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An Empirical Study on Term Structure Models

— Evidence from Sweden

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ABSTRACT

The purpose of this paper is to test for the goodness-of-fit of the term structure models for the short-term interest rate in Sweden. The study period ranges from February 1, 2002 to March 31, 2012. The maximum likelihood method is used to estimate the models and the likelihood ratio test is applied to perform a comparison among the models. The implication of empirical results can be summarized as four issues: Firstly, the volatility of the short-term interest rate in Sweden follows a GARCH process. Secondly, the interest rate might evolve according to a non-mean-reverting process. Thirdly, the volatility of Swedish short-term interest rate is not heavily dependent of the level of current interest rate. Finally, only the models that allows for low-elastic volatility to the current rate can measure Swedish interest rate well.

Key Words: Term Structure, Short-Term Interest Rates, Maximum Likelihood Estimation

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1. Introduction

The term structure of short-term interest rates plays a significant role in financial markets. It can be applied in the pricing and valuation of bonds and other financial instruments, such as mortgage-backed securities. Due to the importance, many researchers have devoted in the study on this subject both theoretically and empirically.

Scholars have developed different models to capture the dynamic behavior of short-term interest rates, for instance, the Constant Elasticity of Variance model (the CEV model thereafter) introduced by Cox (1975), the Cox-Ingersoll-Ross Variable-Rate model (the CIR VR model thereafter) proposed by Cox, Ingersoll and Ross (1980), the Geometric Brownian Motion model (the GBM model thereafter) developed by Marsh and Rosenfeld (1983), and the Cox-Ingersoll-Ross Square Root model (the CIR SR model thereafter) presented by Cox, Ingersoll, and Ross (1985) etc. These models describe the stochastic process of short-term interest rates with different restrictions. Chan et al. (1992a) perform an empirical comparison among these models. To conduct the study, they generalize the models and nest them by the CKLS model in their paper. This is referred to as the CKLS framework. In the CKLS framework the volatility of interest yields is assumed to be time-invariant. Brenner et al. (1996) suggest that the volatility of change in short-term interest rates does not only depend on the current rate, but might be affected by news effects on financial markets as well. They thus develop a new framework, which manages to measure both the effects of the current interest and the impacts of news on the volatility, to model the term structure. This is referred to as the BHK framework. In addition, from the empirical perspective, some researchers adopt a discrete-time approximation to measure these continuous-time models. A drawback of this approach is that it might cause temporal aggregation bias. To overcome this problem, Nowman (1997) develops a new approach of model specification which allows for the use of a precise maximum likelihood estimator. Antoniou et al. (2005) follow Nowman's specification in their research on the short-term interest rate in the UK.

However, little empirical study is performed to investigate the short-term interest rate in

Sweden. The only one known is presented by Dahlquist (1996), with a study period between January 1, 1984 and August 31, 1992. Nevertheless, in this study the author only applies the CKLS framework and does not take the time-varying volatility into account. There has not been a study on Swedish interest rate that accounts for news effects on the volatility, at least as known. Are there news effects on the volatility of Swedish interest rate? How does the yield of Swedish short-term interest rate dynamically evolve and how do the models that have been developed explain the dynamics? Which one of the models is more suitable to capture the behavior of the short-term interest rate in Sweden? To answer these questions, this paper focuses on an empirical test for goodness-of-fit of short-term interest rate models, using daily data from February 1, 2002 to March 31, 2012. The study examines the short-term interest rate in Sweden by taking both the CKLS framework and the BHK framework into account in order to investigate whether there exit news effects on the volatility of Swedish interest rate. The main idea follows the studies conducted by Chan et al. (1992a), Brenner et al. (1996) and Antoniou et al. (2005). In this paper, the maximum likelihood method is applied to estimate the models and the likelihood ratio test is used to test for the validity of restrictions on parameters.

The empirical results imply four issues: Firstly, the results of the likelihood ratio test suggest rejection of the Level model and imply that the short-term interest rate in Sweden is of a time-varying volatility. It follows a GARCH process. Secondly, the estimates of α and β are not significant, which imply that the short-term interest rate in Sweden evolves according to a non-mean-reverting process. Thirdly, the significant γ-estimates are fairly low, indicating that the volatility of Swedish interest rate is of low elasticity to the current interest rate. Fourthly, all models with restrictions on γ are rejected by the likelihood ratio test. This implies that only the models allowing for a low but non-zero value of γ describes the interest rate in Sweden well.

The remainder of the thesis is organized as follows. Section 2 discusses some models that have been proposed to capture the dynamics of short-term interest rates and the intuitions behind them, as well as some empirical evidence on short-term interest rates and term structure models. Section 3 deliberates the estimation method, the model specification and the testing strategy. Section 4 describes the data and presents some summary statistics. The empirical results are reported and interpreted in Section 5. Section 6 provides the conclusion.

2. Theoretical Framework

The term structure of short-term interest rates means that the yield of the short-term interest rate is a function of the time to the maturity. This implies that the interest rate behaves as a stochastic process, i.e. the change in the short-term interest rate depends on both a deterministic term and a diffusion term. Such a stochastic process can be generally written as the following mathematical form

$$dr = \Theta(\mathrm{r})dt + \Sigma(\mathrm{r})dz, \quad (1)$$

where the function $\Theta(\mathrm{r})$ represents the level of the change in the interest rate while $\Sigma(\mathrm{r})$ the volatility, and z is a function of time, t, following a Brownian motion process with the increment dz.[1] The first term of the RHS of Eq. (1) is called the deterministic or the drift term, which captures the level of the change in the short-term rate, while the second term of the RHS is the diffusion term, which describes the volatility of the change in the interest rate.

Many researchers focus on the investigation of such a dynamics and provide a variety of models to capture the behavior of short-term interest rates. The following of this section reviews some classical models as well as some empirical evidence. Subsection 2.1 discusses the CKLS framework presented by Chan et al. (1992a), which consists nine models— the CKLS model, the CIR SR model, the Brennan-Schwartz model, the Vasicek model, the CEV model, the GBM model, the Merton model, the Dothan model, and the CIR VR model. Subsection 2.2 discusses the critique on the CKLS framework and presents the BHK framework proposed by Brenner et al. (1996). Finally, some empirical evidence on short-term interest rates and term structure models is reviewed in Subsection 2.3.

[1] A brief introduction to Brownian motion is presented in Appendix A1.

2.1 The CKLS Framework

The CKLS framework is presented by Chan et al. (1992a) when they conduct an empirical study on term structure models. In their paper the authors compare eight classical short-term interest rate models—the CIR SR model, the Brennan-Schwartz model, the Vasicek model, the CEV model, the GBM model, the Merton model, the Dothan model, and the CIR VR model—and test for the goodness-of-fit of the models. To conduct their research, they nest the eight models by a generalized form—the CKLS model, which can be expressed by the following equation:

$$dr = \kappa(\bar{r} - r)dt + \sigma r^{\gamma} dz, \quad (2)$$

where $\kappa, \bar{r}$ and σ are positive constants. The parameter $\bar{r}$ represents the long-run mean of interest rates and σ is the standard deviation of the change in interest rates. The parameter γ measures the elasticity (i.e. the sensitivity) of volatility to the level of current interest rate, r.

The CKLS model is of mean-reversion, which implies that the short-term interest rate tends to its long-term mean, i.e., given $\kappa > 0$, if the short-term interest rate, r, is higher than the long-run mean, $\bar{r}$, the expected change in the interest rate will be negative, and vise versa. Therefore, the parameter κ plays a role as the speed of adjustment to the long-term mean. It also implies a negative relation between the behavior of the yield and the level of the interest rate, that is, a higher short rate implies a relatively lower change in the interest. The CKLS model can also be expressed as

$$dr = (\alpha + \beta r)dt + \sigma r^{\gamma} dz, \quad (3)$$

where $\beta = -\kappa$ and $\alpha = \kappa\bar{r}$. Therefore, if $\alpha > 0$ and $\beta < 0$, the short-term interest rate is of mean-reversion, otherwise it is not mean-reverting.

