A multi-commodity dynamical model for traffic networks

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Abstract

A dynamical model for traffic networks is proposed and analyzed. In the traffic network, the transportation demands are considered as multi-commodity flows where each commodity has a unique destination. The network is modeled by a multigraph where at each node each commodity splits among the outgoing links in a way such that the drivers are more likely to avoid a road when the density on it increases. It will be shown that if the graph has no cycles, the density of each commodity on each link will converge to a unique limit that does not depend on the initial state.

Network resilience, namely structural robustness of the network with respect to perturbations, is also studied. In particular, it is shown that if all commodities have access to all outgoing links, the network can manage perturbations whose magnitude is less than a quantity which plays the natural role of residual capacity of an equilibrium. If instead not all commodities have access to all links, overreaction of the network to perturbations implies that even small perturbations might be amplified and start a cascade.

Finally, the idea of back-pressure is employed to provide a simple distributed control strategy. Analogously to the single commodity case, such actual strategy is able to back-propagate the information that congestion is happening ahead, thus allowing the drivers to reroute even if their decision is based on local information only.

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List of Notations

$\mathbb{R}$	The set of real numbers.
$\mathbb{R}_+$	The set of non-negative real numbers.
$[x]_+$	The positive part of x.
δ	The total magnitude of a perturbation.
δ_e	The magnitude of a perturbation on link e.
$\mathcal{E}$	The set of links.
$\mathcal{E}^k$	The set of links where commodity k are allowed to appear.
$\mathcal{E}_v^+$	The set of outgoing edges from node v.
$\mathcal{E}_v^-$	The set of incoming edges in node v.
$\mathcal{E}_v^{\mathcal{J}}$	$\bigcup_{k\in\mathcal{K}} \mathcal{E}_v^k$.
$\mathcal{E}_v^k$	The set of outgoing links from node v where commodity k is allowed to appear.
$\mathcal{G}$	The family of distributed routing policies.
$\gamma_1(f^*,\mathcal{G})$	The strong resilience.
$\mathcal{K}$	The set of commodity demands.
λ^k	The vector of inflows of commodity k.
λ_v^k	The inflow of commodity k to node v.
$\mathcal{M}$	A multigraph.
$\mathcal{N}$	A network.

$\Phi_t(x_0)$	The semi-flow at time t for initial state x_0.
$\mathcal{R}$	$\mathbb{R}_+^{\mathcal{E}}$
$\mathcal{R}_v$	$\mathbb{R}_+^{\mathcal{E}_v^+}$
$\mathcal{R}_v^k$	$\mathbb{R}_+^{\mathcal{E}_v^k}$
ρ^*	The limit density $\lim_{t\to\infty}\rho(t)=\rho^*$.
ρ^v	The vector of aggregate densities on links outgoing from a node v.
ρ_e	The aggregate density on link e.
ρ_e^k	The density of commodity k on link e.
ρ_e^{k*}	The limit density of commodity k on link e, $\lim_{t\to\infty}\rho_e^k(t)=\rho_e^{k*}$.
$G_{v\to e}^k(\rho^v)$	The fraction of commodity k routed to link e from node v.
$\mathcal{S}_v$	The simplex over $\mathcal{E}_v^+$.
σ_e	The tail of link e.
τ_e	The head of link e.
$\tilde{\mathcal{N}}$	A perturbed network.
$\tilde{\vartheta}_e(\rho_e)$	The perturbed velocity function.
$\mathcal{V}$	The set of nodes.
$\vartheta_e(\rho_e)$	The velocity function for link e.
C_e	The capacity for link e.
f_e	The aggregate flow on link e.
f_e^k	The flow of commodity k on link e.
$G_v^k(\rho^e)$	The routing policy for commodity k at node v.
$R(\mathcal{N},f^*)$	The minimum node residual capacity.

Contents

1

Introduction

Reliable and efficient transportation systems for people and goods are a fundamental part in the today's society. The transportation demands will continue to grow in the future. According to reports by the European Union, [EU, 2011], the transports of goods will increase by about 40 % by 2030 and about 80 % by 2050. The demands for public transportations will also increase by about 30 % to 2030 and about 50 % to 2050. The transportation networks today are already overloaded and it is estimated that today's congestions make up about one percent of the total European gross domestic product. Construction of models and control strategies for transportation networks are therefore of paramount importance to enhance good usage of transportation networks, giving both economic and environmental benefits.

In this master's thesis, a model that can be used to simulate urban traffic will be developed. It will model the traffic network on a macroscopic level, i.e., the densities on the roads will be seen as continuous and we are not trying to capture the behavior of each single car. We will model the fact that drivers want to go to different destinations using the same roads, as a multi-commodity problem.

1.1 The multi-commodity problem

The multi-commodity network problem is connected with sending different classes of particles, or commodities, from different sources to different destinations. The particles are sent along some kind of link. Links intersect/merge in nodes. Particles approaching a node have the possibility to travel further on a subsequent link of their choice. The multi-commodity network problem has several physical interpretations, to mention a few:

Traffic networks In a traffic network the particles are vehicles that start from different places and want to go to different destinations. The roads are the links that the cars can travel on and the junctions can be seen as nodes.

Data networks In a data network packages with data have to be sent between the clients and severs. The network cables can be interpreted as the links. Incoming and outgoing package from one sever will be routed to different clients, so there are locations in the network where the packages are directed to different links.

Production chains A factory producing different products might use some common machines in the manufacturing. Here the products can be seen as the particles and the machines as links. After a product has been handled by a machine, it might be sent to another one.

Supply chains Products, which can be seen as the particles, from different manufactures should be delivered to the stores. The transports, e.g., ships, trains or trucks, can be seen as links and places where the products are reloaded can be seen as nodes.

In this thesis it will be studied how the densities of the particles on the links in a multi-commodity network evolve in time. This involves analysis of the dynamics of the system, which is considerably different from the static theory usually employed to study transportation networks.

1.2 Problem formulation

The aim of this master thesis is to set up and analyze a model for the dynamical multi-commodity network problem. The model will be interpreted as a traffic routing problem and assumed to be distributed in the sense that the only fact that can affect drivers' decisions on road selection is the density on the outgoing roads from a junction. The model will be analyzed with respect to stability and robustness, i.e resilience.

1.3 Previous work

A comprehensive summary of static equilibrium in traffic networks, and traffic planing in general, can be found in [Patriksson, 1994]. However, this theory does not cover the dynamic properties of how the system converges to an equilibrium.

Another common model for traffic flows is the LWR model, which is based on equations for fluid dynamics and is a Riemann problem. In [Herty et al., 2006] the authors extend the LWR model to the multi-commodity case and propose criterion to handle junctions. The drawback of the LWR model is that it is hard to analyze for a whole traffic network.

During the last century, different control approaches for urban traffic networks have been developed. For a summary of some of them, see [Hamilton et al., 2013]. Two common strategies for traffic management today are TRANSYT and SCOOT, [Robertson and Bretherton, 1991]. TRANSYT bases its decisions on historical data and is an open-loop control. Therefore the performance degrades if the control strategy is not recalculated, and it is not able to react to perturbations or other unpredictable and rare events. SCOOT on the other hand, uses both historical data and measurements on the actual densities to control the traffic lights. In SCOOT, the control strategy needs to be computed in a centralized environment, which can be a disadvantage due to the requirement of information transmission and computational load. Therefore it can be of interest to search for strategies for distributed traffic control, which the model in this thesis might lay the foundation for.

In [Varaiya, 2009] a control strategy for traffic lights is proposed which ensures that the mean queue length stays bounded, if there exists any control strategy to keep it bounded. The proposed strategy is closely related to the ideas proposed for controlling the traffic lights in this thesis, including the possibility to keep the control distributed. However, in contrast to our work the queue lengths are modeled stochastically.

The main idea in [Varaiya, 2009] is very close related to the idea of back-pressure mentioned in [Tassiulas and Ephremides, 1992]. Basically, the idea is that if the outgoing roads from a junction are congested, the inflow will be stopped such that the density will increase on the incoming roads to the junction too. Then the information of a congestion ahead will back-propagate through the network implicitly, since the densities are higher. The back-pressure strategy has also recently been simulated for traffic networks, see [Wongpiromsarn et al., 2012].

The dynamical network problem has previously been studied for the single commodity case in [Como et al., 2013a], [Como et al., 2013b] and [Como et al., 2013c]. Parts of the framework and terminology given in previous references have therefore been used for the framework in this thesis. Some of the results stated in this thesis are modifications of analogous results developed in the single commodity scenario. For completeness, we provide the proofs in Appendix A.

1.4 Basic definitions

Let $\mathbb{R}$ be the set of real numbers and let the set of non-negative numbers be denoted by $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$.

If $x \in \mathbb{R}^n, y \in \mathbb{R}^n$, we define the following relations

$$\begin{aligned} x \leq y &\Leftrightarrow x_i \leq y_i, \\ x < y &\Leftrightarrow x_i \leq y_i, x \neq y, \\ x \ll y &\Leftrightarrow x_i < y_i. \end{aligned}$$

The relations $>$, $\gg$, $\geq$ are defined in an analogous way.

If $\mathcal{A}$ is a finite set, we denote the cardinality of a set $|\mathcal{A}|$, and with $\mathbb{R}^{\mathcal{A}}_{(+)}$ we mean the space of (nonnegative) real-valued vectors of length $|\mathcal{A}|$. We also let $\mathbb{R}^{\mathcal{A}\times\mathcal{B}}$ denote the space of matrices indexed by pairs of $\mathcal{A}\times\mathcal{B}$. If all the entries in the matrix are non-negative, the matrix is denoted $\mathbb{R}_+^{\mathcal{A}\times\mathcal{B}}$. Finally, $[x]_+$ is defined as the positive part of x as follows

$$[x]_+ := \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}.$$

1.5 Basic facts about monotone system

For a dynamical system, where $f(x,t)$ is Lipschitz,

$$\dot{x} = f(x,t),$$

we denote the solution for $t \geq 0$ with initial states x_0 as

$$x(t) = \Phi_t(x_0),$$

where $\Phi_t(x_o)$ is called the semi-flow.

A dynamical system is said to be monotone [Hirsch and Smith, 2005] with respect to the standard positive cone if for all $t \geq 0$ for which the solution exists

$$x_0 \leq y_0 \implies \Phi_t(x_0) \leq \Phi_t(y_0).$$

The definition above says that if we start with an initial state larger than another then this relation among trajectories is kept as time goes by.

The following Theorem, proved in [Kamke, 1932], provides a sufficient condition for a dynamical system to be monotone.

THEOREM 1
Suppose that we have a dynamical system

$$\dot{x}_i = f_i(x,t),$$

and it holds that

$$\frac{\partial f_i}{\partial x_j} \geq 0 \quad \forall i, j\, i \neq j,$$

then the system is monotone. □

EXAMPLE 1—MONOTONE SYSTEM
Consider the two state system

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1 = F_1(x_1, x_2) \\ \dot{x}_2 &= x_1 - x_2 = F_2(x_1, x_2). \end{aligned}$$

Clearly,

$$\frac{\partial}{\partial x_2}F_1 = \frac{\partial}{\partial x_1}F_2 = 1 \geq 0,$$

and according to Kamke's theorem, the system is monotone. The analytical solution to the system, with initial condition $(x_1(0), x_2(0)) = (x_1^0, x_2^0)$, is

$$\Phi_t(x_1^0, x_2^0) = \begin{bmatrix} \frac{x_1^0+x_2^0}{2} + \frac{x_1^0-x_2^0}{2}e^{-2t} \\ \frac{x_1^0+x_2^0}{2} - \frac{x_1^0+x_2^0}{2}e^{-2t} \end{bmatrix}.$$

Now, with a larger initial condition, $(2x_1^0, 2x_2^0) \geq (x_1^0, x_2^0) \geq (0,0)$, the relations between the states are preserved $\forall t \geq 0$,

$$\Phi_t(2x_1^0, 2x_2^0) - \Phi_t(x_1^0, x_2^0) = \begin{bmatrix} \frac{x_1^0+x_2^0}{2} + \frac{x_1^0-x_2^0}{2}e^{-2t} \\ \frac{x_1^0+x_2^0}{2} - \frac{x_1^0+x_2^0}{2}e^{-2t} \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which is consistent with the theory for monotone systems. □

2

A model for multi-commodity dynamical flow networks

In this chapter a model will be defined for multi-commodity dynamical flow networks. The network will be modeled as a graph. For each link we will consider a quantity representing the density of vehicles on the link, and a velocity function used to model the net flow, or average velocity, of vehicles on the link. We follow the literature [Papageorgiou et al., 2007] and assume that the velocity function only depends on the density on the link. When vehicles enter a node, they have to make a decision of which of the outgoing links they should use. To model this decision, a routing policy for each node will be defined.

In the second part of this chapter, a few assumptions will be made which try to capture the real life behavior of drivers and traffic networks.

2.1 Definitions

To describe the structure of a traffic network, i.e., roads and junctions, a multigraph will be used. A directed *multigraph* $\mathcal{M}$ is a pair of a finite set of nodes $\mathcal{V}$ and a finite multiset, i.e., a set were members are allowed to occur more than once, of links $\mathcal{E}$ which contains ordered pairs of nodes. Since the set of edges is a multiset, the definition allows multiple parallel links between two nodes. In the traffic interpretation the links can be seen as roads and the nodes as junctions. Since the links are directed, two links have to be used for modeling a two-way road. Moreover, a graph is said to be *acyclic* if there is no possibility to follow a sequence of the directed edges such that the same node is reached again.

For a link $e = (v_1, v_2) \in \mathcal{E}$ we write $\sigma_e = v_1$ for its tail and $\tau_e = v_2$ for its head. The *set of outgoing links*, $\mathcal{E}_v^+$, for a node $v \in \mathcal{V}$ is defined as $\mathcal{E}_v^+ := \{e \in \mathcal{E} : \sigma_e = v\}$. In the same manner the *set of incoming links* is defined as $\mathcal{E}_v^- := \{e \in \mathcal{E} : \tau_e = v\}$.

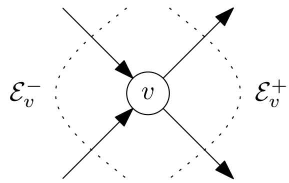

Figure 2.1 The sets $\mathcal{E}_v^-$ and $\mathcal{E}_v^+$ for a node v

The sets, for a node v, are illustrated in Figure 2.1. To make the notation shorter, further on we will denote $\mathcal{R}_v := \mathbb{R}_+^{\mathcal{E}_v^+}$ and $\mathcal{R} := \mathbb{R}_+^{\mathcal{E}}$.

Since the multi-commodity problem is about transporting different amounts of commodities to different destinations, the following set is used to describe the transportation demands.

DEFINITION 1—THE SET OF COMMODITY DEMANDS
Let *the set of commodity demands*, $\mathcal{K}$, be a set of pairs (λ^k, d_k) where $\lambda^k \in \mathbb{R}_+^{\mathcal{V}}$ is a vector with the inflow of the commodity k at each node and $d_k \in \mathcal{V}$ the destination node of commodity k. The element of the vector λ^k that corresponds the inflow to node v is denoted by λ_v^k. □

Example 2 illustrates Definition 1.

EXAMPLE 2—COMMODITY DEMANDS
For the multi-commodity problem showed in Figure 2.2, $\mathcal{K}$ has two elements (λ^A, v_1) and (λ^B, v_5) where $\lambda^A = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 1 & 0 \end{bmatrix}$ and $\lambda^B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0.5 & 0 \end{bmatrix}$. □

In the traffic interpretation, a commodity corresponds to the class of vehicles which have a specific destination. But, the generality of this definition allows several commodities to have the same destination node. This can be useful if the commodities have different preferences on the route to take in order to reach their destination. For instance, we can have both cars and trucks that want to reach a specific location, e.g., a store. Truck drivers might find however a path through, say, the city much less preferable than car drivers, and prefer a fast motorway. Vice versa, car drivers want to minimize the time they spend on the road, so they prefer to remain in the city as much as possible. Vehicles of the two classes will choose the less preferred route only if their first choice is congested and hence impractical.

To ensure that every vehicle of every commodity is able to reach its destination, the following sets are introduced.

