

MSc Finance Thesis

# Barrier Quanto Options in Energy Markets

Author: Adriano Tosi

Supervisors: Karl Larsson and Rikard Green

Lund University

Department of Economics

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# Abstract

Weather derivatives have increased their relevance in energy markets throughout the last years. Among them, quanto options are one of the fairly new contracts that are traded, over the counter, to manage price and volume risk. Their research literature is scarce and few papers have been published so far. The purpose of the thesis is to slightly improve the current quanto options literature, from a theoretical perspective. The thesis has a presentation and discussion of quanto contract features. Then, barrier and parisian quanto options are presented as possible modifications of the basic quanto contract structure. Economic motivations of the latter issues are treated from a price and risk perspective. Then, a put style quanto option closed form pricing formula is derived as an extension of Benth et al. (2012) work [2]. After that, Monte Carlo simulation is used to price barrier and parisian quanto options in a theoretical framework.

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# Chapter 1

## Introduction

### 1.1 Weather Derivatives

The first weather risk management deal was an over the counter (OTC) transaction made in 1997. Since that moment, the instruments used in such types of deals have been defined as weather derivatives. Since then, they have been widely used and their popularity has been increasing throughout the years. As an example, the Chicago Mercantile Exchange (CME) organizes weather futures and options on futures as tradable assets; i.e. weather derivatives are traded on regular market, nowadays, and not only OTC. Another relevant market is the Nordic Power Exchange, which is specialized on energy commodities. The Weather Risk Management Association (WRMA) provides an extensive presentation of the weather derivatives markets on its website.

The traditional weather instruments provided by CME or used in OTC deals, do not have a comprehensive hedge, due to the fact that they mainly focus on volumetric risk. A possible alternative to this imperfect risk management instruments is a Quanto option. It is a fairly new type of weather-energy hybrid derivative, which is mainly traded OTC. This type of derivative is a better risk management tool than a traditional one, e.g. weather futures, because it focuses on volumetric risk and price risk simultaneously.

This Quanto option feature will be easily seen in the presentation of its payoffs function in chapter 2. Furthermore, Quanto option superiority, as risk management tool, is also strongly supported by more well-known papers as Benth et al. (2012) and Caporin et al. (2012), respectively [2] and [4].

Constructing portfolios of basic derivatives instruments, e.g. plain vanilla options and futures, to reach similar hedging strategies of a Quanto option is feasible but not optimal. Quanto option is cheaper and more efficient than a portfolio of plain vanilla options written on the same underlying variables, as Caporin et al. (2012) stated in their paper [4].

Pricing Quanto option is still an open discussion. On the one hand, complex data intensive econometric models are used to simulate possible payoffs scenarios through Monte Carlo, as Caporin et al. (2012) did [4]. On the other hand, closed form pricing formula are derived as Benth et al. (2012) did [2]. Even though, the former is less elegant than the latter solution, it allows to price variations of a Quanto option structure, when no closed pricing formula is available.

It is worth pointing out the unlikeness between insurance contracts and derivatives dealing with weather risk. The former treats events which have really small probability and huge financial impact, whereas, the latter faces more likely events but with less financial damages. For instance, a typical weather insurance contract could be referred to a hurricane, while, a weather derivative could be connected to unexpected temperatures or rain. Even though this paragraph seems to be out of place, it is one of the milestones of this thesis. Nevertheless, its treatment will be in chapter 3.

## 1.2 Purpose of the Thesis

The main purpose of the thesis is to slightly improve current quanto option literature, which is fairly scarce nowadays. Concluding the theoretical quanto option pricing model constructed by Benth et al. (2012) [2], i.e. proving a closed form solution of a put quanto option, is the first step to go beyond in the current literature.

Suggesting possible variations of quanto options structure from an economic and mathematical prospective, i.e. barrier and parisian quanto options, are treated. Then, the thesis will focus on pricing barrier and parisian quanto option through Monte Carlo simulation, by adopting the well-known Schwartz-Smith model as underlying stochastic process.

# Chapter 2

## Quanto Options

Quanto options are weather-energy hybrid derivatives, which are used in energy markets to hedge volume and price risk simultaneously. They are based on two underlying variables, one related to an energy commodity, e.g. natural gas, while the other is related to a weather condition, e.g. cold temperature. As it will be presented successively, the underlying variables are indexes and not simple spot prices or physical measures of the previously defined commodity and weather condition, respectively. Furthermore, it is good to note that energy quanto options studied in this thesis should be not confused with currency quanto options, which are used to hedge exchange rate risk exposure.

### 2.1 Quanto Option Payoffs

#### 2.1.1 Call Payoff

In this section there is a presentation of the call quanto option payoff structure with variables definition. Furthermore, there is a discussion of the payoff function to give an insight of it to the reader.

## Call Quanto Option Payoff Function

$$c(E, I) = \gamma \cdot [\max(E - K_E^C, 0) \cdot \max(I - K_I^C, 0)] \quad (2.1)$$

Another way of writing the call quanto option payoff function is:

$$c(E, I) = \gamma \cdot (E - K_E^C) \cdot (I - K_I^C) \cdot 1_{E > K_E^C} \cdot 1_{I > K_I^C} \quad (2.2)$$

## Variables and Parameters Definition:

$\gamma$ : Volume adjustment factor, e.g. MMBtu.

$E$ : Energy commodity index.

$K_E^C$ : Energy commodity index strike price for call quanto option style.

$I$ : Temperature index.

$K_I^C$ : Temperature index strike measure for call quanto option style.

1: Indicator function, which can assume value either 0 or 1 depending on the subscript condition. If  $E > K_E^C$  or  $I > K_I^C$ , then  $1 = 1$

## Energy Commodity Index

$$E = \frac{1}{T - t} \sum_{i=t}^T S_i \quad (2.3)$$

## Variables and Parameters Definition:

$S_i$ : Energy commodity spot price at time  $i$ , e.g. natural gas.

$T$ : Maturity.

$t$ : Start counting point at time  $t$ .

$[t, T]$  : Considered period, e.g. a month

### Temperature Index

$$I = \sum_{i=t}^T \max[\varphi - \psi_i, 0] \quad (2.4)$$

### Variables and Parameters Definition:

$\varphi$ : Constant parameter, which is defined either as 65 degrees Fahrenheit or 18 degrees Celsius in the US or in Europe, respectively.

$\psi_i$ : Daily average temperature during day  $i$ . It is computed by making the average between the maximum and minimum temperature during day  $i$ .

### Comments:

A call quanto payoff function could be interpreted as product of two vanilla call options, whereas, a put quanto payoff function could be considered as the product of two vanilla put options, equations (2.1) and (2.5), respectively.

As it was mentioned previously, quanto options have the purpose of hedging price and volume risk contemporaneously. This can be easily seen if we analyze equation (2.1) by dividing it in two parts.  $\max(E - K_E^C, 0)$  part represents the hedge against price risk, while,  $\gamma \cdot [\max(I - K_I^C, 0)]$  represents the hedge against volume risk. The latter is a quite standard feature in weather derivatives, e.g. options on weather futures, whereas, the former is a feature of traditional financial options. This combination of crossed product allows quanto options to have flexibility to take into account volume and price changes, simultaneously.

On the one hand, equation (2.3) is simply an average of the spot prices over a determinate period of time  $[t, T]$ . On the other hand, equation (2.4)

is a sum of the defined function  $\max[\varphi - \psi_i, 0]$  over the same time period  $[t, T]$ . The latter function is widely used in weather derivative models as underlying variable for temperature conditions, especially in CME.  $\max[\varphi - \psi, 0]$  is a constructed index, which is called Heating Degrees Days (HDD).  $I$ , equation (2.4), is the cumulative spot HDD over the time period  $[t, T]$ .

The energy commodity index and the temperature index, in equations (2.3) and (2.4) respectively, point out the fact that a quanto option is a path depended derivative, e.g. look back options. I.e. a quanto option can be looked as an Asian style option.

### 2.1.2 Put Payoff

As in the previous subsection, here, there is a presentation of the put quanto option payoff structure with variables definition. Nevertheless, there is no further explanation of the payoff function, because the insight of the put payoff function is the same as the call one.

#### Put Quanto Option Payoff Function

$$p(E, I) = \gamma \cdot [\max(K_E^P - E, 0) \cdot \max(K_I^P - I, 0)] \quad (2.5)$$

Another way of writing the put quanto option payoff function is:

$$p(E, I) = \gamma \cdot (K_E^P - E) \cdot (K_I^P - I) \cdot 1_{E < K_E^P} \cdot 1_{I < K_I^P} \quad (2.6)$$

#### Variables and Parameters Defintion:

$K_E^P$ : Energy commodity index strike price for put quanto option style.

$K_I^P$ : Temperature index strike measure for put quanto option style.

1: Indicator function, which can assume value either 0 or 1 depending on the subscript condition. If  $E < K_E^P$  or  $I < K_I^P$ , then  $1 = 1$

## 2.2 Quanto Option Contracts

Quanto options are tailored made contracts and are usually traded OTC. As a result, their features can change a lot among different deals. Nevertheless, there are some characteristics that persist in all quanto option contracts. For instance, they cover several months of a season, e.g. winter, and their strikes and volume adjustment factors can change over time during different months. These features are analyzed in the following subsections from a call and put type of contract prospective. Moreover, in each subsection there is a practical example, which shows how a quanto contract works.

Furthermore, note that in the entire thesis, the quanto options that are considered are European style and not American style, i.e. they can be exercised only at maturity, for instance at the end of each month of a five months contract.

### 2.2.1 Call Contract

It is important to underline the fact that the following discussion of a quanto contract is one of the  $n$  possible different types of contracts that can be tailor made during OTC deals.

Table 2.1: Call Quanto Option Contract Strucure

	November	December	January	February	March
$K_{E,j}^C$	$K_{E,N}^C$	$K_{E,D}^C$	$K_{E,J}^C$	$K_{E,F}^C$	$K_{E,M}^C$
$K_{I,j}^C$	$K_{I,N}^C$	$K_{I,D}^C$	$K_{I,J}^C$	$K_{I,F}^C$	$K_{I,M}^C$
$\gamma_j$	$\gamma_N$	$\gamma_D$	$\gamma_J$	$\gamma_F$	$\gamma_M$

As it can be easily seen in table 2.1, this call quanto contract structure covers 5 months of a winter season. Each month has its own energy commodity index strike,  $K_{E,j}^C$ , temperature index strike,  $K_{I,j}^C$ , and volume adjustment

factor,  $\gamma_j$ , for  $j \in [N, D, J, F, M]$ . It is worth noting that the above parameters can change over time, i.e. they can assume different values depending on the  $j$  month. Usually, the historical coldest months have higher strikes and volume factor, e.g.  $K_{E,M}^C < K_{E,J}^C$ ,  $K_{I,M}^C < K_{I,J}^C$  and  $\gamma_M < \gamma_J$ . This is due to the fact that higher energy commodity index,  $E$ , and higher temperature index,  $I$ , are expected during the coldest months. Note that in this contract structure, it is assumed that the holder of the call quanto option receives a payment each month, if the contract conditions are satisfied. The holder of a quanto contract receives a payment in month  $j$  if  $E_j > K_{E,j}^C$  and  $I_j > K_{I,j}^C$ . The received amount would correspond to  $\gamma_j \cdot (E_j - K_{E,j}^C) \cdot (I_j - K_{I,j}^C)$ .

Note that if  $E_j \leq K_{E,j}^C$  or  $I_j \leq K_{I,j}^C$ , then, it does not imply that the entire quanto contract is worthless  $\forall j$ .

### Energy Consumer Example

If a certain energy consumer, e.g. government institution, has to warm up some facilities, e.g. administrative offices, during winter time, it needs a certain amount of fuel, i.e. natural gas. Usually, such type of institution has agreements with a set of suppliers for natural gas delivery. Nevertheless, these agreements are made at the beginning of the winter season and they are set on expected consumption and expected energy commodity price during the delivery period. Nevertheless, if the temperature turned out to be colder than what was expected to be, the aggregate demand of natural gas would increase and, accordingly, natural gas spot price would rise too.