The expected value and the conditional variance of short-term interest rate measured by the CKLS model are given by

$$E[dr] = (\alpha + \beta r)dt, \quad (4)$$

and

$$Var[dr] = \sigma^2 r^{2\gamma} dt. \quad (5)$$

Hence, the CKLS model implies that both the expected value and the volatility of the interest yield depend the current rate, and to what extent the volatility is affected is measured by the parameter γ.

Vasicek (1977) proposes a model when he devotes in deriving a general form of the term structure of interest rates. In his study, three assumptions are imposed: a Markov-process interest rate, which implies that the interest rate is only determined by its current value; a spot-rate-determined price of bond, i.e. the value of a discounted bond is only determined by the current assessment of the spot interest rate; and an efficient market, that is, no transaction costs exist and information is available to all agents who act rationally. The Vasicek model can be expressed as

$$dr = \kappa(\bar{r} - r)dt + \sigma dz. \quad (6)$$

As mentioned in the above, the Vasicek model can be rewritten as

$$dr = (\alpha + \beta r)dt + \sigma dz. \quad (7)$$

Therefore, the Vasicek model can be regarded as a special case of the CKLS model, with a restriction $\gamma = 0$. This implies that the volatility of the yield of interest rate is not affected by the current interest.

Brennan and Schwartz (1977, 1979, 1980) present the Brennan-Schwartz model in their papers when analyzing the value of bonds. They assume that the short-term interest rate evolves according to the following stochastic process:

$$dr = \kappa(\bar{r} - r)dt + \sigma r dz. \quad (8)$$

The Brennan-Schwartz model can also be rewritten as the following form

$$dr = (\alpha + \beta r)dt + \sigma r dz. \quad (9)$$

The Brennan-Schwartz model is also a mean-reverting model, but with a difference from the Vasicek model that the volatility of interest yields does depend on the current level of the interest rate because it assumes $\gamma = 1$.

Another mean-reverting model, the CIR SR model, is developed by Cox et al. (1985). They assume that the short-term interest rate evolves according to

$$dr = \kappa(\bar{r} - r)dt + \sigma\sqrt{r}dz. \quad (10)$$

Eq. (10) can also be written as

$$dr = (\alpha + \beta r)dt + \sigma\sqrt{r}dz. \quad (11)$$

The CIR SR model implies that the volatility is dependent of the level of current interest rate, but the impact of the current rate is less than that in the Brennan-Schwartz model since $\gamma = 0.5$ is assumed.

Merton (1973) investigates the option pricing theory and used the Merton model as an instrument to derive the option pricing formula. This model can be generally expressed as

$$dr = \alpha dt + \sigma dz. \quad (12)$$

Eq. (12) implies that the short-term interest rate evolves according to a Brownian motion process, which means that the short-term interest rate, r, satisfies four conditions: First, the original value of the interest is zero, i.e. $r_0 = 0$. Second, the process of the interest rate has independent increments, i.e., for $0 \le a < b \le t < c$, the increments $r_c - r_t$ and $r_b - r_a$ are independent. Third, for $0 \le s < t$, $r_t - r_s$ is normally distributed with mean zero and variance $t - s$, i.e., $r_t - r_s \sim N(0, t - s), if\ 0 \le s < t$. Fourth, the short-term interest rate, r, has continuous trajectories.

Different from the previous models discussed, the Merton model is not of mean-reversion, which implies that the short rate does not tend to its long-term mean. The Merton model indicates that both the level and the volatility of the interest yield are not affected by the current interest rate, and the conditional variance is constant over time.

The GBM model is developed by Mash and Rosenfeld (1983) when they investigate the stochastic process of interest rates and equilibrium prices of bonds. In their research, the authors assume that the interest is lognormally distributed and propose that the short-term interest rate should evolve according to a stochastic process as

$$dr = \beta r dt + \sigma r dz, \quad (13)$$

which implies a geometric Brownian motion process (a GBM process thereafter). A GBM-evolving interest rate indicates that the interest, r, has a stochastic differential equation as follows:

$$\frac{dr}{r} = \beta dt + \sigma dz, \quad (14)$$

$$r_0 > 0.$$

By defining $y = \ln r$ and applying Ito's lemma, it can be shown that

$$dy = \left(\beta - \frac{1}{2}\sigma^2\right) dt + \sigma dz. \quad (15)$$

Hence,

$$y_t - y_0 = \ln r_t - \ln r_0 = \left(\beta - \frac{1}{2}\sigma^2\right) t + \sigma z, \quad (16)$$

and

$$r_t = r_0 . \exp\left\{\left(\beta - \frac{1}{2}\sigma^2\right) t + \sigma z\right\}. \quad (17)$$

Eq. (17) implies that r_t is always larger than zero given $t \geq 0$. It also indicates that the logarithm of r_t is normally distributed, i.e.

$$\ln r_t \sim N\left(\ln r_0 + \left(\beta - \frac{1}{2}\sigma^2\right)t, \sigma^2 t\right).$$

The GBM model indicates that both the level and the volatility of the interest yield are affected by the current rate.

Dothan (1978) derives a valuation formula of default-free bonds by assuming that the short-term interest rate evolves according to the following stochastic process:

$$dr = \sigma r dz. \quad (18)$$

Eq. (18) is the expression of the Dothan model and implies that the short-term interest rate behaves following a GBM process without a drift term.

The CIR VR model is proposed by Cox et al. (1980) when they conduct an analysis of variable rate loan contracts. They propose that the stochastic evolution of the spot interest is given by

$$dr = \sigma r^{3/2} dz. \quad (19)$$

The CIR VR model indicates that the volatility of the change in interests is heavily affected by the current interest rate because

$$Var[dr] = \sigma^2 r^3. \quad (20)$$

Another model that is not of mean-reversion is the CEV model presented by Cox (1975) and by Cox and Ross (1976). The CEV model can be expressed as follows:

$$dr = \beta r dt + \sigma r^\gamma dz. \quad (21)$$

The CEV model implies that both the level and the volatility of the yield are affected by the current short-term interest rate.

Hence, the CKLS framework can be summarized in the following table.

Table 1: Summary of the CKLS Framework

Model	**Functional Form**	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$
CKLS	$dr = (\alpha + \beta r)dt + \sigma r^{\gamma} dz$	-	-	-
Vasicek	$dr = (\alpha + \beta r)dt + \sigma dz$	-	-	0
Brennan-Schwartz	$dr = (\alpha + \beta r)dt + \sigma r dz$	-	-	1
CIR SR	$dr = (\alpha + \beta r)dt + \sigma\sqrt{r} dz$	-	-	0.5
Merton	$dr = \alpha dt + \sigma dz$	-	0	0
GBM	$dr = \beta r dt + \sigma r dz$	0	-	1
Dothan	$dr = \sigma r dz$	0	0	1
CIR VR	$dr = \sigma r^{3/2} dz$	0	0	1.5
CEV	$dr = \beta r dt + \sigma r^{\gamma} dz$	0	-	-

Note: In this table, only the restrictions on parameters are presented. The CKLS model is treated as an unrestricted model and thus no constraint is imposed on its parameters.

The CKLS model is regarded as an unrestricted model while the others as restricted ones. The Vasicek model, the Brennan-Schwartz model and the CIR SR model are only imposed restrictions on the parameter γ, with the restriction values equal to 0, 1, and 0.5. The Merton model and the GBM model are constrained by γ equal to 0 and 1 respectively, but the Merton model is further restricted by $\beta = 0$, whereas the GBM model $\alpha = 0$. The Dothan model and the CIR VR model are all restricted by both α and β equal to zero, but they are constrained by different conditions on γ, where in the Dothan $\gamma = 1$ and in the CIR VR $\gamma = 1.5$. The CEV model is not imposed any restriction on γ, and the unique constraint on it is $\alpha = 0$.

