Branching Avoidance in Kinematic Image Space for Linkage Synthesis

Adam Nilsson, Lund University

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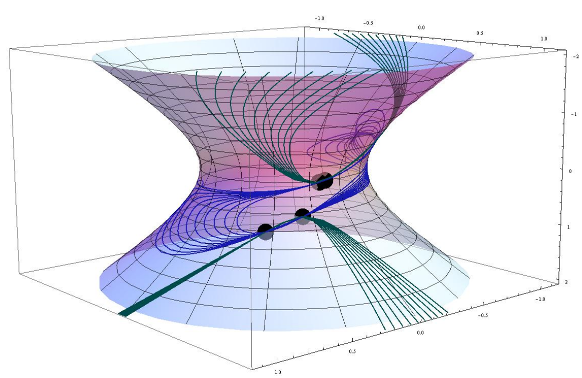


Abstract

We consider a finite position synthesis and branching avoidance of four-bar linkages based on kinematic theory derived from dual quaternions. The branching defect is defined as a situation where all specified task positions are not connected by a continuous motion of the linkage end effector. The workspace of a four-bar linkage is derived to be the intersection of two hyperboloids in the kinematic image space. This is used to develop a fast method to determine branching of a linkage and to find an explicit solution to the end effector trajectory. A new synthesis method is developed where one task position is given a lower priority. Using the new branching analysis method and by synthesizing linkages with various values of one parameter of the linkage, we can determine a range of values for the constrained dimension which will give useful linkages for the four most prioritized task positions. Finally a method to find the closest useful linkage to the last task position is derived.

Sammanfattning

Vi betraktar syntes och grendefekten av fyrlänks-mekanismer utifrån ett begränsat antal task-positioner, baserat på kinematisk teori härledd från duala kvarternioner. Grendefekten är definierad som situationen där alla specificerade task-positionerna inte är sammankopplade av en kontinuerlig rörelse av mekanismens end-effector. Arbetsområdet av en fyrlänks-mekanism är härlett som snittet mellan två hyperboloider i det kinematiska bildrummet. Detta används för att utväckla en snabb metod för att avgöra om grendefekten föreligger för mekanismen, och för att härleda en explicit lösning för mekanismens end effector trajektoria. En ny syntes-metod är utväcklad där en av task-positionerna är given en lägre prioritet. Genom att använda den nya grendefektsanalys-metoden och beräkna länk-mekanismer med varierande värden av mekanismens parametrar, kan vi hitta ett intervall av värden på parametern som ger en användbar mekanism för de första fyra task-positionerna. Slutligen härleder vi en metod för att avgöra vilket värde på parametern som ger den länk-mekanism som når närmast den lägre prioriterade task-positionen.

Contents

Preface

Acknowledgments

The work behind this thesis was conducted at University of California Irvine during the spring 2012. I specially want to thank my supervisor at UC Irvine, Michael McCarthy for giving me the opportunity of working with him, and for our valuable discussions during the work progress. I want to thank my LTH supervisor Stefan Diehl and my lab partners at UC Irvine Mark Plecnik and Kaustubh Sonawale for their support during the process. I also want to thank my father Klas Nilsson and my sister Sofie Nilsson for many valuable comments on the report. Finally I like to thank the international offices at LU and LTH along with the University of California Education Abroad program for giving me the opportunity to come to California and Irvine.

Chapter 1

Introduction

1.1 Linkage Mechanisms

Robots are getting more and more common in industrial environments and are primary used to move objects between a set of positions. The object can be either a tool or the product itself.

To ensure generality in the motion, the arm usually involves several degrees of freedom, so the same robot arm can be used for many different applications. A typical industrial robot has six motors and can move in six degrees of freedom, which is the full flexibility in a 3-dimensional space. A high degree of freedom is good from a general perspective, but might be a disadvantage in other situation.

The automation industry today is generally more interested in robots with flexibility than designing a specific linkage which can only perform the task it was designed for. However when a specific motion is known and no other motion is of interest for the process, it might be much more convenient to constrain the motion by designing a mechanism of less degrees of freedom, such as a mechanical linkage.

Definition 1. *A mechanical linkage is an assembly of bodies connected together to manage forces and movement.*

There are both advantages in terms of performance and cost by using linkage mechanisms. A one degree of freedom linkage would require only one actuator for the same motion as a normal robot would require at least three actuators for. Less actuators gives both lower cost and higher accuracy. Linkages can be found as components in many mechanisms and machines, some examples are the following.

Example, Airplane Flaps Considering the flaps of an airplane wing. Flaps are used to change the aerodynamic property of the wing during start, landing and altitude changes. The optimal positions x, y and θ for the flaps are determined by aerodynamic calculations.

Figure 1.1: The flaps of an airplane changes the aerodynamic property of a wing to allow start, landing and altitude changes. Photograph by Jerome Mervelet [1]

In order to position the flaps at the optimal locations, the flaps may be supported by various types of mechanisms, e.g. based on links or sliding rails. The flaps could be positioned at any positions by robot arm with three degrees of freedom using three actuators. Since each actuator adds mass, cost and unreliability to the system, it might be much more beneficial to design a linkage which together with only one actuator can do the same thing. Two examples of a linkage supporting the flaps are presented in patent [2] which is an example of a multiloop eleven bar linkage and in patent [3] which is an example of a six-bar linkage.

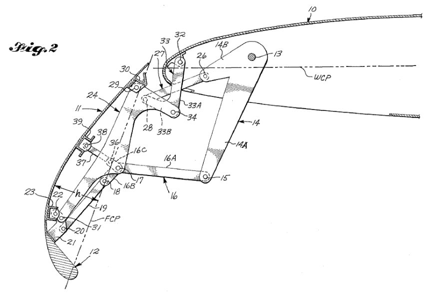


Figure 1.2: This eleven-bar linkage mechanism was patented by Boeing in 1981. It positions the flap at three specified positions [2].

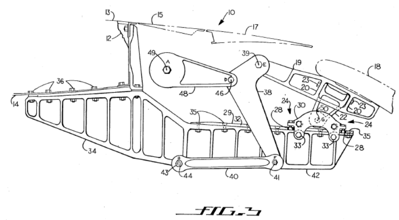


Figure 1.3: A six-bar linkage in a configuration named *Stephenson III* is used to control the motion of the flaps [3].

Example, Race Car Suspension The wheel of a car is required to move while the shocks deflects. Consider a car in straight motion seen from the front. In the view plane, a wheel has three degrees of freedom, x, y and rotation θ.

The wheel requires a supporting mechanism that allows the wheel to move on a one-dimensional trajectory, which is finally controlled by the shocks. If a robot arm with three degrees of freedom was used to support the wheel, it would require at least two actuators together with the shock to constrain the wheel motion. Since there are huge forces involved in a car suspension, the motors would have to produce a lot of force even though the actual work would be minimal. It is much more beneficial to design a support mechanism with only one degree of freedom and let the shock control the final motion.

A double wishbone car suspension, see Figure 1.4, commonly refereed to as double A-arm, is an example of a four-bar linkage when seen in the front plane. The car chassis is considered as one link, the two wishbones or A-arms are one link each and the upright is the fourth link. This mechanism can be found in most competitive race-cars.

This master thesis will primary consider four-bar linkages, which is the most basic linkage mechanism. It is still of interest to study since more complex linkages can be synthesized as a combination of open chains and four-bar linkages [6].

Example, Trunk Closing Mechanism A design example for a linkage is to move an object along a path or through a set of positions. An example of a possible design case is the trunk cover in an automotive.

Figure 1.4: A double wishbone suspension is used in many competitive race-cars such as this Formula Student [4] car LUR5 from LU Racing [5].

Figure 1.5: Chrysler Sebring 2005 has a four-bar linkage to support the trunk cover motion.

Figure 1.5 shows the trunk linkage in a Chrysler Sebring. The design criteria in this example is that the trunk should be positioned in a closed and an opened position. Two intermediate positions could also be specified near the opened and the closed position to obtain a smooth closing motion, the lock mechanism may need to be hooked in from a certain direction while closing.

The alternative to a linkage in this case is a much longer curved bar connected to a hinge inside the trunk, which is the most common design in coupé cars.

1.2 Synthesis of Four-Bar Linkages

A four-bar consists of a ground link, two crank links and a coupler link. In the case where the linkage is driven by one of the cranks, the other crank is called the follower.

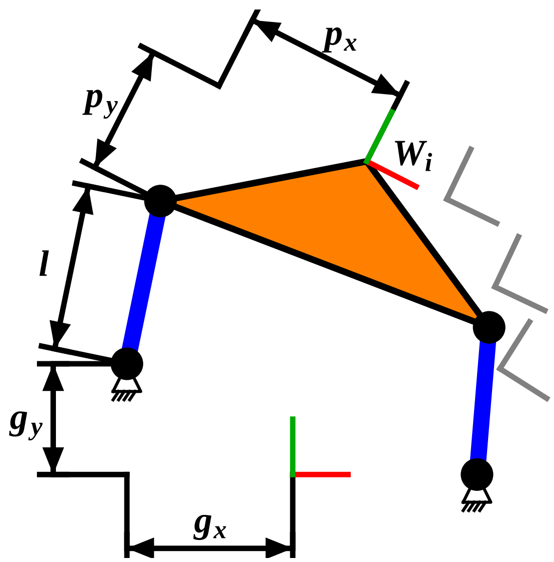


Figure 1.6: Drawing of a four-bar linkage with dimension-annotations for one of the cranks. The two blue lines are called the crank links and the orange triangle is called the coupler link. The fourth link is the ground.

Linkage synthesis is the process of generating a linkage that satisfies some design specifications. In this thesis we will consider synthesis where the dimensions of the linkage is calculated so the end effector can be positioned at a set of task positions.

Definition 2. *A* ***position*** *consist of a translation and a corresponding rotation. A* ***task position*** *is a position used to define the task for a mechanism.*

A crank of a four-bar linkage is defined by its ground pivot, link length and position of the moving pivot relative to the end effector frame. Using homogeneous transformation matrices, the vector along a crank link can be written as:

$$\overrightarrow{l} = W_i \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} - \begin{pmatrix} g_x \\ g_y \\ 1 \end{pmatrix} \tag{1.1}$$

where W_i is the matrix describing task position number i. The length of the link is then given by:

$$l_i^2 = \left[W_i \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} - \begin{pmatrix} g_x \\ g_y \\ 1 \end{pmatrix} \right]^T \left[W_i \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} - \begin{pmatrix} g_x \\ g_y \\ 1 \end{pmatrix} \right] \tag{1.2}$$

The equation has either zero, two or four real solutions $(p_x, p_y, g_x, g_y, l_i)$ [7]. At least two solutions are necessary since we need two different cranks.

1.2.1 Homogeneous Transformation Matrices

For the unoriented reader a quick review of homogeneous transformation matrices is given. If coordinate system W is located with origin in the point $\vec{p}$ and the coordinate axes are given by the vectors $\vec{e_x}$ and $\vec{e_y}$, then the homogeneous transformation matrix is given by:

$$W = \begin{pmatrix} \vec{e}_x & \vec{e}_y & \vec{p} \\ 0 & 0 & 1 \end{pmatrix} \tag{1.3}$$

A vector $\vec{v} = (x, y, 1)^T$ in W or a vector $\vec{v} = (x, y, 0)^T$ in a coordinate system with the same rotation as W and origin at the parent's origin is given in the parent coordinate system by:

$$\vec{v_0} = W\vec{v} \tag{1.4}$$

In other terms, the zero as the last element means that the vector should be considered from an origin while the number one means that the vector should be translated to an origin of an other coordinate system. Several frames can be added in a chain by matrix multiplication. The last element in the vector is useful since the coordinate axes will always start in the origo of the last coordinate system but will be rotated by all parent coordinate systems.

