Master Essay I

AF Bostäder's house allocation problem

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Abstract

This paper evaluates the practicality and efficiency of different solutions to the house allocation problem faced by the company providing housing for students at Lund University. Both static and dynamic mechanisms are considered. A series of simulations show that the problem could be solved more efficiently by altering the booking period length of the currently used mechanism, and even more efficiently by adopting some variant of the top trading cycles mechanism developed by Abdulkadiroglu and Sönmez (1999).

Keywords: house allocation; matching; mechanism design; top trading cycles; serial dictatorship; experiment; AF Bostäder.

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1 Introduction

Imagine a man finding a collection of historical artifacts under a big rock. The man has no interest in archaeology and decides to sell them. But whom should he sell the artifacts? Under most circumstances, this is an easy question to answer. If his motive is profit maximization, he will simply set the prices in such a way that each artifact is only demanded by one buyer. Such prices can easily be found by, for example, holding an auction. Supply then equals demand and the market clears. The person willing to pay the most for an artifact gets to buy it. This particular process of allocating goods among buyers is a type of *price mechanism*. As long as there is a functional market to trade the artifacts on, there is normally no need to consider alternatives to price mechanisms. However, consider a situation in which the market does not clear. Perhaps the government sets a price ceiling on historical artifacts, causing demand to exceed supply. Then the question of whom to sell the artifacts becomes more complicated. They can not simply be sold to the highest bidder, because there is a limit to how high the bids can get. A new kind of procedure to select buyers is needed. If the man does not care who gets the artifacts and he only wants his money as soon as possible, he might select a procedure where the artifacts are sold on a first-come, first-served basis, or one where buyers are picked randomly. Even if he is allowed to use a price mechanism to select buyers, he might still choose not to if he is driven by motives other than profit maximization. Perhaps he recognizes the historical significance of the artifacts and does not wish to sell them to a buyer who would melt them and extract the valuable metals they are made of. He will then choose to select buyers using some alternative procedure. Which procedure he should choose depends on his intentions. If he is concerned with which use the artifacts will be put to, he might construct some procedure that separates the buyers who wish to study them or put them on public display from the buyers who would use them as door stops and subsequently selects a number of buyers of the former type. Or if he wants to sell the artifacts to the buyers who would derive the most utility from them, he might attempt to invent a procedure that ranks all buyers accordingly and selects the top ranking buyers.

Precisely how to create such a procedure, called a *mechanism*, that satisfies the demands of the artifact seller falls under a subfield of game theory called *mechanism design*. Traditional game theory is concerned with how rational agents act within an inherited framework. Mechanism design, on the other hand, is about designing a framework that will yield some desired results. Historical artifacts are seldomly found under rocks, but the problem of designing mechanisms when a price mechanism is either inapplicable or impractical often arises in many real life situations. It would be highly impractical for a big organization to let office space and conference rooms be traded on an internal market. Instead, it needs some kind of mechanism deciding who is assigned what office or conference room at any point in time. It would be possible to trade human organs on markets, but this practice is illegal in many countries, making price mechanisms inapplicable. Some other way to determine who should be given what organ is then required. Both of these are problems where a one-to-one *matching* between agents and indivisible objects must be selected. Such matchings are found using *matching mechanisms*, which is the type of mechanism this paper will focus on.

AF Bostäder (AFB) is a housing company facing a similar problem, tasked with the responsibility of providing housing for students at Lund University. AFB currently owns about 5800 apartments and corridor rooms (AF Bostäder, n.d.), while Lund University had 47 000 active students in 2011 (Lund University, 2012). As is often the case for student housing, profit maxmimization is not the only motive of the supplier. The apartments and corridor rooms also have to be affordable to students living on a very restricted budget. It is the policy of AFB to set rents such that each housing district is self-sufficient, without making additional profits. (Theofanous, 2012) Consequently, the market for student housing does not clear. The city of Lund is also experiencing a general housing shortage (Boverket, 2012). AFB is thus faced with the problem of how to distribute apartments and corridor rooms among students. If each agent is already assigned an accommodation and accommodations are traded with monetary compensation, it is called a *housing market* (Shapley and Scarf, 1974). If all accommodations start out unassigned and there is no market to trade them on, it is called a *house allocation problem*. (Hylland and Zeckhauser, 1979). If some agents are initially assigned accommodations and some agents are not, it is called a *house allocation problem with existing tenants* (Abdulkadiroglu and Sönmez, 1999). A house allocation problem with existing tenants, such as the one facing AFB, can be solved using many different kinds of mechanisms. For example, the housing company can introduce a queue, distribute apartments or corridor rooms randomly or assign them to the earliest applicants. When designing and choosing among different mechanisms, it is common to investigate which *properties* they satisfy. Any true statement describing a mechanism is a property of that mechanism. A property could be that the mechanism always favors applicants whose last names begin with the letter E, or that it reallocates 10 apartments every week. Assuming that the housing company is interested in having satisfied tenants, there are some properties of mechanisms that are generally considered desirable for house allocation problems. If a mechanism is *individually rational*, no one can be made worse off by the mechanism's reallocations, ensuring that there is no incentive not to participate in the allocation process. If it is *Pareto consistent*, then the selected matching can not be altered to make one agent better off without making some other agent worse off. Additionally, a mechanism is *strategy-proof* if it ensures that no agent can be made better off by reporting false preferences over the different corridor rooms and apartments. This paper will consider several different mechanisms as solutions to AFB's house allocation problem. Their properties will be examined and the efficiency of some mechanisms will be estimated by looking at the matchings produced by them in a series of simulations carried out on a fictional population with randomized preferences.

1.1 Purpose

The purpose of this paper is to describe and evaluate different mechanisms as solutions to AF Bostäder's house allocation problem.

1.2 Method

Several different mechanisms are introduced and an examination of which properties they satisfy is carried out. Then three of AF Bostäder's housing districts are chosen and a fictive population is created with randomized preferences. The population and the apartments are subsequently processed through different mechanisms, producing several different matchings. Finally, the efficiency of the resulting matchings are compared using some different measures.

1.3 Related literature

One of the earliest studies of matching problems is Gale and Shapley's (1962) examination of marriage markets and college admission problems. Shapley and Scarf (1974) analysed an allocation problem where a set of indivisible objects are privately owned by a set of agents and introduced the idea of a top trading cycle. Allocation problems where private ownership is not assumed have been examined extensively in later years, with applications in, for example, kidney exchange programmes (Roth, Sönmez and Ünver, 2004) and school choice (Abdulkadiroglu and Sönmez, 2003; Chen and Sönmez, 2006; Erdil and Ergin, 2008). The literature dealing with house allocation problems from a mechanism design perspective is also fairly large, with most of the focus placed on house allocation problems for college students. A large part of the literature is entirely theoretical. (Abdulkadiroglu and Sönmez, 1999; Andersson and Andersson, 2009; Sönmez and Ünver, 2005; Sönmez and Ünver, 2010). There have, however, been some attempts at experimental and empirical papers (Chen and Sönmez, 2003; Guillen and Kesten, 2008). For house allocation problems, dynamic mechanisms have received less attention than static mechanisms. An exception to this is Bloch and Cantala (2011) who considered what they called a dynamic assignment problem, where the set of agents varies over time, in the sense that one agent leaves the set and a new one enters in every period. Kurino (2009) examined the house allocation mechanisms called serial dictatorship and top trading cycles in a dynamic context. Another paper examining house allocation mechanism in a dynamic setting was written by Abdulkadirogly and Loertscher (2007). Dynamic mechanisms have also been examined in the context of other applications, such as kidney exchange programmes (Ünver, 2009).

1.4 Overview

In section 2, the model and the theoretical framework used to analyze different mechanisms is presented, some properties mechanisms can satisfy are defined and examples of different mechanisms are given. Section 3 describes the process by which AF Bostäder currently allocates corridor rooms and apartments among students. In section 4, AF Bostäder's current mechanism is described within the theoretical framework established in section 2, its properties are examined and the simulation processes for the different mechanisms are explained. In section 5, the efficiency measures used to compare the results of the simulations are described and the outcomes are presented. Section 6 provides some concluding remarks on the results.

2 Theory

This section introduces the model that will serve as a theoretical framework for the remainder of the paper. In addition, the concepts and properties that will be used to evaluate AF Bostäder's mechanism and its alternatives are defined. The last subsection gives examples of different mechanisms that could serve as solutions to the AFB's house allocation problem.

2.1 Model

Consider a set of all potential agents $\mathscr{I}$ that could constitute a population and a set of all potential housing units $\mathscr{H}$ that could exist in a sociey. Following Abdulkadiroglu and Sönmez (1999) and Sönmez and Ünver (2010), a *house allocation problem with existing tenants*, sometimes also referred to as simply a *problem*, is defined as a list $\langle I, H, P \rangle$ where:

- $I \subseteq \mathscr{I}$ is a finite set of agents,
- $H \subseteq \mathscr{H}$ is a finite set of housing units, and
- P is a preference profile.

H_O is the set of all occupied housing units, H_V is the set of all vacant housing units and $\{h_0\}$ is the set of all *null houses* h_{0i}. For an agent i to be assigned his or her null house h_{0i} is equivalent to being assigned no housing unit in $H_O \cup H_V$. $H \subseteq \mathscr{H}$ is then defined by $H = H_O \cup H_V \cup \{h_0\}$ and $H_O \cap H_V \cap \{h_0\} = \emptyset$. There are m housing units $h_j \in H \diagdown \{h_0\}$. I_E is the set of all existing tenants, each assigned exactly one $h_j \in H_O$ and I_N is the set of all agents assigned their null houses. $I \subseteq \mathscr{I}$ is then defined by $I = I_E \cup I_N$ and $I_E \cap I_N = \emptyset$. There are n agents $i \in I$. Ω is the preference domain, i.e. the set of all possible ordinal preference relations, defined over some set of indivisible objects. In the present case, Ω is defined over H. Each $i \in I$ has an ordinal *preference relation* $P_i \in \Omega$. The *preference profile* P is a list $(P_i)_{i \in I} \subset \Omega^n$, where Ω^n thus denotes the set of all possible preference profiles P over H, given I. For individual preferences, $x \succsim_i y$ denotes that agent i *weakly* prefers x to y, $x \succ_i y$ denotes that agent i *strictly* prefers x to y and $x \sim_i y$ denotes that agent i is *indifferent* between x and y. Initially, it is assumed that all P_i over H exclusively involve strict preferences. This assumption will eventually be relaxed. P_i is only observed by agent i and is unknown to the housing company. Most mechanisms allow each agent $i \in I$ to send one or more signals to the housing company, which the housing company may use to form beliefs about P_i. The signal or group of signals sent by agent i is called a *strategy* and the set of all strategies available to agent i is denoted by S_i. Agent i's strategy l is denoted by $s_{il} \in S_i$ and the strategy chosen by agent i is denoted by $s_i \in S_i$. $S_i^* \subset S_i$ is the set of all *undominated* strategies. Every s_i is assumed to be an element in

S_i^*, meaning that no agents will play dominated strategies. The *strategy profile* $s \subseteq \prod_{i \in I} S_i$ is the set of every s_i chosen by each $i \in I$. The housing company chooses an ordering f of all the agents according to some criteria. The agent $i \in I$ ranking the highest under f is denoted by $f(1)$, the agent ranking second by $f(2)$ and so on. The ranking at step k in some matching process is denoted by f_k. The outcome of a house matching mechanism is a one-to-one and onto function $\mu : I \to H$ called a *matching*. In other words, under a matching μ, every $i \in I$ is assigned exactly one $h_j \in H$. Each matching is an element of the *matching space* $\mathscr{M}$, which is simply the set of all possible matchings. An agent $i \in I$ being assigned a housing unit $h_j \in H$ under the matching μ will be denoted by $\mu(i) = h_j$. The original allocation is given by the matching $\lambda \in \mathscr{M}$, under which $\lambda(i) \in H$ is the housing unit originally assigned to agent i. The initial assignment of some $i \in I$, $\lambda(i)$, will be referred to as i's *endowment*. No matching mechanism evaluated in this paper will ever reallocate the same $h_j \in H \diagdown \{h_0\}$ twice, making $\lambda(i)$ equivalent to i's assignment prior to any reassignment for each $i \in I$. The preferences of I over $\mathscr{M}$ are characterized by the relationship given in 1, where $\nu \in \mathscr{M}$.

$$\forall i \in I : \quad \mu \succ_i \nu \iff \mu(i) \succ_i \nu(i) \tag{1}$$

This means that an agent will prefer a matching μ to some other matching ν if he or she prefers his assignment under μ to his assignment under ν.