2.2 The BHK Framework

The models that consist of the CKLS framework are called " the Level models", because in these models the volatility of the interest rate is assumed only affected by the level of current interest rate or a constant over time (Antoniou, Brenales and Berermann, 2005), and thus the models are called, for instance "the Level-CKLS model" etc. However, this time-invariant volatility assumption is criticized. Brenner et al. (1996) suggests that the volatility of the interest yield should not only be determined by the current rate, but might be affected by news effects on the market as well. Thus the CKLS framework has a drawback that it cannot capture the volatility of interest rates caused by news effects. To overcome this weakness, they develop a new framework that accounts for news effects on the volatility, which is called the BHK framework. The BHK framework consists of two forms of models. The one, based on the CKLS model, can be expressed as

$$\Delta r_{t+1} = \alpha + \beta r_t + \varepsilon_{t+1}, \quad (22)$$

$$\varepsilon \sim N(0, \sigma_{t+1}^2 r_t^{2\gamma})$$

$$\sigma_{t+1}^2 = a_0 + a_1 \varepsilon_t^2 + b\sigma_t^2. \qquad (23)$$

This model is called the GARCH-CKLS model, because it assumes that the volatility of the interest yield follows a GARCH (1, 1) process, i.e. the volatility of the change in the interest rate in the next period (σ_{t+1}^2) is affected by both the unexpected shock (ε_t^2) and the volatility (σ_t^2) in the current period. Since it assumes $E[\varepsilon_{t+1}^2] = \sigma_{t+1}^2 r_t^{2\gamma}$, the model can capture the effects of both the current interest rate and news at the same time.

The other form in the BHK framework is to allow for asymmetric news effects on the volatility,[2] which can, for example based on the CKLS model, be expressed as follows:

$$\Delta r_{t+1} = \alpha + \beta r_t + \varepsilon_{t+1}, \quad (24)$$

$$\varepsilon \sim N(0, \sigma_{t+1}^2 r_t^{2\gamma})$$

$$\sigma_{t+1}^2 = a_0 + a_1 \varepsilon_t^2 + a_2 \varepsilon_t^2 D_t + b\sigma_t^2, \qquad (25)$$

[2] The meaning of asymmetric effects is that a negative shock imposes larger impacts on the volatility than a positive shock.

where D_t is a dummy variable equal to one if $\varepsilon_t < 0$ and zero otherwise.

Eq. (25) is an expression of a threshold GARCH model (TGARCH model) or a GJR model, named after the authors Glosten, Jagannathan and Runkle (1993). By introducing a dummy variable D_t, this model manages to measure asymmetric reaction of the volatility to shocks. Hence, this model is defined as the GJR-CKLS model.

2.3 Empirical Evidence on Short-term Interest Rates

Short-term interest rates have been empirically studied by a lot of researchers. In Chan et al. (1992a), the authors apply the GMM method to estimate the models and then perform a comparison among them by using interest rates of one-month treasure bills in the US. The authors obtained an estimate of γ approximately equal to 1.5 from the unrestricted model and 1.28 from the CEV model. They thus conclude that the evolution of interest rates is heavily sensitive to the current interest rate, i.e. the value of parameter γ is high, and thus the models that allow for highly current-rate-dependent volatility, for example the CIR VR model and the CEV model, perform better in measuring the dynamic behavior of short-term interest rates. Such a conclusion is verified by an application of their framework in Japan (Chan, Karolyi, Longstaff, Sanders, 1992b), where an estimate of γ approximately equal to 2.4 is obtained.

Tse (1995) uses data from eleven countries and performs an empirical study based on the CKLS framework. The estimates of γ of the interest in the US, Holland and France are respectively 1.73, 1.60 and1.62, and the author thus conclude that in these countries the volatility of short-term interest rates are heavily affected by the current rates. In the meantime, the author obtains low values of γ-estimates in Australia, Belgium, Canada, Germany, Italy, Japan, Switzerland and the UK. In particular in Switzerland, the γ-estimate is only about 0.04. Thus it is concluded that in these countries the volatility of interest rates are of low-elasticity to the current rates.

Dahlquist (1996) applies the CKLS framework and studies the short-term interest rates in

Denmark, Germany, Sweden and the UK. The sample period that he uses for Sweden spans from January 1, 1984 to August 31, 1992. What he finds is that the short-term interest rate in Sweden evolves according to a mean-reverting process and the estimate of γ is approximately equal to 1.15, which indicates that the volatility of Swedish interest is quite sensitive to the current interest rate. Meanwhile, he also concludes that the other three countries, Denmark, Germany and the UK the volatility is less sensitive to the levels of current interests, with the γ-estimates approximately equal to 0.97, 0.39, and 0.16 respectively.

Gray (1996) applies the CKLS framework and studies the evolution of the short-term interest rate in Australia, obtaining an estimate of γ approximately equal to 1.5. The author thus concludes that the volatility of the change in the interest rate in Australia is highly sensitive to the current rate.

Nowman (1998) conducts an empirical study on the term structure models with Euro-currency interest rates. The author obtains an estimate of $\gamma = 1.05$ for the US and 0.98 for Japan. Such results imply a different conclusion in Chan et al. (1992a) and Chan et al. (1992b), where the authors state that the volatility of the changes in the interest rates in America and Japan are highly sensitive to the spot rate.

Bayers and Nowman (1998) investigate the short-term interest rate in the UK and the US by using the continuous term structure models. The results with the interest rate in the UK indicate the γ-estimates of the one-month, three-month, six-month and twelve-month interest rates are approximately 1.43, 1.35, 1.26 and 1.35 respectively, whereas in the US show that the γ-estimates of the one-month, three-month, six-month, and twelve-month interest rates are 0.95, 1.34, 1.31 and 1.15 respectively.

Another study performed by Nowman (2002) tests the term structure models with the short-term interest rate in Japan. In this study, the author uses four different data series and the maximum estimate of γ obtained is about 0.35, which demonstrates that the volatility of Japanese interest is relatively non-sensitive to the level of current rate.

In addition, Nowman (2011) investigates the interest rate in the UK by using monthly spot yields with two, three, five, and ten-year maturities. The sample period ranges from January 1970 to March 2010. The author obtains estimates of γ about 0.70, 0.78, 0.89 and 0.90 for the two, three, five, and ten-year rates respectively. Such results indicate that the elasticity of volatility to the current rate is dependent of the maturity of bonds. The closer to the maturity, the lower the elasticity of the volatility to the current rate.

Brenner et al. (1996) apply the BHK framework in studying the short-term interest rate in the US and compare the results with those obtained from using the CKLS framework. The results indicate that the estimate of γ from the CKLS is higher than that from the BHK framework. They authors thus make a conclusion that the time-invariant volatility assumption in the CKLS framework might cause the value of γ to be overestimated and state that the misspecification of the volatility in the Level models might cause the estimate of γ to be biased.

Brailsford and Maheswaran (1998) also perform an investigation on the dynamics of the short-term interest rate in Australia. They first estimate the models with the time-invariant volatility assumption and obtain an estimate of γ approximately equal to 1.7, which implies the same conclusion as Gray's research (Gray, 1996). Meanwhile, the authors apply the BHK framework and obtain a γ-estimate approximately equal to 1.14, which is lower than the one obtained from the CKLS framework. Therefore, they conclude that the time-invariant volatility assumption might lead to the estimate of γ to be exaggerated, which supports the conclusion in Brenner et al. (1996).

Antoniou et al. (2005) conduct a research on the dynamics of the short-term interest rate in the UK. The authors perform a comparison between the CKLS and the BHK frameworks and obtain a higher γ estimate (about 1.5) in the CKLS than in the BHK framework (about 0.77). They thus conclude that the CKLS framework exaggerates the dependence of volatility on the level of current interest. This is also in line with the result in Brenner et al. (1996).

