. The most important one is that the history will not surely repeat in the future. We do need historical data, but we cannot fully trust the past will happen exactly in the future. This is an obvious thing in financial markets, like a stock return cannot always increase or decrease as the past. It cannot take into account unprecedented events and their effects. In a word, this approach relies more on the history than other models. In financial markets, the correlations, volatility and other infections change all the time. To reflect the real marker information, we need to rely on latest information more than old one. But historical simulation can not reflect this kind of sensitivity to the new information. All of these will bring a lot of problems when applying the method into practice.

2.3.2 Variance-Covariance Method

Variance-covariance method is one of the most used approaches in risk management nowadays, which is based on Markowitz's portfolio theory. It is an analytical estimation of the volatility of asset returns and of the correlations between these asset price series. This method assumes the returns of risk assets are normally distributed, and the correlations between risk assets are constant. To be specific, VaR provides a value of the maximum loss of the portfolio with a given time period and probability. So variance-covariance method observes the movement of portfolio component series over time and uses probability theory to compute this possible maximum loss. This can be done by calculating the standard deviation of the series, and then by assuming normal distribution, the maximum loss under the required confidence level can be estimated.

Assume $R=(r_1,...,r_n)$ is a vector containing returns of n assets in a portfolio, $W=(w_1,...,w_n)$ is a vector containing the weight of each portfolio asset with $\sum_{i=1}^{n} w_i = 1$, and the variance-covariance matrix of the returns is:

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{1n} & \sigma_{2n} & \cdots & \sigma_n^2 \end{bmatrix}. \tag{1.2}$$

So the return of the portfolio is $r_p = WR'$ and the variance of the portfolio is $\sigma_p^2 = W\Sigma W'$. Then the VaR based on the variance-covariance method can be calculated as:

$$VaR_\alpha = E[r_p] + \sigma_p \Phi^{-1}(\alpha), \tag{1.3}$$

where Φ^{-1} is the inverse function of standard normal distribution.

This approach can get the determinant of VaR based on the probability theory. It also represents the co-movements of the series based on standard deviations and correlations. They determine the variance-covariance matrix of the normal distribution of changes in the portfolio assets. The strength of this method is that VaR can be easy to estimate given the assumption of normal distribution.

However, some problems also arise from its assumptions. If the returns are not normally distributed, the estimated VaR will not reflect the real VaR. Actually, many studies have demonstrated that the financial returns are heavy-tailed, skewed, i.e. not normal distributed, which restricts the usage of this method. Another problem is that the approach is limited by the difficulties associated with deriving probability distributions. In addition, the correlation between assets is not usually constant.

2.3.3 Monte Carlo Simulation

Monte Carlo simulation is a non-parametric method. It is more flexible and realistic than the previous two methods, and therefore, it is more likely to be used to estimate VaR. It can use actual historical distributions of returns rather than normal distribution. Rather than computing the covariances and correlations of the returns, a large number of random simulated returns are generated in computer, each simulation will be different but in total the simulations will aggregate to the chosen statistical parameters (historical distributions, volatility and correlation). The estimation of parameters is easy to compute based on different choice of distributions. The strength of Monte Carlo simulation comes from the freedom that one has to pick alternative distributions for variables. For example, if one simulates 10000 data deriving from the portfolio, these simulations can be ranked from lowest to highest, and 95% VaR is correspond for 5% quantile of all the simulations, or the 500th lowest value.

The advantages of Monte Carlo simulations can be seen when compared to the other two approaches for computing Value at Risk. In contrast with the variance-covariance method, we do not have to make assumptions about normality in returns. Comparing with the historical simulation approach, we begin with historical data but we are free to bring in both subjective judgments and other information to improve the predicted probability.

The problem is that if the assumed statistical distribution of the asset returns and the estimates of its parameters are wrong because of the wrong choice of the model, it will lead to errors in the calculated VaR. In addition, as the number of market assets increase and their co-movements become more complicated, we have to estimate the probability distributions for hundreds of risk assets.

In order to overcome these drawbacks of Monte Carlo simulation, some improvements need to be considered to make it better. That is what we should address in this paper based on the ideas from Monte Carlo simulation. As mentioned above, the distribution is an important issue in Monte Carlo method. So if we can get the distribution of the portfolio, we can use Monte Carlo method to simulate many random numbers, and then estimate VaR. However, in financial markets, one cannot find a very good specified distribution for all the return series. So if some methods or models can be fit to the series, we can go further to generate the random numbers from these models, then we can end up with a better estimation of VaR. That is what we will do in this paper.

3 Modeling the Margins – GARCH Model

Recent developments in financial econometrics suggest the use of non-linear time series to model the asset return and risk. Among these non-linear models, ARCH (Autoregressive conditional heteroscedasticity) and GARCH (Generalized autoregressive conditional heteroscedasticity) models are the most widely used, and many studies have showed that ARCH/GARCH models have many advantages in modeling financial returns. Since for financial returns, they usually exhibit volatility clustering. Figure 3.1 shows a simple example of the returns of Hong Kong Hang Seng Index, it can be clearly seen that larger violation happens between 2008 and 2010. Volatility measures variation of returns and changes of price, which plays an important role in financial portfolios since it is commonly used to quantify the risk. In statistics, volatility clustering is called heteroscedasticity, which means the variance is not constant but changes with time and depends on the past (conditional). Traditional time series models cannot captures the volatility clustering well, but ARCH/GARCH models can fit this information well in their models.

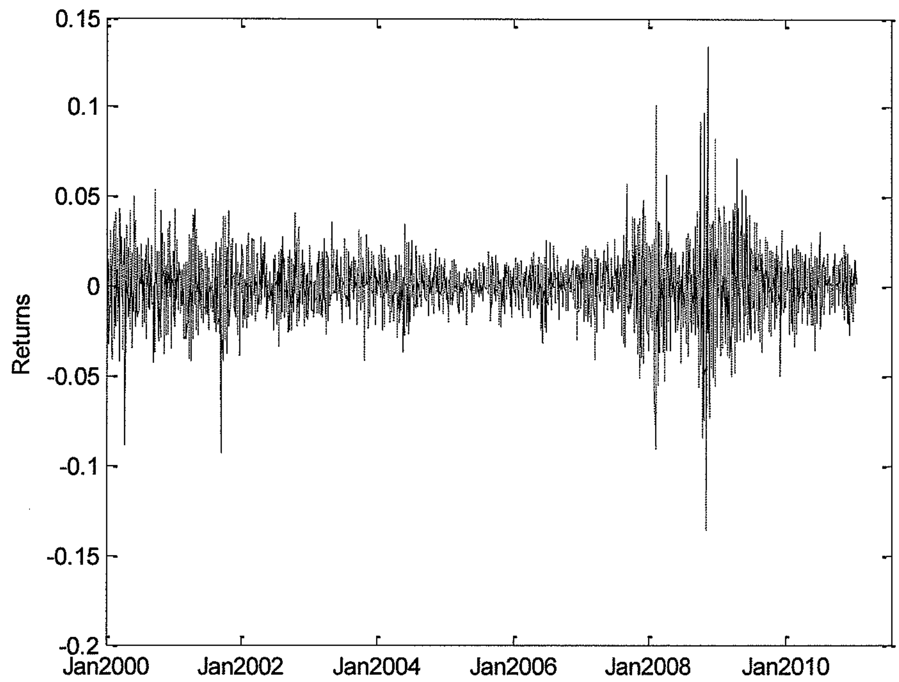


Figure 3.1 Returns of Hong Kong Hang Seng Stock Index

In this paper, we use GARCH type models to filter the margins in order to remove the heteroscedasticity effect of the marginal returns. Since there are so many papers which discuss the theory and principle of GARCH models, we only summarize the basic concepts, and mainly focus on the part of GARCH models that will be used in this paper.

3.1 GARCH Model

GARCH model was first built by generalizing Engle's earlier ARCH model by Bollerslev in 1986. Comparing to ARCH model, GARCH model includes autoregressive (AR) terms as well as moving average (MA) terms. Adding MA terms means that the conditional variance is also a linear function of its own lags, which improves the performance of ARCH model. For GARCH model, the term autoregressive represents a feedback mechanism that incorporates past observations into the present. Conditional indicates that variance has a dependence on the past. Heteroscedasticity means a time-varying volatility. In general, GARCH is a mechanism that uses past variances in the explanation of future variances. More specifically, GARCH is a technique to model the serial dependence of volatility.

The GARCH(p,q) includes p lags of the conditional variance in the linear ARCH(q) conditional variance equation. The variance equation can be generalized as:

$$\sigma_t^2 = \kappa + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 + \sum_{j=1}^{q} \alpha_j \varepsilon_{t-j}^2 \quad . \tag{3.1}$$

Remarks

(1) ε_t is the error terms (innovations) from the mean equation, and it can split into a stochastic piece Z_t and a time dependent standard deviation σ_t characterizing the typical size of the terms so that $\varepsilon_t = \sigma_t Z_t$. In some articles Z_t is also called residual series. Z_t is a zero mean, identical and independent distribution, which is sometimes assumed to have normal distribution, t distribution and skew t distribution (Appendix A).

(2) $\beta_i \geq 0, \alpha_j \geq 0, \kappa \geq 0$, with constrains that $\sum_{i=1}^{p} \beta_i + \sum_{j=1}^{q} \alpha_j < 1$

(3) If and only if $\sum_{i=1}^{p} \beta_i + \sum_{j=1}^{q} \alpha_j < 1$, GARCH(p,q) is weakly stationary. And the closer the sum tends to 1, the closer the parameterized conditional variance converges towards the unconditional variance.

(4) $\sum_{i=1}^{p} \beta_i \sigma_{t-i}^2$ indicates GARCH terms and $\sum_{j=1}^{q} \alpha_j \varepsilon_{t-j}^2$ indicates ARCH terms. The parameters of ARCH terms and GARCH terms, presenting the impact of ARCH effect (past innovations and shocks) and GARCH effect (past volatility) on the conditional variance (volatility) of return now and decay rate of those effects to the coming periods, respectively.

(5) Usually GARCH(1,1) is sufficient to use for financial returns.

Comparing traditional time series model that assume the homoskedastic, GARCH model successfully take the heteroscedastic time-varying variance into account. Due to the widely use of GARCH model, many extensions and modifications of the model have been suggested, some of them are used more than the basic GARCH model nowadays. This has been greatly improved the flexibility of the basic model. One example is that GARCH model often fails to fully capture the fat tails observed in asset returns. Heteroscedasticity can explain some but not all heavy-tails. To compensate for this limitation, heavy-tailed distributions such as t distribution and skew t distribution are applied to GARCH model.

3.2 GJR Model

The main problem of the basic GARCH model is that it is a symmetric variance process, in which ignores the sign of the disturbance. It assumes that the positive or negative information have the same impact on the volatility, which show symmetric effect in the variance equation of the model. But the situation is not like this in the real financial markets.

Studies have shown that when the stock markets get shocks, prices go down, following by the stronger fluctuations. In contrast, when the markets are motivated positively, prices increase following with violations decreasing. In another word, negative shocks often give rise to higher volatility than same magnitude of positive ones do (Christie, 1982 and Schewart, 1989). This can be explained by behavioral finance that the investors feel risk more than satisfaction when facing volatilities.

Actually, this phenomenon that negative shocks are followed by more volatility than the positive are, is referred to the leverage effect in the case of equity returns. A decline in stock returns leads to an increase debt-to-equity and more risk shareholders bear (Brooks, 2009). The leverage effect causes the asymmetries of variance dynamics and points out the drawbacks of GARCH model because of its symmetric effect towards the conditional variance. So some asymmetric GARCH models have been developed. Next we will introduce one of the most widely used asymmetric GARCH models namely GJR-GARCH model.

GJR-GARCH model is build by Glosten, Jagannathan and Runkle in 1993. The variance equation of GJR(p,q) presents as:

$$\sigma_t^2 = \kappa + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 + \sum_{j=1}^{q} \{\alpha_j + \gamma_j \, 1(\varepsilon_{t-j} < 0)\} \varepsilon_{t-j}^2 , \qquad (3.2)$$

where $1(\varepsilon_{t-j} < 0) = 1$ if $\varepsilon_{t-j} < 0$, $1(\varepsilon_{t-j} < 0) = 0$ otherwise. With the constrains:

$$\beta_i \geq 0, \alpha_j \geq 0, \gamma_j \geq 0,$$
$$\alpha_j + \gamma_j \geq 0,$$
$$\sum_{i=1}^{p} \beta_i + \sum_{j=1}^{q} \alpha_j + \frac{1}{2} \sum_{j=1}^{q} \gamma_j < 1.$$

$\sum_{j=1}^{q} \gamma_j 1(\varepsilon_{t-j} < 0)\varepsilon_{t-j}^2$ is called leverage terms in GJR model. So GJR model is a GARCH process that includes leverage terms for modeling asymmetric volatility clustering.

In GJR model, large negative changes are more likely to be clustered than positive changes. The GJR(p,q) model has p GARCH coefficients with lagged variances, q ARCH coefficients with lagged squared innovations, and adding q leverage coefficients with the square of negative lagged innovations. Thus, the leverage coefficients are applied to negative innovations, giving negative changes additional more weight. This can have a better interpretation of the phenomenon that a decreasing price following with a stronger fluctuation. Furthermore, GARCH model can be seen as nesting in the GJR model. If all leverage coefficients are zero, then GJR model reduces to GARCH model. This means one can test a GARCH model against a GJR model using the likelihood ratio test.