As a result, the government institution would need to provide more heating energy to its offices, than what was planned, and it would have to buy such energy on the spot market at a really high price. I.e. the government institution would have an unexpected loss equal to the incremental administrative offices demand of gas, times, the difference between the current gas spot price and the gas price that it would have paid if it had had a more comprehensive supply contract. To avoid this kind of loss, a call quanto contract would be an optimal hedge against volume and price risk faced

by the energy consumer. For instance, if the contract had been defined as follow the government institution would not have had any unexpected loss. Assume the payoff function of this contract example being defined as the one in equation (2.1). If  $K_E^C$  was equal to the gas price that this institution would have paid if it had entered in a supply contract during normal market condition and  $K_I^C$  was equal to an expected cumulative HDD over the considered month, the holder of the quanto option would have received a payment equal to  $\gamma \cdot (E - K_E^C) \cdot (I - K_I^C)$  if  $E > K_E^C$  and  $I > K_I^C$ . As a result, the cash inflow provided by this quanto option would have offset the unexpected losses borne by the government institution over the considered month.

### 2.2.2 Put Contract

The majority of the characteristics of a put quanto contract are fairly similar to the call quanto contract described in the previous subsection. For example, the strikes and volume factor can change among different months as it is shown in table 2.2. Nevertheless, the few different features are pointed out in the following lines. In addition, another example is given but this time from an energy producer prospective. It is fair to state that the following energy producer example is similar to the one made by Benth et al. (2012) [2].

Table 2.2: Put Quanto Option Contract Structure

	November	December	January	February	March
$K_{E,j}^P$	$K_{E,N}^P$	$K_{E,D}^P$	$K_{E,J}^P$	$K_{E,F}^P$	$K_{E,M}^P$
$K_{I,j}^P$	$K_{I,N}^P$	$K_{I,D}^P$	$K_{I,J}^P$	$K_{I,F}^P$	$K_{I,M}^P$
$\gamma_j$	$\gamma_N$	$\gamma_D$	$\gamma_J$	$\gamma_F$	$\gamma_M$

The only difference between the put quanto contract and the call quanto contract is the condition that has to occur to enable the quanto option holder to receive a payment. A put quanto contract holder receives a pay-

ment of  $\gamma_j \cdot (K_{E,j}^P - E_j) \cdot (K_{I,j}^P - I_j)$  if  $E_j < K_{E,j}^P$  and  $I_j < K_{I,j}^P$ , for  $j \in [N, D, J, F, M]$ . However, note that if  $E_j \geq K_{E,j}^P$  or  $I_j \geq K_{I,j}^P$ , then, it does not imply that the entire put quanto contract is worthless  $\forall j$ .

### Energy Producer Example

Assume an energy producer of natural gas that has planned its gas production conditional on its rational expectation of future gas consumption and gas price. If it turned out that the temperature is higher than expected, during winter time, the aggregate demand of natural gas for heating would decline and gas price would drop too. The unexpected loss for the producer would be equal to the decreased demand, times, the difference between the gas price that the producer would have sold, if its customers had bought the energy commodity, and current gas spot price at which the energy producer has to sell the excess gas. A put quanto contract would have been able to offset such unexpected loss.

Assume the payoff function in equation (2.5) as benchmark. If  $K_E^P$  was equal to the gas price that the producer would have sold, if its customers had bought the energy commodity in normal market conditions, and  $K_I^P$  was equal to an expected cumulative HDD over the considered month, the holder of the put quanto contract would have received a payment equal to  $\gamma \cdot (K_E^P - E) \cdot (K_I^P - I)$  if  $E < K_E^P$  and  $I < K_I^P$ . As a result, the cash inflow provided by this put quanto option would have offset the unexpected losses borne by the energy producer over the considered month.

# Chapter 3

## Barrier-Parisian Quanto Options

The option structures presented, in the previous chapter, can be defined as standard quanto options. Nonetheless, there is, in the author's opinion, the possibility to modify these instruments to satisfy some specific market participants' needs. Inserting a barrier on one of the underlying drivers of a quanto contract can be a possible modification to reach such goal. Nevertheless, barrier quanto options have a drawback regarding volatility, but, it can be reduced by using parisian barriers.

These issues are broadly treated in the following subsections with a specific focus on economic motivations for why a barrier should be introduced in a quanto contract. After that, barrier and parisian quanto option payoff functions and contract structures are presented. Note that, in this chapter, the exposition of the topics is done only from a call style prospective. This choice is made to avoid cumbersome repetitions during the analysis. However, everything that is said can be easily extended to the put style case with simple adjustments and modifications of what follows.

### 3.1 Economics Motivations

The following part of the thesis is mainly qualitative oriented and it leads the reader through the logical process of why the author of the thesis con-

siders barrier and parisian quanto option modifications interesting from a research prospective.

### 3.1.1 Full insurance

A standard quanto option, with the features explained so far, can be interpreted as a derivative that fully insures the holder from volume and price risk in energy markets. In other words, it can be state that a standard quanto option is a full insurance against the above mentioned risks. Nonetheless, as it was pointed out in the introduction chapter, there is a remarkable difference between traditional insurance and derivatives contracts. On the one hand, the former deal with catastrophic events with low probabilities, i.e. hurricanes and earthquakes. On the other hand, the latter treat unexpected events with higher probabilities and with lower impact than the insurance ones. Accordingly, there is a distinction between unexpected events, which have relative high probabilities and relative small impact, and extreme events, which have low probabilities and huge impact.

The holder (female) of a standard call quanto option has a hybrid instrument that allows her to face both unexpected and extreme events simultaneously. This extraordinary quanto option peculiarity is really attractive and it makes this instrument a remarkable hedge. Nevertheless, if the quanto contract holder wanted to take advantage of only one of these two possible set of events, i.e. unexpected and extreme scenarios, she would need to eliminate one of them in the contract specification. In other words, if she wanted to hedge her volume and price risk from catastrophic scenarios, she would delete the unexpected ones in the contract stipulation. While, if she desired to have a hedge only against the unexpected events, she would need to cut off the extreme ones.

For example, this could be seen as a butcher (male) that has a full insurance divided in life insurance and injure insurance. If the butcher wanted to take advantages of only one of the two sub insurances, he would get rid of the

other one. For instance, in his job the probability of passing away is really low and it is likely that he is not willing to pay the life premium to insure himself against such event that is really unlikely. However, he is willing to pay the injure premium to avoid potential work inabilities due to knife cuts.

At the same way the quanto option holder could be willing to pay a certain amount of money to insure her from unexpected events, but she would not be willing to pay for the extreme ones. Strictly speaking, she wonders if it is worth paying for a temperature that is almost impossible to see, e.g. mines 30 Celsius degrees in New York down town during January. Similar case would be paying a contract that insures her against an energy commodity price that is way above its usual trend, e.g. a price of ten USD, when its mean is five UDS and standard deviation is one USD. Therefore, if she is concerned about volume and price risk for unexpected scenarios, but she does not want any hedge for extreme events, a standard quanto option can be modified to satisfy her needs. Inserting a barrier in the quanto structure is the suggestion provided by the author. This allows to model the quanto option in such a way that she will have an instrument that provides hedge against those specific risks that she is concerned about.

### 3.1.2 Barrier and Parisin

In this subsection there is a qualitative dissertation of how a normal barrier,  $H^B$ , and then, a parisian barrier,  $H^P$ , should be introduced in a standard call quanto option. However, there will be an in depth treatment of barrier and parisian quanto options from a mathematical prospective, in the following sections of this chapter. The author expects intermediate option theory knowledge from the reader, therefore, basic barrier and parisian concepts are supposed to be known. Nevertheless, if the reader needs basic explanations the author advises to consult the following reference [6].

As written in the previous subsection, the holder of a call quanto option that does not want full insurance, i.e. no hedge against extreme cold tem-

peratures and high prices, can choose to insert an up and out barrier,  $H^B$ , on one of the two underlying variables. In other worlds, she can decide to have an up and out barrier call quanto option (barrier quanto option) set on the energy commodity spot price, during the life of the contract. The only modification in the standard quanto option is made by the following condition. If the commodity spot price touches or acrosses  $H^B$ , at least one time during the predefined period, the barrier quanto option will be worthless and the holder will not receive any payoff at maturity. The latter contract specification still has the purpose of hedging volume and price risk, but by excluding extreme prices and temperatures scenarios during the entire life of the contract.

It is really likely that the reader noticed that the barrier quanto option mentioned before excludes extreme events in prices and temperatures, even though  $H^B$  is set only on the commodity spot price. This is due to the fact that the energy commodity prices and temperatures are highly correlated as Benth et al. (2012) and Caporin et al. (2012) underline it in their papers [2] and [4], respectively. Therefore, if the commodity spot price has a high upward trend, it is really likely that temperatures are really cold. As a result, inserting a barrier on one of the two underlying variables implies a double monitoring, even though only one of them knocks out the barrier quanto option.

No rule is set to decide which of the two underlying variables should be chosen as barrier benchmark. However, the author chose to use the commodity price as barrier benchmark, because it is historically a more standard variable to monitor, rather than a temperature one. Nevertheless, depending on different situations, e.g. clients' preferences, the barrier can be set on either of the two underlying variables. Anyhow, in the author's opinion, it would be better to set the barrier on the underlying variable with less volatility. This is due to the fact that in an up and out barrier quanto option, the higher the volatility is, the higher is the probability of being knocked out.

To attenuate the volatility issue in the barrier quanto option, an up and out parisian call quanto option (parisian quanto option) is suggested by the author. The only difference between a parisian quanto option and a barrier quanto option consists in the following condition. The parisian quanto option is knocked out if the barrier benchmark variable is equal or above  $H^P$  for a predefined amount of time. I.e., if the commodity spot price is equal or above  $H^P$  for a certain period of time during the contract life, the option becomes worthless and no payment will be given at maturity. The only effect that the latter contract modification does is making the barrier quanto option less sensible to volatility. In the parisian case the probability of being knocked out decreases compared to the barrier quanto option case. However, the volume and price risks that a parisian quanto option hedges are similar as the ones hedged by the barrier one. Parisian quanto option still excludes extremes events as the barrier quanto option does; nevertheless, the former allows some commodity spot prices outliers, whereas, the latter does not.

### 3.1.3 Impact on Price

So far, no word has been written about the different impact that this modifications have on the standard quanto option price. Even though this issue will be treated in mathematical terms in the following chapters, it is worth facing the issue from a theoretical qualitative prospective too.

By considering the barrier and parisian quanto options described in the previous subsection, it is pretty straight forward to understand that their prices are lower than a standard quanto option one. This is due to the fact that the barriers  $H^B$  and  $H^P$  delete some scenarios of the commodity price in the two different option cases, respectively.

Note that, if  $H^B$  and  $H^P$  are equal, the price of a barrier quanto option is lower than the price of a parisian quanto option, *ceteris paribus*. The barrier quanto option has more probability of being knocked out compared

to the parisian one. For this reason the price reflects the less chances of reaching maturity.

From a price point of view, barrier and parisian quanto options are more attractive than a standard quanto option, because they are cheaper. Nevertheless, by comparing the three different prices, i.e. standard, barrier and parisian quanto option prices, it is worth noting the tradeoff between price and chances of being knocked out. The more changes there are of being knocked out, the lower the quanto option price is.

## 3.2 Barrier-Parisian Quanto Option Payoffs

In this section there is a presentation of the barrier quanto option and parisian quanto option payoff functions. In each subsection there is a list of variables specifications that will become useful during the pricing chapters of the thesis.

### 3.2.1 Barrier Payoff

This is a up and out barrier option style.

#### Barrier Call Quanto Option Payoff Function

$$bc(E, I) = \left\{ \gamma \cdot [\max(E - K_E^{B,C}, 0) \cdot \max(I - K_I^{B,C}, 0)] \mid a \right\} \quad (3.1)$$

#### Variables and Parameters Defintion:

$\gamma$ : Volume adjustment factor.

$E$ : Energy commodity index as defined in equation (2.3).

$K_E^{B,C}$ : Energy commodity index strike price for barrier call quanto option.