3. Methodology

The main purpose of this section is to present the research methodology used to estimate the models and test for the goodness-of-fit of the models. Subsection 3.1 states the estimation method—the ML estimation, and discusses its advantages against the GMM estimation. This is followed by a description of model specification in Subsection 3.2. At last, the testing strategy is presented in Subsection 3.3.

3.1 Estimation Method

The models discussed in Section 2 can be estimated by either the generalized method of moments (GMM) estimation or the maximum likelihood (ML) estimation.[3] Chan et al. (1992a, 1992b), for instance, perform the CKLS framework and use the GMM estimation because of its distribution-free advantage, whereas Brenner et al. (1996), Nowman (1997) and Antoniou (2005) prefer the ML estimation for the following reasons: First, Broze et al. (1995) propose in their Proposition 3 that the GMM estimation does not work well when $\gamma > 1$ (i.e. when the data series has a volatility with high sensitivity to the current rate). Due to a lot of empirical evidence, for example Chan et al. (1992a, 1992b) and Byers and Nowman (1998), showing $\gamma > 1$, adopting an ML estimator can avoid the problem of using a poorly-behaved estimator. Second, Dahlquist (1996) point out that the cost of using a GMM estimator compared to using an ML estimator is that the statistical tests are of less power.[4] Third, Broze et al. (1995) states that the ML estimation provides a more efficient procedure than the GMM estimation.[5] Another, and the most important, advantage of the ML estimation against the GMM approach is that a precise estimator can be introduced into the model estimation by using the functional form proposed by Nowman (1997). This will be discussed below. Due to these four reasons, this paper follows Brenner et al. (1996), Nowman (1997) and Antoniou et al. (2005), applying the ML approach to estimate the models.

[3] For more details about the GMM and ML estimation, see for example, Greene (2011).

[4] The probability of rejecting the null hypothesis when it is false is known as the power of the test. Therefore, a test of more power can avoid the Type Ⅱ error.

[5] Efficiency means the minimum variance of the estimates. Thus a more efficient estimator gives estimates with lower minimum variance.

3.2 Model Specification

To achieve the estimation, Chan et al. (1992a, 1992b) and Brenner et al. (1996) specify the model as the following form:

$$\Delta r_{t+1} = \alpha + \beta r_t + \varepsilon_{t+1}, \quad (26)$$

$$E[\varepsilon_{t+1}] = 0.$$

This specification is a discrete-time approximation of the continuous-time model. Grossman et al. (1987) and Nowman (1997) state that such an approximation might cause the "temporal aggregation bias" (Granger, 1969; Christiano and Eichenbaum, 1986; Gulasekaran and Abeysinghe, 2002), which have negative effects on the statistical inference of discrete-time models. Temporal aggregation occurs when a process is sampled slower than its natural evolution (Bay, Chrisman, Pohorille, Shrager, 2004). To deal with the temporal aggregation bias, Nowman (1997) presents an alternative specification approach as follows:[6]

$$r_{t+1} = e^{\beta} r_t + \frac{\alpha}{\beta}\left(e^{\beta} - 1\right) + \varepsilon_{t+1}, \quad (27)$$

$$E[\varepsilon_t \varepsilon_s] = 0 \ \ (t \neq s).$$

This functional form allows for the use of an exact ML estimator. The parameters can be estimated by the following assumptions of conditional variance specifications:

Model 1: Level-CKLS

$$E[\varepsilon_{t+1}^2] = \int_t^{t+1} e^{2(t-\tau)\beta} \sigma^2 r_t^{2\gamma} d\tau = \frac{\sigma^2}{2\beta}\left(e^{2\beta} - 1\right) r_t^{2\gamma} \quad (28)$$

Model 2: GARCH-CKLS

[6] For details about the derivation of the model, see Appendix A2.

$$E[\varepsilon_{t+1}^2] = \int_t^{t+1} e^{2(t-\tau)\beta} \sigma_{t+1}^2 r_t^{2\gamma} d\tau = \frac{\sigma_{t+1}^2}{2\beta}\left(e^{2\beta} - 1\right) r_t^{2\gamma} \quad (29)$$

$$\sigma_{t+1}^2 = a_0 + a_1 \varepsilon_t^2 + b\sigma_t^2 \quad (30)$$

Model 3: GJR-CKLS

$$E[\varepsilon_{t+1}^2] = \int_t^{t+1} e^{2(t-\tau)\beta} \sigma_{t+1}^2 r_t^{2\gamma} d\tau = \frac{\sigma_{t+1}^2}{2\beta}\left(e^{2\beta} - 1\right) r_t^{2\gamma} \quad (31)$$

$$\sigma_{t+1}^2 = a_0 + a_1 \varepsilon_t^2 + a_2 \varepsilon_t^2 D_t + b\sigma_t^2 \quad (32)$$

where D_t is a dummy variable equal to one if $\varepsilon_t < 0$ and zero otherwise.

Model 1 is the Level-CKLS model specified using Nowman's approach. Model 2 is the GARCH-CKLS model that captures both the news effects and the impacts of the current rate on the volatility. Model 3 is an extension of Model 2. It is the GJR-CKLS model that allows for asymmetric news effects on the volatility.

To estimate the models, the ML approach is used. With $E[\varepsilon_{t+1}^2] = m_{t+1}^2$, Nowman (1997) defines $L(\theta)$ for the Level-CKLS model as minus twice the logarithm of Gaussian likelihood function:

$$L(\theta) = \sum_{t=1}^{T} \left[2\ln m_{t+1} + \frac{\{r_{t+1} - e^{\beta} r_t - \frac{\alpha}{\beta}\left(e^{\beta} - 1\right)\}^2}{m_{t+1}^2} \right], \quad (33)$$

where $\theta = (\alpha, \beta, \gamma, \sigma^2)$ is a vector of parameters to be estimated. This implies that the maximization of the loglikelihood is equivalent to the minimization of Eq. (33). Thus to estimate the parameters, it needs

$$\min_{\theta} L(\theta) = \sum_{t=1}^{T} \left[2\ln m_{t+1} + \frac{\{r_{t+1} - e^{\beta} r_t - \frac{\alpha}{\beta}\left(e^{\beta} - 1\right)\}^2}{m_{t+1}^2} \right].$$

3.3 Testing Strategy

The main purpose of this paper is to study the goodness-of-fit of the short-term interest rate models, i.e. to test for the validity of restrictions. To conduct this investigation, the test procedure follows two scenarios. The first scenario is to compare the three general models shown in Subsection 3.2 and test for the validity of restrictions on the volatility. The models can be nested by the GJR-CKLS. Therefore, the GJR-CKLS model can be regarded as an unrestricted model while the Level-CKLS model and the GARCH-CKLS model as restricted ones. Table 2 summarizes the parameter restrictions on the models.

Table 2: Parameter Restrictions on the Models

Model	a_1	a_2	b
Level	0	0	0
GARCH	-	0	-
GJR	-	-	-

Note: Only the restrictions on parameters are shown in the table. The GJR-CKLS model is treated as an unrestricted model and thus no constraint is imposed.

The Level-CKLS model is restricted by $a_1 = 0$, $a_2 = 0$ and $b = 0$ since it assumes that the volatility of change in the interest rate is time-invariant, i.e. there are no GARCH or TGARCH effects on the volatility. The GARCH-CKLS model allows for both the impacts of current interests and news on the volatility and the only restriction on it is $a_2 = 0$, which implies that it does not account for the asymmetric news effects on the volatility.

The second scenario is to extend the investigation into the CKLS-nested models discussed in Subsection 2.1. By comparing the three general models in the first stage, the functional form of the volatility can be examined. If, for example, the Level and the GARCH models are rejected, then the GJR form of the volatility is assumed in the second scenario.

