Structured Products Modelled as Stochastic Processes

by

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Abstract

In this thesis, the value of the two composed financial products (structured products), sprinters and auto-calls, are studied and compared to the European call option. The study is preformed using both analytical and numerical solutions, where the underlying is either said to follow a GBM-process (Geometric Brownian Motion) or a GARCH-process (Generalized Autoregressive Conditional Heteroskedasticity).

One of the main results is that the GARCHdt-model is optimistic to a larger extent than the GBM-model. This is shown by a generally higher price of derivatives. Many times this is an effect of the individually high payoffs resulting from the simulations. This effect is clearest seen in the simulations of option, where the GARCHdt-simulations reach payoffs ten times higher than the highest GBM pay off. With the introduction of structured products and several underlies, this effect is somewhat damped, but still substantial. Although the GARCHdt-model gave unreasonable high payoffs occasionally, it many times out-performed the GBM-model. When compared to structured products with real data as underlies, the GARCHdt-simulations showed a higher number of similarities than the GBM-simulations.

Acknowledgement

First of all I would like to thank my supervisor, Sven Åberg, who made it possible for me to write this thesis. I appreciate all the encouragement and help I got from him.

I would also like to thank my girlfriend, Ellinor, for reading the entire report. A report with too many typos, dangling modifiers and other mistakes can be very unpleasant to read. With her help these were reduced.

Finally I would like to thank the Division of Mathematical Physics for having me write my thesis there.

Self-Reflection

In this self-evaluation I will evaluate my work during this course and discuss how it improved my skills. One should keep in mind that the first time I encountered econophysics was in the course Complex Economics in spring 2012. Now, about 10 months later, I have written my master thesis in econophysics.

When writing a report in a new subject for the first time, it is hard to know how to express oneself. There is a special rhythm and vocabulary in texts written by physicists and mathematicians. It took me many years to feel comfortable with this way of writing. But the texts in papers published by economists are quite different. The economists argue in a different way and use completely different words. Many of these words are hard to translate into Swedish, and even if one manages, the Swedish word is often hard to interpret as well. To be able to understand what I was reading, I was forced to read lots of publications, books and homepages. This was really tough the first weeks, but did pay off in the long run. Today I can read easier financial texts and follow easier argumentations. This understanding did of course also improve how I wrote my own texts. Thanks to this, I now can write for a broader group. All branches of science have in one or other way connections far away from their own subjects, so the ability to communicate outside the own field is a valuable skill.

Beside the linguistic barrier there were several other problems. A problem that I encountered early, and still suffer from, is the lack of other reports in a similar field. It is very hard to write something in a new field when there is nothing to compare with and nothing to follow. Hence I felt many times, that I did not know what I was to achieve. I had developed the GARCHdt-model from the already existing GARCH(1,1)-model, but how should I use it? There was no clear answer to this. I tried many different ways of presenting the simulations and comparing the models. Unfortunately my ideas often turned out to be quite useless or uninteresting. This was both a time consuming and annoying part of the course. If all my effort put on bad ideas and theories had been put on those that actually contributed to report, they could have been developed even further making the report over all better. At the same time as I am annoyed by the time spent on bad ideas, I know that testing ideas and assumptions is how the scientific world develops. Investigating ideas and assumptions does also give experience, independently of how good they are.

If I was to write the same thesis again with my current knowledge and experience, I would have done it very much as the first time. While doing so, I would of course try to not do the same mistakes again, but I certainly would like to experiment with the more successful ideas. It would for example be very interesting to investigate the GARCH-model even further, or try to find analytical solutions to more problems.

Contents

1 Introduction

Ever since the humanity began to trade goods and currency between each other, there has been an advantage for those who can tell what to buy now, to sell for a profit in the future. If a person for example bought lots of wheat a year when the harvest was good and the price low, he could sell it to much higher prices the next year if the harvest was bad. If the harvest on the other hand once again was good, he would not be able to sell his old wheat to a high price and might instead loose money. This simple example, that once could have been a true story, is very similar to how the market today works with stocks, derivatives etc.. No-one can be sure how the stock market will behave in the future, but if they could they would be able to profit from it. For several centuries the classical economics has been developed for the purpose to predict and take advantage of the market. Unfortunately the field has developed in a static manner where predictions based on faulty assumptions and simplicity many times is regarded higher than empirical observations [2]. The theories that have been used and taught for a long time are merely a special cases of a more general theory. They might be true in a slowly changing market, but can be fatally wrong when something out of the ordinary happens. Even though some economists are not too keen to change their way to think, academics with roots in physics and mathematics might have another view. With their experience to treat empirical data and to develop existing models to contain new phenomena the new branch *econophysics* was born. The problems observed when studying the market are many times very similar to those arising in physics or mathematics. The GBM-model is for example an application of diffusion processes, something that is studied in natural science.

In recent years, different structured financial products composed by two or several financial instruments are increasingly used. Such products are today used not only by financial institutes and banks, but are also exposed to individuals as alternative to put the savings at bank accounts. The structured products give opportunities to combine safe savings with market possibilities.

In section 2 the two structured products, sprinters and auto-calls are briefly described. An overview of the used mathematical models is given in section 3. In section 4 calculations and analysis of prices of options is performed. The same content, but for sprinters and auto-calls, is found in section 5 and 6. These sections are more detailed and several underlies are introduced. Finally a short summary and discussion is presented in section 7. In addition to these sections, there is also an appendix where basic financial terms are explained.

2 Structured products: sprinters and auto-Calls

The aim of this thesis is to investigate the structured products *sprinters* and *auto-calls* and how they should be priced. In contrast to an option they consist of several financial instruments such as bonds and options and might follow more than one underlying. Just as for an option the pay off can be very different depending on the structured product in question, but both the sprinter and the auto-call have different barriers which determine the pay off. It is always the "worst" underlying, i.e. the underlying with lowest price, that is compared to these barriers and sets a price on the product. Hence the worst underlying will many times be referred to as *the* underlying. There are two very important barriers that determine the price, the *risk barrier* and the *nominal/auto-call barrier* (nominal for sprinters and auto-call for auto-calls). The first barrier acts as a protection of the investment. At day one all the underlies start at 100% of their initial values, but during the term to maturity they will vary up and down. If the risk barrier is set to 50% of the initial value, that means that as long as the underlying is above 50% of the initial value at the expire date, the nominal investment will be paid back. So even if the underlying market is going down the investment is protected down to a 50% drop. But if the underlying is below these 50% the investment is no longer safe. For most products the pay off is paid as if the invested money were directly invested in the underlying itself, i.e. if the index is only 40% of the starting value the investor only gets back 40% of the invested money. In a few special cases there also is another barrier below the risk barrier called the *knock-out barrier*. If the underlying ends below this barrier nothing is paid out at all. The nominal/auto-call barrier is usually set to 100% of the initial value of the underlying. If the underlying of a sprinter ends above this barrier the pay off is boosted proportionally to how high above the nominal barrier it ended. For the auto-call it is a bit more complicated and it would be wise first to settle the differences between an auto-call and a sprinter before discussing this further.

Most important difference between the sprinter and the auto-call is that the sprinter has a specified time to maturity while the auto-call can end at several predetermined dates. A sprinter with a time to maturity 5 years will therefor always end after 5 years and the pay off will be determined from the underlying at that date. An auto-call with the same time to maturity usually have a check date once every year or more often, where the product will end if underlying is below the risk barrier or over the auto-call barrier. So if the underlying of an auto-call is above the auto-call barrier at a check date the product will expire and the holder will get back the nominal investment. But this is of course not the only pay off from the auto-call. Beside the nominal pay back the holder might get coupons every check date the underlying is above the *coupon barrier*, which lies somewhere between the risk barrier and the auto-call barrier. The coupon is usually a percentage of the nominal investment, so if the coupon is 5% of the initial investment of e.g. 10 000 SEK it gives the holder 500 SEK. Sometimes these coupons are paid out at the check date and sometimes they are paid out when the auto-call expires. In figure 1 three pay off scenarios for an auto-call is shown.

In the first scenario, the worst underlying is above the auto-call barrier at the first check point. Hence the product expires and the holder gets back the nominal investment and one coupon. In the second scenario, the underlying stays above the coupon barrier

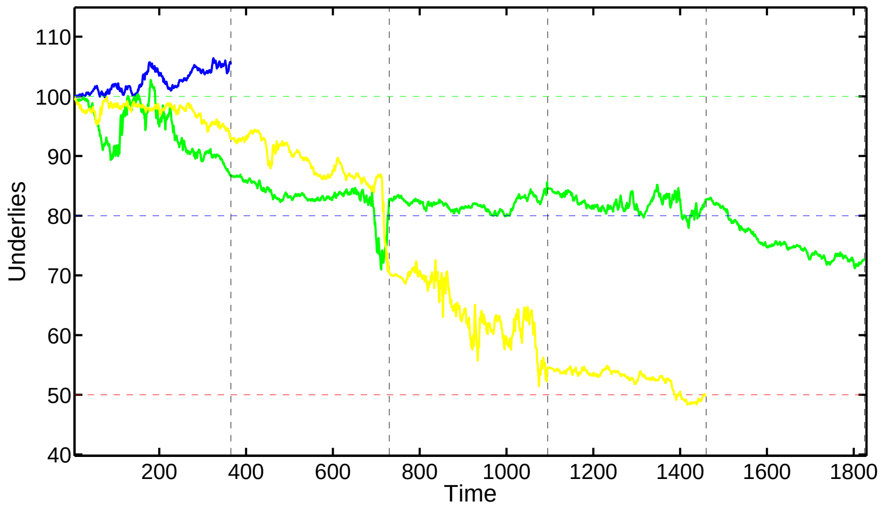

Figure 1: Three scenarios for an auto-call.

four times and ends at the maturity date above the risk barrier but under the auto-call barrier. Therefore the nominal investment and four coupons are paid back. In the last of the three scenarios, the underlying is only above the coupon barrier at one check date. The product ends at the fourth check date when the underlying is just below the risk barrier. Hence only about half of the nominal investment is paid back together with one coupon. As shown in this example both the pay off and the length of the auto-call can vary much. Something to consider before doing the investment.

One common feature of the sprinter and the auto-call is that they guarantee that the nominal investment will be repaid if the products expires with the underlying above the risk barrier. At a first sight it seems to be an incredibly investment opportunity where the invested money is almost safe, especially if the risk barrier is abnormally low. But how is it possible to sell such a product? Well, the strength in the products lies in the way they are compound. Only a fraction of the money the product cost is invested in options and almost all money goes to buy bonds (a fraction of the money is also charged by the company selling the product). If a costumer buys a product for 10 000 SEK, 9 000 SEK is used to buy bonds that will be worth 10 000 SEK at the expire date. By doing so the nominal investment will always be available if it must be paid back (at least if the issuer of the bond can pay at the expire date). The other 1 000 SEK (minus the fee) can then be used to buy options in the underlying market. So if the market is developing in a favourable way, the options will be worth more, and more money can be paid to the owner of the structured product.

3 Mathematical models

The market is a very complicated phenomenon to describe. One reason for this complexity is that it partly follows the expectations of people, something that is influenced by politics, experiences, etc. To derive a model for such a system is hence very challenging. It is however not so far away from physics models as one may think. Indeed one of the most common ways to model the market is by the same means as diffusion is treated in physics, something that will be evident further on. In the first subsection the Geometric Brownian Motion will be introduced. This is a schematic model of the market and will serve as a benchmark. The advantage of this model is that it is easily modelled to price options, which will be seen in the second subsection, and can be solved analytically. The analytical solution is obtained via the famous *Black and Scholes equation* which is introduced in the third subsection. Apart from the Geometric Brownian Motion, not many models can be used to find analytical solutions. Instead they are forced to price options using simulations. As the models become more complicated, so do the way of modelling them. In the last subsection a more realistic model, compared to GBM, is introduced. This is the GARCH-model.

3.1 Geometric Brownian Motion

A good way to understand how a phenomenon works is to model it. The model must not be able to reproduce all the features observed in reality, but is should at least give results close to the reality. One way to see if the model is good enough, is to simulate them on a computer. In the case of a stock, it could be the price that is simulated as a function of time in a so called time series. When simulating such a time series one begins with a starting value, S_0, and from there simulates the next value using an algorithm. This can be done several times, providing a series of the desired length. The Geometric Brownian Motion, GBM, is a model that reproduces many of the features that is important when studying the market [3]. It does e.g. show both *Markov property* and *ergodicity* (the first property says that the next value to be calculated in the series only depends on the previous value, the second property says that any value has a non-zero probability to be reached from any other value). Another important feature is that the GBM always has positive values (a stock price can not be negative). In its standard form the GBM can be expressed

$$dS = S\mu dt + S\sigma\varepsilon\sqrt{dt}. \tag{1}$$

Here S is the value of a stock or something similar, μ is the drift parameter (later the interest), σ the volatility (or standard deviation), ε a normally distributed random number and dt the infinitesimally small size of the step [4]. At a first sight the last term with the square root of dt might seem strange, but has its origin in the theory of *stochastic differential equations*. The first term on the right hand side is as purely deterministic term where the drift is growing linearly with the time. The second part is however the stochastic part and is not linear in time, but linear in the square-root of time. This makes

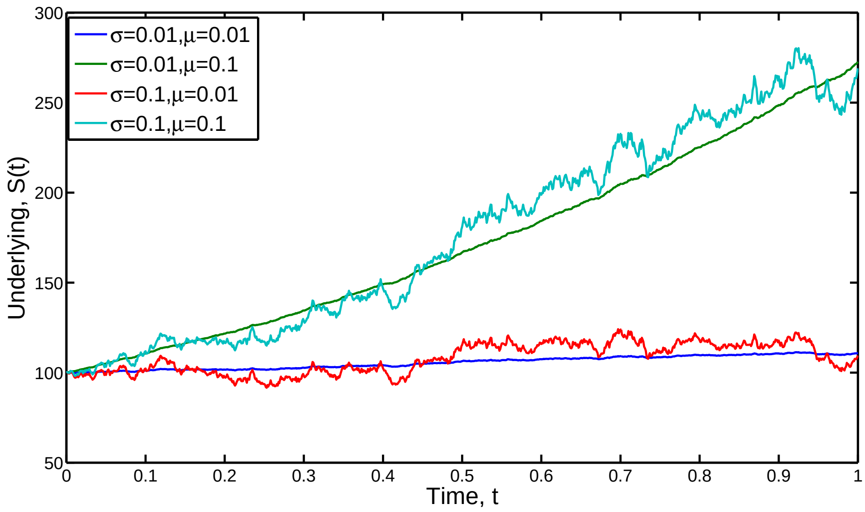

Figure 2: Four time series with different values of σ and μ. The same random number is used for all four series.

both the first and second moment of the GBM-process linear in time[1]. Note that the volatility is here constant, perhaps the major drawback of this model. To use equation 1 to simulate a time series one must first make it discrete. This is done by adding an index to every time dependent parameter in the equation (in this case only S and ε). Before the simulation is made one chooses how long the series shall be, say n time steps. One must also decide how long time these n steps will represent, call this time T. These two parameters n and T will then set the size of dt as $dt = (T - t_0)/n$, where t_0 is the starting time. The evolution of the time series will now be simulated as follows:

1. Draw a normally distributed random number $\varepsilon(i)$.
2. Use this random number to calculate the change in $S(i)$, $dS(i)$.
3. Calculate $S(i+1) = S(i) + dS(i)$.
4. Repeat step 1-3 until the length of the series is as long as desired

After the simulation is done, one obtains a vector with all the values the variable S_i. In figure 2 these values are plotted as a function of time for four sets of parameters, σ and μ. Note that the different parameters make the value of S behave quite differently.

From obtained time series of S it is also possible to calculate the *logarithmic return* [4], from now on only called return. When comparing real data from the market with simulated data this is often a useful tool and is calculated as

$$r(i) = ln(S(i+1)/S(i)). \tag{2}$$

[1]The first moment of the process is linear in time since the first moment of a constant is one and zero of a normally distributed random number. Hence the first moment of the GBM-process is; $\langle dS \rangle = \langle S\mu dt \rangle + S\sigma \langle \varepsilon \rangle \sqrt{dt} = S\mu dt$. The second moment can be calculated using the same arguments; $\left\langle dS^2 \right\rangle = \left\langle S^2\mu^2 dt^2 \right\rangle + 2\mu\sigma \langle \varepsilon \rangle dt^{3/2} + S^2\sigma^2 \left\langle \varepsilon^2 \right\rangle \ dt = S^2\mu^2 dt^2 + S^2\sigma^2 dt \approx S^2\sigma^2 dt$. Here the term involving dt^2 is neglected since dt is proportional to $1/n$ (where n is the number of time steps) and hence small. [4]

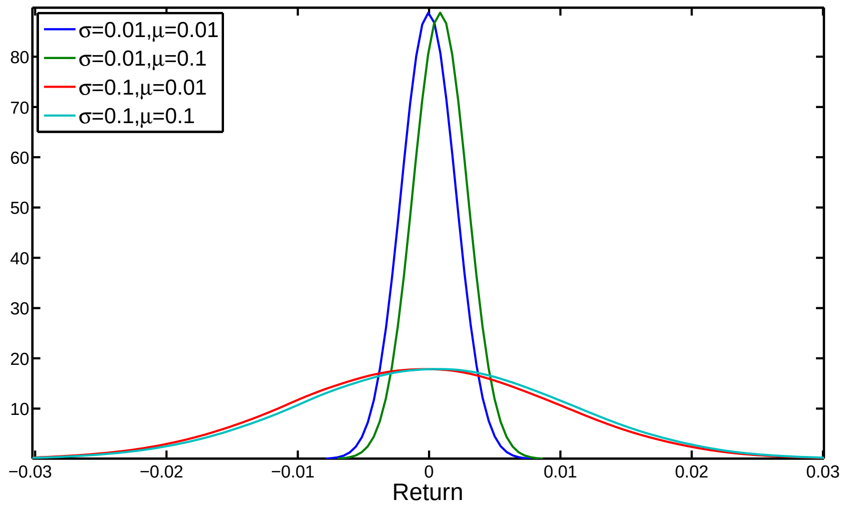

Figure 3: The returns of the four time series. The shift of the green and light blue curve is due to the fact that the drift term is large and positive.

One should notice that the return strongly depends on the size of the time step. Counter-intuitively, high frequency data return will have more high price changes than e.g. daily data [4]. Two other parameters that affect the return are the volatility and the drift. As can be seen in figure 3, where the distribution of the return is shown for the four simulations of figure 2, the volatility sets the broadness of the curve, while the drift shifts it to the left or the right depending on the sign of the drift.

When comparing the return of the simulated GBM series with real return data, they often agree well for small absolute values of the return. But when looked at higher values of the returns (the so called tails of the distribution) the GBM does not agree with the data at all, see Figure 4. The data in figure 4 consists of the daily index of NASDAQ from early 1986 to present.

The reason that the GBM cannot provide returns similar to the real data is that it relies on normally distributed random numbers. The normal distribution works well within the first few standard deviations from the mean, but longer distances from the mean value the distribution goes far to fast to zero. Because of this the probability that a return will be greater than $\pm$ 10% is incredibly small. Contrary to the GBM, the real data possess "fat tails".

3.2 Pricing options using simulations

It does not really matter which algorithm is chosen to simulate the evolution of a time series, any algorithm can be used to estimate the price of options (if the estimate is good or not is another question!). In the previous subsection the GBM-algorithm was introduced, so it will be used as an example where a call option is priced. If only interested in the price at one specific time, t, the price is only affected by S. To simulate the price, S has to be limited to the range 0 to S_{max} and discretized. Then for each value of S between the boundaries, N simulations are made resulting in N end values of S, S_{end}. Each of these end values will then be evaluated using the payoff for the call option

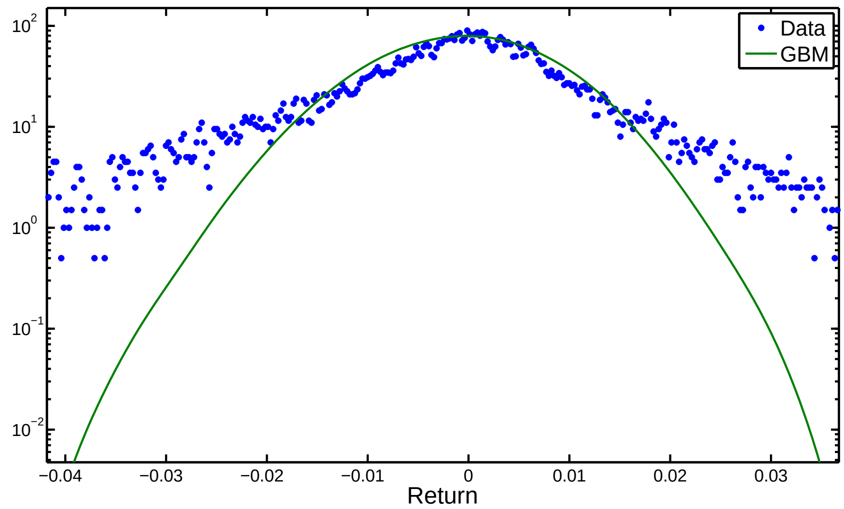

Figure 4: The returns of the GBM compared with the return from real data. The data consists of the daily index of NASDAQ from April 12:th 1986 to May 15:th 2012 [5].

$$G(S_{end}) = max(S_{end} - E, 0), \tag{3}$$

where G is the price of the option and E is the exercise price. The mean value of all the calculated $G(S_{end})$ will then be the estimated price of the option at the chosen starting value of S. To obtain a complete curve of price estimation for the call option, all the starting values from 0 to S_{max} must be used. Note that the price estimation will be different for different values of t. This is because the size of the time step is calculated as $dt = \frac{T-t}{n}$ where T is the total time of the simulation, t is the passed time and n is the number of steps to be made. When all the simulations are made, the pay off from the options are obtained. This is however not the same thing as the price of the option at $T - t$. To find the price at this time, the pay off must be divided by the interest during the time passed. If the simulated option e.g. is considered 1 year before maturity date, the pay off must be divided by one years interest to get the option price. The result for 5 simulations using different values for the volatility is shown in Figure 5. Note that the curves in this figure are slightly oscillating. By increasing the number of simulations, N, this can be dealt with. If 5 simulations with the same volatility, but with different drift terms were made instead, a very similar figure would have been obtained.

3.3 Black and Scholes

In many areas of mathematics and physics the "beauty" of a solution to an equation or of the equation itself is important. This beauty is something that the *Black and Scholes equation* is famous for, but since it is based on the GBM-model it does not agree very well with reality. With this in mind the two physicists Jean-Philippe Bouchaud and Marc Potter wrote an article where they criticised the market for holding on to the B&S, which they merely saw as a special case of a more general model, [6]. They admitted that the B&S model had been improved with the introductions of implied volatility, but they also

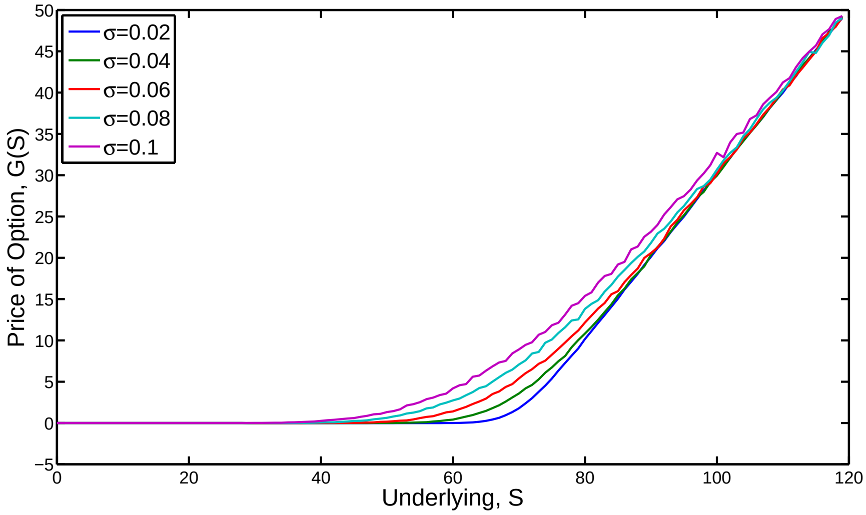

Figure 5: The price of five options plotted as a function of the price of the underlies. The five curves all have r=5% a year, n=100, N=10 000, T=100, t=90 but different volatilities.

pointed at the problem of using a normal distribution to simulate the price development. In one of the last lines they formulated the main problem in a striking allegory: "The use of the Black-Scholes formula with an implied volatility and using the Greeks as rudder is perhaps fine for sailing in a slowly changing environment, but it will without any doubt leave you stranded whenever a storm sets in.". Even though the B&S in many people's eyes serves as a scapegoat, responsible for e.g. market crashes, it has been an important step in pricing options. As such it will be used as a benchmark throughout this thesis, a benchmark to be compared with the GARCH-model. As said before the B&S can be seen as the counterpart of solving the problem of pricing an option using GBM-simulations. No explicit derivation will be executed here, but it was first made by Fischer Black and Myron Scholes in their article *The Pricing of Options and Corporate Liabilities* in 1973 [7]. In their original derivation they used a slightly different nomenclature than presented here, but the formula remains the same and takes the form

$$\frac{\partial G}{\partial t} = rG - rS\frac{\partial G}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G}{\partial S^2}. \tag{4}$$

In this differential equation, S is the price of the underlying, G is the value of the option, t is the time, r is the interest and σ is the volatility. Already when this model was proposed, Black and Scholes knew that it only applied when certain conditions were satisfied. In their paper they wrote these 7 assumptions:

1. The short-time interest rate is known and is constant through time.

2. The stock price follows a random walk in continuous time with a variance rate proportional to the square of the stock price. Thus the distribution of possible stock prices at the end of any finite interval is log-normal. The variance rate of the return on the stock is constant.

3. The stock pays no dividends or other distributions[2].

4. The option is "European," that is, it can only be exercised at maturity.

5. There are no transaction costs in buying or selling the stock or the option.

6. It is possible to borrow any fraction of the price of a security to it or to hold it, at the short-term interest rate.

7. There are no penalties to short selling. A seller who does not own a security will simply accept the price of the security from a buyer, and will agree to settle with the buyer on some future date by paying him an amount equal to the price of the security on that date.

To solve equation 4 for a specific option, the payoff for the option at the maturity, time $t = T$, $G(S(T), T)$, is used as a boundary condition. Following the steps in [4] by changing variables and mapping the problem onto a diffusion equation the solution can be obtained by

$$w(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} exp(-\frac{(x-x')^2}{4\tau}) w(x',0)dx'. \tag{5}$$

where w is connected to G by equation 6. The new variables, τ and x, are introduced as $\tau = \frac{\sigma^2}{2}(T-t)$ and $x = ln(S/E)$ (where T is the life time of the option and E is the exercise price). Once w is found G is obtained as

$$G(x,\tau) = E \cdot exp(-\frac{\lambda-1}{2}x - \frac{(\lambda+1)^2\tau}{4}) w(x,\tau) \tag{6}$$

Here a new constant, λ, has been introduced according to $\lambda = \frac{2r}{\sigma^2}$. In the previous subsection the price for a call option was estimated using a GBM-simulation. To show that this method generates the same solution as the B&S method, the problem for the call option will be solved analytically using B&S. At the maturity date the pay off for the call option was given by equation 3. If this pay off is expressed using τ and x as introduced above, $G(x,0)$ can be written as

$$G(x,0) = E \cdot \Theta(exp(x) - 1)(exp(x) - 1)$$

where Θ is the Heavyside step function. By inserting $G(x,0)$ into equation 6 and solving for $w(x,0)$ one obtains

$$w(x,0) = \Theta(exp(x)-1)exp(\frac{\lambda+1}{2}x) - \Theta(exp(x)-1)exp(\frac{\lambda-1}{2}x) = w_1(x,0) - w_2(x,0).$$

w_1 and w_2 can now be used separately and later be added to form the total solution. First $w_1(x,0)$ is inserted into equation 5 to find w_1 at any arbitrary time.