$I$ : Temperature index as defined in equation (2.4).

$K_I^{B,C}$ : Temperature index strike measure for barrier call quanto option.

$a$ :  $S_i < H^B, \forall i, i \in [t, T]$

$S_i$ : Energy commodity spot price at time  $i$ .

$H^B$ : Barrier for the barrier quanto option case.

$[t, T]$ : Considered period, e.g. a month.

### Comments:

The only remarkable difference from equation (2.1) is the barrier condition  $a$ . This condition does not allow  $S_i$  to assume extreme values during  $[t, T]$ .

## 3.2.2 Parisian Payoff

This is a up and out parisian option style.

### Parisian Call Quanto Option Payoff Function

$$pc(E, I) = \left\{ \gamma \cdot [\max(E - K_E^{P,C}, 0) \cdot \max(I - K_I^{P,C}, 0)] \mid n \leq m \right\} \quad (3.2)$$

### Variables and Parameters Defintion:

$\gamma$ : Volume adjustment factor.

$E$ : Energy commodity index as defined in equation (2.3).

$K_E^{P,C}$ : Energy commodity index strike price for parisian call quanto option.

$I$ : Temperature index as defined in equation (2.4).

$K_I^{P,C}$ : Temperature index strike measure for parisian call quanto option.

$n$ : Number of days  $S_i \geq H^P$ ,  $i \in [t, T]$ .

$S_i$ : Energy commodity spot price at time  $i$ .

$H^P$ : Barrier for the parisian quanto option case.

$[t, T]$ : Considered period, e.g. a month.

$m$ : Predefined number of days during interval  $[t, T]$ .

### Comments:

The only remarkable difference from equation (3.1) is the barrier condition  $n \leq m$ . This condition allows  $S_i$  to assume extreme values during  $[t, T]$ . However, this amount of extreme values has to be lower than  $m$ , otherwise the parisian quanto option will be knocked out.

## 3.3 Barrier-Parisian Quanto Option Contracts

The contract structure of a barrier and a parisian quanto option is ferly similar to the standard quanto option case. As it can be seen in table 3.1, the barrier and parisian quanto contract strikes,  $K_{E,j}^{l,C}$  and  $K_{E,j}^{l,C}$ , and the volume adjustment factor,  $\gamma_j$ , can vary over the contract time length for  $j \in [N, D, J, F, M]$  and for  $l \in [B, P]$ , i.e. they can change each month in both cases.

In addition, both the barrier and the parisian contract structures have a condition  $a_j$  and  $n_j \leq m_j$ , respectively, that must be respected in order not to be knocked out. In table 3.1, it is showed that these conditions can change over time as the other contract parameters. For instance, if  $H_N^B \neq H_J^B$ , then,  $a_N \neq a_J$  in the barrier case, whereas, if  $H_D^P \neq H_F^P$ , then,  $(n_D \leq m_D) \neq (n_F \leq m_F)$  in the parisian case. Where,  $H_j^B$  and  $H_j^P$  are

the barriers in the  $j$  month for the barrier and parisian quanto option case, respectively.

Table 3.1: Barrier and Parisian Contract Structures

	Barrier			Parisian			
Month	$a_j$	$K_{E,j}^{B,C}$	$K_{I,j}^{B,C}$	$n_j \leq m_j$	$K_{E,j}^{P,C}$	$K_{I,j}^{B,C}$	$\gamma_j$
November	$a_N$	$K_{E,N}^{B,C}$	$K_{I,N}^{B,C}$	$n_N \leq m_N$	$K_{E,N}^{P,C}$	$K_{E,N}^{P,C}$	$\gamma_N$
December	$a_D$	$K_{E,D}^{B,C}$	$K_{I,D}^{B,C}$	$n_D \leq m_D$	$K_{E,D}^{P,C}$	$K_{E,D}^{P,C}$	$\gamma_D$
January	$a_J$	$K_{E,J}^{B,C}$	$K_{I,J}^{B,C}$	$n_J \leq m_J$	$K_{E,J}^{P,C}$	$K_{E,J}^{P,C}$	$\gamma_J$
February	$a_F$	$K_{E,F}^{B,C}$	$K_{I,F}^{B,C}$	$n_F \leq m_F$	$K_{E,F}^{P,C}$	$K_{E,F}^{P,C}$	$\gamma_F$
March	$a_M$	$K_{E,M}^{B,C}$	$K_{I,M}^{B,C}$	$n_M \leq m_M$	$K_{E,M}^{P,C}$	$K_{E,M}^{P,C}$	$\gamma_M$

On the one hand, the holder of a barrier quanto option receives a payment equal to  $\gamma_j \cdot (E_j - K_{E,j}^{B,C}) \cdot (I_j - K_{I,j}^{B,C})$ , in month  $j$  at maturity, if  $E_j > K_{E,j}^{B,C}$ ,  $I_j > K_{I,j}^{B,C}$  and  $a_j$  are satisfied.

On the other hand, the holder of a parisian quanto option receives a payment equal to  $\gamma_j \cdot (E_j - K_{E,j}^{P,C}) \cdot (I_j - K_{I,j}^{P,C})$ , in month  $j$  at maturity, if  $E_j > K_{E,j}^{P,C}$ ,  $I_j > K_{I,j}^{P,C}$  and  $n_j \leq m_j$  are satisfied.

Note that usually  $H_j^P \leq H_j^B$ . This is due to the fact that a parisian quanto option is less sensible to volatility compared to a barrier quanto option.

# Chapter 4

## Pricing Quanto Options

Pricing quanto options can be done in different ways. Quanto options can be priced by closed form solutions, which allows to price standard quanto options, or by Monte Carlo method, which allows more complex structures as barrier and parisian cases. The former method is presented in this chapter, whereas, the latter is explained in the following one.

### 4.1 Framework

Benth et al. (2012) had a formidable insight to build a pricing framework for standard quanto options. Their idea was to convert the pricing of the quanto option from an Asian style pricing problem to a European one, by using as underlying variables futures prices instead of indexes. Moreover, they assumed futures prices lognormal dynamics.

They applied this insight to price a call standard quanto option with price  $C_{t^*}$ , at time  $t^*$ , and payoff function represented in equation (2.1). They derived a closed form pricing formula for a call standard quanto option. For further details on their framework and proof see Benth et al. (2012) [2].

### 4.1.1 Put Quanto Option Price

The author of this thesis used Benth et al. (2012) framework and insight to derive a closed form pricing formula for a put standard quanto option,  $P_{t^*}$ , at time  $t^*$ , with payoff function in equation (2.5). The general underlying idea is fairly similar to theirs. However, this chapter can be seen as a completion of the remarkable research improvement made by Benth et al. (2012) [2].

#### Put Quanto Option Price

$$P_{t^*} = \phi(t^*, T) \cdot \mathbb{E}_{t^*}^{\mathbb{Q}} [p(E, I)] \quad (4.1)$$

#### Variables and Parameters Definition:

$P_{t^*}$ : Put Quanto Option Price at time  $t^*$ .

$\phi(t^*, T)$ : Discount factor, i.e.  $e^{-r(T-t^*)}$ .

$r$ : Risk free rate, which is assumed to be constant.

$\mathbb{E}_{t^*}^{\mathbb{Q}}[\cdot]$ : Conditional expectation at time  $t^*$  with respect to the price measure  $\mathbb{Q}$ . Note that  $\mathbb{Q}$  is the risk neutral measure, whereas,  $\mathbb{P}$  represents the real world measure.

$p(E, I)$ : It is defined as the payoff function in equation (2.5).

The idea, now, is to use futures prices in place of the commodity and temperature indexes in equation (4.1). Define futures prices as follows:

#### Futures Prices

$$F_{t^*}^E(\tau_1, \tau_2) = \mathbb{E}_{t^*}^{\mathbb{Q}} \left[ \frac{1}{\tau_2 - \tau_1} \sum_{i=\tau_1}^{\tau_2} S_i \right] \quad (4.2)$$

$$F_{t^*}^I(\tau_1, \tau_2) = \mathbb{E}_{t^*}^{\mathbb{Q}} \left[ \sum_{i=\tau_1}^{\tau_2} \max[\varphi - \psi_i, 0] \right] \quad (4.3)$$

### Variables and Parameters Defintion:

$F_{t^*}^E(\tau_1, \tau_2)$ : Futures price at time  $t^*$ ,  $t^* \leq \tau_2$ , of a future contract written on the commodity price  $S_i$  with delivery period  $[\tau_1, \tau_2]$ .

$F_{t^*}^I(\tau_1, \tau_2)$ : Futures price at time  $t^*$ ,  $t^* \leq \tau_2$ , of a future contract written on HDD index with delivery period  $[\tau_1, \tau_2]$ .

if  $t^* = \tau_2$ , then:

$$F_{\tau_2}^E(\tau_1, \tau_2) = \frac{1}{\tau_2 - \tau_1} \sum_{i=\tau_1}^{\tau_2} S_i \quad (4.4)$$

$$F_{\tau_2}^I(\tau_1, \tau_2) = \sum_{i=\tau_1}^{\tau_2} \max[\varphi - \psi_i, 0] \quad (4.5)$$

and if  $[\tau_1, \tau_2] = [t, T]$ , then:

$$F_{\tau_2}^E(\tau_1, \tau_2) = E \quad (4.6)$$

$$F_{\tau_2}^I(\tau_1, \tau_2) = I \quad (4.7)$$

∴

### Put Quanto Option Price

$$P_{t^*} = \phi(t^*, T) \cdot \mathbb{E}_{t^*}^{\mathbb{Q}} [p(F_T^E(\tau_1, \tau_2), F_T^I(\tau_1, \tau_2))] \quad (4.8)$$

Where,

$$p(F_T^E(\tau_1, \tau_2), F_T^I(\tau_1, \tau_2)) = \gamma \cdot (K_E^P - F_T^E(\tau_1, \tau_2))^+ \cdot (K_I^P - F_T^I(\tau_1, \tau_2))^+ \quad (4.9)$$

Note that, the pricing task is on the terminal values of the futures prices rather than on indexes, now. Moreover, note that  $t = \tau_1$ ,  $T = \tau_2$  and  $t^* \leq \tau_2$ .

### 4.1.2 Log-Normal Assumption

In order to derive a put quanto option closed form solution, there is need of an assumption about the futures prices dynamics with respect to the  $\mathbb{Q}$  measure. As Benth et al. (2012) did in their paper [2], a log-normal assumption is made. This is supported by the fact that the futures are traded assets and such hypothesis is not too unrealistic.

**Assume:**

$$\ln(F_T^E(\tau_1, \tau_2)) - \ln(F_{t^*}^E(\tau_1, \tau_2)) \sim N(\mu_x, \sigma_x^2) \quad (4.10)$$

Where  $N(\mu_x, \sigma_x^2)$  is a normal distribution with  $\mu_x$  as mean and  $\sigma_x^2$  as variance. By following the successive simple steps we reach the needed dynamic,

$$\frac{\ln(F_T^E(\tau_1, \tau_2)) - \ln(F_{t^*}^E(\tau_1, \tau_2)) - \mu_x}{\sigma_x} = \epsilon_x \quad (4.11)$$

Where  $\epsilon_x \sim N(0, 1)$ .

$$F_T^E(\tau_1, \tau_2) = F_{t^*}^E(\tau_1, \tau_2) \cdot \exp(\mu_x + \sigma_x \cdot \epsilon_x) \quad (4.12)$$

By following the previous steps for  $F^I(\tau_1, \tau_2)$  too, the futures pierces dynamics, with respect to log-normal assumption, are:

### Futures prices Dynamics

$$F_T^E(\tau_1, \tau_2) = F_{t^*}^E(\tau_1, \tau_2) \cdot \exp(\mu_x + \sigma_x \cdot \epsilon_x) \quad (4.13)$$

$$F_T^I(\tau_1, \tau_2) = F_{t^*}^I(\tau_1, \tau_2) \cdot \exp(\mu_y + \sigma_y \cdot \epsilon_y) \quad (4.14)$$

### Variables and Parameters Defintion:

$F_T^E(\tau_1, \tau_2)$ : Futures price at time  $T$ , i.e. terinal value, of a future contract written on the commodity price  $S_i$  with delivery period  $[\tau_1, \tau_2]$ .