DEFINITION 2—SET OF ALLOWED OUTGOING LINKS
The set of allowed outgoing links for a node $v \in \mathcal{V}$ and a commodity $k \in \mathcal{K}$ is denoted $\mathcal{E}_v^k := \{e \in \mathcal{E}_v^+ : \text{Particles of commodity } k \text{ on link } e \text{ are allowed}\}$. □

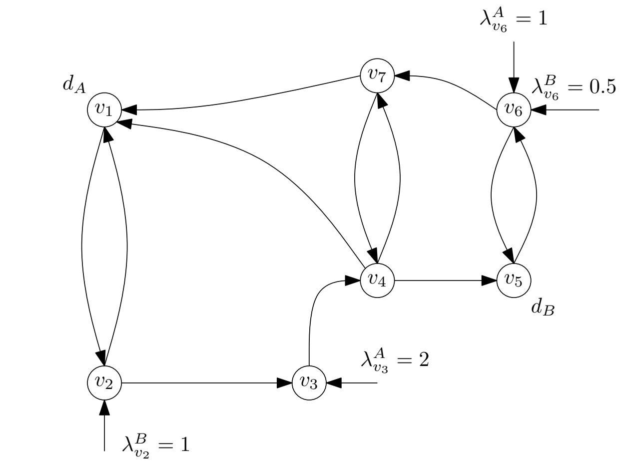

Figure 2.2 A simple network with specified commodity demands.

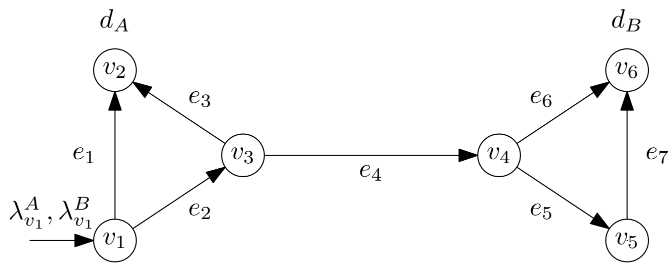

Figure 2.3 Example when not all commodities should appear on all links.

Another simplifying notation is then, $\mathcal{R}^k_v := \mathbb{R}^{\mathcal{E}^k_v}_+$. With $\mathcal{J} \subseteq \mathcal{K}$ we define $\mathcal{E}^{\mathcal{J}}_v := \bigcup_{k\in\mathcal{K}} \mathcal{E}^k_v$. The set of all links that commodity $k \in \mathcal{K}$ is allowed to use can then be defined as $\mathcal{E}^k := \bigcup_{v\in\mathcal{V}} \mathcal{E}^k_v$. We clarify the situation in the following example.

EXAMPLE 3
Imagine two cities connected by a long bridge, see Figure 2.3, where e_4 is the bridge. No matter what happens with the traffic in one of the cities, the drivers with destination in that city will never take the bridge over to the other city. So if we have two commodities A and B with $d_A = v_2$ and $d_B = v_6$, then $\mathcal{E}^A = \{e_1, e_2, e_3\}$ and $\mathcal{E}^B = \{e_2, e_4, e_5, e_6, e_7\}$. □

There are many other reasons for which some links should be blocked. For example, when some roads are reserved for public transportation and when trucks are forbidden to use some roads.

However, in many cases all commodities are allowed to take any route they want from a node, and therefore we introduce the following definition,

DEFINITION 3—FULLY ACCESSIBLE
A graph $\mathcal{M}$ together with a set of commodity demands $\mathcal{K}$ and the sets of allowed links $\mathcal{E}^k$ is fully accessible if for all $k \in \mathcal{K}$ and for all $v \in \mathcal{V}$ either $\mathcal{E}_v^k = \mathcal{E}_v^+$ or $\mathcal{E}_v^k = \emptyset$. □

For example, the network in Example 2 can be fully accessible, as there is not a risk that any vehicle of any commodity will not be able to reach its destination. In some scenarios it might still be conceivable to block roads for some commodities. For example, this might be done to avoid pathological cases such as vehicles running in loop along parallel links.

On every link in the graph, we might have vehicle densities of each commodity. All densities on all links can then be described by the matrix $\rho \in \mathbb{R}_+^{\mathcal{E}\times\mathcal{K}}$. The element corresponding to the density of commodity $k \in \mathcal{K}$ on link $e \in \mathcal{E}$ is then denoted ρ_e^k. The important notation of aggregate density is then defined as $\rho_e := \sum_{k\in\mathcal{K}} \rho_e^k$. This has the intuitive meaning of the full mass of vehicles on link e. The notion of aggregate density will play a fundamental role in th analysis of the system, as it will turn out that in some important cases the dynamics is monotone with respect to the aggregate density, even if this is not the case for the single commodities. Moreover the vector of aggregate densities on the outgoing links from a node $v \in \mathcal{V}$ is denoted $\rho^v = \{\rho_e : e \in \mathcal{E}_v^+\}$.

To model how fast the vehicles propagate forward on a link, a velocity function is assigned to each link. Together with a multigraph they define a network.

DEFINITION 4—NETWORK
A network $\mathcal{N} := (\mathcal{M}, \vartheta)$ is a pair of a directed multigraph $\mathcal{M}$ and a family of velocity functions $\vartheta := \{\vartheta_e : \mathbb{R}_+ \to \mathbb{R}_+\}_{\forall e\in\mathcal{E}}$ such that $\vartheta(\rho_e)$ describes the velocity of the vehicles on every link $e \in \mathcal{E}$. □

The flow of a commodity $k \in \mathcal{K}$ on a link $e \in \mathcal{E}$ is then given by the density times the velocity, such that

$$f_e^k := \rho_e^k \cdot \vartheta_e(\rho_e).$$

In the same way, the aggregate flow on a link is given by $f_e := \sum_{k\in\mathcal{K}} f_e^k = \rho_e \vartheta_e(\rho_e)$. Moreover, the flow capacity on each link $e \in \mathcal{E}$ is defined as

$$C_e := \sup_{\rho_e \geq 0} \rho_e \vartheta_e(\rho_e).$$

In the definition of a network, it is assumed that the only quantity that can affect the velocity is the aggregate density on the link. All other properties, like the the speed limit and size of the road are assumed to be static properties. Moreover, the velocity depends on the aggregate density, hence there is no possibility for one commodity to propagate faster than another. This efficiently captures the behavior on a single lane road, but on a multi lane road, where lanes might be assigned to different destinations, the velocity might be different. Imagine for instance an exit line on a highway, where the flow propagates slowly on that line, but not on the rest. This can be recovered in our model by letting each lane be modeled by a link, even if this would rule out the possibility, for a vehicle, to change lane.

REMARK 1
In [Como et al., 2013a] they also define a network, but they assign a flow function to each link instead. The definition of network has been here changed in this thesis since in the multi-commodity case the quantity which depends on the aggregate density is the velocity on every link, and not the flow. □

When a commodity flow enters a non destination node, it should be split among the outgoing edges, in such a way mass conservation is respected. This means that all incoming flow has to leave the node in the same time as it enters. To this aim, define $\mathcal{S}_v := \{p \in \mathcal{R}_v : \sum_{e \in \mathcal{E}_v^+} \rho_e = 1\}$, where p can be seen as a probability vector. If the flow instead reaches its destination, it should be directed further. We are now in the position to define a distributed routing policy, namely the map which handles such a splitting.

DEFINITION 5—DISTRIBUTED ROUTING POLICY
A distributed routing policy is a family of differentiable functions $\mathcal{G} := \{G_v^k(\rho^v) : \mathcal{R}_v \to \mathcal{R}_v\}_{\forall v \in \mathcal{E}, \forall k \in \mathcal{K}}$, where $G_{v \to e}^k(\rho^v)$ is the component in the codomain $\mathcal{R}_v$ corresponding to link $e \in \mathcal{E}$, for which it holds, for every node $v \in \mathcal{V}$ and every commodity demand $k \in \mathcal{K}$, that

a) if $v \neq d_k$ and $\mathcal{E}_v^- \cap \mathcal{E}^k \neq \emptyset$ then

$$G_v^k(\rho^v) : \mathcal{R}_v \to \mathcal{S}_v \subset \mathcal{R}_v,$$
$$G_{v \to e}^k(\rho^v) \equiv 0, \quad \forall e \in \mathcal{E}_v^+ \setminus \mathcal{E}_v^k.$$

b) if $v = d_k$ or $\mathcal{E}_v^- \cap \mathcal{E}^k = \emptyset$ then

$$G_v^k(\rho^v) \equiv 0.$$ □

We say that such a routing policy is distributed because each map of the family only depends on the aggregate density of the outgoing links. The routing policy is therefore not allowed to use information from non connecting edges when it decides how the vehicles should be routed. Moreover, the routing policy is only able to take the aggregate flow into consideration and is therefore not able to see where the vehicles are going. This assumption is quite natural for the traffic interpretation where a driver is only able to observe the aggregate densities when deciding which road to take. In other applications, for example data networks, it is instead easy to check the destination of the packages when they reach a node. However, it can still be useful to use a distributed routing policy, to avoid using link capacity for communication between nodes.

This definition of the routing policy is an extension of the one defined in [Como et al., 2013a]. In contrast to the single commodity case, the routing policy has to take care of the fact that the network might not be fully accessible and that all the commodities might not have the same destination node.

Using the definitions above we are now able to define a dynamical multi-commodity network.

DEFINITION 6—DYNAMICAL MULTI-COMMODITY NETWORK
A dynamical multi-commodity network is a network $\mathcal{N}$ associated with a family of distributed routing policies $\mathcal{G}$ and a set of commodity demands $\mathcal{K}$, where the dynamics of the network is given by

$$\begin{aligned} \dot{\rho}_e^k &= \Lambda_{\sigma_e}^k \cdot G_{\sigma_e \to e}^k(\rho^v(t)) - \rho_e^k(t)\vartheta_e(\rho_e(t)), \quad \forall e \in \mathcal{E}, \forall k \in \mathcal{K}. \\ \Lambda_v^k &:= \sum_{j \in \mathcal{E}_v^-} \rho_j^k(t) \cdot \vartheta_j(\rho_j(t)) + \lambda_v^k \end{aligned} \tag{2.2}$$

□

The definition above is a direct consequence of conservation laws. Particles that flow into a node have to be spread out on the outgoing links, and the change of densities of a link is the difference between the inflow and the outflow, which is described by the differential equation in the definition.

At last, a definition is needed to decide whenever the dynamical network reaches an equilibrium such that the commodity demands are fulfilled.

DEFINITION 7—FULLY TRANSFERRING
A dynamical network with the set of commodity demands, $\mathcal{K}$, is said to be fully transferring if for every $(\lambda^k, d_k) \in \mathcal{K}$ it holds

$$\liminf_{t \to \infty} \sum_{e \in \mathcal{E}_{d_k}^-} f_e^k = \sum_{v \in \mathcal{V}} \lambda_v^k.$$

□

The definition above says that a network is fully transferring if the limit inflows of a commodity to the destination node for that commodity equals all the static inflows for that commodity.

REMARK 2
One might notice that if all densities converge, then $\lim_{t \to \infty} \dot{\rho}_e^k = 0, \forall e \in \mathcal{E}, \forall k \in \mathcal{K}$ and the inflow to each node equals the outflow. In this situation, it is easy to see that the network is fully transferring at the equilibrium. □

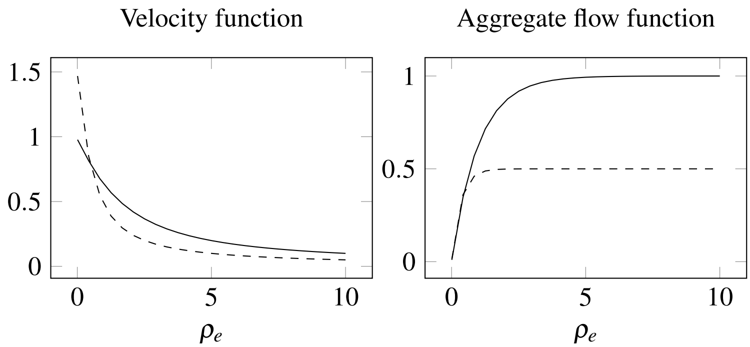

Figure 2.4 An example of a velocity function satisfying Assumption 1 and its corresponding aggregate flow function. The parameters used are $C_e = \mu_e = 1$ (solid line) and $C_e = 0.5$, $\mu_e = 3$ (dashed line).

2.2 Assumptions

In this section a few assumptions will be stated to capture the behavior of a traffic network. These assumptions will also later on turn out to be helpful in the proof.

First, the following assumption of the velocity function is made.

ASSUMPTION 1—VELOCITY FUNCTION
For each link $e \in \mathcal{E}$ the velocity function $\vartheta_e : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuously differentiable function with bounded derivative such that $\rho_e \vartheta_e(\rho_e)$ is strictly increasing and $\sup_{\rho_e \geq 0} \rho_e \vartheta_e(\rho_e) < +\infty$. □

Since the product $\rho_e \vartheta_e(\rho_e)$ is bounded and increasing with ρ_e, $\vartheta_e(\rho_e)$ will be decreasing with ρ_e. This means that a higher density will give a lower velocity as the road will become more congested.

Example 4 shows that there actually exist velocity functions satisfying Assumption 1.

EXAMPLE 4—VELOCITY FUNCTION
The function $\vartheta_e : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\vartheta_e(\rho_e) = C_e \cdot \frac{1 - e^{-\mu_e \rho_e}}{\rho_e},$$

where μ_e and C_e are positive constants, satisfy Assumption 1. This function together with the aggregate flow function $\rho_e \vartheta_e(\rho_e)$ are plotted in Figure 2.4. In the figure the parameters $C_e = \mu_e = 1$ are used for the solid line and $C_e = 0.5$, $\mu_e = 3$ are used for the dashed line. From Figure 2.4 it is possible to conclude that the parameter μ_e defines how sensitive the velocity function is to density changes, and can therefore be used to model the properties of the road. The parameter C_e can then be chosen in such a way that

$$\lim_{\rho_e \to 0+} \vartheta_e(\rho_e) = C_e \mu_e$$

equals the limit speed. □

In road traffic networks it is quite natural that the velocity is at least slightly decreasing when the traffic density increases. Measurements of how the flow depends on the density have been done in a real traffic network, see [Geroliminis and Daganzo, 2008]. According to these measurements, it seems as if the flow is increasing with the density, just like in the assumption. However, in the measurements it also seems like that after a certain point, the flow decreases with the density instead, even if the measurements are more unsure with these high densities. Our model captures well the scenario in which flows are increasing, which in turn well describes the behavior of the network in most of the cases.

Also some assumptions on the graph structure are needed, to ensure that it is possible for every commodity to reach its destination.

ASSUMPTION 2—EXISTENCE OF ORIGIN-DESTINATION PATHS
For every commodity demand $k = (\lambda_v^k, d_k) \in \mathcal{K}$ with $\lambda_k > 0$ it is assumed that $\mathcal{E}_v^k \neq \emptyset$. It is also assumed that for all $k \in \mathcal{K}$ and every $e \in \mathcal{E}^k$ there exists a path in $\mathcal{E}^k$ between τ_e and d_k. □

The first part of this definition guarantees that the static inflow at one node has at least one outgoing link that it can use. The second part states that commodity flows are able to reach their destinations from all of its accessible paths.

The last assumption is connected with the behavior of the routing policy.

ASSUMPTION 3—RESPONSIVE BEHAVIOR OF THE ROUTING POLICY
The routing policy is assumed to satisfy the following properties

a) For every $k \in \mathcal{K}$ and $v \in \mathcal{V}$ it holds that

$$\frac{\partial}{\partial \rho_j} G_{v \to e}^k(\rho^v) \geq 0, \quad \forall e, j \in \mathcal{E}_v^k, e \neq j.$$

b) For every $k \in \mathcal{K}$, $v \in \mathcal{V}$ and for every proper subset $\mathcal{J} \subsetneq \mathcal{E}_v^k$ there exists a continuously differentiable map $\bar{G}_v^k : \mathcal{R}_v \to \mathcal{S}_v$, such that if

$$\rho_e^v \to \infty, \quad \forall e \in \mathcal{E}_v^k \setminus \mathcal{J},$$
$$\rho_j^v \to \rho_j^{\mathcal{J}}, \quad \forall j \in \mathcal{J},$$

then

$$
\begin{aligned}
&G^k_{v\to e}(\rho^v) \to 0, && \forall e \in \mathcal{E}^k_v \setminus \mathcal{J}, \\
&G^k_{v\to j}(\rho^v) \to \bar{G}^k_{v\to j}(\rho^v), && \forall j \in \mathcal{J}. \qquad \square
\end{aligned}
$$

Since the routing policy is a mapping to a probability vector, the first part of the Assumption 3 also implies that

$$
\frac{\partial}{\partial \rho_e} G^k_{v\to e}(\rho^v) \leq 0, \quad \forall e \in \mathcal{E}^k_v.
$$

This means that if the aggregate density increases on one outgoing link, the vehicles are at least not more likely to take that link. For the traffic interpretation this can be seen as the driver tries to avoid roads with high traffic density. This assumption fits quite well with the real life behavior of drivers. Even if the density on the outgoing roads has a very little effect on the drivers decision, it is quite natural that the drivers try to avoid congested roads.