Definition 1. *A (static)* matching mechanism φ *consists of a* strategy space S_i *for every* $i \in I$ *and an* outcome function $\prod_{i \in I} S_i \to \mathscr{M}$.

The outcome of φ is a matching denoted by $\varphi(I, H, s)$ and agent i's assignment under $\varphi(I, H, s)$ is denoted by $\varphi_i(I, H, s) \in H \diagdown H_V$.

Definition 2. *A matching mechanism is a* direct *mechanism if* $\forall i \in I : \quad S_i = \Omega$.

In other words, a direct mechanism is a mechanism where the strategy of each agent i consists of reporting some preference relation $P_i' \in \Omega$. It should be noted that there is no requirement that $P_i' = P_i$. Each agent $i \in I$ has the option to misrepresent his or her true preferences such that $s_i \neq P_i$. A direct matching mechanism is a special kind of *direct revelation mechanism*, as defined by Mas-Colell, Whinston and Green (1995). If H consists of some housing units that are considered equivalent in the eyes of the agents, it is permissible to sort all such units into a *class* of housing units $c_m \subseteq C$, where C is the set of all such classes and $\cap_{c_m \in C} c_m = \emptyset$. The housing units are equivalent and may be sorted into a class $c_m \subseteq C$ if $\forall h_{j,k} \in c_m, i \in I : \quad h_j \sim_i h_k$. If housing units can be sorted into such classes, Ω will be defined over $C \cup \{h_0\}$ rather than over H. This means that a unit h_k for which $\nexists h_j \neq h_k : h_j \sim_i h_k \quad \forall i \in I$ will be the sole element in a class $c_k \subseteq C$. This is how indifference is taken into account in this paper.

2.2 Properties

Not every matching mechanism is equivalent. Mechanisms are commonly evaluated by examining which *properties* they satisfy. This subsection will describe some possible properties of mechanisms that are commonly thought to be desirable if the utility of the agents being matched is under consideration.

Definition 3. *A matching $\mu \in \mathscr{M}$ is* individually rational *if* $\forall i \in I : \quad \mu(i) \succsim_i \lambda(i)$.

This means that a matching μ is not individually rational if some agent prefers his or her endowment to the housing unit he or she is assigned under μ. In models based on von Neumann-Morgenstern utility, it makes sense to distinguish between ex ante and ex post individual rationality. However, this paper does not employ von Neumann-Morgenstern utility and individual rationality will therefore always be used to mean ex post individual rationality.

Definition 4. *A (static) matching mechanism is* individually rational *if it always selects individually rational matchings. (Abdulkadiroglu and Sönmez, 1999)*

A mechanism that always selects individually rational matchings is a mechanism guaranteeing every agent a housing unit that is at least as good as his or her endowment. Consequently, there is no incentive to abstain from participating in the matching process if the mechanism is individually rational. A requirement that a mechanism be individually rational is called a *participation constraint.* (Mas-Colell, Whinston and Green, 1995)

Definition 5. *A direct matching mechanism is* manipulable *if*

$$\exists s_{il} : \varphi(s_{il}, P_{-i}) \succ_i \varphi(P_i, P_{-i}).$$

Definition 6. *A direct matching mechanism is* strategy-proof *if it is not manipulable at any preference profile $P \subset \Omega^n$.*

In other words, a mechanism is strategy-proof if it is always a dominant strategy for every agent $i \in I$ to report his or her true preferences. A strategy-proof mechanism is sometimes also referred to as *truthfully implementable in dominant strategies* or *dominant strategy incentive compatible* (Mas-Colell, Whinston and Green, 1995). In the context of house matching mechanisms, this means that it is always a dominant strategy for each agent $i \in I$ to rank the housing units in his or her application in accordance with his or her preference relation $P_i \in \Omega$.

Definition 7. *A matching $\mu \in \mathscr{M}$ is* Pareto efficient *if*

$$\nexists \nu \in \mathscr{M} : \nu(i) \succsim_i \mu(i) \quad \forall i \in I \textit{ and } \nu(i) \succ_i \mu(i) \textit{ for some } i \in I.$$

In other words, a matching μ is *Pareto efficient* if it is impossible to find a matching ν which is at least as good as μ for every agent under consideration and preferred to μ by at least one agent.

Definition 8. *A matching mechanism φ is* Pareto consistent *if the matching $\varphi(P)$ is Pareto efficient for all preference profiles $P \subset \Omega^n$. (Svensson, 1999; Svensson and Larsson, 2005)*

Pareto consistency is sometimes also referred to as *ex post efficiency* (Mas-Colell, Whinston and Green, 1995) or simply *Pareto efficiency.*

2.3 Examples of mechanisms

2.3.1 Serial dictatorship

Serial dictatorship is a simple direct matching mechanism that has an ordering f of a set of agents I choosing among a set of housing units $G \subseteq H \diagdown \{h_0\}$ and the set of null houses $\{h_0\}$, where $G = H \diagdown \{h_0\}$ at step one of the algorithm. The agent ranking the highest under f, $f(1)$, is assigned his or her top choice in $H \diagdown \{h_0\}$, which is then removed from G. Subsequently, $f(2)$ is assigned his or her top choice among the remaining houses in G. This process continues until either $G = \emptyset$ or every agent $i \in I$ is assigned a housing unit $h_j \in H$. (Abdulkadiroglu and Sönmez, 1999) The reason it is called serial dictatorship may become clearer considering the following definition, based on the definition of a *dictatorial social choice function* by Mas-Colell, Whinston and Green (1995).

Definition 9. *A matching mechanism is* dictatorial *if*

$$\exists i \in I : \varphi(P) \in \{\mu \in \mathscr{M} : \mu(i) \succsim_i \nu(i) \quad \forall \nu \in \mathscr{M}\} \quad \forall P \in \Omega^n.$$

This clearly holds for $i = f(1)$. Now define the matching space at step 2, $\mathscr{M}_{st2}$, as $\mathscr{M} \diagdown \{\{\mu(f(1))\} \cup \{f(1)\}\}$, then $\varphi(P_{-f(1)}) \in \{\mu \in \mathscr{M}_{st2} : \mu(i) \succsim_i \nu(i) \quad \forall \nu \in \mathscr{M}_{st2}\} \quad \forall P \in \Omega^n$ holds for $i = f(2)$. Continuing this line of reasoning, one can see that this will apply to all $i \in f$. Hence, it is *serially* dictatorial.

Example 1. *Consider the problem $\langle I, H, P \rangle$, where $I_E = \emptyset$, $I_N = \{i_1, i_2, i_3, i_4\}$, $H_O = \emptyset$, $H_V = \{h_1, h_2, h_3\}$ and $s = \{P_1, P_2, P_3, P_4\}$. The ordering of the agents is given from highest to lowest by $f = (i_1, i_2, i_3, i_4)$ and the preference profile P is given by the following table.*

P_1	P_2	P_3	P_4
h_2	h_1	h_1	h_3
h_3	h_3	h_2	h_2
h_1	h_2	h_3	h_1
h_{01}	h_{02}	h_{03}	h_{04}

The endowments of I are given by λ, interpreted as i_1 being assigned h_{01}, i_2 being assigned h_{02} and so on.

$$\lambda = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ h_{01} & h_{02} & h_{03} & h_{04} \end{pmatrix}$$

The outcome of φ is found using the following steps.

1. *At step one, agent $f(1) = i_1$ is assigned his or her top choice h_2 among the available housing units in $H_V = \{h_1, h_2, h_3\}$.*

2. *At step two, agent $f(2) = i_2$ is assigned his or her top choice h_1 among the remaining housing units in $H_V = \{h_1, h_3\}$.*

3. *At step three, agent $f(3) = i_3$ is assigned his or her top choice h_3 among the remaining housing units in $H_V = \{h_3\}$.*

Since $H_V = \emptyset$ at the end of step 3, the process terminates, producing the final matching μ.

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ h_2 & h_1 & h_3 & h_{04} \end{pmatrix}$$

Theorem 1. *The serial dictatorship matching mechanism is individually rational (i), strategy-proof (ii) and Pareto consistent (iii).*

Proof. (i) At the beginning of the assignment process, each agent is assigned his or her null house. Provided that each agent prefers any assignment $h_j \in H \diagdown \{h_0\}$ to no assignment, he or she cannot be made worse off. Hence, the mechanism is individually rational.

(ii) At each step, the agent $i \in I_N$ ranking the highest under f is assigned his or her most preferred $h_j \in H_V$ under s_i. Suppose h_1 is agent i's most preferred unit in H_V, which will also be i's assignment if $s_i = P_i$. Then, i can only affect $\varphi_i(s_i, s_{-i})$ by playing some strategy $s_i = P_i'$ under which some $h_j \neq h_1$ is i's most highly ranked alternative in H_V. This $h_j \neq h_1$ will then be i's assignment and $h_j \prec_i h_1$ will hold for any such unit h_j. Consequently, $\forall s_{il} : \quad \varphi_i(s_{il}, s_{-i}) \precsim_i \varphi_i(P_i, s_{-i})$. By relation 1, $\forall s_{il} : \quad \varphi(s_{il}, s_{-i}) \precsim_i \varphi(P_i, s_{-i})$. Hence, the mechanism is strategy-proof.

(iii) A matching is not Pareto efficient iff $\mu(f(i)) \prec_i \mu(f(j))$ and $\mu(f(j)) \prec_j \mu(f(i))$ for at least one pair of agents $f(i), f(j) \in f$. If $i > j$, then $\mu(f(j)) \in G$ at step i and if $j > i$, then $\mu(f(i)) \in G$ at step j. Thus, every $\mu \in \mathscr{M}$ produced by φ is Pareto efficient and the mechanism is Pareto consistent. □

For a more rigorous proof of theorem 1, taking indifference into account, see Svensson (1994). Individual rationality only holds under the condition that h_{0i} be the least preferred choice of each agent in I. Situations where this condition does not hold are thinkable. For example, if a university would employ a serial dictatorship mechanism and force all students to participate in it, some students who prefer off-campus housing might be made worse off under the matching selected by the mechanism. In many real life applications, every $i \in I_E$ is entitled to keep $\lambda(i)$. This means that in spite of its positive properties, serial dictatorship is often not applicable. This problem is addressed in the following matching mechanism.

2.3.2 Serial dictatorship with squatting rights

Serial dictatorship with squatting rights is a matching mechanism similar to serial dictatorship allowing tenants to keep the housing units they are assigned under λ. Each $i \in I_N$ reports a strict preference relation $P_i \in \Omega$ over H and each $i \in I_E$ reports either IN or OUT and a strict preference relation $P_i \in \Omega$ over H. Every $i \in I_E$ reporting OUT is assigned $\lambda(i)$ with certainty. The strategy spaces are $S_i = \Omega$ for every $i \in I_N$ and

$S_i = S_{i1} \times S_{i2} = \{IN, OUT\} \times \Omega$ for every $i \in I_E$. The remaining housing units are then assigned, according to the procedure in random serial dictatorship, to the population $J = I_N \cup \{i \in I_E : s_{i1} = IN\}$, where $G = H_V \cup \{\lambda(i) \in H_O : s_{i1} = IN\}$. If $I_E = \emptyset$, then serial dictatorship with squatting rights simply reduces to serial dictatorship. (Abdulkadiroglu and Sönmez, 1999)

Example 2. *Consider the problem $\langle I, H, P \rangle$ where $I_E = \{i_1, i_2, i_3\}$, $I_N = \{i_4\}$, $H_O = \{h_1, h_2, h_3\}$, $H_V = \emptyset$, $s = \{OUT + P_1, IN + P_2, IN + P_3, P_4\}$ and $f = (i_1, i_4, i_3, i_2)$. P is given by the following table.*

P_1	P_2	P_3	P_4
h_2	h_1	h_1	h_3
h_3	h_3	h_2	h_2
h_1	h_2	h_3	h_1
h_{01}	h_{02}	h_{03}	h_{04}

The endowments of I are given by λ.

$$\lambda = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ h_1 & h_2 & h_3 & h_{04} \end{pmatrix}$$

The outcome of φ is found using the following steps.

1. *At step one, agent $f(1) = i_4$ is assigned his or her top choice h_3 among the available housing units in $H_V = \{h_2, h_3\}$.*

2. *At step two, agent $f(2) = i_3$ is assigned his or her top choice h_2 among the remaining housing units in $H_V = \{h_3\}$.*

Since $H_V = \emptyset$ after step two, the process is terminated, producing the final matching μ.

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ h_1 & h_{02} & h_2 & h_3 \end{pmatrix}$$

It can be seen that $\lambda(2) \succ_2 \mu(2)$. This is one of the major drawbacks of the serial dictatorship with squatting rights mechanism.