$F_T^I(\tau_1, \tau_2)$ : Futures price at time  $T$ , i.e. terinal value, of a future contract written on HDD index with delivery period  $[\tau_1, \tau_2]$ .

$\exp(\cdot)$ : It represents the exponential function.

$\mu_i$ : First moment of  $\ln(F_T^i(\tau_1, \tau_2)) - \ln(F_{t^*}^i(\tau_1, \tau_2))$  normal distribution,  $i \in [x, y]$ .

$\sigma_i$ : Squared root of the second moment of  $\ln(F_T^i(\tau_1, \tau_2)) - \ln(F_{t^*}^i(\tau_1, \tau_2))$  normal distribution,  $i \in [x, y]$ .

$\epsilon_i$ : random variable distributed as  $N(0, 1)$ ,  $i \in [x, y]$ .

$\rho_{x,y}$ : correlation between the assumed bivariate normally distributed random vaiables  $(\sigma_x \cdot \epsilon_x, \sigma_y \cdot \epsilon_y)$ .

### Comments:

Benth et al. (2012) applied such general representation of futures dynamics to a bivariate geometric Brownian motion model and to a Schwartz-Smith two factors model with seasonality, see [2]. Note that the former model is less suitable for a commodity process, while, the latter has better performance in such issue, see [8]. These cases can be easily applied to the put quanto closed form solution too. However, the pricing formula provided in following section investigates only the general case.

## 4.2 Put Closed form Solution

Finding a closed form pricing formula for a put quanto option consists in solving equation (4.8). Its proof is showed in appendix A and its solution is:

$$\begin{aligned}
 P_{t^*} = & e^{-r(T-t^*)} \cdot (K_E^P \cdot K_I^P \cdot \Phi(q_1, q_2, \rho_{x,y}) + \\
 & -F_{t^*}^I(\tau_1, \tau_2) \cdot K_E^P \cdot \exp(\mu_y + \frac{1}{2}\sigma_y^2) \cdot \Phi(q_1^*, q_2^*, \rho_{x,y}) + \\
 & -F_{t^*}^E(\tau_1, \tau_2) \cdot K_I^P \cdot \exp(\mu_x + \frac{1}{2}\sigma_x^2) \cdot \Phi(q_1^{**}, q_2^{**}, \rho_{x,y}) + \\
 & +F_{t^*}^I(\tau_1, \tau_2) \cdot F_{t^*}^E(\tau_1, \tau_2) \cdot \exp(\mu_y + \mu_x + \frac{1}{2}(\sigma_y^2 + \sigma_x^2 + 2\rho_{x,y}\sigma_y\sigma_x)) \cdot \Phi(q_1^{***}, q_2^{***}, \rho_{x,y}))
 \end{aligned}$$

Where,

$$q_1 = \frac{\ln(K_E^P) - \ln(F_{t^*}^E(\tau_1, \tau_2)) - \mu_x}{\sigma_x}, \quad q_2 = \frac{\ln(K_I^P) - \ln(F_{t^*}^I(\tau_1, \tau_2)) - \mu_y}{\sigma_y}$$

$$q_1^* = q_1 + \rho_{x,y}\sigma_y, \quad q_2^* = q_2 + \sigma_y$$

$$q_1^{**} = q_1 + \sigma_x, \quad q_2^{**} = q_2 + \rho_{x,y}\sigma_x$$

$$q_1^{***} = q_1 + \sigma_x + \rho_{x,y}\sigma_y, \quad q_2^{***} = q_2 + \sigma_y + \rho_{x,y}\sigma_x$$

$\Phi(q_1, q_2, \rho_{x,y})$  is a standard cumulative bivariate normal distribution with correlation  $\rho_{x,y}$ .

The solution is quite similar to the call quanto option derived by Benth et al. [2]. However, the difference between the two solutions is among the different quantiles position. Moreover, the author noted that if  $-q_1$  and  $-q_2$  are considered, *ceteris paribus*, the above put quanto formula becomes the solution of the call quanto option price. Note that  $\gamma$  is assumed to be equal to one.

## Chapter 5

# Pricing Barrier-Parisian Quanto Options

This chapter investigates how to price barrier and parisian quanto options from a theoretical prospective. The insertion of a barrier does not allow to use a closed form solution anymore. For this reason, the author of the thesis chose to use Monte Carlo simulation as numerical method to price such exotic modifications of the standard quanto option. The author suggests to model spot gas price and spot HDD through a Schwartz-Smith model with seasonality. In other words, a daily monitoring of the underlying variables is needed, e.g. gas price, to be able to check the barrier condition.

Benth et al. (2012) used an extension of the Schwartz-Smith model with seasonality, i.e. a HJM-style model, in order to model futures prices of gas and HDD, see [2]. Their model construction started from a Schwartz-Smith model with seasonality to end up into a HJM model for futures prices. On the same line of thought, the author of this thesis decided to use the same starting point of Benth et al. (2012), i.e. their Schwartz-Smith model with seasonality, but instead of modeling futures prices in a HJM-style, the author of this thesis chose to model spot gas price and spot HDD.

Furthermore, the author of the thesis made a Monte Carlo algorithm in Python programming language, to be able to price barrier and parisian

quanto options. In order to run the algorithm, the author used Benth et al. (2012), [2], parameters. This is due to the fact that an estimation of the Schwartz-Smith model with seasonality parameters was out of the purpose of the thesis. Nonetheless, such used parameters were estimated on futures gas and HDD prices on an extension of the model used in this thesis. Therefore, a calibration of them was made in order to obtain more feasible simulations of spot gas prices and spot HDD. Such a choice of focusing on the Monte Carlo algorithm rather than on the parameters estimation was done on purpose. However, a correct estimation of the model parameters would have been more appropriate; nevertheless, this could be a new starting point for a future research. Other possible future research issues are pointed out later in the thesis.

## 5.1 Barrier-Parisian Quanto Option Price

To start with, a call style barrier and parisian quanto option price definition is given in the following lines. Furthermore, a call style standard quanto option price definition is given in order to make comparisons with the barrier and parisian ones. Note that a Monte Carlo pricing of put style barrier and parisian quanto options can be done by simple modifications of what follows in this chapter.

Moreover note that, intermediate knowledge of Monte Carlo pricing, see [5], and Schwartz-Smith model, see [8], is supposed to be known by the reader. This choice is due to avoid cumbersome repetitions of well-known concepts in the literature.

### Call Quanto Option Price

$$C_{t^*} = \phi(t^*, T) \cdot \mathbb{E}_{t^*}^{\mathbb{Q}} [c(E, I)] \quad (5.1)$$

### Call Barrier Quanto Option Price

$$BC_{t^*} = \phi(t^*, T) \cdot \mathbb{E}_{t^*}^{\mathbb{Q}} [bc(E, I)] \quad (5.2)$$

### Call Parisian Quanto Option Price

$$PC_{t^*} = \phi(t^*, T) \cdot \mathbb{E}_{t^*}^{\mathbb{Q}} [pc(E, I)] \quad (5.3)$$

Where  $\phi(t^*, T)$  is a discount factor as defined in equation (4.1). The payoff functions  $c(E, I)$ ,  $bc(E, I)$  and  $pc(E, I)$  are defined as in equations (2.1), (3.1) and (3.2), respectively. However,  $\gamma$  is still assumed to be equal to one, as in the closed form solution derived in this thesis.

In a nutshell, the price of the above quanto options is the conditional expectations, with respect to the  $\mathbb{Q}$  measure, of the simulated payoffs, discounted at time  $t^*$ .

## 5.2 Schwartz-Smith model with Seasonality

One of the most famous and widely used models for commodities is the Schwartz-Smith model, see [8]. This well-known model has been expanded, throughout the years, to incorporate seasonality, as suggested by Schwartz-Smith in their paper [8]. Nowadays, this model is a milestone of the literature and it is broadly used in option pricing.

As mentioned before, the suggestion given by the author is to use this famous model not only for the commodity spot price process but also for the spot HDD process. Strictly speaking, such a model could be seen as an alternative to complex econometric models that try to capture temperature dynamics, e.g. see [4]. The thesis model is defined as follow and the notation is fairly standard as in the literature.

## Schwartz-Smith model with Seasonality

$$S_{t,i} = \exp(X_{t,i} + Z_{t,i} + \Lambda_{t,i}) \quad (5.4)$$

$$dX_{t,i} = \left( \mu_i - \frac{1}{2}\sigma_i^2 \right) dt + \sigma_i dW_{t,i} \quad (5.5)$$

$$dZ_{t,i} = -\kappa_i Z_{t,i} dt + v_i dB_{t,i} \quad (5.6)$$

$$\Lambda_{t,i} = \sum_{k=1}^{K_i} (\omega_{k_i} \cos(2\pi kt) + \omega_{k_i}^* \sin(2\pi kt)) \quad (5.7)$$

for  $i = \text{Gas, HDD}$

### Variables Defintion:

$S_{t,i}$ : Represents either the spot gas price or the spot HDD, at time  $t$ .

$X_{t,i}$ : Stochastic variable which represents the equilibrium level of  $S_{t,i}$ .

$Z_{t,i}$ : Stochastic variable which represents the short term deviation of  $S_{t,i}$

$dX_{t,i}$ : Arithmetic Brownian motion.

$dZ_{t,i}$ : Ornstein-Uhlenbeck process, with mean reversion towards zero.

$\Lambda_{t,i}$  : Deterministic Seasonality function,

Before moving ahead, it is worth noting that the above model specification is done with respect to the  $\mathbb{P}$  measure, i.e. real world. In addition, the Brownian motions specified in the model are correlated among them, not only in each process but also between the different processes. This is due to the fact that gas price and HDD have a correlation different from zero. The correlation matrix of the Brownian motions can be defined as follow.

## Correlation Matrix

$$\begin{bmatrix} 1 & \rho_{W_{t,Gas}B_{t,Gas}} & \rho_{W_{t,Gas}W_{t,HDD}} & \rho_{W_{t,Gas}B_{t,HDD}} \\ \rho_{B_{t,Gas}W_{t,Gas}} & 1 & \rho_{B_{t,Gas}W_{t,HDD}} & \rho_{B_{t,Gas}B_{t,HDD}} \\ \rho_{W_{t,HDD}W_{t,Gas}} & \rho_{W_{t,HDD}B_{t,Gas}} & 1 & \rho_{W_{t,HDD}B_{t,HDD}} \\ \rho_{B_{t,HDD}W_{t,Gas}} & \rho_{B_{t,HDD}B_{t,Gas}} & \rho_{B_{t,HDD}W_{t,HDD}} & 1 \end{bmatrix} \quad (5.8)$$

For instance,  $\rho_{W_{t,Gas}W_{t,HDD}}$  is the correlation between  $W_{t,Gas}$  and  $W_{t,HDD}$  Brownian motions.

To be able to simulate the possible paths of spot gas price and spot HDD, in order to compute the quanto options payoffs, an exact formula has to be derived. By applying a Girsanov transformation to the above Schwartz-Smith model with seasonality, it is possible to move from the  $\mathbb{P}$  measure to the  $\mathbb{Q}$  measure. This is a fairly standard step in option pricing to obtain risk neutral processes, see [1]. Then, the author of the thesis solved the stochastic differential equations of the arithmetic Brownian motion and Ornstein-Uhlenbeck process, see appendix B. After the above mentioned common mathematical steps, the obtained formulas are as follow.