REMARK 3
This responsive behavior is also assumed in [Como et al., 2013a], except that we take in consideration the fact that the network might not be fully accessible. □

The example below shows that there actually exists functions satisfying Assumption 3.

EXAMPLE 5—THE LOGISTIC FUNCTION
The logistic function given by

$$
G^k_{v\to e}(\rho^v) = b^k_e \frac{e^{-\beta^k_e \rho_e}}{\sum_{j\in\mathcal{E}^+_v} b^k_j e^{-\beta^k_j \rho_j}},
$$

where

$$
b^k_e = \begin{cases} 0 & \text{if } e \notin \mathcal{E}^{k+}_v \\ 1 & \text{if } e \in \mathcal{E}^{k+}_v \end{cases},
$$

and $\beta^k_e > 0$, satisfies the Assumption 2.

The parameter β^k_e can be used to model routing preferences of vehicles of commodity k. Indeed, small values of β^k_e imply a certain degree of preference for link e, while high values model aversion. □

3

Stability analysis

In this chapter investigation of the global stability for a dynamical multi network will be made. Global stability throughout this paper will be meant in the following way:

DEFINITION 8—GLOBAL STABILITY
A multi-commodity dynamical network $\mathcal{N}$ achieves global stability if there exists a unique limit density ρ^* such that for every initial state $\rho(0) \in \mathbb{R}_+^{|\mathcal{K}|\times|\mathcal{E}|}$ it holds that

$$\lim_{t\to\infty} \rho(t) = \rho^*, \qquad \square$$

With global stability we mean the fact that for every initial state, the system will converge to a unique state. As a first step towards a result for the whole network, a local network, namely a network only consisting of one node, will be investigated. It will be showed that if some capacity constraints for the inflows are satisfied, all the densities on all outgoing links will converge to a unique limit point that does not depend on the initial densities. For the whole dynamical network, a sufficient condition for unique limit densities will also be stated.

3.1 Stability for a local dynamical network

To begin with, the stability properties of the local dynamical network will be investigated. A local network is a network with only one non-destination node, v, and time-varying inflows, $\lambda_v^k(t)$ for each commodity $k \in \mathcal{K}$, see Figure 3.1. The dynamics on the outgoing links in a local network is given by

$$\dot{\rho}_e^k(t) = \lambda_v^k(t) G_{\sigma_e\to e}^k(\rho^v(t)) - \rho_e^k(t)\vartheta_e(\rho_e(t)), \qquad \forall e \in \mathcal{E}_v^+, \forall k \in \mathcal{K}. \qquad (3.1)$$

Since the global network can be seen as an interconnection of local networks, studying the properties of a local network will give us the base from which we could state the stability conditions for the global network.

The main theorem of this section is the following result.

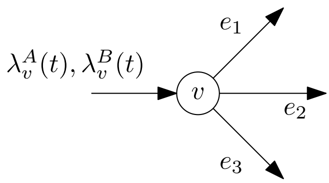

Figure 3.1 A local network with two commodities, A and B, and three outgoing links $\mathcal{E}_v^+ = \{e_1, e_2, e_3\}$.

THEOREM 2—STABILITY FOR A LOCAL DYNAMICAL MULTI-COMMODITY NETWORK

Assume $\mathcal{N}$ be a local multi-commodity dynamical network satisfying (3.1), Assumption 1 and Assumption 3. Assume moreover that the network has a converging inflow such that $\lim_{t\to\infty} \lambda_v^k(t) = \lambda_v^k, \forall k \in \mathcal{K}$. Then it holds that

a) if $\sum_{j\in\mathcal{J}} \lambda_v^j < \sum_{e\in\mathcal{E}_v^{\mathcal{J}}} C_e$ for every nonempty $\mathcal{J} \subseteq \mathcal{K}$, then there exists a finite ρ^* such that $\lim_{t\to\infty} \rho_e^k(t) = \rho_e^{k*}$ for every $e \in \mathcal{E}_v^+$ and $k \in \mathcal{K}$.

b) if there exists a nonempty $\mathcal{J} \subseteq \mathcal{K}$ such that $\sum_{j\in\mathcal{J}} \lambda_v^j \geq \sum_{e\in\mathcal{E}_v^{\mathcal{J}}} C_e$, then $\rho_e \to \infty$ for some $e \in \mathcal{E}_v^{\mathcal{J}}$. □

As a first step, it will be showed that the aggregate density on each outgoing link will converge. For the aggregate system, results from [Como et al., 2013a] can be used.

LEMMA 1—CONVERGENCE OF THE AGGREGATE DENSITIES WITH STATIC INFLOWS

Suppose that the inflows to the local network (3.1) are constant, i.e., $\lambda_v^k(t) \equiv \lambda_v^k$, $\forall k \in \mathcal{K}$, the velocity functions satisfy Assumption 1 and the routing policies satisfy Assumption 3 a). Then there exists a unique aggregate limit density $\rho^* \in \mathcal{R}_v$, such that, for every initial densities $\rho(0) \in \mathbb{R}_+^{\mathcal{E}_v^+ \times \mathcal{K}}$, the local network given by (3.1) satisfies

$$\lim_{t\to\infty} \rho(t) = \rho^*. \qquad \square$$

Proof Consider the aggregate flows on each outgoing link, $\rho_e = \sum_{k\in\mathcal{K}} \rho_e^k$. It is easy to see that the aggregate densities evolve according to

$$\dot{\rho}_e = \sum_{k\in\mathcal{K}} \lambda_v^k G_{\sigma_e\to e}^k(\rho^v(t)) - \rho_e(t)\vartheta(\rho_e(t)), \quad \forall e \in \mathcal{E}_v^+. \tag{3.2}$$

We define the map $H_e(\rho^v) : \mathcal{R}^v \to \mathcal{S}^v$ as follows:

$$H_e(\rho^v) := \frac{\sum_{k\in\mathcal{K}} \lambda_v^k G^k_{\sigma_e\to e}(\rho^v)}{\sum_{k\in\mathcal{K}} \lambda_v^k}.$$

It is easy to see that $H_e(\rho^v)$ satisfies

$$\frac{\partial}{\partial \rho_e} H_j(\rho^v) \geq 0, \quad \forall j, e \in \mathcal{E}_v^+, j \neq e,$$

since each routing policy is satisfying Assumption 3 a). The aggregate dynamics for the local dynamical network (3.2) can be rewritten as

$$\dot{\rho}_e = \left(\sum_{k\in\mathcal{K}} \lambda_v^k\right) H_e(\rho^v) - \rho_e(t)\vartheta(\rho_e(t)), \quad \forall e \in \mathcal{E}_v^+.$$

Now, applying [Como et al., 2013a, Lemma 2] shows that there exists a unique aggregate limit flow f_e^* for each $e \in \mathcal{E}_v^+$. But since the flows are given by $f_e(\rho_e) = \rho_e \vartheta_e(\rho_e)$ and the mappings $\rho_e \to \rho_e \vartheta_e(\rho_e)$ are assumed to be strictly monotone and continuous, the mappings between aggregate flows and aggregate densities are bijective and therefore the densities also will converge to a unique limit. □

The next step is to show that the density of each commodity on every outgoing link also converges. To be able to show that, one must ensure that the aggregate limit densities are bounded. Therefore the following lemma is stated.

LEMMA 2—CONDITION FOR A LOCAL NETWORK TO HAVE FINITE LIMIT DENSITIES

A sufficient and necessary condition for a local network, given by (3.1) and satisfying Assumption 3, to have finite limit densities is that for every subset $\mathcal{J} \subseteq \mathcal{K}$ it holds that

$$\sum_{j\in\mathcal{J}} \lambda_v^j < \sum_{e\in\mathcal{E}_v^{\mathcal{J}}} C_e. \tag{3.3}$$

□

Proof *Sufficiency* Let $\mathcal{I} := \{e \in \mathcal{E} : \rho_e^* = +\infty\}$. Observe then that $\limsup_{t\to\infty} \dot{\rho}_e(t) \geq 0, \forall e \in \mathcal{I}$. For a commodity $k \in \mathcal{K}$ two scenarios are possible. Either $\mathcal{E}_v^k \nsubseteq \mathcal{I}$, but then the property b) in Assumption 3 gives that $G^k_{e\to i}(\rho^{v*}) = 0, \forall i \in \mathcal{I}$ and those commodities are not contributing to the infinite limit density. In the other case, we have that $\mathcal{E}_v^k \subseteq \mathcal{I}$. Introduce the set $\mathcal{J} := \{k \in \mathcal{K} : \mathcal{E}_v^k \subseteq \mathcal{I}\}$ and sum up equation (3.1) over the commodities in $\mathcal{J}$,

$$\limsup_{t\to\infty} \sum_{j\in\mathcal{J}} \sum_{e\in\mathcal{E}_v^j} \left(\lambda_v^k G^k_{v\to e}(\rho^v(t)) - \rho_e^k(t)\vartheta(\rho_e(t))\right) = \sum_{j\in\mathcal{J}} \lambda_v^j - \sum_{e\in\mathcal{E}_v^{\mathcal{J}}} C_e \geq 0.$$

In the equality we use the fact that the image of the routing policy is a simplex is used, together with the fact that the aggregate flow is bounded. Then inequality (3.3) is violated. Hence $\mathcal{I} = \emptyset$.
Necessity Let $\mathcal{J} \subseteq \mathcal{K}$ be a nonempty subset such that

$$\sum_{j\in\mathcal{J}} \lambda_v^j \geq \sum_{e\in\mathcal{E}_v^{\mathcal{J}}} C_e.$$

Then

$$\begin{aligned}\sum_{j\in\mathcal{J}} \sum_{e\in\mathcal{E}_v^{\mathcal{J}}} \dot{\rho}_e^j &= \sum_{j\in\mathcal{J}} \lambda_v^j - \sum_{e\in\mathcal{E}_v^{\mathcal{J}}} \left(\sum_{j\in\mathcal{J}} \rho_e^j\right) \vartheta_e(\rho_e) \\ &\geq \sum_{j\in\mathcal{J}} \lambda_v^j - \sum_{e\in\mathcal{E}_v^{\mathcal{J}}} \rho_e \vartheta_e(\rho_e).\end{aligned}$$

Taking the limit of both sides gives

$$\liminf_{t\to\infty} \sum_{j\in\mathcal{J}} \sum_{e\in\mathcal{E}_v^{\mathcal{J}}} \dot{\rho}_e^j \geq \sum_{j\in\mathcal{J}} \lambda_v^j - \sum_{e\in\mathcal{E}_v^{\mathcal{J}}} \rho_e^* \vartheta_e(\rho_e^*) \geq \sum_{j\in\mathcal{J}} \lambda_v^j - \sum_{e\in\mathcal{E}_v^{\mathcal{J}}} C_e \geq 0,$$

where the central inequality is strict for any $e \in \mathcal{E}_v^{\mathcal{J}}$ such that $\rho_e^* < \infty$. This implies that there exists at least one edge $e \in \mathcal{E}_v^{\mathcal{J}}$ such that $\rho_e \to +\infty$ as $t \to \infty$ and the network is not fully transferring. □

In the case that the aggregate limit densities are finite, we can ensure that the density of each commodity is also finite.

LEMMA 3—CONVERGENCE OF COMMODITY DENSITIES IN A LOCAL DYNAMICAL NETWORK
If the local system (3.1) satisfies the condition given in Lemma 2, then each commodity density on the outgoing links will also converge to a unique finite value, i.e.,

$$\lim_{t\to\infty} \rho_e(t) = \rho_e^* < +\infty \quad \Longrightarrow \quad \lim_{t\to\infty} \rho_e^k(t) = \rho_e^{k*}, \quad \forall e \in \mathcal{E}_v^+, \forall k \in \mathcal{K}.$$

□

Proof Consider an arbitrary edge $e \in \mathcal{E}_v^+$. Since the aggregate densities converge, for every $\varepsilon > 0$ there exists a $t_0 > 0$ such that

$$|\rho_e(t) - \rho_e^*| < \varepsilon, \quad \forall t \geq t_0.$$

Due to the continuity assumption of the routing policy and since the aggregate densities are finite, there also exists a $\xi_k > 0$ such that

$$\left|\lambda_v^k G_{v\to e}^k(\rho^v(t)) - \lambda_v^k G_{v\to e}^k(\rho^{v*})\right| < \xi_k, \quad \forall k \in \mathcal{K},$$

and a $\kappa > 0$ such that

$$|\vartheta_e(\rho_e(t)) - \vartheta_e(\rho_e^*)| < \kappa.$$

Using the local system dynamics from (3.1) together with the inequalities, gives for $t > t_0$

$$\begin{aligned}\dot{\rho}_e^k(t) &= \lambda_v^k G_{v\to e}^k(\rho^v(t)) - \rho_e^k(t)\vartheta_e(\rho_e(t)) \\ &\leq \lambda_v^k G_{v\to e}^k(\rho^{v*}) + \xi_k - \rho_e^k(t)(\vartheta_e(\rho_e^*) - \kappa), \quad \forall k \in \mathcal{K}.\end{aligned} \tag{3.5}$$

Let $\alpha_k = \lambda_v^k G_{v\to e}^k(\rho^{v*})$ and $\beta_k = \vartheta_e(\rho_e^*)$. After applying the affine transformation,

$$\hat{\rho}_e^k(t) = \rho_k^e(t) - \frac{\alpha_k + \kappa}{\beta_k - \xi_k},$$

equation (3.5) can be written as

$$\dot{\hat{\rho}}_e^k(t) \leq -\hat{\rho}_e^k(t) \cdot (\beta_k - \xi_k).$$

Direct use of Gronwalls inequality (see Appendix A.4) yields

$$\hat{\rho}_e^k(t) \leq \hat{\rho}_e^k(0) \cdot e^{-(\beta-\xi)\cdot t} \to 0 \quad \text{when } t \to \infty,$$

if we choose ξ_k such that $\xi_k < \beta_k$. With $\rho_e^{k*} = \lim_{t->\infty} \rho_e^{k*}(t)$, the inequality above can be written as

$$\rho_e^{k*} \leq \frac{\alpha_k + \kappa}{\beta_k - \xi_k}.$$

In the same way it holds

$$\begin{aligned}\dot{\rho}_e^k(t) &= \lambda_v^k G_{v\to e}^k(\rho^v(t)) - \rho_e^k(t)\vartheta_e(\rho_e(t)) \\ &\geq \lambda_v^k G_{v\to e}^k(\rho^{v*}) - \xi_k - \rho_e^k(t)(\vartheta_e(\rho_e^*) + \kappa), \quad \forall k \in \mathcal{K}.\end{aligned}$$

Using the same technique again gives a lower bound on ρ_e^{k*}, and the limit density can be bounded as follows

$$\frac{\alpha_k - \kappa}{\beta_k + \xi_k} \leq \rho_e^{k*} \leq \frac{\alpha_k + \kappa}{\beta_k - \xi_k}.$$

Since we have the freedom to choose κ and ξ_k arbitrarily small, it has been now shown that

$$\lim_{t\to\infty} \rho_e^{k*}(t) = \rho_e^{k*}, \quad \forall e \in \mathcal{E}, \forall k \in \mathcal{K}. \qquad \square$$

Again, since the mapping between densities and flows is bijective, the lemma implies that the limit flows will be unique as well.

The next step is to show that, if the system converges for static inflows, converging inflows are a sufficient condition for the densities to converge.

LEMMA 4—CONVERGING AGGREGATE INFLOW IMPLIES CONVERGING AGGREGATE DENSITIES
Consider the local system (3.1). If $\lim_{t\to\infty} \lambda_k(t) = \lambda_k, \forall k \in \mathcal{K}$, then the aggregate densities on each link $e \in \mathcal{E}_v^+$ will also converge, namely

$$\lim_{t\to\infty} \rho_e(t) = \rho_e^*, \quad \forall e \in \mathcal{E}_v^+. \qquad \square$$

Proof For the aggregate density on the outgoing links, the dynamics is given by (3.2). Denote the right hand side of equation (3.2) by F_e. Then it holds, due to property a) in Assumption 3, that

$$\frac{\partial}{\partial \rho_e} F_j \geq 0, \quad \forall e, j \in \mathcal{E}_v^+; j \neq e,$$

and

$$\frac{\partial}{\partial \lambda_k} F_e \geq 0 \quad \forall e \in \mathcal{E}_v^+, \forall k \in \mathcal{K}.$$

The system is then a controlled monotone system in the sense of Angeli and Sontag [Angeli and Sontag, 2003], so we have monotonicity with respect to both the input variables and the states.

