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Evaluating Capital Allocation below portfolio level

Master's Thesis

PHILIP SANDWALL
& OSCAR ÅGREN

Department of Mathematical Statistics
The Faculty of Engineering at Lund University
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Abstract

This thesis explores the ability for retail banks to allocate economic capital below portfolio level. First, a discussion about capital requirements and risk measures to provide a sound basis for determining the economic capital of the bank. In general, economic capital is allocated to the banks portfolios but not on a more granular level, through a capital allocation method. This study discuss three different approaches for allocation of economic capital below portfolio level; game theory, finance and optimization. Both the game theory and finance approach reach the same conclusion, that the best allocation principle is the gradient of the risk measure. The optimization method allocates economic capital through minimization of a concept called risk residual, which conclude that the optimal allocation is derived from the marginal distribution of a customer. Capital allocation below portfolio level give the management a good overview of risks from different customers. In order to determine the performance of the portfolios in the bank a Risk-Adjusted-Return-On-Capital is used, with economic capital as input. The thesis include some comments about how the choice of capital allocation methods affect the performance measurement. The thesis concludes with an evaluation of the methods by simulations of a fictional bank conducted in the software R.

Key Words: Risk Appetite, Economic Capital, Risk measure, Capital Allocation Methods, Allocation Below Portfolio level, Game theory, Optimization, Marginal Contribution

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Contents

List of Figures

List of Tables

0.1 Acronyms

AVaR - Average Value-at-Risk
BCBS - Basel Committee on Banking Supervision
CAPM - Capital Asset Pricing Model
CL - Confidence Level
CVaR - Conditional Value-at-Risk
EAD - Exposure At Default
EL - Expected Loss
ES - Expected Shortfall
ETL - Expected Tail Loss
EVA - Economic Value Added
EY - Ernst & Young
GPD - General Pareto Distribution
IIF - Institute of International Finance
LGD - Loss Given Default
MES - Marginal Expected Shortfall
MME - Moment Matching Estimation
PD - Probability of Default
QAS - Quantitative Advisory Services
RAF - Risk Appetite Framework
RAROC - Risk-Adjusted-Return-On-Capital
RAS - Risk Appetite Statement
ROC - Return-On-Capital
RWA - Risk Weighted Assets
S&P - Standard & Poors
SSG - Senior Supervisors Group
TVaR - Tail Value-at-Risk
VaR - Value-at-Risk

0.2 Notations

- X denotes the risk the bank is exposed to
- $i = 1,...,n$ are all the portfolios of the bank, with $1,...,n \in N$
- $X_1 +... +X_n$ is the risks of the banks portfolios
- $j^i = 1^i,...,d^i$ are all assets in portfolio i, with $1^i,...,d^i \in D^i$
- $\rho(\cdot)$ is an arbitrary risk measure
- $\widehat{\rho}(\cdot)$ is a coherent risk measure
- $\rho(X) = C_E$ is the amount of economic capital held by the bank
- C_R is the amount of regulatory capital
- c_i is the optimal amount of economic capital allocated to portfolio i
- $r(\cdot)$ is an arbitrary cost function
- L is the total loss of the bank
- L_i is the loss of portfolio i
- l_j^i is the loss of asset j in portfolio i
- x_j^i is the unexpected loss of asset j in portfolio i

1

Introduction

SINCE THE GREAT DEPRESSION of the 1930's the financial markets have experienced rises and falls of fortune, but few events have had such severe consequences as the global financial crisis of 2008. One could argue that the catalyst to the crisis was the collapse of the American housing market in 2006, which set in motion a complex series of events. These nearly broke down the whole financial system and brought the problems of the sub-prime loans into light. The severity of the situation caused fear among investors. Poor confidence by investors towards the stock market created passivity and a global down-turn in the economy. The US Senate's Levin-Coburn Report, which was released in 2011, summarized a number of possible causes for the crisis with the following statement.

> [...]high risk, complex financial products; undisclosed conflicts of interest; and the failure of regulators, the credit rating agencies, and the market itself to rein in the excesses of Wall Street.
> [Levin 11, p.8].

1.1 Background

In the early 2000, during the pre-global banking crisis era, the notion of a Risk Appetite Framework (RAF) was not given much attention. Generally, sophisticated models for risk management as whole was not given much interest. Typically banks had a Risk Appetite Statement (RAS) defining the level of desired risk, but most of them lacked the framework to properly define risk appetite and the structures needed to support these decisions within the organization. According to the Senior Supervisors Group (SSG), one of the problems that many financial firms suffered from was, "[...] a disparity between the risks that the firm took and those that the Board perceived it to be taking." [SSG 09, p.23] The financial crisis brought this disparity into light. The report, released in 2009, emphasized the need to involve the Board more extensively in the risk management process. One of the proposed solutions to the information discrepancy was to implement and work with a RAF. The general definition of risk appetite is, the level of risk that an organization is prepared to accept, before measures are needed to be taken in order to reduce it; and the RAF is thus the framework surrounding and supporting that statement. One of the key points for success when working with the RAS and RAF is to embed it with the culture of the organization. The 2009 IIF report, defines the risk culture as

> [...] the norms and traditions of behaviour of individuals and of groups within an organization that determine the way in which they identify, understand, discuss, and act on the risks the organization confronts and the risks it takes. [IIF SCI 09, p.31]