The likelihood ratio test (LR test thereafter) is applied to test for the validity of restriction conditions (the goodness-of-fit as well). To perform the LR test, the following steps should be carried out: First, estimate the model without restrictions. This gives the unrestricted estimates $\hat{\theta}$ and the loglikelihood function for the unconstrained model, $\ln L(\hat{\theta})$. The second step is to estimate the model with restrictions, which gives the restricted estimates $\tilde{\theta}$ and the loglikelihood function for the constrained model, $\ln L(\tilde{\theta})$. The LR test is to check whether the difference between these two loglikelihood values is significantly different from zero. The intuition is that the imposition of restrictions should not result in a large reduction in the loglikelihood function if they are valid. Therefore, under the null hypothesis that the restrictions are true, the test statistic is of a Chi-squared distribution with K degrees of freedom, i.e.

$$\xi_{LR} = 2[\ln L(\hat{\theta}) - \ln L(\tilde{\theta})] \sim \chi_K^2$$

where K is the number of restrictions.[7]

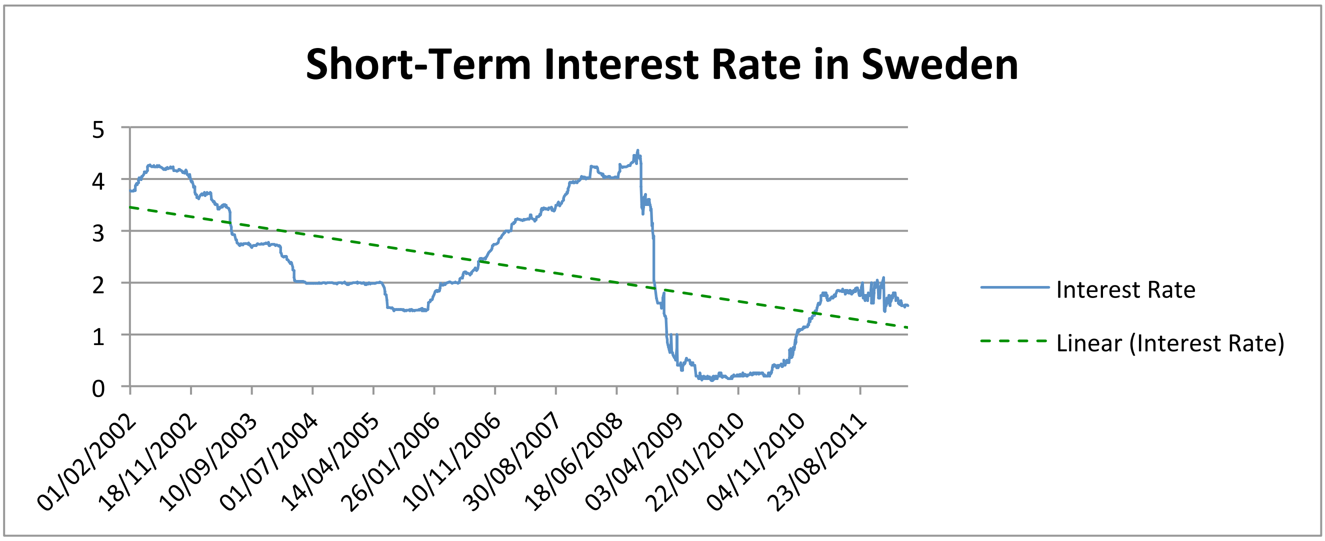
4. Data

This paper focuses on an investigation on the term structure of the short-term interest rate in Sweden. To achieve the study purpose, the interest rate of one-month treasure bills in Sweden is used as the underlying short-term interest rate. Daily data are collected from the database of Sveriges Riskbank, which is the central bank of Sweden. The sample period ranges from February 1, 2002 to March 31, 2012. The data set contains totally 2,557 observations.

Figure 1 illustrates the short-term interest rate in Sweden in the study period. The solid line shows the movement of the interest rate, whereas the dashed line illustrates the general trend of the interest.

[7] For more details about the likelihood ratio test, see for example, Verbeek (2008).

Figure 1: Illustration of the Short-Term Interest Rate in Sweden



From the figure above, it can be seen that the short-term interest rate in Sweden has a generally decreasing trend (illustrated by the dashed line) with some fluctuations. The evolvement of the interest rate can be divided into four stages. The first stage is the period from February 1, 2002 to November 11, 2005, where the interest rate was of a decreasing trend. During this period, the interest rate declined from about 4% to 1.5%. Then it entered the second stage with a piece of sharply increasing trend. This increasing movement lasted until September 8, 2008 and the short rate went up to approximately 4.5%. After that, the interest went into a decreasing stage again until June 30, 2009. The interest rate dramatically plunged into its bottom, at only about 2.5%. The fourth stage started from July 1, 2009. Staying in the valley some days, the interest recovered itself, jumping up to approximately 1.8% and keeping a relatively stable movement, despite some small fluctuations.

The change in the short-term interest rate in Sweden describes the data from another viewpoint, and its movement is illustrated in Figure 2.

Figure 2: Illustration of the Change in the Short-Term Interest Rate in Sweden

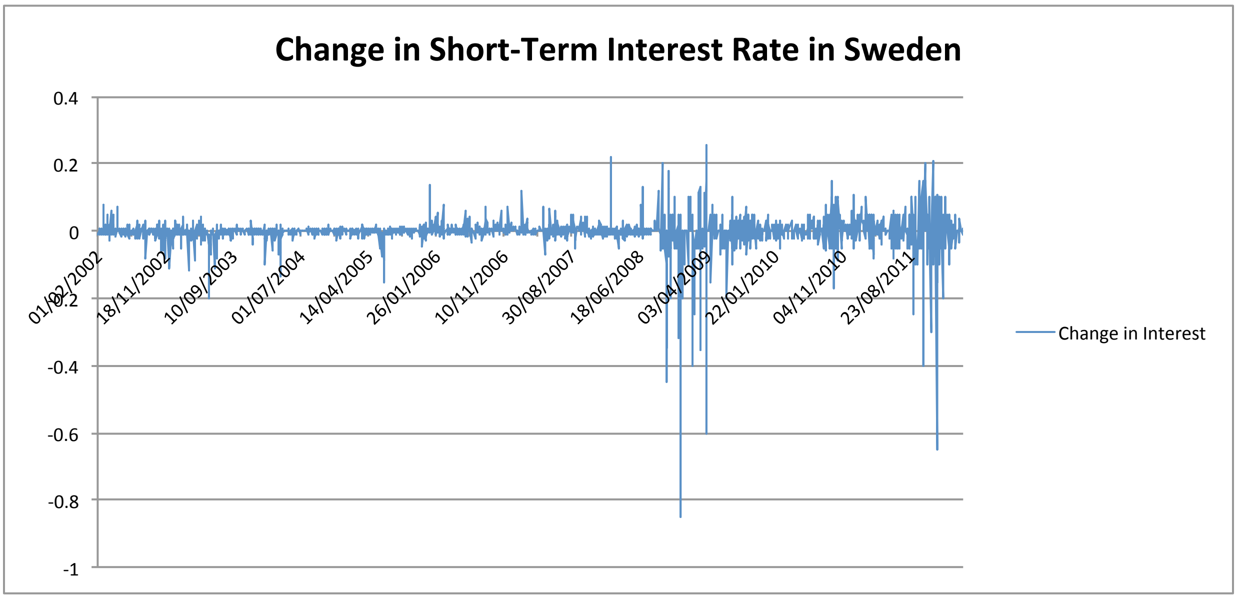


Figure 2 shows that the change in the short-term interest rate in Sweden is of a non-stable fluctuation, in particular after June 2008. There are two most volatile periods. The one started from September 19, 2008 to March 31, 2009, with the maximum absolute change around 0.85%. The other one was between September 2, 2011 and January 9, 2012, which had the largest absolute change about 0.65%.