## Exact formulas

$$S_{t_j,i} = \exp(X_{t_j,i} + Z_{t_j,i} + \Lambda_{t_j,i}) \quad (5.9)$$

$$X_{t_j,i} = X_{t_{j-1},i} + \left( \mu_i - \lambda_{x,i} - \frac{1}{2}\sigma_i^2 \right) (t_j - t_{j-1}) + \sigma_i \sqrt{t_j - t_{j-1}} \epsilon_{w,i} \quad (5.10)$$

$$\begin{aligned} Z_{t_j,i} = & \exp(-\kappa_i (t_j - t_{j-1})) Z_{t_{j-1},i} - \frac{\lambda_{z,i}}{\kappa_i} (1 - \exp(-\kappa_i (t_j - t_{j-1}))) \\ & + \sqrt{\frac{v_i^2}{2\kappa_i} (1 - \exp(-2\kappa_i (t_j - t_{j-1})))} \epsilon_{B,i} \end{aligned} \quad (5.11)$$

$$\Lambda_{t_j,i} = \sum_{k=1}^{K_i} (\omega_{k_i} \cos(2\pi k t_j) + \omega_{k_i}^* \sin(2\pi k t_j)) \quad (5.12)$$

for  $i = \text{Gas, HDD}$

As Schwartz-Smith mentioned in their paper, [8],  $\lambda_{x,i}$  and  $\lambda_{z,i}$  are two constants market price of risk which are subtracted from the drifts of the Brownian motion and the Ornstein-Uhlenbeck process, respectively. Note that, for any size of the time interval  $dt_j = t_j - t_{j-1}$ , the exact formulas hold. To conclude, the noises  $\epsilon_{w,i}$  and  $\epsilon_{B,i}$ , for  $i = \text{Gas, HDD}$ , are multivariate normally distributed with zero mean vector and a positive definite covariance matrix with all the elements of its trace equal to one.

### 5.3 Monte Carlo Algorithm

As mentioned before, the author of the thesis coded a Monte Carlo algorithm in Python, in order to price barrier and parisian quanto options. In this section there is a discussion of the main features of the algorithm and possible alternatives that could have been used in the programming implementation. The Python code is in appendix D and the author sends the reader there for further technical details.

The algorithm could be seen as divided in two parts. The first part does Monte Carlo simulation of spot gas price and spot HDD for a chosen location, e.g. New York, by using the Schwartz-Smith model with seasonality. I.e., the used formulas for the implementation are those from equation (5.9) to (5.12). The second part of the code prices call-style standard, barrier and parisian quanto options, with antithetic variates as variance reduction technique, for a chosen month of the winter season, e.g. January . A quanto option contract can be seen as a sum of five months quanto options, hence, by being able to price one month quanto option it is straight forward to price a quanto contract of five months. As a result, the code can be ran as many time as it is needed in order to price different months with different

conditions, and then sum up the different month prices in a specific point in time. In a nutshell, the algorithm simulates the underlying variables, checks the payoff condition of the respective considered quanto option and then it finds the price and its standard deviation of the different quanto options, for a specific month.

By moving in more technical issues, the author decided to draw the correlated random noises, in the Monte Carlo simulation, from a multivariate normal distribution. However, a Cholesky decomposition could have been a potential alternative to the multivariate normal distribution. P. Glasserman has an extensive treatment of such issue in his famous book Monte Carlo methods in financial engineering, see [5]. As the reader noted from equation (5.9) to (5.12), the code does exact simulation of the stochastic variables. Anyhow, an Euler approximation would have been a potential alternative. However, the author chose to use an exact simulation rather than an Euler approximation, because the former is more robust than the latter regarding the size of the time step interval that is considered, i.e.  $dt_j$ .

In order to check the barrier quanto option condition a Boolean array is used. Its elements assume value true when the condition is satisfy and false when it is violated. Therefore, the barrier quanto option is knocked out if the Boolean array has one element equal to false. To monitor the parisian condition an array containing zeros and ones is built. Where one represents violation of the parisian barrier, whereas, zero represents no violation. If the number of ones exceeds the predefined number of days, i.e.  $m$ , then, the parisian quanto option is knocked out.

Antithetic variates it the variance reduction technique that is used in the code. However, also control variates or some other variance reduction techniques, see [5], could have been potential alternatives. The author chose the antithetic variates because it is widely used in academia and in the industry as well.

In order to reduce computational time in the Monte Carlo simulation, the author decided to use a linear algebra way of programming rather than a computer science one. In other words, the simulations are stored in matrixes rather than in lists. Moreover, generators are used in the "for loops" to further reduce the computational effort of the code. As a result, the program needs only 2 minutes and 20 seconds to run, by using ten thousand simulations.

## 5.4 Theoretical Example

As mentioned at the beginning of this chapter, the author borrowed Benth et al. (2012) parameters, see [2], in order to run the Monte Carlo algorithm and create a theoretical example of pricing barrier and parisian quanto options. Benth et al. (2012) estimated these parameters, with a HJM model, on gas and HDD futures data for New York location. Thereby, these parameters cannot be used immediately in the thesis model but they have to be calibrated. The author calibrated some of them to be able to generate feasible simulations of spot gas price and spot HDD. The calibrated parameters are showed in appendix C with further comments. Even though the calibrated parameters enable to simulate feasible paths of the underlying variables, they should be estimated on their own appropriate model and data to be considered reliable from an empirical prospective.

### Theoretical Example Features

The theoretical example of pricing standard, barrier and parisian quanto options, through the Monte Carlo algorithm, has the following features. To start with,  $K_E^C = K_E^{B,C} = K_E^{P,C} = 5$  UDS and  $K_I^C = K_I^{B,C} = K_I^{P,C} = 990$  cumulate HDD. The barriers are set as  $H^B = H^P = 6.5$  USD. The  $m$  condition for the parisian quanto option is  $m = 5$ , which represents the amount of days at which the commodity spot price can be equal or above  $H^P$ . The simulations starting points for spot gas price and spot HDD are 5 USD and 33 HDD, respectively. The HDD index is defined as  $\max[65 - \psi, 0]$ ,

see equation (2.4) and its comments, and it represents 32 Fahrenheit and 0 Celsius degrees at the starting point. The risk free rate,  $r$ , is assumed to be constant and equal to 0.02. The considered horizon,  $T$ , is one month and the quanto option prices are computed at the beginning of this hypothetical month, e.g. January. The time step,  $dt$ , is one day, in the Monte Carlo simulation.

### 5.4.1 Results

The first result obtained, by running the Monte Carlo algorithm, is the plot of the spot gas price simulation over the considered month, which is assumed to be 30 days. As it can be seen in figure 5.1, as mentioned earlier, the starting point of the simulation is chosen to be 5 USD and the simulation ranges between 2.8 USD and 8.5 USD.

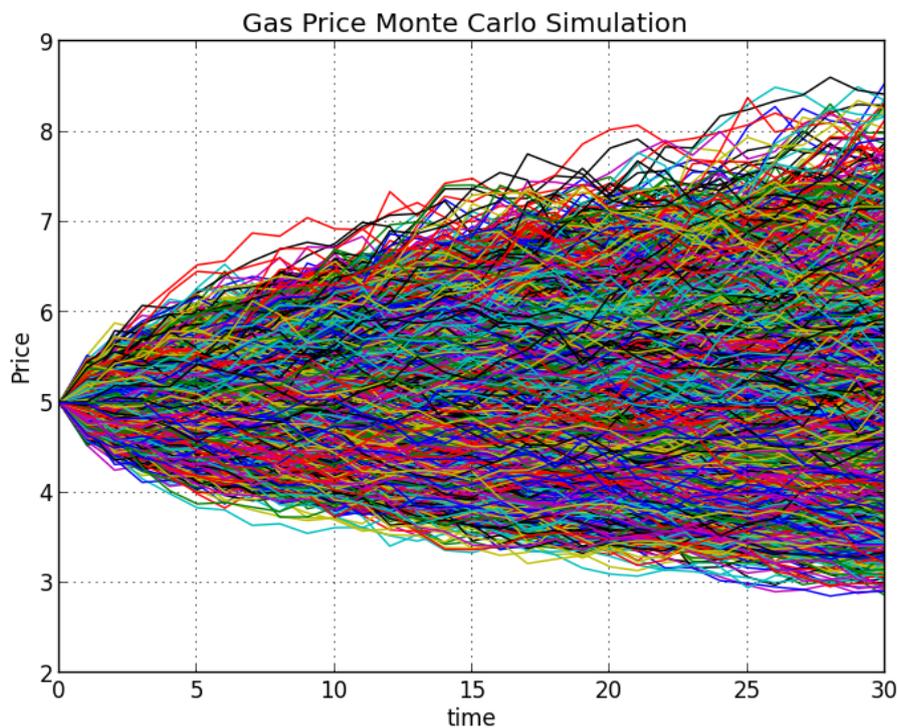


Figure 5.1: Gas Price Monte Carlo Simulation

The second obtained result is the simulation of the spot HDD index over this hypothetical month, figure 5.2. As mentioned before, its starting point is at 33 HDD and the simulation range between 12 and 85 HDD over the entire period. In other words, the simulation fluctuates between 54 and -20 Fahrenheit degrees, which correspond to 12 and -29 Celsius degrees, respectively. As the reader noted, this HDD simulation represents a hypothetical extraordinary cold winter month, for instance in New York City.

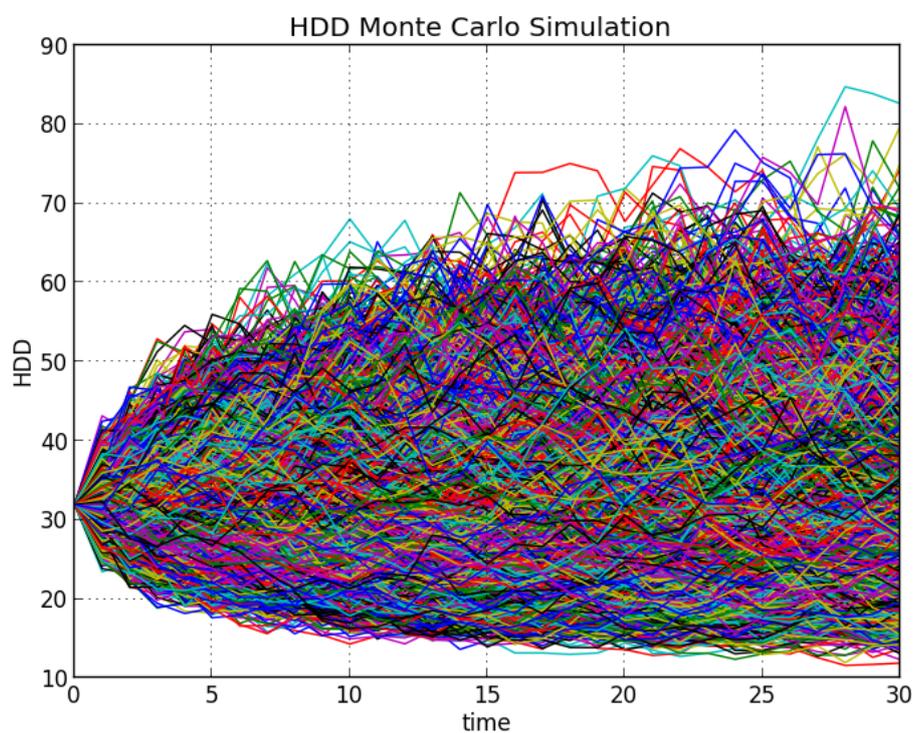


Figure 5.2: HDD Monte Carlo Simulation

To conclude, the numerical algorithm provides the prices of standard, barrier and parisian quanto options as they are defined in equations (5.1), (5.2) and (5.3), respectively. Even though an analysis of the Python code has already been given previously, the reader can examine in depth the pricing part of the code in order to have a full picture of the numerical procedure that was adopted to obtain the prices, see appendix D. Table 5.1 shows the

end results of the prices and respective standard deviations.

Table 5.1: Standard, Barrier and Parisian Quanto Option Prices in USD

	$C_{t^*}$	$BC_{t^*}$	$PC_{t^*}$
Price	15.28	10.27	12.92
Standard Deviation	0.307	0.21	0.254

In the next subsection there is a broad discussion of the obtained quanto option prices results from an economic perspective. Furthermore, a part of the discussion is assigned to analyze pros and cons of the adopted thesis model for pricing barrier and parisian quanto options, i.e. the model from equation (5.4) to equation (5.7).