[2]This assumption means that no part of the earnings of the corporation is distributed to its shareholders

$$w_1(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} exp(-\frac{(x-x')^2}{4\tau})\Theta(exp(x')-1)exp(\frac{\lambda+1}{2}x')dx'$$

With the change of variable $y = \frac{x-x'}{2\sqrt{\tau}}$ and by use of the step function it becomes

$$w_1(x,\tau) = -\frac{1}{\sqrt{2\pi}} exp(\frac{\lambda+1}{2}x) \int_{\frac{x}{2\sqrt{\tau}}}^{-\infty} exp(-y^2 - (\lambda+1)\sqrt{\tau}y)dy.$$

By completing the square in the exponent of the integrand and interchange the limits, the expression can be further simplified to

$$w_1(x,\tau) = \frac{1}{\sqrt{2\pi}} exp(\frac{\lambda+1}{2}x + \frac{(1+\lambda)^2}{4}\tau) \int_{-\infty}^{\frac{x}{2\sqrt{\tau}}} exp(-(y + \frac{(1+\lambda)}{2}\sqrt{\tau})^2 dy.$$

Another change of variable,$z = y + \frac{(1+\lambda)}{2}\sqrt{\tau}$, and introducing the variable $d_1 = \frac{(1+\lambda)\sqrt{\tau}}{2} + \frac{x}{2\sqrt{\tau}}$ then finally gives

$$\begin{aligned} w_1(x,\tau) &= \frac{1}{\sqrt{2\pi}} exp(\frac{\lambda+1}{2}x + \frac{(1+\lambda)^2}{4}\tau) \int_{-\infty}^{d_1} exp(-z^2)dz \\ &= \frac{1}{2\sqrt{2\pi}} exp(\frac{\lambda+1}{2}x + \frac{(1+\lambda)^2}{4}\tau)(1 + \sqrt{\pi} \cdot erf(d_2)). \end{aligned}$$

Here erf is the error function defined by $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x exp(-t^2)dt$ (there are several other definitions of the error function, but this is the one used by MATLAB). After an almost identical calculation $w_2(x,\tau)$ reads

$$w_2(x,\tau) = -\frac{1}{2\sqrt{2\pi}} exp(\frac{\lambda-1}{2}x + \frac{(1-\lambda)^2}{4}\tau)(1 + \sqrt{\pi} \cdot erf(d_2)).$$

Combining these two expressions and inserting into equation (6) finally give

$$G(x,\tau) = \frac{1}{2\sqrt{2\pi}}(S(1 + \sqrt{\pi} \cdot erf(d_1)) - Eexp(-\lambda\tau)(1 + \sqrt{\pi} \cdot erf(d_2))). \tag{7}$$

In figure 6 this solution is plotted together with three of the simulations made in the previous section. The agreement is, as it should, striking.

When using B&S to price something one must be aware of the time scale used. Usually both the volatility and the interest is given per year, but this can not be directly used in B&S if the step size is supposed to be in days. For the interest, r, this problem is solved by dividing the yearly interest by the numbers of open market days a year. The number of open market days a years may be different for different countries and different years, but it is usually around 252 days. The volatility however is a bit more tricky. The yearly

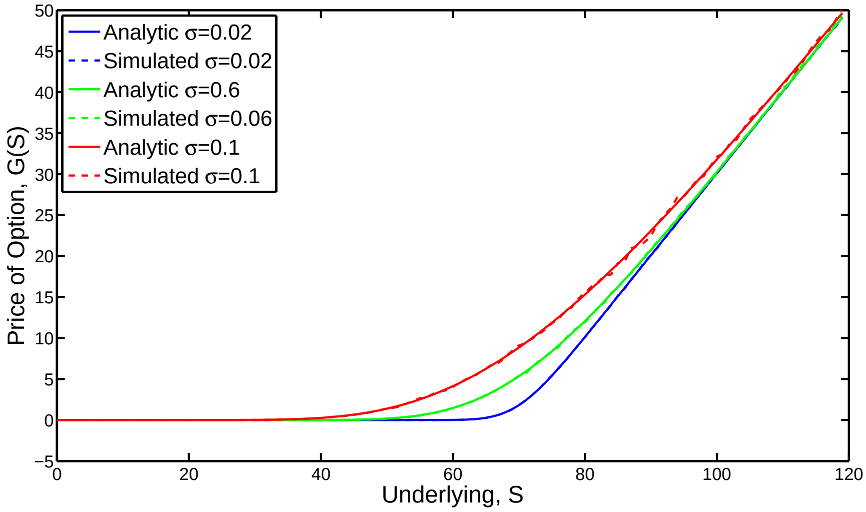

Figure 6: The option price, G, as a function of the underlying S for different values of σ. The other parameters are; interest=5% a year, n=100, N=10 000, T=100 and t=90. The solid lines are the solutions obtained from B&S, while the dashed lines are the simulated.

volatility should now be divided by the square root of open market days. This is because the volatility only comes in as squares.

3.4 GARCH-model

The main problem with the GBM-moded is that the returns are normal distributed. Hence it cannot provide the fat tails of the returns which are observed form real data, see figure 4. Many attempts have been made the last 50-60 years, to construct a model that will provide these fat tails. One of the fruits of these attempts is the GARCH-model (Generalized Autoregressive Conditional Heteroskedasticity) which was proposed by Tim Bollerslev in 1987 [1]. In this model the volatility is not constant, but a stochastic variable developing just as the underlying itself.

In the first publication concerning the GARCH-process, Bollerslev presented a very broad model which can be adjusted to the desired complexity. In this thesis the simplest of these models will be used. This is the so called *GARCH(1,1)-model* (from now on only refered to as GARCH). In Rudi Schäfer and Thomas Guhr's article from 2010 [8] they presented this model in the following way

$$\sigma_k^2(t) = \alpha_0 + \alpha_1 r_k^2(t-1) + \beta_1 \sigma_k^2(t-1) \tag{8}$$

$$r_k(t) = \sigma_k(t)(\sqrt{c(t)}\eta(t)) + \sqrt{1-c(t)}\varepsilon_k(t)). \tag{9}$$

Here r is the return, t is the time, σ is the volatility and η and ε are independent random numbers drawn from a normal distribution. The index k is used to distinguish the k different time series that are simulated. These series are correlated via c. If two series have $c = 1$ they are 100 % correlated (in other cases the correlation matrix must be

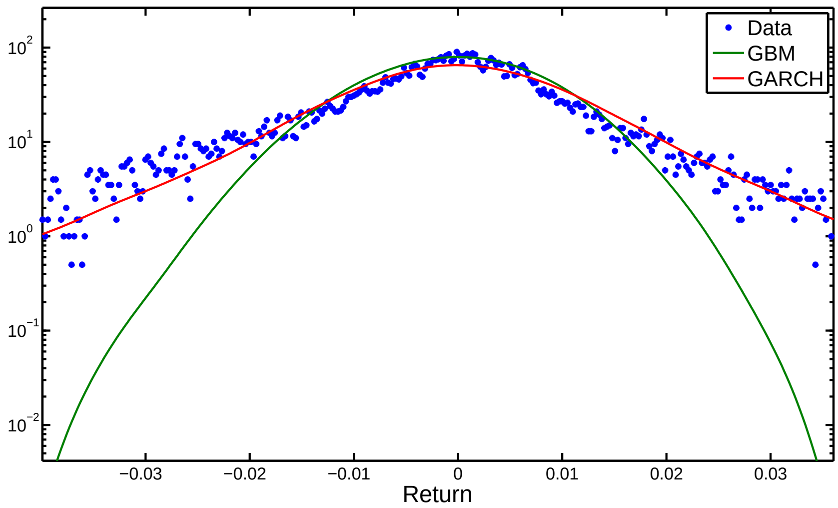

Figure 7: The returns of the GBM and GARCH compared with the return from real data. The data consists of the daily index of NASDAQ from April 12:th 1986 to May 15:th 2012 [5].

calculated to see how they are correlated). Finally α_0, α_1 and β_1 are constants. It does not appear in the article which values Schäfer and Guhr used for these three constants, but by personal contacts with Schäfer, the suggested values are $\alpha_0 = 2.3 \cdot 10^6, \alpha_1 = 0.15$ and $\beta_1 = 0.84$, respectively. If the multiple time series is of little interest it is possible to drop the correlation part and the index k to obtain the cleaner formulas for the GARCH-process

$$\sigma^2(t) = \alpha_0 + \alpha_1 r^2(t-1) + \beta_1 \sigma^2(t-1) \tag{10}$$

and

$$r(t) = \sigma(t)\eta(t). \tag{11}$$

With these two equations the price is updated according to

$$S(t) = S(t-1)exp(r(t)). \tag{12}$$

In figure 7 the returns of a GARCH time series is plotted together with GBM-return and real data return. It is evident that the GARCH-simulation provides much better returns than the GBM-simulation.

The return distribution of the GBM-simulation is simply a Gaussian with mean 0 and a standard deviation equal to the volatility. This is one of many things that makes the GBM-models so easy and tempting to work with. For the GARCH-model on the other hand the return distribution is quite different. The central part of the distribution looks very much as a Gaussian with distinct mean and standard deviation. The tails however are very unlike the tails of a Gaussian. Instead of rapidly go to zero outside the first standard deviations, the GARCH-model gives rise to a return distribution with fat tails which die off as an inverse power law. Indeed, the power of the right tail was found to be approximately -3.8, while the power of the left was -3.7. When the slopes of the left and right tails were averaged the power was found to be -3.7. By combining a symmetric power law distribution for the tails with a Gaussian distribution for the central region, one can obtain an analytic expression for the return distribution of the GARCH-model. The analytic expression will hence be of the form

$$P(r) = \left\{ \begin{array}{ll} N_1 \cdot exp(-\frac{r^2}{2\sigma}), & |r| < \varepsilon \\ \frac{N_2}{\alpha+|r|^\beta}, & |r| \geq \varepsilon \end{array} \right\}$$

where P denotes the return distribution. To fit the Gaussian to the data, N_1 is roughly set to the peak value of the GARCH-distribution. σ is then changed to make the distributions overlap as well as possible in the central region (note that N_1 and N_2 are normalizations constants and have to be changed later). For the power law the value of β is found by plotting the GARCH-distribution in a log-normal plot. By doing this the slope of the GARCH-distribution becomes the power of r, which is the same thing as β. Just like for the Gaussian, the other two parameters, α and N_2, are set to a value so that the two distributions coincide as well as possible. After this the "real" value of N_1 and N_2 are to be found. But before they can be found one must choose a point ε on the positive r-axis (by symmetry this will set the point on the negative r-axis too) where the Gaussian distribution will end and the power law will take over. This point can be found as the intersection between the two un-normalized distributions. At this point $\sigma = 0.0035$, $\alpha = 10^{-8}$, $\beta = 3.7$ and $\varepsilon = 7.5 \cdot 10^{-3}$ is used. To find N_1 and N_2 the following criteria must be fulfilled:

1. The probability distribution must be normalized.

$$N_1 \cdot \int\limits_{|r|<\varepsilon} exp(-\frac{r^2}{2 \cdot \sigma^2})dr + N_2 \cdot \int\limits_{|r|>\varepsilon} \frac{1}{\alpha + r^\beta}dr = 1$$

2. The probability distribution must be continuous.

$$N_1 \cdot exp(-\frac{r^2}{2 \cdot \varepsilon^2}) = \frac{N_2}{\alpha + \varepsilon^\beta}$$

.

Due to the symmetry of the chosen distributions the integrals under 1 can be rewritten to

$$2N_1 \cdot \int\limits_0^\varepsilon exp(-\frac{r^2}{2 \cdot \sigma^2})dr + 2N_2 \cdot \int\limits_\varepsilon^\infty \frac{1}{\alpha + r^\beta}dr = 1$$

The first of these integrals can now be recognized as an error function. When the complete calculations are made the integral becomes approximately $8.5 \cdot 10^{-3} \cdot N_1$. The other integral is much harder to compute. Hence it was calculated numerically on a computer and was found to be approximately $N_2 \cdot 3.2 \cdot 10^5$. The other requirement on N_1 and N_2, the continuity, gives that the constant in front of N_1 is 0.10 and the one in front of N_2 is $7.6 \cdot 10^7$, also these approximative. This gives a system of equations which easily can be solved. The solutions becomes $N_1 = 112.2$ and $N_2 = 1.49 \cdot 10^{-7}$. Hence the probability distribution for the GARCH-model can be expressed analytically by

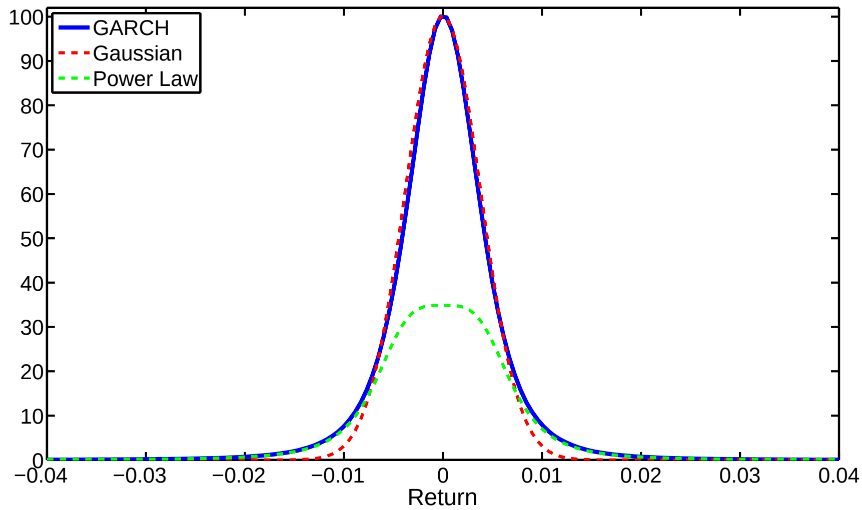

Figure 8: The returns of the GARCH-model can be approximated by a Gaussian for the central part and a power law for the tails.

$$P(r) = \begin{cases} 112.2 \cdot exp(-\frac{r^2}{7 \cdot 10^{-3}}), & |r| < 7.5 \cdot 10^{-3} \\ \frac{1.49 \cdot 10^{-7}}{10^{-8} + |r|^{3.7}}, & |r| \geq 7.5 \cdot 10^{-3} \end{cases}$$

In figure 8 the return distribution for the GARCH-model is shown together with the Gaussian and the power law. In figure 9 the same thing is shown, but in a log-normal plot. At the moment it might seem a bit pointless to find an analytic expression for the return distributions. But it will become a powerful tool to use later on.

One disadvantage of the GARCH-process described by equations 10-11 is that the step size in time is not present. Instead the model assumes that the user only is interested in daily returns, which is not enough when simulating option prices. Another drawback is that the interest is not included either. This is something that must be included to price options. To solve these problems, the updating of the price (equation 12), can be changed to

$$S(t) = S(t-1)(exp(r(t)\sqrt{dt}) + \mu dt). \tag{13}$$

The cost of this is that the return r no longer is the real return for the time series. Instead the return must be calculated through equation 2. From here on the return defined by equation 2 will be denoted r, while the return defined by equation 11 will be denoted by R. This new way to update the price will from now on be referred to as the *GARCHdt-model*. One may now ask how r behaves compared to R. One of the reasons to use the GARCH-model is the fat tails of the R-distribution. The answer to this, is that the two returns are almost the same. When dt=1 (which means that dt is equal to one day) it is impossible to distinguish between the GARCH-return from the GARCHdt-return. Even in a log-plot and in an loglog-plot they look very much alike.

A more interesting case, when comparing the returns from the GARCHdt-model and the GARCH-model is when dt becomes much smaller. To be able to compare the re-

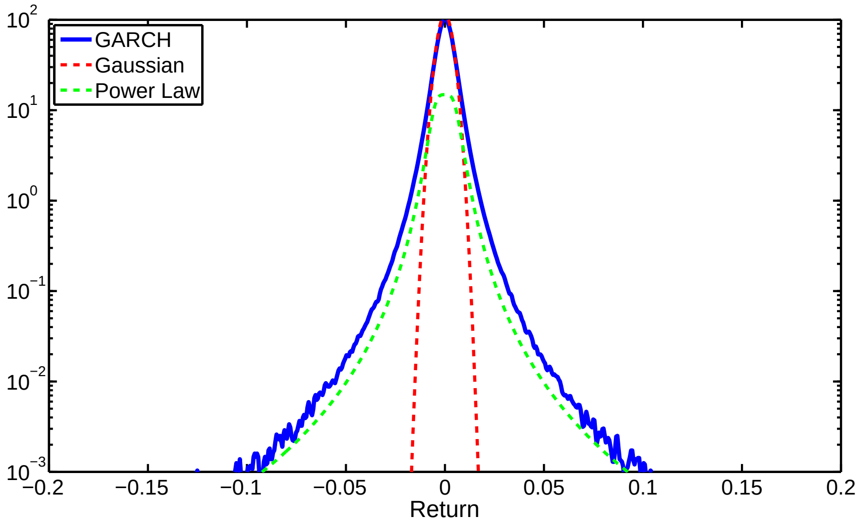

Figure 9: This figure shows the same thing as figure 8, but in a log-normal scale.

turns despite the apparent time mismatch, one must simulate a GARCHdt series which is $1/dt$ times longer than the GARCH series. Then every $1/dt$ price in the GARCHdt series is used to create a new time series. This new series is now a zoom out of the GARCHdt series, so that it looks like a series which has been updated daily instead of four times an hour. The r-distribution for this series can now be directly compared with the R-distribution. In figure 10, dt is equal to 0.01 (approximately 15 minutes), and the distributions agree very well. This agreement can also be seen in the corresponding lognormal-plot and loglog-plot shown in figure 11 and figure 12.

In the GBM and the B&S-model the volatility, σ, is set at the beginning of the calculations and is then kept constant. In the GARCH-models this freedom of choosing volatility is restrained. If the start value of the volatility is 0.02 it can be pretty much anything in the next step. Due to this it is hard to compare the GARCH-models with e.g. the GBM-model, because they will have different volatilities. Instead one is forced to use the historical volatility (the standard deviation of the return) for the GARCH-models. At present there is no way to control what the historical volatility for a time series will be. The only thing that can be done is defining the initial volatility. This is however not good enough as the volatility anyhow will change around some "equilibrium". Where this equilibrium is set, is determined by the constant α_0 in equation 10. So by replacing α_0 by a new parameter σ_0, the equilibrium can be changed and the volatility will start in the equilibrium. This new parameter will be called *equilibrium volatility*. In figure 13 σ_0 is plotted as a function of the historical volatility. This figure will serve as a calibration curve between the initial volatility and the equilibrium volatility. The plot was produced by simulating 10 000 GARCH-simulations for each σ_0. The initial volatility for each of the simulations was then calculated, and their mean is the points shown in figure 13. The interest was set to 5 % a year and dt to 1 day.

In figure 14 the returns for 5 time series are shown. All of them have different volatilities, 0.02, 0.04, 0.06 0.08 and 0.1, with their corresponding equilibrium volatili-

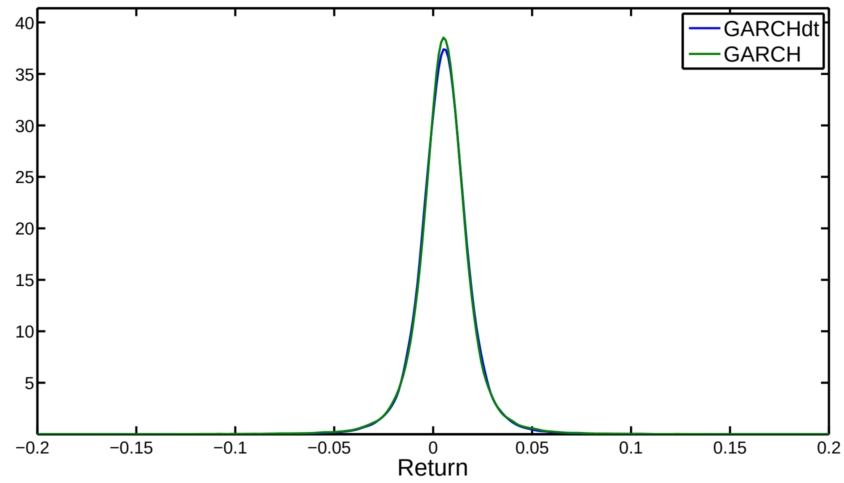

Figure 10: The returns of the GARCHdt and GARCH when dt=0.01. In the simulation 10 000 000 time steps were made for the GARCHdt series and 100 000 for the GARCH series. The interest rate was set to 0 since the GARCH-model does not support interest, and the volatility to 20% per year.

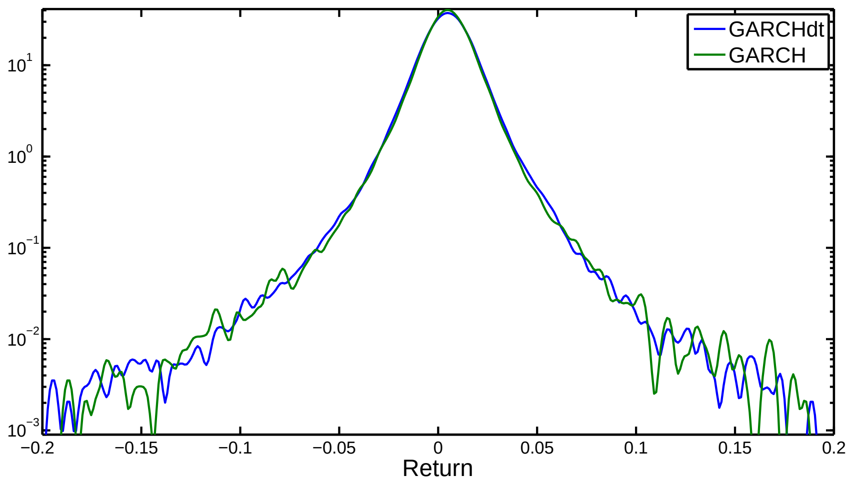

Figure 11: The returns of the GARCHdt and GARCH in a lognormal-plot when dt=0.01. In the simulation 1 000 000 time steps were made for the GARCHdt series and 10 000 for the GARCH series. The interest rate was set to 0 since the GARCH-model does not support interest.

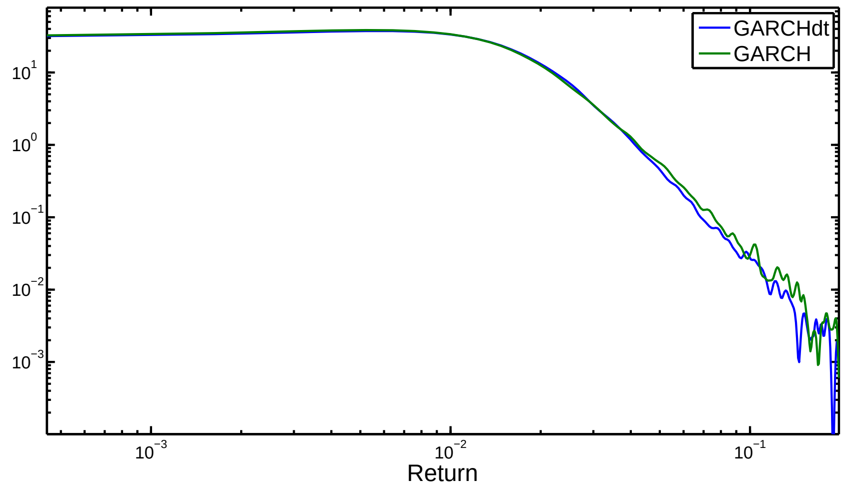

Figure 12: The returns of the GARCHdt and GARCH in a loglog-plot when dt=0.01. In the simulation 1 000 000 time steps were made for the GARCHdt series and 10 000 for the GARCH series. The interest rate was set to 0 since the GARCH-model does not support interest.

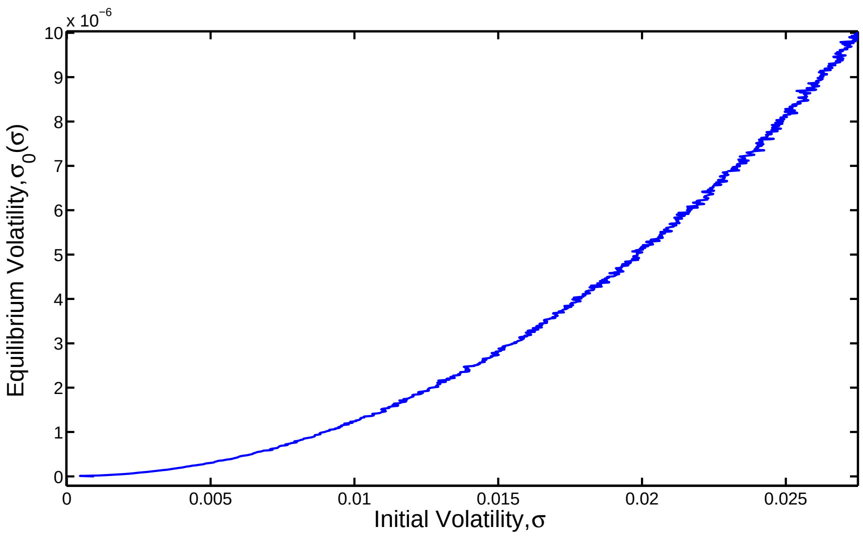

Figure 13: The equilibrium volatility , σ_0, as a function of the initial volatility, σ. Each point was calculated as the mean of 10 000 simulations with $r = 5\%$ a year and $dt = 1$ day.

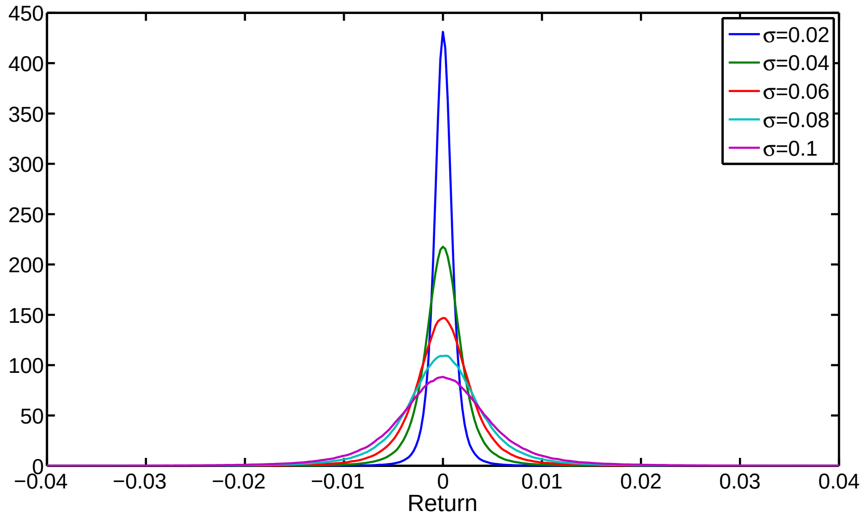

Figure 14: The return distribution of 5 time series with different volatilities. The other parameters are $r = 5\%$ a year, n=1000, T=1000 and dt=1 day.

ties $2.0 \cdot 10^{-8}, 7.8 \cdot 10^{-8}, 1.7 \cdot 10^{-7}, 3.1 \cdot 10^{-7}$ and $4.8 \cdot 10^{-7}$. The actual historical volatilities were calculated to be 0.022, 0.041, 0.059, 0.080 and 0.099.