This statement summarize what role risk culture has in the company and how the RAS and RAF is part of it. The culture of the company can be determined by a few key drivers, Figure 1.1 illustrate four parameters that primarily determine the risk appetite for a bank. With that stated, the RAF is not a static document but an ongoing process

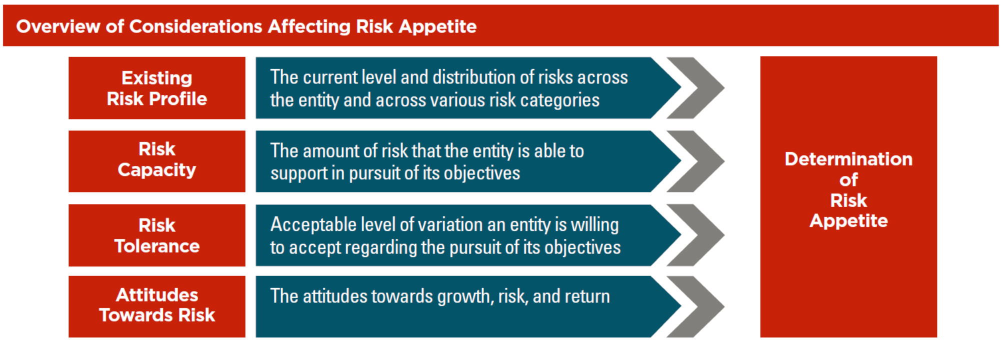

Figure 1.1: Risk Appetite Drivers [p.4][Rittenberg 12]

which is evaluated and evolved depending on how the company performs. There are a few factors that limit the amount of risk a bank can expose itself to. The capacity, determine the maximum amount of risk a bank can handle. The capacity is determined by certain restrictions, called limits. The risk tolerance depends on how much exposure the bank is willing to have and is determined by the RAF of the bank. The tolerance thus needs to be within the limits for the capacity. Depending on the risk tolerance the banks wants to hold an amount of capital that hedge these risks, this is called economic capital, denoted C_E. With the calculations of a risk measure, the economic capital of the bank can be determined. The risk measure determines how much unexpected loss the bank is exposed to, depending on the RAF the bank have determined a quantile of the loss distribution. The quantile determine the amount of economic capital wanted to be held. The credit rating for the bank is based on the quantile, and thereby amount of economic capital held. Figure 1.2 on page 3 indicate the quantile, which corresponds to an AA grade for the bank.

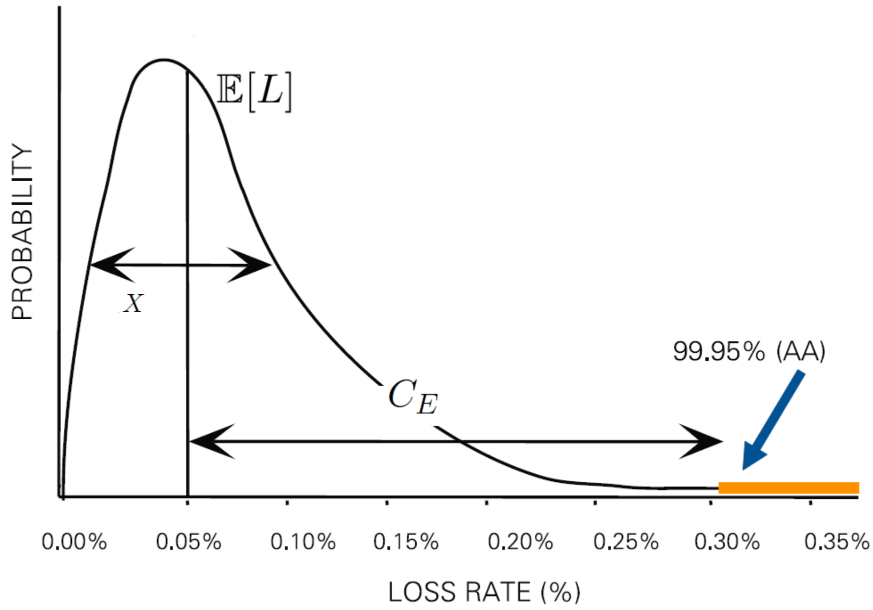

Figure 1.2: Economic Capital and credit rating [p.3][Hashagen 07]

There are a number of different grades the bank can be scored, Table 1.1 on page 3 indicate some of the top grades, together with the required confidence level α of the quantile. The maximum probability of default for the bank, with a certain rating, is determined by $1 - \alpha$.

Rating (S&P)	Maximum Default Probability	Required Confidence level
AAA	0.015 %	99.985 %
AA	0.050 %	99.950 %
A	0.060 %	99.940 %
A-	0.070 %	99.930 %
BBB+	0.110 %	99.890 %

Table 1.1: Credit rating

The banks internally determine their level of economic capital, based on risk profile. In order to prevent insolvency, default and increase the financial stability, banks are required by law to hold a certain amount of capital, called regulatory capital, C_R. For many cases the economic capital is greater than the regulatory capital, but in some situations it could be the other way around, so there is no general relationship between the two capital variables.

$$C_R \lesseqgtr C_E$$

1.2 Problem

The amount of economic capital chosen to be held by the bank depends on the risk appetite of the bank and the amount of unexpected loss the bank is exposed to. It does not exist an industry-wide standard for this calculation. Instead there are various techniques, called risk measures, that can determine the economic capital of a bank. The economic capital should be thought of as a burden for the bank [Denault 01, p.1], therefore the bank wants to allocate it among its portfolios. The goal is that each portfolio hold an amount of economic capital that corresponds to that portfolios risk exposure. In order to complete this procedure a capital allocation method is used. It has two main purposes:

Risk management: By allocating capital to the banks portfolios, the bank can get a holistic view of the risks the bank is exposed to. It gives managers an indication of which portfolios that should be increased or decreased too fit the company risk profile.