1.3 Multiloop Linkages

The four-bar linkage consists of only one kinematic loop. More complex linkages can be constructed using more kinematic loops. A six-bar linkage consists of six links connected by seven joints and can be arranged in several different ways. McCharty presents a method for synthesizing six-bar linkages in [6], see Figure 1.8. The method is to first specify a 3R chain, which has three degrees of freedom and can reach any position within its work space. The 3R chain is constrained by adding more links. Those extra links can by synthesized with the four-bar synthesis methods. Patent [3] is an example of an application of where six-bar linkage is composed from four-bar linkages. The inventor write explicitly in the abstract that it is a six bar linkage built with a four bar linkage as a component.

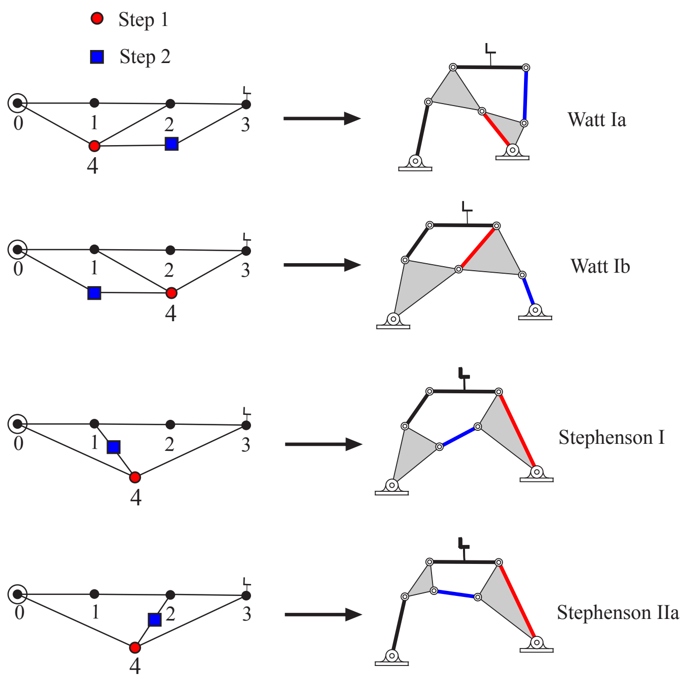


Figure 1.7: The different arrangement of links for a six-bar linkages. From a 3R chain, the red link is synthesized first using the same method as for four-bars. [6]

1.3.1 Synthesis of a Watt Ia six-bar linkage

To synthesis a six-bar Watt Ia linkage, one start with a 3R chain reaching all task positions. Since a 3R chain has 3 degrees of freedom it can reach any positions as long as they are within a certain distance. E.g. if all the links have the length $l = 1$, a position that are further away than three from the base is of course not possible to reach. A similar case occurs for example if the first link has length $l_1 = 10$ and the second and third has the lengths $l_2 = l_3 = 1$, then a position closer than eight from the base is of course never reachable.

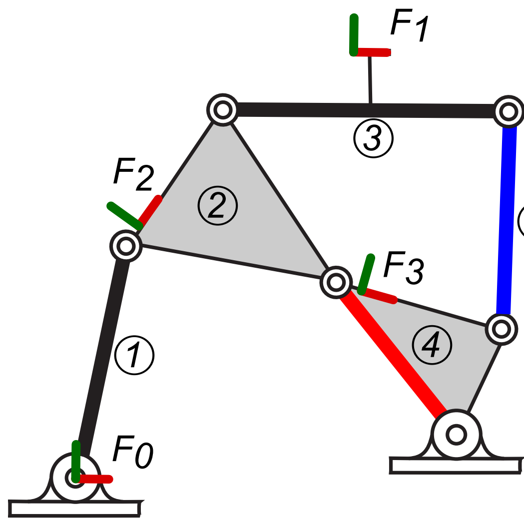


Figure 1.8: A Watt Ia six-bar linkage drawing from [6] with notations added on.

In order to constrain the 3R chain, first the position of frame F_2 in frame F_0 is calculated at all task positions. Then a four-bar can be synthesized with link 1 and link 4 as the cranks and link 2 as the coupler.

A frame F_3 is fixed to link 4 and the position of frame F_1 given in frame F_3 is calculated at all task positions. A four-bar with link 4 as the ground can then be synthesized. Link 2 will be one of the cranks which is already defined at this time. Link 5 will be the second crank in this four-bar.

1.4 Mechanism Branches

Some linkages can be assembled in different configurations, and will depending on the configuration follow different trajectories. Different terminology may exist in different literature. Depending of design requirements the usefulness of a linkage may be different. To avoid confusions or misunderstandings, some definitions are introduced.

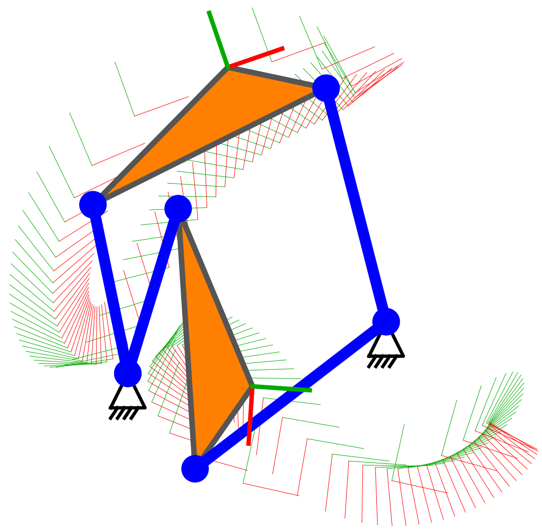


Figure 1.9: Drawing of a branching four-bar linkage in two different configurations. The reachable trajectory is different for the two configurations.

Definition 3. *A configuration of a linkage and its continuous motion is called a* ***branch of the linkage****. When the task positions lie in different branches,* ***the linkage is branching*** *and not useful.*

Definition 4. *A linkage with all task positions in the same branch is called a* ***useful linkage****.*

The existence of a solution to (1.2) only tells that the linkage can be assembled at each of the task position in some configurations, but there does not have to be a continuous motion between the task positions. I.e. the task positions may belong to different branches of the mechanism. It does not have to be an issue when the linkage has two branches as long as all task positions lie in the same branch.

1.5 Problem Statement

It is not yet well known what causes the mechanism to branch. The current approach to find non-branching linkages is to define a tolerance zone for the task positions and randomly choose task positions in the zone until a non branching linkage is found. This process is much time consuming and will give different results each time. This thesis will investigate the conditions for what causes the mechanism to branch and to find a strategy of modifying task positions to generate the best possible non-branching linkage.

1.6 Outline and Approach

- First some kinematic theory that is necessary for the later work will be introduced, and some equations describing the kinematics of 2R chains and four-bar linkages are derived. The equations are referred to as the constraint manifold equations.
- The conditions of when a mechanism branches are analyzed in detail in order to find a fast method to analyze the branching structure of a four-bar linkage.
- The shape of the constraint curve (which is a curve describing the motion of a mechanism) is analyzed and a first approach to find useful linkages will be to project the task positions onto curves with shapes similar to shapes that a single branch constraint curve may take. The first approach did not increase the number of useful linkages, so a second approach was considered.
- The second approach to find useful linkages will be to only consider four task positions at a beginning. Then all possible linkages to those four task positions will be calculated. Using the branching analysis method, all useful linkages to the four task positions will be calculated and a fifth task position will be projected on to the motion of the closest matching linkage.

1.7 Related Work

1.7.1 Kinematic Theory

Linkage synthesis is not a prioritized field and the research is progressing rather slowly. The approach in this thesis is based on the theory of quaternions explored in 1844 by Hamilton [8] and geometry in the kinematic image space presented in 1911 by Grünwald [9]. Not much has been done based on this theory. Bottema and Roth have a rigorous introduction to the theoretical kinematics [10] including dual quaternions and constraint manifold. McCarthy [7] gives an easier introduction to the theoretical kinematics.

1.7.2 Branching

Branching and usefulness of linkages have been approached in different ways. I have defined a linkage as useful if there is a continuous motion between the task positions, and it is assumed that the mechanism can be actuated along the desired motion. Chase [11] and Parrish [12] considered linkages that are driven by rotation of one crank, which introduces singularity configurations where the driving link can not control if the linkage will move in one or another direction. Chase defines a branch as the range of motion which can be uniquely controlled by a rotating input link. What is defined as a branch in this thesis is called a

Circuit by Chase. By writing the kinematic loops for the linkage and analyze the Jacobian of the loop equations at each task position, Parrish [12] can disqualify some linkages immediately, but to classify them as useful he needs to do a closer numeric analysis of the motion.

1.7.3 Branching Analysis in the Kinematic Image Space

The closest related work to this thesis is presented by Schröcker in 2005[13] and 2007[14]. Schröcker uses constraint manifold equations of 2R chains. The branching conditions are determined by the intersection between two constraint manifolds. Schröcker derives the same property of the constraint manifold shapes as is done in this thesis. However Schröcker's derivations are different. The results in [13] and [14] appear to be the same as in Chapter 3.

Chapter 2

Kinematics Theory

The kinematics of an arbitrary linkage is in the general case non-linear. This causes problems for design, analysis and control of such mechanism. An arbitrary non linear algebraic equation has usually no exact explicit solution. However in some special cases when a variable transformation exist, it might be possible to obtain an exact solution.

2.1 Quaternions and Dual Quaternions

In 1844, William Rowan Hamilton[8] explored a mathematical object which he named quaternions. The quaternions was an extension of the complex numbers from one imaginary dimension $\mathbf{i}$ into three imaginary dimensions $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$. They have shown to be very useful to represent rotations and are used in many robotics and flight control systems today.

Theory of dual numbers where later on used to extend the quaternions to dual quaternions, which can be used to represent a displacement with both rotation and translation. A dual quaternion consist of two quaternions, one real and one dual part, which gives a total of eight elements. The real part is only related to the rotation while the dual part is related to a combination of the rotation and translation.

A more common way of representing translations and rotations today is by using vectors and rotation matrices. The quaternions where Vectors where the inspiration for J.W. Gibbs when invented the vectors in 1884 [15].

The reader is referred to Appendix A for an introduction to dual quaternions. In the case of planar kinematics, only four non-zero elements remain of the dual quaternion.

Definition 5. *A planar dual quaternion is the four components of a dual quaternion which are non-zero for a planar displacement.*

A planar displacement (x, y, θ) can be expressed with a planar dual quaternion given by:

$$
\begin{aligned}
Q &= (q_1, q_2, q_3, q_4) = \\
&= \left(\cos\left(\tfrac{\theta}{2}\right), \sin\left(\tfrac{\theta}{2}\right), \tfrac{x}{2}\cos\left(\tfrac{\theta}{2}\right) + \tfrac{y}{2}\sin\left(\tfrac{\theta}{2}\right), \tfrac{y}{2}\cos\left(\tfrac{\theta}{2}\right) - \tfrac{x}{2}\sin\left(\tfrac{\theta}{2}\right)\right)
\end{aligned}
\tag{2.1}
$$

Because of the trigonometric unity, any planar dual quaternion has to satisfy the condition $q_1^2 + q_2^2 = 1$.

Definition 6. *A planar dual quaternion that satisfies the unity condition*

$$q_1^2 + q_2^2 = 1 \tag{2.2}$$

is called a displacement planar dual quaternion.

The unity condition can easily be enforced by the operator

$$P_d(Q) = \frac{Q}{\sqrt{Q_1^2 + Q_2^2}} \tag{2.3}$$

P_d is a projection operator (which means if the operator P_d is applied to a object in the range of P_d, the same object is returned), operating from the four-dimensional domain of all planar dual quaternions into the projection space of all displacement planar dual quaternions.