4 Modeling the Dependence Structure– Copula

In risk management, as well as asset allocation and derivative pricing, the dependence structure among risk assets plays a crucial role. There are some approaches to model dependence, most commonly by the multivariate models. The simplest example is that many researchers use multivariate normal or t distributions to model financial assets. And the multivariate GARCH models are also widely used into modeling and predicting of risk assets. So why we choose copula instead of the multivariate GARCH models to model the dependence in this paper? The most attractive part of copula is that it does not require any assumptions of the margins like normal distributed, and allows to decompose a high dimensional joint distribution into its marginal distributions and then use a copula function to link them together. The idea is straightforward and can be used in high dimensions easily, which is very helpful in financial applications especially when the investors have many assets in a portfolio. In contrast, traditional multivariate distributions require that all random variables have the same marginal distribution. And for multivariate GARCH models, there are many parameters which make the models less flexible and the parameter estimations more difficult. Besides, multivariate GARCH models also have some assumptions of the margins which restraint the usage. Compared to multivariate GARCH models and other multivariate models, copula provides a more flexible dependence structure, which describes the dependence structure of a set of variables. Thus copula has become the standard tool in modeling dependence among financial time series, especially when investors do not want to be limited to the specific assumptions of asset distributions.

Copula theory was first introduced by Sklar in 1959, but it was first used in financial applications by Embrechts in 1999. In probability theory and statistics, copula is a kind of distribution function. The cumulative distribution functions of random variables can be written in terms of marginal distribution functions and a copula. The marginal distribution functions describe the marginal distribution of each component of the random variables and the copula describes the dependence structure between the components. In this chapter, we will introduce some basic concepts of copula.

4.1 Copula Theory

For brevity, we introduce bivariate copula (2-dimensional copula) first, and then we extend the definition of copula to the n-dimensional case.

Definition 4.1.1 (Bivariate copula) A two-dimensional subcopula is a function C' with following properties:

1. Dom $C' = S_1 \times S_2$, where S_1 and S_2 are subsets of $\mathbf{I}$ containing 0 and 1;
2. C' is grounded and 2-increasing;

3. For every u in S_1 and every v in S_2,

$$C'(u,1)=u \quad \text{and} \quad C'(1,v)=v.$$

Note that every (u,v) in Dom C', $0 \le C'(u,v) \le 1$.

Definition 4.1.2 (Bivariate copula) A two-dimensional copula is a 2-subcopula C whose domain is $\mathbf{I}^2$.

Equivalently, a copula is a function C from $\mathbf{I}^2$ to $\mathbf{I}$ with the following properties:

1. For every u, v in **I**,

$$C(u,0)=0=C(0,v),$$
$$C(u,1)=u,\ C(1,v)=v.$$

2. For every u_1,u_2,v_1,v_2 in **I** such that $u_1 \le u_2, v_1 \le v_2$,

$$C(u_2,v_2)-C(u_2,v_1)-C(u_1,v_2)+C(u_1,v_1) \ge 0. \tag{4.1}$$

   This is called rectangle inequality of copula.

Theorem 4.1 (Fréchet-Hoffding bounds in 2-dimensions) Subcopula satisfies the following inequality:

$$\max(u+v-1,0) \le C(u,v) \le \min(u,v), \tag{4.2}$$

for every $u,v \in \mathbb{R}$.

From the bivariate case, it is easy to see that copula function is a two dimensional distribution which combines univariate margins together. Next we extend the dimensions from 2 to n(>2) and introduce its density formula.

Definition 4.2 (N-dimensional copula) An n-dimensional copula is a function $C: A_1 \times A_2 \times ... \times A_n \to \mathbb{R}$, where for each i, $A_i \subset I$ and contains at least 0 and 1, such that

1. C is grounded;
2. Its 1-dimensional margins are the identity function on $I: C_i(u)=u,\ i=1,2,...,n$;
3. C is non-decreasing.

Definition 4.3 (Copula density) The density $c(u_1,u_2,...,u_n)$ associated to a copula $C(u_1,u_2,...,u_n)$ is defined as:

$$c(u_1,u_2,...,u_n)=\frac{\partial^n C(u_1,u_2,...,u_n)}{\partial u_1 ... \partial u_n}. \tag{4.3}$$

In another word, to make it simpler, we can say a copula is a function C such that:

$$F(x_1,x_2,...,x_n)=C(F_1(x_1),F_2(x_2),...,F_n(x_n)), \tag{4.4}$$

for any $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$. We can see that copula is a multivariate distribution with all univariate margins being U(0,1). Similarly, we get the Fréchet-Hoffding bounds in n-dimension easily.

Theorem 4.2 (Fréchet-Hoffding bounds in n-dimensions) For every copula, it satisfies the following inequality:

$$\max(u_1 + u_2 + ... + u_n - 1, 0) \leq C(U) \leq \min(u_1, u_2, ..., u_n), \tag{4.5}$$

for every $U \in I^n$.

Theorem 4.3 (Sklar's theorem) Let F be a joint n-dimensional distribution function with marginal distributions $F_1, F_2, ..., F_n$. Then there exists a copula C such that:

$$F(x_1, x_2, ..., x_n) = C(F_1(x_1), F_2(x_2), ..., F_n(x_n)). \tag{4.6}$$

If $F_1, F_2, ..., F_n$ are all continuous, then C is unique. Conversely, if C is a copula and $F_1, F_2, ..., F_n$ are distribution functions, then F defined by Formula 4.6 is a n-dimensional distribution function with marginal distributions $F_1, F_2, ..., F_n$. Sklar's theorem provides the theoretical foundation for the applications of copula.

Remarks

(1) A copula describes how the margins are tied together in the joint distribution.
(2) The joint distribution can be decomposed into the marginal distributions and a copula function.
(3) We can estimate the marginal distributions and copula separately.
(4) Given a copula, we can obtain many multivariate distributions by selecting different margins.

4.2 Elliptical Copula

Elliptical copulas are the simplest copula of elliptically contoured distributions. Multivariate normal and t distributions are the mostly used elliptical distributions. One of the advantages of elliptical copula is that one can specify different levels of correlation between margins, while the disadvantages are that they do not have closed form expressions and are restricted to have radial symmetry.

4.2.1 Gaussian Copula

Gaussian copula is simply derived from multivariate normal distribution:

$$C_G(u_1, ..., u_d; \Sigma) = \Phi_d(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_d); \Sigma), \tag{4.7}$$

where Φ^{-1} is the inverse function of standardized normal distribution, Σ is the standardized dispersion matrix or correlation matrix that determines the dependence structure. Then Gaussian copula density can be expressed as:

$$c_G = \frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} \exp(-\frac{1}{2}(X-\mu)^T \Sigma^{-1} (X-\mu)). \quad (4.8)$$

And the multivariate distribution is the integral of this density function on all its margins. Figure 4.1 gives an example of the surface of cumulative distribution function of 2-dimensional elliptical copula- Gaussian copula.

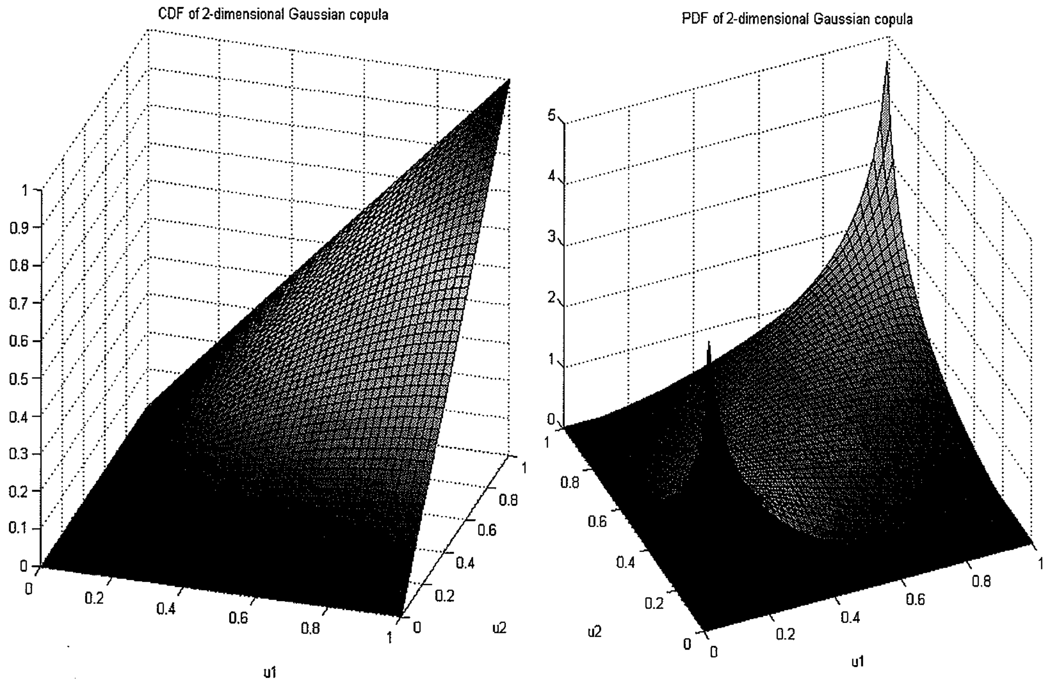


Figure 4.1 Surface of the cumulative distribution function and probability density function of elliptical copula

4.2.2 t Copula

Like Gaussian copula, t copula is derived from multivariate t distribution:

$$C_t(u_1,...,u_d;\Sigma,v) = t_d(t^{-1}(u_1),...,t^{-1}(u_d);\Sigma,v), \quad (4.9)$$

where $t_v^{-1}(\cdot)$ is the inverse function of t distribution with v degree of freedom, Σ is the correlation matrix that determines the dependence structure. Then t copula density can be expressed as:

$$c_t = \frac{\Gamma(\frac{v+d}{2})}{\Gamma(\frac{v}{2})(\pi v)^{d/2} |\Sigma|^{1/2}} (1 + \frac{(X-\mu)^T \Sigma^{-1} (X-\mu)}{v})^{\frac{-(v+d)}{2}}, \quad (4.10)$$

and the multivariate distribution is the integral of this density function on all its margins.

4.3 Archimedean Copula

The class of Archimedean copulas has been named by Ling (1965), but it was recognized by Schweizer and Sklar (1961) in the study of t-norms. This type of copulas falls into the class of so-called Laplace transform Archimedean copulas.

Definition 4.4 (Laplace transform) The Laplace transform of a positive variable γ with distribution function F_γ is defined in the form:

$$\tau(s) = E_\gamma(e^{-s\gamma}) = \int_0^{+\infty} e^{-st} dF_\gamma(t). \tag{4.11}$$

Definition 4.5 (Archimedean copula) A copula is called Archimedean if it can be written in the form:

$$C(u_1,...,u_d) = \varphi^{-1}(\varphi(u_1) + ... + \varphi(u_d)), \tag{4.12}$$

for some strict generator φ with its generalized inverse φ^{-1}:

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t), & 0 \le t \le \varphi(0) \\ 0, & \varphi(0) \le t \le \infty \end{cases}. \tag{4.13}$$

With the generator φ satisfies:

1. $\varphi(1) = 0$;
2. φ is continuous;
3. For all $t \in (0,1)$, $\varphi'(t) < 0$, which means φ is strictly decreasing;
4. For all $t \in (0,1)$, $\varphi'(t) \ge 0$, which means φ is convex.

Archimedean copula has advantages that it is very easy to construct new distributions and nice properties like easy modeling, and disadvantage that it has the limit that there are only one or two parameters to capture dependence structure. Besides, Archimedean copula can measure non-linear correlation and tail dependency. Furthermore, for this class of copula, the inverse of the generator function has a nice representation as a Laplace transform of some function which simplifies the simulation of pseudo observations from these copulas.

According to different generators, we can get different specified Archimedean copulas, such as Clayton copula, Gumbel copula, Frank copula.

4.3.1 Clayton Copula

The generator is given by $\varphi(u) = u^{-a} - 1$, hence $\varphi^{-1}(t) = (t+1)^{-1/a}$, and $a > 0$ to satisfies decreasing. Therefore n-dimensional Clayton copula is:

$$C(u_1,u_2,...,u_n) = [\sum_{i=1}^{n} u_i^{-a} - n + 1]^{-1/a}, \quad a > 0. \tag{4.14}$$

Figure 4.2 gives an example of the surface of cumulative distribution function of 2-dimensional Archimedean copula - Clayton copula.

