Branching Avoidance in Kinematic Image Space for Linkage Synthesis

Adam Nilsson
Master Student
Department of Mathematics
Lund University
Email: adam.nilsson.49@student.lth.se

Finite position synthesis of four-bar linkages is a problem representing issues or designing components in mechanism, which may be found many types of machines, including cars as well as airplanes. The branching defect is defined as a situation where all specified task positions is not connected by a continuous motion for the linkage. The branching defect is the largest limitation for designing useful linkages with the up to day finite position synthesis method. The problem is investigated using kinematic theory based on dual quaternions. We derive the workspace of a four-bar linkage to be the intersection of two hyperboloids in the kinematic image space. This is used to develop an efficient algorithm for determining branching of a linkage. That also gives an explicit solution to the end effector trajectory. To find non-branching linkages, a new synthesis method was developed where one task position is given a lower priority. Useful linkages can then be obtained from a range of values for a constrained dimension of the linkage, by using the new branching analysis method, by synthesizing linkages with various values of one parameter of the linkage. Finally a method to find the closest useful linkage to the last task position is presented.

1 Introduction

Four-bar linkages can be found as components in many machines including cars or automation manipulators. They are useful when something is required to move along a one degree-of-freedom (DoF) trajectory with a minimal number of actuators.

We consider synthesis of planar four-bar linkages from a finite set of task positions. A task position is a translation (x, y) together with a rotation angle θ, which the mechanism has to move through during its task.

A four-bar linkage consists of two cranks, see Fig. 1. Each crank of a four-bar linkage is described by five variables, which are the position of the fixed and the moving pivot and the length of the crank. Five variables implies that up to five task positions can be specified. The synthesis is based on solving the design equations which are derived from the length of a crank. We use the notations from Fig.

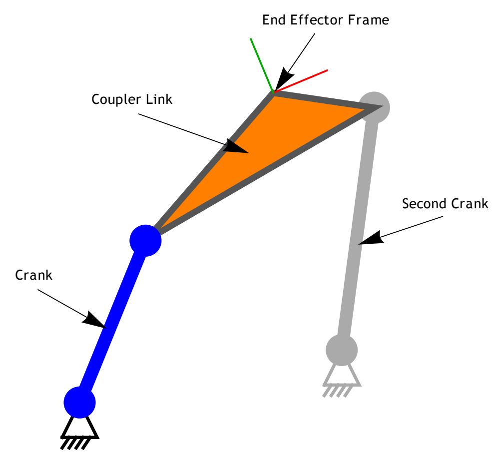

Fig. 1. A four-bar linkage can be considered as two 2R chains with the coupler link in common. The fourth link is the base link which in many cases is the same a the ground.

2, where $W_i \in \mathbb{R}^{3\times 3}$ is a homogeneous transformation matrix representing task position number i. The design equation then yeild

$$l^2 = \left[W_i \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} - \begin{pmatrix} g_x \\ g_y \\ 1 \end{pmatrix} \right]^T \left[W_i \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} - \begin{pmatrix} g_x \\ g_y \\ 1 \end{pmatrix} \right] \quad (1)$$

A solution (l, p_x, p_y, g_x, g_y) to Eqn. 1 guarantees that the crank can reach the task positions W_i. When two cranks are connected together as a four-bar linkage, that linkage can still be positioned at all task positions. However there may not be a continuous motion between all of them any more, the links may need to be decoupled to move from one task position to another. The case where there is not a continuous motion between the task positions is known as the branching defect, which is illustrated in Fig. 3. The up to date strategy to avoid branching linkages has been to randomly choose new task