2.2 Kinematic Image Space

To simplify visualization of a planar dual quaternion, it is useful to project it onto a three-dimensional space. It will show to be convenient to use the space defined by:

Definition 7. *The projection space $\mathbb{S}$ of the operator:*

$$P_{\mathbb{S}}(Q) = \left(\frac{q_1}{q_1}, \frac{q_2}{q_1}, \frac{q_3}{q_1}, \frac{q_4}{q_1}\right) = (1, s_1, s_2, s_3) \tag{2.4}$$

is called the ***kinematic image space****, and was first introduced by [9].*

Definition 8. *The point*

$$S = (s_1, s_2, s_3) \tag{2.5}$$

is called the image point of the displacement defined by the planar dual quaternion Q.

Theorem 1. *The image point of a displacement x, y and rotation θ is given by:*

$$
\begin{aligned}
S = \mathbf{P}_{\mathbb{S}}(Q) = \left[\tan\left(\frac{\theta}{2}\right), \frac{1}{2}\left(x + y\tan\left(\frac{\theta}{2}\right)\right), \frac{1}{2}\left(y - x\tan\left(\frac{\theta}{2}\right)\right)\right] = \\
= \left[s_1, \frac{1}{2}(x + ys_1), \frac{1}{2}(y - xs_1)\right]
\end{aligned}
\tag{2.6}
$$

Proof. The planar dual quaternion of a displacement is given by (2.1) to be:

$$Q = \left(\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right), \frac{x}{2}\cos\left(\frac{\theta}{2}\right) + \frac{y}{2}\sin\left(\frac{\theta}{2}\right), \frac{y}{2}\cos\left(\frac{\theta}{2}\right) - \frac{x}{2}\sin\left(\frac{\theta}{2}\right)\right) \tag{2.7}$$

The image point is obtained by applying the projection operator (2.4)

$$\frac{Q}{q_1} = \left[1, \tan\left(\frac{\theta}{2}\right), \frac{1}{2}\left(x + y\tan\left(\frac{\theta}{2}\right)\right), \frac{1}{2}\left(y - x\tan\left(\frac{\theta}{2}\right)\right)\right] \tag{2.8}$$

The image point is given by the last three components. □

As one can see, the first component is only related to the rotation, while the two last components are a combination of translation and rotation.

2.2.1 Shapes in the Kinematic Image Space

The map between the physical dimensions to the image space is non linear. Some non-linear kinematics in the spatial dimensions takes a nice form after the non linear mapping. A straight line in $\mathbb{S}$ can describe a curved motion in the spatial dimensions.

Rotate around a point in the spatial dimension

It can be observed in (2.6) that a rotation around a point in the physical dimension is described by a line in the kinematic image space.

Translate without rotating

From (2.6), it can also observed that a translation along a straight line without rotation will follow a straight line with a constant x-value, $x = s_1$.

2.3 Constraint Manifolds

A constraint manifold is the set of positions the end effector of a mechanism can reach. For a 2R chain (Figure 2.1) which has two degrees of freedom, the constraint manifold forms a surface. For a one degree of freedom linkage, like a four-bar linkage, the constraint manifold is a curve, called the constraint curve. The constraint manifolds for 2R chains are importance when studying four-bar linkages since the four-bar can be seen as two 2R chains with the coupler link as a common second link. The constraint curve is then the set of points reached by both 2R chains, and therefore the intersection of two 2R chains' constraint manifolds.

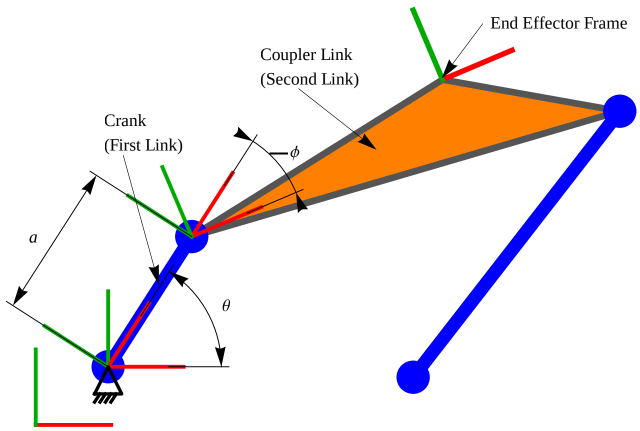


Figure 2.1: A 2R chain consists of two links. The first link is connected to the ground by a rotation joint. The second link is connected to the first with a second rotation joint. The end effector can be located anywhere on the second link.

2.3.1 Constraint Manifold for 2R Chain

The constraint manifold equation of a 2R chain will be derived by writing the kinematics with planar dual quaternions. This gives four equations, one for each component in the planar dual quaternion. The kinematic variables (the joint angles) is eliminated from the equations and the constraint manifold equation is obtained.

Forward Kinematics for 2R chain

Let $T(x, y)$ be a dual quaternion describing a pure translation, $R(\theta)$ be a dual quaternion describing a pure rotation. The dual quaternion Q of the end effector on the second link is obtained by dual quaternion multiplications, denoted by $\otimes$.

$$Q = T(g_x, g_y) \otimes R(\theta) \otimes T(a, 0) \otimes R(\phi) \otimes T(p_x, p_y) \tag{2.9}$$

This can be expanded into:

$$Q = \begin{pmatrix} \cos\left(\frac{\theta+\phi}{2}\right) \\ \sin\left(\frac{\theta+\phi}{2}\right) \\ \frac{1}{2}\left(a\cos\left(\frac{\theta-\phi}{2}\right) + (g_x + p_x)\cos\left(\frac{\theta+\phi}{2}\right) - p_y \sin\left(\frac{\theta+\phi}{2}\right)\right) \\ \frac{1}{2}\left(a\sin\left(\frac{\theta-\phi}{2}\right) + (p_x - g_x)\sin\left(\frac{\theta+\phi}{2}\right) + p_y \cos\left(\frac{\theta+\phi}{2}\right)\right) \end{pmatrix} \tag{2.10}$$

2.3.2 Obtaining the Constraint Manifold Equation

It is desirable to find a constraint manifold equation that works both for all displacement planar dual quaternions and in the kinematic image space.

Theorem 2. *If the constraint manifold equation is a homogeneous equation with all terms of the same order (e.g. q_i^2 or q_iq_j if the order is two), and the dual quaternion Q satisfies the constraint manifold equation, then all multiples of Q (i.e. tQ for $t \in \mathbf{R}$) satisfies the constraint manifold equation.*

Proof. The proof is given for second order terms. This is the only order that will be used later on, but it is easy to make the proof for any other order as well. Let $Q = (q_1, q_2, q_3, q_4)^T$. The constraint manifold equation can be written as the sum of all possible combinations of q_i and q_j.

$$K(q_1, q_2, q_3, q_4) = \sum_{i=1,j=1}^{4,4} c_{i,j} q_i q_j = 0 \tag{2.11}$$

Let $P = tQ$, such that $p_i = tq_i$. Then

$$\sum_{i=1,j=1}^{4,4} c_{i,j} p_i p_j = \sum_{i=1,j=1}^{4,4} c_{i,j} t^2 q_i q_j = t^2 \sum_{i=1,j=1}^{4,4} c_{i,j} q_i q_j \Longleftrightarrow \tag{2.12}$$

$$\sum_{i=1,j=1}^{4,4} c_{i,j} p_i p_j = t^2 \cdot 0 = 0 \tag{2.13}$$

This proves that if the constraint manifold equation is satisfied for a planar dual quaternion Q, it is also satisfied by a planar dual quaternion $P = tQ$. □

There is a linear map between the four trigonometric functions $\cos(\frac{\theta+\phi}{2})$, $\sin(\frac{\theta+\phi}{2})$,$\cos(\frac{\theta-\phi}{2})$ and $\sin(\frac{\theta-\phi}{2})$ and the components of the planar dual quaternion q_1, q_2, q_3 and q_4, since (2.10) can be written as:

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix}_{4\times 4} \begin{pmatrix} \cos(\frac{\theta+\phi}{2}) \\ \sin(\frac{\theta+\phi}{2}) \\ \cos(\frac{\theta-\phi}{2}) \\ \sin(\frac{\theta-\phi}{2}) \end{pmatrix} \tag{2.14}$$

The variables θ and ϕ can be eliminated in (2.10) by first extracting the values of the trigonometric functions in terms of q_i, then square and use the trigonometric unity.

Let:

$$c_1 = \cos(\frac{\theta + \phi}{2}) \tag{2.15}$$

$$c_2 = \sin(\frac{\theta + \phi}{2}) \tag{2.16}$$

$$c_3 = \cos(\frac{\theta - \phi}{2}) \tag{2.17}$$

$$c_4 = \sin(\frac{\theta - \phi}{2}) \tag{2.18}$$

According to the trigonometric unity $\cos^2(x) + \sin^2(x) = 1$:

$$c_1^2 + c_2^2 - c_3^2 - c_4^2 = 1 - 1 = 0 \tag{2.19}$$

By substituting c_i with its values in terms of the components in the dual quaternion ($C = A^{-1}Q$), the constraint manifold equation is obtained as:

$$\begin{split} K(q_1, q_2, q_3, q_4) = q_1^2 + q_2^2 - \frac{1}{a^2}\big[(g_x q_1 + g_y q_2 - p_x q_1 + p_y q_2 - 2q_3)^2 + \\ (g_x q_2 - g_y q_1 + p_x q_2 + p_y q_1 + 2q_4)^2\big] = 0 \end{split} \tag{2.20}$$

Interpreting the Constraint Manifold Equation

It is easy to see that the constraint manifold equation is a second degree polynomial equation of the four variables q_i. However more structures are hidden in the equation.

By writing (2.20) as a quadratic form

$$Q^T \mathbf{M} Q = 0 \tag{2.21}$$

with the symmetric matrix M given by:

$$
\begin{aligned}
M_{1,1} &= -\frac{2(g_x - p_x)^2}{a^2} - \frac{2(p_y - g_y)^2}{a^2} + 2 \\
M_{1,2} &= -\frac{2(g_x + p_x)(p_y - g_y)}{a^2} - \frac{2(g_x - p_x)(g_y + p_y)}{a^2} \\
M_{1,3} &= \frac{4(g_x - p_x)}{a^2} \\
M_{1,4} &= -\frac{4(p_y - g_y)}{a^2} \\
M_{2,2} &= -\frac{2(g_x + p_x)^2}{a^2} - \frac{2(g_y + p_y)^2}{a^2} + 2 \\
M_{2,3} &= \frac{4(g_y + p_y)}{a^2} \\
M_{2,4} &= -\frac{4(g_x + p_x)}{a^2} \\
M_{3,3} &= M_{4,4} = -\frac{8}{a^2} \\
M_{3,4} &= 0
\end{aligned}
\tag{2.22}
$$

It can easily be seen that (2.22) contains 8 different elements or coefficients. The symmetry of M reduces it to 10 of maximum 16 possible coefficients. One coefficient is zero and two of them are equal, which reduces it to 8. Since the equation is homogeneous, there is an infinite number of solutions for the coefficients. By assigning one of the coefficients a value, the number of solutions is reduced to one. Since (2.20) is a homogeneous equation, it is according to Theorem 2 valid for all scale's of planar dual quaternions. To simplify geometric reasoning, the constraint manifold will from now on be analyzed in the kinematic image space $\mathbb{S}$, which gives three dimensions instead of four.