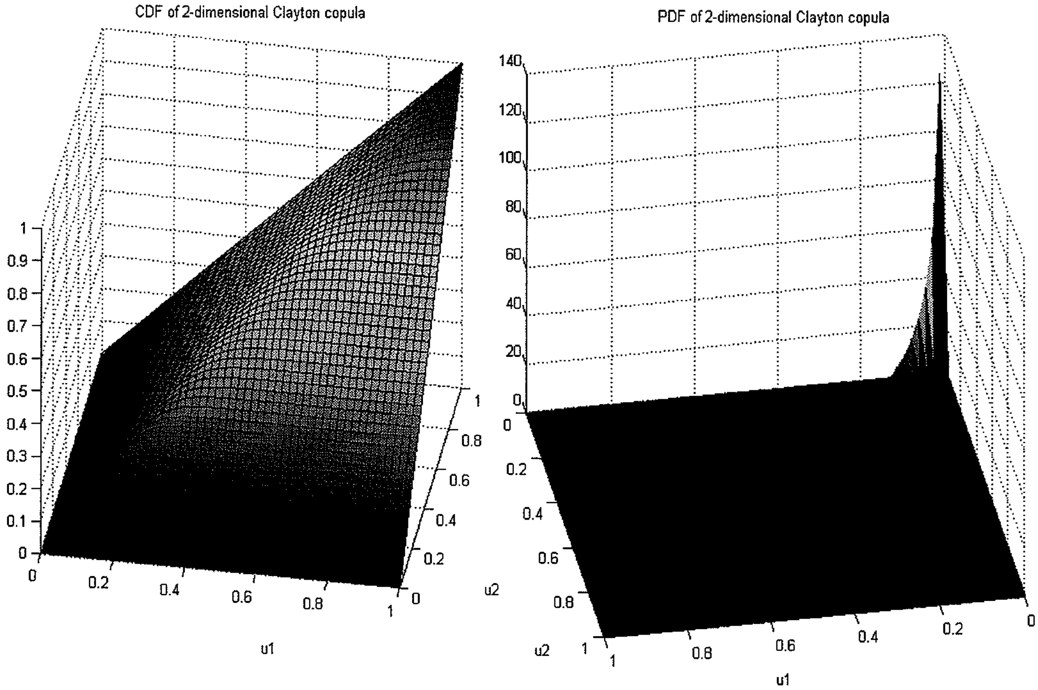


Figure 4.2 Surface of the cumulative distribution function and probability density function of Archimedean copula

4.3.2 Gumbel Copula

The generator is given by $\varphi(u)=(-\ln(u))^a$, hence $\varphi^{-1}(t)=\exp(-t^{1/a})$, and $a>1$ to satisfies decreasing. Therefore n-dimensional Gumbel copula is:

$$C(u_1,u_2,...u_n)=\exp\{-[\sum_{i=1}^{n}(-\ln u_i)^a]^{1/a}\},\ a>1. \tag{4.15}$$

4.3.3 Frank Copula

The generator is given by $\varphi(u)=(\frac{\exp(-au)-1}{\exp(-a)-1})$, hence $\varphi^{-1}(t)=-\frac{1}{a}\ln(1+e^{t}(e^{-a}-1))$, and $a>0$ to satisfies decreasing. Therefore n-dimensional Frank copula is:

$$C(u_1,u_2,...,u_n)=-\frac{1}{a}\ln\{1+\frac{\prod_{i=1}^{n}(e^{-au_i-1})}{(e^{-a}-1)^{n-1}}\},\ a>0\ when\ n\geq 3. \tag{4.16}$$

Finally, Figure 4.3 shows scatter plots of 1000 simulated numbers based on the five 2-dimensional copulas with different parameters.

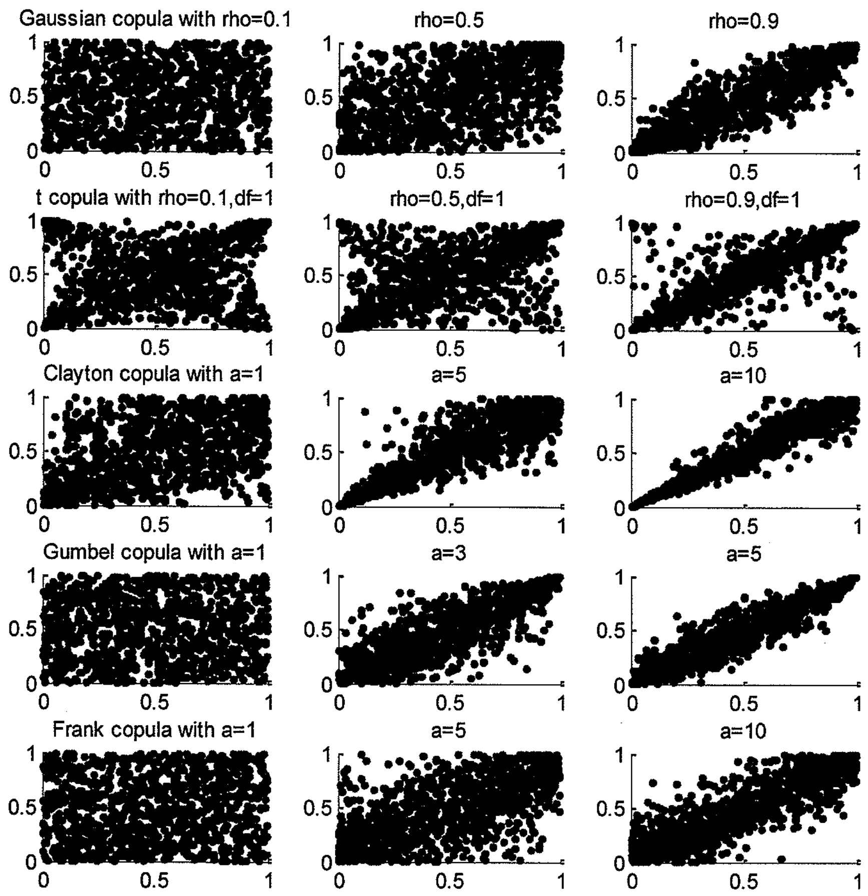


Figure 4.3 Scatters of 1000 simulated numbers from five copula

4.4 Dependence Measures

In this section, we briefly introduce two kinds of dependence measures. Pearson's rho is the most used as the linear correlation to measure dependence. Kendall's tau and Spearman's rho are two mainly used rank correlations. To better explain the idea of the different correlations, 2 dimensions are used for Pearson's rho and Kendall's tau while 3 dimensions are used for Spearman's rho below.

4.4.1 Pearson's (Linear) Rho

Pearson's correlation is the most widely used method to measure correlation. It is defined as:

$$\rho = \frac{Cov(X,Y)}{\sqrt{\operatorname{var}(X)\operatorname{var}(Y)}}. \tag{4.17}$$

Remarks

(1) Sensitive to outliers;

(2) Invariant only under strictly increasing linear transformations;

(3) Independence implies 0 correlation;

(4) Zero correlation does not always imply independence.

4.4.2 Kendall's Tau

Assume (X_1, Y_1) and (X_2, Y_2) are independent and identical distributed random vectors, with a joint distribution function H.

Concordance probability: $P((X_1 - X_2)(Y_1 - Y_2) > 0)$;

Discordance probability: $P((X_1 - X_2)(Y_1 - Y_2) < 0)$.

And Kendall's tau is defined as:

$$\tau = P((X_1 - X_2)(Y_1 - Y_2) > 0) - P((X_1 - X_2)(Y_1 - Y_2) < 0). \qquad (4.18)$$

It means that Kendall's tau is the difference between concordance probability and discordance probability. And it is easy to get $\tau \in [-1, 1]$.

4.4.3 Spearman's Rho

Assume $(X_1, Y_1), (X_2, Y_2)$ and (X_3, Y_3) are three independent random vectors with a common joint distribution function H and a copula C. And Spearman's rho is defined as:

$$\rho_S = 3\{P((X_1 - X_2)(Y_1 - Y_3) > 0) - P((X_1 - X_2)(Y_1 - Y_3) < 0)\}. \qquad (4.19)$$

4.4.4 Copula and Correlation Measurement

To compare different copula functions, we need to have a correlation measurement independent of marginal distributions. Both Kendall's tau and Spearman's rho can be calculated by using a copula function, as follows:

$$\tau = 4\iint C(u,v)dC(u,v) - 1, \qquad (4.20)$$

$$\rho_S = 12\iint C(u,v)\, dudv\ - 3. \qquad (4.21)$$

We can see that both Kendall's tau and Spearman's are only a function of copula. But Pearson's correlation cannot be written as a function of only copula, which also depends on marginal distributions (Lehmann, 1966). Thus, comparisons between results using different copulas should be based on either a common Kendall's tau or Spearman's rho.

4.5 Copula Estimation

4.5.1 Full Maximum Likelihood (FML)

Full maximum likelihood is the most used method in time series. The idea of the method is simple -- to find the parameters that make the likelihood function get its maximum value. It is defined as follows:

$$f(x_1,...,x_n) = c(F_1(x_1),...,F_n(x_n)) \cdot \prod_{j=1}^{n} f_j(x_j), \tag{4.22}$$

$$c(F_1(x_1),...,F_n(x_n)) = \frac{\partial^n (C(F_1(x_1),...,F_n(x_n)))}{\partial F_1(x_1) \dots \partial F_n(x_n)}; \tag{4.23}$$

Let $\{x_{1t},...,x_{nt}\}_{t=1}^{T}$ be the sample data matrix, the log-likelihood can be expressed as:

$$l(\theta) = \sum_{t=1}^{T} \ln c(F_1(x_{1t}),...,F_n(x_{nt})) + \sum_{t=1}^{T}\sum_{j=1}^{n} \ln f_j(x_{jt}). \tag{4.24}$$

Thus, the maximum likelihood estimator is:

$$\hat{\theta}_{MLE} = \max_{\theta} l(\theta). \tag{4.25}$$

The full maximum likelihood is easy to compute when the dimensions are not high. However, the main drawback is that in high dimensions, it could be very computationally intensive, because it is necessary to estimate the parameters of the marginal distributions and the parameters of the dependence structure of copula, simultaneously.

4.5.2 Inference for Margins (IFM)

To overcome the drawbacks of full maximum likelihood method, inference for margins method could be used. We can see from the likelihood function (Formula 4.24), it can be decomposed to two terms: one term involves the margins and all parameters of the copula density, and another term involves the copula density and its parameters. So the main idea of inference for margins is that it estimates the parameters for marginal distributions and copula separately in two steps.

Step 1: Estimate the parameters δ_j from marginal distributions,

$$\delta_j = \arg\max_{\delta} l(\delta_j) = \arg\max_{\delta_j} \sum_{t=1}^{T} \ln f_j(x_{jt};\delta_j). \tag{4.26}$$

Step 2: Estimate the dependence parameter by maximizing pseudo-log-likelihood, it can be defined as:

$$\hat{\theta}_{IFM} = (\hat{\delta}_1,...,\hat{\delta}_n,\hat{\theta})' = \arg\max_{\theta} l(\theta,\hat{\delta}_1,...,\hat{\delta}_n)$$
$$= \arg\max_{\theta} \sum_{t=1}^{T} \ln c(F_1(x_{1t};\hat{\delta}_1),...,F_n(x_{nt};\hat{\delta}_n);\theta). \tag{4.27}$$

4.5.3 Canonical Maximum Likelihood (CML)

The idea of canonical maximum likelihood is that the copula parameters may be estimated without specifying the margins. Canonical maximum likelihood maximizes the pseudo-log-likelihood empirical marginal distributions. It can be expressed as:

$$\hat{\theta}_{CML} = \arg\max_{\theta} l(\theta) = \arg\max_{\theta} \sum_{t=1}^{T} \ln c(\hat{F}_1(x_{1t}),\dots,\hat{F}_n(x_{nt});\theta), \tag{4.28}$$

where $\hat{F}_j(x) = \frac{1}{T+1}\sum_{t=1}^{T} 1(X_{jt} \le x)$, with $1(X_{jt} \le x)$ is the indicator function.

Canonical maximum likelihood can be seen as a full maximum likelihood if the margins are given.

4.5.4 Inversion of Kendall's Tau and Inversion of Spearman's Rho

Copula can be also be estimated by the rank correlations when they are available in closed forms. The inversion of Kendall's tau estimator can be expressed as:

$$\hat{\theta}_K = \arg\min_{\theta} (\hat{\tau}(X,Y) - \tau_\theta)^2 . \tag{4.29}$$

The inversion of Spearman's rho estimator can be expressed as:

$$\hat{\theta}_S = \arg\min_{\theta} (\hat{\rho}(X,Y) - \rho_\theta)^2 . \tag{4.30}$$

4.6 Simulation of Copula

One of the main applications of copula is in Monte Carlo simulations. In this section, we first use a bivariate copula to explain the simulation in a simple way and then extend the case into multivariate case.

Assume a bivariate copula which is set up, the task is to generate (u,v) of observations of [0,1] uniformly distributed random variables. C is the joint distribution of random variables U and V. To get this goal we can use conditional distribution

$$c_u(v) = P(V \le v \mid U = u) \tag{4.31}$$

for random variable V at a given observation u of random variable U. From probability theory, we know that

$$c_u(v) = P(V \le v \mid U = u) = \lim_{\Delta u \to 0} \frac{C(u+\Delta u, v) - C(u,v)}{\Delta u} = \frac{\partial C(u,v)}{\partial u}, \tag{4.32}$$

where $c_u(v)$ is the partial derivative of the copula, and it is a non-decreasing almost everywhere in [0,1].

Thus, we can generate the random number (u,v) in the following steps:

1. Generate two independent random variables u and t from uniform [0,1];
2. Set $v = C_u^{-1}(t)$, where C_u^{-1} is the inverse function of c_u;
3. The pair (u,v) is just the random numbers from the copula.

The idea is the same when extending the simulation to multivariate case. The goal in multivariate case is to simulate $(U_1,\ldots,U_d)$ from the copula $C(u_1,\ldots,u_d)$. The steps are expressed as follows:

1. Generate u_1 from U_1 which are [0,1] uniformly distributed;
2. Set

$$G_2(U_2 \mid U_1 = u_1) = P(U_2 \le u_2 \mid U_1 = u_1) = \frac{\partial C(u_1,u_2,1,\ldots,1)}{\partial u_1}, \tag{4.33}$$

   we put $u_2 = G_2^{-1}(u_2' \mid u_1)$, where $u_2' \sim U(0,1)$;

3. In general,

$$\begin{aligned} G_k(U_k \mid u_1,\ldots,u_{k-1}) &= P(U_k \le u_k \mid U_1 = u_1,\ldots,U_{k-1} = u_{k-1}) \\ &= \frac{\partial C(u_1,\ldots,u_k,1,\ldots,1)}{\partial u_1 \ldots \partial u_{k-1}} \Big/ \frac{\partial C(u_1,\ldots,u_{k-1},1,\ldots,1)}{\partial u_1 \ldots \partial u_{k-1}}, \end{aligned} \tag{4.34}$$

   we put $G_k^{-1}(U_k' \mid u_1,\ldots,u_{k-1})$, where $U_k' \in U(0,1)$.