The actual volatility of the change in the interest can be measured by taking the squared value of the change in the short rate and this is shown in Figure 3.

Figure 3: Actual Volatility of the Change in the Short-Term Interest Rate In Sweden

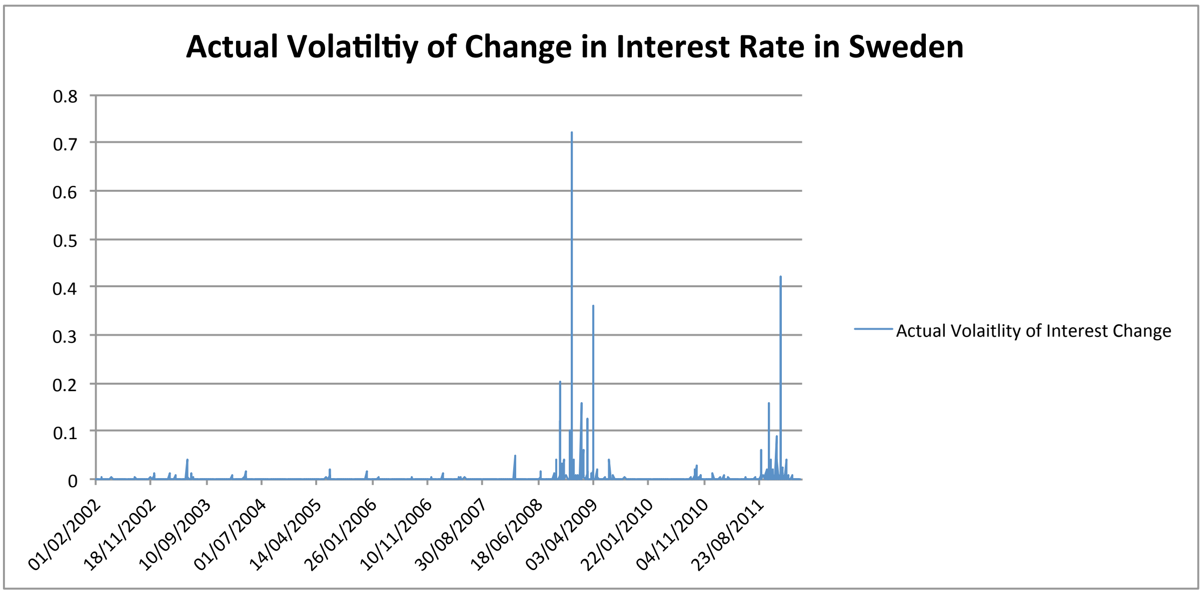


Figure 3 indicates that the actual volatility of the change in the interest rate might not follow a homoscedasticity, especially in the period from September 19, 2008 to March 31, 2009 and the period from September 2, 2011 to January 9, 2012. The maximum squared value of the interest yield was around 0.75% throughout the sample period.

Table 3 shows some descriptive statistics for the short-term interest rate in Sweden, including the mean, the median, the maximum and minimum values, the variance, and the standard deviation.

Table 3: Summary Statistics of Short-Term Interest Rate in Sweden

Statistic	Interest Rate (%)	Change in Interest Rate
Mean	2.2926	-0.0009
Median	2	0
Maximum	4.55	0.253
Minimum	0.11	-0.85
Variance	1.5794	0.0019
Standard Deviation	1.2567	0.0433

The mean level of the interest rate during the sample period is about 2.2926%, while its average change is only around -0.0009%. The maximum and minimum level of the interest is 4.55% and 0.11% respectively. The minimum change in the interest implies the highest absolute change of 0.85%. The variance and standard deviation indicate the volatilities of the interest rate and its yield. The variance of the interest level is approximately 1.5794%, whereas the variance of the interest yield is only about 0.0019%.

5. Empirical Results and Interpretation

In this section, the empirical results are presented and discussed. Firstly, a scenario of a comparison among the three general models—the Level-CKLS model, the GARCH-CKLS model and the GJR-CKLS model is conducted. After that, another scenario of an extensive study is introduced to compare the models among the CKLS-nested models, such as the Vasicek model, the Brennan-Schwartz model, etc. The empirical results of the two scenarios are presented and discussed in the following subsections.

5.1 Results of the Comparison among the General Models

The first scenario of the empirical work is to perform a comparison among the three general term structure models—the Level-CKLS, the GARCH-CKLS and the GJR-CKLS models. In this scenario, the GJR-CKLS model is treated as an unrestricted model while the Level-CKLS and the GARCH-CKLS models are regarded as restricted ones. The restrictions have been shown in Table 2 in Subsection 3.3. The empirical results are shown in Table 4.

Table 4: Empirical Results of Term Structure Models

Parameters	Level-CKLS	GARCH-CKLS	GJR-CKLS
$\boldsymbol{\alpha}$	0.0011*	0.0012	0.0013
$\boldsymbol{\beta}$	-0.0008***	-0.0002	-5.43519E-05
$\boldsymbol{\gamma}$	0.0628***	0.0883***	0.0874***
$\boldsymbol{a_0}$	0	1.37E-05***	1.39E-05***
$\boldsymbol{a_1}$	0	0.1196***	0.1282***
$\boldsymbol{a_2}$	0	0	-0.0158***
$\boldsymbol{b}$	0	0.8852***	0.8851***
LL	6755.16	8019.41	8020.22
$\boldsymbol{\chi^2}$	2530.12***	1.62	-

*Note: ***/**/* denotes 1/5/10% level of significance. LL is the abbreviation of "loglikelihood". The GJR-CKLS model is treated as an unrestricted model and the* χ^2*-statistic is therefore not given.*

The results indicate some issues to be discussed. Firstly, the χ^2-statistics provide the results of the LR test. A robust significant test statistic for the Level-CKLS model implies that it can be significantly rejected against the unrestricted GJR-CKLS model, in spite of its significant estimates of α and β. In the meantime, an extra comparison between the likelihood values of Level-CKLS model and the GARCH-CKLS model is performed, and the result of the LR test suggests that the Level-CKLS model can be rejected against the GARCH-CKLS model.[8] Therefore, the results indicate that the assumption that the volatility of interest rate in Sweden follows a time-invariant process might not be adequate, whereas the time-varying assumption describes the volatility better. The following figure illustrates a comparison among the volatilities, including the actual volatility and the volatilities from the three models.

[8] This is not shown in the Table.