Consider a quadratic form in $\mathbb{S}$ of the same structure with the elements c_i. The constraint manifold equation in the kinematic image space is given by:

$$
\begin{pmatrix} 1 & x & y & z \end{pmatrix}
\begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ m_2 & m_5 & m_6 & m_7 \\ m_3 & m_6 & m_8 & 0 \\ m_4 & m_7 & 0 & m_8 \end{pmatrix}
\begin{pmatrix} 1 \\ x \\ y \\ z \end{pmatrix} = 0
\tag{2.23}
$$

The zeros tell that there is no product between y an z. The coefficient for both y^2 and z^2 is c_8, which generates a circle for a constant value of x. By choosing $m_8 = 1$ we can expand and rewrite (2.23) as:

$$
\begin{aligned}
(y - (m_6 x + m_3))^2 + (z - (m_7 x + m_4))^2 - (m_5 + m_6 + m_7)x^2 - \\
2(m_2 + m_3 m_6 + m_4 m_7)x - (m_1 + m_3^2 + m_4^2) = 0
\end{aligned}
\tag{2.24}
$$

Even though we have seven different coefficients, they are all determined by the five independent linkage parameters. By matching the values of m_1 with the corresponding elements in (2.22), one can observe that:

$$m_5 + m_6 + m_7 = m_1 + m_3^2 + m_4^2 = \left(\frac{a}{2}\right)^2 \tag{2.25}$$

and

$$m_2 + m_3 m_6 + m_4 m_7 = 0 \tag{2.26}$$

Which simplifies (2.24) into:

$$(y - (m_6 x + m_3))^2 + (z - (m_7 x + m_4))^2 = \left(\frac{a}{2}\right)^2 (1 + x^2) \tag{2.27}$$

This shows that the constraint manifold is a hyperboloid centered in

$$(0, m_3, m_4)$$

and with a center line direction

$$(1, m_6, m_7)$$

The radius of the circle at a fixed x is

$$r(x) = \frac{a}{2}\sqrt{1 + x^2}$$

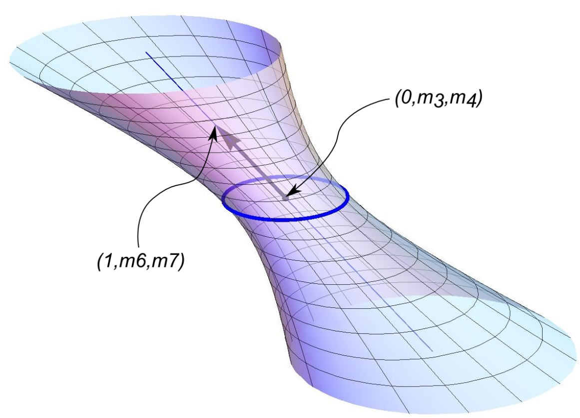


Figure 2.2: A constraint manifold of a 2R chain takes the form of a hyperboloid centered at the point $(0, m_3, m_4)$ and with a centerline direction of $(1, m_6, m_7)$

By matching the values in (2.23) and (2.22), the center line and radius of the hyperboloid is found to be:

$$center(x) = \frac{1}{2}\left(2x, g_x - p_x + x(g_y + p_y), g_y - p_y - x(g_x + p_x)\right) \tag{2.28}$$

$$radius(x) = \frac{a}{2}\sqrt{1 + x^2} \tag{2.29}$$

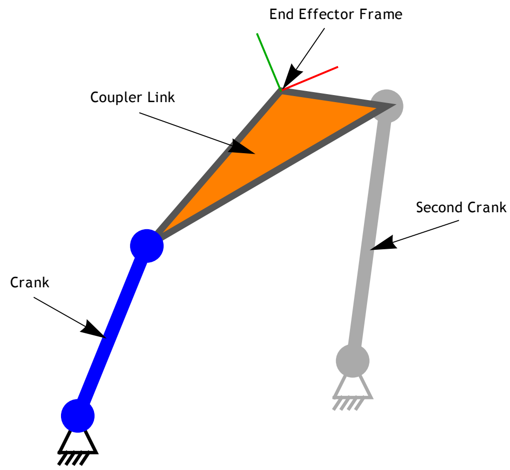


Figure 2.3: A four bar linkage is constructed by two 2R chains with a common coupler link.

2.4 Constraint Curve of a Four-Bar Linkage

Consider a four-bar linkage constructed by a pair of 2R chains, with a common coupler link. Also consider the two constraint manifold equations for the 2R chains. For simplicity, one of the equations is for now assumed to take a simplified shape.

$$y - (m_6 x + m_3))^2 + (z - (m_7 x + m_4))^2 = \left(\frac{a}{2}\right)^2 (1 + x^2) \tag{2.30}$$

$$y^2 + z^2 = \left(\frac{r}{2}\right)(1 + x^2) \tag{2.31}$$

The end effector of the four bar has to satisfy both constraint manifold equations, and is therefore given by the intersection curve.

2.5 Solving for the Constraint Curve

It is possible to find an explicit solution of the constraint curve. Consider two circles with radius R and r with a distance d from each other along the first coordinate axis. The two intersection points, see Figure 2.4, are given by:

$$\left(\frac{d^2 - r^2 + R^2}{2d}, \pm\frac{\sqrt{-d^4 + 2d^2r^2 + 2d^2R^2 - r^4 + 2r^2R^2 - R^4}}{2d}\right) \tag{2.32}$$

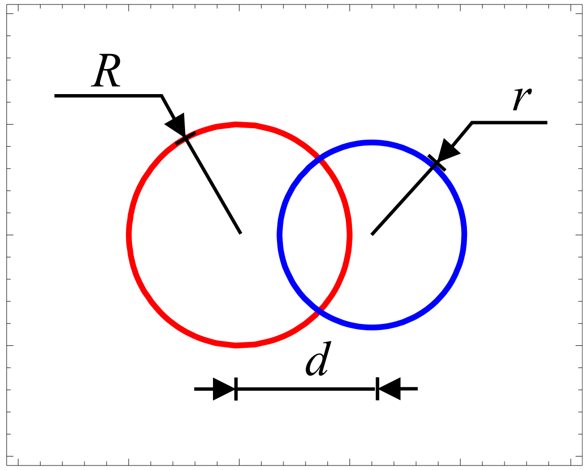


Figure 2.4: Two circles with radius R and r with their centers at distance d from each other. The intersection between the circles is given by (2.32).

Consider two constraint manifold hyperboloids at a fixed x-value. This gives two circles. Align a coordinate system L with origo on the first center line and the y-axis (Green in Figure 2.5) pointing in the direction of the second center line. The first hyperboloid is for now assumed to have the trivial shape. The z-axis (Blue) is chosen to be in the same $x = constant$ plane and orthogonal to the y-axis.

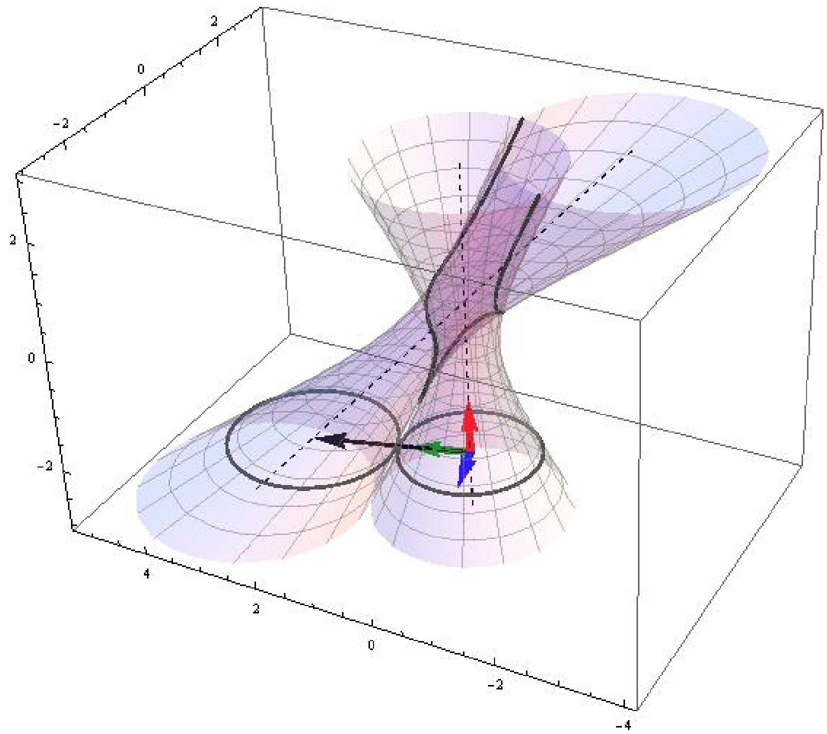


Figure 2.5: At a fixed value of x, the constraint manifolds forms two circles. A coordinate system is placed with the y-axis (Green) pointing from one center line to the other. The hyperboloid intersections can be simplified by the coordinate transformation.

Let $d(x)$ be the vector from the first center line to the second center line. The vector $d(x)$ is given by:

$$\left(0, \frac{1}{2}(g_x + g_y x - p_x + p_y x), \frac{1}{2}(-g_x x + g_y - p_x x - p_y)\right) \tag{2.33}$$

The length of $d(x)$ is:

$$\|d(x)\| = \sqrt{d(x) \cdot d(x)} = \sqrt{\frac{1}{4}(-g_x x + g_y - p_x x - p_y)^2 + \frac{1}{4}(g_x + g_y x - p_x + p_y x)^2} \tag{2.34}$$

The radius r is given by:

$$r(x) = \frac{a}{2}\sqrt{1 + x^2} \tag{2.35}$$

By substituting (2.34) and (2.35) into (2.32) the coordinates of the intersection curve in the coordinate system L is obtained.

Transform Coordinates

The coordinate transformation from L to an initial coordinate system is described by the matrix $L(x)$

$$L(x) = \frac{1}{\|d(x)\|} \begin{bmatrix} \|d(x)\| & 0 & 0 \\ 0 & d_y(x) & -d_z(x) \\ 0 & d_z(x) & d_y(x) \end{bmatrix} \tag{2.36}$$

Since the x-axis remains the same, the y-axis is given by the vector $d(x)$ and the z-axis is the vector $d(x)$ rotated a quarter revolution.

By substituting the values of R, r and d into (2.32), one obtain the intersection in the coordinate system L:

$$C_L(x) = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \tag{2.37}$$

Where:

$$s_1 = x \tag{2.38}$$

$$s_2 = \frac{a^2\left(-\left(x^2+1\right)\right) + (x(g_x+p_x) - g_y + p_y)^2 + (g_x + x(g_y+p_y) - p_x)^2 + x^2 + 1}{4\sqrt{(x(g_x+p_x) - g_y + p_y)^2 + (g_x + x(g_y+p_y) - p_x)^2}} \tag{2.39}$$

$$s_3 = \pm\sqrt{\frac{a^4\left(-\left(x^2+1\right)^2\right) + 2a^2\left(x^2+1\right)\cdot\left((x(g_x+p_x) - g_y + p_y)^2 + (g_x + x(g_y+p_y) - p_x)^2 + x^2 + 1\right) - \left((x(g_x+p_x) - g_y + p_y)^2 + (g_x + x(g_y+p_y) - p_x)^2 - x^2 - 1\right)^2}{4\sqrt{g_x^2\left(x^2+1\right) + 2g_x\left(p_x\left(x^2-1\right) + 2p_y x\right) + g_y^2\left(x^2+1\right) - 2g_y\left(2p_x x - p_y x^2 + p_y\right) + \left(x^2+1\right)\left(p_x^2+p_y^2\right)}}} \tag{2.40}$$

The constraint curve in a original coordinate system is finally obtained by the matrix multiplication:

$$C(x) = L(x)C_L(x) \tag{2.41}$$

2.5.1 Arbitrary Hyperboloid Intersection

In the case of two arbitrary constraint manifolds, the coordinate transformation gets a bit more complex. The vector between the center lines is obtained in the same way but will now include more coefficients. The new x-axis will take the direction of the first center line. A basis vector for the x-axis can for example be obtained by:

$$e_x = center_1(1) - center_1(0) \tag{2.42}$$

The new coordinate system is translated from $(x, y, z) = (x, 0, 0)$ into the position given by the first center line. The vector between the center lines is obtained by:

$$d(x) = center_2(x) - center_1(x) \tag{2.43}$$

The e_y vector is obtained by normalizing the vector $d(x)$

$$e_y(x) = \frac{center_2(x) - center_1(x)}{\|center_2(x) - center_1(x)\|} \tag{2.44}$$

The e_z vector can be obtained by rotating e_y by the angle $\pi/2$.

$$e_z(x) = R_{\{x,\frac{\pi}{2}\}} \cdot e_y(x) \tag{2.45}$$

The constraint curve is then given by:

$$C(x) = center_1(x) + [e_x, e_y(x), e_z(x)]\, C_L(x) \tag{2.46}$$

There is no real reason for writing out the expression for the constraint curve since it will cover more than a page. However we have shown that an explicit solution does exist.