5 Empirical Results

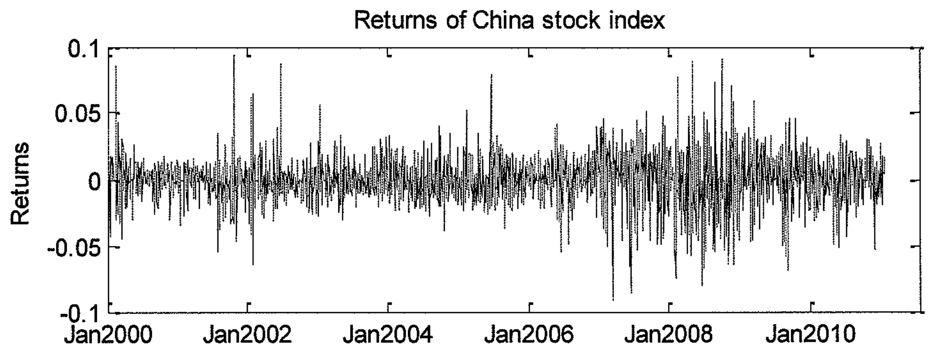
5.1 Data Analysis

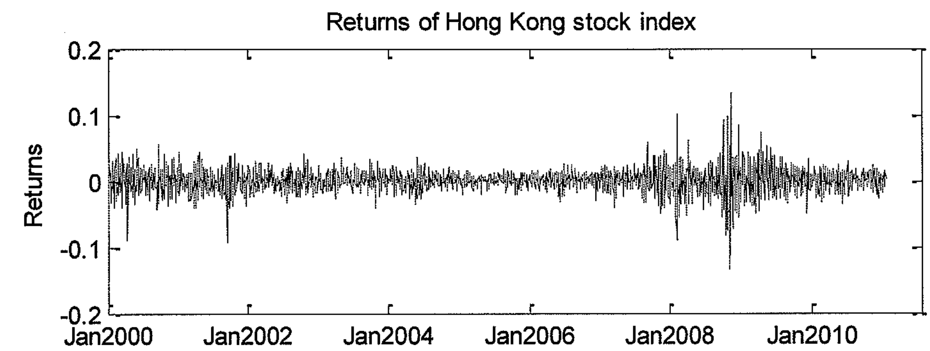
With a main interest in Asian stock markets, we choose five major Asian stock market indices to study in this chapter. They are Shanghai Stock Exchange Composite Index (China), Hong Kong Hang Seng Index (Hong Kong), Taiwan Capitalization Weighted Stock Index (Taiwan), Kuala Lumpur Composite Index (Malaysia), Nikkei Stock Average (Japan). The data (download from Datastream) cover the daily stock indices at the first 11 years of the 21st centaury, from January 3rd 2000 to December 31st 2010. The total observations during this period are 2870. Log-returns of the five indices are defined as

$$r_{i,t} = \ln(\frac{P_{i,t}}{P_{i,t-1}}),\ i = 1,...,5 \quad , \tag{5.1}$$

where $P_{i,t}$ is the i-th stock index at time t. In this case i=1:5 corresponds to the stock market of China, Hong Kong, Taiwan, Malaysia and Japan, respectively.

Figure 5.1 shows the log-returns of the five Asian stock indices and Figure 5.2 shows the histograms of the log-returns. Table 5.1 shows the results of some basic statistical summaries of the returns, whose concepts are briefly expressed in Appendix B.





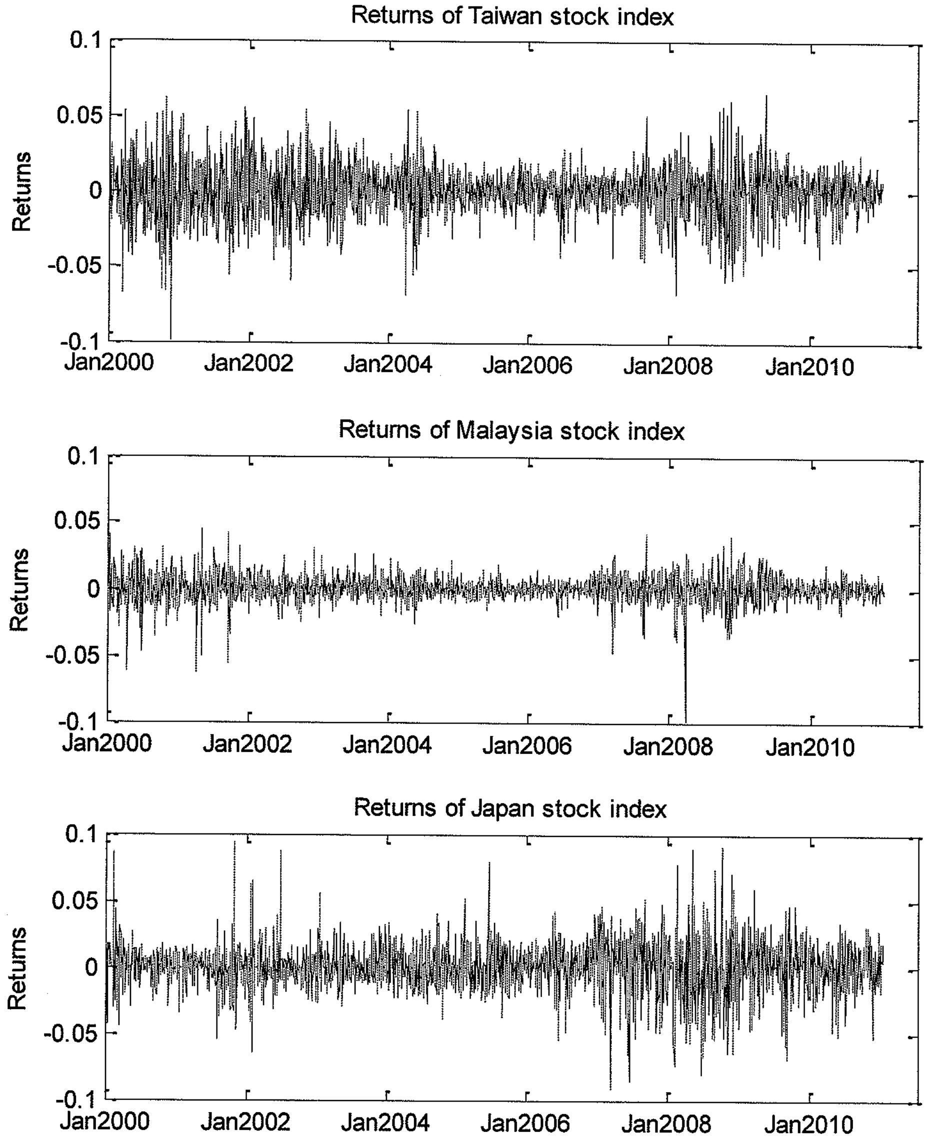


Figure 5.1 Log-returns of 5 Asian stock indices

Table 5.1 Basic statistics of the returns

	China	Hong Kong	Taiwan	Malaysia	Japan
Mean	0.000246	0.000107	0.0000210	0.000218	-0.000227
Max	0.0940	0.134	0.0652	0.0450	0.129
Min	-0.0926	-0.136	-0.0994	-0.0998	-0.100
Std. Dev	0.0165	0.0163	0.0156	0.00910	0.0141
Skewness	-0.1002	-0.0461	-0.215	-0.875	-0.248
Kurtosis	7.415	11.239	5.640	12.819	9.269
Jarque-Bera	2335.4	8118.4	857.846	11896	4728.5
J-B p-value	0.001	0.001	0.001	0.001	0.001

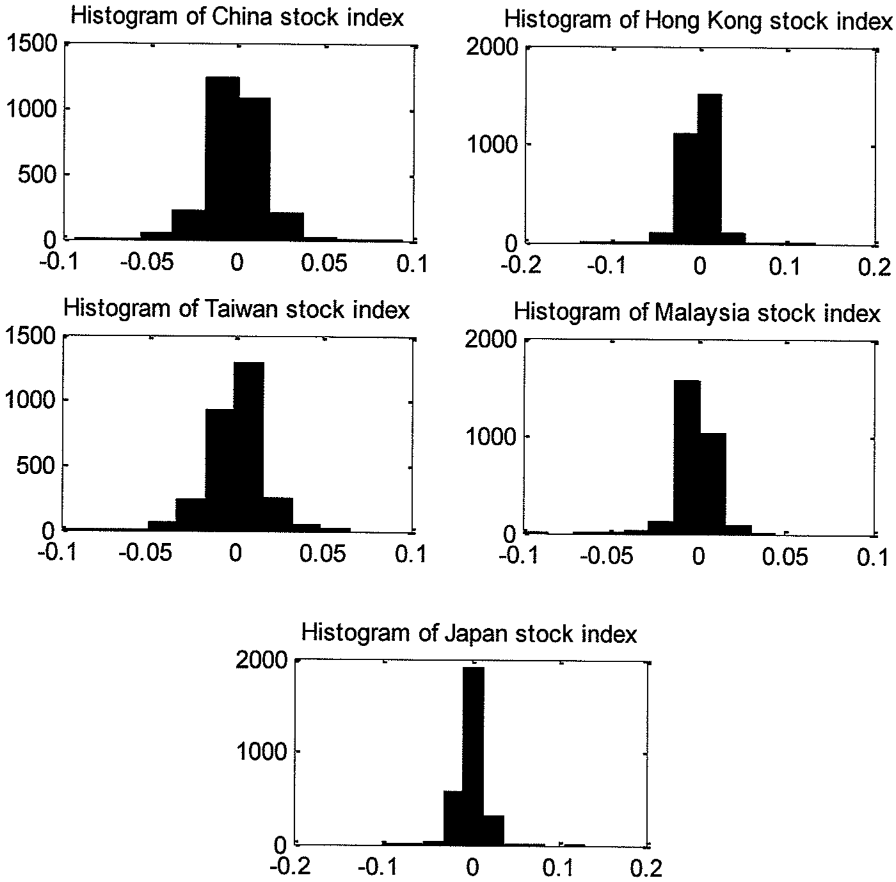


Figure 5.2 Histograms of the log-returns

5.1.1 Checking for Correlation in Return Series

Table 5.1 shows basic statistical summaries of the five stock indices, from which it can be seen the returns are not normal distributed. From Figure 5.1, it can be roughly seen that all the returns have volatility clustering (conditional heteroscedasticity). To prove the phenomena in statistics, autocorrelation function (ACF) and partial autocorrelation function (PACF) are used, whose definitions cover in Appendix C. Figure 5.3 – Figure 5.7 show the application of ACF and PACF to the five index returns. Besides, we usually use Ljung-Box test to check whether the autocorrelations with different lags are zero. Ljung-Box test statistic is defined as

$$Q_{LB} = T(T+2)\sum_{k=1}^{K}\frac{\tau_k^2}{T-k}, \tag{5.2}$$

where τ_k^2 is autocorrelation at lag k, and K is the number of autocorrelations has been tested. For a significant level a, the critical region for rejection is