At this point the GARCH(1,1)-model has been extended to the GARCHdt-model. This new model has two obvious advantages over the old one. The first is the neat way of setting a time scales and the second is the ability to choose scale for volatility. With this new model it is possible to price options using the algorithm explained earlier in this section. But sometimes it is important to simulate several time series at once and be able to control how they are correlated. Therefore the correlation parts that once were omitted will be brought back again. This is accomplished by changing equation 10 to

$$\sigma_k^2(t) = \sigma_{0,k} + \alpha_1 r_k^2(t-1) + \beta_1 \sigma_k^2(t-1). \tag{14}$$

This differs from Schäfer and Guhr's equation 8 [4] only by the change of α_0 to $\sigma_{0,k}$, where k now denotes which series is meant. This means that it is possible to simulate several series at once, but with different volatilities. The correlation itself will however only be present in the calculations of R, which yet again is given by equation 9. To see how the correlation shall be interpreted, 4 time series are simulated in figure 15. The blue and green index are correlated to 100% (c=1 for both series), but the blue should have a volatility of 0.02 while the green should have a volatility of 0.04. The red index has c=0.5 and an expected volatility of 0.02. Finally the light blue index has c=0 and a volatility about 0.04. When the historical volatilities were calculated for each of the indices they were 0.024, 0.041, 0.021 and 0.041 respectively, in good agreement with the expectations.

To see how the indices are correlated the correlation matrix is calculated. The matrix elements are defined by

$$C(i,j) = \frac{\langle S(i,t)S(j,t))\rangle - \langle S(i,t)\rangle\langle S(j,t)\rangle}{\sqrt{(\langle S_i^2\rangle_t - \langle S_i\rangle_t^2)(\langle S_i^2\rangle_t - \langle S_i\rangle_t^2)}}. \tag{15}$$

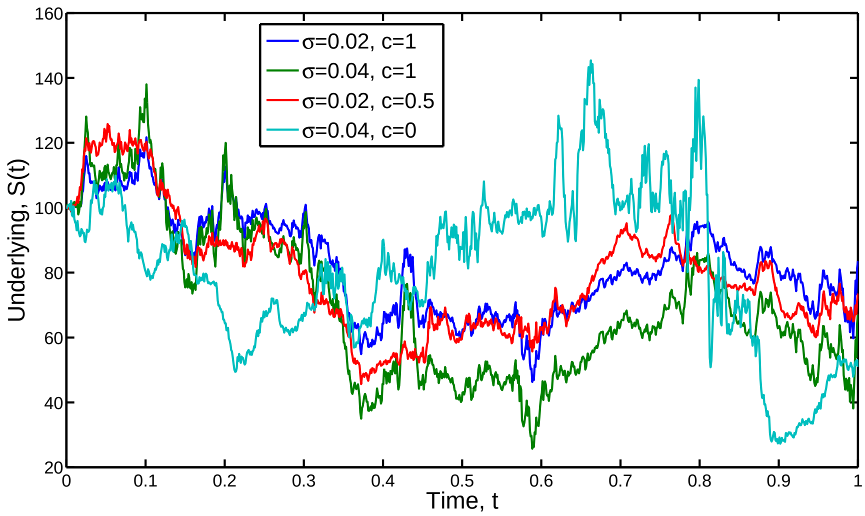

Figure 15: 4 time series with different volatilities and correlations as functions of time. The volatilities and the correlations are given in the figure. Beside these parameters the others were $r = 5\%$ a year, $n = 100$, T=1000 and dt=1 day.

Here $S(i,t)$ is the i:th time series, $\langle ... \rangle$ denotes the mean value. There are two limiting values that the elements of the correlation matrix can take, namely 1 and -1. If the element is 1 the two series are correlated to 100% and if it is -1 the series are anti-correlated, i.e. when one index is going up, the other index is going down. If the element is 0 the series are not correlated at all, at least not in the time window the series shows. The complete correlation matrix for the series in figure 15 is calculated as

$$\begin{pmatrix} 1 & 1 & 0.88 & -0.02 \\ 1 & 1 & 0.88 & -0.02 \\ 0.88 & 0.88 & 1 & 0.24 \\ -0.02 & -0.02 & 0.24 & 1 \end{pmatrix}.$$

Here two of the properties of the correlation matrix are evident. It is symmetric and has ones on the diagonal. The first feature comes from the fact that the i:th series is correlated to the j:th just as much as the j:th is correlated to the i:th. The second property comes from the fact that the i:th series naturally is 100% correlated to itself. Since the first and second time series in figure 15 are 100% correlated, the first and second row are identical. More interesting is that the $c = 0.5$ for the third series gives a high correlation (88%) with the first and second. If another simulation is made there is no reason to expect it to be correlated in a similar way again. If the simulations were done again the correlation this time could equally well be 20%. The fourth series which had $c = 0$ happens to be weakly anti-correlated with the two first series (-2%) and relatively strongly correlated with the third (24%). This example shows the difficulties with the correlation, it is quite hard to simulate series with the desired correlations. One way to solve this is to simulate thousands of series with the same values of c and try to find one set up for c that gives the desired correlation; more on this topic will be discussed in section 5.7. If the correlation matrix is calculated for every simulation the simulation might be discarded if the correlation is far away from the intended correlation.

3.5 Problems with the GARCH-model

The GARCHdt-model is supposed to be a better market model than the GBM-model, much because of the fluctuating volatilities and the broad tails of the return distribution. This is at the cost of a less intuitive and a less controlled model. In theory the GBM-series can end at any value between the two limiting values 0 and infinity. The probability for these extreme outcomes is however very small, but could change the value of the moments of a given sample. Take for example the first moment, the mean value, of 10 000 outcomes of a GBM-simulation. 9999 of the simulations ends up normally distributed with a mean of 105% of the starting value. But one of the simulations ends at 500% of the starting value and with this value included the new mean becomes almost 110%, i.e. one single value gives the mean value a large boost in the positive direction. A value such as this is called an outlier and might, depending on the circumstances, make the mean a very bad way to describe a set of data. The GBM-simulations that are produced from normally distributed random numbers have no problems with outliers because they are so unlikely. But for the GARCHdt-simulations they can have a very large impact. Therefore the mean no longer gives reliable information about the simulated series.

The actual problems with the outliers for the GARCH-simulations begins with the returns. If the return once happen to be 4.0 the index will rise 400% in one single step. Such a rise is of course not theoretically impossible, but must be rejected in a model that aims to simulate the market (as such a jump is incredibly rare). Therefore one solution would be to truncate the return distribution of GARCHdt-simulation at some value α, the *truncation parameter*. To do this, every single return is controlled before the index is updated. If the absolute value is lower than α, the index will be updated as usual. But if the absolute value is higher than α the return is rejected and a new return must be suggested. First when a return with absolute value lower than α is suggested, it will be accepted and the update will be made. In the probability distribution of the return this will look like the distribution has been cut off at $\pm\alpha$, see figure 16.

At this point the probability distribution of the GARCH-model return can be used to see how large percent of the distributions that is rejected. If α for example is set to 0.1, one can integrate the probability distribution from -0.1 to 0.1 and multiply by 100 to see how many percent of the distribution that actually is used. The remaining percentage is of course the rejected percentage. The only problem is that the distribution must be found for the specific parameter configuration before it can be used. The distribution in figure 16 is the truncated distribution of the one in figure 9 with $\alpha = 0.02$. Hence the analytic expression derived earlier can be used to see how much of the distribution that was rejected; in this case about 4 ‱was rejected.

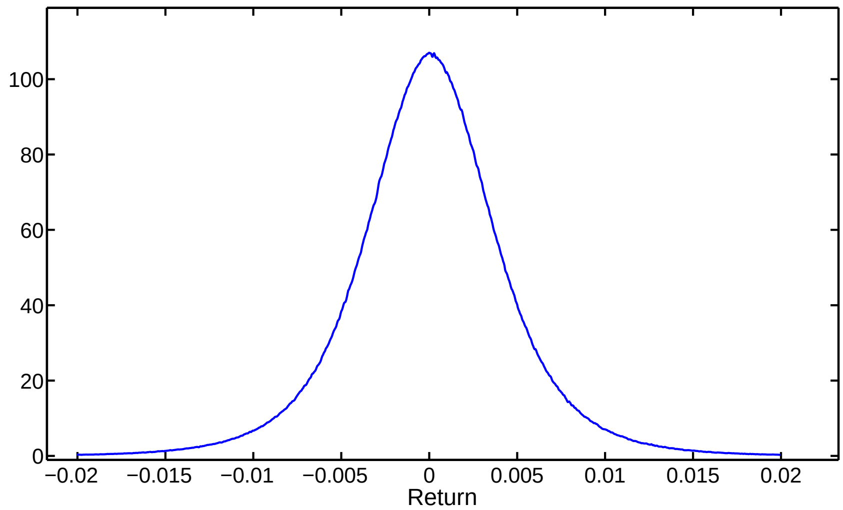

Figure 16: A return distribution from a GARCHdt-simulation. The distribution is truncated at ± 0.02 to avoid extreme outliers.

4 Options

4.1 Introduction

The main subject of this thesis concerns the two structured products sprinters and autocalls. They are unfortunately rather complex products, which makes it hard to understand the content they are involved in. Therefor this section will describe the same content, but from the perspective of an option, which will make the content easier to understand. The call options have already been touched upon several times in Mathematical Models (Section 3.5), where it has been priced both using simulations and analytically. In this section the option price set by B&S will be analysed and discussed, and later on the pay off of an option will be compared to a bank deposition.

4.2 Analysis of the price

In section 3.5 an expression for the price of a call option was derived using B&S. This price is set in such a way that the option at the expire date will be worth exactly as much as the invested money would have been worth after the same time in a bank. This is something that becomes quite clear if the option price is calculated when σ is set to zero, while r is non-zero. Note that this means that the underlying develops in a deterministic way, where only the interest is affecting it. In other words, the end value of the underlying can be calculated analytically according to

$$S(T) = S(t) \cdot (1 + r)^{(T-t)}$$

Here T is the maturity date, t the present date and r the yearly interest. When the updating of the underlying is daily, a more useful formulation of this is

$$S(T) = S(t) \cdot (1+)^{\frac{r}{252}} \cdot d \tag{16}$$

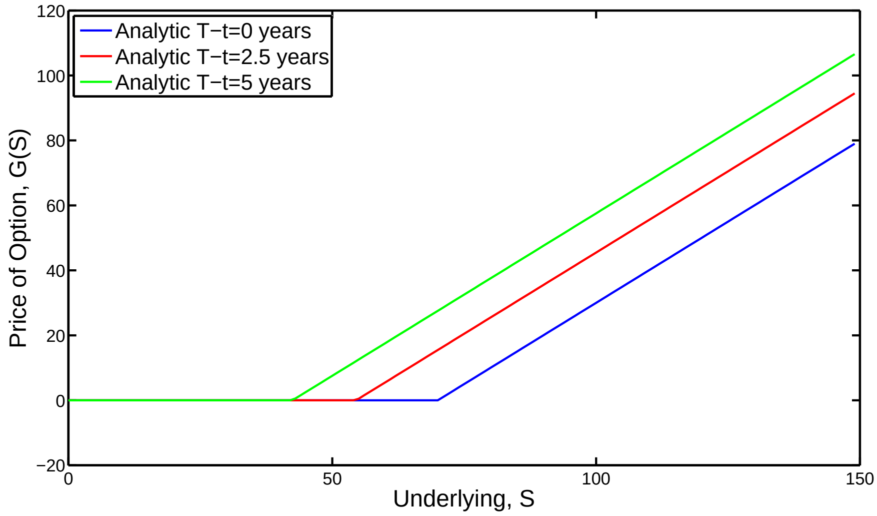

Figure 17: The price of an option at three different times without any volatility. The strike price is set to 70, T to 5 years and r to 10% a year.

where the yearly return is divided by the number of trading days a year and the difference $T - t$ is expressed in days, d. In figure 17 three solutions with $\sigma = 0$ are shown. They also have a strike price, E, set to 70 and $r = 10\%$ per year. If only focusing on the green solution , with $T - t = 5$ years, this is strictly zero until the underlying S is 42.5. This is because 42.5 is the lowest price that will end higher than the strike price at the maturity date according to equation 16, $42.5 \cdot (1 + \frac{0.1}{252})^{5 \cdot 252} \approx 70$. To the right of this point the option always has a price, and this price is proportional to the price of the underlying (and hence also to the interest). To see once more that B&S actually sets a fair price, one can consider this example. According to figure 17, the price of the option, when the underlying is 65, is roughly 22.5. After 5 years the underlying will be worth a little more than $107 \approx 65 \cdot (1 + \frac{0.1}{252})^{5 \cdot 252}$ which would mean that the option will give a profit of 107-70=37. If the option never was bought and money instead was put in a bank they would have been worth $22.5 \cdot (1 + \frac{0.1}{252})^{5 \cdot 252} \approx 37$.

In the next figure, figure 18, the σ is set to 10% a year while the interest is zero. This solution is much smoother since the underlying is not developing in a deterministic way. The indeterministic nature of the underlying namely forces the analytic solution to average the end price of the underlying in a statistical manner, from where the smooth appearance emerges. To the left in this figure the option is worth nothing, since there in theory is almost no chance at all that the underlying will end above the strike price at the maturity. At some point around 45 (for the green solution) the price is going up. This is because at this point there might be a chance that the underlying will end above the strike price. As the starting value of the underlying is increasing, the price of the option gets even higher because the chance of earning money from the option increases too. In this region the price is increasing rapidly, since the option can go from being worthless to being worth something. This changes after the starting value of the underlying becomes higher than the strike price, since there now is a risk that the option worth something actually could end up being worthless. At even higher start prices the green solution approaches

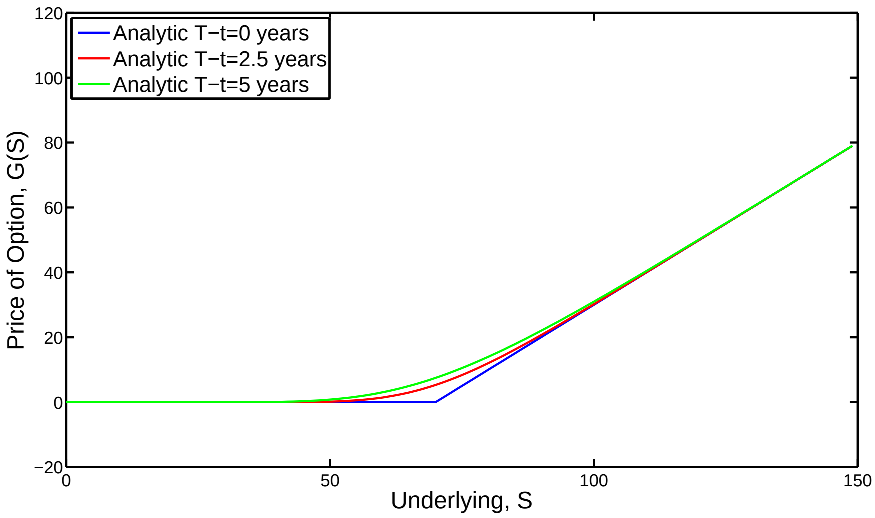

Figure 18: The price of an option at three different times without any interest. The strike price is set to 70, T to 5 years and σ to 10% a year.

the blue $T - t = 0$ solution. This is because the underlying is more or less certain to stay above the strike price and there is no statistical preference for the underlying to go up or down.

4.3 Option vs Bank Deposition - GBM

There is a big difference between investing in an option compared to putting the money in a bank. The option can produce lots of money for the holder if he/she is lucky, but can also end up being worth less at the expire date. Nothing about this is known beforehand. The bank deposition on the other hand is considered to be a safe investment, where the money is expected to grow with the given interest rate from day 1 to the expire date. Despite their differences it is interesting to compare which of the two investments that in general produces more money. How the money in the bank develops is known when the time period, interest rate and amount of money are known. The only problem is to know how much money that should be deposited. This bank deposition should namely be equal to the price of the option at time $t \neq T$. If the option is priced correctly the average pay off from the option should be equal to the bank deposition at the expire date. With the aid of equation 7 the price of a call option with $E = 70$, $r = 10\%$ a year, $\sigma = 10\%$ per year, $T - t = 1$ years and an underlying of 80 should be set to approximately 18.2. With this option price a pay off simulation with $N = 100000$ was made. To compare the bank deposition and the option, the option pay off is normalized to the bank pay off, the resulting histogram can be seen in figure 19 together with 5 other simulations with other volatilities.

The mean value of the histogram for $\sigma = 0.1$ in figure 19 is 1.0, which means that 18.2 was the fair price of the option at the time $T - t = 1$ years. There is however not many payoffs at precisely 1.0 and the distribution is rather broad, which also can be understood by computing the standard deviation of the pay off, which in this case was 0.32. None

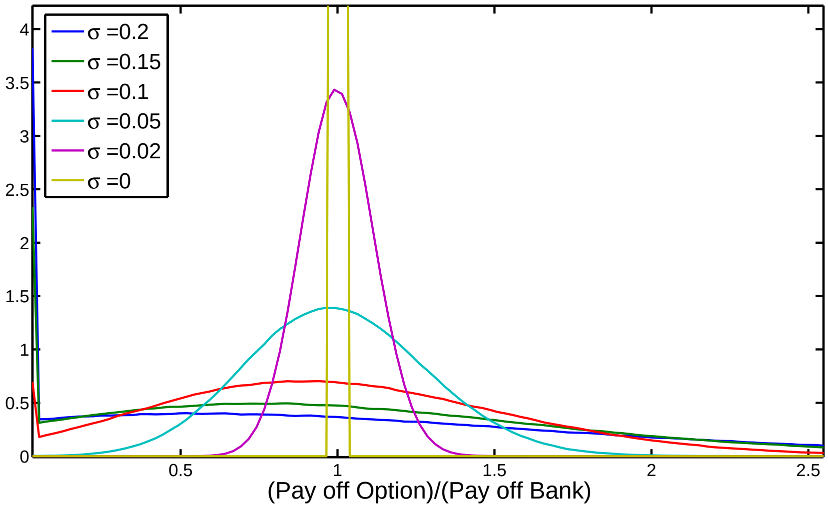

Figure 19: The pay off for an option with an underlying of 80 normalized to the fair price money deposited in a bank. The simulation was made with a GBM-simulation where $N = 1000000$, $dt = 1$ day, $T - t = 1$ years, $E = 70$, $r = 0.1\%$ a year and six different volatilities. The fair price of the options at day one was found using the analytic solution from equation 7. The yellow curve has $\sigma = 0$ and is because of this just a peak at 1. As such, it does not provide any interesting information and is only partly seen in this figure.

Volatility	Option Price	STD Pay off	Pay off > 1 (%)	Max/Min Pay off
0.2	18.2	0.96	42	10.0/0
0.15	17.3	0.78	44	7.1/0
0.1	16.8	0.56	47	4.9/0
0.05	16.7	0.28	49	2.6/0
0.02	16.7	0.11	49	1.5/0.4
0	16.7	0	na	1/1

Table 1: A few parameters for six GBM-simulations displayed in figure 19. Since the mean pay off for each of the simulations was 1.0 they are not displayed here. The left most column displays the volatility the simulations were done with. The next ones show how much the option was worth at $T - t = 1$ years, the standard deviation of the payoffs, what percentage of the investments that gave back more money than a bank deposition and the last column shows the maximum and minimum payoffs in ratios of the bank payoffs.

of these measurements really gives any information whether it is a good idea to invest in an option or not. To answer that question one could count how many times the option payoffs were higher than the bank payoffs at the maturity and vice versa. If this is done one comes to the conclusion that in 47 % of the times the option payoffs were higher than the bank payoffs, and in 53 % of the times the bank payoffs was higher. In other words it is more likely to loose money with the option than gaining. One should however keep in mind that the option can be worth much more than the bank deposition at the maturity date. The smallest value in figure 19 is for example 0 while the highest is 4.9. In table 1 the same data is presented for the other simulations shown in figure 19. Also the maximum and minimum pay off for each simulation is shown here. It is clear that B&S provides the fair price of the option, since the normalized mean is 1.0 for all simulations. But one should also remember that this just is a mean of 1 000 000 simulations. Indeed more than half of the simulations gave a result that would be a loss. The higher volatility, the higher risk that the option will be worth less than a bank deposition at the expire date. But it is also evident that an option with high volatility under lucky circumstances can generate much more than a similar option with lower volatility.

For this case the underlying of the option had a starting value of 80. Since the underlying is a continuous variable on the interval $[0, \infty)$ the staring value could have been anything. If the starting value would have been changed, the simulations would have been carried out in the same way but with a somewhat different result.

4.4 Option vs Bank Deposition - GARCHdt

For the GARCH-simulation there is no B&S to provide the fair price. But instead one could produce a price curve such as the one produced by the GBM in figure 5 to provide the fair price. In figure 20, six such curves for the GARCHdt-simulations are compared to the corresponding B&S-curves. The parameters where set precisely as they were in the previous section, with the addition of α, the truncation parameter, which was set to 0.2.

It is interesting to note how the GARCHdt price deviates from the B&S. They agree quite well for low values of S, but differs more at higher values. The deviation is purely an affect by the volatility, since the deviation is not observed with the volatility set to zero. That the price is higher according to the GARCHdt-simulations means that B&S undervalues the option, except at very low $S-$values.

With the GARCHdt price of the opions it is now possible to generate the same histogram as was generated for the GBM-simulations. Once again the parameters were the same as for the GBM-simulation with the addition of α. The result is seen in figure 21. By comparing this figure to the corresponding GBM-figure one can see that the peaks in the GARCHdt-figure are pushed to the left. One can also see that the GARCHdt-distributions have high peaks and decay slowly in the tails, while the GBM-distributions are normaly distributed. In table 2 a few properties of the GARCHdt-simulations are shown. The standard deviation of the pay off is higher for all the GARCHdt-simulations compared to the GBM-simulation. The maximum pay off from the GARCHdt is also several times higher than the corresponding pay off for the GBM. This would have shifted

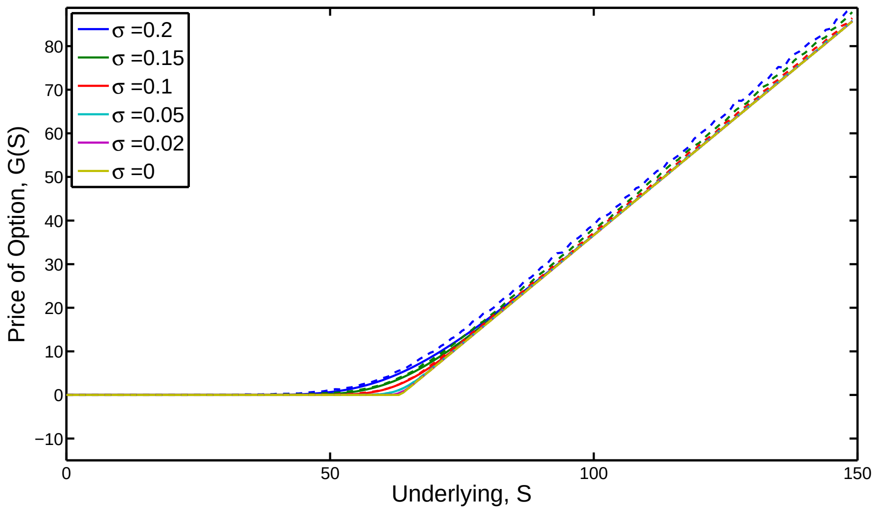

Figure 20: The price curve of an option, calculated with the GARCHdt-algorithm, compared to the analytic B&S solution, as a function of the underlying. The parameters were set to the same value as for the GBM-simulaion, with the addition of $\alpha = 0.2$.

the GARCHdt-simulation away from the mean pay off at 1 if it was not for the fact that more of these simulations ended under 1, something that also can be seen in the table.

To summarize the comparison between the GARCHdt- and the GBM-model for options, the GARCHdt seems to value the far out-of-the-money options just like the GBM. But the other options are valued higher because of the stochastic volatility, which more often give opportunities for the underlying to rise very much. On the other hand, more option investors must loose money to make room for those few who earns more, else the fair price would be shifted.

Volatility	**Option Price**	**STD Pay off**	**Pay off > 1 (%)**	**Max/Min Pay off**
0.2	20.3	1.10	41	151.1/0
0.15	18.7	0.86	44	118.7/0
0.1	17.3	0.61	47	48.5/0
0.05	16.8	0.31	48	24.3/0
0.02	16.7	0.14	49	8.3/0
0	16.7	0	na	1/1

Table 2: A few parameters for six GARCHdt-simulations displayed in figure 21. Since the mean pay off for each of the simulations was 1.0 they are not displayed here. The left most column displays the volatility the simulations were done with. The next ones show how much the option was worth at $T - t = 1$ years, the standard deviation of the payoffs, what percentage of the investments that gave back more money than a bank deposition and the last column shows the maximum and minimum payoffs in ratios of the bank payoffs.

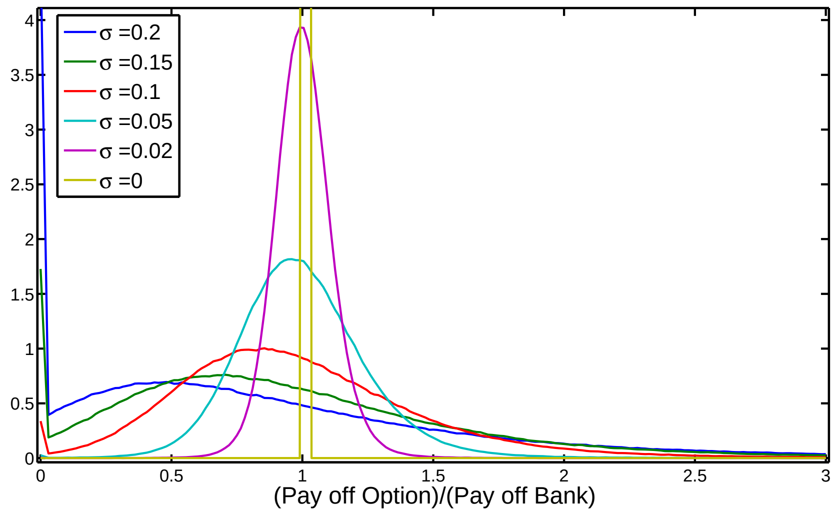

Figure 21: The pay off for an option with an underlying of 80 normalized to the same amount of money deposited in a bank, calculated using the GARCHdt-algorithm. The simulation was made with $N = 1000000$, $dt = 1$ day, $T - t = 1$ year, $E = 70$, $r = 0.1\%$ a year, $\alpha = 0.2$ and six different volatilities. The fair price of the options at day one was found in figure 20 which was produced using the GARCHdt-algorithm.

5 Sprinters

Imagine a product that gives a 20% interest rate per year if the market is doing well. And if it is doing bad it still gives back the nominal investment as long as it has not crashed below 50%. Sounds like a good deal, does it not? This is the way the sprinters are presented for the costumers. In the first subsection the sprinters are discussed in more details; how they work, what the restrictions are, what types there are, etc.. In the second subsection a limiting case of sprinters are priced using B&S, and in the following two subsections priced using GBM-simulations and GARCH-simulations respectively. In the fifth subsection a real sprinter are investigated. The real underlying index is used to find correct volatilities, correlations, etc. to finally investigate the product.