Theorem 2. *The serial dictatorship with squatting rights matching mechanism is not individually rational (i) or Pareto consistent (ii).*

Proof. (i) Since $s_{i1} = IN$ requires i to give up $\lambda(i)$, he or she may be assigned a housing unit such that $\lambda(i) \succ_i \mu(i)$, as has been shown in the example above. The mechanism is thus not individually rational.

(ii) Consider the problem $\langle I, H, P \rangle$ where $I_E = \{i_1, i_2\}$, $I_N = \emptyset$, $H_O = \{h_1, h_2\}$, $H_V = \emptyset$ and $s = \{OUT + P_1, OUT + P_2\}$. The preference profile is given by the following table.

P_1	P_2
h_2	h_1
h_1	h_2
h_{01}	h_{02}

Since $H_V = \emptyset$ at step one, the process terminates immediately. This means that the original allocation λ also gives the final matching μ.

$$\lambda = \mu = \begin{pmatrix} i_1 & i_2 \\ h_1 & h_2 \end{pmatrix}$$

The matching μ is Pareto dominated by ν.

$$\nu = \begin{pmatrix} i_1 & i_2 \\ h_2 & h_1 \end{pmatrix}$$

Hence, the mechanism is not Pareto consistent. □

As no $i \in I_E$ is guaranteed a housing unit h_j satisfying $h_j \succ_i \lambda(i)$, there are incentives to keep one's current housing unit and abstain from participating in the matching process. This implies that the mechanism may lose out on some benefits from trade. Serial dictatorship with squatting rights is not a direct matching mechanism, which means that the concept of strategy-proofness is not applicable.

2.3.3 Serial dictatorship with waiting list

Serial dictatorship with waiting list is a direct matching mechanism remedying some of the problems with the serial dictatorship with squatting rights mechanism.

Definition 10. $\{h_j \in H_V : h_j \succ_i \lambda(i)\}$ *is the set of* acceptable *housing units for* $i \in I$.

All agents $i \in I$ report their preferences over H. At step one, the agent i ranking the highest under f_1 among those agents for which $\{h_j \in H_V : h_j \succ_i \lambda(i)\} \neq \emptyset$, i.e. among the agents for which there is at least one available and acceptable housing unit, is assigned his or her most preferred available and acceptable housing unit. The agent is then removed from the ordering f_2 and his or her assigned unit $\mu(i)$ is removed from the set H_V. If $i \in I_E$, then $\lambda(i) \in H_O$ is tranferred into H_V for step two. This process continues until $\forall i \in f_T : \{h_j \in H_V : h_j \succ_i \lambda(i)\} = \emptyset$. In other words, the process terminates when there no longer are any acceptable housing units available to any $i \in f_T$. When the process is terminated, any agent $i \in I$ not re-assigned any unit $h_j \in H$ gets to keep the unit $h_j \in H_O \cup \{h_{0i}\}$ he or she is assigned to under λ.

Example 3. *Consider the problem* $\langle I, H, P \rangle$*, where* $I_E = \{i_1, i_2\}$*,* $I_N = \{i_3\}$*,* $H_O = \{h_1, h_2\}$*,* $H_V = \{h_3\}$*,* $S = \{P_1, P_2, P_3\}$ *and* $f = (i_1, i_2, i_3)$*.*

P is given by the following table.

P_1	P_2	P_3
h_2	h_3	h_2
h_3	h_2	h_1
h_1	h_1	h_3
h_{01}	h_{02}	h_{03}

The endowments are given by λ.

$$\lambda = \begin{pmatrix} i_1 & i_2 & i_3 \\ h_1 & h_2 & h_{03} \end{pmatrix}$$

The outcome of φ is found using the following steps.

1. *At step one, agent i_1 is assigned his or her only acceptable unit h_3 in $H_V = \{h_3\}$.*
2. *At step two, agent i_2 has no acceptable unit in $H_V = \{h_1\}$. Agent i_3 is assigned his only acceptable unit h_1 in $H_V = \{h_1\}$.*

After step two, $H_V = \emptyset$, which implies that $\{h_j \in H_V : h_j \succ_i \lambda(i)\} = \emptyset \quad \forall i \in f_3$ and the process terminates, producing the final matching μ.

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 \\ h_3 & h_2 & h_1 \end{pmatrix}$$

Theorem 3. *The serial dictatorship with waiting list matching mechanism is individually rational.*

Proof. If $s_i = P_i$, the agent will only be assigned units in $\{h_j \in H_V : h_j \succ_i \lambda(i)\} \cup \{\lambda(i)\}$. Consequently, $\mu(i) \succsim_i \lambda(i)$ will always hold. □

Theorem 4. *The serial dictatorship with waiting list matching mechanism is not strategy-proof (i) or Pareto consistent (ii).*

Proof. (i) Consider the problem $\langle I, H, P\rangle$, where $I_E = \{i_1, i_2\}$, $I_N = \{i_3\}$, $H_O = \{h_1, h_2\}$, $H_V = \{h_3\}$ and $f = (i_1, i_2, i_3)$. P_{-1} is given by the following table.

P_2	P_3
h_3	h_2
h_2	h_1
h_1	h_3
h_{02}	h_{03}

Next, consider the two following strategies for agent i_1, where P_1 is his or her true preferences.

P_1	P_1'
h_2	h_2
h_3	h_1
h_1	h_3
h_{01}	h_{01}

If $s_1 = P_1$, then agent i_1 is assigned h_3 at step one. If $s_1 = P_i'$, then the following steps take place.

1. At step one, there is no acceptable housing unit in H_V for agent i_1. The process moves on to agent i_2, who is subsequently assigned h_3 and $\lambda(i_2) = h_2$ is sorted into H_V for step 2.

2. At step two, agent i_1 is assigned h_2.

As $h_2 \succ_1 h_3$, $\varphi_1(P_1', P_{-1}) \succ_1 \varphi_1(P_1, P_{-1})$. This means that the mechanism is manipulable at preference profile P. Hence, it is not strategy-proof.

(ii) Consider the problem $\langle I, H, P \rangle$ described in example 3. It was shown that the serial dictatorship with waiting list mechanism selected the final matching μ.

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 \\ h_3 & h_2 & h_1 \end{pmatrix}$$

The preference profile was given by the following table.

P_1	P_2	P_3
h_2	h_3	h_2
h_3	h_2	h_1
h_1	h_1	h_3
h_{01}	h_{02}	h_{03}

The matching μ is Pareto dominated by ν.

$$\nu = \begin{pmatrix} i_1 & i_2 & i_3 \\ h_2 & h_3 & h_1 \end{pmatrix}$$

Hence, the mechanism is not Pareto consistent. □

2.3.4 Top trading cycles

Top trading cycles is a direct matching mechanism allowing existing tenants to trade assignments with one another. It was developed by Abdulkadiroglu and Sönmez (1999) and the top trading cycles algorithm used to find the matching selected by the mechanism is influenced by the top trading cycles mechanism discussed by Shapley and Scarf (1974), attributed to David Gale. The main difference between the two is that Gale's top trading cycles algorithm, in the terminology of this paper and the context of house allocation problems, was developed for problems where $H_V \cup I_N = \emptyset$. The top trading cycles algorithm works in the following way. At each step, each agent $i \in I$ points to his or her most preferred housing unit $h_j \in H$ under the preferences announced in $s_i \in S_i$. Each $h_j \in H_O$ points to its current tenant, each $h_j \in H_V$ points to the highest ranking agent $f(1)$ and each h_{0i} points to the corresponding agent i. A *cycle* is an ordered list of agents and housing units $(j_1, j_2, ..., j_k) \subset I \cup H$, where each j points to the next j in the list and j_k points to j_1. Agent i pointing to housing unit h_j will be denoted by $\{i\} \rightarrow \{h_j\}$. At any step k, each agent participating in a cycle is assigned the unit he or she is pointing to, after which all agents in the cycle are removed from the process along with their assignments. Whenever $f(1)$ points to a housing unit $h_j \in H_V$, a cycle is formed with only these two elements. If at step k, in this type of cycle, $\lambda(f(1)) \in H_O$, then it is sorted into H_V for step $k+1$. Whenever there are no cycles left, the algorithm moves to the next step. The process continues as long as there is at least one available housing unit and one remaining agent. When the process terminates, every remaining agent is assigned his or her null house h_{0i}. Abdulkadiroglu and Sönmez (1999) remark that if there are several cycles at some step k and only one of them is removed, this would not alter the outcome of the mechanism. The reason for this is that any cycle not removed at step k remains a cycle at step $k+1$.

Example 4. *Consider the problem* $\langle I, H, P \rangle$, *where* $I_E = \{i_1, i_2, i_3, i_4\}$, $I_N = \{i_5, i_6\}$, $H_O = \{h_1, h_2, h_3, h_4\}$, $H_V = \{h_5\}$, $S = \{P_1, P_2, P_3, P_4, P_5, P_6\}$ *and* $f = (i_1, i_3, i_5, i_4, i_6, i_2)$. P *is given by the following table.*

P_1	P_2	P_3	P_4	P_5	P_6
h_2	h_4	h_2	h_1	h_1	h_5
h_5	h_2	h_5	h_3	h_3	h_4
h_1	h_1	h_3	h_4	h_5	h_2
h_3	h_5	h_1	h_5	h_2	h_1
h_4	h_3	h_4	h_2	h_4	h_3
h_{01}	h_{02}	h_{03}	h_{04}	h_{05}	h_{06}

The endowments are given by λ.

$$\lambda = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 \\ h_1 & h_2 & h_3 & h_4 & h_{05} & h_{06} \end{pmatrix}$$

The outcome of φ *is found using the following steps.*

1. *At step one,* $\{i_1, i_3\} \rightarrow \{h_2\}$, $\{i_2\} \rightarrow \{h_4\}$, $\{i_4, i_5\} \rightarrow \{h_1\}$, $\{i_6\} \rightarrow \{h_5\}$, $H_V = \{h_5\} \rightarrow \{f_1(1)\} = \{i_1\}$ *and* $h_i \in H_O \rightarrow \{i\}$. *The only cycle that is formed is* $(h_1, i_1, h_2, i_2, h_4, i_4)$. i_1 *is assigned* h_2, i_2 *is assigned* h_4 *and* i_4 *is assigned* h_1 *and they are removed from the process along with their assignments.*

2. *At step two,* $\{i_3, i_6\} \to \{h_5\}$, $\{i_5\} \to \{h_3\}$, $H_V = \{h_5\} \to \{f_2(1)\} = \{i_3\}$ *and* $\{h_3\} \to \{i_3\}$. *The only cycle that is formed is* (h_5, i_3). i_3 *is assigned* h_5 *and both are removed from the process.* $\lambda(i_3) = h_3$ *is sorted into* H_V *for step three.*

3. *At step three,* $\{i_5, i_6\} \to \{h_3\}$ *and* $H_V = \{h_3\} \to \{f_3(1)\} = \{i_5\}$. *The only cycle that is formed is* (h_3, i_5). i_5 *is assigned* h_3 *and both are removed from the process.*

At the end of step three $H \setminus \{h_0\} = \emptyset$ *and the process is terminated, yielding the final matching* μ.

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 \\ h_2 & h_4 & h_5 & h_1 & h_3 & h_{06} \end{pmatrix}$$

Theorem 5. *The top trading cycles mechanism is individually rational.*

Proof. Every $\lambda(i)$ points to agent i until he or she is removed from the process. An agent $i \in I$ will therefore never point to and consequently never be assigned any housing unit $\mu(i) \in \{h_j \in H : h_j \prec_i \lambda(i)\}$. (Abdulkadiroglu and Sönmez, 1999) □

In order to prove that the mechanism is also strategy-proof, lemma 1 is needed. The whole proof is due to Abdulkadiroglu and Sönmez (1999).

Lemma 1. *Suppose that all agents except agent* i *announce the partial preference profile* P_{-i} *and that* i *leaves the process at step* k *under* P_i *and at step* k' *under* P_i', *where* $k \leq k'$. *Then the remaining agents and housing units are the same at the beginning of step* k *regardless of whether agent* i *announces* P_i *or* P_i'.

Proof. Since agent i participates in no cycle prior to step k, the same cycles form regardless of whether i announces P_i or P_i' and the same agents and housing units are removed from the process before step k. □

Theorem 6. *The top trading cycles matching mechanism is strategy-proof.*

Proof. Suppose that all agents except agent i announce the partial preference profile P_{-i}. By announcing the preferences P_i', i joins the cycle $(h_j, j_1, j_2, ..., j_p, i)$, leaves the process at step k and is assigned h_j. By announcing his or her true preferences P_i, agent i leaves the process at step k'. There are two cases to consider.