Chapter 3

Analysis of Branching in Kinematic Image Space

In the case of real solutions of the constraint equations, there is a linkage that goes through all the task positions. However the different task positions may belong to different branches of the linkage.

3.1 Determine Branches

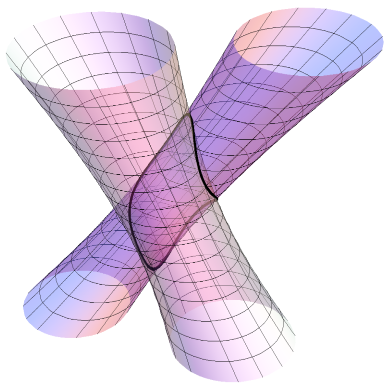
A today's approach to analyze branching of linkages is to form the equations of a closed kinematic loop and solve the end effector position for a discrete set of values for one of the links angles.

3.1.1 New Approach

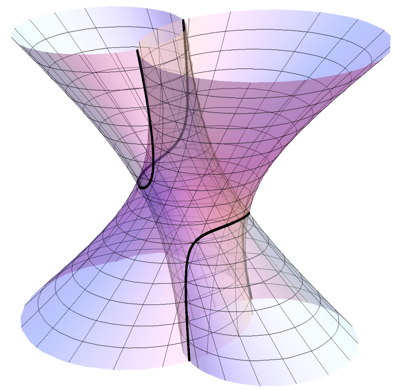
With our new explored knowledge, the branching analysis can be done in a much faster way. By solving for the x-values were the constraint manifolds intersects in only one point, a set of intervals, which are candidates to include continuous constraint curves, is obtained. By just checking one point inside each interval we will know wherever there is a continuous curve inside the whole interval or not. The task positions must all lie in the same interval for the linkage to be useful.

The structure of the constraint curve was investigated by Schröcker et al. in [14]. They state that the constraint curve has two affinely finite branches, one branch or two affinely infinite branches. This is not the whole truth about the constraint curve. The constraint curve can take one of the five forms shown in Figure 3.1.

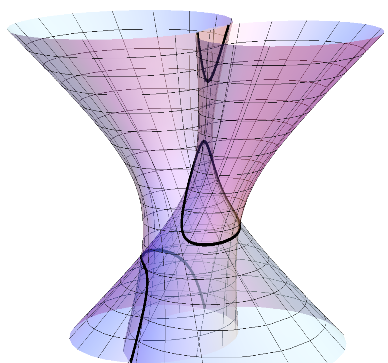
Consider a fixed value of x. As shown previously, each manifold forms a circle for a fixed value of x. The circles intersects in either zero, one, two or all points. Intersection in all points only occurs when both the center lines



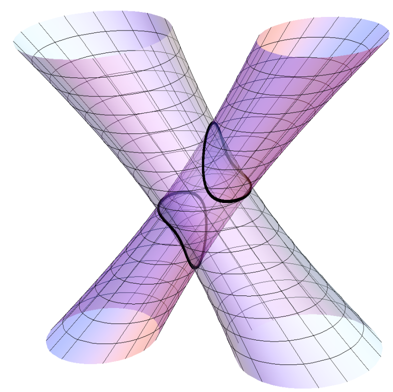
(a) One closed affinely finite constraint curve.



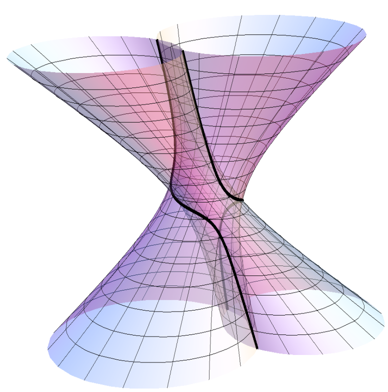
(b) Two half open infinite constraint curves.



(c) One closed affinely finite, and two half open infinite constraint curves.



(d) Two closed affinely finite constraint curves.



(e) Two infinite constraint curves.

Figure 3.1: Illustration of the possible structures of the constraint curve.

intersects and the two radius are equal at the same time, which is the case for an ideal parallel linkage (Figure 3.2). This case is not of big interest for us.

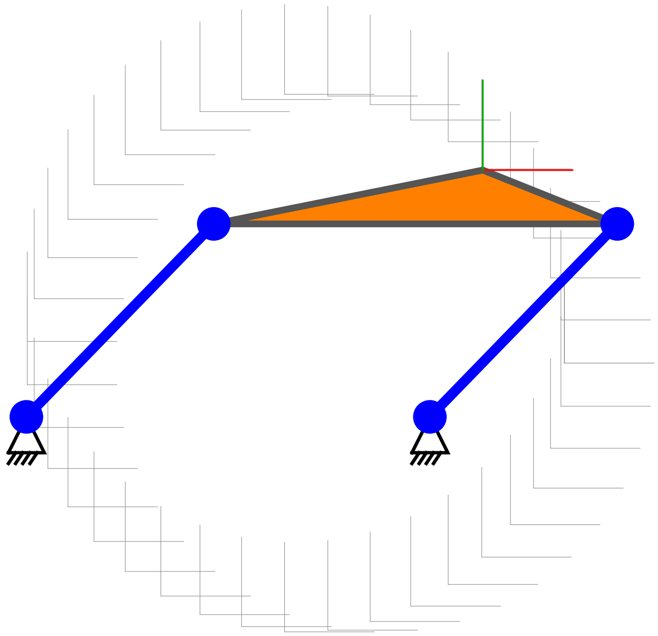


Figure 3.2: An ideal parallel linkage. The end effector orientation remains constant while the position moves on a circle.

Two circles intersect at one point if the distance between the center lines is the same as either the difference in radius or the sum of radius. Those conditions can be formulated as

$$\|\Delta r(x)\| = \|center_2(x) - center_1(x)\| \tag{3.1}$$

$$\|\Sigma r(x)\| = \|center_2(x) - center_1(x)\| \tag{3.2}$$

Two intersection points occur when the distance between the center lines is in between the sum and the difference of radius.

$$\|\Delta r(x)\| < \|center_2(x) - center_1(x)\| < \|\Sigma r(x)\| \tag{3.3}$$

The absolute values can be eliminated by squaring the equations.

$$\Delta r(x)^2 = (center_2(x) - center_1(x)) \cdot (center_2(x) - center_1(x)) \tag{3.4}$$

$$\Sigma r(x)^2 = (center_2(x) - center_1(x)) \cdot (center_2(x) - center_1(x)) \tag{3.5}$$

Those equations are second degree polynomials since the center lines are given by first degree polynomials, and the radius are square roots of a second degree polynomial.

Note It is important to mention that the product of two square roots does not form a polynomial of the same order as the square root's arguments in a general case. However it does in this case since the square roots have the same arguments.

It is of interest to study for which x values one solution occurs. Since the equations are quadratic, they can have zero, one or two distinct real solutions. Since there are two equations, there can be a maximum of four x values where the constraint manifolds intersects in only one point. Those points will be referred to as the **split points** since they splits the image space into intervals I_s that either includes no intersection points or includes two intersection points for each x-value. This means that for an interval I_s, there is either a continuous constraint curve covering the whole interval or no constraint curve at all, except at the boundary which is shared with the neighbor intervals.

The four split points x_i form four intervals for possible constraint curves. The first instinct would suggest either three or five intervals, which are:

$$]-\infty, x_1], [x_1, x_2], [x_2, x_3], [x_4, x_5], [x_5, \infty[$$

or

$$[x_1, x_2], [x_2, x_3], [x_4, x_5]$$

but certain attention has to be considered at infinity.

The case $x \to \infty$

The first dimension in $\mathbb{S}$ is represented by $x = \tan\left(\frac{\theta}{2}\right)$. This means that $x \to \pm\infty$ corresponds to the angle $\theta \to \pm\pi$. It was shown in section 2.2.1 that a line of the form

$$\left[s_1, \frac{1}{2}(y + zs_1), \frac{1}{2}(z - ys_1)\right] \tag{3.6}$$

describes all position with the translation (y, z) in the physical dimensions. The ends at infinity of such a line will connect to each other in the physical dimensions, since the angle $\theta = -\pi$ gives the same position as $\theta = \pi$.

A linkage that includes $\theta = \pi$ in its work space will have a constraint curve that goes to infinity and connects to an other curve at minus infinity. There will be a line of form (3.6) tangent to the constraint curves on the ends that are connected together.

The reason for four intervals is that $\pm\infty$ are connected together and therefore the last and first value forms one interval. The four intervals are:

$$[x_1, x_2] \tag{3.7}$$
$$[x_2, x_3] \tag{3.8}$$
$$[x_3, x_4] \tag{3.9}$$
$$[x_4, x_1] \tag{3.10}$$

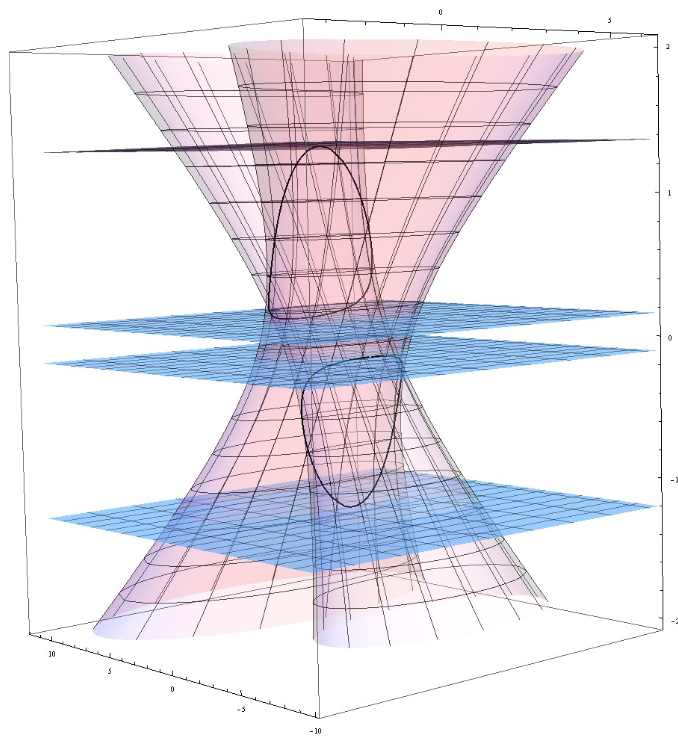


Figure 3.3: The image space is divided into a maximum of four regions which either contains a constraint curve for all x or no constraint curve at all.

Physical Interpretation

The physical interpretation of this is that a linkage movement can be limited by a maximum angle and a minimum angle. At the maximum and minimum angle there is only one translation the end effector can take. For any angle in between there are two different translations. It also means that a four-bar linkage can not have more than two translations with the same orientation.

3.1.2 Determine the Side of Infinite Curves

In the case of two infinite constraint curves there are no split points which implies that all task positions lie in the same interval. However it also implies that there are two decoupled constraint curves since they never join each other in a single point. To determine which one of the curves an image point belongs to, the image point can be transformed to the coordinate system L. The sign of the z-component will then determine which one of the curves an image point belongs to.