Performance measurement: One of the most prominent methods for determining the performance of the banks portfolios, in a fair way, is Risk-Adjusted-Return-On-Capital, RAROC. This method adjust for the risk of the portfolios by using the economic capital allocated to a portfolio as input in calculations.

A further discussion of uses for capital allocation can be found in e.g. [IIF SCI 09]. The problem with allocation of economic capital to portfolios has its origin in the diversification effect. By gathering many risky assets in a bank, the total risk of the assets becomes less than the sum of their individual risks. This is due to correlation where risks cancel each other out. The process of capital allocation begins with aggregating the risks in order to determine the economic capital, which is done with a risk measure, and then allocating capital to the portfolios corresponding to their risk. The difficulty with capital allocation is thus to determine how to distribute the effects of diversification. Since the financial crisis the interest for risk management have increased, which the following statement give proof of "[...]while the financial crisis has affected banks in different ways, the desire to improve risk management is universal." [IIF SCI 09, p.8] The following statement from the IFF report, highlights the need of improvement of the level of accuracy and methods used by banks before the financial crisis.

> Deficiencies in risk methodologies and reporting are being addressed through:
>
> ***i)*** Improving risk models;
>
> ***ii)*** Collecting more and better data
>
> [...] [IIF SCI 09, p.9]

For the bank to not have accurate data and a liable allocation method there is a possibility that the allocation does not reflect the 'true' risk exposure. By adopting a more sophisticated approach towards allocation of capital the banks will aspire to a more efficient allocation. By improving the economic capital allocation the bank can more easily determine which portfolios of the bank that pose risks. One commonly state that an asset with high risk need to be priced higher than an asset with low risk, because the asset poses a higher threat to the bank and thereby motivates a higher return. By having a holistic view of risks, the banks have incentives for changing their pricing structure, to a risk-based-pricing. For the risk-based-pricing structure to be considered for implementation by banks, the allocation methods need to fulfil three properties:

Intuitive: It is important that the model can be understood by relevant staff and therefore should be built on some meaningful intuitive concept of risk e.g. unexpected loss.

Stable: In the sense that the method provides a robust result i.e no major changes of the allocation due to an extra simulation. It is also preferred that the model is not excessively sensitive to underlying assumptions.

Transparent: As the method should be used by the whole bank it is important that the model is built on logic and can easily be used. It is desired that the calculations made by the method can be easily understood and backtracked.

Currently most banks apply an allocation method which allocate capital on a portfolio level. The aim of this paper is to investigate if it would be possible to find a risk measure and a capital allocation method, which would enable banks to allocate capital on a more granular level; below portfolio level. By allocating capital on a customer level, the bank will have information of the risk of each individual customer. Thereby the bank have the possibility to price each customer according to that customers individuals risk exposure.

Our hypothesis is that, in general, banks are not pricing customers according to the risk the customers are exposing the bank to. By changing the pricing structure, some of its high-risk customers will be charged with a higher price and those with low risk, a lower. In the worst case scenario, by implementing a risk-based pricing, the bank lose a critical mass of their business because many customers end up with a higher price and therefore take their business elsewhere; a scenario which is not significantly probable. When introducing a more complex pricing strategy the calculations for determining individual risk will become more advanced, than on a portfolio level. The trade-off between more time-consuming calculations and accuracy of the model is an important aspect to take into account when choosing method. Another interesting aspect is whether the more complex allocation method will suit as performance measures. To address these concerns, three research questions have been stated for this paper.

1.3 Research Questions

i) How granular can a capital allocation method perform?

ii) Can a trade-off between increased calculation and accuracy of the method be determined, how does it affect the allocation methods?

iii) Which of the allocation methods works well as a performance measure?

1.4 Limitations

This paper have some structural limitations. First, there are six different categories of risk; market-, credit-, operational-, insurance-, liquidity- and group- risk. This paper will focus on credit risk since that will be the main risk driver for the allocation procedure. Secondly, the theory in the paper can be applied to any bank, but the simulations are developed too imitate a retail bank; with large portfolios of customer loans. These simulations will be divided into two scenarios, a normal and a stressed scenario. Thirdly, three different classes of capital allocation methods can be identified; those based on the Vasicek model, approximation methods and Monte-Carlo simulations, all summarised by Eva Lütkebohmert in [Lutkebohmert 08]. Approximation methods generally suffice on assets with correlated loss data, while Vacisek and Monte-Carlo simulations requires additional data such as probability of default, modelling of exposures and loss given default, Monte-Carlo is also computationally challenging [Mausser 08]. This paper will explore three different capital allocation methods which will be presented in Chapter 2. By limiting the paper to approximations methods, it ensures that all methods can be applied to a coherent loss data structure and simulated in a feasible manner.