Figure 4: Volatility Illustration of Short-Term Interest Rate in Sweden

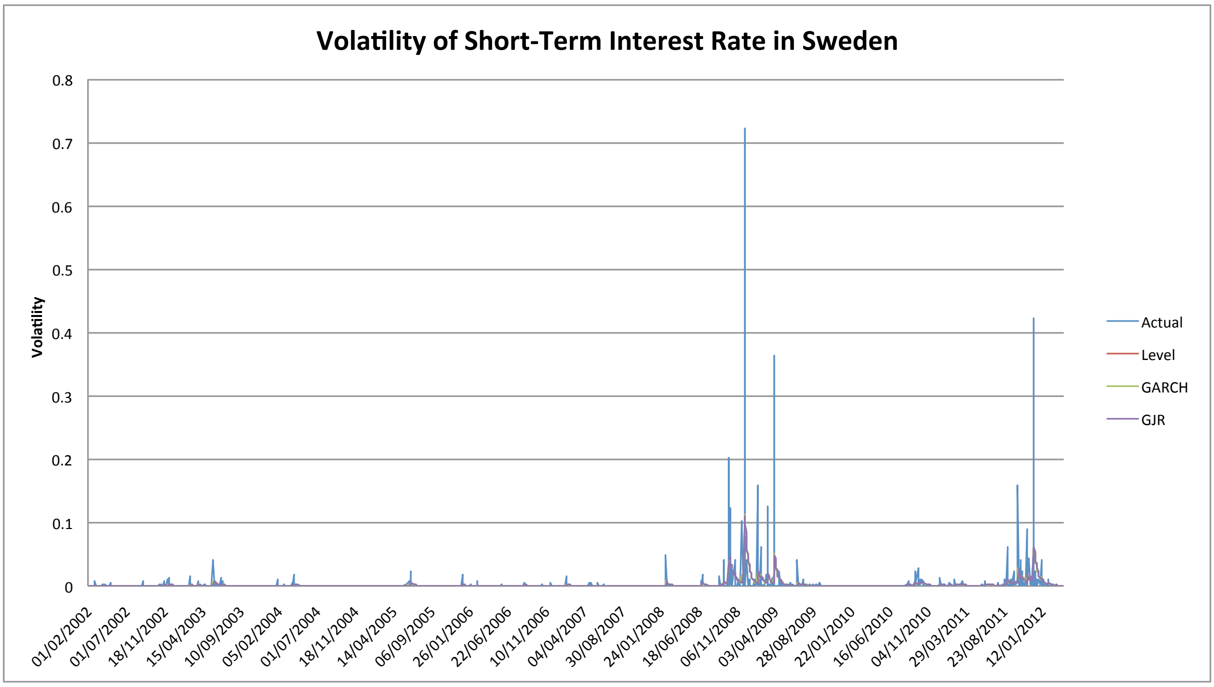


Figure 4 compares the volatility of the change in Swedish interest rate simulated by the Level-CKLS, the GARCH-CKLS and the GJR-CKLS models with the actual volatility. The actual volatility is measured by the squared value of the interest yield, while the simulated volatilities are measured by the conditional variances obtained from the Level, GARCH, and GJR-CKLS models. From the illustration, it can be seen that the volatility obtained by Level-CKLS model does not simulate the actual volatility well, whereas the time-variant volatility models—the GARCH-CKLS and the GJR-CKLS models do.

At the same time, the comparison between the GARCH-CKLS and the GJR-CKLS models provides a test statistic for the GARCH-CKLS model of insignificance, which implies that the GARCH-CKLS model cannot be rejected against the GJR-CKLS model and thus the volatility of Swedish interest rate might not be affected by asymmetric news effects.

In addition, the insignificant estimates of α and β in the GARCH-CKLS model (and the GJR-CKLS model as well) imply that the interest rate in Sweden do not evolve according to a mean-reverting process. From the theoretical perspective, mean-reversion does make sense because from the economic theory, when the interest rate is high, the demand for loans and other investments decrease. This will lead to the level of interest rate goes down. In contrast, when the interest rate is low, the demand for loans and investments increase, resulting in the interest rate climb up. However, in empirical work, this does not always hold because in reality financial agents do not always expect “the mean-reversion hypothesis”. For instance, when the interest rate is high at present, financial agents might expect that it would be higher in the future and thus increase their demand for loans and investments, which lead to a heavier increase in the interest. Thus in practice, investors sometimes think that the interest rate moves according to a mean-reverting process, but sometimes not. Their financial behavior depends on what they are thinking about the movement of the interest rate and this might result in a random walk process in the interest evolvement. The implication of the empirical results is consistent with what is obtained by Antoniou et al. (2005) in their research on British interest rate.

Finally, the fact that the estimate of γ from the GARCH-CKLS model is significantly different from zero and the value is around 0.0883, which indicates fairly low elasticity of the volatility to the level of current interest. What is notable is that the estimate of γ obtained from the Level-CKLS model is about 0.0628, lower than that from the GARCH-CKLS and the GJR-CKLS models. This is not in accordance with what Brenner et al. (1996) and Antoniou et al. (2005) conclude in their research, where they state that the time-invariant assumption causes the estimate of γ to be overestimated. Until now, there has not been relevant literature showing the same result, at least as known.

5.2 Results of the Comparison among the CKLS-Nested Models

By the comparison conducted in the previous subsection, the results of the LR test suggest that the short-term interest rate in Sweden might be of a GARCH process. However, up to now it has not been examined whether the restrictions on parameters of the CKLS-nested models are valid. To do that, another scenario that extends the GARCH effects into the CKLS-nested models and compares them with an unrestricted model is introduced. In this scenario, the GARCH-CKLS model is treated as an unconstrained model, while the Vasicek model, the CEV model, etc., are treated as restricted ones. The restrictions are the same as what have been summarized in Table 1. The only difference is that in this scenario, all the models are imposed GARCH effects on the volatility and called, for example, the GARCH-Vasicek model. The empirical results are presented in Table 5.

Table 5: Empirical Results of Term Structure Models

Model	α	β	γ	a_0	a_1	b	LL	χ^2
Vasicek	0.0010	-4.56E-10	0	2.06E-05***	0.1581***	0.8628**	8007.03	24.75***
Brennan-Schwartz	0.0021	-0.0002*	1	4.87E-08***	0.0163	0.9804***	5808.78	4421.27***
CIR SR	0.0014	-0.0002*	0.5	4.87E-08***	0.0052***	0.9917***	7327.12	647.59***
Merton	0.0010	0	0	2.05E-05***	0.1543***	0.8650***	8007.03	24.76***
GBM	0	-3.28E-11	1	8.97E-08***	0.0038***	0.9941***	6917.32	2204.17***
Dothan	0	0	1	7.49E-08***	0.0034***	0.9949***	6926.37	2186.07***
CIR VR	0	0	1.5	0.0099***	1.3607***	-0.0106***	2307.57	11423.68***
CEV	0	-5.44E-10	0.0843***	1.45E-05***	0.1245***	0.8818***	8018.96	0.91
CKLS	0.0012	-0.0002	0.0883***	1.37E-05***	0.1196***	0.8852***	8019.41	-

*Note: All the models are imposed GARCH-effects on their volatility terms. Thus them models are actually, say "GARCH-Vasicek" model, etc. ***/**/* denotes 1/5/10% level of significance. The GARCH-CKLS model is treated as the unrestricted model, so the χ^2-statistic is not given.*

The table in the above shows the results of estimation and LR test. All the χ^2-statistics are robustly significant, except the GARCH-CEV model, which indicates that the unique restricted model not rejected is the GARCH-CEV model. It is worth noting that all the models rejected are imposed restrictions on γ, whereas the GARCH-CEV model is the only one that is not. Therefore, the results of LR test deliberate indication that the restrictions on γ might not be valid and the models with such restrictions do not describe Swedish interest rate well. In addition, it can be seen that the estimate of γ from the GARCH-CKLS model is about 0.0883 while from the GARCH-CEV model about 0.0843. They are quite close to zero but are of robust significance. Such results indicate that the short-term interest rate in Sweden is of a low-elastic volatility to the current interest level. Hence, it might be concluded that for Swedish interest rate only the models that allow for low but non-zero value of γ can describe the dynamics well. However, compared with what is found in Dahlquist (1996), where the author applies the CKLS framework and obtains γ for Swedish interest rate equal to 1.15, the results in this paper convey quite distinctive implication.[9]

It is also notable that the GARCH-CEV model is one that is not of mean-reversion. As mentioned in the Subsection 5.1, the estimates of α and β obtained from the GARCH-CKLS model are insignificant, implying that the interest rate in Sweden evolves according to a non-mean-reverting stochastic process. Therefore, the result of non-rejection of the GARCH-CEV model provides an indication in line with what is concluded in that scenario.

Hence, the implication of empirical results can be summarized as follows: First, the results imply that the volatility of the interest rate in Sweden is not time-invariant. It follows a GARCH process. Second, the interest rate does not evolve according to a meant-reverting process. Third, the low γ suggests that the volatility of Swedish interest rate is not sensitive to the level of current rate. Fourth, all models that are imposed restrictions on the parameter γ do not describe Swedish short-term interest rate well and only the ones that allow for the volatility that has low elasticity to the current rate do.