## 5.4.2 Discussion

### Price Discussion

From table 5.1, it is easy to see that the price of the standard quanto option is the most expensive one,  $C_{t^*} = 15.28$  USD. While, the parisian quanto option price is cheaper than the standard quanto option case, with  $PC_{t^*} = 12.92$  USD. As expected, the cheapest price is the barrier quanto option case,  $BC_{t^*} = 10.27$  USD.

$$BC_{t^*} \leq PC_{t^*} \leq C_{t^*} \quad (5.13)$$

From a risk prospective, the standard quanto option is a full insurance and it covers for extreme and unexpected events simultaneously. As a result, it has the highest price among them. On the other hand, the price of the parisian case is lower, because it allows only few extreme events during its life time and it can be considered as a partial insurance. Lastly, the barrier

quanto option does not allow any extreme events, and thus it covers less risky scenarios during its life time. As a result, the barrier case is a partial insurance much stricter than the parisian one, hence, it has the lowest price.

To explain the difference in the three quanto option prices a comparison among the different simulated payoffs can be made. For instance, in the barrier quanto option case, the amount of simulated payoffs that become zero at maturity is the biggest, whereas, for the standard quanto option it is the smallest. This is due to the fact that the contract conditions of the barrier case are much stricter than the standard quanto option one. As a result, the price of the barrier quanto option is lower than the standard quanto option one, because it has less simulated payoffs different from zero. On the same line of thought, the parisian quanto option has more simulated payoffs different from zero than the barrier one, but less than the standard quanto option case. Accordingly, Its price is between the previous two, as it is showed in the inequality relation (5.13).

Furthermore, by analyzing the price standard deviations in table 5.1, it is easy to see that they have the same pattern as the respective prices. The standard quanto option has the largest, 0.307 USD, then the parisian, 0.254 USD, and lastly the barrier, with 0.21 USD. This is due to the fact that the variance of the discounted payoffs of the different quanto options, *ceteris paribus*, has different dispersion around its mean, which is the price. In other words, the standard quanto option discounted payoffs have the largest variance because no upward condition is set. On the other hand, the barrier quanto option discounted payoffs must be more concentrated around their mean because upward condition is set on them. Therefore, their variance is lower than the former case. Lastly, the parisian case is a hybrid of the two previous cases, as a result also its variance, and thus its standard deviation, is between the barrier and standard quanto option cases.

## Model Discussion

Since the beginning of this chapter, the model used to simulate the spot gas price and the spot HDD index is the model defined from equation (5.4) to equation (5.7). The Schwartz-Smith model with seasonality is well-known as commodity stochastic process but it is a new idea to consider it as a temperature index forecasting model. In the author's opinion, such model used in the thesis, i.e. from (5.4) to (5.7), is a good model for the coldest months of the year, such as January, February and December, but it is less appropriate for intermediate months such as March and November. Owing to the fact that the model is an exponential of the form  $\exp(x)$  it can assume value zero only if  $x$  goes to minus infinity. As a result, the likelihood of obtaining a simulation of  $\text{HDD} = 0$  is really small. This can be acceptable in January, when the temperature is unlikely to exceed 65 Fahrenheit degrees, e.g. in New York, but it would not be acceptable in March.

In the author's opinion, there is a possible fix to such drawback for these intermediate months. It consists in using the model from equation (5.4) to equation (5.7) to simulate spot gas price and spot temperature, rather than spot HDD. This latter solution would be a potential model for months such as March and November, but, not for months like January, February and December, for which the used thesis model is theoretically correct. In other words, if the temperature was simulated, in a month like March, they could fluctuate between 0 and, for instance, 90 Fahrenheit degrees, which correspond to a HDD fluctuation between 65 and 0, respectively. In such latest model specification of spot temperature, the likelihood of obtaining an  $\text{HDD} = 0$  is higher than in the model used in the thesis. Therefore, it could be a good model for March and November.

Anyhow, either the model specified from equation (5.4) to (5.7), i.e. spot HDD, or the potential modification with the spot temperature, i.e. the model for March and November, or both should be tested empirically with

real world data. As it will be written successively, this is one of the main future research issues that can be undertaken as an extension of this thesis.

On the same line of thought, potentially weather forecasting model alternatives are ARMA time series models or physical numerical models. Where the first statistical based models are more appropriate for mid-long term forecasting, whereas, the seconds are more suitable for short-term forecasting. However, these issues could be one of the potential improvements of the thesis. In other words, it would be interesting to see, in an empirical study, if the model suggested in the thesis is comparable to those used in practice, e.g. ARMA. It is likely that the author will undertake such empirical study in his future research.

# Chapter 6

## Conclusion

The main purpose of the thesis was to slightly improve the current literature of quanto options from a theoretical prospective. A put style standard quanto option closed form pricing formula was derived as an extension of Benth et al. (2012) work. The solution is based on log-normal assumption and it is given in general form.

Then, barrier quanto options were investigate from an economic point of view as partial insurances. Such idea was thought in order to satisfy some potential specific market participants' needs. In the author's opinion, barrier quanto options have some drawbacks with respect to the underlying variables volatility; therefore, parisian quanto options were suggested to offset such shortcoming.

After that, a Schwartz-Smith model with seasonality was introduced as stochastic process for spot gas price and spot HDD. Then, this model was implemented in a Monte Carlo algorithm in order to price barrier and parisian quanto options. A theoretical example of the pricing was given, but, it is not reliable from an empirical prospective, because no parameters estimation was made. However, such latest issue is one of the future research topics pointed out in the following section.

## 6.1 Future Work

Quanto option is a new research field in the literature and this thesis opened a door for future research in such interesting field. To start with, in the author's opinion a closed pricing formula for barrier quanto option could be derived. It would be an interesting challenge to investigate such topic in the near future.

Furthermore, an estimation of the Schwartz-Smith model, with seasonality, parameters would be an appropriate empirical study. In other words, appropriate data should be collected and then used to estimate the needed parameters with Kalman Filter techniques.

Another interesting empirical study might be comparing the suggested thesis model with alternative ARMA or physics models for weather forecasting.

From an option pricing prospective, barrier options are affected by discretization errors in the monitoring of the underlying asset. As a result, further improvements can be made in the Monte Carlo algorithm in appendix D. For example, a Brownian bridge could be applied in order to reduce such discretization error, e.g. see [3].

To conclude, the Monte Carlo algorithm, in appendix D, could be easily modified to be inserted in an object oriented programming environment in Python. For instance, a class called Quanto option could be made with different methods. The methods could be divided in the Monte Carlo simulation, the pricing of the standard, barrier and parisian quanto options. In other words, a class with four different methods would be built. In order to further reduce the computational time, the algorithm could be written in more advanced programming language such as C++ or Java.

# Appendices

# Appendix A

## Closed form Solution

*Proof.* As it was mentioned before, to price a put quanto option the following equation has to be solved,

$$P_{t^*} = \phi(t^*, T) \cdot \mathbb{E}_{t^*}^{\mathbb{Q}} [p(F_T^E, F_T^I)] \quad (\text{A.1})$$

Where the payoff function can be written as,

$$p(F_T^E, F_T^I) = \gamma \cdot (K_E^P - F_T^E) \cdot (K_I^P - F_T^I) \cdot 1_{F_T^E < K_E^P} \cdot 1_{F_T^I < K_I^P} \quad (\text{A.2})$$

Note that  $F_T^E = F_T^E(\tau_1, \tau_2)$  and  $F_T^I = F_T^I(\tau_1, \tau_2)$ . The main step is to solve the conditional expectation.

$$\begin{aligned} \mathbb{E}_{t^*}^{\mathbb{Q}} [p(F_T^E, F_T^I)] &= \mathbb{E}_{t^*}^{\mathbb{Q}} \left[ K_E^P \cdot K_I^P \cdot 1_{F_T^E < K_E^P} \cdot 1_{F_T^I < K_I^P} \right] \\ &\quad - \mathbb{E}_{t^*}^{\mathbb{Q}} \left[ K_E^P \cdot F_T^I \cdot 1_{F_T^E < K_E^P} \cdot 1_{F_T^I < K_I^P} \right] \\ &\quad - \mathbb{E}_{t^*}^{\mathbb{Q}} \left[ K_I^P \cdot F_T^E \cdot 1_{F_T^E < K_E^P} \cdot 1_{F_T^I < K_I^P} \right] \\ &\quad + \mathbb{E}_{t^*}^{\mathbb{Q}} \left[ F_T^E \cdot F_T^I \cdot 1_{F_T^E < K_E^P} \cdot 1_{F_T^I < K_I^P} \right] \end{aligned} \quad (\text{A.3})$$

### First Conditional Expectation

The first conditional expectation,  $\mathbb{E}_{t^*}^{\mathbb{Q}} \left[ K_E^P \cdot K_I^P \cdot 1_{F_T^E < K_E^P} \cdot 1_{F_T^I < K_I^P} \right]$ , is one of the drivers of the proof. Actually, it is the only main difference between the solution of a put quanto option price formula, and the call quanto option derived by Benth et al. (2012) [2].

The first conditional expectation can be seen as,

$$K_E^P \cdot K_I^P \mathbb{Q}_{t^*} (F_T^E < K_E^P \cap F_T^I < K_I^P) \quad (\text{A.4})$$

Which can be written as,

$$K_E^P \cdot K_I^P \mathbb{Q}_{t^*} \left( \exp(\mu_x + \sigma_x \cdot \epsilon_x) < \frac{K_E^P}{F_{t^*}^E} \cap \exp(\mu_y + \sigma_y \cdot \epsilon_y) < \frac{K_I^P}{F_{t^*}^I} \right) \quad (\text{A.5})$$

By applying simple algebra rules we obtain,

$$K_E^P \cdot K_I^P \mathbb{Q}_{t^*} \left( \epsilon_x < \frac{\ln \left( \frac{K_E^P}{F_{t^*}^E} \right) - \mu_x}{\sigma_x} \cap \epsilon_y < \frac{\ln \left( \frac{K_I^P}{F_{t^*}^I} \right) - \mu_y}{\sigma_y} \right) \quad (\text{A.6})$$

Which is equivalent to,

$$K_E^P \cdot K_I^P \Phi(q_1, q_2, \rho_{x,y}) \quad (\text{A.7})$$

Where,

$$q_1 = \frac{\ln \left( \frac{K_E^P}{F_{t^*}^E} \right) - \mu_x}{\sigma_x} \quad (\text{A.8})$$

$$q_2 = \frac{\ln \left( \frac{K_I^P}{F_{t^*}^I} \right) - \mu_y}{\sigma_y} \quad (\text{A.9})$$

While,  $\Phi(q_1, q_2, \rho_{x,y})$  is a standard cumulative bivariate normal distribution with correlation  $\rho_{x,y}$ .

## Second Conditional Expectation

The second conditional expectation,  $\mathbb{E}_{t^*}^{\mathbb{Q}} \left[ K_E^P \cdot F_T^I \cdot 1_{F_T^E < K_E^P} \cdot 1_{F_T^I < K_T^P} \right]$ , can be solved by factorizing a standard bivariate normal distribution function. For details of how to factorize such well-known function see the following reference [7]. However, also this insight is taken from Benth et al. (2012) [2].