5.1 Introduction

The sprinter is a structured product whose pay off is directly connected to one or more underlies. In the case of more than one underlying, it is to a large extent the worst of the underlies that sets the price. If the worst underlying is higher than the corresponding starting value at the maturity date, the investor gains money proportionally to how the whole set of underlies evolved, multiplied by a predetermined pay off factor, β. If the worst underlying on the other hand is lower than the starting value at the maturity date, the investor only gets back the money he invested, provided it still is higher than 50% of the starting value. If it is lower than 50% of the starting value the investor only gets back as much as if the invested money were directly invested in the underlying itself, i.e. if the end value is 40% of the starting value, the investor only gets back 40% of his money. The limit at 50% in this example is not always set at 50%, but can range between any two

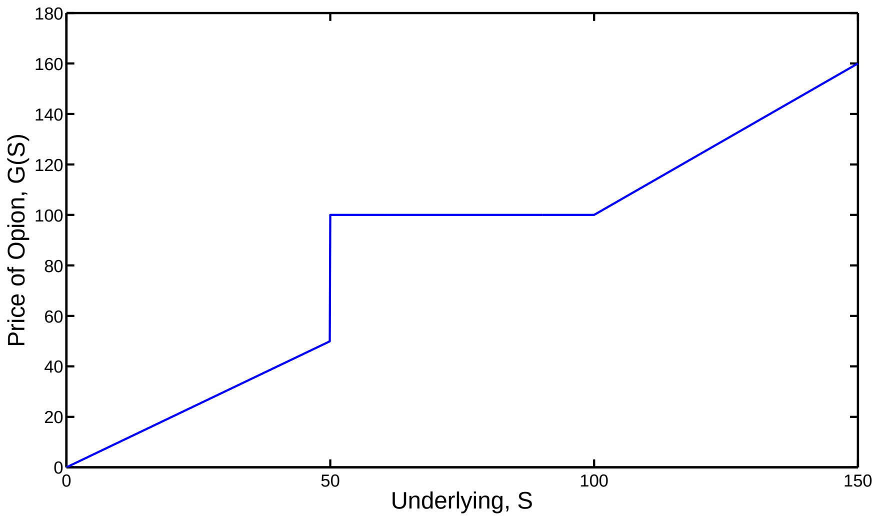

Figure 22: The pay off for a sprinter as a function of the underlying with $\gamma_1 = 50\%$, $\gamma_2 = 100\%$ and $beta = 1.2$ at the maturity date. Note that the right part is increasing slightly faster than the left one because of β .

numbers depending on the sprinter in question. This is also true for the limit at 100%, which can be higher or lower. These two limits will in the simulations be parameters labelled γ_1 for the lower limit and γ_2 for the higher. The pay off for a sprinter at $t = T$ with $\gamma_1 = 50\%$, $\gamma_2 = 100\%$ and $\beta = 1.2$ is shown in figure 22.

The sprinter is not one product itself, but rather a combination of an obligation and several options. Most of the money the investor pays for the sprinter goes to buy the obligation, which at the expire of the sprinter will provide the 100% pay back if necessary. The rest of the money (except for expenditure) is used to buy call options in the underlying market. In that way, the seller of the sprinter can pay back more than the invested money if the market develops profitably.

The pricing of a sprinter depends on several different parameters. In the easiest case when the sprinter depends on one index only, the parameters are reduced to the volatility of the underlying, σ, the interest rate, r, the value of γ_1, γ_2 and β. If more than one underlying are used, all there volatilities must be included in the analysis together with the correlation between them. Hence the pricing becomes dramatically harder with an increasing number of underlies. Due to this the sprinter will only be priced by B&S in the one-index limit.

5.2 Priced analytically with B&S

Although the B&S will not be used to price sprinters with more than one underlying, it will serve as a starting point for the simulations. By comparing the limiting cases of the simulations where only one underlying is used for the sprinters, the GBM-simulation should coincide with the analytical solution. It is also to expect that the GARCHdt-simulation should differ slightly from the analytical solution due to the broader tails of the returns. To price the sprinter analytically the procedure from section 3.3 is used.

The pay off for the sprinter at the maturity , $t = T$, can in the one limit case be divided into three parts according to the three equations below. Each of the three parts cover one interval each along the S-axis.

$$G(S, t = T) = G_1(S, t = T) + G_2(S, t = T) + G_3(S, t = T)$$

$$G_1(S, t = T) = \Theta(\gamma_1 - S(T)) \cdot S(T)$$

$$G_2(S, t = T) = \Theta(S(T) - \gamma_1) \cdot \Theta(\gamma_2 - S(T)) \cdot G_0$$

$$G_3(S, t = T) = \Theta(S(T) - \gamma_2) \cdot ((S(T) - \gamma_2) \cdot \beta + \gamma_2)$$

Here γ_1 and γ_2 no longer is a percentage, but a certain percent of the initial value of the underlying . To find the pay off for the sprinter at a time $t \neq T$ the functions $G_i, i = 1, 2, 3,$ must be found for arbitrary t. This can be done by employing the same procedure as was used to price options. First to be solved is G_1. If S and t are transformed according to $x = ln(S/\gamma_1)$ and $\tau = \frac{\sigma^2}{2}(T - t)$, G_1 becomes:

$$G_1(x, 0) = \Theta(1 - exp(x)) \cdot \gamma_1 \cdot exp(x).$$

By the use of equation 6, G_1 is given at any time t according to:

$$G_1(x, \tau) = \gamma_1 \cdot exp(-(\lambda - 1)\frac{x}{2} - (\lambda + 1)^2 \frac{\tau}{2}) \cdot w_1(x, \tau)$$

Here λ is a ratio between the interest and the volatility given by $\lambda = \frac{2r}{\sigma^2}$. At $t = T$, or $\tau = 0$, the previous equation becomes:

$$G_1(x, 0) = \gamma_1 \cdot exp(-(\lambda - 1)\frac{x}{2}) \cdot w_1(x, 0)$$

Combined with the first expression for $G_1(x, 0)$ it is possible to solve for $w_1(x, 0)$, and after a some algebra $w_1(x, 0)$ reads:

$$w_1(x, 0) = \Theta(1 - exp(x)) \cdot exp((\lambda - 1)\frac{x}{2})$$

To find $w_1(x, \tau)$ at arbitrary τ the expression for $w_1(x, 0)$ is inserted into equation 5

$$w_1(x, \tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} exp(-\frac{(x - x')^2}{4\tau}) \cdot \Theta(1 - exp(x')) \cdot exp((\lambda + 1)\frac{x'}{2})dx' =$$

$$= \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{0} exp(-\frac{(x - x')^2}{4\tau} + (\lambda + 1)\frac{x'}{2})dx'.$$

With the change of variable $y = \frac{x - x'}{2\sqrt{\tau}}$ this becomes

$$\frac{1}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{\tau}}}^{\infty} exp\left(-y^2 + (\lambda + 1)\frac{x - 2\sqrt{\tau}y}{2}\right) dy.$$

If the square in the exponent is completed, the expression above can be written as

$$\frac{1}{\sqrt{\pi}} \int\limits_{\frac{x}{2\sqrt{\tau}}}^{\infty} exp(-(y + \frac{(\lambda+1)\sqrt{\tau}}{2})^2 + (\lambda+1)\frac{x}{2} + (\lambda+1)^2\frac{\tau}{4})dy.$$

Another change of variable, $z = y + \frac{(\lambda+1)\sqrt{\tau}}{2}$, finally gives

$$\frac{1}{\sqrt{\pi}} \int\limits_{\frac{x}{2\sqrt{\tau}}+\frac{(\lambda+1)\sqrt{\tau}}{2}}^{\infty} exp(-(z)^2 \cdot exp((\lambda+1)\frac{x}{2} + (\lambda+1)^2\frac{\tau}{4})dz =$$

$$= \frac{1}{2}(1 - erf(d_1) \cdot exp((\lambda+1)\frac{x}{2} + (\lambda+1)^2\frac{\tau}{4})$$

where

$$d_1 = \frac{x}{2\sqrt{\tau}} + \frac{(\lambda+1)\sqrt{\tau}}{2}.$$

Here erf is the error function defined by $erf(x) = \frac{2}{\sqrt{\pi}}\int\limits_0^x exp(-t^2)dt$. When this result is inserted in equation 6, the solution to the problem reads

$$G_1(x,\tau) = \gamma_1 \cdot exp(-(\lambda-1)\frac{x}{2} - (\lambda+1)^2\frac{\tau}{2}) \cdot \frac{1}{2}(1 - erf(d_1)) \cdot exp((\lambda+1)\frac{x}{2} + (\lambda+1)^2\frac{\tau}{4}) =$$

$$= \frac{\gamma_1}{2}(1 - erf(d_1)) \cdot exp(x.)$$

After a change back to the variables S and T, $G_1(x,\tau)$ finally reads

$$G_1(x,\tau) = \frac{S}{2}(1 - erf(d_1)) \tag{17}$$

where

$$d_1 = \frac{1}{2\sqrt{\frac{\sigma^2}{2}(T-t)}}(ln(\frac{S}{\gamma_1}) + (r + \frac{\sigma^2}{2})(T-t)) \tag{18}$$

This function is plotted in figure 23 for $T = 300$, $\gamma_1 = 50$, $\sigma = 0.02$, $r = 5\%$ a year and several values of t. The solutions to the other two terms, G_2 and G_3, can be obtained in a similar way as $G_1(x,\tau)$. The second term, $G_2(x,\tau)$, is however a bit more complicated than the other two terms since it contains two step functions. Therefore a derivation of the second term will be presented while the third will be given more or less mediately. The second term $G_2(x,\tau)$ at $t = T$ is described by

$$G_2(S, t = T) = \Theta(S(T) - \gamma_1) \cdot \Theta(\gamma_2 - S(T)) \cdot G_0.$$

Here it is more suitable to set $x = ln(S)$ rather than involving γ_1 or γ_2. After this, G_2 can be written as

$$G_2(x, 0) = \Theta(exp(x) - \gamma_2) \cdot \Theta(\gamma_1 - exp(x)) \cdot G_0.$$

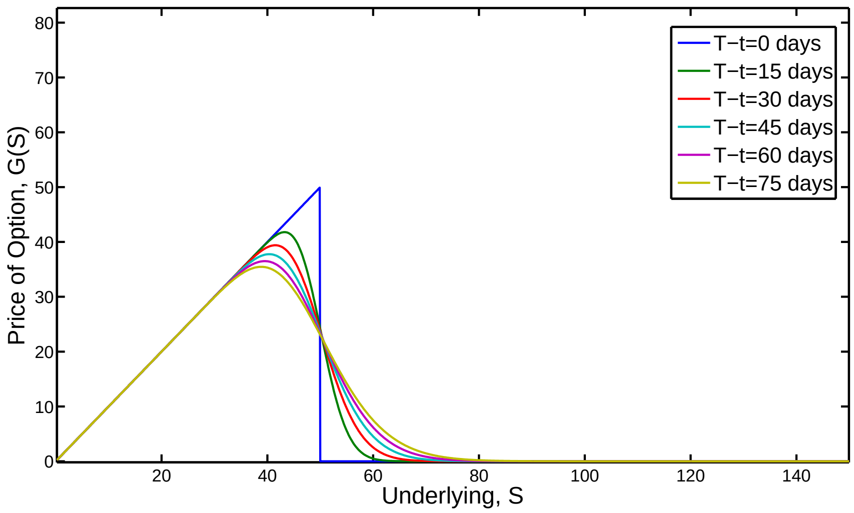

Figure 23: The first part of the sprinter pay off, G_1 given by equation 17-18, plotted as a function of the underlying, S, for different values of t, $T = 300$, $\gamma_1 = 50$, $\sigma = 0.02$ and $r = 5\%$ per year.

In this step, t was replaced by $\tau = \frac{\sigma^2}{2}(T - t)$ so $G_2(x, t = T)$. Again it is straight forward to use equation 6 to obtain another expression for $G_2(x, 0)$. When this is done $w_2(x, 0)$ comes out as

$$w_2(x,0) = \frac{G_0}{E} \cdot \Theta(exp(x) - 2\gamma_2) \cdot \Theta(\gamma_1 - exp(x)) \cdot exp((\lambda - 1)\frac{x}{2}).$$

By inserting this expression into equation 5 and by doing approximately the same calculations as for $w_1(x, \tau)$, $w_2(x, \tau)$ becomes

$$w_2(x,\tau) = \frac{G_0}{2E}(erf(d_2) - erf(d_3) \cdot exp((\lambda - 1)^2\frac{\tau}{4} + (\lambda - 1)\frac{x}{2})).$$

Here the arguments for the error functions are

$$d_2 = \frac{x - ln(\gamma_1)}{2\sqrt{\tau}} + \frac{(\lambda - 1)\sqrt{\tau}}{2}$$

and

$$d_3 = \frac{x - ln(\gamma_2)}{2\sqrt{\tau}} + \frac{(\lambda - 1)\sqrt{\tau}}{2}.$$

Inserted back into equation 6 this gives the final result for $G_2(x, \tau)$

$$G_2(x,\tau) = \frac{G_0}{2}(erf(d_2) - erf_{d_3}) \cdot exp(-\lambda \cdot \tau).$$

In the variables S and t, this reads

$$G_2(S,t) = \frac{G_0}{2}(erf(d_2) - erf_{d_3}) \cdot exp(-(T - t)r) \tag{19}$$

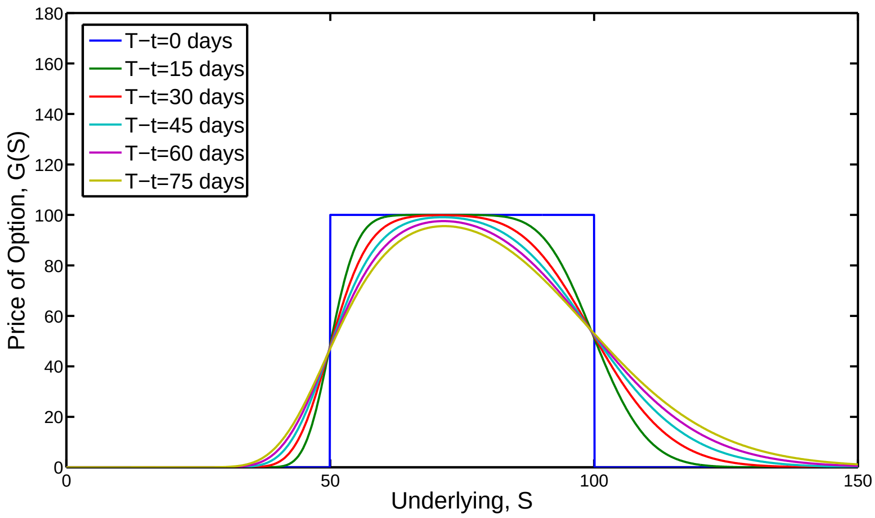

Figure 24: The second part of the sprinter pay off, G_2 given by equation 19-21, plotted as a function of the underlying, S, for different values of t, $T = 300$, $\gamma_1 = 50$, $\sigma = 0.02$ and $r = 5\%$ per year.

and d_2 and d_3 becomes

$$d_2 = \frac{1}{2\sqrt{\frac{\sigma^2}{2}(T-t)}}(ln(\frac{S}{\gamma_1}) + (r - \frac{\sigma^2}{2})(T-t)) \tag{20}$$

and

$$d_3 = \frac{1}{2\sqrt{\frac{\sigma^2}{2}(T-t)}}(ln(\frac{S}{\gamma_2}) + (r - \frac{\sigma^2}{2})(T-t)). \tag{21}$$

When plotted with the parameters $T = 300$, $\gamma_1 = 50$,$\gamma_2 = 100$, $\sigma = 0.02$, $r = 5\%$ a year the solution is given by figure 24. After very similar computations to those done to obtain $G_1(S,t)$, $G_3(S,t)$ can be expressed as

$$G_3(S,t) = \frac{(1-\beta)\cdot\gamma_2}{2}(1 - erf(d_4)) \cdot exp(-(T-t)\cdot r) \tag{22}$$

where

$$d_4 = \frac{1}{2\sqrt{\frac{\sigma^2}{2}(T-t)}}(ln(\frac{S}{\gamma_1}) + (r + \frac{\sigma^2}{2})(T-t)) \tag{23}$$

The result when this function is plotted for $T = 300$, $\gamma_2 = 100$, $\sigma = 0.02$, $r = 5\%$ a year and various t is shown in figure 25. At last the complete function, $G(S,t)$, is shown in figure 26 where the parameters are set according to $T = 300$, $\gamma_1 = 50$,$\gamma_2 = 100$, $\beta = 1.2$, $\sigma = 0.02$, $r = 5\%$ a year and again varying t.

5.3 Analysis of the price

With the analytic solution settled, it is now straight forward to compare to simulated solutions. But before doing so, it can be good to understand why the analytical solution

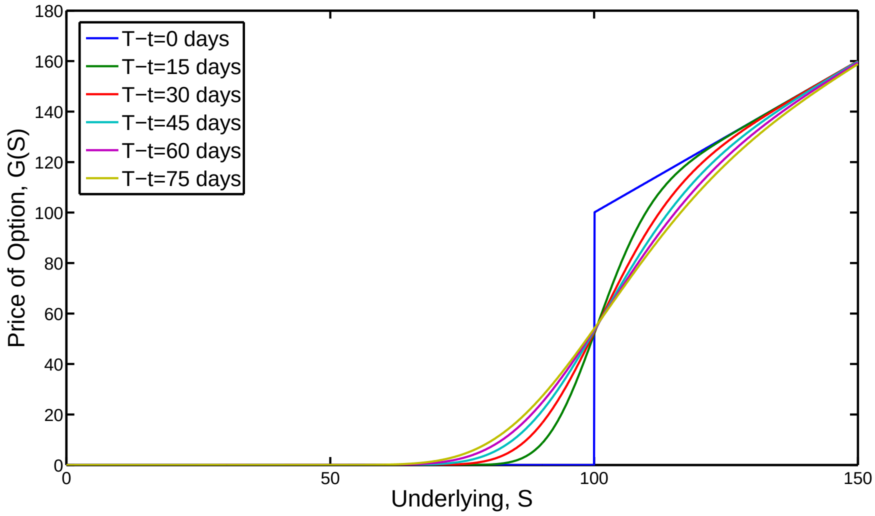

Figure 25: The third part of the sprinter pay off, G_3 given by equation 22-23, plotted as a function of the underlying, S, for different values of t, $T = 300$, $\gamma_1 = 50$, $\sigma = 0.02$ and $r = 5\%$ per year.

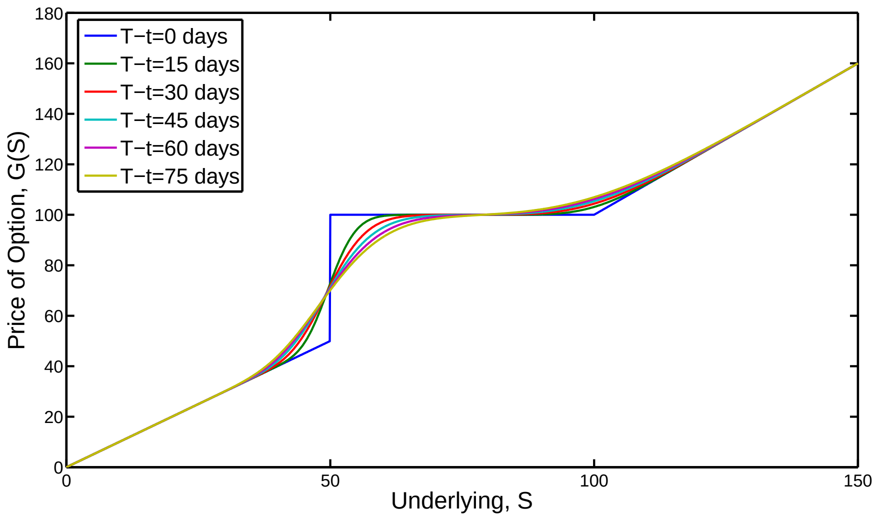

Figure 26: The compleate sprinter pay off, $G = G_1 + G_2 + G_3$ given by equation 17-23, plotted as a function of the underlying, S, for different values of t, $T = 300$, $\gamma_1 = 50$, $\sigma = 0.02$ and $r = 5\%$ per year.

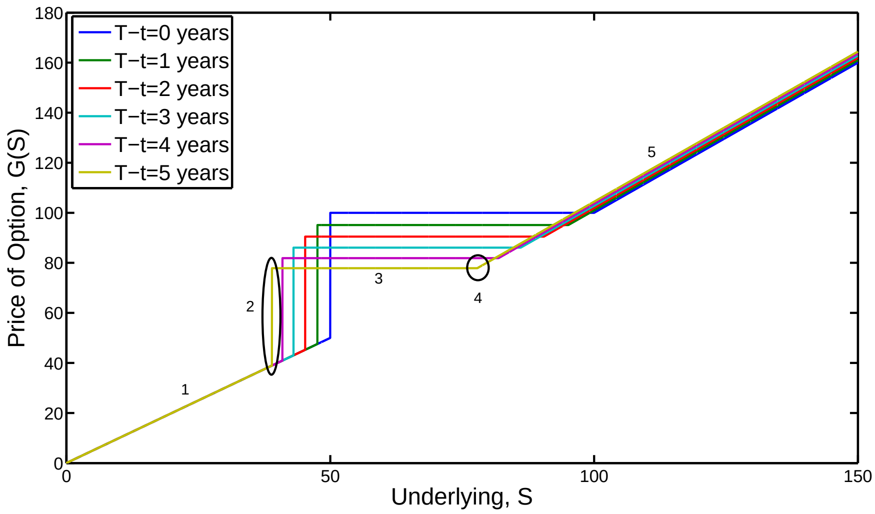

Figure 27: The option price as a function of S and t with $\sigma = 0$ and $r = 5\%$ a year.

looks as it does. To do this in an easy way, one can study two limiting cases; $\sigma = 0$ or $r = 0$. The two cases are shown in figure 27 and figure 28 respectively. There are six different values of $T - t$, but in the further discussion only the $T - t = 5$ years will be concerned. In the first figure, where $\sigma = 0$ and $r = 5\%$ a year, the price has two well defined points where the price drastically changes (numbered 2 and 4 in the figure) as well as three intervals where the price is either a constant or a linear function of the underlying (the intervals 1, 3 and 5 in the figure). This specific appearance of the price is no coincidence. In fact the price for this specific case is very logical as is described in the five points which corresponds to the five numbers in the figure.

1. In this interval the price of the sprinter is connected to the underlying in a linear way. As $\sigma = 0$ it is only the interest that affects the price of the underlying. Hence the value of the sprinter increases just as a normal bank deposition.

2. At this point the underlying is worth approximately 39. By use of equation 16 $39 \cdot (1 + \frac{0.05}{252})^{5 \cdot 252} \approx 50$. That means that the underlying will precisely reach the value of γ_1 at the expire date and the sprinter will hence be worth much more than it otherwise would.

3. For all points between 2 and 4 the underlying will end between γ_1 and γ_2. Therefore the price becomes constant for all these values.

4. This is the point, with the lowest value that will be worth more than γ_2 at the expire date. By inverting equation 16, one can see that this value should be $\frac{100}{(1+\frac{0.05}{252})^{5 \cdot 252}} \approx 78$, which also is observed.

5. Since all values of the underlying to the right of 4 will be worth more than γ_2 at the expire date, the price of the sprinter now starts to rise again. This region is very much like 1, with the exceptions that the slope is now β instead of 1 and that this region has no upper boundary.

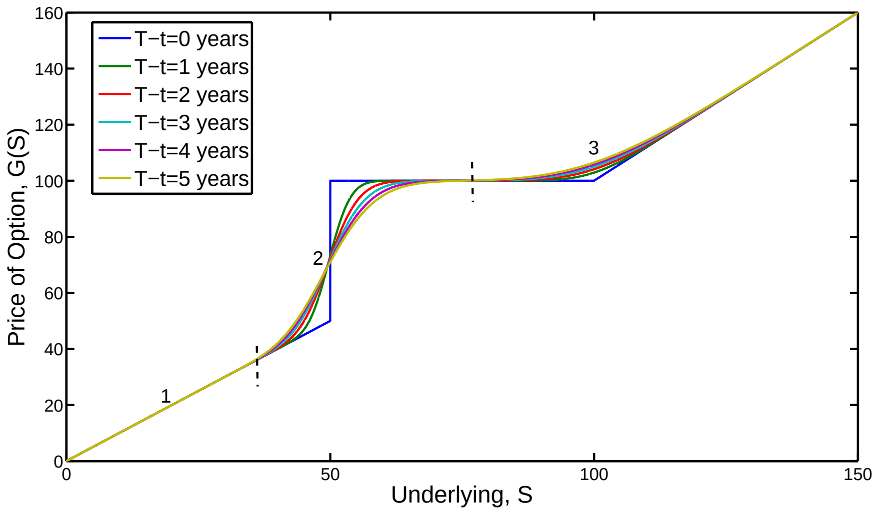

Figure 28: The option price as a function of S and t with $r = 0$ and $\sigma = 5\%$ a year.

In the second figure, figure 28, where the interest is zero but the volatility is 5 % a year, a much smoother price curve is obtained. The smooth appearance is a consequence from the fact that it is produced using normally distributed random numbers. Note that the price of the sprinter in this figure is the same as the pay off since there is no interest. Once more the curve is divided, this time into 3 regions to make the explanation of its appearance more clear.

1. In this region there is in principle no chance at all that the underlying will reach γ_1. Due to this, the price of the sprinter is in a linear relationship with the underlying, with slope 1

2. As the underlying is getting closer to γ_1, the chance to reach it gets larger. Hence the price gets higher than the underlying. The price does however not reach γ_1 until the right boundary of region 2. Before this point the risk of the underlying falling below γ_1 is too large.

3. At the boundary of region 2 and 3 the probability of the underlying to stay between γ_1 and γ_2 is the largest. To the right of this point, region 3, the price gets higher as the invested money is considered to be safe at the same time as there is a chance to gain even more money. In the limit when the value of the underlying is very high, the price and the underlying once more have a linear relationship, this time with β as slope.

5.4 Simulated with GBM - One Underlying

In the limit when the number of time steps in the GBM-simulation goes to infinity, the obtained pricing solution should approach the analytical B&S-solution. This is at least the case when the sprinter only depends on one underlying; if there are more underlies, B&S becomes to complex too handle (except for in some extreme cases). So in this section the

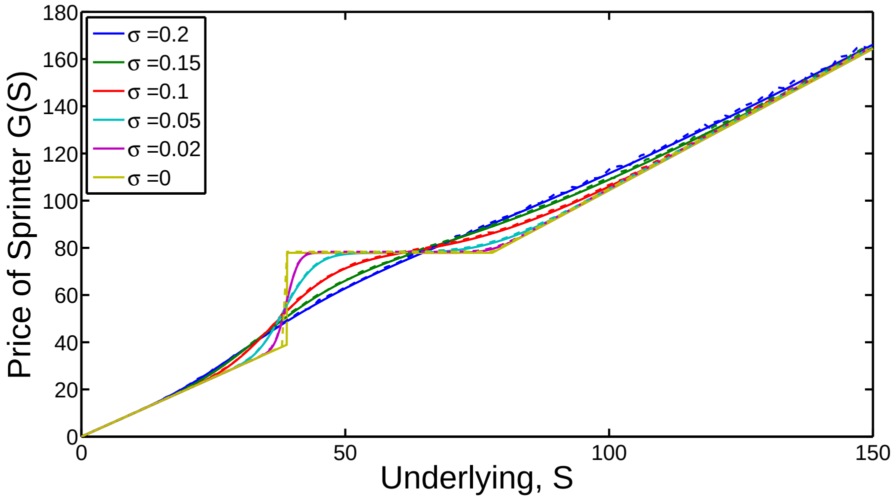

Figure 29: $G(S,t)$ for different values of σ at $T-t=5$ years. The analytical solution is shown as a solid line while the simulated price is shown as a dashed line. The other parameters were: $\gamma_1 = 50, \gamma_2 = 100, \beta = 1.2, r = 5\%$ per year,$N = 10000$ and $dt = 1$ day.

goal is to see that the limiting case gives back the analytical solution, and to investigate a few additional properties.