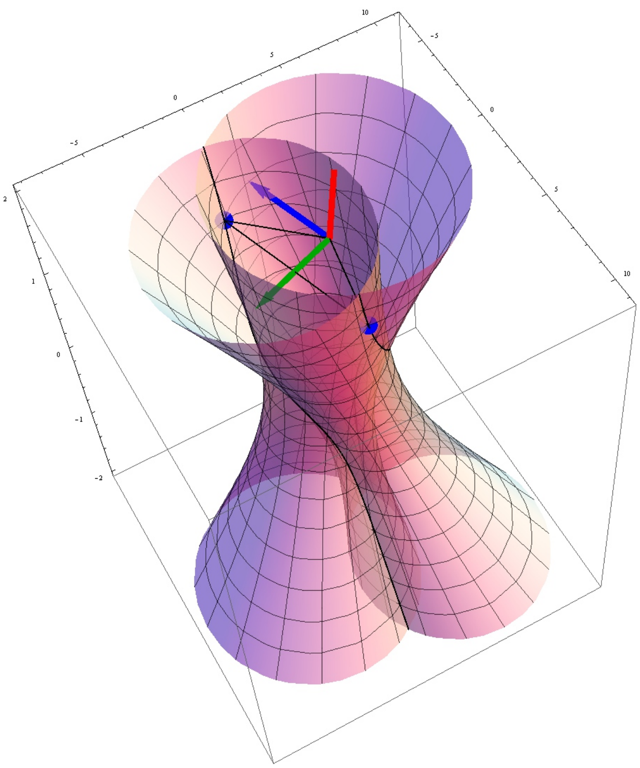


Figure 3.4: Two infinite constraint curve. By analyzing an image point in the coodinate system $L(x)$, the sign of the z-component (Blue axis) will determine which one of the curves the point belongs to.

3.1.3 The Side of a Closed Curve

Chase[11] considers a linkage useful if it can be driven by rotating one input link and still reach all task positions. At some values of the input angle a linkage reaches a maximum or minimum value of the end effector rotation. There are two different trajectories leading to this position and it may not be possible to control which one of the trajectories the end effector will follow when leaving the position. Since the end effector rotation is directly related to the x-value in the kinematic image space, the minimum and maximum values of the end effector trajectory will occur at the split points.

Chase[11] defines a linkage as useful if all task positions can be reached when the linkage is actuated by rotation of one of the links. Therefore it is of interest to determine which side of a closed curve an image point belongs to. To determine the side, the same method as in Section 3.1.2 can be applied. Even if the usefulness defined by Chase[11] is not primary considered in this theses, the method is still able to use Chase's definition.

Chapter 4

Modifying Task Positions

The first approach to avoiding branching was to identify some primitive approximate shapes for the constraint curve and adjust the task positions to mach one of those primitive shapes.

4.1 Intersection Shapes

By experimenting with moving two cylinders it is observed that the shape of the intersection goes from one ellipse to a guitar shape, a figure eight and two ellipses. Those shapes are just approximations and hyperboloids may form much more complex intersections. It is still reasonable to think that five task positions that lie close to one of those shapes should have a non branching linkage.

This is investigated by matching an ellipse (which is the most simple case) to four of the task positions and projecting the fifth position onto the ellipse.

An general ellipse is determined by five coefficients. When the ellipse is matched to four points, there is an infinite number of solutions. In the experiments, the ellipse closest to a circular shape where selected.

4.2 Projection Methods

4.2.1 Projecting on Curve given by an Implicit Equation

Consider a curve C defined by $g(y, z) = k$, i.e. the curve is a contour curve to the function $g(y, z)$. Also consider the square distance function $\mathbf{d}_p(y, z)$ from a point p to an arbitrary point (y, z), given by:

$$\mathbf{d}_p(y, z) = y^2 + z^2 \tag{4.1}$$

The minimum distance from point p to C occurs when the contour curves of $\mathbf{d}_p(y, z)$ tangent C. To find the tangential point we look for where the gradients of g and d are parallel. This will give us a curve for which the tangential point is the intersection with C.

After running a large number of syntheses from randomly selected task positions it was not possible to see a significant difference of the number of useful linkages after the modification verses before. When analyzing the constraint manifolds after synthesizing from the modified positions, the intersection curve did not take shape of the ellipse which it was supposed to.

4.3 Discussion

This gives the conclusion that it is not good enough to project the task positions onto an approximated shape to avoid branching. Instead the exact shape of the constraint curves would need to be calculated.

Synthesizing could be done by matching the hyperboloids to the task positions in the image space. Fitting a general hyperboloid given by 7 coefficients is just to solve a linear equation, but the solution is not guaranteed to be a hyperboloid, it might be a ellipsoid instead. Also the extra relations between the 7 coefficients determined by the 5 linkage parameters introduce many nonlinearities.

An other approach will be introduced in the next chapter.

Chapter 5

Four-Point Synthesis

A crank of the linkage is defined by five parameters, thus up to five task positions can be specified. If one of the linkage parameters is specified, only four task positions remains to be specified.

Instead of specifying five positions and synthesizing a linkage by an iterative process that goes through the tolerance zone, four task positions will be specified and all possible motions that go exactly through those four task positions will be calculated. With that information, a fifth task position could be moved to the closest possible point.

The approach to calculate all possible motions to a set of four task positions is:

* Chose one of the linkage parameters (e.g. g_y) and find the values where the constraint curve is changing structure and form a set of intervals with the same structure in the whole interval.

* Analyze the branching conditions for each interval and determine the useful structure intervals.

Definition 9. *An interval for a linkage parameter that gives the same structure of the constraint curve in the whole interval is called a* ***structure interval****.*

Definition 10. *The values of a linkage parameter where the structure of the constraint curve changes are called the* ***separation points****.*

5.1 Determine the Structure Intervals

Consider a four-bar linkage with one linkage parameter constrained, e.g. $g_y = value$. From four task positions, the synthesis algorithm will give the remaining linkage parameters and the constraint curve can then be calculated. Using the branching analysis from Chapter 3, the structure of the constraint curve is easily known.

Let G be a vector with a set of values for the linkage parameter g_y. If the structure of the constraint curve is the same for two neighbor values G_i, G_{i+1}, the constraint curve is assumed to have the same structure in the interval in between. This is of course an approximation but it is true for small enough distances between G_i and G_{i+1}. If the structure is different between two values G_i and G_{i+1}, a refined search is made in the interval $[G_i, G_{i+1}]$ until the difference between the separation points is within a specified precision tolerance.

During experiments it was observed that in some intervals the structure could change in a stochastic or at least unknown way. Those intervals are ignored in favor to the robustness of the algorithm.

5.2 Analyze the Branching Conditions

A fast closed loop method for investigating the branching was previously developed in Chapter 3. For each structure interval it can easily be determined whether the four task positions lies in the same branch. If they do, the structure interval is useful.

Hypothesis 1. *If one linkage synthesized from four task positions and a value of a linkage parameter (e.g. g_y) in a structure interval is useful, then all linkages synthesized from the same four task positions and any value of the linkage parameter in the structure interval is useful.*

Hypothesis 1 is not proved to be true but with some geometric reasoning, it can be fairly well motivated.

Motivation Recall the equations of the constraint manifold center line and radius.

$$center(x) = \left(x, \frac{1}{2}(g_x + g_y x - p_x + p_y x), \frac{1}{2}(-g_x x + g_y - p_x x - p_y)\right) \quad (5.1)$$

$$radius(x) = \frac{a}{2}\sqrt{1 + x^2} \quad (5.2)$$

An infinite change of one or more variables will give an infinitely close constraint manifold, and thus an infinitely close constraint curve. In other words, the constraint curve is a continuous function of the linkage parameters. This would imply that two finitely close constraint curves with the same structure will approximate a continuous surface generating infinitely many constraint curves in between.

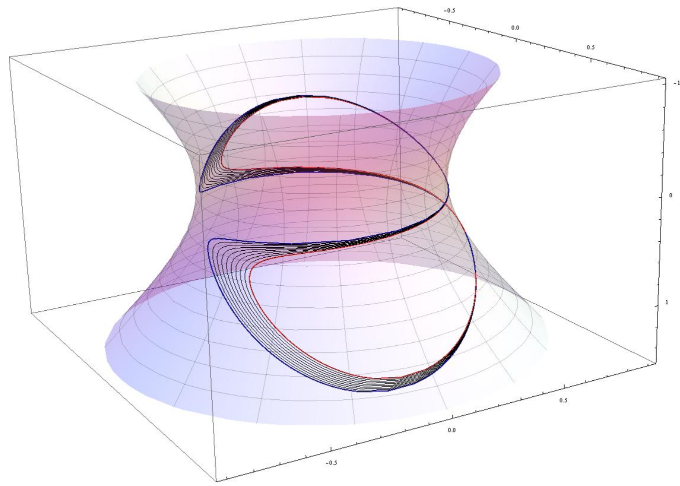


Figure 5.1: Two close constraint curve approximates a surface in between that includes an infinitely number of constraint curves in between.

In the case of only one single branch, all task positions will always be in the same branch. Consider the case of two branches where the constraint curve consists of two closed curves, and all task positions lies on the same branch. The linkage parameters are slightly modified so a near by constraint curve with the same structure is obtained. Then there is according to our assumptions a continuous surface with constraint curves of the same structure in between. For a task position to leave its current branch, the constraint curve has either to separate into two new curves, or the two existing curves have to join each other at the task position. But those cases imply a new structure of the curve.

Conclusion This means that, as long as one linkages in a structure interval is useful, then all linkages in that structure interval are useful.

5.3 Multiple Solutions

Multiple solutions occurs for some values of g_y, i.e. there are many linkages with the same value of g_y. The center point curve is the curve that describes all positions of the fixed pivot for linkages synthesized to four task positions. It is a continuous cubic curve for four task positions [16]. Multiple solutions introduce an issue when dividing the values of g_y into interval with different branching structures. If the values of g_y where either the branching structure or the number of solutions is calculated, the points (g_x, g_y) can be plotted to show the structure of the center point curve.

The separator values are defined as the values where the structure of the constraint curve changes. The structure is considered to change if either the number of solutions changes or the branching structure of the constraint curve for any of the multiple solutions is changing. Let the separator values be the values where either the constraint curve changes structure, or where the number of solutions are changing. The center point curve may look like Figure 5.2.

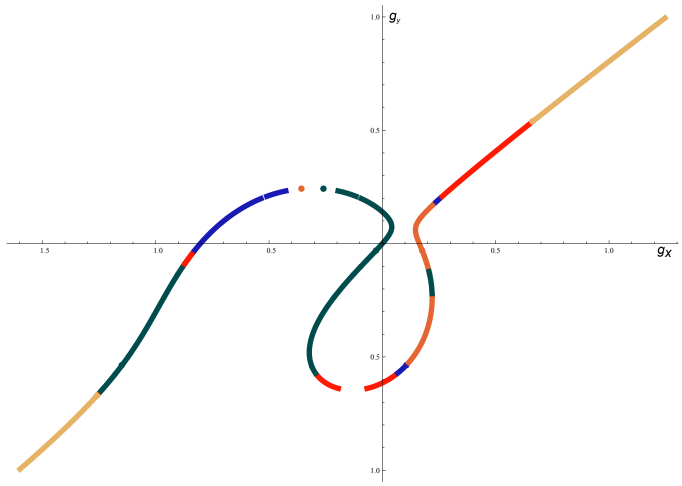


Figure 5.2: Illustration of a center point curve where the different branching conditions are indicated with different colors. The curve is generated from the trunk cover example.

5.4 Example

Recall the example in section 1.1. In this example let the value of g_y be in the interval $g_y \in [-2, 2]$. The iterative search is used to obtain the values of g_y where the constraint curve changes structure. The following values were obtained in this example

$$G^{sep} = (-0.7901, 0.2469, 0.6420, 0.8395) \tag{5.3}$$

The Branching cases for the intervals of g_y values in this example are

$g_y \in [-2, -0.7901]$ Two closed finite curves, all positions did not happen to be on the same branch.