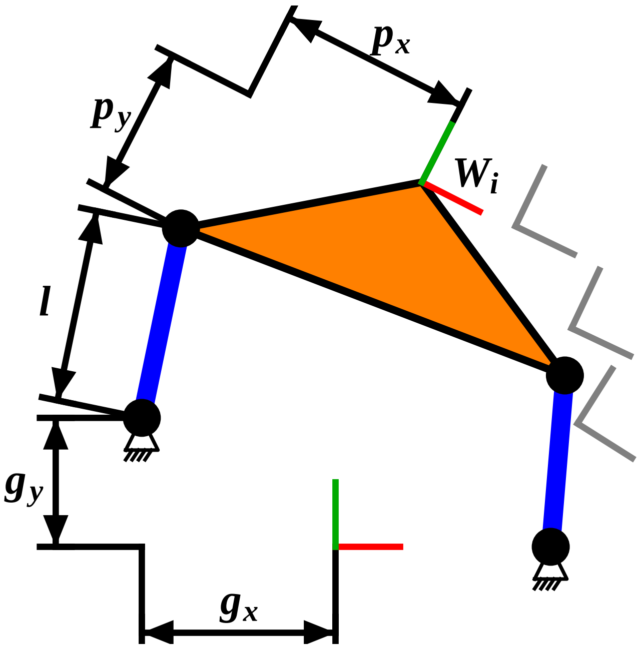

Fig. 2. A four-bar linkage is described with the linkage parameters for each crank. The figure shows the annotations for the left crank.

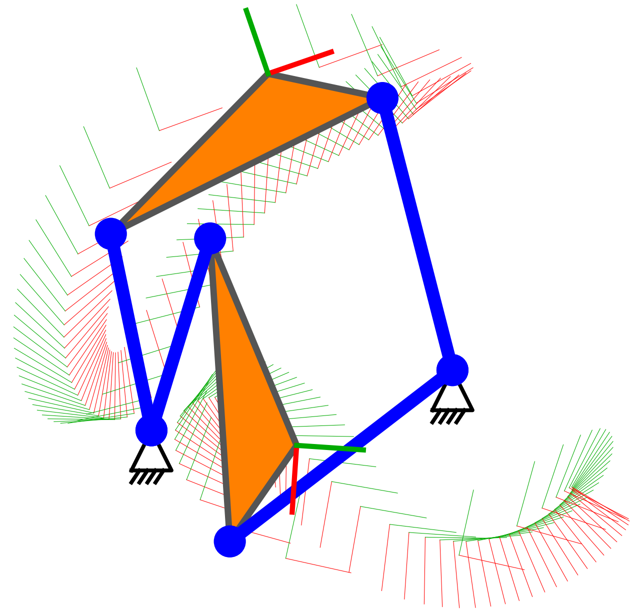

Fig. 3. Illustration of the branching defect of a four-bar linkage. The linkage can be assembled in two different configurations. Each configuration has its own continuous end effector trajectory.

positions close to the initial specified set, and synthesis linkages until a non-branching linkage is found.

Contribution We develop a method based on kinematic analysis in *the kinematic image space* to determine the branching conditions of a four-bar linkage in an efficient way. The kinematic image space is a representation of planar displacements and was introduced by [1] in 1911. It is based on the same fundamental theory as planar dual quaternions which is a restriction of dual quaternions. We use our new branching determination method together with the traditional finite-position synthesis from [2] to calculate all four-bar linkages which are non branching for four out of five task positions. Finally we provide a method to select the optimal four-bar linkage with respect to a fifth task position.

1.1 Linkage Meachanisms

We start with defining some important terminology.

Definition 1. *A **2R chain** is a mechanism consisting of two links where: the first link is connected to ground by a revolution joint, the two links are connected to each other by a revolution joint and an end effector frame is fixed on the second link.*

Definition 2. *A **four-bar linkage** is a mechanism consisting of four links connected to each other by a total of four revolution joints. Each link is connected by two of the joints. An end effector frame is fixed on the coupler link which is the link opposite to the ground link.*

Definition 3. *The links connecting the coupler link and the ground link are called the **cranks** of the four-bar linkage.*

1.2 Dual Quaternions

Our kinematic analysis is based on planar dual quaternions which is a planar restriction of dual quaternions. Dual quaternions are dual numbers of quaternions which were invented by Hamilton [3] in 1844, as an extension to the complex numbers into a four-dimensional hyper-complex number with three imaginary dimensions and one real dimension. Quaternions can be used to describe a rotation in a three dimensional space [4]. To represent a displacement we need to use a dual quaternion which has a total of eight components. When considering planar geometry only four non-zero components of the dual quaternion remains.