$$Q_{LB}(K) > \chi^2_{1-\alpha}(K). \tag{5.3}$$

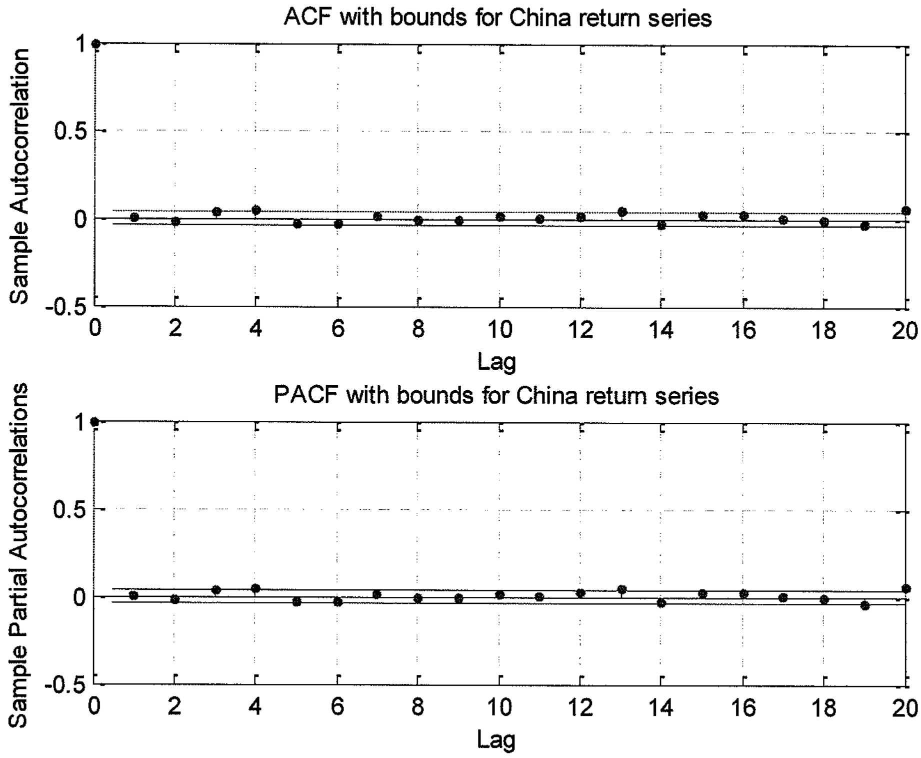


Figure 5.3 ACF and PACF of China returns

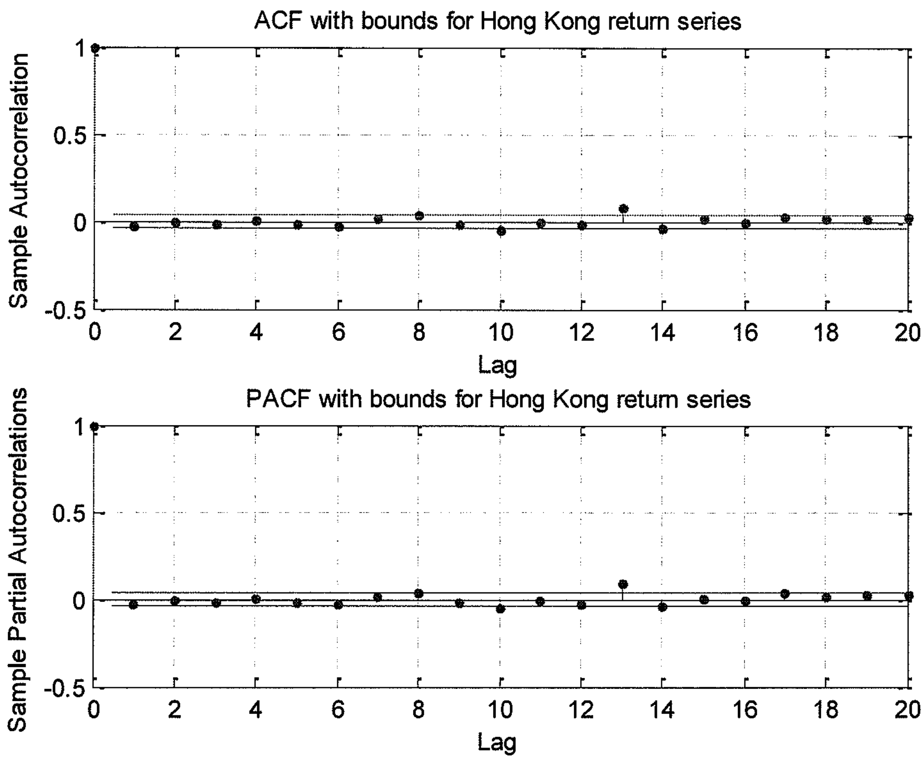


Figure 5.4 ACF and PACF of Hong Kong returns

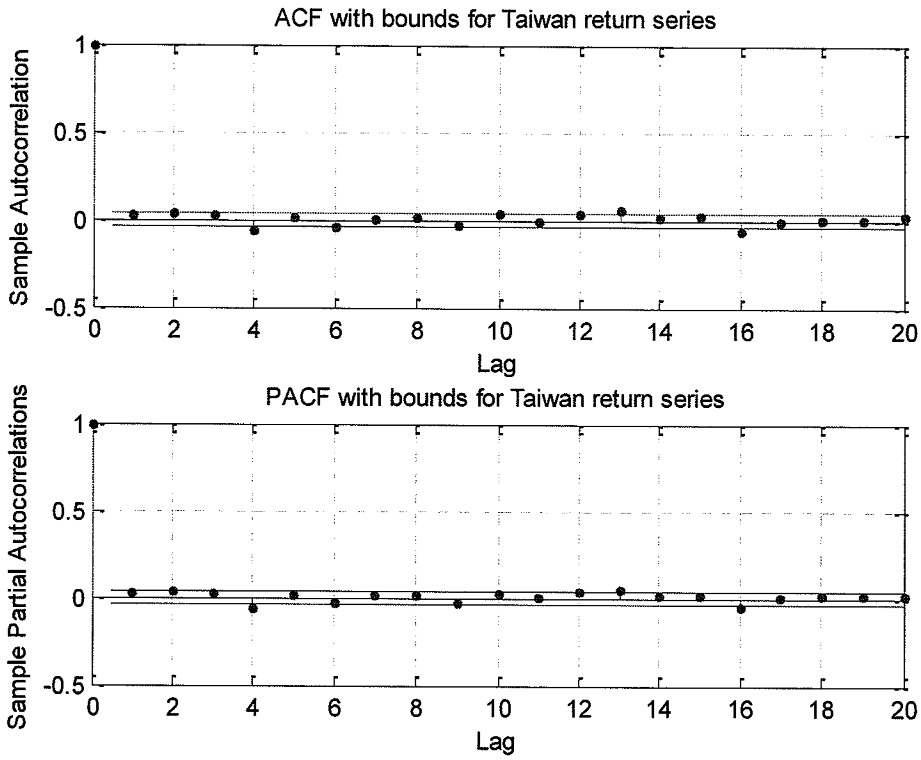


Figure 5.5 ACF and PACF of Taiwan returns

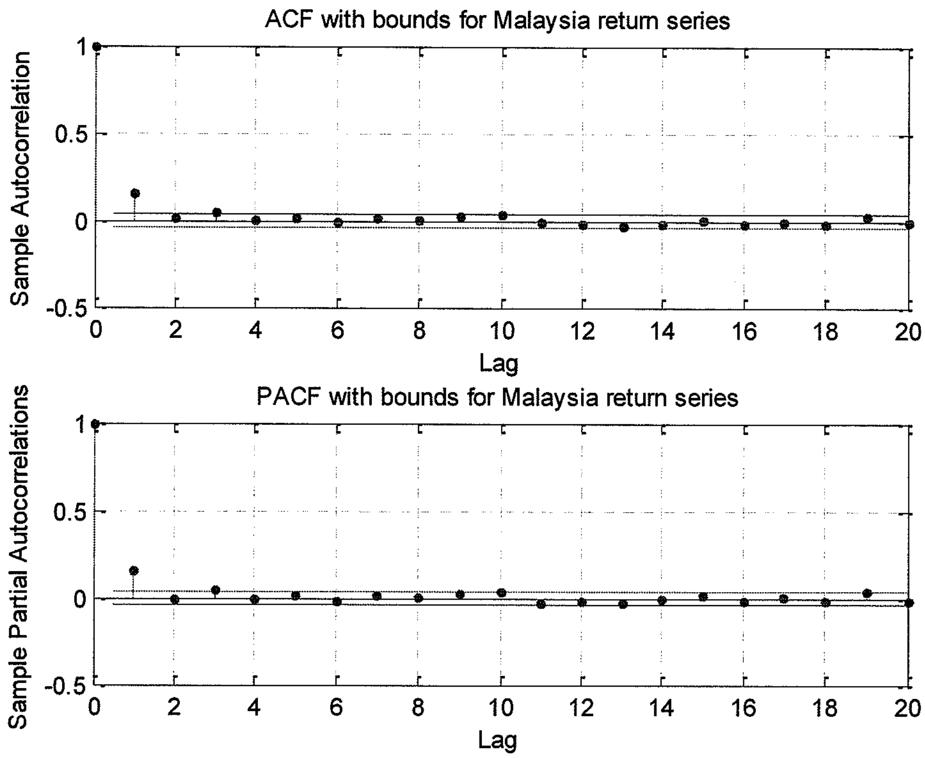


Figure 5.6 ACF and PACF of Malaysia returns

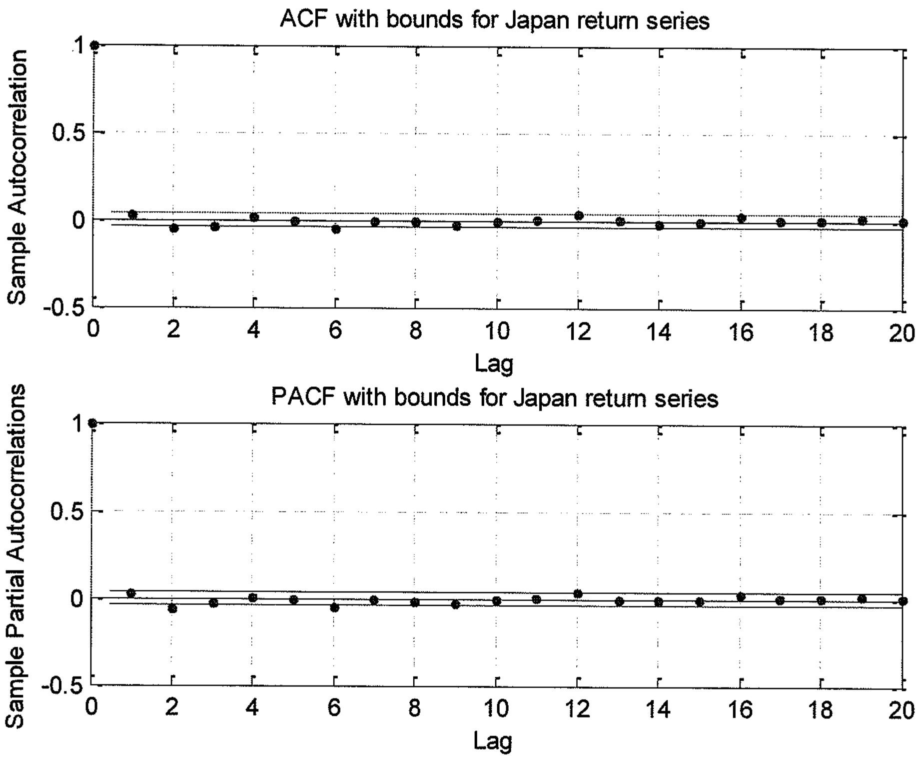


Figure 5.7 ACF and PACF of Japan returns

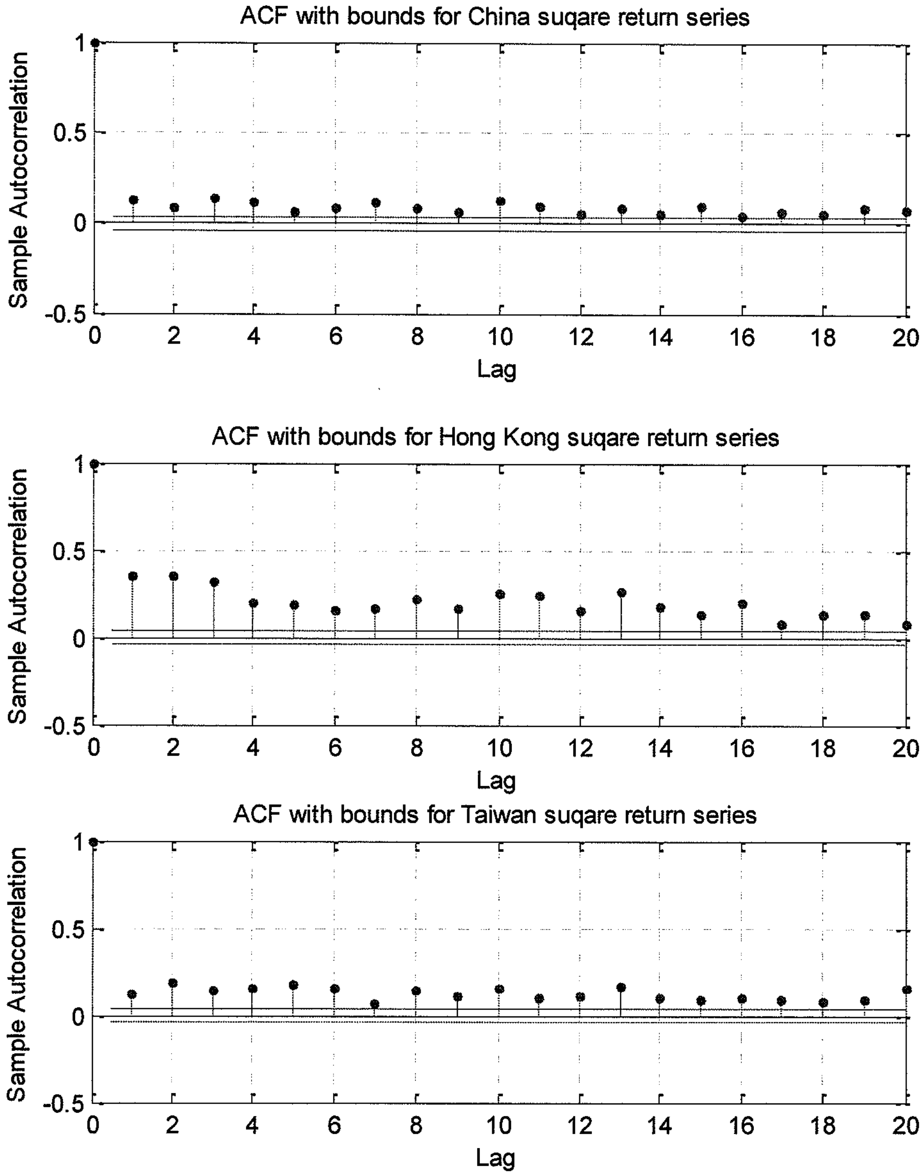
Table 5.2 P-value of Ljung-Box test for return series at 5% confidence level

	China	Hong Kong	Taiwan	Malaysia	Japan
1	0.9008	0.2011	0.0993	0.0000	0.1994
2	0.8286	0.3952	0.0193	0.0000	0.0064
3	0.1445	0.4819	0.0224	0.0000	0.0040
4	0.0136	0.6460	0.0005	0.0000	0.0083
5	0.0137	0.6312	0.0010	0.0000	0.0170
6	0.0133	0.5115	0.0005	0.0000	0.0030
7	0.0155	0.5195	0.0011	0.0000	0.0060
8	0.0261	0.3206	0.0018	0.0000	0.0102
9	0.0394	0.3568	0.0012	0.0000	0.0075
10	0.0502	0.1095	0.0007	0.0000	0.0123
11	0.0729	0.1469	0.0013	0.0000	0.0196
12	0.0734	0.1443	0.0003	0.0000	0.0078
13	0.0155	0.0002	0.0000	0.0000	0.0125
14	0.0153	0.0001	0.0000	0.0000	0.0174
15	0.0115	0.0001	0.0000	0.0000	0.0260

From the results of Ljung-Box test (Table 5.2) and the Figure 5.3 – Figure 5.7, we can see that all the other country return series show autocorrelations at some lags at 5% confidence level.

5.1.2 Checking for Correlation in Square Return Series

After checking the correlations of returns, their second-order moments (variance) should also be checked. From time series theory, we can get the variance series by squaring their corresponding return series. Figure 5.8 shows the autocorrelation functions of the variance series. We can see from the figures that the variance processes are largely correlated for each square return series. It indicates that there exists conditional heteroscedasticity of the index series. To better model these series with heteroscedasticity, GARCH models are applied to the variance processes.