1. $k' \geq k$.

   Consider step k when agent i has announced P_i. According to lemma 1, the same agents and housing units remain at the beginning of step k regardless of whether i announces P_i' or P_i. Thus, h_j points to j_1, j_1 points to j_2, ..., j_p points to i at step k and they keep doing so as long as agent i remains in the process. Under P_i, agent i is then assigned some $\varphi_i(P_i) \succsim_i h_j$ or joins the mentioned cycle and is assigned h_j.

2. $k' < k$.

   According to lemma 1, the same housing units and agents remain at the beginning of step k' regardless of whether agent i announces P_i' or P_i. Since agent i is assigned his or her most preferred choice remaining at step k' under P_i and h is an available option at k', $\varphi_i(P_i) \succsim_i h_j$ must hold.

□

Theorem 7. *The top trading cycles matching mechanism is Pareto consistent.*

Proof. Any agent removed from the process at step one is assigned his or her top choice. Any agent removed from the process at step two is assigned his or her top choice among the housing units that are still part of the matching process. Since preferences are strict, the agent cannot be made better off without making some agent removed from the process at step one worse off. The argument can be extended to show that no agent removed from the process at some step k can ever be made better off without making some other agent removed from the process at a preceding step worse off. (Abdulkadiroglu and Sönmez, 1999) □

A practical problem with the top trading cycles mechanism is that for larger sets H and I, it may be difficult to locate the cycles that are formed.

2.3.5 You request my house - I get your turn (YRMH-IGYT)

The YRMH-IGYT mechanism uses an alternative algorithm that was developed by Abdulkadiroglu and Sönmez (1999) as well. In this mechanism, if an agent's most preferred housing unit is some $h_j \in H_V$, he or she points to that unit, and if the most preferred housing unit is some $h_j \in H_O$, he or she points to the agent the unit is currently assigned to. The algorithm starts by assigning $f(1)$ his or her top choice $h_j \in H$ and subsequently removing both from the process. After that, $f(2)$ is assigned his or her top choice, and so on, until an agent requests an $h_j \in H_O$. When this happens, the agent he or she points to is moved to the top of the ordering f and the process continues. A *loop* is formed when there is an ordered list of agents $(i_1, i_2, i_3, ..., i_k)$ such that i_1 demands the housing unit currently occupied by i_2, i_2 demands the unit occupied by i_3, ..., and i_k demands the unit occupied by i_1. Loops are always formed by agents in I_E and the first agent in the list is always the highest ranking agent under f. Whenever a loop is formed, all agents in the loop are assigned their demanded units and are removed from the process along with their assignments. The algorithm terminates whenever either no agents or no houses remain in the process.

Theorem 8. *For any ordering f, the YRMH-IGYT matching mechanism selects the same matching as the top trading cycles mechanism.*

Proof. The algorithm can locate and remove a loop in one of two possible ways.

1. There is an ordered list of agents $(i_1, i_2, ..., i_k)$, where $f(1) = i_1$ at the start of the process. In this list, i_1 demands $\lambda(2)$, i_2 demands $\lambda(3)$, ..., i_{k-1} demands $\lambda(k)$ and i_k demands some $h_j \in H_V$. As i_{k-1} demands $\lambda(k)$, i_k is moved to the top of the ordering f and is assigned the demanded $h_j \in H_V$. Thereby, $\lambda(k)$ is made available and is assigned to i_{k-1}, $\lambda(k-1)$ is assigned to i_{k-2}, and so on until $\lambda(2)$ is assigned to i_1. Here, $(h_j \in H_V, i_1, \lambda(2), i_2, ..., \lambda(k), i_k)$ is a cycle as defined in section 2.3.4.

2. There is a loop of agents $(i_1, i_2, ..., i_k)$. Agent i_1 is assigned $\lambda(2)$, agent i_2 is assigned $\lambda(3)$, ..., agent i_k is assigned $\lambda(1)$. Here, $(\lambda(1), i_1, \lambda(2), i_2, ..., \lambda(k), i_k)$ is a cycle as defined in section 2.3.4.

   The algorithm locates cycles and lets the agents trade assignments accordingly. Since the step at which a cycle is located and removed does not affect the outcome of the matching process, the YRMH-IGYT mechanism will select the same matching as the top trading cycles mechanism. (Abdulkadiroglu and Sönmez, 1999)

□

Example 5. *Consider the same problem $\langle I, H, P \rangle$ as in example 4, where $I_E = \{i_1, i_2, i_3, i_4\}$, $I_N = \{i_5, i_6\}$, $H_O = \{h_1, h_2, h_3, h_4\}$, $H_V = \{h_5\}$, $S = \{P_1, P_2, P_3, P_4, P_5, P_6\}$ and $f = (i_1, i_3, i_5, i_4, i_6, i_2)$. P is given by the following table.*

P_1	P_2	P_3	P_4	P_5	P_6
h_2	h_4	h_2	h_1	h_1	h_5
h_5	h_2	h_5	h_3	h_3	h_4
h_1	h_1	h_3	h_4	h_5	h_2
h_3	h_5	h_1	h_5	h_2	h_1
h_4	h_3	h_4	h_2	h_4	h_3
h_{01}	h_{02}	h_{03}	h_{04}	h_{05}	h_{06}

The endowments are given by λ.

$$\lambda = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 \\ h_1 & h_2 & h_3 & h_4 & h_{05} & h_{06} \end{pmatrix}$$

The outcome of φ is found using the following steps.

1. *At step one, $f_1(1) = \{i_1\} \to \{i_2\}$, at which point i_2 becomes $f_1(1)$. Continuing in this way, $\{i_2\} \to \{i_4\}$ and $\{i_4\} \to \{i_1\}$, forming a loop (i_1, i_2, i_4). Then, i_1 is assigned h_2, i_2 is assigned h_4, i_4 is assigned h_1 and they are removed from the process, along with their assignments.*

2. *At step two, $f_2(1) = \{i_3\} \to \{h_5\}$. Then, i_3 is assigned h_5 and h_3 is sorted into H_V.*

3. *At step three, $f_3(1) = \{i_5\} \to \{h_3\}$ and i_5 is assigned h_3.*

At the end of step three, no housing units remain in the process and it is terminated, yielding the final matching μ.

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 \\ h_2 & h_4 & h_5 & h_1 & h_3 & h_{06} \end{pmatrix}$$

Note that this is exactly the same matching selected by the top trading cycles mechanism.

When the agents have preferences over $C \cup \{h_{0i}\}$ rather than over H, Abdulkadiroglu and Sönmez (1999) suggest the following tie-breaking rule. Given a weak preference relation R_i over H - or equivalently - a strict preference relation R_i over $C \cup \{h_{0i}\}$, construct a strict preference relation P_i over H such that for any $i \in I$:

1. Given two housing units $h_j \in c_m$ and $h_j \notin c_m$, the unit belonging to the class preferred under R_i is strictly preferred under P_i.
2. Given two housing units of the same type, $h_{j,k} \in c_m$,

   (a) if both units are occupied, then the unit whose owner is higher ranking under f is preferred under P_i.
   (b) if $h_j \in H_O$ and $h_k \in H_V$, then $h_j \succ_i h_k$ under P_i.
   (c) if $h_{j,k} \in H_V$, then the unit with the lower index number is preferred under P_i.

This tie-breaking rule may be used for the top trading cycles matching mechanism as well.

Theorem 9. *The YRMH-IGYT matching mechanism is individually rational, strategy-proof and Pareto consistent.*

Proof. By theorem 8, the YRMH-IGYT mechanism always selects the same matching as the top trading cycles matching mechanism. Hence, it also satisfies the same properties. Consequently, by theorems 5, 6 and 7, it is individually rational, strategy-proof and Pareto consistent. □

The benefit of using the YRMH-IGYT mechanism rather than the top trading cycles mechanism is that automates the process of locating the cycles in each step, which makes it easier to deal with in practice. Additional properties of the YRMH-IGYT mechansim not discussed in this paper have been examined by Sönmez and Ünver (2010).

2.3.6 Dynamic mechanisms

All of the matching mechanisms described so far in chapter 2 have been *static* matching mechanisms. *Dynamic* mechanisms have only attracted some attention in recent years. For this reason, there appears to be no consensus on the proper definition of a dynamic matching mechanism. Furthermore, most examinations of dynamic mechanisms have been restricted to direct mechanisms. To define a dynamic matching mechanism, some new concepts are needed. A dynamic matching mehanism may reallocate housing units over several *periods* in time, where t_0 will denote the beginning of the matching process and T will denote the end thereof. The time frame of the process is given by $[t_0, T]$ and each discrete time period is denoted by $t \in [t_0, T]$. These periods are not to be confused with the steps taken in the different algorithms. When a matching mechanism is a dynamic process, it involves several intermediate matchings, together constituting a *matching plan* μ^p. The matching in period t is denoted by μ_t, agent i's assignment in period t is denoted by $\mu_t(i)$ and $\mu(i)$ will be interpreted as $\mu_T(i)$. $\mathscr{M}_t$ denotes the matching space in period t and $\prod_{t=t_0}^{T} \mathscr{M}_t$ thus denotes the *matching plan space*. Following Kurino (2009), with some modifications to allow for dynamic non-direct mechanisms and to ensure terminological consistency, a dynamic matching mechanism will be defined as follows.

Definition 11. *A* dynamic matching mechanism φ *consists of a strategy space* S_i *for each* $i \in I$ *and an outcome function* $\prod_{i \in I} S_i \to \prod_{t=t_0}^{T} \mathscr{M}_t$ *selecting a* matching plan $\mu^p \in \prod_{t=t_0}^{T} \mathscr{M}_t$ *for each strategy profile* $s \in \prod_{i \in I} S_i$.

Kurino (2009) identifies two different kinds of dynamic direct mechanisms.

Definition 12. *A dynamic matching mechanism is a* spot mechanism *if, in every period, agents are asked only to reveal preferences for the current period.*

Definition 13. *A dynamic matching mechanism is a* futures mechanism *if agents are asked to reveal their preferences over every period.*

An example would be a serial dictatorship (with squatting rights or waiting list) spot mechanism, in which each period constitutes a static serial dictatorship (with squatting rights or waiting list) mechanism, or a top trading cycles spot mechanism, in which each period constitutes a static top trading cycles mechanism. Abdulkadiroglu and Loertscher (2007) use a different definition of dynamic mechanisms, under which Kurino's spot mechanism would be classified as a static mechanism. Kurino (2009) also introduces a dynamic counterpart to the individual rationality property.

Definition 14. *A dynamic matching mechanism is* acceptable *if*

$$\forall i \in I, t \in [t_0, T] : \quad \mu_{t+1}(i) \succsim_i \mu_t(i).$$

In other words, a dynamic matching mechanism satisfies acceptability if each agent is made weakly better off as time progresses. For dynamic spot mechanisms, this is equivalent to a requirement that every period constitute an individually rational static mechanism. In this paper, it is assumed that preferences are *time-invariant*, meaning that all preferences remain the same in each period. It is also assumed that the matching process is significantly short not to be affected by the time preferences, i.e. discount rates, of any agents $i \in I$. Consequently, the preferences of I over $\mathscr{M}$ are also characterized by relation 1, as well as the following relation.

$$\forall i \in I : \quad \mu \sim_i \nu \iff \mu_T(i) = \nu_T(i)$$

This means that it is irrelevant in which period $t \in [t_0, T]$ i is assigned $\mu(i)$ or $\nu(i)$. An individual matching will always be evaluated in period T. Kurino (2009) has shown that the top trading cycles mechanism loses its Pareto consistency and partially loses its strategy-proofness when transformed into a spot mechanism. This applies to the YRMH-IGYT mechanism as well, as it is only an alternative algorithm leading to the same matchings.

3 Background

This section will provide the context of AF Bostäder's house allocation problem and explain how housing units are currently being allocated among students by AFB. If no other source is given, the information in this section was either retrieved from AFB's official website (AF Bostäder, n.d.) or supplied in a personal interview with the students' elected representative at AFB. (Theofanous, 2012).

AF Bostäder is a company providing housing for students at Lund University. At this date, AFB owns roughly 5800 accommodations of different types, located in the city of Lund. About half of the accommodations consist of corridor rooms and the other half of apartments with one to four rooms. Each of the accommodations is located in one of eleven housing disctricts, named Delphi, Gylleholm, Kämnärsrätten, Klosterängen, Magasinet, Parentesen, Sparta, Studentlyckan, Tomegapsgården, Ulrikedal and Vildanden. Lund University had 47 000 active students in 2011 (Lund University, 2012), contributing to a relatively high demand for housing. To ensure that students on a restricted budget can afford housing, it is the policy of AFB to set rents such that each housing district is self-supporting, rather than setting them to maximize profits. With this in mind, it is not surprising that the market for student housing does not clear, contributing to the general housing shortage in the city of Lund. (Boverket, 2012) This creates a need for selecting a mechanism to allocate apartments and corridor rooms among the students.