Definition 4. *The four non-zero elements of a dual quaternion describing a planar displacement is called a **planar dual quaternion**.*

Dual quaternions as well as planar dual quaternions can be applied after each other using dual quaternion multiplication. For multiplication of planar dual quaternions we use the symbol $\otimes$.

2 Constraint Manifolds

Considering a 2R chain, which is a part of a four-bar linkage. An end effector frame W is fixed to the second link. The planar dual quaternion of the end effector frame is given by a function of the two joint angles ϕ and θ from Fig. 4. Let $T(x,y)$ be a planar dual quaternion for a translation (x,y), and let $R(\theta)$ be a planar dual quaternion for a rotation of angle θ.

Definition 5. *A constraint manifold for a linkage mechanism is the set of reachable positions for the end effector.*

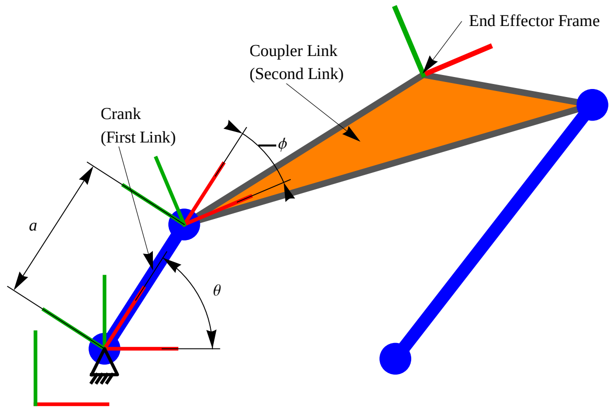

Fig. 4. The end effector frame is obtained by multiplying a sequence of primitive planar dual quaternions. The end effector fram is a function of the two joint parameters θ and ϕ.

The constraint manifold equation for a simplified case of a 2R chain is derived in [5]. We make a more general derivation and obtain the planar dual quaternion of the end effector frame W by multiplications of the sequence of planar dual quaternion given by Eqn. 2.

$$E(\theta,\phi) = T(g_x,g_y)\otimes R(\theta)\otimes T(a,0)\otimes R(\phi)\otimes T(p_x,p_y) \quad (2)$$

Equation 2 is expanded into Eqn. 3 which can be seen as a parameterization of the constraint manifold.

$$E(\theta,\phi) = \begin{pmatrix} \cos\left(\frac{\theta+\phi}{2}\right) \\ \sin\left(\frac{\theta+\phi}{2}\right) \\ \frac{1}{2}\left(a\cos\left(\frac{\theta-\phi}{2}\right)+(g_x+p_x)\cos\left(\frac{\theta+\phi}{2}\right)-p_y\sin\left(\frac{\theta+\phi}{2}\right)\right) \\ \frac{1}{2}\left(a\sin\left(\frac{\theta-\phi}{2}\right)+(p_x+g_x)\sin\left(\frac{\theta+\phi}{2}\right)+p_y\cos\left(\frac{\theta+\phi}{2}\right)\right) \end{pmatrix} \quad (3)$$

Parameterizations are efficient in forward kinematics since it gives the end effector explicit from the joint values, but less efficient to check validity of positions. We want to derive an implicit constraint manifold equation where the parameters θ and ϕ are eliminated. Such equation could be used to check if an arbitrary position is reachable by checking if the position satisfies the equation. In a general case, we can say that the end effector position is specified by the four components of the planar dual quaternion, $Q(\theta,\phi) = (q_1,q_2,q_3,q_4)^T$. To eliminate the joint values in the equations, we observe that the joint values only occur in the four trigonometric functions. We substitute $c_1 = \cos\left(\frac{\theta+\phi}{2}\right)$, $c_2 = \sin\left(\frac{\theta+\phi}{2}\right)$,$c_3 = \cos\left(\frac{\theta-\phi}{2}\right)$and $c_4 = \sin\left(\frac{\theta-\phi}{2}\right)$ into Eqn. 3 and obtain the following:

$$\begin{cases} q_1 = c_1 \\ q_2 = c_2 \\ q_3 = \frac{1}{2}(ac_3+(g_x+p_x)c_1-p_yc_2) \\ q_4 = \frac{1}{2}(ac_4+(p_x+g_x)c_2+p_yc_1) \end{cases} \quad (4)$$

From the determinant of the system, we find it non-singular for $a \neq 0$. Since $a > 0$ holds (all links have non-zero length), we can always solve for the variables $\{c_i\}$ in terms of $\{q_i\}$.