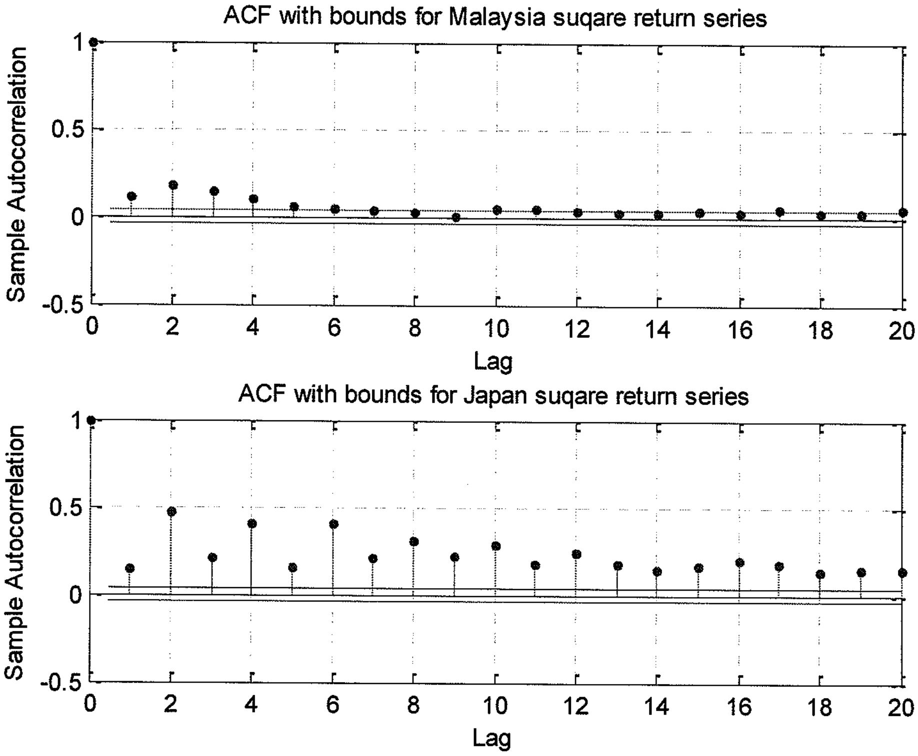


Figure 5.8 ACF of square return series

5.2 Marginal Modeling

Our analysis in Section 5.1 shows that both the mean and the variance of the returns are correlated. To better model these series, AR-GARCH models can be good choice. To be specific, AR (Autoregression) models are used to model mean equation of the return equation (Formula 5.4) and GARCH type models are used to model the variance equation (Formula 3.1 and Formula 3.2). Based on the theory discussed in Chapter 3, we choose AR(p)-GARCH(1,1) and AR(p)-GJR(1,1) models for the marginal distributions both with innovations of normal distribution, t distribution and skew t distribution. The parameter estimations are estimated by maximum likelihood method in Matlab 2010b, and the results are summarized in Table 5.3 - Table 5.7.

Mean equation:

$$Y_t = C + \sum_{i=1}^{p} \varphi_i Y_{t-i} + e_t . \tag{5.4}$$

Where Y_t is the return of the asset, φ_i is the parameter of the mean equation, and e_t is a white noise.

Table 5.3 China return series modeling

	AR(4)-GARCH(1,1)			AR(4)-GJR(1,1)		
	Normal	t	Skew t	Normal	t	Skew t
C	0.00031	0.0005	0.0002	0.00012	0.0004	0.0001
AR(1)	0.00869	0.0122	0.0106	0.01332	0.0168	0.016
AR(2)	-0.0171	0.0068	0.0015	-0.01340	0.0111	0.0074
AR(3)	0.03504	0.051	0.0495	0.03878	0.0556	0.0549
AR(4)	0.03424	0.0303	0.0297	0.03980	0.0345	0.0349
K	0	0	0	0	0	0
ARCH(1)	0.06706	0.0915	0.0931	0.04615	0.0635	0.0648
GARCH(1)	0.92493	0.9085	0.9069	0.92543	0.8966	0.8953
gamma					0.0744	0.0753
d.f.		3.671	3.6782		3.7062	3.7223
lambda			-0.0457			-0.046
LLF	8022.99	8183.581	8185.762	8029.68	8189.52	8191.73
AIC	-16030	-16349.16	-16351.52	-16043	-16359.0	-16361.4
BIC	-15982	-16295.50	-16291.90	-15996	-16299.4	-16295.8

Table 5.4 Hong Kong return series modeling

	AR(13)-GARCH(1,1)			AR(13)-GJR(1,1)		
	Normal	t	Skew t	Normal	t	Skew t
C	0.00052	0.0005	0.0005	0.00021	0.0004	0.0003
AR(1)	0.01788	0.007	0.0043	0.02450	0.0135	0.011
AR(2)	-0.01828	-0.0134	-0.0148	-0.0132	-0.0085	-0.0095
AR(3)	0.02324	0.0275	0.0259	0.03986	0.0366	0.0359
AR(4)	-0.0056	-0.0028	-0.0037	-0.0024	0.0007	0
AR(5)	-0.0151	-0.0159	-0.0163	-0.0113	-0.0121	-0.012
AR(6)	-0.03198	-0.0256	-0.026	-0.0296	-0.0246	-0.0243
AR(7)	0.01272	0.009	0.0085	0.01970	0.0134	0.0142
AR(8)	0.00519	0.0068	0.0066	0.01321	0.0113	0.0121
AR(9)	-0.00443	0.0016	0.0026	0.00261	0.0065	0.0083
AR(10)	-0.02051	-0.0175	-0.0176	-0.0144	-0.0129	-0.0122
AR(11)	-0.01790	-0.0156	-0.0159	-0.0115	-0.0125	-0.0122
AR(12)	-0.03166	-0.0217	-0.0215	-0.0235	-0.0197	-0.019
AR(13)	0.06814	0.0687	0.0689	0.070	0.0719	0.0727
K	0	0	0	0	0	0
ARCH(1)	0.06257	0.0549	0.055	0.02111	0.0198	0.019
GARCH(1)	0.93262	0.9451	0.945	0.93073	0.9392	0.9391
gamma					0.0679	0.0693
d.f.		6.4896	6.5648		7.4299	7.5807
lambda			-0.0262			-0.0305
LLF	8341.5	8384.547	8385.168	8362.98	8398.014	8398.86

AIC	-16649	-16733.09	-16732.33	-16692	-16757.72	-16758.02
BIC	-16548	-16625.77	-16619.0	-16591	-16644.74	-16638.47

Table 5.5 Taiwan return series modeling

	AR(4)-GARCH(1,1)			AR(4)-GJR(1,1)		
	Normal	t	Skew t	Normal	t	Skew t
C	0.00059	0.0007	0.0006	0.000328	0.0006	0.0005
AR(1)	0.0411	0.0241	0.0217	0.045295	0.0284	0.0265
AR(2)	0.00556	0.006	0.0047	0.011949	0.0112	0.0103
AR(3)	0.0190	0.0106	0.0094	0.020831	0.0143	0.013
AR(4)	-0.0195	-0.0254	-0.0269	-0.00851	-0.0205	-0.0214
K	0	0	0	0	0	0
ARCH(1)	0.06830	0.0539	0.054	0.028232	0.0218	0.0226
GARCH(1)	0.92603	0.9453	0.946	0.91973	0.943	0.9432
gamma					0.0583	0.0565
d.f.		5.7366	5.7738		6.1338	6.2849
lambda			-0.0293			-0.033
LLF	8223.87	8285.204	8285.95	8243.63	8296.36	8297.345
AIC	-16432	-16552.40	-16551.90	-16471	-16572.72	-16572.68
BIC	-16384	-16498.74	-16492.27	-16424	-16513.09	-16507.10

Table 5.6 Malaysia return series modeling

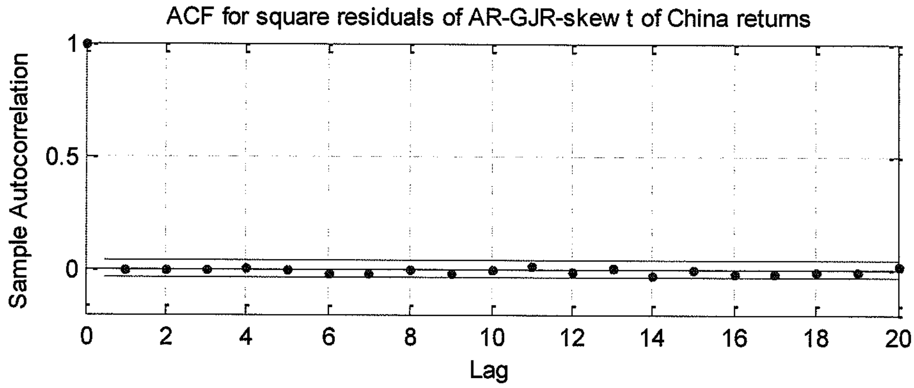
	AR(1)-GARCH(1,1)			AR(1)-GJR(1,1)		
	Normal	t	Skew t	Normal	t	Skew t
C	0.00046	0.0004	0.0004	0.00032	0.0003	0.0003
AR(1)	0.1676	0.1263	0.1270	0.17109	0.1281	0.1285
K	0	0	0	0	0	0
ARCH(1)	0.09770	0.1168	0.1174	0.06176	0.0861	0.0862
GARCH(1)	0.89829	0.8771	0.8767	0.89603	0.8711	0.8705
gamma					0.0687	0.0696
d.f.		4.5274	4.5108		4.6091	4.6009
lambda			0.0190			0.0209
LLF	9829.61	9966.535	9966.85	9842.52	9971.807	9972.189
AIC	-19649	-19921.07	-19919.69	-19675	-19929.61	-19928.37
BIC	-19619	-19885.29	-19877.96	-19645	-19887.87	-19880.68

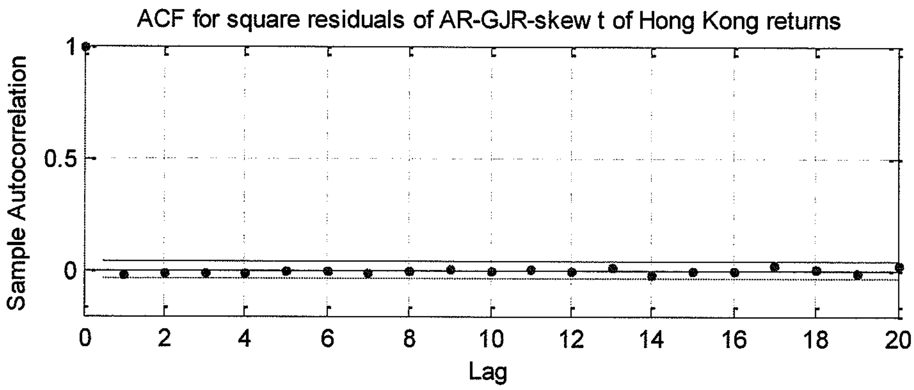
Table 5.7 Japan return series modeling

	AR(1)-GARCH(1,1)			AR(1)-GJR(1,1)		
	Normal	t	Skew t	Normal	t	Skew t
C	0.000248	0.0003	0.0002	-0.0001	0.0001	-0.0001

AR(1)	0.038912	0.0307	0.0247	0.04751	0.0381	0.0336
AR(2)	-0.01693	-0.0215	-0.0282	-0.0103	-0.0135	-0.0187
K	0	0	0	0	0	0
ARCH(1)	0.08586	0.0754	0.0752	0.02231	0.021	0.0215
GARCH(1)	0.9017	0.9161	0.9151	0.90103	0.9098	0.9083
gamma					0.0999	0.1005
d.f.		8.2710	8.8396		9.2819	9.8746
lambda			-0.0673			-0.0743
LLF	8522.26	8559.87	8563.498	8551.67	8579.721	8584.229
AIC	-17033	-17105.74	-17110.99	-17091	-17143.44	-17150.45
BIC	-16997	-17064.00	-17063.29	-17056	-17095.74	-17096.79

From the results, we can make a general conclusion for each margins that GJR models usually have a better fit of the observations, since they have the largest log-likelihood functions (LLF) and smallest AIC and BIC. Besides, the models with the innovations of t and skew t distributions have a better fit than the innovations of normal distribution. In general, among these models, AR(p)-GJR(1,1) with skew t distributed innovations models can best fit each margin. Then check the standardized residuals from these models to see whether the residuals have correlated or not.