AFB is currently assigning accommodations to applicants using a queueing mechanism. There is a single general queue for all accommodations in all housing districts. In order to be placed in the queue, called a *housing list*, it is a requirement for the applicant to be an active student at Lund University or employed within Akademiska Föreningen (from which AFB's name is derived) and to be a paying member of Akademiska Föreningen. If the membership and study requirements are not met at any time, the applicant's place on the housing list is forfeited. There is also a residence limit, restricting the total duration of tenancy to a maximum of six years. In addition to this, there is an upper age limit of 35 years for signing a tenancy contract for a corridor room. Students who have been accepted to Lund University for the first time are exempt from the study and membership requirements, and new students who do not live within commuting distance are eligible to apply for *new student housing*, which is a lottery granting a limited number of new students priority access to corridor rooms.

Students submitting a valid housing application are placed on the housing list. The date of submission of a valid housing application is called the *list date*, which is personal and can not be transferred to another applicant. Available accommodations are advertised on AFB's website, where they can be booked by applicants during a *booking period*. The duration of the booking period is always three days. Each applicant is limited to a maximum of three simultaneous bookings at any point in time. Bookings are binding, and if a booking is cancelled after being assigned a corridor room or apartment at the end of the booking period, the applicant loses his or her place in the queue. All applicants are *ranked* based on their list date and any priority access that may have been granted. After the end of the booking period, the accommodations are allocated among the applicants who have booked the individual corridor rooms and apartments in accordance with their *rankings*. Upon signing a tenancy contract, the tenant's position in the queue is forfeited. Upon booking a corridor room, a person granted priority access is placed at the top of the ranking of those who booked the room in question. When browsing available accommodations, the applicant is shown how many people have currently booked each available accommodation and what his or her ranking among those who have currently booked the accommodation would be. The applicant with the highest ranking among the applicants who have booked each accommodation is offered to sign a tenancy contract for the corridor room or apartment in question. If an applicant is the highest ranking applicant for more than one accommodation, he or she is offered a tenancy contract for the accommodation he or she booked the earliest. The tenant must give notice one month in advance before revoking the tenancy contract.

4 Analysis

In subsection 4.1, AF Bostäder's mechanism described in section 3 will be placed in the context of the theoretical framework developed in section 2. The properties of AFB's current mechanism will then be examined in subsection 4.2 and potential sources of inefficiency therein will be identified in subsection 4.3. Finally, the simulations carried out are described in subsection 4.4.

4.1 Description of AF Bostäder's mechanism within the theoretical framework

There are approximately 5800 housing units $h_j \in H \setminus \{h_0\}$ owned by AF Bostäder. These can, under some indifference assumptions, be sorted into several classes $c_m \subset C$. In this context, I_N denotes the set of agents on the housing list currently assigned their null houses and I_E denotes the set of agents currently assigned a housing unit $h_j \in H \setminus \{h_0\}$. Each agent $i \in I$ has a preference relation $P_i \in \Omega$ over C. S_i is restricted to rankings of i's top alternatives in $H_V \cup \lambda(i) \cup h_{0i}$, which means that the mechanism is not a direct mechanism. The agent selects up to three housing units $h_j \in H_V$ in any period t. The agents are ordered under f in accordance with their list dates, where a higher list date corresponds to a higher ranking. At the end of the booking period, the unit is assigned to the agent ranking the highest under f among those who booked the unit. If an agent is the highest ranking for more than one booked unit, he or she is assigned the unit that was booked first. The remaining unit will then be assigned to the second highest ranking agent among those who booked it, and so on. This implies that each $h_j \in H_O$ is assigned only to one $i \in I$, even in the cases where a housing unit is shared by several people. Any cohabitant i will be characterized by $i \notin I$ for simplicity. The units can be booked and unbooked freely during the booking period, making it possible for the agent to rank his or her top three units $h_j \in H$ according to his or her preference relation $P_i \in \Omega$ over C. An agent $i \in I_E$ is guaranteed his or her current assignment, $\lambda(i)$, and a tenancy contract may be revoked at any time, with one month's notice. This means that h_{0i} is always a guaranteed option as well. For an agent $i \in I_N$, $\lambda(i) = h_{0i}$, which is also a guaranteed option. For each $i \in I_E$ not assigned any $h_j \in H_V$ by φ that did not revoke his or her contract, $\mu(i) = \lambda(i)$. Similarly, for each $i \in I_N$ not assigned any $h_j \in H_V$ by φ, $\mu(i) = \lambda(i) = h_{0i}$. The strategies $s_{il} \in S_i$ thus consist of rankings of at most the top five elements in $H_V \cup \lambda(i) \cup h_{0i}$ for each $i \in I_E$ and of at most the top four elements for each $i \in I_E$. The difference is due to the fact that $\forall i \in I_N : \quad \lambda(i) = h_{0i}$. If $\lambda(i)$ is preferred to any of the relevant top elements in $H_V \cup h_{0i}$, s_i will consist of a ranking of the elements in $\{h_j \in H_V \cup \lambda(i) \cup h_{0i} : h_j \succsim_i \lambda(i)\}$ and if $\lambda(i)$ is preferred to all the relevant top elements, then $s_i = \lambda(i)$, i.e. the strategy will consist of keeping the agent's current assignment.

The mechanism is a continuously ongoing process, with new housing units being assigned to new agents every day, and consequently, new units being sorted into H_V every day. This means that AFB's mechanism is a dynamic matching mechanism. If an agent $i \in I_E$ is assigned a housing unit $h_j \in H_V \cup h_{0i}$ in period t, his previously assigned housing unit $\lambda(i) \in H_O$ is then sorted into H_V in period $t+1$. An agent $i \in I$ who is assigned some $h_j \in H_V$ in period t is subsequently removed from the process. It is technically possible for the agent to immediately place him- or herself at the bottom of f by once again filling out a valid housing application, but this option will be ignored for simplicity. Additionally, it will be shown that the mechanism is not sufficiently fast to assign or reassign any housing units to those at the very bottom of f, making this a theoretical simplification of no practical importance. Due to the housing shortage in Lund (Boverket, 2012), it is safe to assume that $|H \setminus \{h_0\}| < |I|$ and $|H_V| < |I_N|$ in any period $t \in [t_0, T]$. Thus, the matching process is unlikely to terminate as long as $H_V \neq \emptyset$, assuming that h_{0i} is the least preferred alternative of the agents.

Example 6. *Consider the problem $\langle I, H, P \rangle$, where $I_E = \{i_1, i_2\}$, $I_N = \{i_3, i_4\}$, $H_O = \{h_1, h_2\}$, $H_V = \{h_3\}$, $f = (i_1, i_3, i_2, i_4)$ and $s = \{P_1^*, -, P_3^*, P_4^*\}$, where P_i^* is interpreted as agent i booking his or her top alternatives in H_V, but no more than three units, that are preferred to his or her endowment and $-$ is interpret as not booking any housing units under any circumstances. P is given by the following table.*

P_1	P_2	P_3	P_4
h_2	h_2	h_1	h_3
h_1	h_1	h_3	h_2
h_3	h_3	h_2	h_1
h_{01}	h_{02}	h_{03}	h_{04}

The endowments are given by λ.

$$\lambda = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ h_1 & h_2 & h_{03} & h_{04} \end{pmatrix}$$

There are two periods $t \in [t_0, T]$, which means that the matching process is six days long. Agent i_2's contract is revoked in the first period. The outcome of φ is found using the following steps.

1. *In the first period, i_2's contract is revoked and he or she is assigned h_{02}, after which both are removed from the process. Agent i_1 prefers no unit in $H_V = \{h_3\}$ to $\lambda(1)$, which means that i_1 will not book any units and keeps his or her current assignment. Agents i_3 and i_4 book $h_3 \in H_V$. As i_3 ranks higher than i_4 under f, meaning that he or she has an earlier list date, h_3 is assigned to i_3 and both are removed from the process. After this, $H_V = \emptyset$ and the first period ends.*

2. *In the second and last period, h_2 is announced to be available. Agents i_1 and i_4 book $h_2 \in H_V$. As i_1 ranks higher than i_4 under f, h_2 is assigned to i_1 and both are removed from the process. After this, $H_V = \emptyset$ and the second period ends, yielding the final matching μ.*

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ h_2 & h_{02} & h_3 & h_{04} \end{pmatrix}$$

Under μ, h_1 is left unassigned.

4.2 Properties of AF Bostäder's mechanism

AFB's mechanism shares many similarities with the serial dictatorship with waiting list mechanism described in section 2.3.3. Since the mechanism guarantees each agent his or her current assignment, it satisfies acceptability, just like the serial dictatorship with waiting list mechanism satisfies individual rationality.

Theorem 10. *AF Bostäder's current mechanism is acceptable.*

Proof. Since $\mu_t(i)$ is guaranteed in period $t+1$ for all $i \in I$, any $s_{il} \in S_i$ at $t+1$ including an element in $\{h_j \in c_m : c_m \prec_i \mu_t(i) \text{ or } c_m \prec_i h_{0i}\}$ is a weakly dominated strategy. Consequently, at $t+1$,

$$\forall s_{il} \in S_i^* : \quad \mu_t(i) \precsim_i h_j \in s_{il} \text{ and } \forall s_{il} \in S^* : \quad \mu_{t+1}(i) \in s_{il}.$$

As this holds for each $t \in [t_0, T]$, AFB's mechanism is acceptable. □

Just like the similar serial dictatorship with waiting list mechanism, AFB's mechanism is also not Pareto consistent.

Theorem 11. *AF Bostäder's mechanism is not Pareto consistent.*

Proof. Consider the problem $\langle I, H, P \rangle$, where $I_E = \{i_1, i_2\}$ $I_N = \{i_3\}$, $H_O = \{h_1, h_2\}$, $H_V = \{h_3\}$, $f = (i_3, i_1, i_2)$, $s = (P_1^*, P_2^*, P_3^*)$ and there is one period $t \in [t_0, T]$. Let P be given by the following table.

P_1	P_2	P_3
h_2	h_1	h_1
h_3	h_3	h_2
h_1	h_2	h_3

The initial matching is given by λ.

$$\lambda = \begin{pmatrix} i_1 & i_2 & i_3 \\ h_1 & h_2 & h_{03} \end{pmatrix}$$

At step one, h_3 is assigned to i_3. As $H_V = \emptyset$ at the end of step one, the process terminates, resulting in the final matching μ.

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 \\ h_1 & h_2 & h_3 \end{pmatrix}$$

The matching μ is Pareto dominated by ν.

$$\nu = \begin{pmatrix} i_1 & i_2 & i_3 \\ h_2 & h_3 & h_1 \end{pmatrix}$$

Hence, the mechanism not Pareto consistent. □

A difference between AFB's current mechanism and the serial dictatorship with waiting list mechanism is that the former is a non-direct mechanism. The non-directness property comes from the fact that the strategies of agents do not consist of supplying a preference relation over every type of housing unit, which they do for the similar serial dictatorship with waiting list mechanism. Instead, the agents can only declare preferences by booking up to three of the units available in H_V in any period, which they, by theorem 10, prefer to their current assignments. Furthermore, as the concept of strategy-proofness is inapplicable to non-direct mechanisms, AFB's mechanism can not be said to satisfy this property. The non-directness property also imposes a restriction on AFB's mechanism. As a dynamic mechanism, the reassignment takes place in multiple periods and as the mechanism only permits agents to apply and thus declare preference for housing units in H_V, the algorithm can only make as many reassignments in each period as there are elements in H_V. If it were a direct mechanism, such as the serial dictatorship with waiting list mechanism or its dynamic spot mechanism counterpart, the housing company would have access to the reported preferences over all types of housing units for all agents, allowing the algorithm to go on for as many steps as it takes for the process to terminate in any period. One implication of this is that the housing units sorted into H_V in period t remain there until period t+1 when AFB's current mechanism is used. Therefore, when the resulting matching is evaluated at some end point T, I_N and H_V may simultaneously be non-empty. This could not happen for the serial dictatorship with waiting list mechanism, assuming that h_{0i} is the least preferred option of each $i \in I_N$. Although this is a minor issue as any end point T will be arbitrary. In reality, AFB's mechanism does not have an end point, but is rather an indefinitely ongoing process. AFB's matching mechanism is similar to a dynamic spot mechanism, as agents are only permitted to apply and thus declare preference for housing units available in the current period. However, as it is not a direct mechanism, it can not be properly defined as such. Changing the agents' strategy sets to consist of reported preference relations over all units in H would transform AFB's mechanism into a serial dictatorship with waiting list spot mechanism.