1.5 Purpose

The purpose of the paper is to compare three types of capital allocation methods and investigate if any of the proposed methods would be suitable for banks to consider. The main concern being how the methods will handle the diversification effect below portfolio level. In order for the allocation method to work in a real-life scenario and for banks to be interested in implement it, the model needs to be both stable, intuitive and transparent. Currently banks can use allocation methods to determine the performance of their portfolios, e.g. Risk-Adjusted-Return-On-Capital, another aspect of this thesis will address if there will be any change to performance measurement by implementing a more granular allocation method.

1.6 Data

All data is fictional if not otherwise stated.

1.6.1 Sources of Information

The research has been based on academic papers and financial journals with relevance to the banking industry, please see the References section for further information.

1.7 Structure of paper

Chapter 2 provides a general background to capital requirements, risk measures, capital allocation and specifically present the theory that concern the three capital allocation methods that have been selected for further investigation. Chapter 3 explain the method by which the allocation methods have been tested. Chapter 4 provides the results of from the simulations. Section 5 will summarize some final conclusions and discuss the problem questions posed. Figure 1.3 summarize the structure of the thesis in a flowchart.

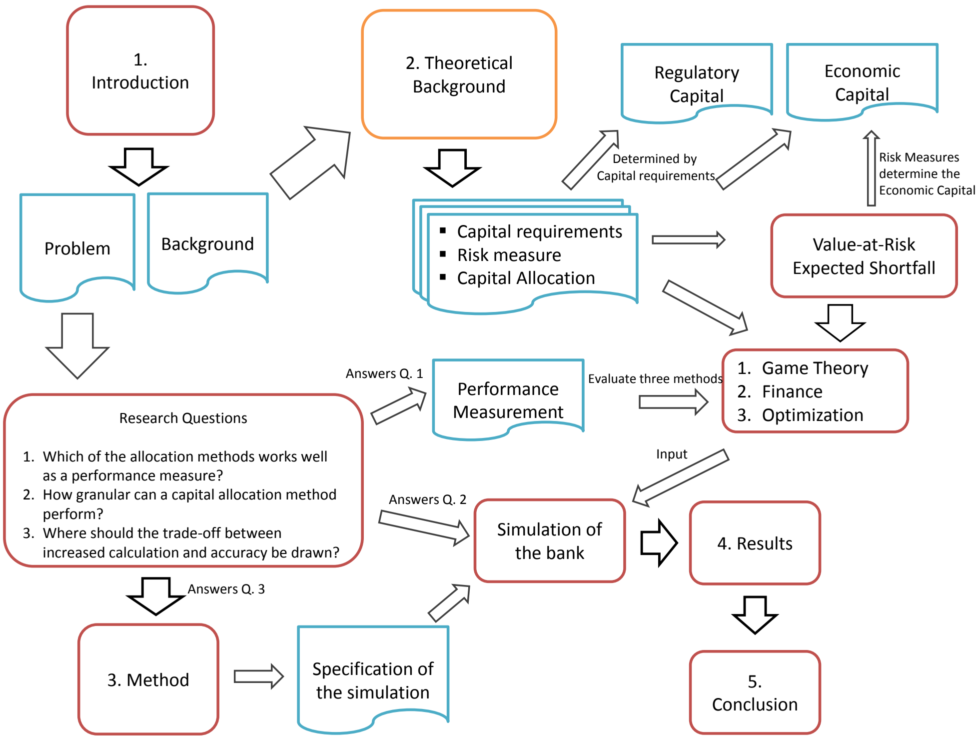

Figure 1.3: Disposition of thesis

2

Theoretical Background

THE AIM OF THIS CHAPTER is to introduce the theory concerning risk measures, capital allocation and performance measurements. In order to do this, first capital requirements for the bank need to be defined, and subsequently the difference between regulatory- and economic capital need to be determined. Regulatory capital is defined by law and economic capital is determined by the bank, with use of a risk measure. This paper will discuss two different risk measures called Value-at-Risk and Expected Shortfall. When the economic capital of the bank is determined, it needs to be allocated between the banks portfolios. Boonen, de Waeganaere and Norde [Boonen 12] state that the approximative capital allocation methods, that are concerned in this paper, can be divided into three sub-categories; game theory, finance and optimization. In order to get a holistic view of the existing methods, one method from each sub-category have been chosen for further investigation.

The first approach, game theory, as the name suggests applies the allocation problem to a game situation with players and costs of participation. The basic assumption stems from the conflict of cooperation and competition, which is clearly expressed by the following quote "People cooperate to organize corporations, then compete with other corporations for business and with each other for positions of power within the corporation." [Aumann 74, p.3] The game is therefore the set of interactions between people and the people that participate are called players, with outcomes of the game called a pay-off.

The second approach, the finance method, to the capital allocation problem will take its origin in the portfolio theory, first presented by Markowitz in 1952 [Markowitz 52]. The theory use expected value and standard deviation, risk, as measures for ranking different investment opportunities against each other. The 1952 theory have been updated into an axiomatic approach, but still based on the original assumptions. The capital allocation is determined by differentiation of a risk measure.

The third approach, the optimization method, to capital allocation models a minimization problem. The capital allocated to each portfolio is assumed to be exogenously given. The allocated capital is compared with the risk of each portfolio, each positive risk amount is called a risk residual. The allocation method tries to minimize the positive exposure by allocating capital corresponding to the marginal risk contributions of the portfolios.