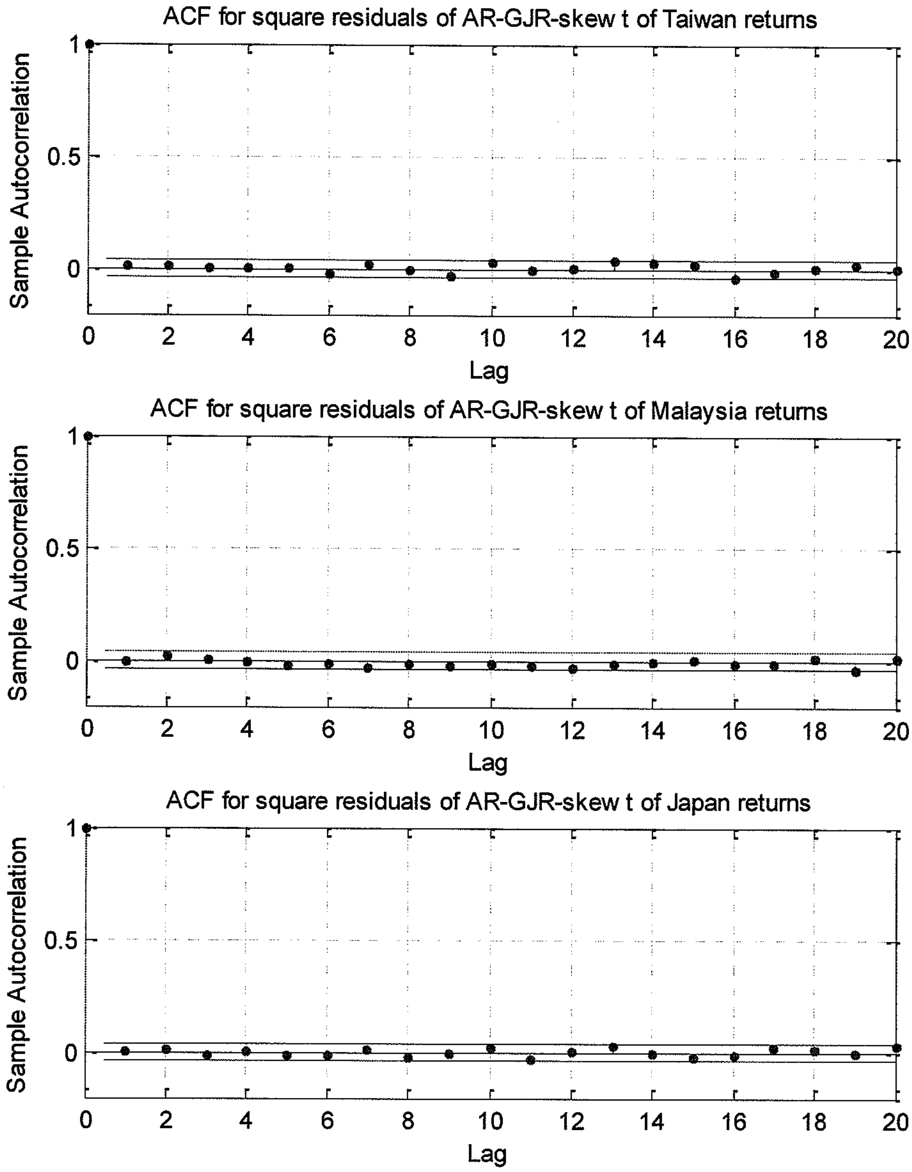


Figure 5.9 ACF of AR-GJR-skew t residuals

The standardized residuals show no correlation in Figure 5.9. Compared the ACF of the square standardized residuals with the ACF of the square returns in Figure 5.8, it shows that AR-GJR-skew t models sufficiently explain the heteroscedasticity effect in each marginal returns, thus we can say that the models fit the margins well. We can also use Ljung-Box test to examine the correlations of the standardized residuals with different lags. The results are in accordance with Figure 5.9 that the standardized residuals have no correlation.

5.3 Dependence Structure Modeling

Next step is to model the margins with copulas. Since the margins of the copulas have assumed in uniform distribution, the transformation of the marginal standardized residuals into uniform distribution should be done before modeling. In this section, only the residuals from the AR-GJR-skew t models are applied, since they fit the margins best. Then by using the estimation method presented in Section 4.5, the dependence models can be estimated straightforwardly.

Table 5.8 Dependence structure modeling

	Gaussian copula	t copula	Clayton copula	Gumbel copula	Frank copula
Parameter		15.6877	0.357218	1.209073	1.668704
LLF	1352.183	1284.297	864.7739	687.1049	746.3269
AIC	-2700.3655	-2566.5949	-1727.558	-1372.210	-1490.654
BIC	-2688.4414	-2560.6328	-1721.586	-1366.248	-1484.692
Margins filtered by AR(p)-GJR(1,1)-SkT					

The log likelihood, AIC and BIC can reflect whether the models are good or not in some way, but not that persuasive in this case because of the different structures of copulas. To check the fit of these copulas in value at risk estimation, backtesting method will be implemented in Section 5.5.

5.4 Empirical Estimation of Value at Risk

After the marginal and dependence structure modeling separately, value at risk (VaR) can be estimated based on the GARCH-copula models. The steps are described as follows:

1. Use Monte Carlo simulation to generate a large number N (choose 5000) based on the five estimated copulas (Table 5.8). So we can get a 5000×5 matrix R containing the simulated numbers. More specifically, $R=[U_1,U_2,U_3,U_4,U_5]$, where U_i ,$i=1,...5$ is a 5000×1 vector of marginal simulated numbers and $i=1,...5$ represents the stock market of China, Hong Kong, Taiwan, Malaysia and Japan, correspondingly.
2. Since the simulated numbers come from copula which are defined on [0,1] uniform distribution, we need to transform the numbers into the original scales of returns. That is $R_i = F_i^{-1}(U_i)$, $i=1,...,5$, which represents the simulated returns of each corresponding margins.
3. Since $r_{i,t}=\ln(P_{i,t}/P_{i,t-1})$, $i=1,...,5$, we can derive that $P_t = e^{R_t}P_{t-1}$. Where $P_{i,t}$ is the i-th stock index at time t. Choose the equal weight of each stock index in a portfolio, that is $w_i = 1/5 = 0.2$. Define the portfolio value as:

$$V_t = w_1 P_{1,t} + ... + w_5 P_{5,t} \quad ,$$

$$\Rightarrow V_{t+1,j} = w_1 P_{1,t} e^{R_{1,j}} + ... + w_5 P_{5,t} e^{R_{5,j}}, \; j = 1:5000 , \qquad (5.5)$$

where $R_{i,j}$ is the i-th index simulated numbers in step 2.

4. Calculate the portfolio profit and loss function at time t as,

$$L_{t,j} = V_{t+1,j} - V_{t,j} = w_1 P_{1,t}(e^{R_{1,j}} - 1) + \ldots + w_5 P_{5,t}(e^{R_{5,j}} - 1). \tag{5.6}$$

   The series $\{L_{t,j}\}$ can be either positive or negative: positive means profit and negative means loss. The distribution of series $\{L_{t,j}\}$ is the distribution of the profit and loss function of the portfolio at time $[t, t+1]$.

5. The one day $VaR_\alpha(t)$ at time t is just the given quantile α (0.01, 0.05 and 0.10) of the distribution of series $\{L_{t,j}\}$, $j = 1:5000$.

Table 5.9 gives an example of the portfolio one day VaR estimated by different copulas at different confidence level on the last sample date (December 31st 2010).

Table 5.9 One day VaR on December 31st 2010

	Gaussian copula	T copula	Clayton copula	Gumbel copula	Frank copula
VaR_{99}	258.9044	267.4211	282.5482	229.1573	226.1687
VaR_{95}	159.2455	155.0506	156.8051	140.2462	136.6899
VaR_{90}	114.0213	109.3902	106.2760	101.1945	100.9601

5.5 Backtesting of VaR

Backtesting is an important issue in estimating VaR. Backtesting method is used to both test the reasonability of VaR estimation from the models, and compare the goodness-of-fit for the models. We choose a moving-window of 870 samples out of the total 2870 observations to do the in-sample test. It means there are 2000 observed profit and loss series, and 2000 one-day VaRs with confidence level 99%, 95% and 90%. In this section, two main backtesting methods are used.

5.5.1 Kupiec's Test

The most widely used backtesting method is the frequency of tail loss proposed by Kupiec (1995), which is based on binomial theory. Kupiec's test tests the difference between the observed and expected number of VaR exceedances of the portfolio loss. In empirical analysis, we observe N losses in excess of VaR out of T samples, thus N/T is the rate of excessive loss. The Kupiec's test answers the question whether N/T is statistically significant different from the given confidence level p of VaR.

Following the binomial theory, the probability of the empirical N failures (loss exceeds the estimated VaR) out of T samples is $(1-p)^{T-N} p^N$. Under the null hypothesis the estimated model is correct and the observed frequency of failures is consistent with the expected frequency, the likelihood ratio test statistic is given as:

$$LR_K = -2\ln[(1-p)^{T-N} p^N] + 2\ln[((1-N/T)^{T-N}(N/T)^N)], \qquad (5.6)$$

which is distributed as $\chi^2(1)$ under null hypothesis. P is the probability of exceedances under null hypothesis, N is the exceedances and T is the total numbers.

If the statistical p-value is below the significance level (1%, 5% or 10%), we reject the model and conclude that the model is not good. We can conduct this test for loss exceedances to determine how well the model predicts the frequency of loss excess VaR. But as the author mentions in the paper, the test presents low power in small samples.

5.5.2 Christoffersen's Test

Christoffersen's test is a conditional test which was made by Christoffersen (1998). The test develops a likelihood ratio test to the joint assumption of unconditional coverage and independence of failures. The test also suggests a procedure to evaluate the precision of predictions in confidence intervals, which tries to capture the VaR estimative conditionality. For example, if volatilities are low in some periods and high in other periods, VaR forecast should respond to the clustering phenomena. The Christoffersen's test enables to separate clustering effects from the assumption effects.

The null hypothesis of the unconditional coverage test is that $\mathrm{I_t} \sim \mathrm{i.i.d.\,Bernoulli(p)}$ against the alternative that $\mathrm{I_t} \sim \mathrm{i.i.d.\,Bernoulli}(\pi)$ where $\mathrm{I_t}$ is the hit sequence of VaR violations, p is the confidence level and π is equal to the ratio between the number of observations and the size of the sample. So the null hypothesis of unconditional coverage is $\pi = p$. Then the test assumes that the hits are independent and the hit series follow a first order Markov process, with switching probability matrix:

$$\begin{bmatrix} 1-\pi_{01} & \pi_{01} \\ 1-\pi_{11} & \pi_{11} \end{bmatrix},$$

where π_{ij} is the probability of an i on day t-1 followed by a j on day t, $i, j = 0,1$. So the null hypothesis of independence is $\pi_{01} = \pi_{11}$.

By combing the two null hypothesis, the final null hypothesis of conditional coverage is $\pi_{01} = \pi_{11} = p$. And the statistics of the test is given as:

$$LR_C = -2\ln[(1-p)^{T-N} p^N] + 2\ln[(1-\pi_{01})^{n_{00}} \pi_{01}^{n_{00}} (1-\pi_{11})^{n_{10}} \pi_{11}^{n_{11}}], \qquad (5.7)$$

where n_{ij} is the number of observations with value i followed by j for $i, j = 0,1$, and $\pi_{ij} = n_{ij} / \sum_j n_{ij}$ are the corresponding probabilities. Under $H_0 : \pi_{01} = \pi_{11} = p$, the test statistics are distributed as $\chi^2(2)$.

The advantage of Christoffersen's test is that it evaluates the dynamic behavior of exceedance process, apart from their frequency estimative. It can reject a VaR model that generates either too many or too few clustered violations, although it needs more observations in order to be accurate.

5.5.3 Backtesting Results

Based on the theoretical analysis of backtesting, the empirical results of the five estimated copulas for the portfolio are summarized in Table 5.10. Then the figures of estimated VaRs at different confidence levels are plotted in Figure 5.10 - Figure 5.12.

Table 5.10 Backtesting results for the five copulas

Gaussian copula	VaR_{99}	VaR_{95}	VaR_{90}
Number of exceptions	31	99	165
Kupiec's p-value	0.0222	0.9182	0.0073
Christoffersen's p-value	0.2443	0.0984	0.0372
t copula	VaR_{99}	VaR_{95}	VaR_{90}
Number of exceptions	28	103	177
Kupiec's p-value	0.0900	0.7593	0.0810
Christoffersen's p-value	0.1744	0.0691	0.0148
Clayton copula	VaR_{99}	VaR_{95}	VaR_{90}
Number of exceptions	25	99	186
Kupiec's p-value	0.2794	0.9182	0.2916
Christoffersen's p-value	0.1573	0.0984	0.1123
Gumbel copula	VaR_{99}	VaR_{95}	VaR_{90}
Number of exceptions	50	119	198
Kupiec's p-value	0	0.0581	0.8813
Christoffersen's p-value	0.0308	0.0998	0.00813
Frank copula	VaR_{99}	VaR_{95}	VaR_{90}
Number of exceptions	50	124	197
Kupiec's p-value	0	0.0174	0.8227
Christoffersen's p-value	0.0308	0.0875	0.0015