4.3 Sources of inefficiency

Even though AFB's current mechanism has been shown not to be Pareto consistent, this does not mean it is equivalent to all mechanisms that are not Pareto consistent. Clearly, a matching not satisfying Pareto consistency where many agents are assigned their most preferred housing units but there is some room for Pareto improvement through trade is preferable to a matching where every agent is assigned his or her null house, or a matching where every agent is assigned his or her least preferred housing unit in $H \setminus \{h_0\}$. Different measures of efficiency are discussed in section 5.1. Three possible sources of inefficiency in AFB's current mechanism are:

1. The fact that it is not a direct matching mechanism.
2. The length of the booking period.
3. The lack of trade between tenants.

The first point is of relevance for the reasons brought up in section 4.2. The length of the booking period is only of relevance because the mechanism is dynamic, with agents being assigned housing units in different periods. The longer the booking period, the larger H_V will tend to be in any period. The larger H_V is, the more options there are for each agent applying for housing, making it more likely that the agent will be satisfied with his or her assigned housing unit. However, since $\lambda(i)$ will be sorted into H_V in period $t+1$ for each $i \in I_E$ assigned a housing unit $h_j \in H_V$ in period t, a shorter booking period gives the mechanism more "roundaboutness", which means that it will reallocate a larger number of housing units in a shorter period of time. Thus, a short booking period has both good and bad properties relative to a long booking period. The third point is of relevance due to the welfare improvements all trade entails.

The impacts of these possible sources of inefficiency will be investigated by performing simulations of AFB's current mechanism and of alternative mechanisms in section 4.4 and subsequently comparing the resulting matchings in section 5. The impact of the relatively short booking period will be investigated by comparing the resulting matching of AFB's current mechanism to the resulting matching of an identical matching mechanism where only the length of the booking period is altered. The impact of the lack of trade between tenants will be investigated by comparing the previously mentioned matchings with the matching produced by the YRMH-IGYT mechanism and a dynamic spot mechanism version thereof, which both allow for trade between tenants. The impact of the non-directness of AFB's mechanism was initially investigated by performing the simulation with a serial dictatorship with waiting list direct spot mechanism. However, in order to limit the scope and size of the paper, those findings were cut from the final version.

4.4 Simulations

To simulate the different mechanisms in the context of AFB's house allocation problem, the focus of this paper will be limited to three housing districts, called Gylleholm, Klosterängen and Studentlyckan. These disctricts were chosen to give a relatively even and varied distribution of different housing types. $H \diagdown \{h_0\}$ consists of 344 housing units sorted into 10 classes $c_m \subset C_n \subset C$, where C_n is one of the three main classes: C_1 consisting of one to one-and-a-half room apartments, C_2 consisting of two room apartments and C_3 consisting of three room apartments, where $\cap_{n=1}^{3} C_n = \emptyset$. The requirement that $\forall h_{j,k} \in c_m, i \in I: \quad h_j \sim_i h_k$ holds by construction in the simulations. However, in reality, it may not hold for the actual applicants. The distribution of $H \diagdown \{h_0\}$ is shown in table 1.

Table 1: $H \diagdown \{h_0\}$

	Gylleholm	Klosterängen	Studentlyckan	Sum
$1 - 1\frac{1}{2}$ rooms	48	112	8	168
2 rooms	0	0	154	154
3 rooms	0	0	22	22
Sum	48	112	184	344

C is described in table 2. Since all two and three room apartments are located in Stu-

Table 2: C

$1 - 1\frac{1}{2}$ room	2 rooms	3 rooms
1g	2a	3a
1k	2b	3b
1s	2c	-
-	2d	-
-	2e	-

dentlyckan and since there are no or only minor differences between one room apartments in the same housing districts, one room apartments are sorted into three classes $c_m \subset C_1$ based on which district the housing unit is located in. The two and three room apartments are sorted into five classes $c_m \subset C_2$ and two classes $c_m \subset C_3$ respectively, based on rent information supplied by AFB. While the number of housing units $h_j \in C_2$ and $h_j \in C_3$ is known, the number of housing units $h_j \in c_m$ is unknown for each $c_m \subset C_2 \cup C_3$. For simplicity, the housing units are assumed to be distributed evenly among the different classes within C_2 and C_3.

As explained previously, each $h_j \in H$ is assigned to at most one $i \in I$. The preferences of 450 fictional agents $i \in I$, composed of 344 existing tenants $i \in I_E$ and 106 non-occupant applicants $i \in I_N$, were randomized. It is thus assumed that each $h_j \in H \diagdown \{h_0\}$ is matched to some $i \in I_E$ and $H_V = \emptyset$ before the start of the matching process, prior to any contract being revoked. The randomization was performed using the following script in Stata/IC 12.0.

```
forvalues i = 1/450 {
shuffle8 1 M

replace sing_mult = r(list) in ‘i’
}

forvalues j = 1/450 {
shuffle8 G K S

replace area = r(list) in ‘j’
}

forvalues k = 1/450 {
shuffle8 2 3

replace two_three_r = r(list) in ‘k’
}

forvalues l = 1/450 {
shuffle8 2a 2b 2c 2d 2e

replace tworoom = r(list) in ‘l’
}
forvalues m = 1/450 {
shuffle8 3a 3b

replace threeroom = r(list) in ‘m’
}
```

This script gives an output of the form displayed in table 3 for agent i_{153}. The boxes give a preference relation going from left to right, such that a symbol for strict preference $\succ_{153}$ could be placed between the elements in each box. The numbers and letters are interpreted

Table 3: $P_{153} \in \Omega$

$i \in I$	Size	District	2-3 rooms	2 rooms	3 rooms
153	M 1	S G K	3 2	2e 2b 2c 2a 2d	3a 3b

in the following way. 'M 1' means that a multiple room solution is preferred to a one to one-and-a-half room solution. 'S G K' gives the preferences over different housing districts. Studentlyckan is preferred to Gylleholm, and Gylleholm is preferred to Klosterängen. '3 2' means that a three room apartment is preferred to a two room apartment. '2e 2b 2c 2a 2d' and '3a 3b' give the preferences of i_{153} over C_2 and C_3 respectively. The preference for either a one to one-and-a-half room apartment or a multiple room apartment trumps the other preferences. This means that the 'District' column reduces to the agent's preferences over C_1. Table 3 could thus be rewritten as in table 4.

Table 4: $P_{153} \in \Omega$

$i \in I$	C_1 vs. $C_2 \cup C_3$	C_1	C_2 vs. C_3	C_2	C_3
153	M 1	S G K	3 2	2e 2b 2c 2a 2d	3a 3b

The classes $c_m \subset C$ are ranked internally within each $C_n \subset C$ rather than within C for each $i \in I$.

Definition 15. *A preference relation over some $c_m \subset C$ is* internal *to each $C_n \subset C$ when*

$$\forall c_{m,n} \subset C_A \cup C_B, C_n \subset C : \quad c_a \subseteq C_A \succ_i c_b \subseteq C_B \iff c_m \subseteq C_A \succ_i c_n \subseteq C_B.$$

This means that if an agent prefers one type of two room apartments to one type of three room apartments, he or she also prefers *all* two room apartments to *all* three room apartments. There is no overlap in the preferences such that some $c_m \in C_n$ is preferred to some $c_n \in C_o$ while some $c_o \in C_o$ is preferred to some $c_p \in C_n$. This is how the preferences were constructed, but it is not necessarily the case that the actual preferences of real life applicants follow this model. It does seem quite likely, however, considering how small the differences are within the two room apartment and three room apartment sets. Table 4 thus represents the preferences relation given by table 5. In table 5, s, g and k denote which housing district the one or one-and-a-half room apartment is located in. The preference relation $P_i \in \Omega$ of each agent $i \in I$ is complete and transitive by construction. Single-peakedness is not satisfied because of the special case where '1 M' and '3 2'. If agents think of housing units in general terms of, for instance, size, then it might not make sense for an agent to, as an example, prefer a three room apartment to a two room apartment, but at the same time prefer a one room apartment to a three room apartment. An assumption of single-peaked preferences would then be in order. A justification for randomizing preferences not satisfying single-peakedness is given by the fact that one room apartments are seldomly shared with other cohabitants, while two and

Table 5:

P_{153}
$3a$
$3b$
$2e$
$2b$
$2c$
$2a$
$2d$
$1s$
$1g$
$1k$

three room apartments often are. Single and multiple room solutions may thus involve very different living situations, arguably making the special case where $c_m \in C_1 \succ_i c_n \in C_3 \succ_i c_o \in C_2$ realistic. It needs to be pointed out, however, that preference relations need not follow any formal rules in the real world and may take any form. The purpose of the simulations is to investigate how different mechanisms operate on a fictional population. For real populations, preferences over C need not be estimated, but can simply be supplied directly by the applicants.

The number of elements in I_N at t_0 was chosen somewhat arbitrarily. The purpose of this paper is to provide a comparison between different matching mechanisms, rather than some nominal measure of efficiency. The size of I_N would not affect the ordinal efficiency rankings of different mechanisms. The size of I_N was chosen to be large enough to make it impossible for every $i \in I$ to be assigned a housing unit $h_j \in H \diagdown \{h_0\}$. The number of elements in I_E before the start of the matching process and before any contracts were revoked equals the total number of housing units $h_j \in H \diagdown \{h_0\}$ and the size of $H \diagdown \{h_0\}$ equals the number of housing units in the three housing districts. Each $h_j \in H \diagdown \{h_0\}$ is initially randomly assigned to one of the agents $i \in I_E$ such that each $i \in I_E$ is assigned exactly one $h_j \in H \diagdown \{h_0\}$. In practice, this is done by randomly sorting all of the elements $h_j \in H \diagdown \{h_0\}$ together with 106 h_{0i} using the random number generator in LibreOffice Calc. This list is then pasted next to the list of every agent in I. Each h_j in the list is then matched to the agent $i \in I$ found on the same row, and this matching defines I_E and I_N. It is assumed that some $i \in I_E$ choose to revoke their contracts and be assigned h_{0i} during the course of $[t_0, T]$. This is not equivalent to the agents preferring to be assigned no housing unit at all to their current housing units. The null house h_{0i} simply represents any housing situation outside of the three housing districts. A more reasonable interpretation would be that $h_j \notin H \diagdown \{h_0\} \succ_i \mu_t(i) \in H_O$ for some $i \in I_E$ in some period $t \in [t_0, T]$, where $h_j \notin H \diagdown \{h_{0i}\}$ is some attainable alternative. Upon being assinged h_{0i}, the agent is removed from the set I. By construction, h_{0i} is the least preferred alternative for all the remaining agents $i \in I$. The contracts being revoked is simulated by randomizing a list of 450 elements in $\{0, 1\}$, where 1 means that the agent $i \in I$ keeps his or her assigned $h_j \in H$ under λ and 0 means that the agent $i \in I_E$ is assigned h_{0i} and removed from the process. If one or more $0 \in \{0, 1\}$ is matched to some agent $i \in I_N$, those elements are once again randomized until every $0 \in \{0, 1\}$ is matched to an agent $i \in I_E$. The current assignments $h_j \in H_O$ of those agents $i \in I_E$ who are matched to a $0 \in \{0, 1\}$ are then sorted into H_V in some period $t \in [t_0, T]$. Exactly when they are sorted into H_V depends on the construction of the simulated mechanism.