To follow the steps in section 3.2 one must first obtain a pay off equation. This equation is rather complicated (the general pay off can be seen as equation 24 in the section where several underlies will be considered), but can be simplified in the one-underlying limit. In this simpler form it looks like

$$G(S,T) = \begin{cases} S & S < \gamma_1 \\ \gamma_2 & \gamma_1 \le S < \gamma_2 \\ (S-\gamma_2)\cdot\beta + \gamma_2 & \gamma_2 \le S \end{cases},$$

In this expression G_0 is the starting value of the sprinter. After a simulation with $T = 5 \cdot 252$, $r = 5\%$ a year, $\sigma = 0.1$ a year, $N = 10000$, $n = 5 \cdot 252$, $\gamma_1 = 50$, $\gamma_2 = 100$ and $\beta = 1.2$ the result can be seen in figure 29.

As predicted the simulation and analytic solution agree well this time too. Everything in figure 29 is however mean values and does not say anything at all about how the actual payoffs are distributed. One way to get this information is to simulate several underlies, normalize the corresponding pay off to the interest and plot the histogram, the very same procedure as for the options. If such a simulation is done with $N = 1000000$, $dt = 1$ day, $T-t = 5$ years, $E = 70$, $r = 5\%$ a year, $\gamma_1 = 50\%$, $\gamma_2 = 100\%$ and six different volatilities, the resulting histogram is seen in figure 30. Note that the fair price once more has been inserted into the algorithm to get the correct interest normalization. This time the fair price was read at $S = 100\%$, the starting value of the underlying, which would mean that the sprinter in question is bought at day one of its life time. Due to the fact that the volatility affects the fair price, the curves are normalized to different starting values. This is the reason that the high 100% pay back peaks are shifted.

It is also interesting to see the standard deviation in the pay off, the loss to win ratio, the maximum and minimum pay off for each simulation. This is shown in table 3 together

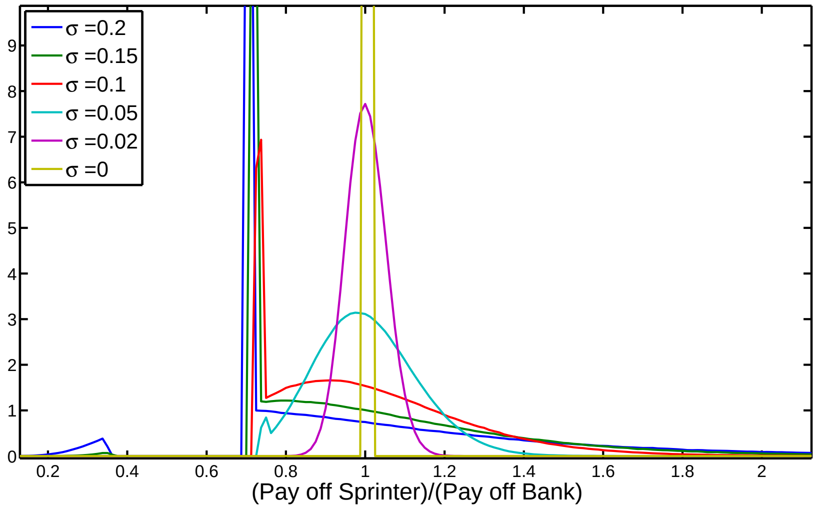

Figure 30: The pay off for six sprinters with an underlying of 100 normalized to the fair price money deposited in a bank. These histograms consists of 1 000 000 GBM-simulations for each volatility, where the parameters are the same as in figure 29. The figure is cut to make the tails more visible.

with the associated volatility and the fair price. The standard deviation in the pay off is very dependent on the starting value of the underlying. A starting value in the middle of the plateau in the pay off curve for the sprinter, has for example low standard deviation in the end price compared with a value on the edge of the plateau. In figure 31 the standard deviation is plotted as a function of S and σ with the other parameters the same as in figure 29. It is actually not the standard deviation that is plotted here, but the standard deviation divided by the mean of the payoffs. As expected the standard deviation increases with the volatility, and the risk of the investment depends much on the starting value of S. The yellow curve is the same as a bank deposition (zero volatility) and has hence zero standard deviation for all values of S. The second lowest volatility has the purple curve, which has zero standard deviation only in the middle of the pay off plateau. The other

σ	**Sprinter Price**	**STD Pay off**	**Pay off $>$ 1 (%)**	**Max/Min Pay off**
0.2	111.5	0.44	41	8.5/0
0.15	108.9	0.32	44	4.6/0
0.1	106.0	0.22	47	3.2/0
0.05	104.5	0.12	48	1.8/0
0.02	104.4	0.05	49	1.3/0
0	104.4	0	na	1/1

Table 3: A few parameters for six GBM-simulations displayed in figure 30. Since the mean pay off for each of the simulations was 1.0 they are not displayed here. The left most column displays the volatility the simulations were done with. The next ones show how much the sprinter was worth at $T-t=5$ years, the standard deviation of the payoffs, what percentage of the investments that gave back more money than a bank deposition and the last column shows the maximum and minimum payoffs in ratios of the bank payoffs.

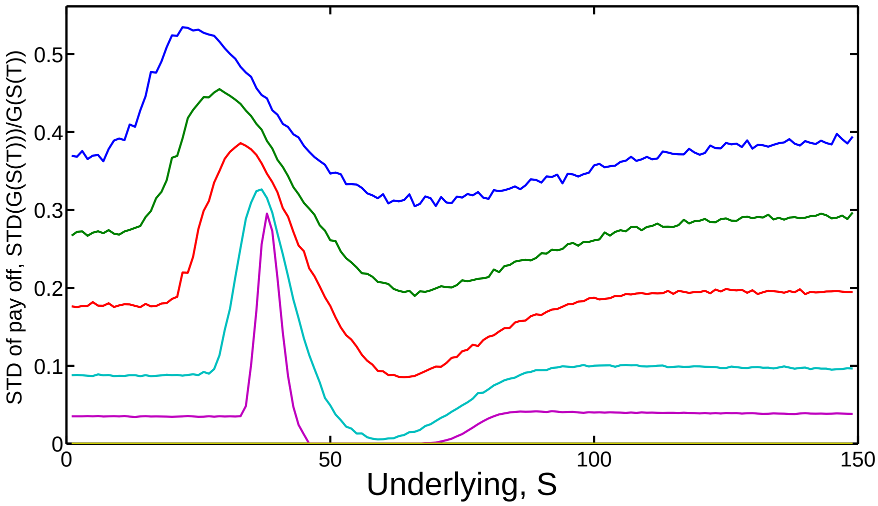

Figure 31: The standard deviation in the pay off divided by the mean pay off for 10 000 GBM-simulations. The parameters were set to the same values as for the simulations to produce figure 29

simulations with higher volatilities also have dramatic drops in the standard deviation at the plateau. The standard deviation is however not zero, since there is a chance that the underlying reaches below the 50%-limit or above the 100%-limits. In particular, starting values around the 50%-limit have high peaks in their standard deviation. This is because the large difference between an end value larger or equal to 50% compared to one less than 50%. It is also clear that this peak comes at lower values of S for simulations with higher volatilities. This since the higher volatility gives the underlying a better opportunity to end at a high value.

5.5 Simulated with GARCHdt - One Underlying

This section will follow almost the same path as the previous one. Hence it will be less detailed, and only the differences will be discussed in closer details. The previous section emphasized once more that the GBM-simulations in the 1-underlying limit gave back the B&S-solution. Therefore it seems unnecessary to distinguish between the B&S and the GBM solutions, and they will be refereed to alternately.

To produce the same histogram for the GARCHdt-simulations as was done for the GBM-simulations, the fair price must first be found. With $\gamma_1 = 50$, $\gamma_2 = 100$, $\beta = 1.2$, $T - t = 5$ years, $N = 10000$, $dt = 1$ day and $r = 5\%$ a year, six price curves with varied volatilities are produced. They are shown together with the analytic B&S-solutions in figure 32. Just as for the options, the simulations with relatively low volatilities agree quite well with the B&S-solution. Those with higher volatilities on the other hand deviates more, especially for high S-values. With these price curves, the fair price can be found and the simulations for the histograms can be made. The result can be seen in figure 33.

If these histograms are compared to those in figure 30, one can see that the GBM-

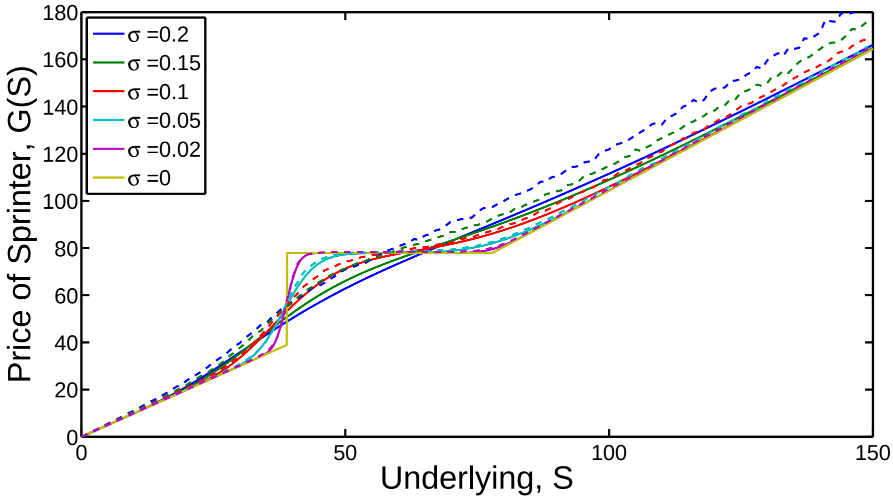

Figure 32: The price curves for six sprinters with varying volatilities. The dashed curves are produced by GARCHdt-simulations while the solid ones are the analyical B&S-solutions. The parameters were: $\gamma_1 = 50$, $\gamma_2 = 100$, $\beta = 1.2$, $T - t = 5$ and $dt = 1$ day.

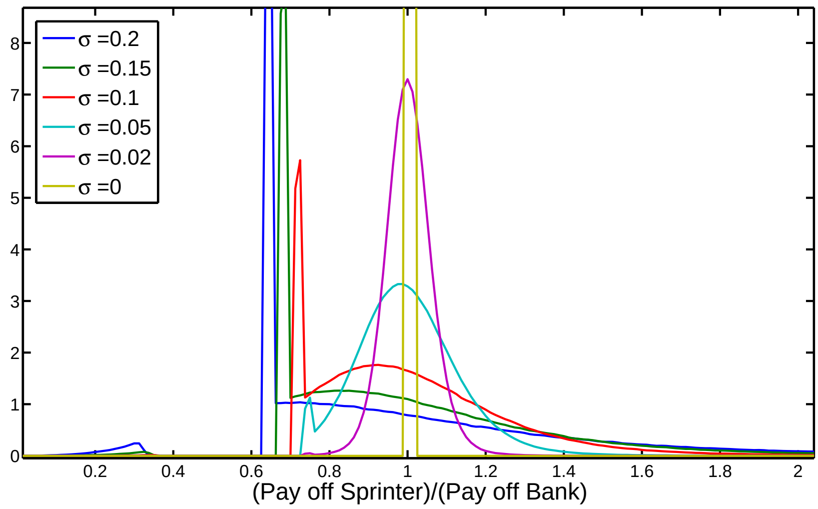

Figure 33: The pay off for six sprinters with an underlying of 100 normalized to the fair price money deposited i a bank. These histgrams consist of 1 000 000 GARCHdt-simulations for each volatility where the parameters are the same as for figure 32. The figure is cut to make the tails more visible.

Volatility	Sprinter Price	STD Pay off	Pay off > 1 (%)	Max/Min Pay off
0.2	121.8	0.61	38	127.1/0.01
0.15	115.0	0.43	40	47.9/0.02
0.1	109.2	0.28	43	26.8/0.04
0.05	106.0	0.14	47	21.7/0.08
0.02	105.2	0.06	49	9.2/0.13
0	104.4	0	na	1/1

Table 4: A few parameters for six GARCHdt-simulations displayed in figure 33. Since the mean pay off for each of the simulations was 1.0 they are not displayed here. The left most column displays the volatility the simulations were done with. The next ones show how much the sprinter was worth at $T-t=5$ years, the standard deviation of the payoffs, what percentage of the investments that gave back more money than a bank deposition and the last column shows the maximum and minimum payoffs in ratios of the bank payoffs.

simulations have thicker peaks than the GARCHdt-simulations. But as emphasized many times before, the GARCHdt-simulations have tails that decrease much slower. This is not particularly visible in the histograms, unless one can zoom in the tail regions, but can be understood any way by comparing the table 4 with the corresponding table for the GBM. Not only is the standard deviation higher for all the GARCHdt-simulations, but the highest and lowest values reach far beyond those of the GBM.

To compare the risks between the GBM and the GARCHdt-simulations one can compare figure 34 and figure 31. The first striking difference is that the GARCHdt-simulations fluctuate a lot more, although the same number of simulations are made. This indicates that the GARCHdt-simulations give extreme end values at a much higher rate than the GBM-simulations. Because of this, the standard deviation in two adjacent S-values can differ quite a lot, e.g. if one of the two series gets a few extreme values and the other none. One can also see that all GARCHdt-simulations with non-zero volatility has a little higher standard deviation than the GBM-simulation. It is interesting to point out that not even the purple line, with $\sigma = 0.02$ a year, is sure to end at a specific value at the price plateau.

5.6 Simulating a Sprinter - Several Underlies

When going from one underlying to several underlies, it becomes too complicated (perhaps impossible) to handle the method to analytically solve the problem. But from everything that has been done so far, the GBM and the B&S have given exactly the same results. In fact the GBM-simulations have provided more information than the analytic solution has, and it will now serve as the new benchmark to be compared to the GARCHdt-simulations.

Before the simulations can be done the complete pay off equation for the sprinter must be presented.

$$G(\mathbf{S},T) = \begin{cases} min(\mathbf{S}), & \text{if } min(\mathbf{S}) < \gamma_1 \\ \gamma_2, & \text{if } \gamma_1 \le min(\mathbf{S}) \le \gamma_2 \\ \gamma_2(\theta \cdot \beta + 1), & \text{if } \gamma_2 \le min(\mathbf{S}) \end{cases} \tag{24}$$

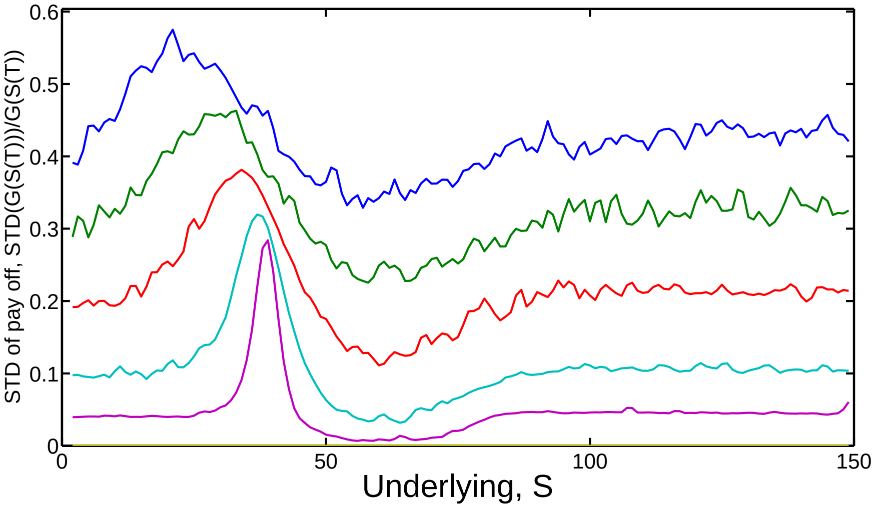

Figure 34: The standard deviation in the pay off divided by the mean pay off for 10 000 GARCHdt-simulations. The parameters were set to the same values as for the simulations to produce figure 32.

Note that the scalar S is here replaced by the vector $\mathbf{S}$ which contains all the end prices of the underlies. Also the parameter θ is new. This parameter is a measure of how well the composition of the stocks has developed. If n underlies are used in the sprinter, θ is given by

$$\theta = \sum_{i=1}^{n} \omega_i \cdot \frac{\mathbf{S}(T,i)}{\mathbf{S}(0,i)} - 1.$$

Here the ω_i simply is the weight of each of the underlies and the sum of all ω_i should equal to one. If nothing else is mentioned the weight is evenly distributed between the underlies. When all the underlies are higher than their starting value, the sum above will be greater than one. In this case the sum minus one will be greater than zero so that θ becomes greater than zero. This is the only case when θ will be used, since a negative θ would imply that at least one of the underlies ended lower than it started. If this is the case $min(\mathbf{S}(T))$ would end between γ_1 and γ_2 which by the definition only gives the nominal investment back.

Another thing that complicates the simulations when the number of underlies is increased is the correlation. If all the underlies are 100% correlated they can be seen as just one, and the one underlying-solution should then be obtained once more. This is however not an interesting case as it is the one that already has been investigated. To see how the end price of the sprinter is affected by the correlation, the end price is calculated in figure 35 as a function of number of underlies, n, and the correlation C. Both GBM-simulations and GARCHdt-simulations are presented in this figure, the GBM as solid lines while the GARCHdt as dashed lines. To be able to compare this figure with those for the one underlying, all the end prices have been normalized to how much they would have been worth if they were put in a bank instead (if they only were to gain interest). A value of 1 will therefore mean that the sprinter generated just as much as a bank would have, while a lower value than 1 means worse and a higher better pay off than the bank.

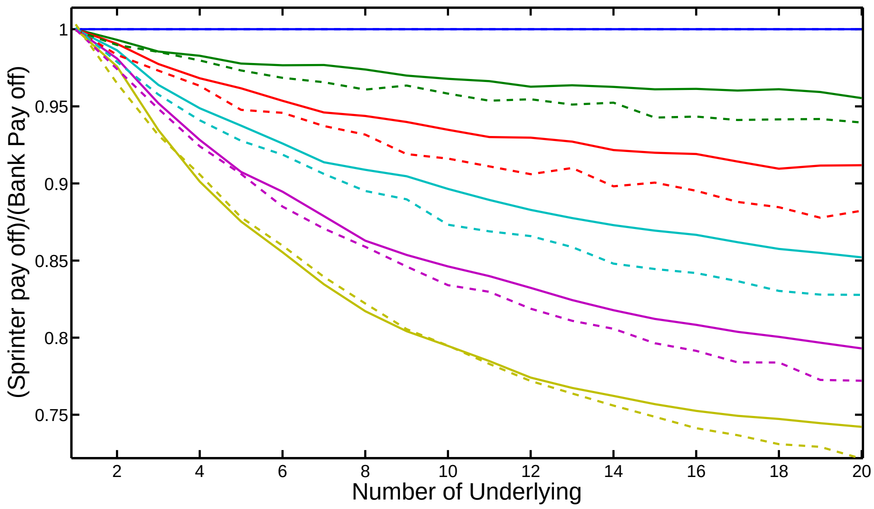

Figure 35: The end price of a sprinter as a function of the number of index and the correlations. The price axis is scaled in percent of the cost of the sprinter. The solid lines represent the GBM-simulation and the dashed lines the GARCHdt-simulations. The colours represent different correlations, blue for $C = 1$, green for $C = 0.8$, red for $C = 0.6$, cyan for $C = 0.4$, magenta for $C = 0.2$ and yellow for $C = 0$. 10 000 simulations were made with $\sigma = 0.1$ per year, $r = 0.05$ a year,$\gamma_1 = 50$, $\gamma_2 = 100$, $\beta = 1.2$, $T - t = 5$ years and $dt = 1$ day.

Although the correlation is of most interest, other parameters can be varied to produce similar figures. Figure 36 does for example show the end price as a function of σ and n. To produce the figures the parameters were set to; $N = 10000$, $\sigma = 0.1$ per year, $r = 0.05$ a year, $\gamma_1 = 50$, $\gamma_2 = 100$ and $C = 0.5$, except when the specific parameter was the one being varied. The update was daily during 5 years. One should also mention that the simulation starts the day the sprinter becomes active, i.e. the underlying starts at 100%. If the simulations instead were started on a later time, $t \neq 0$, the outcome could be quite different.

In all of the figures, it is seen that a large number of underlies has a negative impact on the price (with the exception for when all underlying are correlated to 100% which by rights only is one underlying). One can also see that a higher volatility becomes a real disadvantage at a higher number of underlies. The fact that a high volatility has such a negative impact on the end value of the underlies is very intuitive, the worst of several underlies should rarely be higher than one randomly chosen underlying. This affects the price of the product in such a way that the higher volatility, the lower price. One should however keep in mind that all the products here are priced as their corresponding 1-underlying product. Hence it is not the actual price that is shown here, rather a schematic picture of how the number of underlies affect the 1-underlying product.

In the two previous subsections the standard deviation in the pay off, the loss to win ratio, the maximum and minimum pay off for the GBM and the GARCHdt-simulations were displayed both as figures and tables. The same thing can now be done for the sprinter with four underlies. Indeed, one can make one with different correlations and one

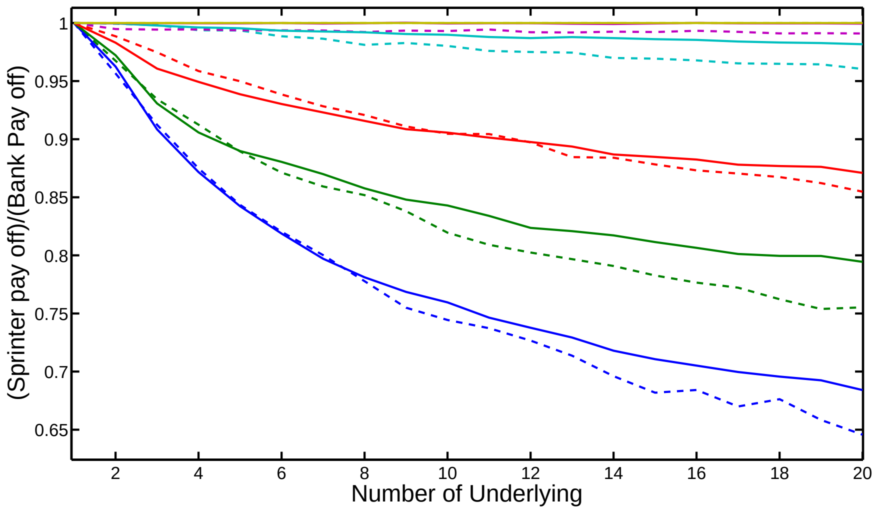

Figure 36: Same as figure 35, with the exception that the volatility and not the correlation is varied. The colours represent different volatilities, blue for $\sigma = 0.2$, green for $\sigma = 0.15$, red for $\sigma = 0.1$, cyan for $\sigma = 0.05$, magenta for $\sigma = 0.02$ and yellow for $\sigma = 0$.

Correlation	**Sprinter Price**	**STD Pay off**	**Pay off > 1 (%)**	**Max/Min Pay off**
1	106.2	0.24	43	2.6/0.34
0.8	105.1	0.23	45	2.5/0.34
0.6	103.5	0.23	47	2.3/0.33
0.4	100.9	0.22	48	2.3/0.35
0.2	98.2	0.21	48	2.2/0.37
0	95.6	0.20	47	1.8/0.35

Table 5: Parameters for the six GBM-simulations displayed in figure 38. The left most column displays the correlation the simulation was done with. The other columns show how much the sprinter was worth at $T - t = 5$ years, the standard deviation of the pay off, what percentage of the investments that gave back more money than a bank deposition and the last column shows the maximum and minimum pay off in ratios of the bank pay off.

with different volatilities. But to do so, the fair price must once again be found. To find the fair price the same procedure that was used for the GARCHdt-simulation is applied. With $N = 10000$, $\sigma = 10\%$ per year, $r = 5\%$ a year, $\gamma_1 = 50$, $\gamma_2 = 100$, $\beta = 1.2$, $T - t = 5$ years, $dt = 1$ day and $\alpha = 0.2$ (for the GARCHdt) the price curves are simulated for six correlations between 0 and 1. These curves can be seen in figure 37.

From these price curves the fair prices for the underlies at 100 % is used to simulate the pay off for 1000 000 GBM and GARCHdt-simulations. The distributions of the pay off (normalized to the bank pay off) for various values of the correlation for both models are shown in figure 38. The solid lines are the GBM distributions and the dashed lines the GARCHdt distributions. The standard deviation and the rest of the parameters for each model are shown in table 5 and table 6, respectively.

To discuss figure 38 further, the distributions have a rather interesting appearance.

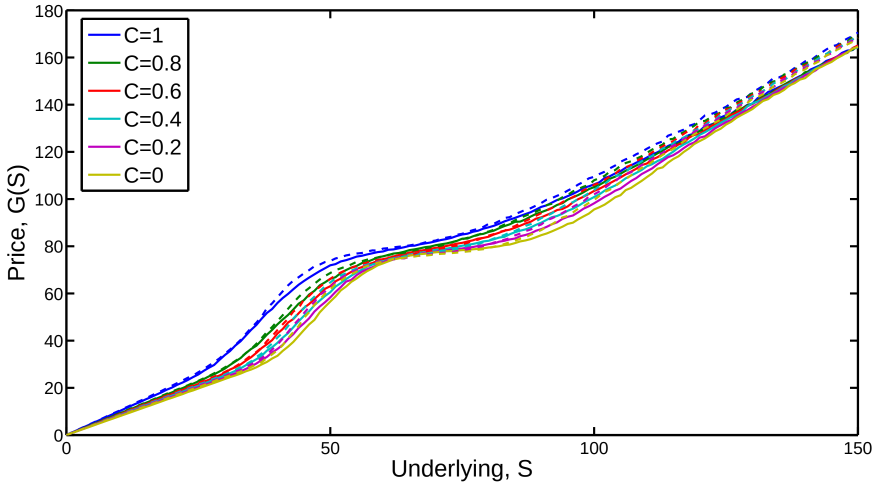

Figure 37: The price curves for a sprinter, modelled both with the GBM (solid lines) and the GARCHdt-model (dashed lines). Each model simulates the price with $N = 10000$, $\sigma = 0.1$ per year, $r = 0.05$ a year,$\gamma_1 = 50$, $\gamma_2 = 100$, $\beta = 1.2$, $T - t = 5$ years and $dt = 1$ day for six values of C. For the GARCHdt-simulation α was set to 0.2.