The chapter will conclude with a discussion about how capital allocation methods can be used for performance measurement. The heart of the discussion will concern Risk-Adjusted-Return-On-Capital, RAROC, and how the methods suits as performance measurements.

2.1 Capital requirements

Theory concerning capital requirements can be found in e.g. reports from BCBS either [BCBS 05] or [BCBS 09], which discuss the use of the Basel framework and also desired characteristics of different risk measured used to determine capital requirements. Hashagen and Demmel [Hashagen 07] provide a concise summary of the business practises around capital requirements, risk measures and capital allocation methods. The IFF report [IIF SCI 09] provide some insight to how the banking industry is evolving after the financial crisis. Rittenberg and Martens [Rittenberg 12] provide a holistic view of how companies can work with risk appetite and addresses how that affect the capital requirements. Nilsson's [Nilsson 12] lecture notes from the course NEKN83 at Lund University provide a good discussion about risk management.

2.1.1 Regulatory Capital

The regulatory capital determines the minimum amount of capital a bank needs to hold according to regulations, e.g. the Basel III reform. Regulatory capital is a function determined by the market- and credit risk exposure. Currently the rules state that "[...] the own funds in the bank must be at least 8% of the risk-weighted assets in the bank; this is the so called capital ratio." [Nilsson 12, p.6] Risk weighted assets is a way to adjust for the risk of assets, depending on the risk that they are exposed too; a mortgage loan that has a house as collateral is less risky than an unsecured loan and thus will be weighted as a less risky loan. According to the BCBS the minimum regulatory capital is determined by the following inequality

$$\frac{C_R}{\mathrm{RWA}} > 8\%$$

The value of the regulatory capital, C_R, which is given as a percentage can be decided by the following equation

$$C_R = \left[N \left(\frac{N^{-1}(PD) + \sqrt{R} N^{-1}(0.999)}{\sqrt{1-R}} \right) \times LGD - PD \times LGD \right] \times \frac{1 + (M - 2.5)b}{1 - 1.5b}$$

where N denotes a normal distribution, PD, the Probability of Default which indicates the probability that a loan defaults. R denotes the default correlation between the portfolios. 0.999 indicates that the model uses a confidence interval with $\alpha = 99.9\%$; since the time horizon is by default calculated on a yearly basis, the 0.001% quantile the measure uses indicates losses that occur only once per 1000 years. LGD, Loss-Given-Default, determines how much of the loan that is lost if it defaults. M stands for the effective maturity of the loan. b is a maturity adjustment factor, determined as follows

$$b = (0.11852 - 0.05478 \times \ln(PD))^2$$

The default correlation R is determined by the following equation

$$R = 0.12 \times (1 - e^{-50 \times PD})/(1 - e^{-50}) + 0.24 \times (1 - (1 - e^{-50 \times PD}))/(1 - e^{-50})$$

The function has two limits which depends on the PD. With PD equal to 100% the highest correlation is 0.12 and with PD equal to 0% the limit is 0.24. Together with the limits, the dependence between correlation and PD for corporate exposures is shown in Figure 2.1 on page 11. The value $'50'$ is called the k-factor. It determines how fast the exponential function decrease, it is generally set to 50 for corporate exposures, Figure 2.1 also give an indication of the shape of the curve for varying k-factor values. The Risk-Weighted-Assets can be derived from the following formula

$$RWA = C_R \times 12.5 \times EAD$$

Exposure-at-Default, EAD, denotes the amount of an investment which could be lost of the loan defaults. C_R is given as a percentage and therefore needs to be multiplied with the EAD. 12.5 is the reciprocal of 8%, (1/0.08 = 12.5). Since the parameters PD, LGD, EAD and M, may be calculated by the bank, given that the models used are approved by the regulators, the input parameters can easily be manipulated by the bank, which pose a problem. A further discussion about the Basel formulas for regulatory capital can be found in e.g. [BCBS 05]