At this point, the agents $i \in I$ are to be processed through different mechanisms, the matchings produced by which are be evaluated in period T. The preference profile P and the original matching λ are to be identical in every simulation. The starting point t_0 and the end point T will also be the same in every simulation, but the amount of elements in $[t_0, T]$ will vary depending on the specifications of the different mechanisms. For the simulations, the time frame chosen is 180 days and it is assumed, for simplicity, that one contract is revoked every two days. This means that 90 units $h_j \in H \diagdown \{h_{0i}\}$ are sorted into H_V during $[t_0, T]$, for each simulated mechanism. The amount of revoked contracts during $[t_0, T]$ is the same for every mechanism, but since different mechanisms involve different numbers of periods, the number of contracts being revoked in each period $t \in [t_0, T]$ is larger the fewer elements there are in $[t_0, T]$. The revoked contracts are evenly distributed among each $t \in [t_0, T]$. If the number of contracts being revoked per period is not an integer under a completely even distribution, it will alternate between being rounded up and being rounded down to create an even stream of units into H_V. The number will always be rounded up in the first period. For example, if there are four periods and 18 contracts are revoked in total, giving $4.5 \notin \mathbb{Z}$ contracts being revoked per period, the first and third periods will have 5 new units sorted into H_V and the second and fourth periods will have 4 new units sorted into H_V. Each agent $i \in I_E$ revoking his or her contract during $[t_0, T]$ is assumed to be uninterested in all $h_j \in H \diagdown \{h_{0i}\}$ for the remainder of the 180 day period. For the simulations, each $i \in I_E$ revoking his or her contract is therefore simply removed at the beginning of the matching process. Their original assignments in H_O are then continuously placed into H_V, evenly distributed across $[t_0, T]$. In reality, the amount of contract that are revoked varies over time and would not reflect the even distribution in these simulations. Nevertheless, what is of importance is that the distribution of revoked contracts over the entire time frame $[t_0, T]$ is the same in every simulation. A necessary assumption for these simulations is that the periods $t \in [t_0, T]$ are treated as a series of static games by all $i \in I$. The strategies for each period $t \in [t_0, T]$ are independent of one another. There are no dynamic strategies, in which some agents let guesses of which housing units will be available in the upcoming periods influence their actions. Consequently, no $i \in I$ will reject an assignment of any $h_j \in H \succ_i \lambda(i)$ in any period $t \in [t_0, T]$.

4.4.1 AF Bostäder's current mechanism

AFB's current mechanism was the first to be simulated using the randomized population. The booking period is three days long, which gives 60 periods $t \in [t_0, T]$. In each period, one or two housing units $h_j \in H \diagdown \{h_0\}$ are taken out of the stock of units for which the original tenant revoked his or her contract during the 180 day period and sorted into H_V. Each $h_j \in H_V$ is then immediately assigned to some $i \in I$. If a housing unit $h_j \in H_V$ is assigned to an agent $i \in I_E$ in period t, $\lambda(i) \in H_O$ will be available in H_V in period $t+1$. In any period, the process begins by assigning $f(1)$ his or her most preferred unit $h_j \in H_V \cup \{\lambda(f(1))\}$. Upon being assigned a unit $h_j \in H_V$, the agent is removed from the process. If $f(1)$ is assigned no $h_j \in H_V$, $\lambda(f(1))$ remains assigned to the agent, who keeps his or her position in f. Next, $f(2)$ is assigned his or her most preferred unit $h_j \in H_V \cup \{\lambda(f(2))\}$ according to the same procedure as $f(1)$. This continues until $H_V = \emptyset$, at which point $t+1$ begins and the procedure is repeated. The whole process ends after period $T = t_{60}$.

4.4.2 AF Bostäder's mechanism with longer booking period

The second mechanism to be simulated is identical to AFB's current mechanism, with the exception that the booking period is extended from three days to nine days, giving 20 periods $t \in [t_0, T]$.

4.4.3 YRMH-IGYT

The fourth mechanism to be simulated is the YRMH-IGYT mechanism. The matching is carried out on the last day of the 180 day time frame, so that all the 90 housing units previously assigned to the agents who revoked their contract during this time frame are made available in H_V for reallocation. Thus, there is only one period $t \in [t_0, T]$. The procedure is carried out exactly as described in section 2.3.5 and terminated once $H_V = \emptyset$.

4.4.4 YRMH-IGYT spot mechanism

The last measure to be simulated is a dynamic spot mechanism version of the YRMH-IGYT mechanism. It is simulated in the same way as the static YRMH-IGYT mechanism, only that the same mechanism is repeated in 20 periods $t \in [t_0, T]$. Any agent i pointing to his or her own unit keeps the position in f and keeps pointing to $\lambda(i)$ until some $h_j \succ_i \lambda(i)$ becomes available. Any agent assigned a new housing unit is simply removed from the process and assumed to not want to change housing again during the remainder of the matching process. However, for real life applications, such as AFB's house allocation problem, any agent $i \in I_E$ may at any time rejoin the process. As this possibility is assumed away in the present simulation, all of the loops are located in the first period, after which the mechanism reduces to the serial dictatorship with waiting list spot mechanism.

5 Results

Subsection 5.1 will introduce the efficiency measures used to evaluate the matchings produced in the simulations. In subsection 5.2, the outcomes of the simulations will be presented in terms of the different efficiency measures. Finally, the efficiency of the different mechanisms will be compared in subsection 5.3.

5.1 Efficiency measures

Comparing the resulting matchings requires some common measure of efficiency. The different mechanisms are compared using the following four efficiency measures.

The first measure of efficiency is given by the size of $\{i \in I : \mu \succ_i \lambda\}$, where a larger set is more efficient. In other words, it is given by the number of agents who prefer their new assignments to their original assignments. This measure is best suited for individually rational static mechanisms and acceptable dynamic mechanisms, as they ensure that $\{i \in I : \mu \prec_i \lambda\} = \emptyset$. If the mechanism is not individually rational or acceptable, a way to account for the agents that are made worse off needs to be incorporated. All mechanisms simulated in this paper are either individually rational or acceptable.

The second measure of efficiency is the number of housing units $h_j \in H \diagdown \{h_0\}$ assigned to some $i \in I_E$ as that agent's most preferred alternative and the third measure of efficiency is the number of housing units $h_j \in H \diagdown \{h_0\}$ assigned to some $i \in I_E$ as one of that agent's top three alternatives.

The fourth measure combines the ideas of the previous measures into a single measure. The general idea of the measure is a summation of the ranking of $\mu(i)$ for each $i \in I_E$, where the *ranking* is 1 if the housing unit is the agent's most preferred alternative, 2 if it is the agent's second most preferred alternative, and so on. This means that a lower score will imply a more efficient allocation. As each preference profile P_i is defined over $C \cup \{h_0\}$, rather than H, the ranking of each $h_j \in H \diagdown \{h_0\}$ is given by the ranking of the class c_m the unit belongs to, according the preferences of the agent assigned that unit. The ranking of some $h_j = \mu(i) \in c_m$ is given by the position of that c_m in the ordinal ranking P_i of $C \cup \{h_0\}$. Each unassigned $h_j \in H_V$ under μ is included in the summation as well, given a ranking of 11, which is one step lower than the lowest ranking $c_m \subset C$ for any $i \in I$. This means that it is the efficiency of the allocation of each $h_j \in H \diagdown \{h_0\}$ that is being measured, rather than how satisfied the population I is. One argument for this would be that the size of I shouldn't affect a measure of how efficiently housing units are allocated. The measure does not use the actual ranking of each $h_j \in H \diagdown \{h_0\}$, but rather the square root of each ranking. This decision was based on an assumption of increasing negative marginal utility of being assigned a lower ranking $h_j \in H \diagdown \{h_0\}$. Note that this transformation in no way affects the ordinal structure between rankings. In another experiment designed to evaluate different house matching mechanisms, Chen and Sönmez (2003) used the fraction of the aggregate utility of the μ induced by φ over the aggregate utility of the Pareto efficient matching as an efficiency measure of the matching mechanism. It would theoretically be possible to use a similar measure to evaluate the mechanisms in section 4. However, due to the complexity of the example chosen, there are most likely several different Pareto efficient matchings, and no easy way to determine what they are, let alone decide which of them represents the social optimum. For this reason, a different but similar approach was chosen, where the sum of the square root of the rankings is divided by the sum of the square root of the ranking of all $h_j \in H \diagdown \{h_0\}$ when they are put to their least efficient use, i.e. if they are all left unassigned to any $i \in I$. In this example, it means that the sum of the square root of the actual rankings would be divided by $\sqrt{11} \times m$, yielding a number in

$$\left[\frac{344}{\sqrt{11} \times m}, 1\right].$$

Here, a score of one equals the efficiency of a matching $\mu \in \mathscr{M}$ where

$$\{\mu(i) : \mu(i) \in H \diagdown \{h_0\}\} = \emptyset.$$

In other words, a score of one equals the least efficient matching between $H \diagdown \{h_0\}$ and I imaginable. To produce a neater and more intuitive measure, each ranking is subtracted by one, yielding a number in $[0, 1]$, where 1 retains its original interpretation and 0 corresponds to a matching $\mu \in \mathscr{M}$ where each $h_j \in H_O$ is assigned to an $i \in I_E$ as his or her top choice and $H_V = \emptyset$. The final formula for the fourth measure is given by expression 2.

$$\frac{\sum_{j=1}^{m} \sqrt{(\text{ranking of } h_j) - 1}}{\sqrt{10} \times m} \tag{2}$$

It should be noted that this measure is inferior to the measure employed by Chen and Sönmez (2003) in the sense that their measure actually compares the outcome to a social optimum that is attainable for all $P \subset \Omega^n$, whereas for the measure employed in this paper, a score of 0 is only attainable for some small number of $P \subset \Omega^n$ for any $H \subseteq \mathscr{H}$. The measure used here is therefore not suitable for comparisons across experiments with different $I \subseteq \mathscr{I}$ and $H \subseteq \mathscr{H}$. However, for the purpose of this paper, it provides a measure that can be used to compare and evaluate the different mechanisms in section 2.3 in this particular context. To summarize, the scores in the following four efficiency measures is be reported in section 5.2.

1. $|\{i \in I : \mu \prec_i \lambda\}|$

2. The number of agents assigned his or her most preferred alternative.

3. The number of agents assigned one of his or her top three alternatives.

4. $\frac{\sum_{j=1}^{m} \sqrt{(\text{ranking of } h_j)-1}}{\sqrt{10} \times m}$

These measures will be referred to as the first, second, third and fourth measure respectively.

5.2 Outcomes

5.2.1 The original random matching

Under the original matching λ, 30 housing units, about 8.7 % of $H \setminus \{h_0\}$, were the top choices of the agents to whom they were assigned and 86 units, about 25.0 % of $H \setminus \{h_0\}$, were one of the top three choices of the agents to whom they were assigned. The fourth efficiency measure gave λ a score of 0.716.

5.2.2 AF Bostäder's current mechanism

The matching mechanism currently employed by AFB reallocated 255 housing units and left five units unallocated in period T. As theorem 10 states, AFB's current mechanism is acceptable. This means that every $i \in I_E$ for which $\mu(i) \neq \lambda(i)$ is an element in $\{i \in I : \mu \succ_i \lambda\}$. The size of this set is thus 255 agents $i \in I_E$. 98 housing units, about 28.5 % of $H \setminus \{h_0\}$, were the top choices of the agents to whom they were assigned and 220 units, about 64.0 % of $H \setminus \{h_0\}$, were one of the top three choices of the agents to whom they were assigned. The fourth efficiency measure gave AFB's current mechanism a score of 0.407.

5.2.3 AF Bostäder's mechanism with longer booking period

When the booking period of AFB's current mechanism was extended from three days to nine days, 239 housing units were reallocated and eight units were left unallocated. The proof of theorem 10 is independent of the length of the booking period, making the mechanism with prolonged booking period acceptable as well. There are thus 239 elements $i \in I_E$ in $\{i \in I : \mu \succ_i \lambda\}$. 114 housing units, about 33.1 % of $H \setminus \{h_0\}$, were the top choices of the agents to whom they were assigned and 231 units, about 67.2 % of $H \setminus \{h_0\}$, were one of the top three choices of the agents to whom they were assigned. The fourth efficiency measure gave the mechanism with a prolonged booking period a score of 0.375.

5.2.4 YRMH-IGYT

The YRMH-IGYT mechanism reallocated 297 housing units and no units were left unassigned. By theorem 9, YRMH-IGYT satisfies individual rationality. There are thus 297 elements $i \in I_E$ in $\{i \in I : \mu \succ_i \lambda\}$. 195 housing units, about 56.7 % of $H \setminus \{h_0\}$, were the top choices of the agents to whom they were assigned and 334 units, about 97.1 % of $H \setminus \{h_0\}$, were one of the top three choices of the agents to whom they were assigned. The fourth efficiency measure gave the YRMH-IGYT mechanism a score of 0.179.

5.2.5 YRMH-IGYT spot mechanism

The YRMH-IGYT spot mechanism reallocated 300 housing units and no units were left unassigned. As the YRMH-IGYT is acceptable, there are 300 elements in $i \in I_E$ in $\{i \in I : \mu \succ_i \lambda\}$. 167 housing units, about 48.5 % of $H \setminus \{h_0\}$, were the top choices of the agents to whom they were assigned and 284 units, about 82.6 % of $H \setminus \{h_0\}$, were one of the top three choices of the agents to whom they were assigned. The fourth efficiency measure gave the YRMH-IGYT spot mechanism a score of 0.251.