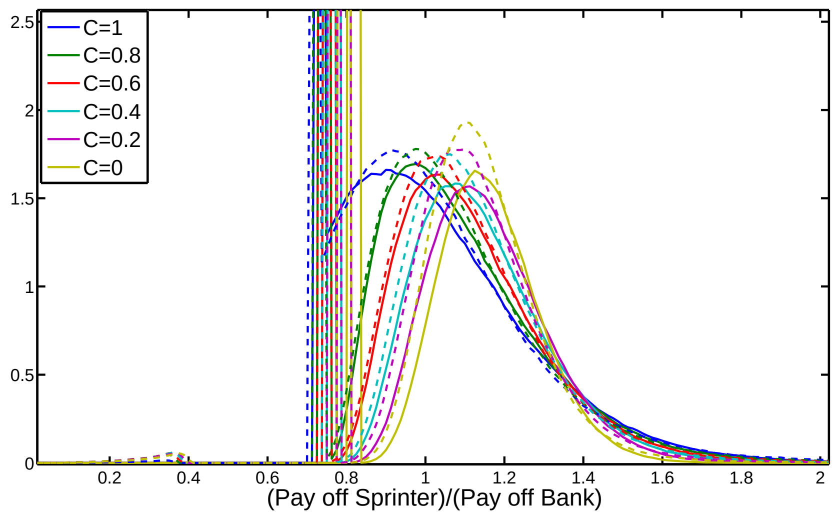

Figure 38: The end price distribution of a sprinter, modelled both with the GBM (solid lines) and the GARCHdt-model (dashed lines). Each model simulates the price with $N = 10000$, $\sigma = 0.1$ per year, $r = 0.05$ a year, $\gamma_1 = 50$, $\gamma_2 = 100$, $\beta = 1.2$, $T - t = 5$ years and $dt = 1$ day for six values of C. For the GARCHdt-simulation α was set to 0.2.

Correlation	Sprinter Price	STD Pay off	Pay off > 1 (%)	Max/Min Pay off
1	109.4	0.29	43	29.0/0.08
0.8	108.0	0.25	46	8.2/0.07
0.6	106.0	0.25	48	7.7/0.05
0.4	103.9	0.24	50	9.9/0.03
0.2	102.0	0.23	51	7.2/0.07
0	99.6	0.22	52	4.5/0.07

Table 6: Parameters for the six GARCHdt-simulations displayed in figure 38. The left most column displays the correlation the simulation was done with. The other columns show how much the sprinter was worth at $T - t = 5$ years, the standard deviation of the pay off, what percentage of the investments that gave back more money than a bank deposition and the last column shows the maximum and minimum pay off in ratios of the bank pay off.

The peaks ranging from 0.7-0.8 correspond for example to those investments that only gave the nominal investment back, i.e. those sprinters with lowest underlying ended between 50-100% of the starting value. The reason that they are dispersed is that a sprinter with high correlation costs more than one with low correlation, if everything else is kept constant. Indeed the peak for the $C = 1$ GBM-distribution is centred around 0.74 since $100/(106.2 \cdot 1.05^5) \approx 0.74$ and the $C = 0$ distribution peak at 0.82 since $100/(95.6 \cdot 1.05^5) \approx 0.82$, which is also seen in the figure. The denominator in this expression is simply the 5% interest a year over 5 years that the bank was supposed to pay. Since the prices for the GARCHdt-simulated series is slightly higher than the GBM-simulated, these distributions are shifted a bit to the left compared to the GBM-distributions. To the left in the figure there are tiny peaks corresponding to those sprinters whose lowest index ended below the 50% threshold. This peak should theoretically start at zero and end at 0.41 ($50/(95.6 \cdot 1.05^5) \approx 0.41$) for the $C = 0$ GBM-distribution. The volatility is however too low to produce such a broad peak and only values between 0.35 and 0.41 can be observed. This is unfortunately hard to see in the figure, and therefore the maximum and minimum values are displayed in the tables. In agreement with how the correlations should work, one can see that the blue curve, $C = 1$, has fewest values below the 50% range, while the other curves have more and the yellow curve, $C = 0$, has the most values below 50%. To the right of the high 100 % peaks, all the sprinters with an underlying higher than 100% ended. All these sprinters had their pay off boosted both by β and θ. Therefore it looks like a second distribution on the right hand side of these peaks. The reason that this part of the distributions are so different can be explained with the following example: Suppose that all the underlies for the yellow curve (with $C = 0$) ends over 100%, then there is a low probability that all the underlies will end at a value slightly higher than 100%. It is much more probable than one or two underlies ends at a much higher value, such as 120% or 130%. This would mean that the θ for this sprinter will be relatively large and so the pay off is boosted to a high value. If the same reasoning is done for the blue curve (where all the underlies are 100% correlated), it is obvious that all the underlies can end slightly above 100% relatively often. Hence the pay off for this sprinter will not be boosted by the same amount, since θ is close to one. Because of this, the yellow distribution is almost zero next to the 100% peak, while the blue distribution has a peak. On the left side of the 100% peak, it was observed that

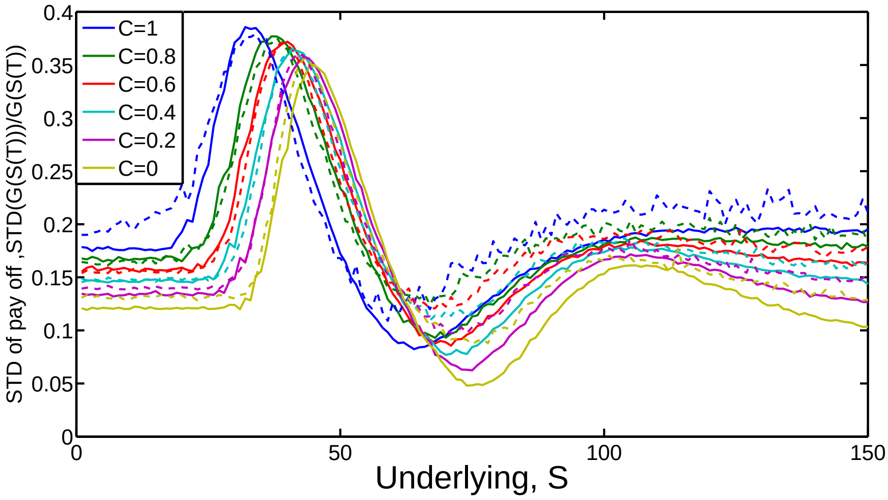

Figure 39: The standard deviation in the price for a sprinter, modelled both with the GBM (solid lines) and the GARCHdt-model (dashed lines). Each model simulates the price with $N = 10000$, $\sigma = 0.1$ per year, $r = 0.05$ a year, $\gamma_1 = 50$, $\gamma_2 = 100$, $\beta = 1.2$, $T - t = 5$ years and $dt = 1$ day for six values of C. For the GARCHdt-simulation α was set to 0.2. The blue curve represents $C = 1$, the green $C = 0.8$, the red $C = 0.6$, the cyan $C = 0.4$, the magenta $C = 0.2$ and the yellow $C = 0$.

the tails were dominated by the uncorrelated series. In the same way the right tails are dominated by highly correlated series. If one of the underlies in a highly correlated series ends at a high value, the other underlies are probable to end at a high value as well. This in contrast to a series with low correlation, where one underlying may drop below the 100%-barrier. Hence the right tail of the highly correlated distributions are dominant.

Another side of the correlation is that a higher correlation gives a higher standard deviation. This can be seen both in table 5, table 6 and in figure 39. This can be understood by considering the two examples given above for correlation of 0 and 1. Since a sprinter that only follows one underlying has an equal probability to go up and down, it also has a higher variance than a sprinter that follows the lower out of 4 underlies. The lowest out of 4 uncorrelated underlies are namely much more likely to be lower than the starting value. To go back to figure 39 it is the counterpart to figure 38, simulated with the very same parameters but displaying the standard deviations in end price instead of the price. From the figure, one can clearly see that the standard deviation for the GARCHdt-simulations are always higher. This was observed already in the 1-underlying simulations. The new interesting feature here is the correlation impact on the standard deviation. The most prominent feature is that the correlation shifts the standard deviation to the right at the same time as it lowers it. Another feature to notice is that the standard deviations are decreasing for values of the underlying above 100 for all but the $C = 1$-curve. This was not seen in the corresponding 1-underlying figure, not for the GBM nor for the GARCHdt. Both of these features can be explained by the same methodology as already has been discussed.

The next thing to investigate is how the volatility changes the simulations when the

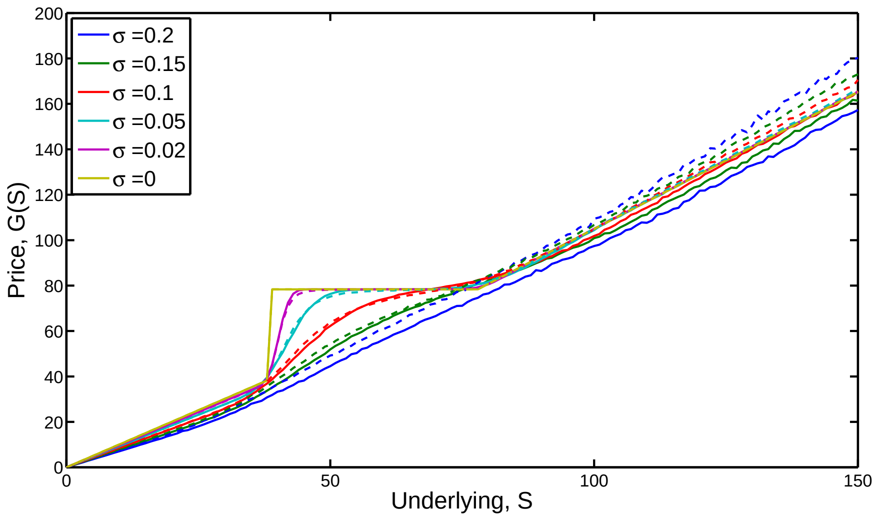

Figure 40: The price curves for a sprinter, modelled both with the GBM (solid lines) and the GARCHdt-model (dashed lines). Each model simulates the price with $N = 10000$ simulations, $C = 0.5$, $r = 0.05$ a year,$\gamma_1 = 50$, $\gamma_2 = 100$, $\beta = 1.2$, $T - t = 5$ years and $dt = 1$ day for six values of σ. For the GARCHdt-simulation α was set to 0.2.

underlies are not 100% correlated. In figure 40 it can be seen that the two models behave quite differently. For high values of S the GARCHdt-model suggests that a higher volatility should have a higher price, while the GBM-model suggest the opposite. This changes somewhere around 75 %, where both models suggest that a higher volatility implies a lower price. This could probably be an affect of θ. For values of S lower than 75 %, the probability that all the underlies will end above the nominal-barrier is rather low. But over for S over 75%, the interest itself almost pushes the underlies above the nominal-barrier. Hence the probability that they all will end above the nominal-barrier increases. From this figure the fair prices used to simulate the histograms in figure 41 are found. All the figures concerning volatilities have the same parameters as the correlation figures had, with the exception that C is 0.5 and that the volatilities are varied.

To analyse the distributions in figure 41, they should be compared to their corresponding one underlying distributions. If done so, one clearly sees that the smaller peak to the right of the nominal peak is much more confined. This is to some extent true for both models, and is once again a consequence of the four underlies to not end at extreme values at the same time. Instead they compensate for each other, which results in a more distinct average. This is seen as a decrease in the standard deviation in table 8 compared to its one underlying counterpart. It is a bit surprising to see that this effect can be observed for the GARCHdt-simulations, but not at all for the GBM-simulations. For the GBM, the standard deviations are almost the same for the four underlies simulations as for the one underlying simulation. One possible explanation could be that the GBM in itself already is so stable around its mean that adding a few more underlies does not change the result very much for the simulations as whole. Another difference, when going from one to several underlies, that affects both models is that there are more underlies that can end below the risk barrier. This is also seen in figure 41 since the left tails are

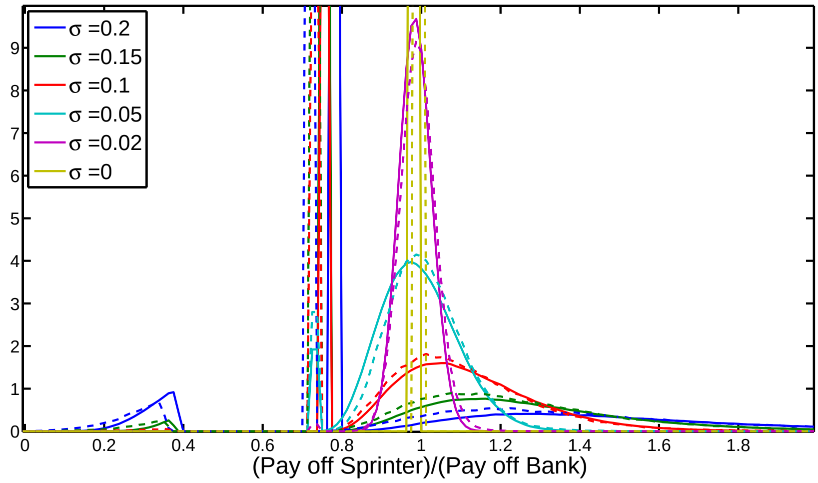

Figure 41: The end price distribution of a sprinter, modelled both with the GBM (solid lines) and the GARCHdt-model (dashed lines). Each model simulated the price with $N = 100000$ simulations, $C = 0.5$, $r = 0.05$ a year,$\gamma_1 = 50$, $\gamma_2 = 100$, $\beta = 1.2$, $T - t = 5$ years and $dt = 1$ day for six values of σ. For the GARCHdt-simulation α was set to 0.2.

Volatility	**Sprinter Price**	**STD Pay off**	**Pay off > 1 (%)**	**Max/Min Pay off**
0.2	97.9	0.46	31	6.0/0.09
0.15	100.8	0.32	39	3.9/0.15
0.1	101.7	0.23	48	2.6/0.30
0.05	104.9	0.11	47	1.6/0.74
0.02	104.9	0.04	45	1.2/0.82
0	104.9	0	na	1/1

Table 7: Parameters for the six GBM-simulations displayed in figure 41. The left most column displays the correlation the simulation was done with. The other columns show how much the sprinter was worth at $T - t = 5$ years, the standard deviation of the pay off, what percentage of the investments that gave back more money than a bank deposition and the last column shows the maximum and minimum pay off in ratios of the bank pay off.

much fatter than in their one underlying distributions. These two effects, the high pay off being more confined and the low pay off being increased in number, does without any doubt have a negative impact on the price. In the tables one also can see that the number of sprinters that gave a pay back ration higher than one, actually increased for the GARCHdt-simulations and decreased for the GBM-simulations. This can to some part be due to the lower prices of the products, which would explain the better pay off for the GARCHdt-simulations, but it can not be the whole truth since the GBM-simulations do not show the same tendencies. Perhaps the long right tails of the GARCHdt distributions balance the increased left tails in a better way than the GBM distributions does. This as an effect of the GBM-simulations fast decaying return distribution.

Before simulating a real sprinter, the normalized standard deviations in the end prices as a function of the underlying will be discussed. This is shown in figure 42. In this

Volatility	Sprinter Price	STD Pay off	Pay off > 1 (%)	Max/Min Pay off
0.2	107.6	0.52	37	15.4/0.01
0.15	105.4	0.37	43	10.4/0.02
0.1	104.9	0.24	50	8.1/0.05
0.05	104.9	0.12	52	2.7/0.11
0.02	104.9	0.05	51	2.1/0.17
0	104.9	0	na	1/1

Table 8: Parameters for the six GARCHdt-simulations displayed in figure 41. The left most column displays the correlation the simulation was done with. The other columns show how much the sprinter was worth at $T - t = 5$ years, the standard deviation of the pay off, what percentage of the investments that gave back more money than a bank deposition and the last column shows the maximum and minimum pay off in ratios of the bank pay off.

figure one can see that the peaks now are more or less aligned around $S = 40$, instead of being shifted as they were in the one underlying simulation. This alignment could be due to the fact that the four underlies make the simulations more stable. In this case the stabilization means that it is more probable that one out of four underlies will stay between the risk barrier and the nominal barrier, than that one underlying out of one will. The same arguments can be used for values of S below 40, where at least one of the underlies is very probable to end below the risk barrier. As a consequence of this the standard deviation for S lower than 40 is rather stable and much lower than in the one underlying simulation. Just like in figure 39 the standard deviation decreases at high S-values.

5.7 Simulating a real sprinter

In this section a real sprinter sold by *SIP Nordic Fondkommission AB*[3] will be simulated. The sprinter in question is *Sprinter Asien 3* and follows four Asian indices during 5 years with maturity the 3:rd of may 2016. The indices are Hang Seng Index (China), MSCI Taiwan Index (Taiwan), Kospi 200 Index (Korea) and MSCI Singapore Cash Index (Singapore). For this sprinter the parameters are $\gamma_1 = 50$, $\gamma_2 = 100$ and $\beta = 2.8$. Their correlations and volatilities must however be found from historical data, preferably data from the past five years since it makes sense to use these data to find the parameters. The correlation can easily be found by computing the correlation matrix presented in section 3.5 as equation 15. If the indices are ordered in the same way they were presented above (with the columns from the left to the right being Hang Seng, MSCI Taiwan, Kospi and MSCI Singapore) the correlation matrix becomes

$$\begin{pmatrix} 1.00 & 0.84 & 0.66 & 0.83 \\ 0.84 & 1.00 & 0.50 & 0.95 \\ 0.66 & 0.50 & 1.00 & 0.37 \\ 0.83 & 0.95 & 0.37 & 1.00 \end{pmatrix}.$$

[3]SIP Nordic Fondkommission AB is a Swedish investment bank which is specialised in structured products. More information can be found on their home page [9].

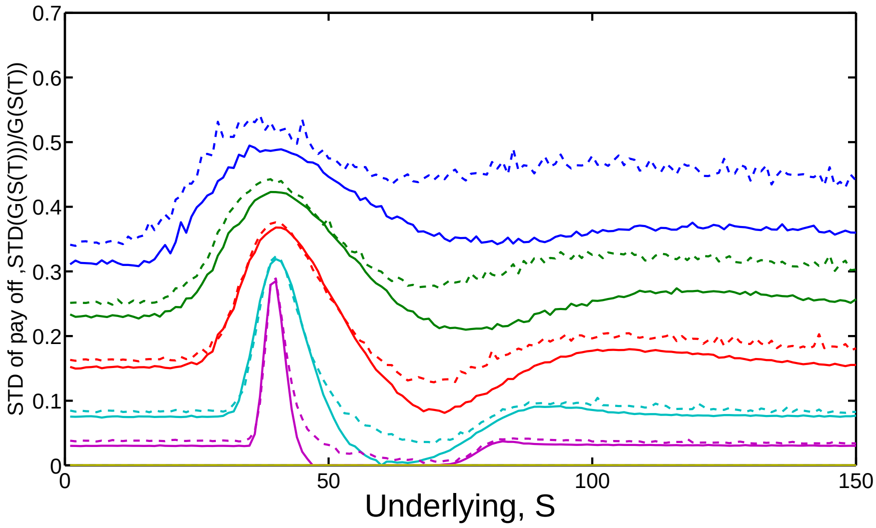

Figure 42: The standard deviation in the price for a sprinter, modelled both with the GBM (solid lines) and the GARCHdt-model (dashed lines). Each model simulates the price with $N = 10000$, $C = 0.5$, $r = 0.05$ a year, $\gamma_1 = 50$, $\gamma_2 = 100$, $\beta = 1.2$, $T - t = 5$ years and $dt = 1$ day for six values of σ. For the GARCHdt-simulation α was set to 0.2. The blue curve represents $\sigma = 0.2$, the green $\sigma = 0.15$, the red $\sigma = 0.1$, the cyan $\sigma = 0.05$, the magenta $\sigma = 0.02$ and the yellow $\sigma = 0$.

From this it can be seen that Hang Seng, MSCI Taiwan and MSCI Singapore follow the same trends to a very large extent. Kopsi 200 on the other hand is much lower correlated with the other indices, but does still follow the general trends of the others rather than being anti-correlated. Since the general correlation matrix has n^2 elements, where n is the number of indices, and there only are n (the $C_i' s$) parameters to alter, this is a bit tricky. A brute force approach would be to go through all the possible configurations C could take. But since the $C_i' s$ are continuous between 0 and 1, one would have to discretize them into very small segments preferably as small as 0.01 to even get fairly close to cover the entire C-space. Even with a step size of 0.01 it would take over 10^{12} simulations to go through the C-space and make a few thousand simulations for each configuration. Combined with the number of steps for each simulation, this would take far to much time. A faster way to find a good configuration would be by using *simulated annealing* [10]. This is a metropolis algorithm which updates the configuration in a stochastic way. If the update is better than the previous configuration it will be kept. If it on the other hand is worse it might be kept anyway, this to avoid being stuck in a local minimum. This acceptance rate, at which worse configurations are accepted, decreases as time goes and at the end only better configurations can be accepted. This means that the algorithm freezes at a good configuration and gives this as the output at the end. Each configuration was tested 1 000 times before it was evaluated. After such an algorithm the C was found to be best described by $C = [0.84\ 0.97\ 0.50\ 0.87]$. This is not necessary the best configuration for C, but it should be one of the best. To make sure that all simulations that will be used have the right correlation, one can add a requirement that the deviation from the "real" correlation matrix can not be greater than an upper limit, say Δ. This deviation can be found by taking the absolute value of the difference between the real

Index	Hist. Volatility (%/day)	Hist. Volatility (%/year)
Hang Seng	2.04	32.38
MSCI Taiwan	2.23	35.32
Kospi 200	3.40	54.05
MSCI Singapore	2.20	35.00

Table 9: The historical volatilities of the indecis used in Asien Sprinter 3. The data used was daily during five years and was downloaded from Yahoo!Finance [5].

correlation matrix and the simulated, and then summing all the elements of the resulting matrix. If the deviation is higher than Δ it will be rejected, otherwise kept. The lower Δ, the better the agreement between the simulations and the desired result. With the C from the metropolis algorithm the rejection rate should be as low as possible. For those simulations that simulate real products, the upper limit of the deviation will be set to 1. Since only the non-diagonal elements can differ from the real matrix elements (the diagonal is by definition 1 for all matrices) this means that the deviation can be spread over 12 elements. Hence the average error for each elements should be about 8%. As previously said a lower Δ would accept a lower average error, but at a high time cost. Already at $\Delta = 1$ the simulations take many times longer time to simulate than the same simulation without any Δ. Since the time taken is not linear in Δ , it would take much longer than twice the time to go from $\Delta = 1$ to $\Delta = 0.5$.

To calculate the volatilities one can use the formula for the *historical volatility*. This is the same thing as the standard deviation of the logarithmic return. For the four indices the historical volatilities can be seen in table 9, calculated from the five years daily data. Since the data is daily the calculated historical volatility is per day.

Apart from the general sprinters simulated in the previous section, Sprinter Asien 3 has a little twist to the calculation of θ. Instead of using only the end values of the indices at the maturity day to calculate θ, the monthly mean of θ during the last year is used. According to SIP Nordic Fondkommission AB this is to stabilize the product during the last year. To get the mean, one must calculate the first θ thirteen months before the product expires. After this θ is calculated every month until the product expires, which would give 13 values of θ. The θ used to calculate the pay off is then the mean of all the monthly θ's.

The last parameter that is needed to simulate the product is the starting indices. To do this as easily as possible a new product will be simulated. This means that all indices start at 100%. With $N = 100000$, $C = [0.84\ 0.97\ 0.48\ 0.96]$, $r = 0.05$ a year, $\gamma_1 = 50$, $\gamma_2 = 100$, $\beta = 2.8$, $T - t = 5$, $dt = 1$ day, $\sigma = [25.44\ 28.24\ 39.57\ 27.59]\%$ a year, $\alpha = 0.2$ and the special twist applied to θ, the product was simulated both with the GBM and the GARCHdt-model. The pay off distributions are plotted in figure 43. The mean values, standard deviation, maximum and minimum pay off together with a percent that describes the fraction of products that gained more money than a bank deposition are shown in table 10.

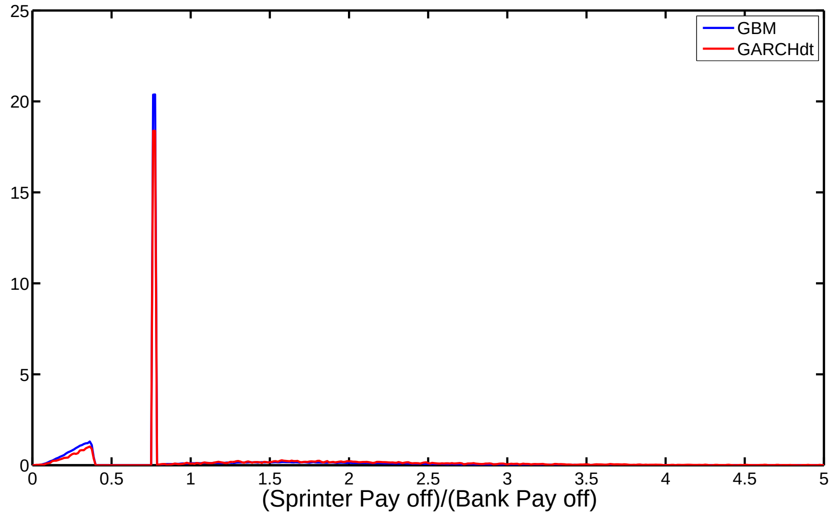

Figure 43: The end price histograms for Sprinter Asien 3. The parameters were $N = 100000$, $C = [0.84\ 0.97\ 0.48\ 0.96]$, $r = 0.05$ a year, $\gamma_1 = 50$, $\gamma_2 = 100$, $\beta = 2.8$, $T - t = 5$, $dt = 1$ day, $\sigma = [25.44\ 28.24\ 39.57\ 27.59]$ % a year, $\alpha = 0.2$ and the special twist applied to θ.

Model	**Mean Pay Off**	**STD Pay off**	**Pay off > 1 (%)**	**Max/Min**
GBM	0.98	0.76	25.4	12.01/0.02
GARCHdt	1.24	1.04	37.9	17.36/0.02

Table 10: A few parameters for the two end price histograms in figure 43. The column second to the left displays the mean pay off, the middle column the standard deviation of the pay off, second from the right what percent of the investments that gave back more money than a bank deposition and the right most column shows the maximum and minimum pay off in ratios of the bank pay off.

Underlying	Hist. Volatility (%/day)	Hist. Volatility (%/year)	C_i
Ford	2.45	38.89	0.82
NASDAQ	1.40	22.22	0.83
S&P 500	0.97	15.40	1.00
NYSE	0.97	15.40	0.89

Table 11: Four underlies with their historical volatilities and correlations. The data used was monthly from 1976 to 2012 and was downloaded from Yahoo!Finance [5].

From the figure and the table one can see that the GARCHdt model has a more positive view of the market than the GBM model. Not only is the right tail of the GBM distribution larger, but the nominal peak is higher too, while the GARCHdt distribution has a longer and fatter right tail. It is also seen that the GBM simulation only gave a higher pay off than the bank in around 1 of 4 times, while the GARCHdt gave a higher pay off in more than 1 out of 3 times. A very interesting thing to notice is the mean pay off of the GBM which is very close to 1. If the mean pay off is 1, the price of the product is the fair price, or in other words, the risk to loose money is in balance to the chance to gain money. This might be a complete coincidence, but could also be so because the investment bank made similar GBM estimations themselves. According to the GARCHdt simulations on the other hand the product should cost a lot more, about 25% more. This underestimation is not solely a consequence of the high pay off ratios, but it has without any doubt a great impact. This impact should perhaps not be as important as it is, because if the calculations are run backwards one finds that the highest ratio of 17.36 would suggest that all the four indices averaged over a rise of 500% during the last year. This is a very unlikely scenario, which perhaps should not be allowed in any model. But as a counter argument to this, the GBM simulation had a highest ratio of 12.01, which implies that the four indices averaged over a rise of 300% during the last year. Even though the two models both are affected by such improbable results, one should keep in mind that the GBM has been used to price products for several decades.