If $\lambda(t) \in \mathbb{R}_+^{\mathcal{K}}$ is converging to λ^*, for each $\varepsilon > 0$, there exists a $t_0 > 0$, such that $|\lambda(t) - \lambda^*| < \varepsilon$ for $t > t_0$. Due to monotonicity of the system we have that

$$\Phi_t(\lambda^* - \varepsilon, \rho(0)) \leq \Phi_t(\lambda(t), \rho(0)) \leq \Phi_t(\lambda^* + \varepsilon, \rho(0)), \quad t \geq t_0,$$

where $\Phi_t : (\mathbb{R}_+^{\mathcal{K}}, \mathcal{R}, \mathbb{R}_+) \to \mathcal{R}_v$ is the semiflow. Given the inflow λ, we denote the corresponding limit density by $\rho^*(\lambda)$. As $t \to \infty$ it holds that

$$\rho_e^*(\lambda^* - \varepsilon) \leq \lim_{t\to\infty} \rho_e(\lambda(t)) \leq \rho_e^*(\lambda^* + \varepsilon),$$

where Lemma 2 guarantees that $\rho_e^*(\lambda^* - \varepsilon)$ and $\rho_e^*(\lambda^* + \varepsilon)$ will converge. This implies that the limit flow $\rho_e^*(\lambda)$ depends continuously on λ (see Appendix A.1) and by letting $\varepsilon \to 0$, the aggregate density on each edge will converge to the limit density generated by λ^*, which is unique. $\square$

This fact is also proven in a more general way in [Angeli and Sontag, 2003].

Since the commodity flows depends continuously on the densities, the outgoing commodity flows will also converge and Lemma 4 together with Lemma 2 give

$$\lim_{t\to\infty} \lambda_k(t) = \lambda_k \implies \lim_{t\to\infty} f_e^k(t) = f_e^{k*}, \quad \forall k \in \mathcal{K}, \forall e \in \mathcal{E}_v^+$$

We are now in the position to prove Theorem 2.

Proof of Theorem 2 The lemmas stated in this section make up the sufficiency part of Theorem 2. The necessity condition in Lemma 2 gives the second part. □

3.2 Stability for the global acyclic dynamical network

The next step will be proving a result about converging densities for the global dynamical network. Knowledge of the stability of the local dynamical network together with induction will be used. In order to use induction, we first need to define the notion of topological ordering for a graph.

DEFINITION 9—TOPOLOGICAL ORDERING
[Gabow et al., 2003] A topological ordering of a directed graph, is an ordering of the nodes $v \in \mathcal{V}$, $v_1, v_2, .., v_n$, such that for every edge $e = (v_a, v_b) \in \mathcal{V}$ it holds that $a < b$. □

Topological ordering is not a property of every graph. However, this is the case for acyclic graphs.

LEMMA 5—TOPOLOGICAL ORDERING OF AN ACYCLIC GRAPH
If the graph is acyclic, there exists a (not necessarily unique) topological ordering of the nodes. □

Proof Proved in [Cormen et al., 2003, Theorem 22.12]. □

We state first a condition for the global network to have finite densities on all links. Using the fact that the local networks need to have finite limit densities together with the fact that there exists a topological ordering we are able to state a sufficient condition for finite limit densities for the global network. In the proof, we shall consider the network node by node. In order to take into account commodities that have not reached theirs destinations only, we introduce the following set.

DEFINITION 10—THE SET OF PRESENT COMMODITIES
For a node $v \in \mathcal{V}$ the set of present commodities is given by $\mathcal{K}_v := \{j \in \mathcal{K} : \mathcal{E}_v^j \neq \emptyset\}$. □

Now a sufficient condition for the global network to have finite limit densities can be stated.

PROPOSITION 1—SUFFICIENT CONDITIONS FOR A DYNAMICAL NETWORK TO HAVE FINITE LIMIT DENSITIES
A sufficient condition for an acyclic dynamical network satisfying the assumptions in Chapter 2 to have finite limit densities is that for every subset $\mathcal{J} \subseteq \mathcal{K}$ it holds that

$$\max_{v \in \mathcal{V}} \sum_{w \leq v} \sum_{j \in \mathcal{J} \bigcap \mathcal{K}_v} \lambda_w^j - \sum_{e \in \mathcal{E}_v^{\mathcal{J}}} C_e < 0,. \tag{3.6}$$

□

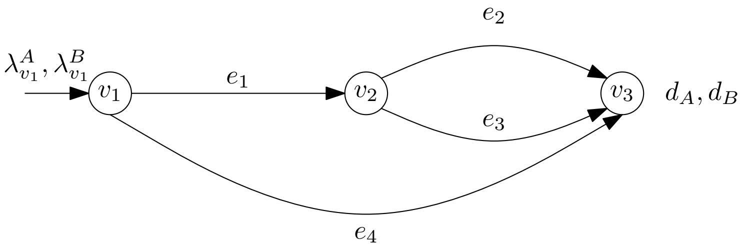

Figure 3.2 The graph used in Example 6. Link e_4 gives the flows the possibility to bypass some edges.

Proof Since the graph is acyclic due to Assumption 2, Lemma 5 gives us the possibility to do an induction proof over the graph. Number the nodes $v = 0, 1, .., n-2$. For $v = 0$ Lemma 2 guarantees that the network is fully transferring. Now consider an arbitrary node $0 < w < n-1$. Only flows that are present on the outgoing links can violate the condition and it is therefore enough to only consider commodities in the set $\mathcal{J} \bigcap \mathcal{K}_v$. Since the inflow of a commodity to node w can not be greater than the total static inflow up to node w, it follows from Lemma 2 that the limit flows are finite for node w. □

The inequality in Proposition 1 can sometimes be too conservative, as Example 6 illustrates.

EXAMPLE 6—PROPOSITION 1 DOES NOT STATE A NECESSARY CONDITION
Consider the dynamical network based on the graph shown in Figure 3.2 below, with two commodities, A and B. Let $\mathcal{E}^A = \{e_1, e_2, e_4\}$, $\mathcal{E}^B = \{e_1, e_3, e_4\}$, $C_{e_1} = C_{e_2} = C_{e_4} = 5$ and $C_{e_3} = 1$. The rest of the simulation parameters are given in Appendix B.1. Since the capacity of e_3 is $C_{e_3} = 1$, Proposition 1 gives a sufficient condition that $\lambda^B_{v_1} < 1$. However, as the plots in Figure 3.3 show, higher inflow, $\lambda^B_{v_1} = 2$ in this case, can be handled, since most of the flow of commodity B chooses link e_4. □

Given that the limit flows are finite and the fact that a topological ordering exists, the following result for existence of unique limit flows for the global network can be stated.

COROLLARY 1—EXISTENCE OF UNIQUE LIMIT FLOWS FOR THE GLOBAL NETWORK
Consider an acyclic multi-commodity network satisfying all assumptions in Chapter 2. If for every subset $\mathcal{J} \subseteq \mathcal{K}$ it holds that

$$\max_{v \in \mathcal{V}} \sum_{w \leq v} \sum_{j \in \mathcal{J} \bigcap \mathcal{K}_v} \lambda^j_w - \sum_{e \in \mathcal{E}^{\mathcal{J}}_v} C_e < 0,$$

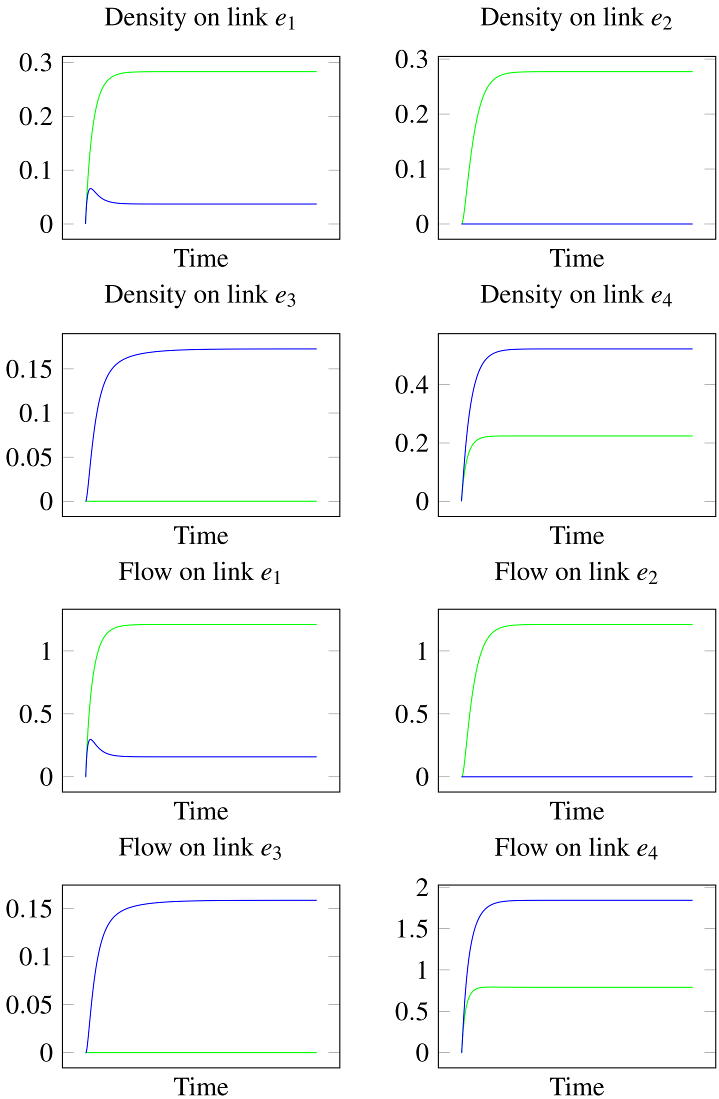

Figure 3.3 The propagation of densities and flows for Example 6. Flows/densities of commodity A are plotted in green and flows/densities of commodity B are plotted in blue.

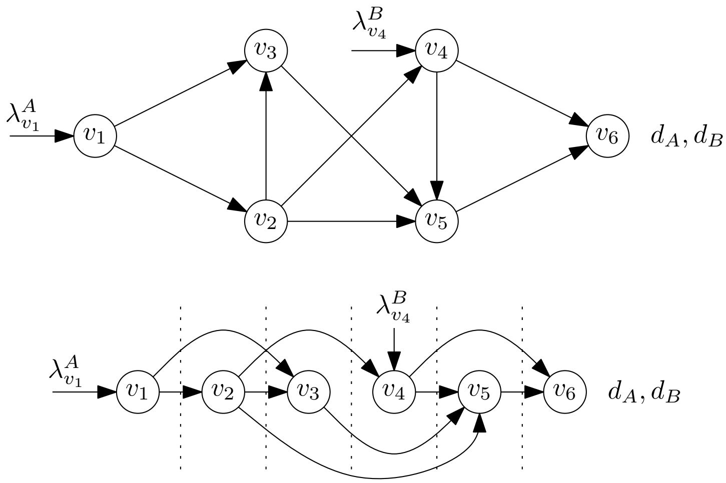

Figure 3.4 The induction argument used to prove Theorem 2. Since the graph is acyclic, if can be formed as a chain of local networks, separated in the figure with dotted lines.

then for every commodity $k \in \mathcal{K}$ and every link $e \in \mathcal{E}$ in the dynamical multi-commodity network given by (2.2) there exists unique limit flows $f_e^{k*}, \forall k \in \mathcal{K}, \forall e \in \mathcal{E}$, which depends on the demands λ_k, but not the initial states, such that

$$\lim_{t \to \infty} f_e^k(t) = f_e^{k*}, \quad \forall k \in \mathcal{K}, \forall e \in \mathcal{E} \qquad \square$$

Proof Since there exists a topological ordering, it is possible to proceed by induction over the nodes, numbered from $v = 0$ to $v = n-1$. In fact, Proposition 1 guarantee that the limit densities will be finite. For the first node $v = 0$, the inputs are only static, so Lemma 1 and Lemma 3 guarantees that the outflows of the first node converge. Suppose now that the outflow is converging for all nodes up to node number w, where $1 \leq w \leq n-2$. Since the inflow to node w only depends on the nodes $v \leq w$ and possibly some static inflows at node w, all inflows to w are converging. Then Lemma 4 implies that the outflows from node w also converge. $\square$

A schematic sketch of the topological ordering and induction argument is shown in Figure 3.4. To illustrate Corollary 1, a numerical example is shown in Example 7.

EXAMPLE 7
Suppose that we have a simple graph with one node and two outgoing links, see Figure 3.5. Let $\lambda_v^A = 1$ and $\lambda_v^B = 2$. Further, let the velocity functions for the edges

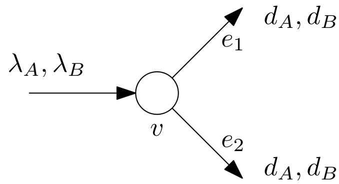

Figure 3.5 A local network with two incoming commodities and two outgoing links.

be

$$\vartheta_1(\rho_1) = 2 \cdot \frac{1 - e^{-2\rho_2}}{\rho_1}, \qquad \vartheta_2(\rho_2) = 2 \cdot \frac{1 - e^{-5\rho_2}}{\rho_2}$$

and the routing policies be

$$G^A_{v\to e_1}(\rho_1,\rho_2) = \frac{e^{-3\rho_1}}{e^{-3\rho_1} + e^{-8\rho_2}}, \qquad G^B_{v\to e_1}(\rho_1,\rho_2) = \frac{e^{-7\rho_1}}{e^{-7\rho_1} + e^{-2\rho_2}},$$
$$G^A_{v\to e_2}(\rho_1,\rho_2) = \frac{e^{-8\rho_2}}{e^{-3\rho_1} + e^{-8\rho_2}}, \qquad G^B_{v\to e_2}(\rho_1,\rho_2) = \frac{e^{-2\rho_2}}{e^{-7\rho_1} + e^{-2\rho_2}}.$$

The flow dynamics for two different initial states is shown in Figure 3.6. The first initial state is when all densities are zero. The second initial state is $\rho^A_{e_1} = 1.5$, $\rho^B_{e_1} = 0.5$, $\rho^A_{e_2} = 0.5$ and $\rho^B_{e_2} = 1$.

Figure 3.6 shows that both initial states converge to the same limit densities, which is exactly what Corollary 1 states. □

3.3 Graphs with cycles

For graphs with cycles, it is not possible to find a topological ordering and use an induction argument like in the acyclic case. However, for the single commodity case, it is proved in [Como et al., 2013c] that the existence of an unique limit flow is ensured even for graphs with cycles. The theorem is based on the monotonicity property of the system. In the multi-commodity case the monotonicity is lost, even for the local network. In fact, denote the right hand side of (3.1) by F^k_e. Then

$$\frac{\partial F^k_e}{\partial \rho^j_e} = \lambda_k(t) \underbrace{\frac{\partial}{\partial \rho^j_e} G^k_{\sigma(e)\to e}(\rho^v(t))}_{\leq 0} - \rho^k_e(t) \underbrace{\vartheta'_e(\rho_e(t))}_{\leq 0},$$

and it is not possible to claim that $\frac{\partial F^k_e}{\partial \rho^j_e} \geq 0, \forall k \neq j, k, j \in \mathcal{K}$. Hence, Kamke's Theorem is not necessary fulfilled and the system might not be monotone. This can also

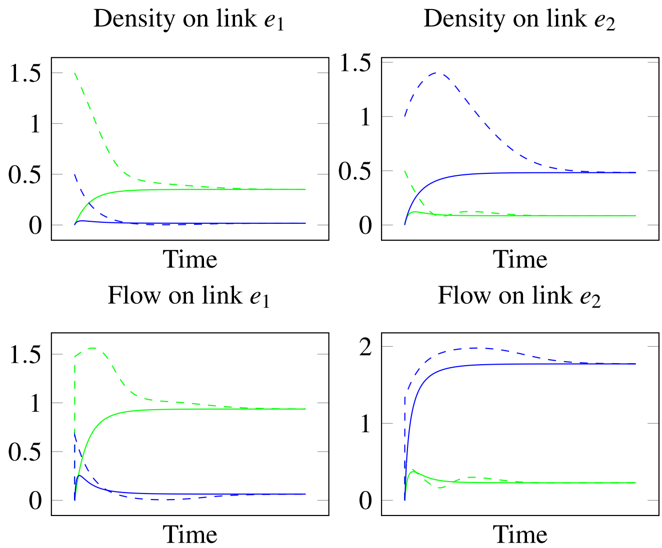

Figure 3.6 How the densities and flows of commodity A (green) and B (blue) in the local network propagates. The solid lines represent the case with zero initial density on all links. The dashed lines represent the solution with the initial densities $\rho^A_{e_1} = 1.5$, $\rho^B_{e_1} = 0.5$, $\rho^A_{e_2} = 0.5$ and $\rho^B_{e_2} = 1$.

be seen in simulations. In Example 8 a simulation is run where one commodity is allowed to go in a cycle.