The second conditional expectation can be written as,

$$K_E^P \cdot F_{t^*}^I \cdot \exp(\mu_y) \cdot \mathbb{E}_{t^*}^{\mathbb{Q}} \left[ \exp(\sigma_y \cdot \epsilon_y) \cdot 1_{\epsilon_x < q_1} \cdot 1_{\epsilon_y < q_2} \right] \quad (\text{A.10})$$

Given the results in the first conditional expectation and by factorizing the bivariate normal distribution, (A.10) becomes,

$$K_E^P \cdot F_{t^*}^I \cdot \exp(\mu_y) \cdot \int_{-\infty}^{q_2} \int_{-\infty}^{q_1} \exp(\sigma_y \cdot \epsilon_y) \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2}\epsilon_y^2\right) \cdot \frac{1}{\sqrt{2\pi}\sqrt{1-\rho_{x,y}^2}} \cdot \exp\left[-\frac{1}{2(1-\rho_{x,y}^2)}(\epsilon_x - \rho_{x,y}\epsilon_y)^2\right] d\epsilon_x d\epsilon_y \quad (\text{A.11})$$

Through some algebra, the exponent of (A.11) becomes,

$$-\frac{\epsilon_x^2 + \epsilon_y^2 - 2\sigma_y(1-\rho_{x,y}^2)\epsilon_y - 2\epsilon_x\epsilon_y\rho_{x,y}}{2(1-\rho_{x,y}^2)} \quad (\text{A.12})$$

Then, by applying the following substitutions,  $w = -\epsilon_x + \rho_{x,y}\sigma_y$  and  $z = -\epsilon_y + \sigma_y$ , (A.11) can be written as,

$$\begin{aligned}
& K_E^P \cdot F_{t^*}^I \cdot \exp\left(\mu_y + \frac{1}{2}\sigma_y^2\right) \int_{-\infty}^{q_2^*} \int_{-\infty}^{q_1^*} \frac{1}{2\pi\sqrt{1-\rho_{x,y}^2}} \cdot \\
& \exp\left[-\frac{1}{2(1-\rho_{x,y}^2)}(w^2 + z^2 - 2wz\rho_{x,y})\right] dw dz
\end{aligned} \tag{A.13}$$

Which is equivalent to,

$$K_E^P \cdot F_{t^*}^I \cdot \exp\left(\mu_y + \frac{1}{2}\sigma_y^2\right) \Phi(q_1^*, q_2^*, \rho_{x,y}) \tag{A.14}$$

Where,

$$q_1^* = q_1 + \rho_{x,y}\sigma_y \tag{A.15}$$

$$q_2^* = q_2 + \sigma_y \tag{A.16}$$

### Third Conditional Expectation

The third conditional expectation,  $\mathbb{E}_{t^*}^{\mathbb{Q}} \left[ K_I^P \cdot F_T^E \cdot 1_{F_T^E < K_E^P} \cdot 1_{F_T^I < K_I^P} \right]$ , can be solved as the second conditional expectation. Anyhow, the few difference between the two solutions are pointed out in the following steps.

The third conditional expectation can be written as,

$$K_I^P \cdot F_{t^*}^E \cdot \exp(\mu_x) \cdot \mathbb{E}_{t^*}^{\mathbb{Q}} \left[ \exp(\sigma_x \cdot \epsilon_x) \cdot 1_{\epsilon_x < q_1} \cdot 1_{\epsilon_y < q_2} \right] \tag{A.17}$$

Given the results in the first conditional expectation and by factorizing the bivariate normal distribution, (A.17) becomes,

$$\begin{aligned}
& K_I^P \cdot F_{t^*}^E \cdot \exp(\mu_x) \cdot \int_{-\infty}^{q_2} \int_{-\infty}^{q_1} \exp(\sigma_x \cdot \epsilon_x) \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2}\epsilon_x^2\right) \cdot \\
& \frac{1}{\sqrt{2\pi}\sqrt{1-\rho_{x,y}^2}} \cdot \exp\left[-\frac{1}{2(1-\rho_{x,y}^2)}(\epsilon_y - \rho_{x,y}\epsilon_x)^2\right] d\epsilon_x d\epsilon_y
\end{aligned} \tag{A.18}$$

Then, by applying the following substitutions,  $w = -\epsilon_x + \sigma_x$  and  $z = -\epsilon_y + \rho_{x,y}\sigma_x$ , (A.18) can be written as,

$$\begin{aligned}
& K_I^P \cdot F_{t^*}^E \cdot \exp\left(\mu_x + \frac{1}{2}\sigma_x^2\right) \int_{-\infty}^{q_2^{**}} \int_{-\infty}^{q_1^{**}} \frac{1}{2\pi\sqrt{1-\rho_{x,y}^2}} \cdot \\
& \exp\left[-\frac{1}{2(1-\rho_{x,y}^2)}(w^2 + z^2 - 2wz\rho_{x,y})\right] dw dz
\end{aligned} \tag{A.19}$$

Which is equivalent to,

$$K_I^P \cdot F_{t^*}^E \cdot \exp\left(\mu_x + \frac{1}{2}\sigma_x^2\right) \Phi(q_1^{**}, q_2^{**}, \rho_{x,y}) \tag{A.20}$$

Where,

$$q_1^{**} = q_1 + \sigma_x \tag{A.21}$$

$$q_2^{**} = q_2 + \rho_{x,y}\sigma_x \tag{A.22}$$

#### Fourth Conditional Expectation

The fourth conditional expectation,  $\mathbb{E}_{t^*}^{\mathbb{Q}} \left[ F_T^E \cdot F_T^I \cdot 1_{F_T^E < K_E^P} \cdot 1_{F_T^I < K_I^P} \right]$ , can be solved as the previous conditional expectations.

The fourth conditional is equivalent to,

$$F_{t^*}^E \cdot F_{t^*}^I \cdot \exp(\mu_x + \mu_y) \cdot \mathbb{E}_{t^*}^{\mathbb{Q}} \left[ \exp(\sigma_x \cdot \epsilon_x + \sigma_y \cdot \epsilon_y) \cdot 1_{\epsilon_x < q_1} \cdot 1_{\epsilon_y < q_2} \right] \quad (\text{A.23})$$

As the other expectations it becomes,

$$\begin{aligned} & F_{t^*}^E \cdot F_{t^*}^I \cdot \exp(\mu_x + \mu_y) \cdot \int_{-\infty}^{q_2} \int_{-\infty}^{q_1} \exp(\sigma_x \cdot \epsilon_x + \sigma_y \cdot \epsilon_y) \cdot \\ & \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2}\epsilon_y^2\right) \cdot \\ & \frac{1}{\sqrt{2\pi}\sqrt{1-\rho_{x,y}^2}} \cdot \exp\left[-\frac{1}{2(1-\rho_{x,y}^2)}(\epsilon_x - \rho_{x,y}\epsilon_y)^2\right] d\epsilon_x d\epsilon_y \end{aligned} \quad (\text{A.24})$$

Then, by applying the following substitutions,  $w = -\epsilon_x + \sigma_x + \rho_{x,y}\sigma_y$  and  $z = -\epsilon_y + \sigma_y + \rho_{x,y}\sigma_x$ , (A.24) can be written as,

$$\begin{aligned} & F_{t^*}^E \cdot F_{t^*}^I \cdot \exp\left(\mu_x + \mu_y + \frac{1}{2}(\sigma_x^2 + \sigma_y^2 + 2\sigma_x\sigma_y\rho_{x,y})\right) \cdot \\ & \int_{-\infty}^{q_2^{***}} \int_{-\infty}^{q_1^{***}} \frac{1}{2\pi\sqrt{1-\rho_{x,y}^2}} \cdot \\ & \exp\left[-\frac{1}{2(1-\rho_{x,y}^2)}(w^2 + z^2 - 2wz\rho_{x,y})\right] dw dz \end{aligned} \quad (\text{A.25})$$

Which is equivalent to,

$$\begin{aligned} & F_{t^*}^E \cdot F_{t^*}^I \cdot \exp\left(\mu_x + \mu_y + \frac{1}{2}(\sigma_x^2 + \sigma_y^2 + 2\sigma_x\sigma_y\rho_{x,y})\right) \cdot \\ & \Phi(q_1^{***}, q_2^{***}, \rho_{x,y}) \end{aligned} \quad (\text{A.26})$$

Where,

$$q_1^{***} = q_1 + \sigma_x + \rho_{x,y}\sigma_y \quad (\text{A.27})$$

$$q_2^{***} = q_2 + \sigma_y + \rho_{x,y}\sigma_x \quad (\text{A.28})$$

Given the resolution of the four conditional expectations, the solution of equation (A.1) can be easily computed.

□

# Appendix B

## Stochastic Differential Equations

In this appendix there is no distinction between a gas spot price process and a spot HDD process. In other words, the stochastic differential equation solutions are derived in general terms.

### Arithmetic Brownian Motion

By applying a Girsanov transformation to equation (5.5), it is straight forward to obtain the following risk neutral process.

$$dX_t = \left( \mu - \lambda_x - \frac{1}{2}\sigma^2 \right) dt + \sigma dW_t^* \quad (\text{B.1})$$

Integrate the above process in a general time interval  $[t, T]$ ,

$$\int_t^T dX_s = \int_t^T \left( \mu - \lambda_x - \frac{1}{2}\sigma^2 \right) ds + \int_t^T \sigma dW_s^* \quad (\text{B.2})$$

Hence, it ends in the following formula,

$$X_T = X_t + \left( \mu - \lambda_x - \frac{1}{2}\sigma^2 \right) (T - t) + \sigma\sqrt{T - t}\epsilon \quad (\text{B.3})$$

where,  $\epsilon \sim N(0, 1)$

### Ornstein-Uhlenbeck process

As for the arithmetic Brownian motion, by applying a Girsanov transformation to equation (5.6), the obtained risk neutral process looks like,

$$dZ_t = -(\lambda_z + \kappa Z_t) dt + v dB_t^* \quad (\text{B.4})$$

Assume a function  $f(\cdot, \cdot)$  defined as,

$$f(Z, t) = \exp(\kappa t) Z_t \quad (\text{B.5})$$

Apply Ito's lemma,

$$df(Z, t) = (\exp(\kappa t) (-\lambda_z - \kappa Z_t) + \exp(\kappa t) Z_t \kappa) dt + \exp(\kappa t) v dB_t^* \quad (\text{B.6})$$

Integrate the above process in a general time interval  $[t, T]$ ,

$$\int_t^T d(\exp(\kappa s) Z_s) = \int_t^T -\exp(\kappa s) \lambda_z ds + \int_t^T \exp(\kappa s) v dB_s^* \quad (\text{B.7})$$

Hence,

$$\begin{aligned} \exp(\kappa T) Z_T - \exp(\kappa t) Z_t &= -\frac{\lambda_z}{\kappa} (\exp(\kappa T) - \exp(\kappa t)) \\ &\quad + \int_t^T \exp(\kappa s) v dB_s^* \end{aligned} \quad (\text{B.8})$$

Then, it results in the following solution,

$$\begin{aligned} Z_T &= \exp(-\kappa(T-t)) Z_t - \frac{\lambda_z}{\kappa} (1 - \exp(-\kappa(T-t))) \\ &\quad + \exp(-\kappa T) \int_t^T \exp(\kappa s) v dB_s^* \end{aligned} \quad (\text{B.9})$$

In order to solve the stochastic integral of equation (B.9), the variance of  $Z_T$  is computed in the following lines,

$$Var [Z_T] = Var \left[ \exp(-\kappa T) \int_t^T \exp(\kappa s) v dB_s^* \right] \quad (\text{B.10})$$

Therefore,

$$Var [Z_T] = \frac{\exp(-2\kappa T) v^2}{2\kappa} \int_t^T \exp(2\kappa s) 2\kappa ds \quad (\text{B.11})$$

Integrate,

$$Var [Z_T] = \frac{v^2}{2\kappa} (1 - \exp(-2\kappa (T - t))) \quad (\text{B.12})$$

By combining equation (B.9) and equation (B.12) the exact formula is obtained,

$$\begin{aligned} Z_T &= \exp(-\kappa (T - t)) Z_t - \frac{\lambda_z}{\kappa} (1 - \exp(-\kappa (T - t))) \\ &\quad + \sqrt{\frac{v^2}{2\kappa} (1 - \exp(-2\kappa (T - t)))} \epsilon \end{aligned} \quad (\text{B.13})$$

where,  $\epsilon \sim N(0, 1)$

# Appendix C

## Parameters

The following parameters were calibrated by the author, starting from those estimated by Benth et al. (2012), see [2]. Only some of the parameters in table C.1 are modified, while the parameters of the correlation matrix, (C.1), are taken from Benth et al. (2012) as they are. As the reader can easily see from table C.1, the choice of the seasonality function is borrowed from Benth et al. (2012). I.e.,  $K_{Gas} = 2$ , whereas,  $K_{HDD} = 1$ , in equations (5.7) and (5.12).

The Parameters in table C.1 are used in the equations from (5.9) to (5.12). While those in the correlation matrix, (C.1), are used as input in the multivariate normal distribution in the Python algorithm.