To get a hint of how well the two models recreate the market, it would be good to have historical data to compare with. The indices used in Sprinter Asien 3 is unfortunately not suitable for this since no data older than 10 years could be found. Instead four other underlies with over 35 year old data will be used. That means of course that all the parameters once again must be extracted to fit the new conditions. The four underlies are presented in table 11 together with their parameters.

In total these data were used to produce 380 sprinters, all shifted by one month in starting date. These are shown as the red line in figure 44. The other two lines are GBM (yellow) and the GARCHdt (blue) simulations. These were produced with exactly the same parameters as those used to produce figure 43, with the exception of C and σ, which were re-calculated to fit the new data. Even with this long set of real data, the 380 sprinters following them only give a crude estimation of how the distribution should look like. One can however see that the nominal peak lies somewhere between the GBM and the GARCHdt simulation. One can also see that the GBM hugely over estimates the risk to end below the risk barrier. In the same region the GARCHdt under-estimates the

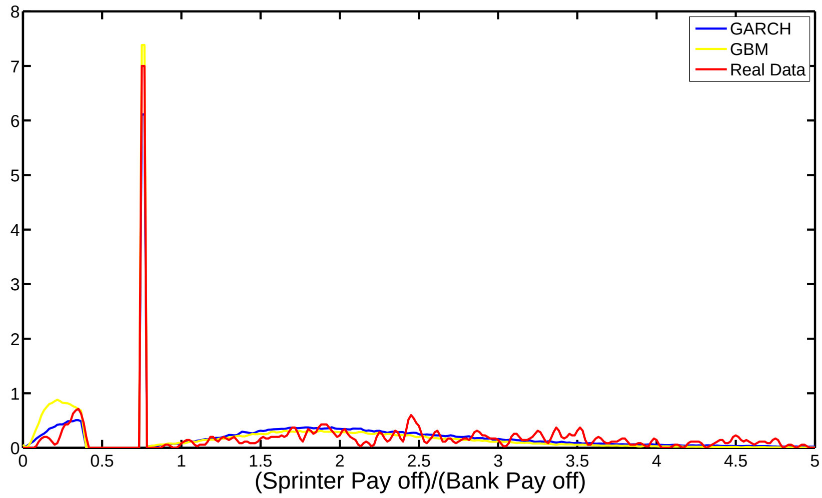

Figure 44: The pay off distribution of a sprinter, simulated both with the GBM-model and the GARCHdt-model togheter with the pay off for the product having real data as underlies. The pay off is shown in ratios of the bank pay off.

Model	**Mean Pay Off**	**STD Pay off**	**Pay off > 1 (%)**	**Max/Min**
GBM	1.40	1.07	51.19	21.41/0.02
GARCHdt	1.92	1.63	67.16	55.11/0.01
Real	2.22	1.51	66.04	6.84/0.11

Table 12: A few parameters for the three end price histograms in figure 44. The column second to the left displays the mean pay off, the middle column the standard deviation of the pay off, second from the right what percent of the investments that gave back more money than a bank deposition and the right most column shows the maximum and minimum pay off in ratios of the bank pay off.

risk to end just below the barrier, but over-estimates the risk to end further below the barrier. On the other side of the nominal peak the two simulations agree better. The distribution of the GARCHdt model is a little higher than the GBM distribution, which agrees better with the real data at high ratios but worse at lower. In general both simulations shows similarities with the real data, but both with obvious errors in certain areas.

A more statistical way to see how well the models reproduce the data is by comparing their parameters in table 12. The GARCHdt model recreates the mean and the standard deviation of the pay off well, and is also almost spot on for the pay off >1. It does however fail enormously in the maximum and minimum pay off. The maximum pay off is a factor 10 to high and the minimum pay off a factor of 10 too small. The GBM model is slightly better here than the GARCHdt, but is still far from the real value. In the other three parameters the GBM model is worse than the GARCHdt, once again hinting that the GARCHdt model might reproduce the real market better.

6 Auto-Call

In recent years the auto-call has become a very popular structured product. The high coupons together with the seemingly low risk barriers make the product seem both profitable and safe, a dream for any investor. For the person more interested in analytical solutions rather than an investment opportunity, it will however be a nightmare. The complex nature of the auto-calls make them incredibly hard to price analytically with common means. Due to this one has to rely on simulations to investigate this kind of structured products. In the first subsection the auto-call is discussed in more details; how does it work?, what are the restrictions? etc.. In the second subsection the pay off is discussed. For the sprinters the maturity date was known from the beginning, which made it easy to see if the out come was profitable or not. For the auto-call with an unknown expire date this is much harder, and hence one should look at it in another way. Since the impact of several underlies, their correlation and volatilities have been emphasized many times in earlier sections, there is no point in doing so here too. Hence the third subsection goes straight forward to simulate a real product.

6.1 Introduction

As said above the auto-calls are very popular nowadays. As a consequence of this the product has developed in many directions, resulting in a wide variety of products, all under the name auto-calls. They all follow one or more underlying. The lowest of these underlies are at several dates read to be compared to different barriers. The pay off is based on how the underlying has developed in relation to these barriers. The barriers are the *auto-call barrier*, the *coupon barrier*, the *risk barrier* and eventually the *knock-out barrier*. How the pay off is calculated can be quite different depending on how the product is composed. Sometimes the coupons barrier is at the same level as the auto-call barrier, resulting in that only one coupon can be paid. Other times the auto-call barrier is above the coupon barrier, so that one coupon is paid every time the underlying is above the latter. How the risk barrier is used can also differ. For some products the risk is only activated at the last check date. This means that the underlying can be below the risk barrier at several check dates, but still has a chance to be above the risk barrier at the last check date. When the underlying is below the risk barrier and expires, the pay off is only the initial investment in proportion to how bad worst underlying developed. If the worst underlying ended at -60%, only 40% of the invested money is paid back. How many check dates there are can also vary. Some products have yearly checks while some have more frequent checks.

In figure 45, the procedure of the pay off for the auto-call is shown. This product has the auto-call barrier at 100%, the coupon barrier at 80% and the risk barrier at 50%. For convenience the product can be said to have an annual check date, although no time is mentioned and there is no real need for any. Anyhow, the product starts at year 1 with all underlies at 100% of their initial values. After one year the first check date has come. If the value of the lowest underlying, X, is above 100 % (of the starting value), the product will gain one coupon and expire, meaning that the invested money will be paid back together with one coupon. If it on the other hand is below 100%, but above

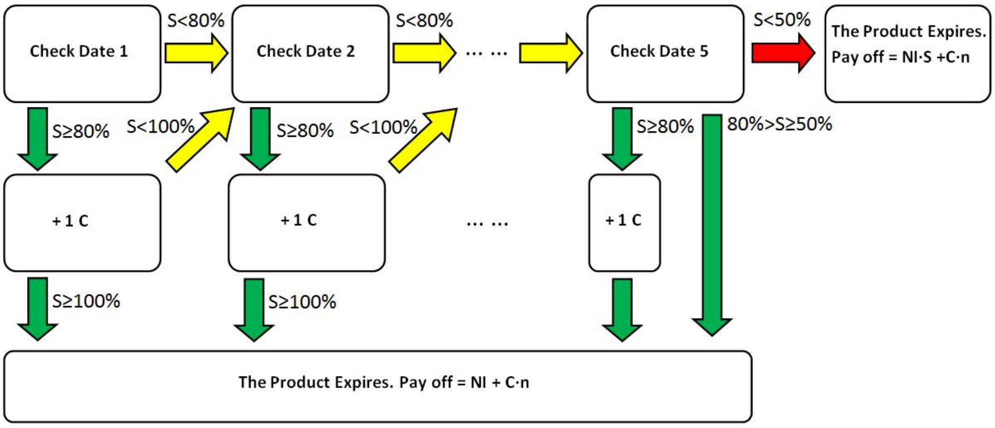

Figure 45: This is a pay off schedule for an auto-call with 5 check-dates. The auto-call barrier is 100%, the coupon barrier 80% and the risk barrier at 50%. The risk is only activated at the last check date. The green arrows indicate positive movements, while the yellow are more neutral. The red arrow indicates when the risk is activated and is the worst scenario possible. X stands for the value of the worst underlying in percent of its own starting value. NI stands for the nominal investment, C for coupon and n for the number of coupons the product has accumulated.

80% it will only gain one coupon but continue to the next check date. The worst case, so far, is that X is lower than 80% and hence only continue without any coupon. This update is repeated until the product expires or until the last check date. In this case the last check date could be after 5 years. If the product lasts to the last check date, the update is performed in another way. If X now is higher than 80% the product gains one coupon and expires. If X is lower than 80% but higher than 50%, it only expires and the invested money together with the earned coupons are paid back. If X on this date is below 50% the risk is activated. Of the money that was invested in the product from the beginning, only X% of the money is paid back together with the earned coupons. This is not necessary an enormous loss, since the coupons can increase the pay off somewhat. It is however the worst possible scenario, where potentially all money can be lost.

6.2 The Pay Off

So far all the products that have been simulated had a specified maturity date. This is not the case for the auto-call, a fact that makes the payoffs a poor way to compare how successful a product is. If e.g. one coupon is paid at the first check date and the product expires the very same date, the pay off is $X_i \cdot (1+\lambda)$, where X_i is the nominal investment and λ is the percentage the coupon gives. The same pay off would be given if the same product failed to generate any coupons until the last check date. These two scenarios give the same amount of money, but they differ in time by four years. To circumvent this problem it is better to compare *yearly returns* rather than payoffs. The yearly return namely averages the change between the initial investment and the pay off over the whole time window. So the first product in the example would have a yearly return of λ since the product gave back $\lambda \cdot X_i$ more money than was invested. The second product on the other hand only had a yearly return of $(1+\lambda)^{1/4}-1$, since it took four years to get $\lambda \cdot X_i$

more money back. A strict definition of the yearly return can be defined as

$$y = \left(\frac{X_e}{X_i}\right)^{\frac{1}{T}} - 1$$

where X_p is the pay off and T is the number of years it took for the product to expire. This definition does however not take the interest into account. Since the invested money is tied to the product for a time T, it would be reasonable to subtract the interest from y to get the *adjusted yearly return* ζ. In other words this is purely the return from the product and can be compared directly to the yearly interest a bank would give. If ζ is larger than 0 the product gains more money than the bank deposition, less if ζ is smaller than 0 and precisely the same if ζ is equal to 0.

$$\zeta = \left(\frac{X_e}{X_i}\right)^{\frac{1}{T}} - 1 - r$$

To clarify how the adjusted yearly return works, a simple example of an auto-call will be explained. For this auto-call the time to expire is 4 years and the coupon is paid yearly with a value of 10% of the nominal investment. Every time a coupon is paid, the simulation will assume that the coupon money is deposited in a bank to gain interest; this unless the product expires. By doing so the products pay off will be maximized during its life time. The auto-call barrier is at 100%, the coupon barrier at 80% the risk barrier at 50% and the interest rate 5% a year. The product will also be free of fees, i.e. there will not be any additional costs besides those directly invested in the product. If 100 000 simulations with these settings are done and plotted as a histogram, several distinct peaks are seen, see figure 46. It is no coincidence, where and why, these peaks appear, indeed they are a bit like the spectral lines in a spectrum. They all correspond to products with an underlying that expires above the risk barrier. If five products expire at the second check date with coupons paid at both times they all have the same ζ, which then gives a peak at this value. The same is true for all possible configurations of expire dates and coupons, they all have discrete ζ-values. This is because the pay off is connected only to the underlying implicitly. If the underlying ends below the risk barrier, the pay off is directly connected to the underlying. This in turn means that the pay off below the risk barrier is continuous, and therefore no values are preferred over others.

In figure 46 there should be 23 distinct peaks. All of these peaks correspond to products that expired after different times or with different numbers of coupons. In table 13 all the 23 combinations of expire dates and coupons for this specific auto-call are shown. Each of the four check dates have one C and one E each. The C stands for coupon and the E for expire. If there is ✓in the C box this means that the product gained one coupon at this check date, otherwise the C box is filled with -. For the E box there are two other symbols,→ and +. The → means that the product did not expire at this check date but continued and the + means that it did expire. Note that it is impossible for the product to expire at one date without gaining a coupon at the same date. The only exception for this is when the product reaches the final check date. Hence there are 1+2+4+8· 2 ways the product can expire without ending below the risk barrier.

Check 1		Check 2		Check 3		Check 4		
C	**E**	**C**	**E**	**C**	**E**	**C**	**E**	**Adj. Yearly Return** (%)
✓	+							5.00
✓	→	✓	+					4.77
✓	→	✓	→	✓	+			4.56
✓	→	✓	→	✓	→	✓	+	4.37
✓	→	✓	→	✓	→	-	+	2.41
✓	→	✓	→	-	→	✓	+	2.31
✓	→	-	→	✓	→	✓	+	2.20
-	→	✓	→	✓	→	✓	+	2.09
✓	→	-	→	✓	+			1.57
-	→	✓	→	✓	+			1.41
✓	→	✓	→	-	→	-	+	0.23
✓	→	-	→	✓	→	-	+	0.11
✓	→	-	→	-	→	✓	+	0.01
-	→	✓	→	✓	→	-	+	-0.01
-	→	✓	→	-	→	✓	+	-0.11
-	→	✓	+					-0.12
-	→	-	→	✓	→	✓	+	-0.23
-	→	-	→	✓	+			-1.77
✓	→	-	→	-	→	-	+	-2.22
-	→	✓	→	-	→	✓	+	-2.35
-	→	-	→	✓	→	-	+	-2.47
-	→	-	→	-	→	✓	+	-2.59
-	→	-	→	-	→	-	+	-5.00

Table 13: In this table it is seen that ζ has a maximum value of 5% when the product expires at the first check date. After that the products which gained all coupons gathers and after these a wide variation between expire dates and coupons. Worst is the product that never gained any coupons, which end at -5%.

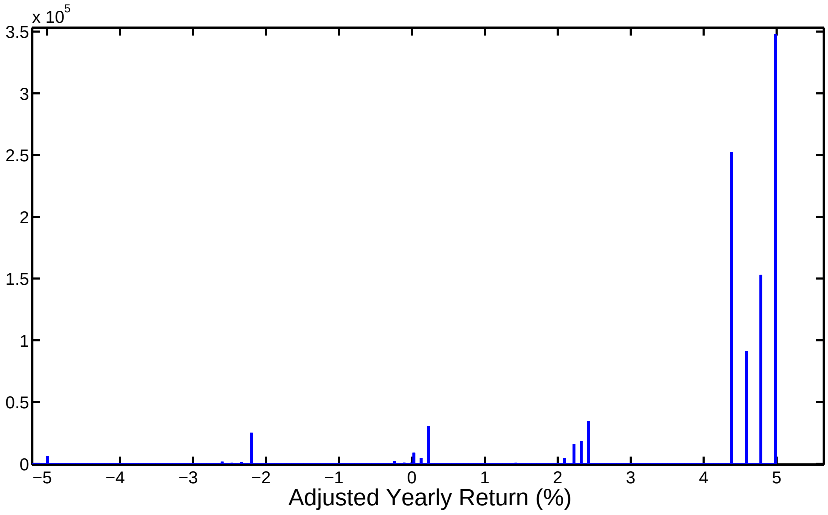

Figure 46: When ζ is plotted in a histogram it is clearly seen that some discrete values are preferred. This is because the pay off for an auto-call only depends implicitly on the underlies, as long as they are above the risk-barrier.

It is of course interesting to use the adjusted yearly return to evaluate the performance of an auto-call, but it is not the only way to do it. Instead of normalizing the pay off with respect to the time, one could say that the pay off must be paid at the last check date, i.e. the investor gets the money on the last check date no matter what. If the product expires before the last check date, they are deposited in a bank to gain interest until the last check date. If done in this way, the money should be maximized during the time to maturity. With this aligned time scale for all the products it is easy to compare them, and one could also normalize them with respect to interest in the same manner that was done for the options and the sprinters.

6.3 Simulating a Real Auto-Call

The real product that will be modelled is a product sold by *Garantum Fondkommission*[4] and is called *GS AC Svenska bolag PlusMinus 913*. This auto-call follows four underlies during a maximum of five years, with two annual check dates. The auto-call barrier is at 100%, the coupon barrier at 80% and the risk barrier at 50%. The risk can only be activated at the last check date and the coupons have a value of 11% of the initial investment. The underlies are four stocks listed on the Swedish *Stockholm Stock Exchange*. They are the clothing company Hennes & Mauritz, the bank Nordea, the telecommunication company Teliasonera and the car/truck manufacturing company Volvo. The historical volatilities of the stocks are shown in table 14, and were calculated from daily data over a period of five years.

[4]Garantum Fondkommission is a corporation specialized in structured products. For more information see their homepage [11].

Stock	Hist. Volatility (%/day)	Hist. Volatility (%/year)
HM	1.86	29.59
Nordea	2.89	45.81
Teliasonera	1.89	30.07
Volvo	2.64	41.97

Table 14: The historical volatilities of the stocks used in GS AC Svenska bolag PlusMinus 913. The data used was daily during five years and was downloaded from Yahoo!Finance [5].

With the same five years data the correlation matrix was calculated to be

$$\begin{pmatrix} 1.00 & 0.80 & 0.86 & 0.73 \\ 0.80 & 1.00 & 0.48 & 0.53 \\ 0.86 & 0.48 & 1.00 & 0.79 \\ 0.73 & 0.53 & 0.79 & 1.00 \end{pmatrix}.$$

By using the simulated annealing algorithm, which was introduced in the previous section, this correlation was found to be best described by $C = [0.93\ 0.34\ 0.80\ 0.96]$.

With all parameters found, the simulations can now be performed. Besides the already mentioned parameters the interest was set to 5% a year, $dt = 1$ day and $N = 100000$ for the GBM and GARCHdt-simulation. For the GARCHdt-simulation the additional parameter α was set to 0.2. In total there are three figures from these simulations. Figure 47 displays the time it took for the products to expire, figure 48 the pay off normalized to the bank pay off and figure 49 a histogram over ζ. The first of these figures shows that the majority of the auto-calls expire at the first or at the last check date. The reason that so many products expire at the first check date is explained by the fact that the stocks have their best possible starting values when the product is new. Then they all start at 100%, which is not possible to have as starting value at any other interval. Hence the starting values are lower for all the intervals after the first one, since at least one stock must be under 100%. After the first check date about 30 to 40 % of the products expires, leaving fewer products to expire at the second check date. And after the second check date, another 15% expires, leaving even fewer products to expire at the third check date. This pattern goes on until the penultimate check date when almost no products expires at all. Left are those products, whose lowest underlying have been below the auto-call barrier at all previous check dates, and they must expire at the last check date. About 30 to 45% of the products expire at this date. Once again one can see that the GARCHdt model is more optimistic than the GBM model. Here it is seen since the GARCHdt simulations expire earlier than the GBM simulations, which implies that those on average should develop more favourable. Also the second figure, figure 48 implies that the GARCHdt has a more optimistic view of the future than the GBM-simulations. This figure shows the pay off from 100 000 simulated auto-calls, normalized to the bank deposition. Hence a ratio of 1 means that the product at the expire date was worth just as much as the money in the bank. In the right and central regions, the two models follow each other rather well, with the exceptions of the peaks, where GARCHdt distributions are distinctly higher. These peaks are combinations of products that expire early with only one or a few coupons. As the products expire later, with more coupons, they become worth more when they

Model	Mean Pay Off	STD Pay off	Pay off > 1 (%)	Max/Min
GBM	0.88	0.37	65.10	1.64/0.01
GARCHdt	0.97	0.30	74.16	1.71/0.01

Table 15: A few parameters for the three end price histograms in figure 44. The column second to the left displays the mean pay off, the middle column the standard deviation of the pay off, second from the right what percent of the investments that gave back more money than a bank deposition and the rightmost column shows the maximum and minimum pay off in ratios of the bank pay off.

expire and ends at a higher ratio. To the left of the distributions, products with long expire time but few coupons ends. Most of the products which activated the risk ended here. A product activating the risk is however not equivalent to a financial failure. It is theoretically possible that a product which activates the risk, already have gained enough coupons to turn the pay off ratio higher than 1. If a product for example gains the first five coupons, and nothing more, that is enough to provide a break-even if the products activates the risk, and 50% of the invested capital is lost. In addition to figure 48, table 15 shows the mean pay off, the standard deviation in the pay off together with the maximum and minimum pay off. Also the percentage of products that gave a higher pay off than the bank is listed in this table. Similar tables were presented for the options and the sprinters. The new phenomenon here is that the auto-call has a pay off that is bounded between 0 and approximately 1.74 (for this auto-call). This has naturally a large impact on the maximum pay off, but the standard deviation and the distribution as whole is affected too. With a smaller range of the pay off the standard deviation becomes very low, but more interestingly, a much higher amount of the products must gain money to make the average close to one. The options and sprinters could in some sense rely on the extremely high payoffs to keep the average pay off close to 1, while the major part of the products give relatively low payoffs. Hence the percentage of options and sprinters paying more than the bank rarely was higher than 40%. For the auto-call it is completely different. This product must be designed so that enough products gains more than the bank, without any extremely high payoffs. Hence the percentage of the products that pay more than the bank is very high, approximately 65% for the GBM and 75% for the GARCHdt. Even though both models have a very high win ratio, the average GBM pay off is lower than expected. If the mean pay off is 1 the price of the product is said to be the fair price, i.e. the price of the product is such that the losses and gains cancel each other. With the somewhat lower mean of 0.90 the GBM-simulations say that the price of the auto-call is too high, or more correctly, the conditions of the products are not good enough. A solution to this problem would be to increase the value of the coupons from 11% to 15%. The GARCHdt-simulations on the other hand is almost spot on when it comes to the fair price. The mean of these simulations was 0.99, which in principle means that the product has been priced by a model similar to the GARCHdt. Even when the agreement is very good, one must keep in mind that it is highly improbable that the product has been priced according to the same procedure.

All ratios above 1 in figure 48 is of course a gain in comparison to the bank deposition, but a higher ratio is not necessarily better than a lower one. For this reason the adjusted

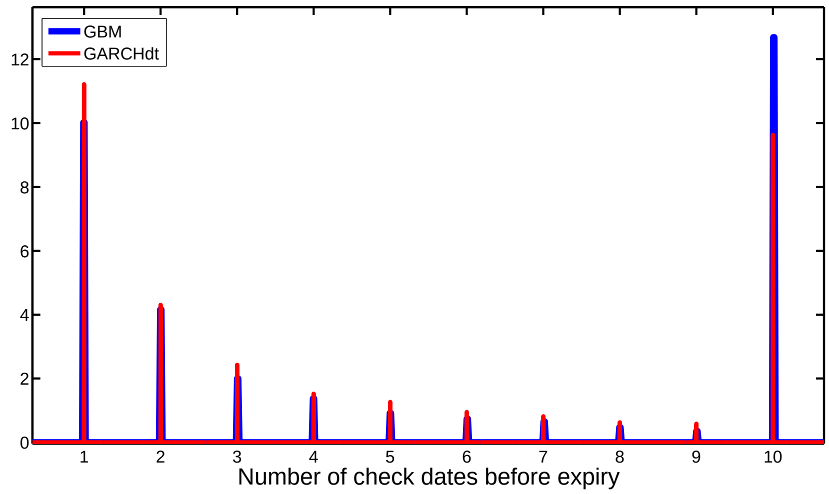

Figure 47: This histogram shows how frequently the products expire at different check dates. Since the total time to maturity is five years, the first check date corresponds to half a year, the second one year, etc.. The simulations were performed with r =5% a year, dt = 1 day and N = 100000, the auto-call barrier at 100%, the coupon barrier at 80%, the risk barrier at 50%, C = [0.93 0.34 0.80 0.96] and the volatility according to table 14. For the GARCHdt-simulation the additional parameter α was set to 0.2.

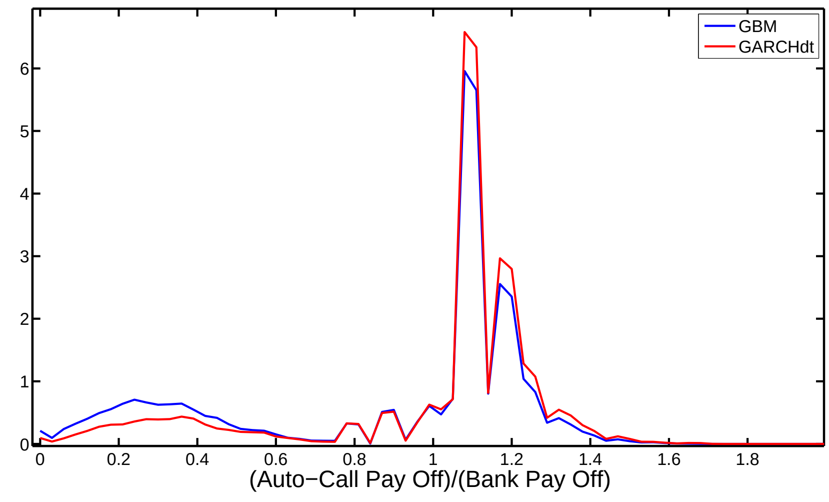

Figure 48: The pay off distribution of an auto-call, simulated both with the GBM-model and the GARCHdt-model. The pay off is shown in ratios of the bank pay off. This figure was produced using the same parameters as for figure 47

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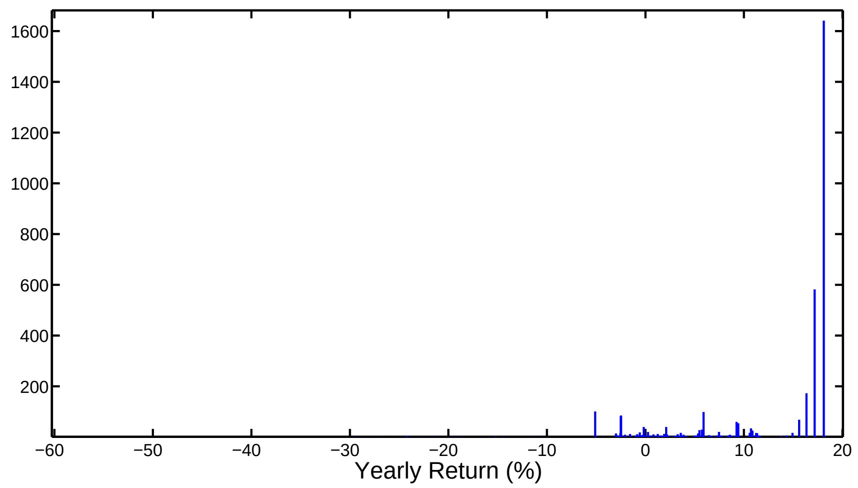

Figure 49: A histogrm over ζ, produced by the GBM-model. The parameters were the same as in figure 47.

yearly return, ζ, was defined. ζ is in some sense the effective interest minus the normal interest, i.e. the annual change in percentage beside the underlying market increase. The strict definition of ζ was defined in the previous section, together with an example where it was applied to a simplified product. Figure 49 shows ζ for GS AC Svenska bolag Plus-Minus 913. Since it is incredibly hard to distinguish the histogram produced with the GBM-model from the GARCHdt-model, only the GBM-models histogram is shown. It is no surprise that there are distinct peaks in these figures. As was discussed earlier, all products ending above the risk barrier have discrete values. If a product expires at the first check date it will give a specified amount of money, without any regard to how the underlies developed. Hence all products expiring at the first check date have the same payoffs, resulting in a peak in the ζ-plot. In the same way, all products not activating the risk, get one of the discrete payoffs, which results in the peaks. Some payoffs are improbable, resulting in low peaks, while others are probable, resulting in high peaks. The highest value of ζ is 14.88 and is obtained when the product expires at the first check date. As earlier was seen it is very probable that the product expires early, hence this peak is the highest of them all. Going from this peak to the left, the peaks become smaller and their coupon gains more complex. One could produce a table for all the possible combinations, as was done in table 13, but with all check dates, this table would have been enormous. In table 16 the mean value and various other parameters for the GBM and GARCHdt-simulations can be seen. A product that is priced correctly, to the fair price, should have the mean value of ζ=0. Then the product on average develops exactly as the bank deposition. The mean value for the GBM-model is close to 0. In fact it is a little higher than 0, which implies that it is priced too low. This goes against the conclusion that was made when analysing the normalized payoffs. Also the GARCHdt-simulations are boosted in the ζ-frame, with a net return almost twice as high as the normal interest. It might seem odd that the same simulations give two contradicting results, but an intellectual experiment can hopefully shine some light on the problem. ζ-wise, the best possible outcome for the product is that it expires as early as possible. Then the invested money increases as much as possible in the shortest time possible. But in terms of the

Model	Mean ζ (%)	STD ζ (%)	Max/Min (%)
GBM	0.71	16.50	14.88/-63.43
GARCHdt	4.64	13.44	14.88/-57.12

Table 16: A few parameters for the three end price histograms in figure 44. The column second to the left displays the mean pay off, the middle column the standard deviation of the pay off, second from the right what percent of the investments that gave back more money than a bank deposition and the rightmost column shows the maximum and minimum pay off in ratios of the bank pay off.

normalized pay off, this means that the product gains only one coupon and wastes the rest of the time. The pay off does certainly gain interest for the whole period of five years, but this increase disappears when it finally becomes normalized to the bank pay off. Hence the first way of presenting the data is normalized to a flexible time, while the second is normalized to a fixed time. Apparently, there is a gap between the two normalizations, but it is hard to tell which is the better one.