EXAMPLE 8—DYNAMICAL NETWORK WITH CYCLES
The graph for the dynamical network, with two commodities A and B, is shown in Figure 3.7. Moreover $\mathcal{E}^A = \{e_1, e_3, e_4, e_5, e_6, e_8\}$ and $\mathcal{E}^B = \{e_1, e_2, e_3, e_5, e_6, e_7\}$, and therefore commodity B is able to go in the cycle $v_4 \to v_1 \to v_5 \to v_4$. The rest of the simulation parameters are stated in Appendix B.2. The trajectory of the densities starting from zero initial densities is shown in Figure 3.8. Three trajectories for different non-zero initial states are also shown in Figure 3.9.

In Example 8 it is possible to see that the monotonicity is lost for both the aggregate densities and for each single commodity density. Notice now that for a monotone system with a zero initial state it holds that

$$\Phi_{t+s}(0) = \Phi_t(\Phi_s(0)) \geq \Phi_t(0),$$

where the inequality follows from the fact that $\Phi_s(0) \geq 0$ and that the system is monotone. Hence, each state in a monotone system will be non-decreasing if the

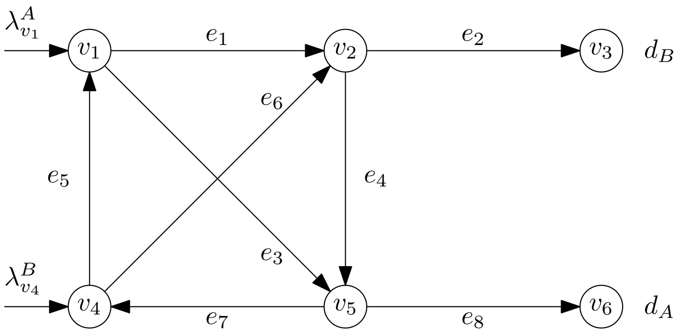

Figure 3.7 A simple multi-commodity network with two commodities, *A* and *B*, where commodity *B* is allowed to go in a cycle.

initial state is zero. However, in Example 8 it is clear that the aggregate flows on edges e_4 and e_6 are decreasing.

When running numerical simulations on the network in Example 8, one can notice that the densities converge to a unique limit which does not seem to depend on the initial state. However, a clear analysis of this scenario must rely on more evaluated facts than those employed in this thesis, and is hence left for future research.

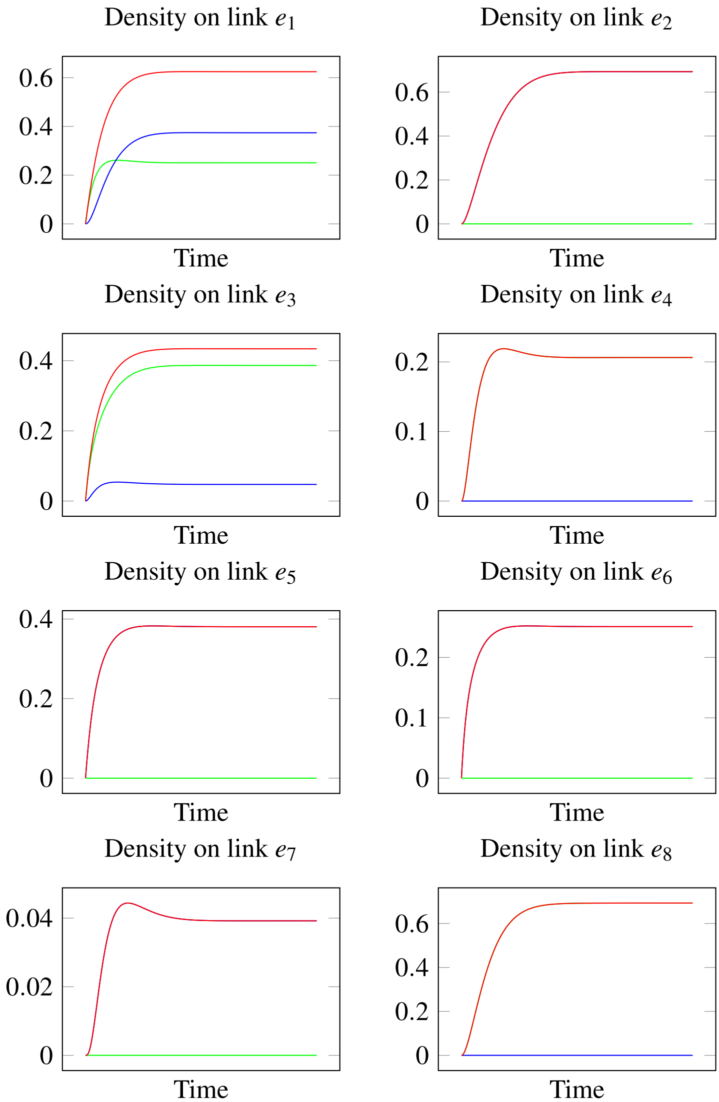

Figure 3.8 How the densities propagates with time for the dynamical network, with two commodities A (green) and B (blue), given in Example 8. The aggregate densities are also plotted in red.

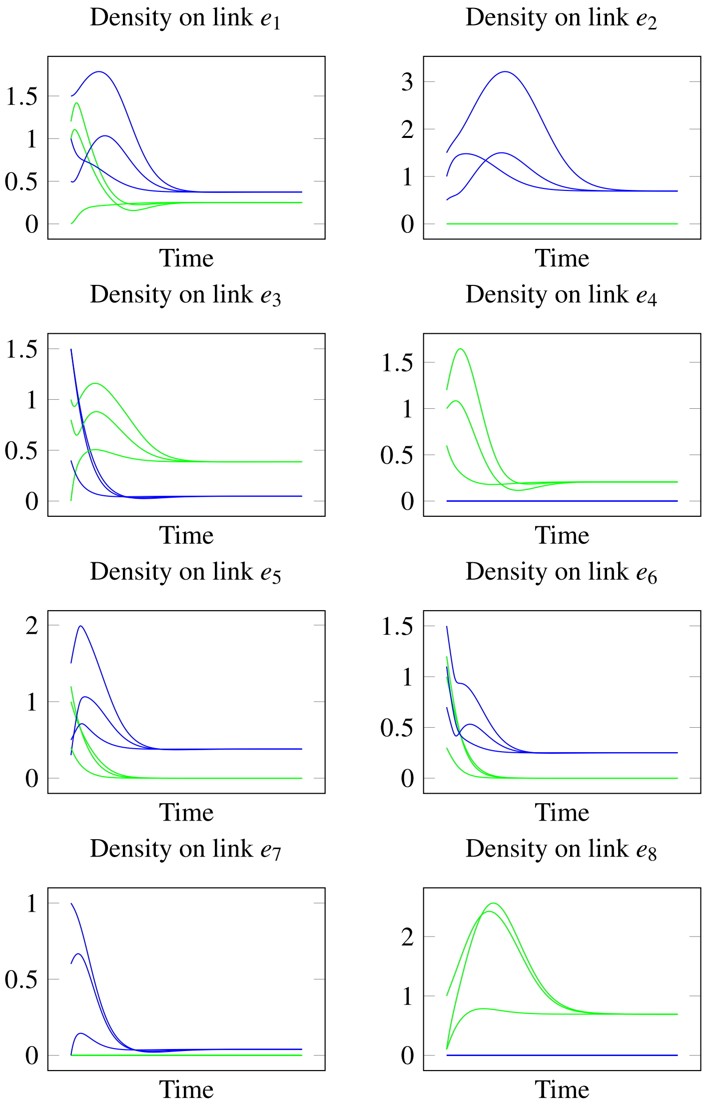

Figure 3.9 How the densities and flows propagates with time for the dynamical network, with two commodities A (green) and B (blue) and three different initial densities, given in Example 8.

4

Resilience

The goal of this chapter is to study network's resilience, i.e., how sensitive the dynamical networks are to perturbations. In particular, we will be interested in perturbations which reduce the capacity on one or many links. In traffic networks, this can be interpreted as car accidents or road works. It is important for the robustness of the network that a small perturbation at one place in the network does not cause large flow changes somewhere else.

First, perturbation for the dynamical network will be formally defined. Then it will be showed that there exists an upper bound on how large perturbations the network can resist such that it is still fully transferring. If the network is fully accessible, it will also be showed that the upper bound is tight. However simulations will show that a small perturbation might be able to affect the flow of one commodity much larger than the magnitude of the perturbation.

4.1 Perturbed dynamical network

To begin with, we have to define what we mean by a perturbation.

DEFINITION 11—ADMISSIBLE PERTURBATION
An admissible perturbation of a network $\mathcal{N} = (\mathcal{G}, \vartheta)$ is a network $\tilde{\mathcal{N}} = (\mathcal{G}, \tilde{\vartheta})$ where $\tilde{\vartheta}$ is a family of perturbed velocity functions $\tilde{\vartheta} := \{\tilde{\vartheta}_e : \mathbb{R}_+ \to \mathbb{R}_+\}_{e \in \mathcal{E}}$ such that

$$\tilde{\vartheta}_e(\rho_e) \leq \vartheta_e(\rho_e), \quad \forall \rho_e \geq 0. \qquad \square$$

REMARK 4
Again this definition differs from the one in [Como et al., 2013b], where it is stated for the flow functions instead. □

For the traffic interpretation a perturbation can be seen as a velocity decreasing action, like an accident or roadworks, and it is quite natural that the velocity at least does not increase when something like that happens.

The magnitude of the perturbation, which is measured with respect to the flow, for a link $e \in \mathcal{E}$ can then be defined as

$$\delta_e := \sup_{\rho_e \geq 0} \left(\rho_e \vartheta_e(\rho_e) - \rho_e \tilde{\vartheta}_e(\rho_e) \right).$$

The total magnitude of all perturbations is then defined as

$$\delta := \sum_{e \in \mathcal{E}} \delta_e.$$

An example of a perturbed velocity function is shown below.

EXAMPLE 9—PERTURBED VELOCITY FUNCTION
Consider again the velocity function

$$\vartheta_e(\rho_e) = C_e \cdot \frac{1 - e^{-\mu_e \rho_e}}{\rho_e},$$

where $C_e > 0$ and $\mu_e > 0$. A perturbed variant is then

$$\tilde{\vartheta}_e(\rho_e) = \frac{C_e}{2} \cdot \frac{1 - e^{-\mu_e \rho_e}}{\rho_e},$$

and the magnitude of the perturbation is

$$\delta_e = \sup_{\rho_e \geq 0} \left(\rho_e \vartheta(\rho_e) - \rho_e \tilde{\vartheta}(\rho_e) \right) = \frac{C_e}{2} \cdot \sup_{\rho_e \geq 0} \left(1 - e^{-\mu_e \rho_e} \right) = \frac{C_e}{2}. \qquad \square$$

With the definitions above, the definition of a perturbed dynamical network can now be stated.

DEFINITION 12—PERTURBED DYNAMICAL NETWORK
For a dynamical network, given in Definition 6, together with an admissible perturbation, given in Definition 11, the corresponding perturbed dynamical network is given by the following dynamics

$$\begin{aligned} \frac{d}{dt} \tilde{\rho}_e^k &= \tilde{\Lambda}_{\sigma_e}^k G_{\sigma_e \to e}^k (\tilde{\rho}^v(t)) - \tilde{\rho}_e^k(t) \tilde{\vartheta}_e(\tilde{\rho}_e(t)), \quad \forall e \in \mathcal{E}, \forall k \in \mathcal{K}, \\ \tilde{\Lambda}_v^k &= \sum_{j \in \mathcal{E}_v^-} \tilde{\rho}_j^k(t) \cdot \tilde{\vartheta}_j(\tilde{\rho}_j(t)) + \lambda_v^k. \end{aligned} \tag{4.2}$$

$\square$

4.2 An upper bound on the strong resilience

A point of interest is how much the network can be perturbed without loosing any throughput, therefore the following concept of *strong resilience* is introduced.

DEFINITION 13—STRONG RESILIENCE
The *strong resilience* , $\gamma_1(f^*, \mathcal{G})$, is defined as the infinitum magnitude over all admissible perturbations such that the network is not fully transferring anymore with respect to an initial flow with f^* as limit flow. □

Moreover a definition is needed of how much spare capacity a dynamical network with a limit flow f^* has.

DEFINITION 14—MINIMUM RESIDUAL CAPACITY
For a multi-commodity dynamical network $\mathcal{N}$, together with a limit flow f^*, the minimum residual capacity is defined as

$$R(\mathcal{N}, f^*) := \min_{v \in \mathcal{V}} \min_{\mathcal{J} \subseteq \mathcal{K}} \sum_{e \in \mathcal{E}_v^{\mathcal{J}}} \left(C_e - \sum_{j \in \mathcal{J}} f_e^{j*} \right). \qquad \square$$

If the network is fully accessible the definition can be rewritten as

$$R(\mathcal{N}, f^*) = \min_{v \in \mathcal{V}} \sum_{e \in \mathcal{E}_v^+} (C_e - f_e^*).$$

The definition is then consistent with the one given in [Como et al., 2013b] for the single commodity case.

EXAMPLE 10—MINIMUM RESIDUAL CAPACITY
Consider the network with three commodities and the limit flows in Figure 4.1. The minimum residual capacity, 2, is attained at node v_2 and commodity subset $\mathcal{J} = \{A\}$.

Now an upper bound on the strong resilience can be stated, i.e., how large perturbation needed for guaranteeing that the dynamical network not fully transferring any more.

LEMMA 6—UPPER BOUND ON THE STRONG RESILIENCE
Suppose that $f^* \ll C$ is a limit flow for a dynamical network, then it holds that

$$\gamma_1(f^*, \mathcal{G}) \leq R(\mathcal{N}, f^*). \qquad \square$$

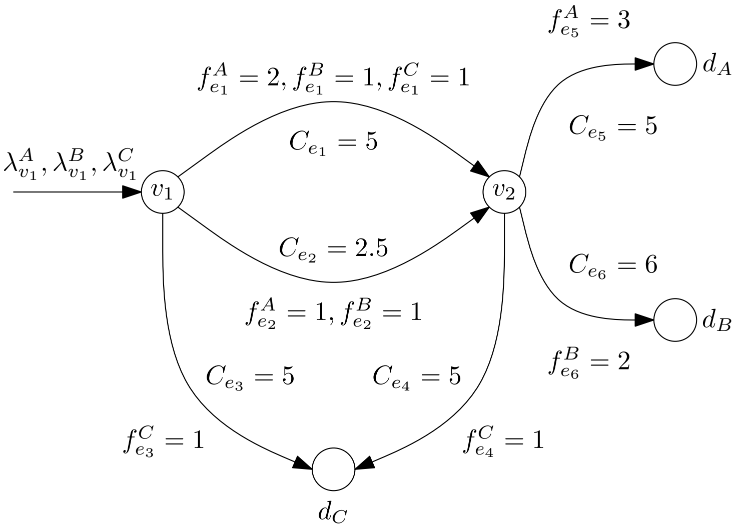

Figure 4.1 Dynamical network with limit flows

Proof Suppose that there exists a node $v \in \mathcal{V}$ and a subset of commodities $\mathcal{J} \subseteq \mathcal{K}$ such that

$$\gamma_1(f^*, \mathcal{G}) = \sum_{e \in \mathcal{E}_v^{\mathcal{J}}} \left(C_e - \sum_{j \in \mathcal{J}} f_e^{j*} \right) = \sum_{e \in \mathcal{E}_v^{\mathcal{J}}} C_e - \sum_{j \in \mathcal{J}} \lambda_v^j = \delta,$$

we used the fact that ρ^* is finite, and hence the inflow equals the outflow for every node. Introduce $\kappa := \sum_{e \in \mathcal{E}_v^{\mathcal{J}}} C_e$. We need to find a perturbation with magnitude δ such that the network is not fully transferring. To this scope, we can perturb the edges as follows

$$\tilde{\vartheta}_e(\rho_e) = \frac{\kappa - \delta}{\kappa} \vartheta_e(\rho), \quad \forall e \in \mathcal{E}_v^{\mathcal{J}},$$
$$\tilde{\vartheta}_e(\rho_e) = \vartheta_e(\rho_e), \quad \forall e \in \mathcal{E}_v^+ \setminus \mathcal{E}_v^{\mathcal{J}}.$$

In fact, we have

$$\sum_{e \in \mathcal{E}_v^{\mathcal{J}}} \tilde{C}_e = \sum_{e \in \mathcal{E}_v^{\mathcal{J}}} \frac{\kappa - \delta}{\kappa} C_e = \sum_{e \in \mathcal{E}_v^{\mathcal{J}}} C_e - \left(\sum_{e \in \mathcal{E}_v^{\mathcal{J}}} C_e - \sum_{j \in \mathcal{J}} \lambda_v^j \right) = \sum_{j \in \mathcal{J}} \lambda_v^j,$$

and the necessary condition given in Lemma 2 for the dynamical network to be fully transferring is violated. □

In the following section we introduce the notion of diffusivity, which will be instrumental to study tightness of this upper bound.