From the trigonometric unity we get $c_1^2+c_2^2 = c_3^2+c_4^2 = 1$, which is equivalent to:

$$c_1^2+c_2^2 = 1 \quad (5)$$
$$c_1^2+c_2^2-c_3^2-c_4^2 = 0 \quad (6)$$

Definition 6. *The constraint manifold equation for a 2R chain is defined as* $c_1^2+c_2^2-c_3^2-c_4^2 = 0$.

By substituting $\{c_i\}$ with their values in terms of $\{q_i\}$ Eqn. 6 gives a homogeneous polynomial equation in q with each term of second degree. The first equation only gives $q_1^2+q_2^2 = 1$ which is a constraint that can easily be satisfied by scaling.

In order to visualize and reason about the constraint manifold we project it to the kinematic image space. The projection is given by the projection operator Eqn.7.

$$P_S(q_1,q_2,q_3,q_4) = \frac{(q_1,q_2,q_3,q_4)}{q_1} = \left(1,\frac{q_2}{q_1},\frac{q_3}{q_1},\frac{q_4}{q_1}\right) = (1,s_1,s_2,s_3) \quad (7)$$

We can project back to the space of planar dual quaternions with the inverse projection operator Eqn.8.

$$P_S^{-1}(s_1,s_2,s_3) = \frac{1}{1+s_1^2}(1,s_1,s_2,s_3) \quad (8)$$

Lemma 7. *If the constraint manifold equation is satisfied by a planar dual quaternion* $Q = (q_1,q_2,q_3,q_4)$*, then the projection of* Q *onto the kinematic image space also satisfies the constraint manifold equation.*

Proof. The constraint manifold equation can be written as:

$$Q^TMQ = 0 \quad (9)$$

The projection of Q onto the kinematic image space is $P_S(Q) = \frac{Q}{q_1}$. By substituting Q by $P_S(Q) = \frac{Q}{q_1}$ into the left hand side of Eqn. 9 we get:

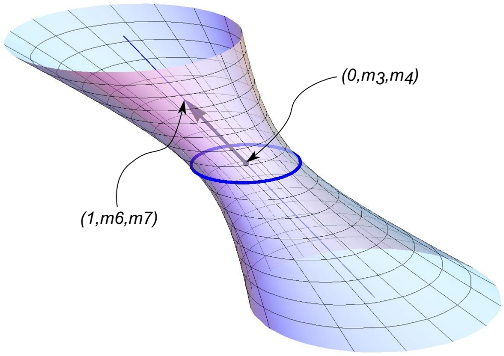

Fig. 5. The constraint manifold is characterized by a centerline and a radius

$$\frac{Q^T}{q_1} M \frac{Q}{q_1} = \frac{1}{q_1^2} Q^T M Q = 0$$

The constraint manifold equation in the kinematic image space is a hyperboloid with the centerline and radius given by:

$$\text{centerline}(x) = \frac{1}{2} \begin{bmatrix} 2x \\ g_x - p_x + (g_y + p_y)x \\ g_y - p_y - (g_x, + p_x)x \end{bmatrix} \tag{10}$$

$$\text{radius}(x) = \frac{1}{a}\sqrt{1 + x^2} \tag{11}$$

3 Branching Analysis

Now when the shape of the constraint manifolds are known, we can start to think about how this knowledge can be used for branching analysis.