6. Conclusion

Although there have been many studies on short-term interest rates and the term structure models, little is known to focus on the short-rate in Sweden, especially with consideration on time-varying volatility. This paper is to investigate the short-term interest rate in Sweden and to test for the goodness-of-fit of the term structure models, by taking both the CKLS framework and the BHK framework into account. To conduct the study, the interest rate of one-month treasure bills in Sweden is used as the underlying interest rate and the study period ranges from February 1, 2002 to March 31, 2012, containing 2,557 daily observations. The ML approach is applied to estimate the models and the LR test is used to perform the comparison among the corresponding models.

[9] This might not give a formal conclusion because the sample periods used are different.

The study is conducted with two scenarios. First, a comparison among the three general models—the Level-CKLS model, the GARCH-CKLS model and the GJR-CKLS model is performed. In this scenario the GJR-CKLS model is treated as an unrestricted model. The result of the LR test implies that the time-invariant volatility assumption might not be valid and thus the Level-CKLS model is rejected, despite its significant α and β. In the meantime, the GARCH-CKLS cannot be rejected against the GJR-CKLS, which indicates that the volatility of the interest in Sweden might follow a GARCH process. The insignificant estimates of α and β imply a non-mean-reverting process of the short-term interest rate in Sweden. This is consistent with what is found in Antoniou et al. (2005), where the authors obtain the same result that the Level-CKLS model suggests mean-reversion whereas the GARCH-CKLS and the GJR-CKLS models do not.

Meanwhile, the second scenario extends the GARCH-volatility to the CKLS-nested models and the LR test is used to test for the validity of the restrictions on the parameters. In this scenario the GARCH-CKLS model is treated as an unrestricted model. The results of the LR test indicate that only the GARCH-CEV model cannot be rejected. Such results suggest that the restrictions on γ might not be valid in modeling the short-term interest rate in Sweden. The estimates of γ are quite low, but are of robust significance, implying that the volatility of short-term interest rate in Sweden is not very sensitive to the level of current interest rate. Therefore, it might be concluded that only the models which allow for quite low elasticity of the volatility to the current rate can describe the short-term interest rate in Sweden well. In addition, because the GARCH-CEV model is of non-mean-reversion, the results are in accordance with the conclusion obtained from the first scenario that the interest rate in Sweden evolves according to a non-mean-reverting process.

Finally, it is worth noting that the estimate of the parameter γ obtained from the Level-CKLS model is lower than that from the GARCH-CKLS and the GJR-CKLS models. This is not in line with the conclusion obtained in Brenner et al (1996) and Antoniou (2005), where the authors state that the value of γ is overestimated under the CKLS framework. Up to now, there has not been relevant literature showing the same result, at least as known.

In spite of the findings above, there are some limitations in this study. Firstly, this study is conducted by using daily data in order to include more observations in the sample set. A drawback of using daily data is that it ignores potential weekday effects on the behavior of the interest rate. If there exist weekday effects, the estimation results would be biased.

Additionally, the models investigated in this paper are all specified as linear-drift models, i.e. the drift terms are all in linearity. However, as Ait-Sahalia (1996) points out, the diffusion in the CEV model does not match with the linearity in the drift term. Thus the linear form in the drift term might be a source of model misspecification and lead to the estimation results to be biased. Therefore, for further studies it is suggested to account for models with nonlinear specification in the drift terms.

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APPENDIX

A1: An Introduction to Brownian Motion

This section presents a brief introduction to Brownian motion for those who are not familiar with basic concepts of Brownian motion. For more details, see for example, Pennacchi (2006).

A stochastic process x is referred to as a Brownian motion or a Wiener process if it satisfies the following conditions:

1. $x(0) = 0$.
2. The process x has independent increments, i.e. if $0 \leq a < b \leq t < c$, $x(c) - x(t)$ and $x(b) - x(a)$ are independent variables.
3. For $0 \leq s < t$, $x(t) - x(s) \sim N(0, (t - s))$.
4. x has continuous trajectories.

A process X is said to follow a geometric Brownian motion (GBM) if it has the following stochastic differential equation (SDE):

$$\frac{dX}{X} = \mu dt + \sigma dz,$$

$$X_0 = x_0 > 0.$$

By defining $Y = \ln X$ and applying Ito's lemma, it can be shown that

$$dY = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dz.$$

Hence,

$$Y_t - Y_0 = \ln X_t - \ln X_0 = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma z,$$

and

$$X_t = x_0 . \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma z\right\}.$$

By using Ito's lemma on $V_t = x_0 . \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma z\right\}$, it can be shown that V_t follows the same SDE as X_t. Because of the same initial value on V_t and X_t, it must be that $V_t = X_t$ for all $t \geq 0$. Since $X_0 = x_0 > 0$, it must be that $X_t > 0, \forall t \geq 0$. In addition, X_t is of a lognormal distribution, i.e.

$$\ln X_t \sim N\left(\ln X_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t, \sigma^2 t\right).$$

A2: Derivation of Nowman's Specification

This section presents the derivation of Nowman's model specification shown in Section 3. The text below follows Nowman (2002) and Antoniou et al. (2005), Appendix 1.

The Level-CKLS model follows the SDE

$$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r^{\gamma}(t)dZ. \quad (A1)$$

Nowman (1997) assumes that as an approximation to the true underlying model given by Eq. (A1), and that over the interval $[0, T]$, $r(t)$ satisfies the following SDE:

$$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma\{r(t' - 1)\}^{\gamma}dZ, \quad (A2)$$

where $t' - 1$ is the largest integer less than t. He also assumes that volatility changes at the beginning of a unit observation period and then remains constant through the period until the next period. Hence, Eq. (A2) is interpreted as meaning that $r(t)$ satisfies the stochastic integral equation:

$$r(t) - r(t' - 1) = \int_{t'-1}^{t}\{\alpha + \beta r(s)\}ds + \sigma\{r(t' - 1)\}^{\gamma}\int_{t'-1}^{t}dZ(s), \quad (A3)$$

for all *t* in $[t' - 1, t']$ where $t' - 1 < t \leq t'$ and $\int_{t'-1}^{t}dZ(s) = Z[t' - 1, t]$.

Nowman (1997) uses Theorem 2 in Bergstrom (1984) to obtain the discrete model corresponding to Eq. (A3), which is given by

$$r(t) = e^{\beta}r(t-1) + \frac{\alpha}{\beta}\left(e^{\beta} - 1\right) + \varepsilon_t, \quad (A4)$$

where ε_t satisfies the following condition:

$$E[\varepsilon_t^2] = \int_{t-1}^{t} e^{2(t-\tau)\beta}\sigma^2\{r(t-1)\}^{2\gamma}d\tau = \frac{\sigma^2}{2\beta}\left(e^{2\beta} - 1\right)\{r(t-1)\}^{2\gamma}. \quad (A5)$$

A Taylor expansion for e^{β} gives $e^{\beta} = 1 + \beta + \frac{\beta^2}{2!} + \cdots$. Substitute e^{β} in Eq. (A4) by the first two term of Taylor expansion gives

$$r(t) = (1+\beta)r(t-1) + \frac{\alpha}{\beta}(1+\beta-1) + \varepsilon_t,$$

$$r(t) - r(t-1) = \alpha + \beta r(t-1) + \varepsilon_t. \quad (A6)$$

Hence, Eq. (A6) is the approximation of the functional form in Eq. (A4).

Applying the same procedure into the conditional variance specification in Eq. (A5) gives

$$\frac{\sigma^2}{2\beta}(1+2\beta-1)\{r(t-1)\}^{2\gamma} = \sigma^2\{r(t-1)\}^{2\gamma}. \quad (A7)$$

Eq. (A7) gives the approximation of the conditional variance given by Eq. (A5).