4.3 Diffusivity

The central idea of the proof for the tight bound of the strong resilience is that no perturbation is able to increase the outflow from a node more than the magnitude of the perturbations for that local node. So if a perturbation of the minimum residual capacity is made on a node before the one that is limiting the residual capacity, the increase of inflow to that limiting node will not be greater than the minimum residual capacity.

LEMMA 7—DIFFUSIVITY PROPERTY FOR THE LOCAL DYNAMICAL SYSTEM
Consider a fully accessible local dynamical network, $\mathcal{N}$, satisfying the Assumptions 1 and 3, with a inflow λ such that

$$\sum_{k\in\mathcal{K}} \lambda_v^k < \sum_{e\in\mathcal{E}_v^+} C_e.$$

Let f^* denote the limit flow for this network. Moreover, let $\tilde{\mathcal{N}}$ be an admissible perturbed network with inflow $\tilde{\lambda}$ such that

$$\sum_{k\in\mathcal{K}} \tilde{\lambda}_v^k < \sum_{e\in\mathcal{E}_v^+} \tilde{C}_e.$$

Let $\tilde{f}_j^*(\tilde{\lambda})$ denote the limit flow for the perturbed network, with the inflows $\tilde{\lambda}$. Then for every $\mathcal{J} \subseteq \mathcal{E}_v^+$ it holds that

$$\sum_{j\in\mathcal{J}} \left(\tilde{f}_j^*(\tilde{\lambda}) - f_j^*\right) \leq \left[\sum_{k\in\mathcal{K}} \tilde{\lambda}_k - \lambda_k\right]_+ + \sum_{e\in\mathcal{E}_v^+} \delta_e. \qquad \square$$

Proof See Appendix A.2. □

Lemma 7 states an upper bound for the changes on the limit flows both when the inflow increases and when the capacity on the outgoing links are reduced by a perturbation. An example of how the limit flows changes when a perturbation occurs is shown in Example 11.

EXAMPLE 11—CAPACITY REDUCTION
Consider a simple local network with one node and two outgoing links, see Figure 4.2.

Let $\lambda_A = 1.35$ and $\lambda_B = 1.35$. Further, let the velocity functions for the edges be

$$\vartheta_1(\rho_1) = c_1 \cdot \frac{1-e^{-14\rho_1}}{\rho_1}, \qquad \vartheta_2(\rho_2) = c_2 \cdot \frac{1-e^{-14\rho_2}}{\rho_2},$$

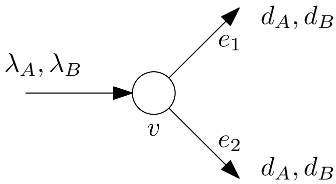

Figure 4.2 A graph for a local network with two incoming commodities and two outgoing links.

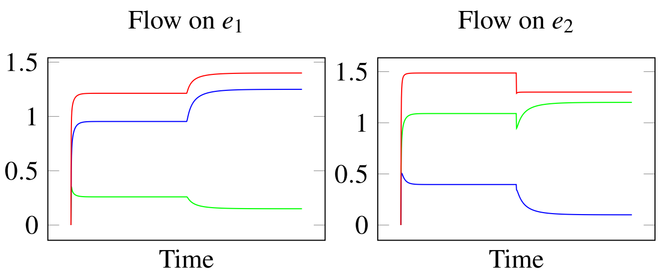

Figure 4.3 How the densities and flows of commodity A (green) and commodity B (blue) changes when the capacity on one link is decreased at half the simulation time. The aggregate flow is also plotted (red).

and

$$G^A_{v\to e_1}(\rho_1,\rho_2) = \frac{e^{-15\rho_1}}{e^{-15\rho_1}+e^{-1\rho_2}}, \qquad G^B_{v\to e_1}(\rho_1,\rho_2) = \frac{e^{-4\rho_1}}{e^{-4\rho_1}+e^{-4\rho_2}},$$
$$G^A_{v\to e_2}(\rho_1,\rho_2) = \frac{e^{-1\rho_2}}{e^{-15\rho_1}+e^{-1\rho_2}}, \qquad G^B_{v\to e_2}(\rho_1,\rho_2) = \frac{e^{-4\rho_2}}{e^{-4\rho_1}+e^{-4\rho_2}}.$$

Moreover, we let $c_1 = c_2 = 1.5$. When half the simulation time have passed, we decrease $c_2 = 1.3$.

The flow dynamics is shown in Figure 4.3. The differences in limit flows are shown is Table 4.1. The routing policies are constructed such that commodity A prefers link e_2 much more than link e_1 and commodity B has no particular preference. Due to commodity A's preference for link e_2 the density on that link will increase, which results in that most of commodity B takes link e_1. After the perturbation, commodity A prefers e_2 even more compared to commodity B, so a fraction larger than the perturbation of commodity B will be rerouted to e_1. □

Example 11 shows that the aggregate flow on each link does not increases more than the magnitude of the perturbation that occurs, just as Lemma 7 claims. However the example also shows that for each commodity it is not true, the flow of commodity

	f^*	$\tilde{f}^*$	$\tilde{f}^* - f^*$
Edge e_1			
Commodity A	0.26	0.15	−0.11
Commodity B	0.95	1.25	0.30
Aggregate $A+B$	1.21	1.40	0.19
Edge e_2			
Commodity A	1.09	1.20	0.11
Commodity B	0.40	0.10	−0.30
Aggregate $A+B$	1.49	1.30	−0.19

Table 4.1 The limit flows before and after perturbation.

B on link e_1 increases more than the magnitude of the perturbation. Figure 4.3 also illustrates the central part in the proof of Lemma 7, that the aggregate limit flow after perturbation is always equal to or greater than the flow when the perturbation has just occurred.

In the next example it is showed how the local system reacts to an inflow increment. Again, the diffusivity property holds for the aggregate flow but not for the single commodities.

EXAMPLE 12—INFLOW INCREMENT

Suppose that we have a simple graph with one node and two outgoing links, see Figure 4.2. Let $\lambda_A = 1.35$ and $\lambda_B = 1.35$. Further, let the velocity functions for the edges be

$$\vartheta_1(\rho_1) = c_1 \cdot \frac{1 - e^{-14\rho_1}}{\rho_1}, \qquad \vartheta_2(\rho_2) = c_2 \cdot \frac{1 - e^{-14\rho_2}}{\rho_2},$$

and

$$G^A_{v\to e_1}(\rho_1,\rho_2) = \frac{e^{-15\rho_1}}{e^{-15\rho_1} + e^{-1\rho_2}}, \qquad G^B_{v\to e_1}(\rho_1,\rho_2) = \frac{e^{-4\rho_1}}{e^{-4\rho_1} + e^{-4\rho_2}},$$
$$G^A_{v\to e_2}(\rho_1,\rho_2) = \frac{e^{-1\rho_2}}{e^{-15\rho_1} + e^{-1\rho_2}}, \qquad G^B_{v\to e_2}(\rho_1,\rho_2) = \frac{e^{-4\rho_2}}{e^{-4\rho_1} + e^{-4\rho_2}}.$$

Moreover, we let $c_1 = c_2 = 1.5$. When half the simulation time have passed, we set $\lambda_A = 1.65$ and $\lambda_B = 1.25$.

The flow dynamics is shown in Figure 4.4. The differences in limit flows are shown is Table 4.2. □

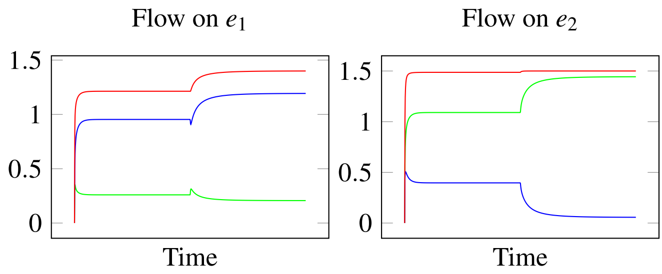

Figure 4.4 How the densities and flows of commodity A (green) and commodity B (blue) changes when the inflow of commodity A is increased and the inflow of commodity B is decreased at half the simulation time. The aggregate flow is also plotted (red).

	f^*	$\tilde{f}^*$	$\tilde{f}^* - f^*$
Edge e_1			
Commodity A	0.26	0.21	-0.05
Commodity B	0.95	1.19	0.24
Aggregate $A+B$	1.21	1.40	0.19
Edge e_2			
Commodity A	1.09	1.44	0.35
Commodity B	0.40	0.06	-0.34
Aggregate $A+B$	1.49	1.50	0.01

Table 4.2 The limit flows before and after inflow increment.

4.4 Tight bound on the strong resilience

Fully accessible network

If we have a fully accessible network, we know that all commodities affect the minimum residual capacity. Since the diffusivity property holds for the aggregate system, the following theorem can be stated.

THEOREM 3
For a fully accessible acyclic dynamical network it holds that

$$\gamma_1(\mathcal{G}, f^*) = R(\mathcal{N}, f^*). \qquad \square$$

Proof Lemma 8 in Appendix A.3 shows that until $\gamma_1(\mathcal{G}, f^*) < R(\mathcal{N}, f^*)$ the network remains fully transferring, hence it must be that $\gamma_1(\mathcal{G}, f^*) \geq R(\mathcal{N}, f^*)$. But Lemma 6 gives that $\gamma_1(\mathcal{G}, f^*) \leq R(\mathcal{N}, f^*)$ and therefore $\gamma_1(\mathcal{G}, f^*) = R(\mathcal{N}, f^*)$. $\square$

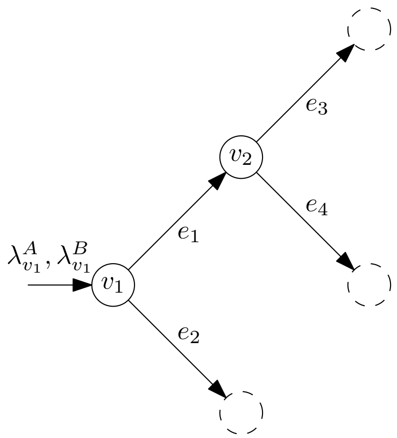

Figure 4.5 Example of a (part of) not fully accessible network for which the upper bound on the strong resilience is not tight.

Not fully accessible network

As Example 11 shows, the diffusivity property does not hold for each commodity flow. Therefore it is quite easy to construct examples with a not fully accessible network when the strong resilience property does not hold as well.

EXAMPLE 13—FAILING OF STRONG RESILIENCE
Let the local network in the Example 11 be the first part of a larger network, see Figure 13. For the node v_1 the minimum residual capacity is then $c_1 + c_2 - \lambda_A - \lambda_B = 0.3$. However let $e_3 \in \mathcal{E} \setminus \mathcal{E}^B$ and $e_4 \in \mathcal{E} \setminus \mathcal{E}^A$, so that commodity A is only allowed on link e_3 and commodity B is only allowed on link e_4. Then choosing $c_4 = f_{e_1}^{B*} + 0.25$ and $c_3 > f_{e_1}^{A*} + 0.3$ will give $R(\mathcal{N}, f^*) = 0.25$, if we choose capacities for the rest of the network to be sufficiently large. But, as Example 11 shows, applying a perturbation of magnitude $0.2 < R(\mathcal{N}, f^*)$ on e_2 will make the inflow to e_4 larger than the capacity, $\tilde{f}_{e_1}^{B*} = f_{e_1}^{B*} + 0.3 > c_4 = f_{e_1}^{B*} + 0.25$ and the perturbed network turns out to be not fully transferring.

4.5 Cascade increment of one commodity

Since the diffusivity property does not hold for each commodity, employing the insights given by Example 11 and Example 12 it is possible to build an example where the increase of one commodity flow is much larger than the perturbation. By letting the perturbation at one node give an increment in inflow to the next node, and then eventually add static inflow, such that the next node will react more on the increment and amplify it. This is done in Example 14.

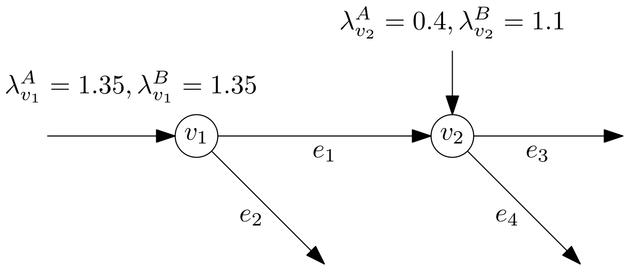

Figure 4.6 Cascade failure.

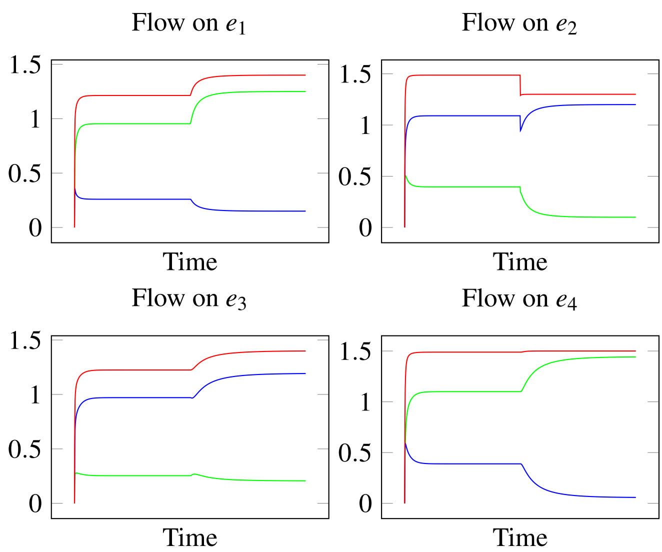

Figure 4.7 How the densities and flows of commodity A (green) and commodity B (blue) changes when the capacity on one link is decreased at half the simulation time. The aggregate flow is also plotted (red).

EXAMPLE 14—CASCADE INCREMENT
Consider the network with inflows in Figure 4.6. Let $c_1 = c_2 = c_3 = c_4 = 1.5$. The rest of the simulation parameters are given in Appendix B.3. Perturb the network in such a way that $c_2 = 1.3$, thus having a perturbation magnitude $\delta = 0.2$. Table 4.3 shows that after v_1 this gives an increase of commodity A on link e_1 of 0.30 and an increase of commodity A on link e_3 of 0.34. Hence the perturbation occurring at the first node gets amplified after the second.

	f^*	$\tilde{f}^*$	$\tilde{f}^* - f^*$
Edge e_1			
Commodity A	0.95	1.25	0.30
Commodity B	0.26	0.15	−0.11
Aggregate $A+B$	1.21	1.40	0.19
Edge e_2			
Commodity A	0.40	0.10	−0.30
Commodity B	1.09	1.20	0.11
Aggregate $A+B$	1.49	1.30	−0.19
Edge e_3			
Commodity A	0.25	0.21	−0.04
Commodity B	0.97	1.19	0.22
Aggregate $A+B$	1.22	1.40	0.18
Edge e_4			
Commodity A	1.10	1.44	0.34
Commodity B	0.39	0.06	−0.33
Aggregate $A+B$	1.49	1.50	0.01

Table 4.3 The limit flows before and after perturbation on edge e_2.

Notice that, since we know that the diffusivity property holds for the aggregate, this kind of cascade increment can only happen when there is another commodity flow on the link which can then decrease and compensate for the commodity that increases more than the perturbation.