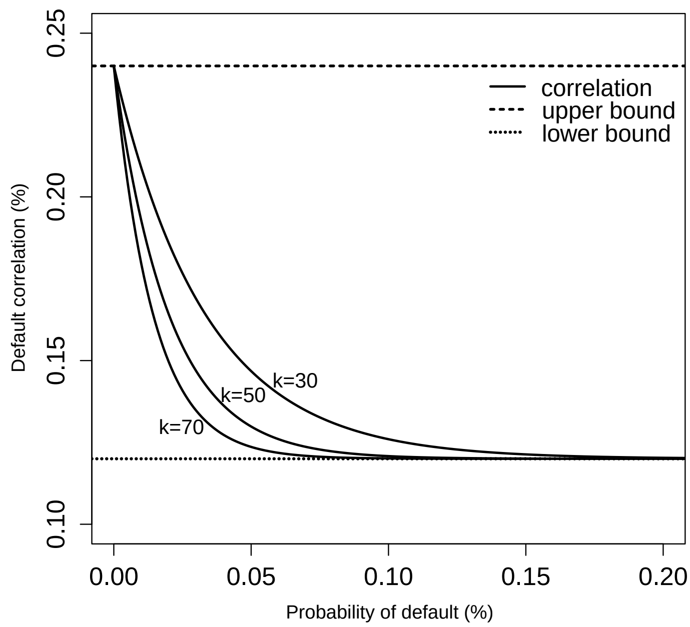

Figure 2.1: Probability of Default correlation with varying k-factor

2.1.2 Economic Capital

The economic capital serve as a hedge against unexpected losses, which depends on the risk of the banks assets. Difficulties arise because the value of the bank's exposures are never constant, it varies depending on the future value of its assets. A risk measure aims to quantify these uncertainties related to the bank's future value. A number of approaches have been developed to quantify the future values into an amount of economic capital. The group of measures that perform this calculation are called risk measures. This paper will address two quantile based risk measures, called Value-at-Risk and Expected Shortfall. The size of the quantile is determined by the confidence interval, $\alpha \in [0,1]$. A higher α result in a larger amount of economic capital and therefore also a higher credit rating. Thus a high α pose both benefits and drawbacks. A large α pose a large burden for the bank but also a large benefit, since a good credit rating will attract customers, shareholders, employees etc.

Since the economic capital is determined by the exposure against unexpected losses, therefore the set of data needs to be adjusted to unexpected losses. Each asset in portfolio i produces a loss, positive or negative, denoted l_1^i,...,l_d^i. The expected loss, EL, can then be calculated by $\mathbb{E}[l_1^i]$,...,$\mathbb{E}[l_d^i]$. In order to transform the expected losses to unexpected losses, x_j^i, the following calculation is made

$$x_j^i = l_j^i - \mathbb{E}[l_j^i] \ , j = 1,...,d$$

This ensures that all assets and portfolios have expectation 0. Under the conditions that $\sum_{j=1}^{d} l_j^i = L_i$ and $\sum_{j=1}^{d} x_j^i = X_i$, this is called to centralise variables.

2.2 Risk measures

This section will present the two most commonly used risk measures, Value-at-Risk and Expected Shortfall. A further discussion about these risk measures and other alternatives, such as Standard Deviation and Expected Loss, can be found in e.g. Martin [Martin 04] provide a comprehensive discussion of risk, risk measures and how to perform capital allocation. Artzner, Delbaen, Eber and Heath [Artzner 97] and [Artzner 99] develop a set of axioms, which a risk measure should fulfil in order to be called coherent. Jarrow, Janosi and Yildirim [Jarrow 02] discuss risk measures and a credit risk model which incorporates both liquidity risk and correlated defaults. In order to develop the concept of risk measures the following definition is made.

Definition 1 *A risk measure $\rho(\cdot)$ is defined as the mapping of a random set, $\mathbb{C}$, of variables, X, into real numbers*

$$\rho : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$$

The set of finite states of nature is defined by Ω and let $\mathbb{C}$ be the set of all risks. Thereby $\mathbb{C}$ is the set of all real-valued functions of Ω. The random variable X corresponds to the risks of the banks and can be thought of as a net worth of an element of Ω. The number $\rho(X)$ assigned by the measure $\rho(\cdot)$ to the risk X will be interpreted as the amount of economic capital, C_E desired to be held by the bank. In order to evaluate the different risk measures and determine their respective strength and weaknesses some common desired attributes are needed. Artzner, Delbaen, Eber and Heath have provided a set of characteristics that can be of assistance for the evaluation. In [Artzner 97] and [Artzner 99] four axioms are presented. If all are fulfilled by the risk measure a coherent and sensible quantification of risk is ensured.

Monotonicity:

$$\rho(X) \leq \rho(Y), \forall X, Y \in \mathbb{C} \text{ with } X \leq Y$$

Positive Homogeneous:

$$\rho(\lambda X) = \lambda \rho(X), \forall X \in \mathbb{C} \text{ and } \forall \lambda \geq 0, \lambda \in \mathbb{R}$$

Translation invariance:

$$\rho(X + \alpha) = \rho(X) - \alpha, \forall X \in \mathbb{C} \text{ and } \alpha \in \mathbb{R}^n$$

Sub-additivity:

$$\rho(X+Y) \leq \rho(X) + \rho(Y), \forall X, Y \in \mathbb{C}$$

Monotonicity states that if an asset has a higher loss, thereby a higher risk, than another asset, the economic capital should be higher for the riskier asset. Positive homogeneity ensures that the risk measure is independent of the currency in which it is measured. Translation invariance define that no extra capital is required when adding an asset that is risk-free. Sub-additivity states that a merger of portfolios does not produce any extra risk.

2.2.1 Value-at-Risk

The most commonly used risk measure is the Value-at-Risk, VaR [Balog 10]. The method is based on a confidence interval α, which determine the size of a quantile, described below.

$$q_\alpha(X) = \sup_{x \in \mathbb{R}} P(X \leq x) \geq \alpha$$

The $1-\alpha$ most extreme losses of the distribution are part of the quantile. VaR is defined as the best of the values from the quantile of the $1-\alpha$ worst losses. In other words, the method determines the limit where every loss that is greater than the VaR is an extreme loss and thus part of the quantile.

$$VaR_\alpha = q_\alpha(X)$$

The VaR is the minimal loss in the $1-\alpha$ worst cases of our portfolio. Defined for a certain significance level, $1-\alpha$, time horizon (usually one year) and portfolio. Some of the attractions with VaR is that it applies to all asset classes, allows for risk aggregation, and it is intuitive, communicative and easy to compute.