5.3 Comparisons

The results for the first measure; $|\{i \in I : \mu \succ_i \lambda\}|$, are given in figure 1. The YRMH-IGYT spot mechanism received the highest score of 300. The static YRMH-IGYT mechanism performed slightly worse than its spot mechanism counterpart, but only by a margin of 3 agents. AFB's current mechanism performed better than the mechanism with a longer booking period, receiving scores of 255 and 239 respectively. This was to be expected, due to the difference in roundaboutness mentioned in section 4.3.

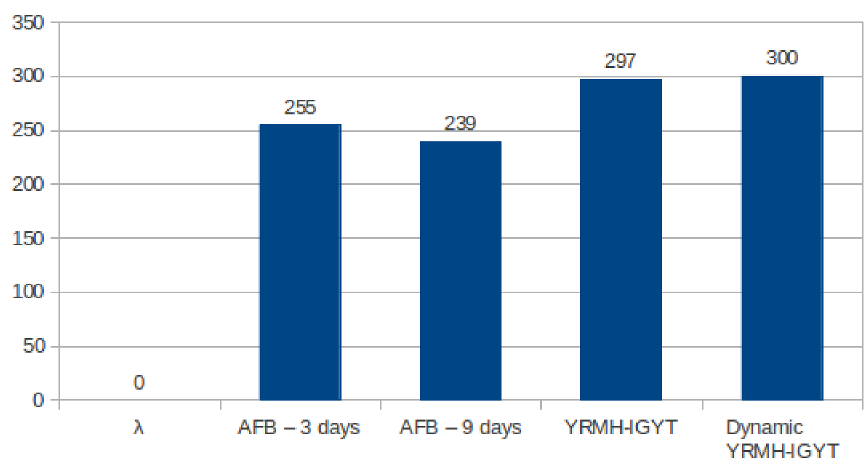

Figure 1: $|\{i \in I : \mu \succ_i \lambda\}|$

The results for the second measure; the number of housing units assigned as the agent's most preferred alternative, are given in figure 2. The YRMH-IGYT mechanism received the highest score of 195 housing units, 56.7 % of $H \setminus \{h_0\}$. The YRMH-IGYT spot mechanism received the second highest score of 167 units, 48.5 % of $H \setminus \{h_0\}$, and the mechanism with a longer booking period performed better than AFB's current mechanism, with percentage scores of 33.1 % and 28.5 % respectively.

The results for the third measure; the number of housing units assigned as one of the agent's top three alternatives, are given in figure 3. Once again, the YRMH-IGYT mechanism received the highest score of 334 housing units, 97.1 % of $H \setminus \{h_0\}$. The YRMH-IGYT received the second highest score of 284 housing units, 82.6 % of $H \setminus \{h_0\}$ and the mechanism with a longer booking period performed better than AFB's current mechanism, with percentage scores of 67.2 % and 64.0 % respectively.

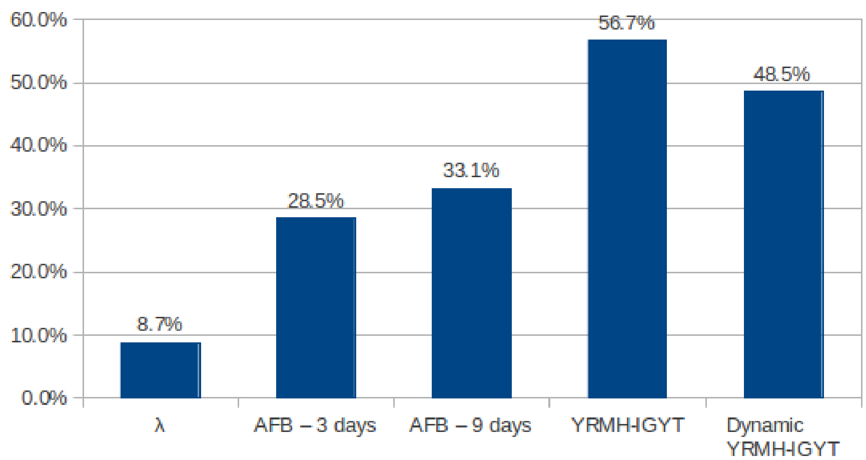

Figure 2: Share of $H \diagdown \{h_0\}$ assigned as agent's top choice.

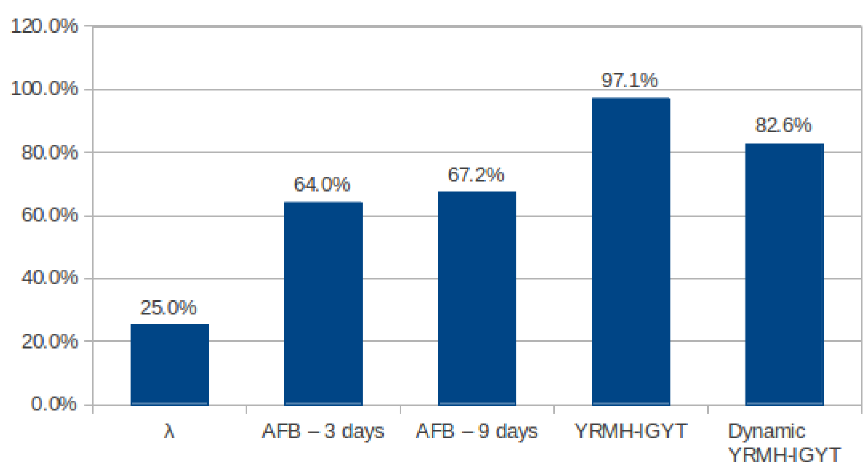

Figure 3: Share of $H \diagdown \{h_0\}$ assigned as one of agent's top three choices.

The results for the fourth measure are given in figure 4. Even in this measure, the YRMH-IGYT mechanism receives the best score of 0.179. The YRMH-IGYT receive the second best score of 0.251 and the mechanism with a longer booking period performed better than AFB's current mechanism, with scores of 0.375 and 0.407 respectively.

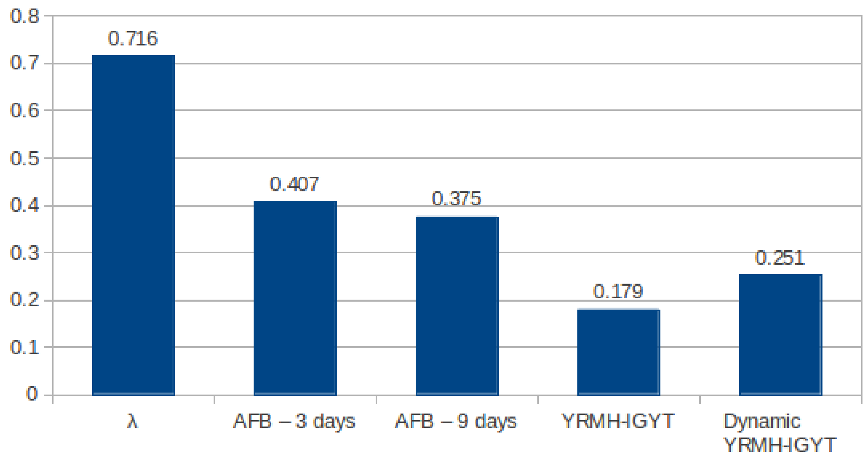

Figure 4: $\frac{\sum_{j=1}^{m}\sqrt{(\text{ranking of } h_j)-1}}{\sqrt{10}\times m}$

All of the results are summarized in table 6, where the different mechanisms receive a ranking for each measure, first being the most efficient.

Table 6: Results

	AFB - 3 day BP	AFB - 9 day BP	YRMH-IGYT	Dynamic YRMH-IGYT
Measure 1	3rd	4th	2nd	1st
Measure 2	4th	3rd	1st	2nd
Measure 3	4th	3rd	1st	2nd
Measure 4	4th	3rd	1st	2nd

6 Conclusion

The purpose of this paper was to evaluate different mechanisms as solutions to AFB's house allocation problem. Some of the described mechanisms have been shown to be unsuitable in AFB's particular context. The serial dictatorship mechanism is unable to handle existing tenants and the serial dictatorship with squatting rights mechanism is not individually rational. The remaining mechanisms of interest were then different variants of the serial dictatorship with waiting list mechanism and the top trading cycles mechanism. AFB's current mechanism was shown to be a non-direct variant of the dynamic serial dictatorship with waiting list spot mechanism. The static variants of the top trading cycles mechanism have been shown to satisfy some desirable properties which the variants of the serial dictatorship with waiting list mechanism do not satisfy, namely strategy-proofness and Pareto consistency. To evaluate the suitability of AFB's current mechanism as a solution to AFB's house allocation problem, simulations were performed, letting a fictional population with randomized preferences and a set of housing units based on three of AFB's housing districts be processed through AFB's mechanism and three additional mechanisms. The simulations showed that more agents would be assigned their top choices and more agents would be assigned one of their top three alternatives if the booking period of AFB's current mechanism were extended. The extended booking period would also yield a better score of 0.375 compared to AFB's current mechanism's score of 0.407 in the fourth measure. To remind the reader of the interpretation of these numbers, a score of 0 represents the best imaginable matching and a score of 1 represents the worst imaginable matching. The intuition of this result is that the longer the booking period is, the more units are available for booking at any point in time, allowing for a better match between agents and housing units. The downside of such an adjustment is related to the non-directness property of AFB's current mechanism. Non-directness restricts the amount of steps that can be carried out in each period and under such a restriction, a shorter booking period gives the mechanism more "roundaboutness". This means that the population is processed through the mechanism faster, allowing it to make a larger amount of reallocations, albeit less efficiently. The decision of how long the booking period should be would then have to weigh these two opposing effects in accordance with the preferences of the decision maker.

The static YRMH-IGYT mechanism, based on the top trading cycles mechanism should in theory, due to its Pareto consistency and directness, perform better than both variants of AFB's current mechanism in all of the four measures. The YRMH-IGYT mechanism also performed accordingly in the simulations. Compared to AFB's current mechanism, the YRMH-IGYT mechanism improved the situation for 297, as opposed to 255 agents. 56.7 %, as opposed to 28.5 % of the apartments were the top choices of their tenants and 97.1 %, as opposed to 64.0 % of the apartments were one of the top three alternatives of their tenants. Moving from AFB's current mechanism to the YRMH-IGYT mechanism improved the score of the fourth measure from 0.407 to 0.179. The downside of this mechanism is the fact that it is a static mechanism, making it impractical for AFB's house allocation problem. Its static nature requires all units to be reallocated at a single point in time. In other words, there can only be one period. Transforming it into a multi-period dynamic spot mechanism solves this problem, but in doing so it partly loses the theoretical properties making it superior to the different variants of the serial dictatorship with waiting list mechanism. Nevertheless, the simulations showed that some of the efficiency of the static mechanism was inherited by the dynamic spot mechanism variant. While scoring lower than its static counterpart on all but the first measure, the YRMH-IGYT spot mechanism still outperformed both variants of AFB's current mechanism in all four measures. Compared to AFB's current mechanism, the YRMH-IGYT spot mechanism improved the situation for 300, as opposed 255 agents. 48.5 %, as opposed to 28.5 % of the apartments were the top choices of their tenants and 82.6 %, as opposed to 64.0 % of the apartments were one of the top three alternatives of their tenants. Moving from AFB's current mechanism to the YRMH-IGYT mechanism improved the score of the fourth measure from 0.407 to 0.251. These results indicate that AFB's current mechanism is a suboptimal solution to the house allocation problem it faces. More efficient matchings could be found using some variant of the top trading cycles mechanism, such as the YRMH-IGYT spot mechanism.

For future examinations of similar house allocation problems, some possible alterations to method and theory should be considered. The model used in this paper lets agents leave the population continuously, but all of the new arrivals enter the population at the beginning of the matching process. It might be a realistic assumption when it comes to student housing, as students tend to move in at the start of a semester. However, for some applications, it might be more appropriate to allow agents to enter the population continuously as well. Additional mechanisms could also be examined, such as the MIT mechanism, which outperformed the theoretically superior top trading cycles mechanism on some accounts in an experimental working paper by Guillen and Kesten (2008). Furthermore, the mechanisms could be evaluated further by examining additional desirable properties, such as the nonbossiness and neutrality properties described by Svensson (1999) or the weak neutrality property described by Sönmez and Ünver (2010). Finally, it could be illuminating to develop models for simulations of dynamic mechanisms based on the theoretical framework of Abdulkadiroglu and Loertscher (2007) or Bloch and Cantala (2011), rather than that of Kurino (2009).

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