5

Traffic lights

In the previous chapters, the steady assumption was that all incoming vehicles to a node shall be routed to the outgoing links, even if all the outgoing links are congested. We introduce in this chapter the notion of back-pressure, proposed in the early 90's in [Tassiulas and Ephremides, 1992] in the context of data networks. Specifically, we will allow a fraction of vehicles to remain on a road if the following roads are too congested. This simple local property is able to back-propagate the information that a congestion is happening ahead and allows vehicles to reroute to less congested branches of the network. The goal of this chapter is to show that it is indeed possible to control the traffic in a distributed manner.

5.1 Dynamical network with traffic lights

A traffic light should stop vehicles to leave the road if all the outgoing links are congested. Moreover, if the incoming is congested but not the outgoing it should allow the cars to go on. This intuitive behavior is captured by the following definition.

DEFINITION 15—TRAFFIC LIGHT FUNCTION

A traffic light function is a continuous mapping $h_e(\rho_e^v) : \mathbb{R}_+^{e \cup \mathcal{E}_v^+} \to [0,1]$ where $\rho_e^v = \begin{bmatrix} \rho_e & \rho^v \end{bmatrix}$.

$$
\begin{aligned}
h_e(\rho^v) &\to 1, \quad \text{when } |\rho^v| \to \infty \text{ and } \rho_e < \infty \\
h_e(\rho^v) &\to 0, \quad \text{when } \rho_e \to \infty \text{ and } |\rho^v| < \infty
\end{aligned}
$$

The family of traffic light functions is defined as $\mathcal{H} := \{h_e(\rho_e^v) : \mathbb{R}_+^{e \cup \mathcal{E}_v^+} \to [0,1]\}_{\forall e \in \mathcal{E}}$ □

An example of a function satisfying Definition 15 is shown below.

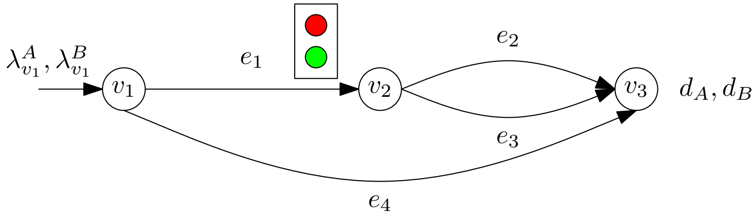

Figure 5.1 A simple network where the traffic lights can improve the throughput.

EXAMPLE 15—TRAFFIC LIGHT FUNCTION

$$h_e(\rho_e^v) = \frac{\rho_e e^{-\rho_e}}{\rho_e e^{-\rho_e} + e^{-\sum_{\mathcal{E}_v^+} \rho_e}} \qquad \square$$

In our model, traffic lights can perceive aggregate densities, but can not distinguish between commodities. As such, flow of every commodity on a link is influenced in the same way by the action of a traffic light on that link. This modification of the dynamics of the system is formally stated in the following definition.

DEFINITION 16—DYNAMICAL MULTI-COMMODITY NETWORK WITH TRAFFIC LIGHTS

A dynamical multi-commodity network with traffic lights is a network $\mathcal{N}$ associated with a family of distributed routing policies $\mathcal{G}$, a set of commodity demands $\mathcal{K}$, and a family of traffic light functions $\mathcal{H}$, where the dynamics of the network is given by

$$\begin{aligned}
\dot{\rho}_e^k &= \Lambda_{\sigma_e}^k \cdot G_{\sigma_e \to e}^k(\rho^v(t)) - \rho_e^k(t)\vartheta_e(\rho_e(t))(1 - h_e(\rho_e^{\tau_e})), \quad \forall e \in \mathcal{E}, \forall k \in \mathcal{K} \\
\Lambda_v^k &:= \sum_{j \in \mathcal{E}_v^-} \rho_j^k(t) \cdot \vartheta_j(\rho_j(t)) \cdot h_j(\rho_j^v) + \lambda_v^k
\end{aligned} \qquad \square$$

EXAMPLE 16

Consider the graph in Figure 5.1. Let $\mathcal{E}^A = \{e_1, e_2, e_4\}$, $\mathcal{E}^B = \{e_1, e_2, e_3, e_4\}$, $C_{e_1} = C_{e_4} = 3$, $C_{e_2} = C_{e_3} = 0.1$ and $\lambda_A = \lambda_B = 1$. The rest of the simulation parameters are given in Appendix B.4. When the drivers reach node v_1 they have no information about the low-capacity roads after node v_2, so they do not take this fact into account when choosing road e_1 or e_4. Without traffic lights, too many drivers choose e_1, which then makes the densities on road e_2 and e_3 increase unbounded. By using the traffic light function proposed in Example 15, the traffic light at the entrance to node v_2 will stop a fraction of cars to go further. Then the density on e_1 will increase and the drivers are more likely to choose e_4. A simulation with traffic is shown in Figure 5.2 and without traffic lights in Figure 5.3. In the plots, we see that since the traffic light give the same fraction of red time to all commodities, there will be some spare capacity on link e_3.

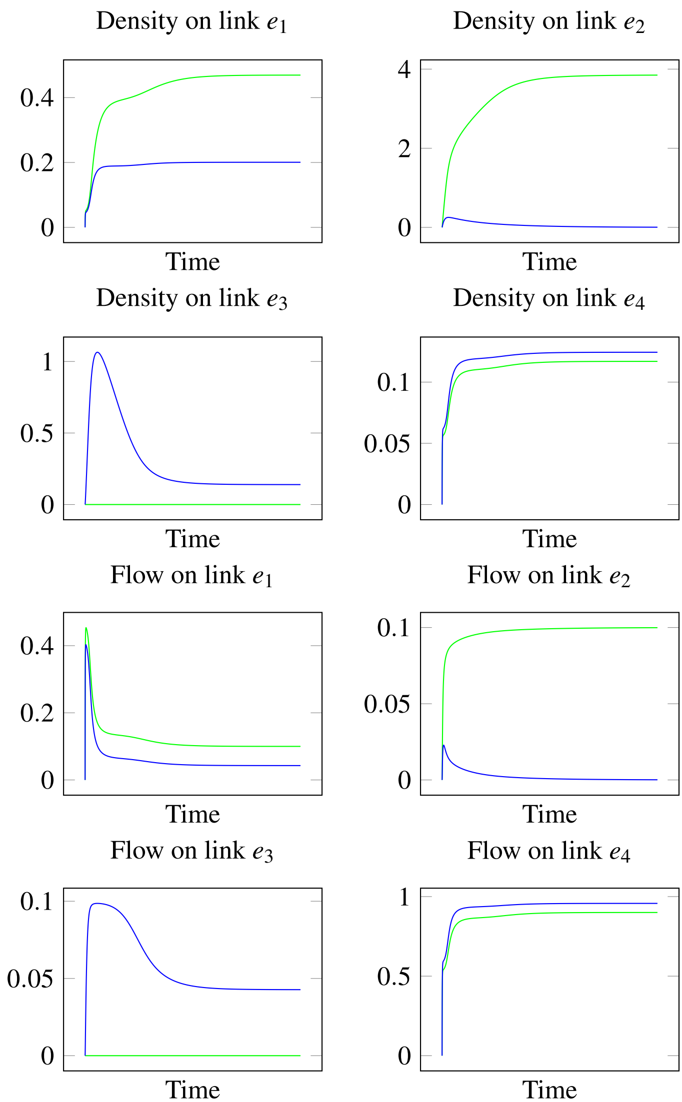

Figure 5.2 Simulation of dynamical network with traffic lights and two commodities A (green) and B (blue). The simulation shows that the flow on e_3 at equilibrium is not maximal, due to the fact that traffic light are given red light to avoid link e_2 to get congested.

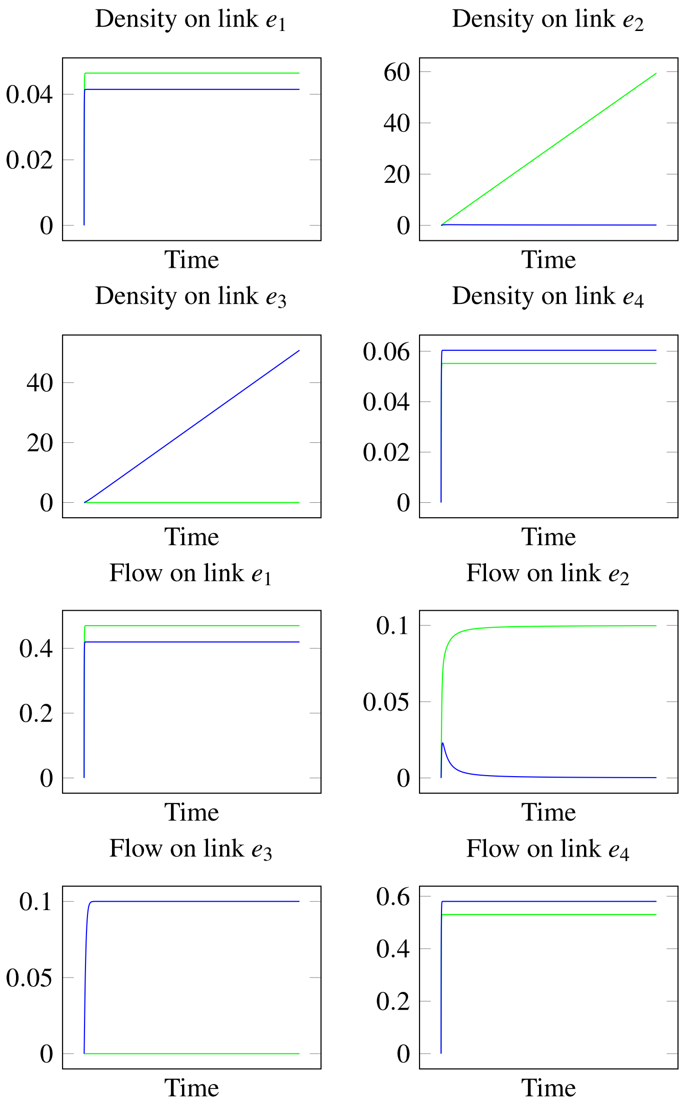

Figure 5.3 Simulation of dynamical network without traffic lights and two commodities A (green) and B (blue). The simulation shows that without traffic lights the densities grow unbounded on links e_2 and e_3.

6

Discussion and future work

The model presented and discussed in this thesis provides a first step towards a comprehensive theory of multi-commodity dynamical networks. Some important results have been established and some others have been conjectured on the basis of numerical simulations. In this final chapter, we will discuss some of the weak points of the proposed theory, altogether with possible future lines of research to overcome them.

6.1 Modeling

An improved model for a junction In the model presented in this thesis, no consideration has been taken to how vehicles travel near and in a junction. For instance, near a junction with assigned lanes, the velocity in the lines might be different. However, one can model junction by seeing each junction as an independent dynamical multi-commodity network, where some of paths through the junction are modeled by links, but it deserves more investigation.

Buffer capacity on roads In the model, the fact that there is only room for a finite amount of cars on each road has not been taken into account. [Como et al., 2013c] introduces a buffer capacity on each link too for the single commodity case, and it seems like this can easily be implemented in this multi-commodity model as well.

Velocity function [Daganzo, 1994], [Daganzo, 1995] suggest a structure of the velocity function, that unlike the structure in our model, also depends on the state of the following link. This seems like a more realistic model for the velocities, and allows a possibility to back-propagate through the network.

Time varying inflows In our model all inflows are static or converging to a fixed value. But in reality the inflows vary with the time of the day and the day of the week. For instance, commuters exhibit a quite periodic behavior. Then the constraints, for the network to be fully transferring, might be violated during a short period of a time. However, after the peak there might be time enough time to drain the network, in such a way that the densities remain stable in the long term. This is a subject for both further mathematical analysis and simulations.

6.2 Stability analysis

Graph with cycles No proof for the convergence of the densities with cyclic graphs has been provided in this thesis. Simulations suggest however the existence of a unique limit point in this case too. Convergence is a quite essential property for the usefulness of this model, and should therefore be investigated in a much deeper way in the near future.

6.3 Robustness

Bound on the diffusivity in each commodity In Chapter 4 we saw that when a perturbation occurs, the flow of one commodity can increase more than the magnitude of the perturbation. One topic for future investigation can therefore be to prove an upper bound of the magnitude of this overreaction. The result can then be used for stating resilience properties of a not fully accessible network.

6.4 Traffic lights

More realistic model for traffic lights In the simple model for traffic lights proposed in Chapter 5 we ignored the fact that a real traffic light switches the green light between the incoming edges. A better model, where the traffic light have some constraints on how the green light time should go in cycles among the incoming roads should therefore be preferable. Improvement of the traffic light model can be seen as a subproblem of modeling junctions, and will be addressed accordingly.

Maximal throughput can be achieved with traffic lights Another topic for investigation would be to analyze how close one can come to the maximal throughput for the network when traffic lights are used. This requires a much deeper investigation of how the traffic lights should be designed. In the single commodity case, it is proved in [Como et al., 2013c] that the network can be fully transferring if the demands are smaller than the min-cut capacity for the graph. For the multi-commodity problem, it is much harder to state how large demands the network is able to fully transfer. In fact, in the multi-commodity case the network structure plays a role in the relation between maximum flow and min cut capacity, as it is shown in [Günlük, 2008].

Design traffic lights for throughput optimality Assuming that traffic lights guarantee that the network is used maximally, it could be of interest to develop strategies that do not delay the vehicles more than necessary. However, this could be hard to solve in a distributed manner, since intuitively it feels like information about the whole network is needed for an optimal solution. First step towards this goal is to understand how close to the optimum it is possible to come in a distributed control environment.

A

Additional theorems and proofs

A.1 Continuous dependence of the aggregate densities on the inflow

THEOREM 4
For a local network $\mathcal{N}$, satisfying Assumption 3 and Assumption 1, the aggregate densities, whose dynamics are given by

$$\dot{\rho}_e(t) = \sum_{k\in\mathcal{K}} \lambda_k G^k_{\sigma(e)\to e}(\rho^v(t)) - \rho_e(t)\vartheta(\rho_e(t)), \quad \forall e \in \mathcal{E}_v^+, \tag{A.1}$$

depend continuously on the inflows, λ_k, $\forall k \in \mathcal{K}$. □

Proof Using the same idea as in proof of [Como et al., 2013a, Lemma 3], we denote the right side of equation (A.1) $F_e(\rho^v, \lambda)$. Then it holds that

$$\begin{aligned}
\frac{\partial}{\partial \rho_e} F_e(\rho^v, \lambda) &= \sum_{k\in\mathcal{K}} \lambda_k \frac{\partial}{\partial \rho_e} G^k_{\sigma(e)\to e}(\rho^v) - \frac{d}{d\rho_e}(\rho_e \vartheta(\rho_e)) \\
&= -\sum_{k\in\mathcal{K}} \sum_{j\neq e} \lambda_k \frac{\partial}{\partial \rho_e} G^k_{\sigma(e)\to j}(\rho^v) - \frac{d}{d\rho_e}(\rho_e \vartheta(\rho_e)) \\
&< -\sum_{j\neq e} F_j(\rho^v, \lambda),
\end{aligned}$$

where the equality follows from the fact that the distributed routing policy codomain is a probability vector. The inequality in the last step follows from the assumption that $\rho_e\vartheta(\rho_e)$ is strictly increasing. The assumption of the cooperative property of the routing policy also guarantees that $\partial F_j(\rho^v, \lambda)/\partial \rho_e \geq 0$. Above we also showed that $\partial F_e(\rho^v, \lambda)/\partial \rho_e < 0$, and therefore the following holds

$$\left|\frac{\partial}{\partial \rho_e} F_e(\rho^v, \lambda)\right| = -\frac{\partial}{\partial \rho_e} F_e(\rho^v, \lambda) > -\sum_{j\neq e} \frac{\partial}{\partial \rho_j} F_e(\rho^v, \lambda) = \sum_{j\neq e} \left|\frac{\partial}{\partial \rho_j} F_e(\rho^v, \lambda)\right|.$$

Therefore the Jacobian matrix is strictly diagonal dominated, and then [Horn and Johnson, 1990, Theorem 6.1.10] gives that it is also invertible. Then the implicit function theorem, see e.g, [Renardy and Rogers, 2004], gives that $\rho^*(\lambda)$ depends continuously on λ. □

A.2 Proof of the diffusivity lemma

Proof This proof is based on the proof in [Como et al., 2013b]. First, introduce $\hat{\lambda} := \max(\lambda, \tilde{\lambda})$ where max() applies component-wise. Moreover let $\hat{\rho}$ be the solution of the local aggregate system (A.1) after perturbation with the inflow $\hat{\lambda}$ and initial condition $\hat{\rho}(0) = \rho^*$, where ρ^* is the limit density before perturbation. As a first step, we will prove that

$$\hat{f}_e(t) \geq \rho_e^* \tilde{\vartheta}_e(\rho_e^*), \quad \forall t \geq 0, \quad \forall e \in \mathcal{E}_v^+. \tag{A.2}$$

Consider a point in the space for the aggregate densities $\hat{\rho} \in \mathcal{R}_v$, such that $\hat{\rho} > \rho^*$ and there exits an edge $i \in \mathcal{E}_v^+$ such that $\hat{\rho}_i = \rho_i^*$. Then [Como et al., 2013a, Lemma 1] implies that $G_{v\to i}^k(\hat{\rho}) \geq G_{v\to i}^k(\rho^*), \forall k \in \mathcal{K}$. Since we also know that $\hat{\lambda} \geq \tilde{\lambda}$ and $\hat{\rho}_i \tilde{\vartheta}_i(\hat{\rho}_i) \leq \hat{\rho}_i \vartheta_i(\hat{\rho}_i) = \rho_i^* \vartheta_i(\rho_i^*)$, it holds that

$$\sum_{k\in\mathcal{K}} \hat{\lambda}^k G_{v\to i}^k(\hat{\rho}) - \hat{\rho}_i \tilde{\vartheta}_i(\hat{\rho}_i) \geq \sum_{k\in\mathcal{K}} \lambda_k G_{v\to i}^k(\rho^*) - \rho_i^* \tilde{\vartheta}_i(\rho_i^*) = 0.$$

Let $\Omega := \{\hat{\rho} \in \mathcal{R}_v : \hat{\rho}_e \geq \rho_e^*, \forall e \in \mathcal{E}_v^+\}$ and $\omega \in \mathcal{R}_v$ the unit outpointing normal vector to the boundary of the set Ω. Then

$$\frac{d}{dt}(\hat{\rho} \cdot w) = \left[\sum_{k\in\mathcal{K}} \lambda^k G_{v\to e}^k(\hat{\rho}) - \hat{\rho}_e \tilde{\vartheta}_e(\hat{\rho}_e) \right]_{\forall e \in \mathcal{E}_v^+} \cdot \omega \leq 0 \quad \forall \rho \in \partial\Omega, \forall t \geq 0.$$

Hence Ω is an invariant set, see Figure A.1, so that the aggregate densities will always be larger or equal to the limit densities for the unperturbed system. This proves the inequality in (A.2).