We consider a four-bar linkage as two 2R chains (from now referred to as the cranks) connected together with a common coupler link (Fig. 3). The end effector frame for the four-bar linkage is the same as the end effector frames for the two cranks. Each crank restrict the end effector position to its constraint manifold, which implies that a four-bar linkage is restricted to the intersection of the two 2R chains' constraint manifolds. We define the intersection curve as the *constraint curve*. A non branching linkage must then have all task positions on the same curve.

Let C_1 and C_2 be two constraint manifolds. Consider a plane in the kinematic image space for a fixed $x = s_1$ coordinate. We ask whether the constraint curve has any points in this plane. From Eqn.10-11 we know that the shape of a constraint manifolds in the plane is a circle. Two circles intersect if the center distance is between the sum and the difference of the radii, which gives Eqn.12.

$$d(x)^2 \in \left[(r_1(x) - r_2(x))^2, (r_1(x) + r_2(x))^2 \right] \tag{12}$$

From Eqn.10 it follows that the squared distance $d(x)^2$ between two constraint manifolds is given by a quadratic polynomial, and from Eqn.11 it follows that the square of the sum or difference of the radiuses gives a quadratic expression well.

The regions of x which satisfies Eqn.12 is obtained by solving the inequality equations Eqn.13-14.

$$d(x)^2 \geq (r_1(x) - r_2(x))^2 \tag{13}$$

$$d(x)^2 \leq (r_1(x) + r_2(x))^2 \tag{14}$$

The right-hand sides of Eqn. 13 and Eqn. 14 are continuous functions since they are quadratic polynomial and we can therefore solve the inequalities by solving for the x-values where equality prevails and evaluate for any point in between to determine if the inequality equation is satisfied or not.

The equality cases of Eqn.13 and Eqn.14 gives two x-values each, which may be real or complex. Since both equations have only real coefficients, either both x-values are real or both complex. This tells us that we have either zero, 2 or 4 real solutions in total.

From that we can find five different branching cases for the four-bar linkage, which are illustrated in Figs.6-10. A similar analysis was done in [6] and [7] but only the cases in Fig. 6, Fig. 9 and Fig. 10 were presented.

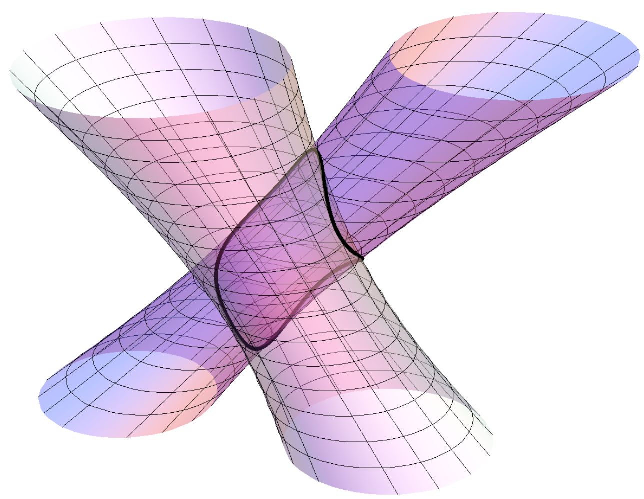

Fig. 6. One single branch of one closed constraint curve.

4 The Four-Bar Constraint Curve

In some cases the whole motion of the four-bar and not only the branching is of interest. To find an explicit expres-

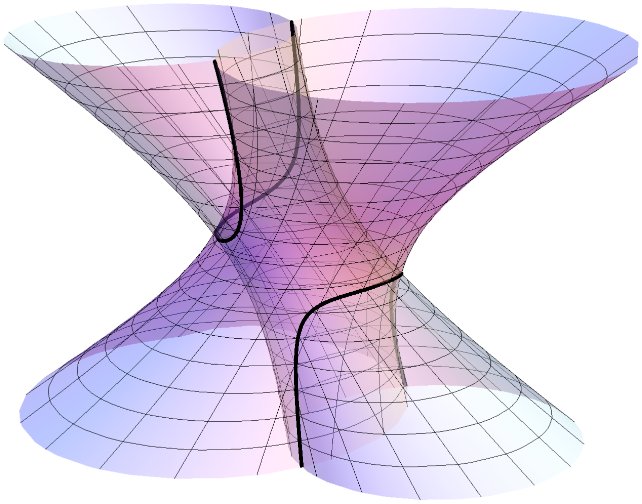

Fig. 7. One single branch of two half open constraint curves.