Some of the criticism towards the risk measure include that it is silent about tail events and that it is not coherent. VaR is not a coherent risk measure since it not sub-additive, this indicates that the stand alone risk of two assets can be lower than the two combined. This is shown by the following example introduced by [Artzner 99]:

2.2.1.1 VaR is not Sub-additive

Lets say we want to calculate the 99% VaR for two binary options A and B, written on the same stock with strike prices $K_A < K_B$ and loss distributions for the short options are

$$\text{Option A} = \begin{cases} 1000 & \text{if } S_T \leq K_A \\ 0 & \text{otherwise} \end{cases} \quad (2.1)$$

$$\text{Option B} = \begin{cases} 1000 & \text{if } S_T \geq K_B \\ 0 & \text{otherwise} \end{cases} \quad (2.2)$$

where S_T is the stock price at the maturity of the option. Assume that $P(S_T \leq K_A) = 0.8\%$ and $P(S_T \geq K_B) = 0.8\%$. when calculating a 99% interval the probability of an outcome should be at least 1% or higher, otherwise the outcome is not registered. In this case the $0.8\% \ngtr 1\%$, therefore the $VaR_{0.99}$ of the option is 0.

$$VaR_{0.99}(L_A) = 0$$

by symmetry the

$$VaR_{0.99}(L_B) = 0$$

hence the VaR of the two individual options are 0. The combined portfolio, $VaR_{0.99}(L_A + L_B)$ will have the $0.8\% + 0.8\% = 1.6\%$ chance for a loss of 1000. Since $1.6\% > 1\%$ the $VaR_{0.99}$ of the combined options are 1000

$$VaR_0.99(L_A + L_B) = 1000$$

Consequently, $VaR_0.99(L_A + L_B) > VaR_{0.99}(L_A) + VaR_{0.99}(L_B)$ thereby violating the sub-additive condition. This simple example serve as proof that VaR is not coherent. For a more holistic discussion about VaR Jorion [Jorion 01] or Tasche and Acerbi [Tasche 02] is recommended.

2.2.2 Expected Shortfall

In recent time an increased interest for coherent and convex methods have been found within the industry. Therefore one method which gains much acceptance is Expected Shortfall (ES), a measure also called Conditional-Value-at-Risk (CVaR), Average Value-at-Risk (AVaR), Tail-Value-at-Risk (TVaR) or Expected Tail Loss (ETL). This risk measure is also quantile based, but looks further into the tail distribution, and could be defined as the average of VaR, mathematically we define it as

$$ES_\alpha(X) = \frac{1}{\alpha}\{\mathbb{E}(X\mathbb{1}_{\{X \leq q_\alpha(X)\}}) + q_\alpha(X)(\alpha - (\mathbb{P}(X \leq q_\alpha(X))))\}$$

ES gives an estimate of the *size* of the loss, given that an extreme event have occurred. Main advantages of the method is that is it both sub-additive and coherent, thus it encourages diversification. For a further discussion of ES we recommend [Martin 09]. One should also note that

$$ES_\alpha(L) \geq VaR_\alpha(L) \tag{2.3}$$

Figure 2.2 on page 15 shows the size of ES versus the minimum loss of the $q_\alpha(X)$ that VaR propose. Figure 2.2 shows clearly how intuitive equation 2.3 is.

2.2.3 Evaluation of risk measures

The two methods presented make use of the same quantile based approach to measuring risk, which ensures that both methods give a result of the actual economic capital needed to support the risk X. There exist other risk measures that do not give the same

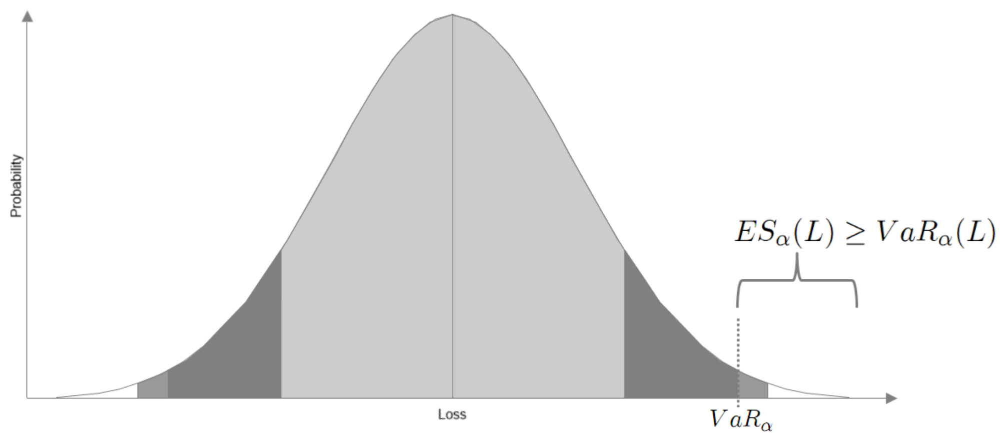

Figure 2.2: Value-at-Risk vs Expected Shortfall

evaluation of economic capital needed, but rather is a mapping of the quantification of the risk e.g standard deviation. Since these methods do not give the same holistic view of economic capital needed, they have been excluded from the analysis. Other than the above mentioned significantly important characteristics, according to BCBS [BCBS 09], banks require the following properties to be fulfilled by a risk measure.

Stable: In the sense that the method provides a robust result i.e no major changes of the allocation due to an extra simulation. It is also preferred that the model does not be too sensitive to underlying assumptions.