The sprinter-section was concluded with a comparison between real data and the models. The same will conclude this section. Just as for the indices involved in the sprinter being simulated there, the stock involved in GS AC Svenska bolag PlusMinus 913 has too short backward data to perform anything. Hence the same underlies that was used then will be used here to, namely Ford, NASDAQ, S&P and NYSE. Their volatilities and correlation can be found in the last section of section 5.7. Since the data for these underlies are 35 years old, and one auto-call lasts for at most five years, it is possible to construct 380 different auto-calls. The expire times of these products can be seen in figure 50 together with the two models. The amount of real products that expires on early dates are striking. Almost 40 % expires at the first check date, while only 24% and 28% expires for the GBM and GARCHdt-model respectively. Otherwise the other expire intervals agree quite well, where the three peaks alter between being the highest and lowest. The exception is of course the last check date, where the real data peak is much lower than the two others. It is most likely those who activated the risk that are left at this check date. At the third check date the real peak is relatively low as well, but this is probably due to the low statistics rather than any interesting feature. The most valuable information gained in this figure is that an auto-call often expires during the first check dates or at the last. The two models recreate this fairly well, with the GARCHdt-model being the slightly better one.

The last figure in this section, figure 51, shows the normalized pay off for the real data together with the GBM and GARCH-products. The GARCHdt-model has a tendency to follow the real data better than the GBM-model in almost every part of this figure. None of the two models manage to follow the two high data peaks between 1 and 1.5, but this is not necessarily a weakness. As discussed before there are not enough real data to produce a perfect picture of the market. If more data were collected, these two peaks would probably be somewhat lower and instead the left tail of the distribution, between 0 and 0.75, would be smoother. Figure 51 has also a corresponding table, table 17, where the usual parameters are shown. Both the mean payoffs and the standard deviations of

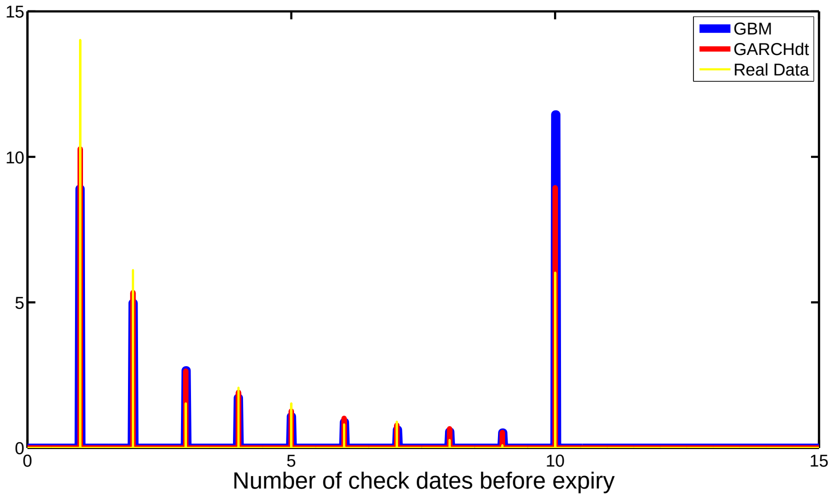

Figure 50: This histogram shows how frequently the products expire at different check dates. Since the total time to maturity is five years, the first check date corresponds to half a year, the second one year, etc.. The figure shows both the expire time for the GBM-model and the GARCHdt-model togheter with the expire time for the product having real data as underlies. This figure was produced using the same parameters as for figure 47, with the exceptions for C and σ, which were changed to fit the data.

Model	**Mean Pay Off**	**STD Pay off**	**Pay off > 1 (%)**	
GBM	1.01	0.30	77.59	1.71/0.03
GARCHdt	1.03	0.27	79.38	1.71/0.04
Real Data	1.05	0.21	85.98	1.44/0.21

Table 17: A few parameters for the three end price histograms in figure 51. The column second to the left displays the mean pay off, the middle column the standard deviation of the pay off, second from the right what percent of the investments that gave back more money than a bank deposition and the rightmost column shows the maximum and minimum pay off in ratios of the bank pay off.

the payoffs seem to be well described by both the GBM and the GARCHdt-simulations. When it comes to the payoffs larger than 1, they are not far behind either, but the higher percentage of the real data can once more be questioned on the same basis as before. In the maximum and minimum columns the same thing that was observed for the sprinter is seen; the models reache values further from the mean than the real data. This time the maximum value is bound by 1.71. Thanks to this, the maximum values of the GBM and GARCHdt-simulations stay relatively close to the real data. In contrast to the sprinter, where the maximum values for the simulations were 3 to 10 times higher than the maximum of the real data, it actually seems very good. Since there are no extremely high outliers, the mean value of the payoffs becomes more relevant. The minimum values however are still wrong, being about a factor 6 too small.

From the real data also a ζ-plot was produced. It is however hard to see anything interesting just by looking at the plot, with all the peaks representing different coupon

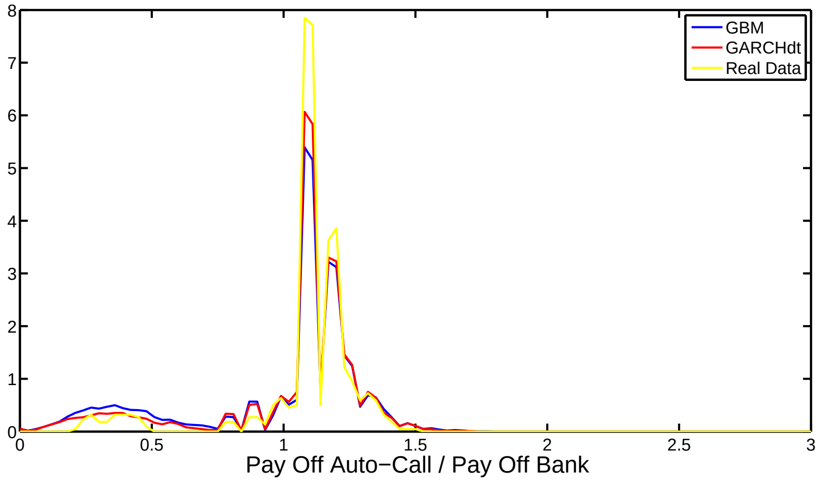

Figure 51: The pay off distribution of an auto-call, simulated both with the GBM-model and the GARCHdt-model togheter with the pay off for the product having real data as underlies. The pay off is shown in ratios of the bank pay off. This figure was produced using the same parameters as for figure 50., with the exceptions for C and σ, which were changed to fit the data.

Model	**Mean ζ (%)**	**STD ζ (%)**	**Max/Min (%)**
GBM	5.92	12.47	14.88/-54.45
GARCHdt	6.71	11.24	14.88/-49.59
Real Data	8.75	9.44	14.88/-27.60

Table 18: A few parameters for the three end price histograms in figure 51. The column second to the left displays the mean pay off, the middle column the standard deviation of the pay off, second from the right what percent of the investments that gave back more money than a bank deposition and the rightmost column shows the maximum and minimum pay off in ratios of the bank pay off.

and expire date combinations. Hence only the table showing the mean, standard deviation, maximum and minimum of ζ is presented, this in table 18. The means of ζ are very high, but one should remember that this in not the real product, but a real product with other underlies. If the product actually had these underlies, the value of the coupons would have been lot lower, or the coupon barrier higher. Nevertheless, the GBM and GARCHdt-simulations give similar results, although they disagree with the real data. The difference is not particularly large, but the real data give a better ζ at a more stable and high rate. In the worst model-simulations, around 50% of the investment was lost a year, while the worst real data scenario only lost about 28% a year.

To conclude the comparison with real data, the GARCHdt-simulations was the best model to reproduce the market. It was not directly seen in the individual simulations, where the GBM gave almost the same result. But in almost all comparisons the GARCHdt-model was a little closer to the real data. Particularly in the expire date figure, figure 50, the GARCHdt-model showed similarities with the real data, that the GBM did not.

7 Summary and Discussion

Options, sprinters and auto-calls have been investigated with two different models, the GBM-model and the GARCHdt-model. For all cases the GARCHdt model priced the products higher than the GBM-model. This is remarkable since the major part of today's products are priced according to the GBM. Which of the two models that sets the correct price is another question, but the GARCHdt-model was supposed to be the better model, providing a better description of e.g. the tails of the return distribution.

The GARCHdt-model was derived from the GARCH(1,1)-model, which was altered to handle different time scales and volatilities. This was done in a successful way, in the sense that the fat tails of the return distribution was conserved. The most outstanding weakness was (ironically) the high frequency of big market changes. This high frequency increased the chance of extreme value, so called outliers, which can alter the results. The main issue is that the underlies are bound from below by 0, but has no upper limit. Hence there is no balance between the low and high outliers. One high outlier can for example with ease make up for several low outliers, creating a skew mean value. This problem was partly solved by introducing a truncation parameter, α, to truncate the return distribution. By doing so, one could choose a maximum value of the return. With the introduction of α, the GARCHdt-simulations became more stable, but how the value of this new parameter should be chosen was not discussed. The value that later was used was more or less chosen arbitrarily by observing data. Perhaps the model would have benefited from a deeper investigation of this parameter.

The option section was more or less intended to be an introductory section for the structured products. As such, it was important to show that the GBM-simulations are equivalent to solving the equation analytically with B&S. This was indeed shown, and it was also shown that the simulations could give more information than B&S. The simulations could for example provide uncertainties in the prices, which the deterministic B&S-solution could not. This was also the first product the GARCHdt-model was applied to. Over all, the price differences between the GBM and the GARCHdt-simulations were not particularly high and neither the standard deviation in the price or the win/loss-ratio did differ much. The big difference was the maximum pay off, where the GARCHdt had a maximum 10 times as high as the GBM. Nonetheless, the GARCHdt seems to value the far out-of-the-money options just like the GBM. The differences in price was only observed for the in-the-money options, where the GARCHdt set a slightly higher price.

The sprinter follows several underlies. This means that new parameters such as the number of underlies and correlation were introduced, which in turn affected the other parameters. To see how an increased number of underlies influenced the price, simulations with the price as a function of the number of underlies and correlation/volatility were performed. Since products following several underlies in general are priced according to the worst underlying, it was rather obvious that an increased number of low-correlated underlies meant a low price. If the correlation was high instead, the underlies behaved more like one underlying and approached that solution. Another somewhat intuitive effect when increasing the number of underlies, was that a higher volatility meant a lower pay off. This should not be a surprise, since the lowest of several underlies always is lower or equal to one of the underlies. As the number of underlies increases, the probability that one underlying will end at a very low value increases. At low volatilities this effect is not very visible, but can be seen for higher volatilities. When pricing the sprinter, one of the most interesting features was that products following several underlies have more stable payoffs. It was clearly seen that a higher correlation gave a higher standard deviation than a lower correlation. This effect was strongest for the GARCHdt-simulation, but could also be seen for the GBM-simulations. The decrease in standard deviation was of course no coincident, but a consequence of a decrease in the maximum pay off. As the correlation was lowered, the maximum pay off was lowered as well. When going from a completely correlated sprinter to a completely uncorrelated, the maximum pay off decreased by 30% for the GBM and by 85% for the GARCHdt. Simultaneously the price of the product decreased and the win/loss-ratio increased. When changing the volatility, with a constant correlation, the two models completely disagreed on how to price the product. The GBM-model said that a high volatility implies a low price, while the GRACHdt-model said the opposite, at least for the in-the-money sprinters. This was without any doubt the major disunity throughout the entire thesis, but can perhaps be understood by the following reasoning. In the one-underlying simulation the GBM-model agreed with the GARCHdt-simulation, they both suggested that a higher volatility should lead to a higher price. The GARCHdt-simulations did however at this stage suggest a much higher price for the products, since the maximum payoffs were much higher. With the introduction of several underlies these maximum payoffs were suppressed, resulting in lower prices. The maximum payoffs for the GRACHdt-simulations were however still high enough to interpret the high volatility as an opportunity. For the GBM-simulation on the other hand the maximum payoffs became so low that a high volatility instead became a risk to fall below the risk barrier. The next application of the models was to simulate a real product, Sprinter Asien 3. To do this as precisely as possible, historical data for the involved underlies were downloaded from Yahoo!Finance. From this data both the volatilities and the correlations were calculated, to later be used in the models. According to the GBM-simulation, this product was priced spot on. The win/loss-ratio was however very bad. The chance to gain more money by buying the product, compared to a bank deposition, was only 25%. The GARCHdt-simulation on the other hand, had a better ratio of 38%, but then it also said that the product was heavily under-priced. The last sprinter application of the models were to simulate old data. Hence old data was downloaded from Yahoo!Finance. This data was then used as underlies for Sprinter Asien 3. In total about 400 sprinters were calculated this way and compared to the models. In several aspects the GARCHdt-model recreated the market better than the GBM-model, but gave far to extreme maximum and minimum values.

The second studied product following several underlies was the auto-call. This type of products can expire at several check dates, with a longest time to maturity of five years. Due to this, one can not just take the mean payoffs to represent the products, the time must be involved in some way too. In order to solve this problem, one new and one semi-new method of comparing payoffs was introduced. In the new method the change in the pay off was normalized to the time to maturity. The result of doing so is an annual effective interest, which was called the adjusted yearly return, denoted by ζ. The adjusted yearly return is defined in such a way that it is additive with the normal interest. Because of this, ζ is zero if a product gains the same amount of money a bank deposition would. The semi-new method is very similar to the methods that was used for the options and the sprinters. Instead of normalizing the pay off to the time to maturity, all products is said to last for five years. If a product expires on a earlier check date, the pay off is put into the bank until the five years has passed. With all payoffs having the same maturity date, it is possible to divide them by the five years bank deposition. In this way the payoffs becomes normalized to the bank pay off. The two ways of analysing the performance of the auto-call, turned out to be quite different. At the same time as one of them said that the product was over-priced, the other could say that the product was under-priced. The problem here is how the time is valued. The ζ-method does in some sense treat the time as a valuable parameter, while the other method treat the time as something that just pass. It is anyhow interesting to analyse the products in both frames, since it is unknown how the corporations sets the price.

When the real product was simulated and analysed in the ζ-frame, the product was correctly priced according to the GBM-model, but under-priced according to the GARCHdt-model. This time, the GBM-model also gave a high probability for the product being a gain compared to the bank deposition. This was of course not a proof, but could be a hint that the corporation valued their product in a similar way. When comparing the models to products that had real data as underlies, the GARCHdt-model once again proved to be the better model. When the models were used to simulate the real product in the other frame, the price was correct according to the GARCHdt-model, but over-priced according to the GBM-model. This contradicted the result the from the ζ-frame, but as said above, the two frames values time differently, leading to such contradictions. When the simulations were compared to the products that had real data as underlies, the GARCHdt-simulations was better in this frame as well. On top of these money-orientated analyses, an additional time analysis was performed as well. This was basically a histogram of the expire dates. The GARCHdt-simulations did on average expire earlier than the GBM simulations. This turned out to be a good feature, as the real data products had a tendency to expire early to. They did however expire early at a much higher rate then the GARCHdt-simulations, but nonetheless did the GARCHdt-model also out-preform the GBM-model in this analysis.

To make a decision whether the GBM or the GARCHdt-model is the better one, there are two questions to focus at. The first is how well the models can describe the real market. This question is more or less already answered several times, where the GARCHdt-model is concluded to be the best. The other question is weather it matters what the individual simulations say. This is a very critical question for the GARCHdt-model, which many times gives completely unreasonable results. That an underlying can rise by several hundred percent in only five years, should perhaps not be as common as it is. The same is true for the high crashing-rate, where an underlying drops to less than 1% of its starting value. With this said, the GARCHdt-model might be a better market model than the GBM-model, but not necessarily a more realistic one.

8 Appendix

8.1 Basic financial terms

Interest A very basic concept encountered in finance is *interest*. This is a fee that is paid by the *borrower* to the *lender*. A common case is when the borrower is a company or a person and the lender is a bank, or the other way around. Consider e.g. the case when a costumer of the bank wants to borrow money for investing in a property. Then the person and the bank have to sign a contract where they specify how the person repays the bank. This payback is usually performed during several months or years, until the debt is paid. But since the borrower also has to pay the bank interest, he will at the end pay the bank more than the money he originally borrowed. If the example is reversed, i.e. if the costumer is depositing money into the bank, he is the one that should gain interest from the bank. This is an important feature that always should be considered when different financial decisions are made.

Bond When e.g. a state or a large company needs money to finance a project or for a development of products, they have the choice to issue bonds. If they do so, they become *issuer* of the bonds. The bonds can then be bought by an investor, which becomes the *holder* of the bonds. The agreement between the issuer and the holder of the bond is that the holder will loan an amount of money to the issuer, which after a specified time will be paid back to the holder. Since the holder would lose money if the same amount of money was paid back (he would have gained money by putting them in a bank and thereby gaining interest) the issuer must also pay the holder some additional money to make the bond attractive. This can be done in a few different ways. One way is for the issuer to pay the holder coupons periodically until the loan is repaid. This will then serve as the interest a bank would have given the holder. A bond which generates coupons for the holder in this way is said to be a *coupon bond*. Another way paying the additional money is by increasing the payback. The holder would then e.g. give the issuer 9 000 SEK, but would get back 10 000 SEK. If bond pays interest like this it is said to be a *zero-coupon bond*. Since the payout for a zero-coupon bond is fixed at the maturity, the holder of the bond can be sure that he will get that fixed amount of money. Hence he can buy a zero-coupon bond for 9 000 SEK and some other financial instrument for 1 000 SEK, and still be sure to get at least 10 000 SEK back independently of how the market behaved. This is an important property of the bonds which gives opportunities to form several kinds of structured products.

Forward Contract For one or another reason it can sometimes be of interest for one part to write a contract with another part, where they agree to buy or sell an asset at a specified future date at a specified price. Such a contract is called a *forward contract*, or a *forward* for short. A forward can be used when a company wants to neutralize the risk they take when dealing with exchange rates. If the company has a contract with a costumer where they will get (or have to pay) 200 000 £, but are uncertain that the exchange rate between SEK and £ will develop beneficially, they may want to hedge the exchange risk. They can do this through selling a forward to buy 200 000 £ to the equivalent amount of SEK of today's exchange rate. Doing so is considered to use the forward in a defensive manner.

Forwards can however also be used in a more offensive manner. Consider e.g. an investor who is certain that the price of a specific asset will raises from this day to a year later. Then the investor is keen to get hold of the asset before the price raise too much. If he however doesn't have enough money to pay for the asset at the moment, he can enter a forward contract to buy the asset (assuming he will have the money on the maturity day). After one year, at the trading day, the investor gets the asset and may have earned or lost money depending on how the price of the asset developed.

From this last example, it is easily understood that the expectations of the underlying asset will influence the price the buyer must pay the seller. If a majority of the market believes that the asset will be worth more in the future than it is today, the price of the forward to buy should be more expensive and vice versa. Another thing to keep in mind when setting a price in the forward contract, is that the seller in some aspect locks up money for keeping the asset during the contract time. If he instead could have sold the asset right away, he could have deposited the money into the bank and earned interest instead.

Option An option works in a similar way to the forward in the aspect that a contract to buy or sell an asset in the future is signed. The person buying an option is called the *holder* of the option, while the seller of the option is called the *writer* of the option. The difference between the option and the forward is that the holder of the option is not forced to buy (or sell) the asset at the expiry date. This is however not true for the writer who must adapt to the holder decision. An option where the holder is given the choice to buy an asset is called a call option, while it is called a put option if the holder is given the choice to sell.

One of the most basic options is the European option. For this option a maturity date, $t = T$, and an exercise price (or strike price), E, is specified. But since the writer has a weaker position than the holder, the holder must also pay the writer a fee, P, for the option. This is a price the holder must pay the writer whether or not the asset is bought at the maturity date. Due to this the value of the asset, S, must be higher than $E + P$, otherwise the holder will lose money. In Figure 52 the payoff, G, (the money the holder gain if he sells the asset right away after he bought it) of a European call option is shown where the strike price is 70 SEK and the fee is 10 SEK.

The S-axis is divided into three regions which represents three distinct cases:

- In the left-most case the price of the asset is below the strike price E. Then there is no reason for the holder to buy the asset since he will lose money doing so. An example of this could be if the value of the asset is 50 SEK at the maturity. Then the holder may buy the asset for 70 SEK and pay the fee for 10 SEK for a total cost of 80 SEK. If the asset then is sold immediately the holder gains 50 SEK. This will result in 30 SEK loss for the holder. If the holder instead does not buy the asset he just pays the fee of 10 SEK and ends up with a loss of 10 SEK.

- In the right-most case the asset is at least 10 SEK higher than the strike price. Here the holder should buy the asset since he can sell it immediately with a profit. Here the asset price could be 150 SEK at maturity. Once again the holder must pay both

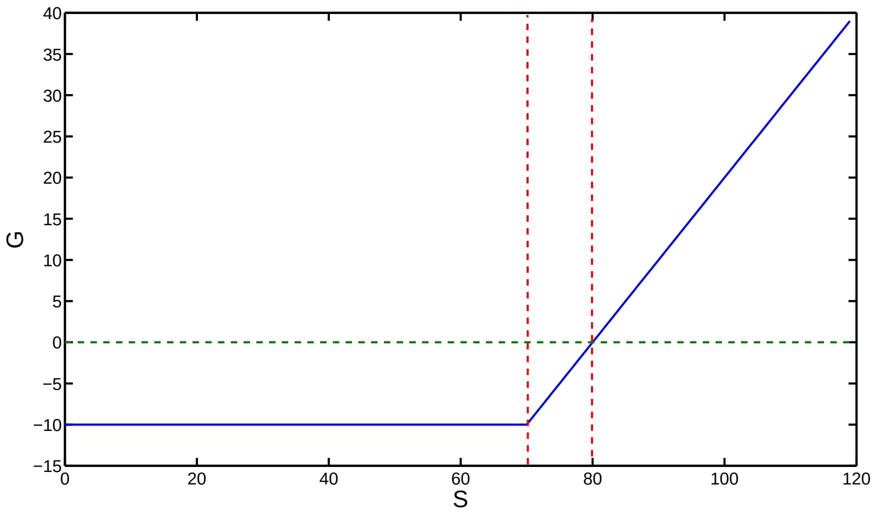

Figure 52: The money the holder of the option gains as a function of the price of the underlying asset.

the strike price and the fee for a total cost of 80 SEK, but this time he can sell the asset immediately for 150 SEK and gain 70 SEK.

- In the middle case the asset price is between the strike price and the strike price plus the fee. Here the holder should buy the asset to minimize the loss. If the asset is worth 75 SEK he can buy it and sell it again with a loss of 5 SEK. Only at the boundary when the asset price is exactly the strike price the loss will be 10 SEK whatever the holder does.

From this example it is obvious that the holder can gain a large amount of money but never lose more than the fee price. Even though the asset itself never can be bought before the maturity date the holder may choose to sell the option. Assume that the price of the underlying asset of a call option is above the strike price, then there should be a greater probability that the option will generate money. The option is said to be *in-the-money*. In this case the holder may want to sell the option rather than wait to buy the asset at maturity. If the price however is the under the strike price the option is said to be *out- of-the-money* and the option is worthless. How the price of the option fluctuates depends on a few different parameters. Some of them are the time left until expire date, the current price of the asset, the interest and the volatility. How the time left before expire date affects the price of the option is shown in Figure 53 .

In the figure the number of days between the day the option was written and expire date was 100 days. As can be seen, an option is worth more the earlier it is bought. This is because the price of the underlying has longer time to raise. To compute these price curves the *Black and Scholes equation* for pricing options was used. This equation has been a powerful tool to price options and is given a more complete review in section 3.3. So far everything has concerned European options, but there are many other types of options that are not so strictly governed by the maturity date and the exercise price. This is a feature that makes them harder to describe mathematically. But the point is

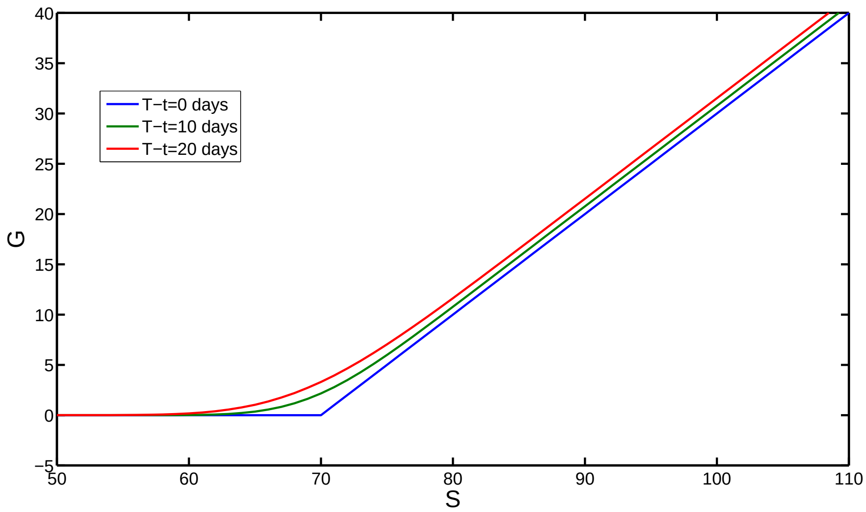

Figure 53: The payoff as a function of the price of the underlying asset and the time left to expire date.

that options can be constructed in such a broad number of ways, that they can be used in almost any portfolio and to be a part of several financial instruments.

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