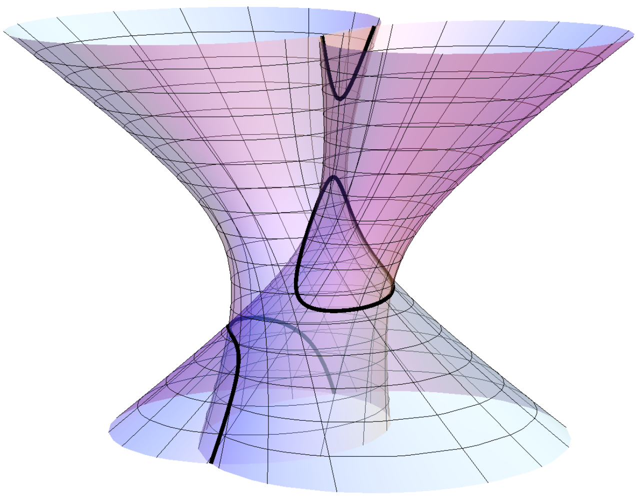

Fig. 8. Two branchse of one closed constraintcurve and two half open constraint curves.

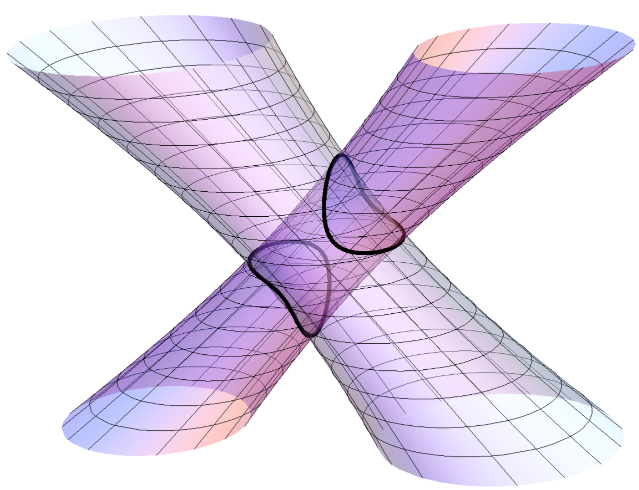

Fig. 9. Two branches of two closed constraint curves.

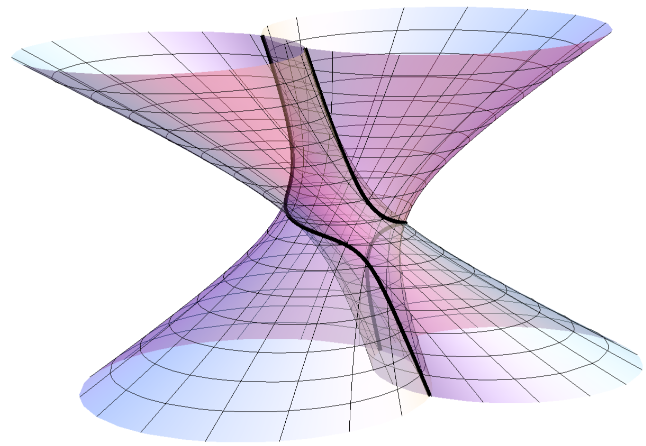

Fig. 10. Two branches of two independent infinite constraintcurves.

sion for the four-bar constraint curve, we introduce a new coordinate system $L(x)$ (see Fig. 11) with:

origin on the centerline of C_1 at position x
e_x aligned with the centerline of C_1
e_y and e_z are in a plane of constant x
e_y pointing towards the centerline of C_2
e_z is orthogonal to e_y