Table C.1: Calibrated Parameters for the Schwartz-Smith model with Seasonality

	$\sigma$	$\kappa$	$\nu$	$\mu$	$\lambda_x$	$\lambda_Z$	$\omega_1$	$\omega_1^*$	$\omega_2$	$\omega_2^*$
GAS	0.2342	0.6116	0.6531	0.0	0.0	0.0	0.05	0.0406	0.0128	0.0270
HDD	0.02	17.0	1.5	0.0	0.0	0.0	0.9044	0.8104	<i>N/A</i>	<i>N/A</i>

The covariance matrix defined in equation (5.8) becomes as follow,

**Correlation Matrix**

$$\begin{bmatrix} 1.0 & -0.6803 & -0.2843 & 0.0 \\ -0.6803 & 1.0 & 0.0 & 0.1817 \\ -0.2843 & 0.0 & 1.0 & -0.6066 \\ 0.0 & 0.1817 & -0.6066 & 1.0 \end{bmatrix} \quad (\text{C.1})$$

# Appendix D

## Python Code

---

```
1 """
Created on Thu May 02 11:05:42 2013
3
@author: Adriano Tosi
5 """
from __future__ import division
7 from scipy import *
from matplotlib.pyplot import *
9 import random
11 """ Introduction """
13 """The first part of this algorithm provides \n
Monte Carlo simulation \n
15 of spot gas price and spot HDD for New York location. \n
In the second part of the script, \n
17 there is a code for pricing standard, barrier \n
and parisian quanto options.\n
19 The program needs just 2 minutes and 20 seconds to run. \n
"""
21
""" Notation """
23
```

```

    """ The Notation in the code is slightly different \n
25 from the rest of \n
    the thesis but it is consistent with itself.\n
27 For example, c stands for commodity, thus gas variable, \n
    I stands for index, hence HDD. \n
29 x stands for Brownian motion process \n
    and z stands for Ornstein–Uhlenbeck process """
31

33 """ Parameters """

35 """ Gas Parameters """

37 k_c = 0.6116
    sigma_c = 0.2342
39 v_c = 0.6531
    omega_1_c = 0.05
41 omega_1_star_c = 0.0406
    omega_2_c = 0.0128
43 omega_2_star_c = 0.0270

45 """ HDD Parameters """

47 k_I = 17.0
    sigma_I = 0.02
49 v_I = 1.5
    omega_1_I = 0.9044
51 omega_1_star_I = 0.8104

53 """ Correlation between the Brownian motions """

55 rho_c = -0.6803
    rho_I = -0.6066
57 rho_x = -0.2843
    rho_z = 0.1817

```

```

59     """ Covariances of the noises """
61
62     cov_c = rho_c
63     cov_I = rho_I
64     cov_x = rho_x
65     cov_z = rho_z
66
67     """ Covariance Matrix """
68
69     row_0 = [1, cov_c, cov_x, 0]
70     row_1 = [cov_c, 1, 0, cov_z]
71     row_2 = [cov_x, 0, 1, cov_I]
72     row_3 = [0, cov_z, cov_I, 1]
73
74     mean_vec = array([0, 0, 0, 0])
75     cov_matrix = array([row_0, row_1, row_2, row_3])
76
77     """ Assumptions """
78
79     """The starting values are chosen \n
80 to start form a gas price of 5 \n
81 and a HDD of 33 which represents \n
82 32 Fahrenheit and 0 Celsius. """
83
84
85     X0_c = 0.773
86     Z0_c = 0.773
87     X0_I = 2.16
88     Z0_I = 0.4
89     seas0_c = 0.0628
90     seas0_I = 0.904
91     K_E = 5
92     K_I = 990
93     r = 0.02

```

```

    T = 30/360
95  N = 30
    dt = T/N
97  H_B = 6.5
    H_P = 6.5
99  m = 5

101
    """ Monte Carlo Simulation \n
103 with Antithetic Variates as \n
    Variance Reduction Technique """
105
    simulations = 10000
107 simulations_2 = 2*simulations

109 M_X0_c = zeros([simulations_2,(N+1)])
    M_Z0_c = zeros([simulations_2,(N+1)])
111 M_X0_I = zeros([simulations_2,(N+1)])
    M_Z0_I = zeros([simulations_2,(N+1)])
113 M_seas_c = zeros([simulations_2,(N+1)])
    M_seas_I =zeros([simulations_2,(N+1)])
115
    M_X0_c[:,0] = X0_c
117 M_Z0_c[:,0] = Z0_c
    M_X0_I[:,0] = X0_I
119 M_Z0_I[:,0] = Z0_I
    M_seas_c[:,0] = seas0_c
121 M_seas_I[:,0] = seas0_I

123
    def even_integers(number):
125         for x in xrange(number):
            if x % 2 == 0:
127                 yield x

```

```

129 g = even_integers(simulations_2)

131 for i in g:
    for j in xrange(1, (N+1)):
133
        e = np.random.multivariate_normal(mean_vec, cov_matrix)
135
        X0_c_1 = (-1/2*sigma_c**2)*dt
137 X0_c_2 = sigma_c*sqrt(dt)
        M_X0_c[i, j]= M_X0_c[i, j-1]+X0_c_1+X0_c_2*e[0]
139 M_X0_c[i+1, j]= M_X0_c[i+1, j-1]+X0_c_1+X0_c_2*(-e[0])

        Z0_c_1 = exp(-(k_c*dt))
        Z0_c_2 = sqrt((v_c**2/(2*k_c))*(1-exp(-2*k_c*dt)))
143 M_Z0_c[i, j]= M_Z0_c[i, j-1]*Z0_c_1+Z0_c_2*e[1]
        M_Z0_c[i+1, j]= M_Z0_c[i+1, j-1]*Z0_c_1+Z0_c_2*(-e[1])
145
        X0_I_1 = (-1/2*sigma_I**2)*dt
147 X0_I_2 = sigma_I*sqrt(dt)
        M_X0_I[i, j]= M_X0_I[i, j-1]+X0_I_1+X0_I_2*e[2]
149 M_X0_I[i+1, j]= M_X0_I[i+1, j-1]+X0_I_1+X0_I_2*(-e[2])

        Z0_I_1 = exp(-(k_I*dt))
        Z0_I_2 = sqrt((v_I**2/(2*k_I))*(1-exp(-2*k_I*dt)))
153 M_Z0_I[i, j]= M_Z0_I[i, j-1]*Z0_I_1+Z0_I_2*e[3]
        M_Z0_I[i+1, j]= M_Z0_I[i+1, j-1]*Z0_I_1+Z0_I_2*(-e[3])
155
        a = j/360
157 seas_1_c = omega_1_c*cos(2*pi*(a))
        seas_2_c = omega_1_star_c*sin(2*pi*(a))
159 seas_3_c = omega_2_c*cos(2*pi*2*(a))
        seas_4_c = omega_2_star_c*sin(2*pi*2*(a))
161 M_seas_c[i, j]= seas_1_c+seas_2_c+seas_3_c+seas_4_c
        M_seas_c[i+1, j]= seas_1_c+seas_2_c+seas_3_c+seas_4_c
163

```

```

        seas_1_I = omega_1_I*cos(2*pi*(a))
165         seas_2_I = omega_1_star_I*sin(2*pi*(a))
        M_seas_I[i,j]= seas_1_I+seas_2_I
167         M_seas_I[i+1,j]= seas_1_I+seas_2_I

169
        C_matrix = exp(M_X0_c + M_Z0_c + M_seas_c)
171
        I_matrix = exp(M_X0_I + M_Z0_I + M_seas_I)
173

175 """ Plotting """

177 close('all')
        figure()
179 for i in xrange(simulations_2):
            plot(C_matrix[i, :])

181
        xlabel('time')
183 ylabel('Price')
        title('Gas_Price_Monte_Carlo_Simulation')
185 grid()
        show()

187

189 close('all')
        figure()
191 for i in xrange(simulations_2):
            plot(I_matrix[i, :])

193
        xlabel('time')
195 ylabel('HDD')
        title('HDD_Monte_Carlo_Simulation')
197 grid()
        show()

```

199

201 """ Pricing Standard Quanto Option """

203 mean\_c\_matrix = zeros([simulations\_2, 1])

sum\_I\_matrix = zeros([simulations\_2,1])

205

for i in xrange(simulations\_2):

207 mean\_c\_matrix[i, 0] = mean(C\_matrix[i, :])

sum\_I\_matrix[i,0] = sum(I\_matrix[i,:])

209

211 quan\_payoffs\_pv = zeros([simulations\_2,1])

213 for i in xrange(simulations\_2):

if mean\_c\_matrix[i, 0] > K\_E and sum\_I\_matrix[i,0] > K\_I:

215 quanto\_1 = mean\_c\_matrix[i, 0] - K\_E

quanto\_2 = sum\_I\_matrix[i,0] - K\_I

217 quan\_payoffs\_pv[i,-1] = ((quanto\_1\*quanto\_2)\*exp(-r\*T))

219 quan\_payoffs\_pv\_2 = []

221 g = even\_integers(simulations\_2)

223 for i in g:

y\_1 = quan\_payoffs\_pv[i,-1]

225 y\_2 = quan\_payoffs\_pv[i+1,-1]

quan\_payoffs\_pv\_2.append((y\_1+y\_2)/2)

227

quanto\_price = mean(quan\_payoffs\_pv\_2)

229 quanto\_SD = std(quan\_payoffs\_pv\_2)/sqrt(len(quan\_payoffs\_pv\_2))

231 print 'quanto\_price\_=', quanto\_price

print 'quanto\_SD\_=', quanto\_SD

233

```

235 """ Pricing Barrier Quanto Option """

237 bar_quan_payoffs_pv = zeros([simulations_2,1])

239 for i in xrange(simulations_2):
    if (C_matrix[i, :] < H_B).all():
241         if mean_c_matrix[i, 0] > K_E and sum_I_matrix[i,0] > K_I:
                bar_1 = mean_c_matrix[i, 0] - K_E
243                 bar_2 = sum_I_matrix[i, 0] - K_I
                bar_quan_payoffs_pv[i, -1]=((bar_1*bar_2)*exp(-r*T))

245
    bar_payoffs_pv_2 = []

247
    g = even_integers(simulations_2)

249
    for i in g:
251         y_1 = bar_quan_payoffs_pv[i, -1]
                y_2 = bar_quan_payoffs_pv[i+1, -1]
253         bar_payoffs_pv_2.append((y_1+y_2)/2)

255
    bar_quan_price = mean(bar_payoffs_pv_2)
257 bar_quan_SD = std(bar_payoffs_pv_2)/sqrt(len(bar_payoffs_pv_2))

259 print 'bar_quan_price_=', bar_quan_price
    print 'bar_quan_SD_=', bar_quan_SD

261

263 """ Pricing Parisian Quanto Option """

265 s = shape(C_matrix)

267 parisian_matrix = zeros([s[0], s[1]])

```

```

269 par_quan_payoffs_pv = zeros([simulations_2, 1])

271 for i in xrange(s[0]):
    for j in xrange(s[1]):
273         if C_matrix[i, j] >= H_P:
                parisian_matrix[i, j] = 1
275         else:
                parisian_matrix[i, j] = 0
277         if sum(parisian_matrix[i, :]) <= m:
                if mean_c_matrix[i, 0] > K_E and sum_I_matrix[i,0] > K_I:
279                 par_1 = mean_c_matrix[i, 0] - K_E
                        par_2 = sum_I_matrix[i, 0] - K_I
281                 par_quan_payoffs_pv[i, -1]=((par_1*par_2)*exp(-r*T))

283 par_payoffs_pv_2 = []

285 g = even_integers(simulations_2)

287 for i in g:
        y_1 = par_quan_payoffs_pv[i, -1]
289        y_2 = par_quan_payoffs_pv[i+1, -1]
                par_payoffs_pv_2.append((y_1+y_2)/2)
291
        par_quan_price = mean(par_payoffs_pv_2)
293 par_quan_SD = std(par_payoffs_pv_2)/sqrt(len(par_payoffs_pv_2))

295 print 'par_quan_price_==', par_quan_price
        print 'par_quan_SD_==', par_quan_SD

```

---

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