Introduce $\mathcal{J} \subseteq \mathcal{E}_v^+$ and $\mathcal{I} = \mathcal{E}_v^+ \setminus \mathcal{J}$. Since there exists an equilibrium for the

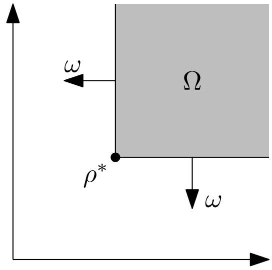

Figure A.1 Schematic sketch of the invariant set used in the proof of the diffusivity lemma.

perturbed system it follows that

$$\begin{aligned}
\sum_{j\in\mathcal{J}} \hat{f}_j^* &= \sum_{k\in\mathcal{K}} \hat{\lambda}^k - \sum_{i\in\mathcal{I}} \hat{f}_k^* \\
&\leq \sum_{k\in\mathcal{K}} \hat{\lambda}^k - \sum_{i\in\mathcal{I}} \rho^* \tilde{\vartheta}_i(\rho_i^*) \\
&= \sum_{k\in\mathcal{K}} \left(\hat{\lambda}^k - \lambda^k\right) + \sum_{j\in\mathcal{J}} f_j^* + \sum_{i\in\mathcal{I}} \rho_i \vartheta_i(\rho_i^*) - \sum_{i\in\mathcal{I}} \rho_i^* \tilde{\vartheta}_i(\rho_i^*) \\
&\leq \left[\sum_{k\in\mathcal{K}} \hat{\lambda}^k - \lambda^k\right]_+ + \sum_{j\in\mathcal{J}} f_j^* + \sum_{i\in\mathcal{I}} \delta_i \\
&\leq \left[\sum_{k\in\mathcal{K}} \hat{\lambda}^k - \lambda^k\right]_+ + \sum_{j\in\mathcal{J}} f_j^* + \sum_{e\in\mathcal{E}_v^+} \delta_e.
\end{aligned}$$

Since the aggregate system is monotone and $\lambda \leq \hat{\lambda}$ it also follows that

$$\tilde{f}_e^*(\lambda) \leq \tilde{f}_e^*(\hat{\lambda}) = \hat{f}_e^*, \quad \forall e \in \mathcal{E}_v^+,$$

which implies

$$\sum_{j\in\mathcal{J}} \tilde{f}_e^*(\lambda) \leq \sum_{j\in\mathcal{J}} \hat{f}_j^*. \qquad \square$$

A.3 Proof of the strong resilience

LEMMA 8
If a perturbation of magnitude

$$\delta < R(\mathcal{N}, f^*)$$

is made on a fully accessible network, then the network is still fully transferring. □

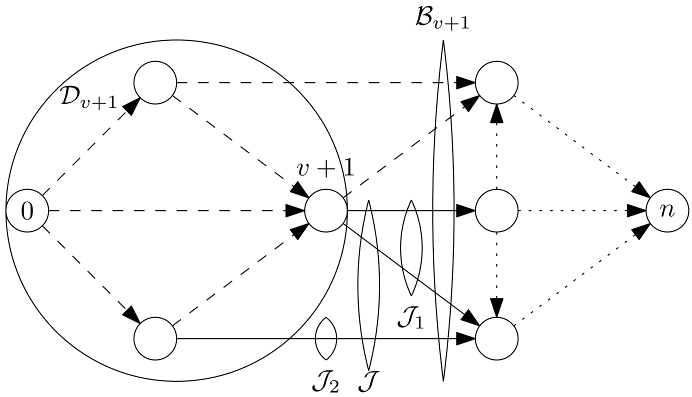

Figure A.2 The sets used in the induction proof. This figure is based on [Como et al., 2013b, Figure 7].

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Proof This proof is largely the same as [Como et al., 2013b, Proof of Lemma 2], but since minor notation changes are made we restate it here for completeness. Due to the existence of topological ordering, the nodes can be numbered $v = 0, 1, .., n-1$. To make the notations simpler, let

$$\lambda_v^* := \sum_{e \in \mathcal{E}_v^+} f_e^*, \quad \tilde{\lambda}_v^* := \sum_{e \in \mathcal{E}_v^-} \tilde{f}_e^*, \quad \tilde{\lambda}_v^{max} := \sum_{e \in \mathcal{E}_v^+} \tilde{C}_e.$$

Also the following set of links are introduced

$$\mathcal{D}_v := \bigcup_{0 \le u \le v} \mathcal{E}_u^+,$$

$$\mathcal{B}_v := \{(u,w) \in \mathcal{E} : 0 \le u \le v, v < w \le n\},$$

see Figure A.2. By induction on $u = 0, 1, .., n-1$ it will now be proved that

$$\sum_{e \in \mathcal{J}} \left(\tilde{f}_e^* - f_e^*\right) \le \sum_{e \in \mathcal{D}_u} \delta_e, \quad \forall \mathcal{J} \subseteq \mathcal{B}_u. \tag{A.3}$$

The inequality above states that the increase of flow on the edges that originates or passes by node u, will not be greater than the magnitude of perturbations on the edges before or at node u.

For the first node $u = 0$ it holds that

$$\sum_{e \in \mathcal{E}_0^+} \delta_e \le \delta < R(\mathcal{N}, f^*) \le \sum_{e \in \mathcal{E}_0^+} (C_e - f_e^*),$$

and it follows that $\lambda_0 := \sum_{k \in \mathcal{K}} \lambda_0^k < \sum_{e \in \mathcal{E}_0^+} C_e$. The diffusivity lemma can then be applied to the first node, $u = 0$, and therefore (A.3) holds in this case.

Now, let $v \leq n-2$ and assume that (A.3) holds for every $u \leq v$. Take a subset $\mathcal{J} \subseteq \mathcal{B}_{v+1}$ and let $\mathcal{J}_1 := \mathcal{J} \cap \mathcal{E}_{v+1}^{+}$ and $\mathcal{J}_2 := \mathcal{J} \setminus \mathcal{J}_1$. For the set $\mathcal{J}_1$ the diffusivity lemma gives

$$\sum_{e \in \mathcal{J}_1} \left(\tilde{f}_e^* - f_e^*\right) \leq \left[\tilde{\lambda}_{v+1}^* - \lambda_{v+1}^*\right]_+ + \sum_{e \in \mathcal{E}_{v+1}^{+}} \delta_e. \tag{A.4}$$

Since both $\mathcal{J}_2 \subseteq \mathcal{B}_v$ and $\mathcal{E}_{v+1}^{-} \subset \mathcal{B}_v$, (A.3) can be applied for the sets $\mathcal{J}_2$ and $\mathcal{J}_2 \cup \mathcal{E}_{v+1}^{-}$ (notice that $\mathcal{J}_2$ and $\mathcal{E}_{v+1}^{-}$ are disjoints). For $\mathcal{J}_2$, (A.3) gives

$$\sum_{e \in \mathcal{J}_2} \left(\tilde{f}_e^* - f_e^*\right) \leq \sum_{e \in \mathcal{D}_v} \delta_e, \tag{A.5}$$

and for $\mathcal{J}_2 \cup \mathcal{E}_{v+1}^{-}$

$$\sum_{e \in \mathcal{J}_2} \left(\tilde{f}_e^* - f_e^*\right) + \sum_{e \in \mathcal{E}_{v+1}^{-}} \left(\tilde{f}_e^* - f_e^*\right) \leq \sum_{e \in \mathcal{D}_v} \delta_e. \tag{A.6}$$

Now, if $\tilde{\lambda}_{v+1}^* \leq \lambda_{v+1}^*$ we add (A.4) and (A.5) up, and if $\tilde{\lambda}_{v+1}^* > \lambda_{v+1}^*$ we add (A.4) and (A.6) up. In both cases we get

$$\begin{aligned}
\sum_{e \in \mathcal{J}} \left(\tilde{f}_e^* - f_e^*\right) &= \sum_{e \in \mathcal{J}_1} \left(\tilde{f}_e^* - f_e^*\right) + \sum_{e \in \mathcal{J}_2} \left(\tilde{f}_e^* - f_e^*\right) \\
&\leq \sum_{e \in \mathcal{E}_{v+1}^{+}} \delta_e + \sum_{e \in \mathcal{D}_v} \delta_e \\
&= \sum_{e \in \mathcal{D}_{v+1}} \delta_e,
\end{aligned}$$

which proves A.3 for the node $v+1$, and the induction step is now proved.

Now for a fixed v in $1 \leq v < n$, inequality (A.3) implies that

$$\begin{aligned}
\tilde{\lambda}_v^* &= \sum_{e \in \mathcal{E}_v^{-}} \tilde{f}_e^* \\
&\leq \sum_{e \in \mathcal{E}_v^{-}} f_e^* + \sum_{e \in \mathcal{D}_{v-1}} \delta_e \\
&\leq \sum_{e \in \mathcal{E}_v^{+}} f_e^* - \lambda_v + \sum_{e \in \mathcal{E}} \delta_e - \sum_{e \in \mathcal{E} \setminus \mathcal{D}_{v+1}} \delta_e \\
&\leq \sum_{e \in \mathcal{E}_v^{+}} f_e^* + \delta - \sum_{e \in \mathcal{E}_v^{+}} \delta_e,
\end{aligned}$$

where λ_v is the constant inflow to node v. Above we have used the fact that we have finite limit density for the unperturbed system such that

$$\sum_{e \in \mathcal{E}_v^{-}} f_e^* + \lambda_v \leq \sum_{e \in \mathcal{E}_v^{+}} f_e^*.$$

Then

$$\begin{aligned}\tilde{\lambda}_v^* + \lambda_v &\leq \sum_{e\in\mathcal{E}_v^+} f_e^* + \delta - \sum_{e\in\mathcal{E}_v^+} \delta_e \\ &< \sum_{e\in\mathcal{E}_v^+} f_e^* + R(\mathcal{N}, f^*) - \sum_{e\in\mathcal{E}_v^+} \delta_e \\ &\leq \sum_{e\in\mathcal{E}_v^+} f_e^* + \sum_{e\in\mathcal{E}_v^+} (C_e - f_e^*) - \sum_{e\in\mathcal{E}_v^+} \delta_e \\ &= \sum_{e\in\mathcal{E}_v^+} (C_e - \delta_e) \\ &= \sum_{e\in\mathcal{E}_v^+} \tilde{C}_e,\end{aligned}$$

and Lemma 2 yields

$$\tilde{f}_e^* < \tilde{C}_e, \quad \forall e \in \mathcal{E}_v^+,$$

for all $1 \leq v < n-1$. But as stated before $\lambda_0 < \sum_{e\in\mathcal{E}_0^+} \tilde{C}_e$ so Lemma 2 gives that

$$\tilde{f}_e^* < \tilde{C}_e, \quad \forall e \in \mathcal{E}_0^+,$$

and therefore $\tilde{f}_e^* < \tilde{C}_e, \forall e \in \mathcal{E}$ and the system is fully transferring. □

A.4 Gronwalls lemma

THEOREM 5—GRONWALLS LEMMA
[Evans, 1998] If the non-negative function $u(t)$ is continuous and differentiable on an interval $I = [a,b]$, $\beta(t)$ is a non-negative continuous function and the inequality

$$u'(t) \leq \beta(t)u(t)$$

holds, then it also holds that

$$u(t) \leq u(a)e^{\int_a^t u(\sigma)d\sigma}, \quad \forall t \in I.$$ □

B

Simulation parameters

In all examples, the velocity function

$$\vartheta_e(\rho_e) = C_e \cdot \frac{1 - e^{-\mu_e \rho_e}}{\rho_e}, \quad \forall e \in \mathcal{E},$$

where $C \in \mathcal{R}$ and $\mu \in \mathcal{R}$. The distributed routing policy used is

$$G^k_{v \to e}(\rho^v) = b^k_e \cdot \frac{e^{-\beta^k_e \rho_e}}{\sum_{j \in \mathcal{E}^+_v} b^k_j e^{-\beta^k_j \rho_j}}.$$

B.1 Example 6

The inflows are $\lambda_A = \lambda_B = 2$ and all initial densities are zero.

Edge	e_1	e_2	e_3	e_4
C_e	5	5	1	5
μ_e	1	1	1	1

Table B.1 Velocity functions properties for Example 6.

Edge	e_1	e_2	e_3	e_4
β^A_e	1	1	-	1
β^B_e	10	-	1	1

Table B.2 Routing policy parameters for Example 6. $-$ corresponds to $b^k_e = 0$ and all other values corresponds to $b^k_e = 1$.

B.2 Example 8

The inflows are $\lambda_A = \lambda_B = 1$.

Edge	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
C_e	1	1	1	1	1	1	1	1
μ_e	1	1	1	1	1	1	1	1

Table B.3 Velocity functions properties for Example 8.

Edge	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
β_e^A	5	1	6	5	7	7	7	1
β_e^B	1	4	6	7	5	9	8	10

Table B.4 Routing policy parameters for Example 8. $-$ corresponds to $b_e^k = 0$ and all other values corresponds to $b_e^k = 1$.

B.3 Example 14

The inflows are $\lambda_{v_1}^A = \lambda_{v_1}^B = 1.35$, $\lambda_{v_2}^A = 0.4$ and $\lambda_{v_2}^B = 1.1$.

Edge	e_1	e_2	e_3	e_4
C_e	1.5	1.5	1.5	1.5
μ_e	14	14	14	14

Table B.5 Velocity functions properties for Example 14.

Edge	e_1	e_2	e_3	e_4
β_e^A	4	4	1	15
β_e^B	15	1	4	4

Table B.6 Routing policy parameters for Example 14.

B.4 Example 16

The inflows are $\lambda_{v_1}^A = \lambda_{v_1}^B = 1$.

Edge	e_1	e_2	e_3	e_4
C_e	3	0.1	0.1	3
μ_e	1	1	1	1

Table B.7 Velocity functions properties for Example 16.

Edge	e_1	e_2	e_3	e_4
β_e^A	4	1	-	2
β_e^B	5	2	1	1

Table B.8 Routing policy parameters for Example 16. $-$ corresponds to $b_e^k = 0$ and all other values corresponds to $b_e^k = 1$.

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