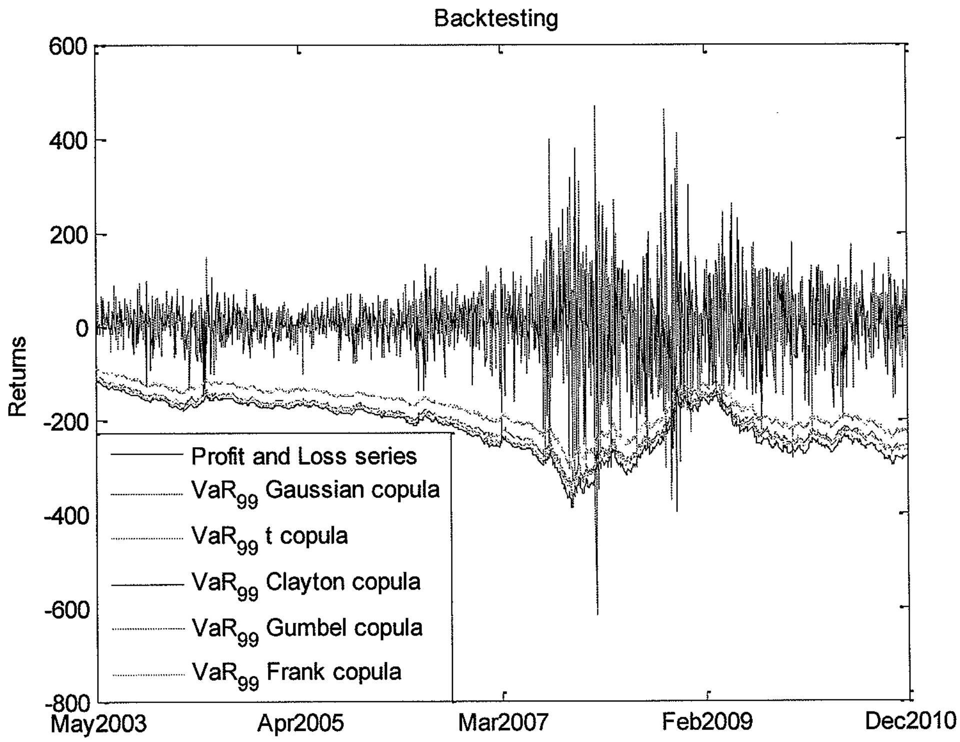


Figure 5.10 VaR_{99} based on five copulas V.S. Real profit and loss

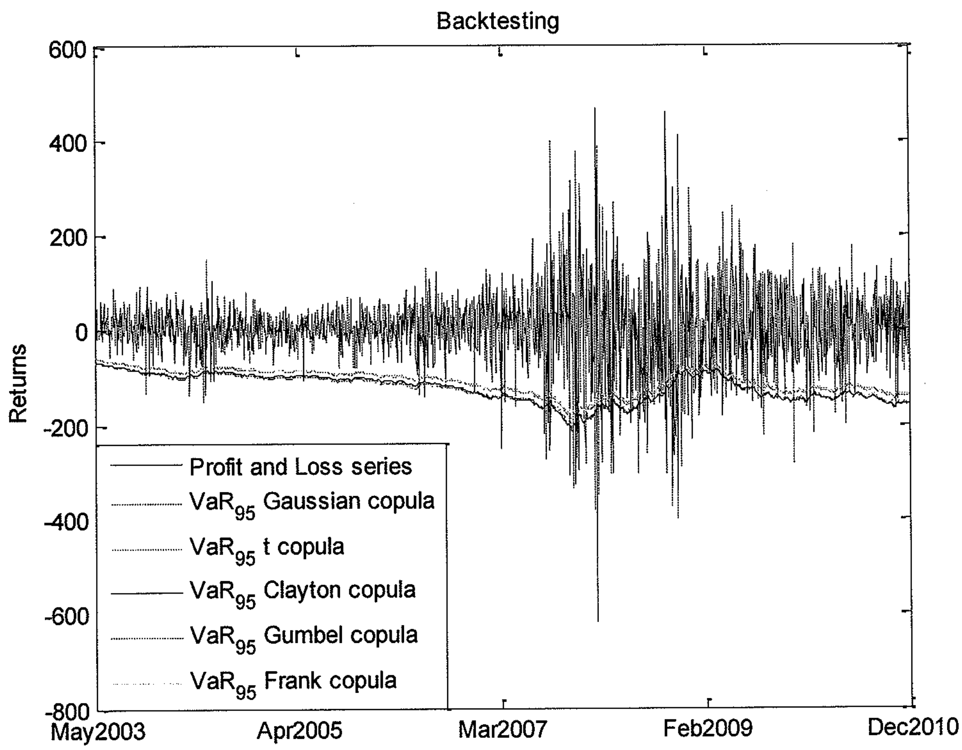


Figure 5.11 VaR_{95} based on five copulas V.S. Real profit and loss

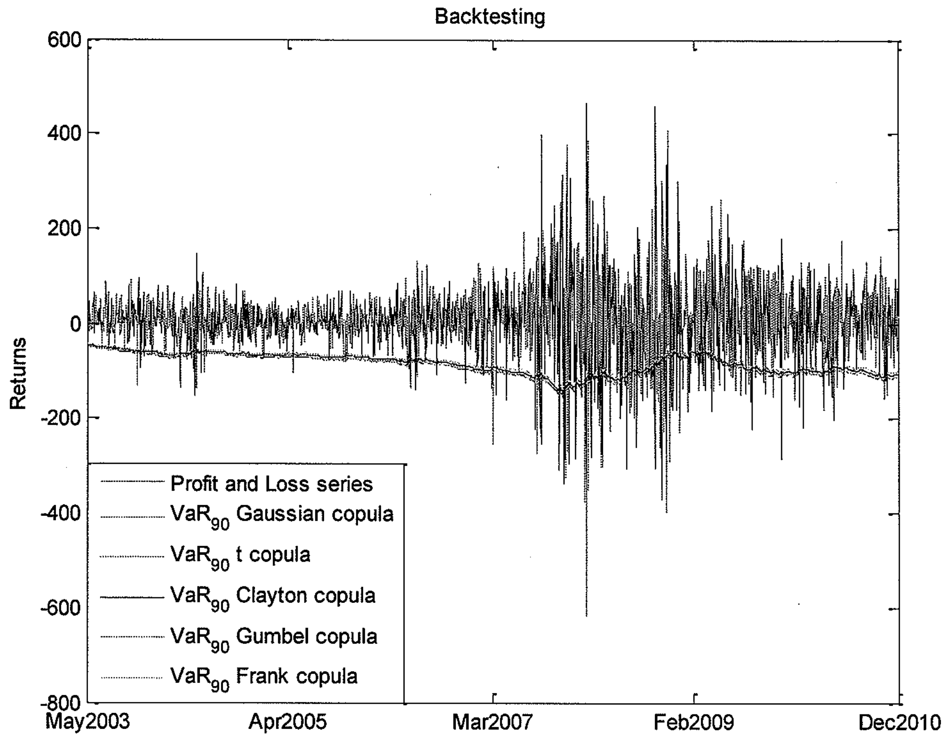


Figure 5.12 VaR_{90} based on five copulas V.S. Real profit and loss

From Table 5.10, we can see that most copulas have a good performance in backtesting of VaR. Most of the exceptions of the simulated VaRs are near the expected exceptions with the given confidence level. If the empirical exceptions are much larger than the expected ones, the models are said to have a poor performance in prediction. If the empirical exceptions are much smaller than the expected ones, the models are failed to capture the information of historical observations. This is what Kupiec's test does. Furthermore, Christoffersen's test judges the dynamic behavior of exceedances, which can reject the model that has either too many or too few clustered violations.

To be specific, for one-day VaR_{99}, Gaussian copula, t copula and Clayton copula have a good performance, while Gumbel copula and Frank copula fail to pass the Kupiec's test because of too many exceptions (both have 50). Among the three good copulas, Clayton copula is the best because of its p-value of the test and the least exceedances. For one-day VaR_{95}, all the copulas have a goodness-of-fit except Frank copula. Gaussian copula and Clayton copula have a better performance in the test, followed by t copula. For one-day VaR_{90}, the three Archimedean copulas are good at estimating exceptions than elliptical copulas. But only Clayton copula has a good performance in Christoffersen's test.

6 Summary and Conclusion

Value at risk (VaR) is an important issue in financial risk management. To estimate VaR, many models have been developed. Traditional VaR models can be applied into practice easily, but have been proved to be problematic when the financial series is not under specified distributions. As a result, some advanced mathematical and statistical models and techniques have been developed. In this paper, GARCH-copula framework with Monte Carlo simulation is used to estimate VaR of a portfolio. The portfolio consists of five main Asian stock market indices in the first 11 years of the 21^{st} centaury.

We first do the basic analysis of the five stock index return series, and find all of them are not normally distributed. There are also some correlations in each return. More importantly, there exists a strong volatility clustering as most financial series. So GARCH type models are chosen to filter these returns. GARCH models have been widely used since 1990s because they can capture the volatility clustering, which is also called heteroscedasticity. GARCH(1,1) model is proved adequate by many papers to evaluate heteroscedasticity in stock returns. Besides, GARCH models always have an assumption of normal distributed innovations, while t distribution and skew t distribution of the innovations are used to improve the ability to fit financial returns. But there exists a problem of the original GARCH model, which has a symmetric variance equation. It ignores the sign of disturbance, which wrongly assumes the positive and negative information have the same impact on the volatility. To overcome this problem, asymmetric GARCH models, such as GJR model are developed. Based on these considerations, we finally choose AR(p)-GARCH(1,1) and AR(p)-GJR(1,1) both with normal, t and skew t distributed innovations to model the five margins. To be specific, AR models are for mean specification, while GARCH models and GJR models are for volatility modeling. By comparing the log-likelihood functions and AIC, we choose the best models fitting the margins. The empirical results are consistent with the theory, where GJR models with skew t innovations have the largest log-likelihood and smallest AIC. Finally, AR(p)-GJR(1,1) with skew t models are used to filter the five margins, and can get their standardized residuals.

The next step is to model the dependency structure of the portfolio. We use copula to model the dependency, because of its flexibility in modeling the multivariate distribution. It allows to model the appropriate margins separately and then use an appropriate copula to link these margins together. The idea is straightforward and can be used in high dimensions easily, which is very helpful in financial applications especially when the investors have many assets in one portfolio. Because of these advantages, copula has been applied into financial modeling more and more in the last 15 years. In this paper, we choose five widely used copulas, which are two elliptical copulas - Gaussian copula and t copula, and three Archimedean copulas - Clayton copula, Gumbel copula and Frank copula. The input of copulas are the filtered uniform distributed residuals transformed from standardized residuals of AR(p)-GJR(1,1)-skew t models. Then based on the copula estimation method (IFM are mainly used), the parameters of copulas can be estimated directly in computer. Till this step, the GARCH-copula models have been built successfully.

In the last step, we use the built GARCH-copula models to estimate VaRs based on Monte Carlo simulation. To generate a large number (5000 in this paper) from the five estimated copulas, VaRs are just the quantile of the profit and loss series at a given time point. One thing has to be mentioned that the VaRs in this paper are calculated at a long position and with a time horizon of one day. Another important thing is the backtesting. We use in-sample test by choosing a moving window of 870 observations (total 2870 observations) to estimate VaRs at different confidence levels, comparing to the real loss. Thus, we calculate 2000 real losses and 2000 VaRs at three given confidence levels. Through Kupiec's test and Christoffersen's test, we get the p-value corresponding to the different copulas and can use them to accept or reject the null hypothesis. Finally, we find that Gaussian copula, t copula and Clayton copula have a good performance for VaR_{99}; Gaussian copula, t copula, Clayton copula and Gumbel copula have a goodness-of-fit for VaR_{95}; Clayton copula, Gumbel copula and Frank copula are good at estimating the exceptions of VaR_{90}, but only Clayton copula has a good performance in Christoffersen's test. In general, for this portfolio Clayton copula is the best for all the three VaRs, and the simple Gaussian copula and t copula are enough to use for estimating VaRs of large confidence levels (95% and 99%).

Even though this paper proves that GARCH-copula framework can be successfully used to estimate VaR in stock markets, some further works can be done to perfect the framework. First of all, the extreme value theory (EVT) can be added into this framework. We can model the margins by using GARCH models to filter the volatility of margins and EVT to model the tails of the marginal residuals. And then we use a copula to link the margins together. The additional EVT part can help to improve the ability of the framework to model the tails and capture the extreme events. Secondly, higher dimensions can be applied in a portfolio to check whether the framework can have a good performance or not. Finally, we can extend the static copulas to time-varying copulas.

Appendix

A Skew T Distribution

Hansen firstly introduced skew t distribution for a GARCH model in 1994. He expressed a generalized student t distribution which allows the distribution to be asymmetric while maintaining the assumption of zero mean and unit variance. Its density function is defined as:

$$f(z;\eta,\lambda)=\begin{cases} bc\left(1+\frac{1}{\eta-2}\left(\frac{bz+a}{1-\lambda}\right)^2\right)^{-\frac{\eta+1}{2}} & if\ z<-\frac{a}{b} \\ bc\left(1+\frac{1}{\eta-2}\left(\frac{bz+a}{1+\lambda}\right)^2\right)^{-\frac{\eta+1}{2}} & if\ z\geq-\frac{a}{b} \end{cases}$$

where $a\equiv 4\lambda\eta\frac{\eta-2}{\eta-1}$, $b^2\equiv 1+3\lambda^2-a^2$, and $c\equiv\frac{\Gamma(\frac{\eta+1}{2})}{\sqrt{\pi(\eta-2)}\Gamma(\frac{\eta}{2})}$.

η and λ represent the degree of freedom parameter and the asymmetry parameter, respectively.

B Jarque-Bera Test

Jarque–Bera test is a goodness-of-fit test of whether sample data have the skewness and kurtosis matching a normal distribution. The test is named after Carlos Jarque and Anil Bera. The test statistic JB is defined as

$$JB=\frac{T}{6}\left(S^2+\frac{(K-3)^2}{4}\right),$$

where T is the number of observations. S and K are the sample skewness and kurtosis, respectively. They are defined in the form:

$$S=\frac{\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^3}{\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2\right)^{3/2}},$$

$$K=\frac{\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^4}{\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2\right)^2}.$$

Skewness and kurtosis both describe the shape of the time series distribution. To be specific, skewness measures the asymmetry of the distribution around return mean and kurtosis measures the peak of the distribution of the return time series. If the sample data come from a normal distribution, the JB statistic asymptotically has a chi-squared distribution with two degrees of freedom. The null hypothesis is a joint hypothesis of the skewness being zero and the kurtosis being three.

C ACF and PACF

Autocorrelation function (ACF) describes the correlation between values of the process at different time points. Let X_t be a stochastic process, s and t are some time points that s< t. Suppose μ_s and μ_t are the corresponding mean while σ_s and σ_t are the corresponding variances, the autocorrelation function can be expressed as:

$$R(s,t)=\frac{E[(X_t-\mu_t)(X_s-\mu_s)]}{\sigma_t\sigma_s}.$$

Partial autocorrelation function (PACF) is a conditional correlation, which is defined as:

$$\phi_{kk}=Cor[X_t,X_{t+k}\mid X_{t+1},\dots,X_{t+k-1}].$$

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