$g_y \in [-0.7901, 0.2469]$ One closed finite curve, all points are always on the one and only branch.

$g_y \in [0.2469, 0.6420]$ Two half open curves, all points are always on the one and only branch.

$g_y \in [0.6420, 0.8395]$ One closed finite and two half open curves, all positions did not happen to be on the same branch.

$g_y \in [0.8395, 2]$ Two closed finite curves, all positions did not happen to be on the same branch.

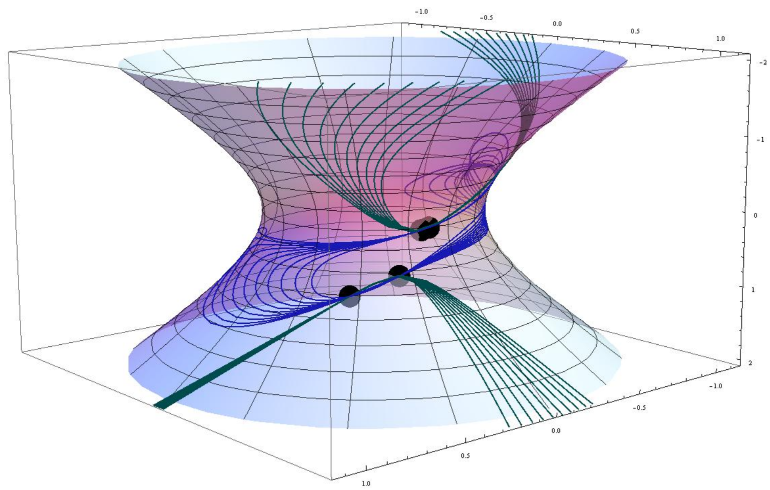


Figure 5.3: All possible useful constraint curves to the four task positions from the car trunk demo.

5.5 Modifying the fifth task position

The initial problem where posted to find a strategy to modify a fifth task position. However the exploration of all possible image curves to four positions opens up new possibilities. An example of a task description could be to move through four task positions and avoid one or more points on the way. Then the line in the image space that describes this point for all orientations can be calculated, and the constraint curve that goes as far away from this line could be used.

If it is still desirable to have a fifth task position it can be done as well. In the classical synthesis methods the task positions are specified together with a tolerance zone for x, y and θ. The tolerance zone is no longer of the same interest since it from now on is possible to find the closest reachable task position.

The task positions in our case are assumed to be specified by the joint values of the first crank. Thus they will all lie on the first crank's constraint manifold. Since the position consists of both translation and rotation, the closest distance has to be defined. The importance of translation verses rotation will be specified by two weights $w1$ and $w2$.

It is assumed that a fifth task position should be reachable by the first specified crank. Therefore it has to lie on the surface of its constraint manifold. If the fifth task position is moved along the parametric curve defined by:

$$p_5(t) = center(t) + \begin{pmatrix} x_0 \pm w_1 t \\ r(x_0 \pm w_1 t)\cos(\theta_0 + w_2 t) \\ r(x_0 \pm w_1 t)\sin(\theta_0 + w_2 t) \end{pmatrix} \tag{5.4}$$

the point will automatically stay on the constraint manifold. This curve is only one example of possible curves to use for modifying the fifth task position, any other curves that lies on the first constraint manifold surface could be used as well.

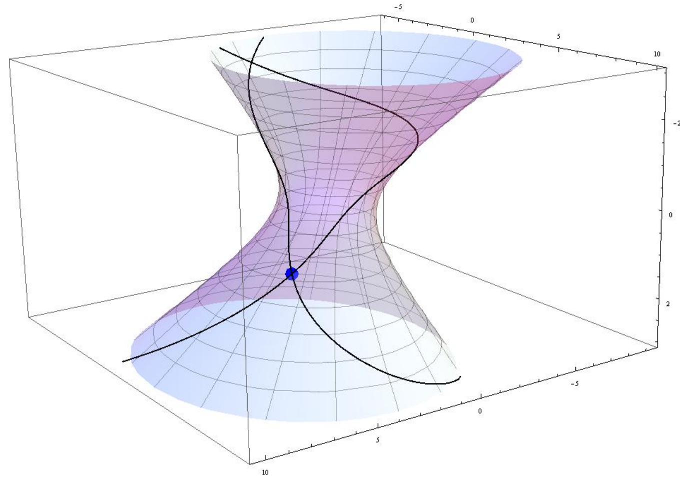


Figure 5.4: A curve defined by (5.4) lies on the surface of a constraint manifold. The blue point illustrates the image point of a fifth task position. There are two similar curves rotating in opposite directions.

To find the intersection with a constraint curve, it is enough to find the intersection with the second constraint manifold that generates the constraint curve. The constraint manifold satisfies the implicit constraint manifold equation (2.20). The values for x, y and z in (5.4) can be inserted into the constraint manifold equation, and the zeros can be solved for.

The `NSolve` function in Mathematica had problems solving this equation. Instead a simple systematic search algorithm were used. The search algorithm finds all zeros within a limited interval for the curve parameter. The algorithm checks the value of the constraint manifold equation (2.20) for a set of parameter values and searches for the values where the sign changes, since the equation is zero at the intersection. When two neighbor values are found, a refined search is made in the interval in between until the second constraint manifold equation is satisfied.

Chapter 6

Discussions

6.1 Adjusting to Primitive Shapes

The idea of adjusting the task positions to match some primitive shapes of intersection curves did not show to be working. It did not show any significant differences in branching before and after adjusting the task positions. Also since the shapes were determined by looking at the intersection of two cylinders, the intersection of two hyperboloids will probably not take the exact same shape. The constraint curve for linkages synthesized did not either take a shape similar to the supposed primitive shape.

6.2 Structure interval

The linkage parameter g_y was used to determine the structure intervals. As seen on the center point curve (Figure 5.2), the curve segments with small changes of g_y are not completed. This could possibly be solved by synthesizing by varying another linkage parameter such as g_x instead. There is still much research to be done before the most efficient and robust way of calculating all possible linkages to four task positions is found.

6.3 Constraint Manifolds

The most interesting question is how to apply this theory on multiple-loop linkages. For a multiple-loop linkage, the constraint manifolds for each four-bar loop in the linkage could be analyzed. This would give a set of constraint curves in the image space. The constraint manifolds and constraint curve only tells which positions are reachable at some joint values. Since the joint values are eliminated when deriving the constraint manifolds, if the constraint manifolds for each four-bar loop in the linkage are analyzed, it would give a set of constraint manifolds which may intersect in more positions than are reachable.

6.3.1 Order of Constraint Manifolds

The constraint manifold for a 2R chain is given by one equation. A curve in a two-dimensional space is defined by one equation of any two coordinates variables defining the plane. A plane in a three-dimensional space is defined by one equation of any three coordinate variables defining the space. One equation constraints one variable, so one equation in a four-dimensional space will define a three-dimensional subspace which is the case for the 2R chain constraint manifold in the planar dual quaternion domain. While deriving the constraint manifold equation, the kinematics was described using two variables (the joint values) and four equations where given. From the first two equations $q_1 = \cos\left(\frac{\theta}{2}\right)$ and $q_2 = \sin\left(\frac{\theta}{2}\right)$, the condition $q_1^2 + q_2^2 = 1$. A similar condition could be obtained from the third and fourth equation. The two variables where eliminated by this and two equations remained. Those equations where subtracted from each other to obtain the final constraint manifold equation. However one of the original two equations has to be saved. Only one of those where considered as the constraint manifold equation but the trigonometric unity condition has to be satisfied as well to be a reachable position.

6.4 Synthesis of Non-Branching Six-Bar Linkages

Exactly how this work can be applied on higher order of linkages is still unknown. For a multiloop linkage consisting of many four-bar linkages. The branching analysis can be applied to each four-bar. A constraint manifold equation can also be derived for a 3R chain. However since a 3R chain has three degrees of freedom, the constraint manifold is a 3 dimensional region instead of a 2 dimensional surface.

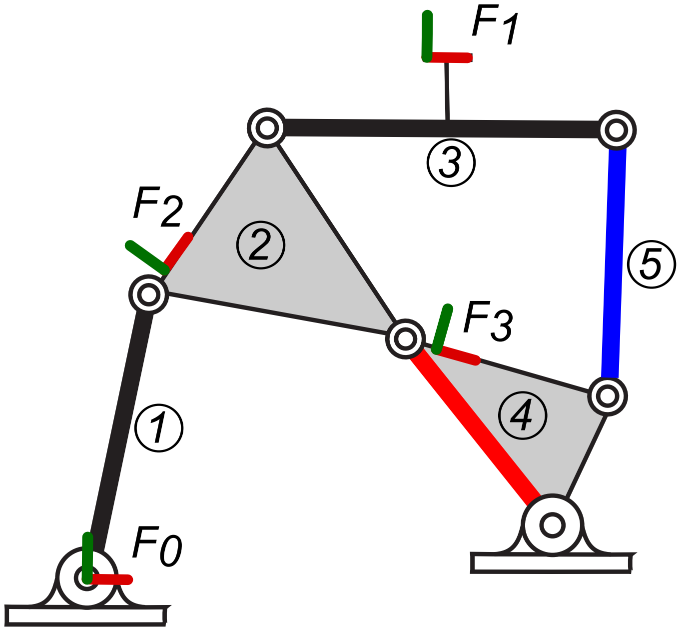


Figure 6.1: A Watt Ia six-bar linkage drawing from [6] with notations added on.

If the two constraint manifold regions C_{1-2-3} for link $1-2-3$ and C_{4-5-3} for link $4-5-3$ are calculated, the intersection would give the work space for the mechanism without link 2 and 4 connected.

A 3R chain $4-2-3$ is considered, the intersection between the regions C_{1-2-3} and C_{4-2-3} would be greater then the actual work space of the mechanism. The reason is that the joint values of link 1 and link 4 is constrained by the four-bar generated by links $1-2-4$

It is an interesting future research topic how the constraint manifolds for four-bar linkages stapled on top of each other will look. An approach I can suggest for analyze the branching of a six bar is based on an assumption that no four-bar sub-linkage in the assembly is allowed to branch.

Chapter 7

Conclusions

7.1 Kinematic Theory

7.1.1 Constraint Manifolds

The constraint manifold equation of a 2R chain was found to be a second degree homogeneous polynomial equation. Since the constraint manifold equation is a homogeneous equation, if it is satisfied by a planar dual quaternion Q, it is also satisfied by all multiples of the planar dual quaternion $P = tQ$. Therefore the constraint manifold equation can be projected onto the kinematic image space. The constraint manifold equation describes a hyperboloid in the kinematic image space. The hyperboloid can be described by a center line and a radius. For each x-value the constraint manifold forms a circle.

7.1.2 Constraint Curve of a Four Bar Linkage

A four-bar linkage can be constructed by two 2R chains with a common coupler link (Figure 2.3). The constraint curve has to satisfy both constraint manifolds of the two 2R chains. Since a constraint manifold forms a circle for a given x-value, the constraint curve at that x-value (which is the intersection of two hyperboloids) will be the intersection of the two hyperboloids' circles at that x-value. Thus the constraint curve have zero, one or two points for each x-value.

Number of Intersection Points

There is exactly one intersection point at an x-value if the distance between the two center lines is the same as the difference or sum of the radius at the same x-value.

Solving for the constraint Curve

The constraint curve can be found by placing a coordinate system with origo on one of the constraint manifold's center lines, with one coordinate axis pointing at the other constraint manifold's center line. In this coordinate system the intersection points can easily be solved for and the result can be transformed back to an inertial coordinate system.

7.2 Mechanism Branching

Between two x-values (split points) where the constraint curve has only one intersection point, there is either no intersection of the manifolds between those split points, or two intersection points for every x-value in between. Thus the branching structure of a four-bar linkage can be determined by finding the split points and checking the structure for only one point in each possible interval.