The four-bar constraint curve in coordinate system $L(x)$ can be obtained by solving for the intersection of two circles at a distance $d(x) = \|\text{centerline}_2(x) - \text{centerline}_1(x)\|$ from each other and with radius $R(x)$ and $r(x)$ given by Eqn. 11. The equations for the intersection in coordinate system $L(x)$ is given by Eqn. 15.

$$\begin{cases} y^2 + z^2 &= R(x)^2 \\ (y - d(x))^2 + z^2 &= r(x)^2 \end{cases} \tag{15}$$

thus the four-bar constraint curve is given by:

$$\begin{aligned} y(x) &= \frac{R(x)^2 - r(x)^2 + d(x)^2}{2d(x)} \\ z(x) &= \pm\sqrt{R(x)^2 - y(x)^2} \end{aligned} \tag{16}$$

A parameterized constraint curve in a global coordinate system can then be calculated by forming the homogeneous transformation matrix for coordinate system $L(x)$ and multiplying with the coordinates in the local coordinate system. This gives a very long expression which we do not write out here, but it can easily be used by a computer.

5 Four-Point Synthesis of Linkages

We consider linkage synthesis from five task positions. Instead of specifying a tolerance zone and randomize the task positions in a tolerance zone, which has been the up to day

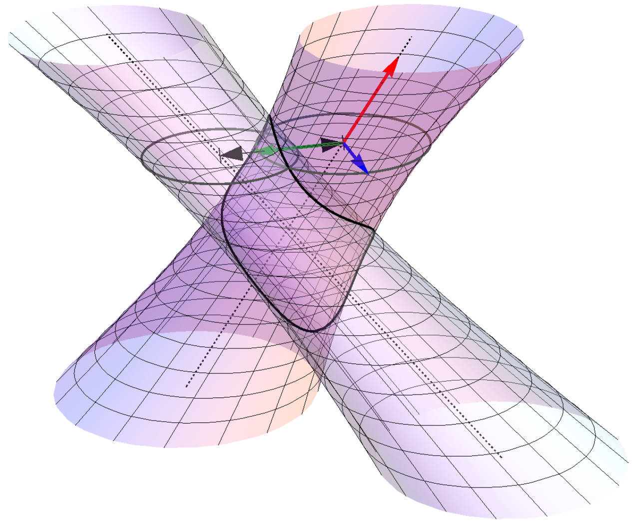

Fig. 11. Coordinate system $L(x)$.

strategy, we give a higher priority to task position number one to four and we are looking for a minimal modification to the fifth task position to get a useful linkage.

Problem 8. *Given five task positions $W_1, \ldots, W_5$ reachable by the first crank $\mathcal{M}_1$, we synthesis a four-bar linkage $(\mathcal{M}_1, \mathcal{M}_2)$ which moves exactly through the first four task positions $W_1, \ldots, W_4$ and as close as possible to the fifth task position W_5.*

The design equations (Eqn. 1) have five unknown variables and therefore up to five task positions may be specified. If we constraint one variable by assigning it with a value, then we may only specify four task positions. If we were only to specify four task positions it would give us the freedom to select one of the linkage variables. For a given value of a linkage variable we can determine whether that value gives a non-branching linkage in an efficient way using our new branching analysis algorithm.

We now suggest a new synthesis strategy which we choose to call four point synthesis:

1. Find the set of useful values of a linkage variable for which the linkage synthesized from the first four task positions is non-branching.
2. Find the reachable region for any useful values of the linkage variable.
3. Select the value of the constrained variable which is reaching closest to the fifth task position.

5.1 Continuous reachable region

We want to determine intervals for the constrained linkage parameter for which the synthesized linkage is non-branching. Such approach is only valid if the set of constraint curves associated with a continuous set of values of the constrained variable is a continuous region as well.

From an infinitesimal change of one linkage variable the constraint manifold will change infinitesimally. Let $\mathcal{M}$ be a constraint manifold. Let $\mathbf{p}$ be the vector of the unknown linkage parameters for the synthesis strategy to solve for and let t be a specified linkage parameter.

Lemma 9. *The constraint manifold for a crank synthesized from four task positions $W_1, \ldots, W_4$ and one specified linkage parameter t is continuous with respect to t.*

Proof. Let the design equation be Eqn. 17 and $(\mathbf{p_0}, t_0)$ be a solution to Eqn 17.

$$F(\mathbf{p}, t) = 0 \tag{17}$$

For an arbitrary t for which Eqn. 17 has at least one solution, there is at least one corresponding value of $\mathbf{p}$.

Let $\mathbf{p}$ and t be functions of a parameter ξ, such that $t(\xi) = t_0 + \xi$ and $(\mathbf{p}(\xi), t(\xi))$ is a solution to Eqn. 17. We prove the existence of such function $\mathbf{p}(\xi)$ by differentiating the function F with respect to ξ.

Since Eqn. 17 is already satisfied for $(\mathbf{p}(0), t(0))$ the derivative of F with respect to ξ is required to be zero.

$$\frac{\partial F}{\partial \xi} = \frac{\partial F}{\partial \mathbf{p}} \frac{\partial \mathbf{p}}{\partial \xi} + \frac{\partial F}{\partial t} \frac{\partial t}{\partial \xi} = 0 \tag{18}$$

Equation 18 can be guaranteed if the jacobian matrix of F for the variables $\mathbf{p}$ span the derivative of F with respect to t.

$$\frac{\partial F}{\partial t}(\mathbf{p_0}, t_0) \in \text{Span}\left[\frac{\partial F}{\partial \mathbf{p}}(\mathbf{p_0}, t_0)\right]$$

Which proves that the constraint manifold is continuous with respect to a linkage parameter t.

5.2 The closest curve to a point

To find the constraint curve to a point, we need to define a notation of distance on a constraint manifold which is not trivial since the dimensions are combinations of translation and rotation. Let S_5 be the image point of the fifth task postion.

An arbitrary image point on the surface of a constraint manifold can be parameterized by:

$$S(\psi_1, \psi_2) = \text{center}(\psi_1) + \begin{bmatrix} 0 \\ \cos(\psi_2) \\ \sin(w\psi_2) \end{bmatrix} \text{radius}(\psi_1) \tag{19}$$

We consider the shortest distance to be along a curve defined by the weight w and:

$$\begin{cases} \psi_1(\delta) & = \delta + \alpha_0 \\ \psi_2(\delta) & = w\delta + \beta_0 \end{cases}$$

To find the distance to the constraint curve defined by the constraint manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$, we insert $S_5(\delta)$ into the constraint manifold equation of $\mathcal{M}_2$ and solve for δ. The constraint manifold equation of $\mathcal{M}_1$ is automatically satisfied by the structure of Eqn. 19.

We already know that the constraint curves between two structure points form a continuous surface. If we also assume the surface to be bounded by the constraint curves at the structure points, then we only need to evaluate the distance to the constraint curves at the separation points.

6 Conclusions

We derived kinematic relations from a four-bar linkages in the physical dimensions to the kinematic image space and used it to develop a strategy to analyze and determine branching of a four-bar linkage. An important concept was the constraint manifolds of 2R chains which we discovered took the form of a hyperboloid with centerlines and radius which we found simple explicit equations for. We also developed a new branching analysis strategy which can, in an almost explicit way, tell if a four-bar linkage is branching or not. The constraint curve of a four-bar linkage is the intersection of two constraint manifolds of two 2R chains. We derived an explicit expression for the constraint curve through a coordinate transformation.

We considered a modified synthesis problem where the first four task positions was given a higher priority than the fifth one. An algorithm to calculate all useful four-bar linkages reaching the first four task positions was developed together with a method to determine which of all useful four-bar linkages reaches closest to the fifth task position.

7 Discussions and Future Research

Synthesis of four-bar linkages is a central component of synthesizing linkages of higher complexity, since those can be considered as multiple connected four-bar linkages. Synthesis of a six-bar linkage involves synthesis of two four-bar linkages which both must be non-branching for the six-bar to be non branching. An interesting problem is for the first four-bar sub-linkage, synthesis a set of useful linkages and then do the same for the second sub-linkage to finally find the optimal combination.

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