The split points divides the kinematic image space into a maximum of four regions, which either includes a constraint curve for all x-values in the region, or does not include a constraint curve at all.

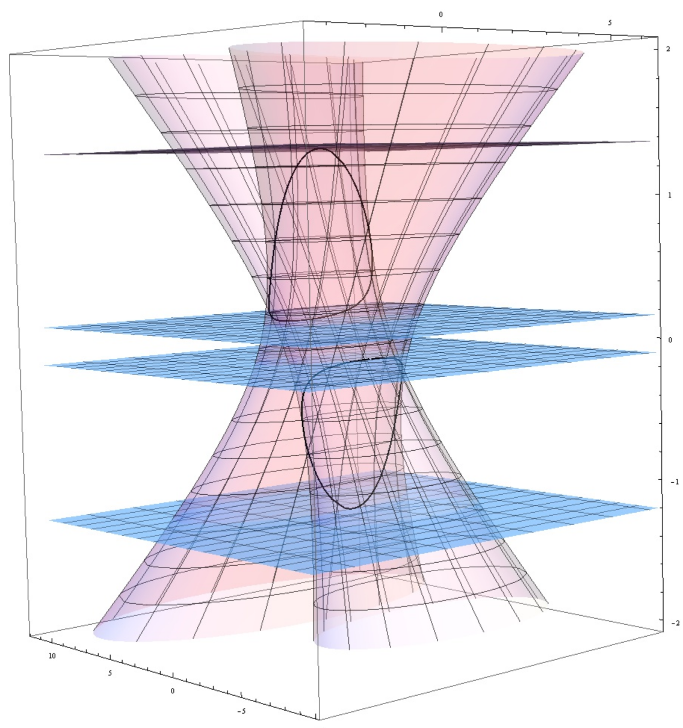


Figure 7.1: The image space is divided into a maximum of four regions which either contains a constraint curve for all x or no constraint curve at all.

It can be determined whether a linkage is useful by checking if all the task positions lie in the same region, if they do, the linkage is useful.

The region below the lowest plane and above the top plane is connected together in infinity and is considered as one and the same region.

7.2.1 The Side of the Constraint Curve

In the case of two infinite curves, the branching is determined by which one of the curves the positions lies on. There can also be situations for closed curves where the side of the constraint curve is of interest. Which side a point lies on can be determined of transforming to a coordinate system located on one center line pointing to the other. The side is determined by the sign of one of the components.

7.3 Extruding all Possible Constraint Curves

By restricting the task specification to only allow four exact task positions, a large set of possible linkages could be found. If one linkage parameter is specified, it can be determined for which values of the parameter the structure of the constraint curve changes. The structure intervals can then be calculated for the linkage parameter. If one linkage in a structure interval is useful, either all linkages in this structure interval are useful, or no linkages in the structure interval are useful. The usefulness of a structure interval can then easily be determined by applying the branching test for any linkage in the interval.

7.4 Modifying a Fifth Task Position

A fifth task position has to lie on one of all possible constraint curves. When all possible constraint curves are calculated, the image point of a fifth task position could be moved along a curve until it intersects with a possible constraint curve. The curve to move along should be a curve on the first crank's constraint manifold, and is preferable determined by some weights constants determining the importance of the rotation verses translation of the task position.

7.5 Future Work and Research

It is still not well known which linkage parameter that is best to vary while finding the structure intervals. g_y was used in this thesis but other parameters might give higher accuracy. The next step would be to calculate all possible constraint curves without holding one crank fixed. Also a lot of computer programming has to be done to provide a robust algorithm. The suggested strategy of specifying zones or points to avoid by the mechanism could be implemented. The next interesting problem is to synthesize multiple loop linkages. If the synthesis starts with specifying one chain and then constraining the chain with four-bar linkages, then the branching of the four bars could be analyzed for different dimension of the first chain. The goal would be to find all dimensions for which the first four-bar is not branching, then proceed to the second four bar and find the dimensions for which the second is not branching either.

Appendix A

Dual Quaternions

This chapter contains my notes from learning and understanding dual numbers and dual quaternions. There are many papers written about quaternion and dual quaternions, however I found most of them very confusing and complicated. To understand the dual quaternions I found it useful to start with the most basic building blocks and build up the understanding from there on.

A.1 Quaternions

The quaternions are mainly used for representing rotations, however they are at the same time a mathematical object which only represents a valid rotation. It is still of importance to start with the fundamental mathematics.

The quaternions will later on be considered as four-dimensional vectors. However the fundamental theory is based on the quaternions being scalar numbers in the same as a complex number is a scalar number even though it has two dimensions and could be represented as a vector. The quaternions are four dimensional hypercomplex numbers.

A.1.1 The Imaginary Dimensions

In the same way the imaginary unit i is introduced by the definition $i^2 = -1$, two new imaginary dimensions will be introduced. Hamilton defined[8] the relationship between the products of the imaginary units to follow the rule:

$$i^2 = j^2 = k^2 = -1 \tag{A.1}$$

$$ij = k, jk = i, ki = j \tag{A.2}$$

$$ji = -k, kj = -i, ik = -j \tag{A.3}$$

An arbitary quaternion can be expressed as:

$$Q = q_0 + q_1 i + q_2 j + q_3 k \tag{A.4}$$

or in a vector form:

$$Q = (q_0, q_1, q_2, q_3) = (s, \vec{v}) \tag{A.5}$$

where:

$$\vec{v} = (q_1, q_2, q_3) \tag{A.6}$$

Some publications have the real part as the fourth element when represented as a vector.

Calculation laws

The calculation rules for quaternions follow by the standard calculation rules together with definition (A.1). The quaternion product expressed in a vector notation is then:

$$Q_1 \circ Q_2 = (s_1 s_2 - \vec{v}_1 \cdot \vec{v}_2, \vec{v}_1 s_2 + s_1 \vec{v}_2 + \vec{v}_1 \times \vec{v}_2) \tag{A.7}$$

The conjugate of a quaternion is (like the conjugate for a complex number) the imaginary parts inverted:

$$Q^* = (q_0, -q_1, -q_2, -q_3) = q_0 - q_1 i - q_2 j - q_3 k \tag{A.8}$$

From (A.7), it follows that

$$Q \circ Q^* = \left(q_4^2 - \vec{v} \cdot (-\vec{v}, \vec{v} q_4 + q_4(-\vec{v}) + \vec{v} \times \vec{-v})\right) = (0, \|Q\|^2) \tag{A.9}$$

where $\|Q\|$ is the euclidean norm of Q. All quaternions multiplied with it is conjugate is a real number. If $\|Q\| = 1$ the quaternion is a unit quaternion.

Application

A quaternion can be used for representing a three-dimensional rotation [17].

A.2 Dual Numbers

A dual number is a number involving the dual unit and consists of a real and a dual part. It is similar to the complex numbers except that the dual unit ε has the property

$$\varepsilon^2 = 0 \tag{A.10}$$

The dual number also follows the commutative law. Let $z_1 = a_1 + b_1\varepsilon$ and $z_2 = a_2 + b_2\varepsilon$ be two dual numbers, and λ be a scalar. The following calculation rules follows from (A.10)

$$z_1 + z_2 = a_1 + a_2 + (b_1 + b_2)\varepsilon \tag{A.11}$$

$$\lambda z = \lambda a + \lambda b \varepsilon \tag{A.12}$$

$$z_1 z_2 = a_1 a_2 + (a_1 b_2 + a_2 b_1)\varepsilon + 0 \tag{A.13}$$

Functions of dual numbers

To work with dual quaternions, we need to know about some functions of dual numbers. The fundamental one is the exponential function. Taking the exponential of a pure dual number and expanding in a power series we get:

$$e^{b\varepsilon} = 1 + (b\varepsilon) + \frac{(b\varepsilon)^2}{2!} + \frac{(b\varepsilon)^3}{3!} + \cdots \tag{A.14}$$

By definition in (A.10), all terms of power two or higher is zero, and we obtain:

$$e^{b\varepsilon} = 1 + b\varepsilon \tag{A.15}$$

The exponential of an arbitrary dual number is

$$e^{a+b\varepsilon} = e^a e^{b\varepsilon} = e^a(1 + b\varepsilon) \tag{A.16}$$

The trigonometric functions of a dual number can be derived from the property

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{A.17}$$

Together with (A.16) we can write

$$e^{i(a+b\varepsilon)} = (\cos(a) + i\sin(a))(1 + bi\varepsilon) \tag{A.18}$$

The dual valued functions of cos and sin can then be obtained from the real and imaginary part

$$\cos(a + b\varepsilon) = Re(e^{i(a+b\varepsilon)}) = \cos(a) - b\varepsilon\sin(a) \tag{A.19}$$

$$\sin(a + b\varepsilon) = Im(e^{i(a+b\varepsilon)}) = \sin(a) + b\varepsilon\cos(a) \tag{A.20}$$

A.3 Screw Displacement

An arbitrary three-dimensional displacement can be described by a screw. The screw consist of a unit screw axis, an angle and a distance. The displacement corresponding to a screw with screw axis S, angle θ and distance d is obtained by rotating the initial frame around the axis S by the angle θ at the same time as the frame is translated along the axis by the distance d. We shall note that the screw axis does not have to go through the origin. Thus it cannot be represented by a single vector. Instead, the axis is represented by the plucker coordinates for its line.

Plucker coordinates

A line specified by one arbitrary point C on the line and a direction vector $\vec{S}$ is transformed to plucker coordinates by:

$$L = (\vec{S}, \vec{C} \times \vec{S}) \tag{A.21}$$

The vector $\vec{C} \times \vec{S}$ is a vector which is normal to a plane that contains the line and it has the length of the shortest distance from origo to the line.

A.4 Dual Quaternions

To represent an arbitrary three-dimensional displacement we can use a dual quaternion.

Such a displacement can be described by a rotation around the unit axes S by an angle θ, and a translation along vector $D = (x, y, z, 0) = xi + yj + zk$. The dual quaternion for this displacement is

$$\hat{S} = S + \frac{\varepsilon}{2} D \circ S \tag{A.22}$$

Given a dual quaternion $\hat{S} = A + \varepsilon B$ where A and B are quaternions. With use of (A.9), the translation vector can be found by:

$$D = 2B \circ A^* \tag{A.23}$$

Screw to quaternion

If the screw axis S intersects with origo, and the angle around S is θ and displacement along S is d, then the dual dumber for the screw is $\hat{\Theta} = \theta + \varepsilon d$. A dual quaternion describing the screw is:

$$\hat{S} = \cos(\frac{\hat{\Theta}}{2}) + S \sin(\frac{\hat{\Theta}}{2}) \tag{A.24}$$

The angle $\hat{\Theta}$ is a dual angle. We use (A.19) and (A.20) to expand this to:

$$\hat{S} = S \sin(\frac{\theta}{2}) + \cos\frac{\theta}{2} + \frac{d\varepsilon}{2}\left(S \cos(\frac{\theta}{2}) - \sin(\frac{\theta}{2})\right) \tag{A.25}$$

In case of a screw axis not intersecting origo, the vector S has to be the plucker coordinates for the screw axis. This is the case for the screw axis intersecting origo as well but in that case the dual part of the screw axis S is zero. Formula (A.24) is valid in both cases but for an arbitrary screw axis it is expanded into a more complex expression.

Displacement Chain

In many cases, e.g. robot kinematics, we want to apply a number of transformations after eachother to finaly reach the end effector position. By having each transformation represented by the dual quaternions Q_i, the end effector position Q_{TCP} is obtained by:

$$Q_{TCP} = \prod_{i=1}^{n} Q_i \tag{A.26}$$

A.5 Notations

$\cdot$	dot product (euclidean norm inner product)
$\circ$	quaternion multiplication
$\otimes$	dual quaternion multiplication
$\times$	cross product (Vector product)
Q^*	conjugate of quaternion

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