Coherent: The risk measure should be able to divide the economic capital into portfolios of the bank and be able to take the diversification effects into account.

Intuitive: The risk measure should be in line with some meaningful intuitive concept of risk.

Easy to compute: The risk measure should be easy to compute; especially by using a more complex model there should be proof of improved accuracy to the results.

Easy to understand: It is the bank's management that will make the decision about implementation and therefore it is important that they understand it. If they do not comprehend the model there is a large chance that the model is not implemented.

Table 2.1 show how VaR and ES satisfy the proposed conditions.

	VaR	ES
Intuitive	Yes	Sufficiently intuitive
Stable	No, depends on assumptions about loss distribution	Depends on the loss distribution
Easy to compute	Sufficiently easy (requires estimate of loss distribution)	Sufficiently easy (requires estimate of loss distribution)
Easy to understand	Yes	Sufficiently
Coherent	No	Yes

Table 2.1: Risk Measures [BCBS 09, p.27]

The two methods have quite similar characteristics, when it comes to the five properties. As shown before and made apparent of Table 2.2, VaR is not coherent due failure of the sub-additivity characteristic. When taking this into account the ES seems to be the preferred method. For a nice loss distribution the method manages to fulfil all five properties. Because of this ES will be the preferred risk measure used in the later part of the paper.

	Homogenity	Translation	Monotonicity	Sub additivity
VaR	✔	✔	✔	✗
ES	✔	✔	✔	✔

Table 2.2: Risk Measure Properties [BCBS 09, p.21]

2.3 Capital Allocation

There exist various approximative approaches for allocating capital. According to [Boonen 12] the different methods can be divided into three sub-categories depending on which basic concept they apply to the allocation problem; game theory, finance and optimization. Later in the paper, one method from each sub-category will be chosen and evaluated in order to determine if there are any differences in output by the approaches. By using a coherent risk measure to determine the economic capital of the bank, the sub-additivity property ensure that, the economic capital for the whole bank is lower than for the some of the individual portfolios alone. The reason for this is that the individual risks of a portfolio are in general not perfectly correlated and therefore some hedge effects will be in play when combining risks. The allocation method determines the rule for how these benefits should be distributed between the different portfolios. The general definition of a capital allocation method is:

Definition 2 $\Pi(\cdot)$ *is an arbitrary allocation method that assigns the amount of economic capital* $\Pi(X_i)$ *to each of the risks* X_i*, for the portfolios* $i = 1,...,n$*; while satisfying*

$$\sum_{i=1}^{n} \Pi(X_i) = C_E$$

Which is subject to $n \in N$ portfolios.

The goal of the allocation method is to allocate economic capital to the different portfolios of the bank, corresponding to the risk of that portfolio. The definition also ensures that all economic capital is allocated to the portfolios. There are two main reasons for a bank to allocate capital. First, the allocation of capital to business units are important when making decisions. By allocation of capital to the portfolios of the bank the management have a holistic view of the risks of the assets in a portfolio. Different hedging possibilities can be explored by increased or decreased exposure to specific portfolios. A certain asset as a stand-alone can pose a huge risk, but combined with the rest of the assets of a portfolio it might serve as a hedge against other risks. The attractiveness of a specific asset is typically evaluated by a risk-return trade-off. Therein lays the second reason for allocation of economic capital, performance measurement. In general, the bank has a structure of portfolios and sub-portfolios with managers deciding on strategic decisions for the portfolios. In general, managers are being evaluated on the return of the risk that their portfolio is exposed to. In order to give the managers a fair assessment it is important that the economic capital is allocated according to the individual risk of the portfolio. The return of the portfolio might then be evaluated against the allocated economic capital. A general discussion about performance measurement and RAROC specifically, can be found at the end of the chapter. In addition to diversification, the bank prefer the allocation method to fulfil the following three properties.

Intuitive: For the allocation method to be understood by the whole organization, it is valuable that relevant staff understands it and that it is based on a common measure e.g. expected loss.

Stable: Small changes, for example an extra simulation, should not result in drastic changes of the allocation.

Transparent: As the method should be used by the whole bank it is important that the model is built on logic and can easily be used.

If the allocation method fulfils these characteristics the attractiveness for the bank to consider to implement it increases. The stability property ensures that the result from the allocation can be trusted and that further simulations will reach the same or similar results; the method is expected to reach a single optima. For the method to be transparent, the calculations should be based on logical assumptions, which can be followed by the user. There exist allocation methods that are black-box solutions, for those cases the managers have no control of what the model actually calculates. Sometimes this can be unfortunate, examples can be found were allocation models change the risks for car loans in Denmark due to a power plant accident in Japan. These events are totally uncorrelated and therefore should not affect the risk on car loans; the allocation of economic capital for car loans. The intuitive property is preferred in order for the method to actually be implemented and understood by the staff of the bank. These three characteristics will serve as reference points for the comparison between the allocation methods. Where all three characteristics are equally important. The next section will make some restrictions of the general definition of an allocation method, by defining a set of axioms.

2.3.1 Coherent Allocation

Shapley [Shapley 53], have developed a set of three axioms which when fulfilled by an allocation method state a coherent allocation. These are similar to the Artzner, Delbaen, Eber and Heath [Artzner 99] axioms for a coherent risk measure. By evaluating the allocation method beforehand and determine that it fulfils the coherence property, the computations may be simplified. Denault [Denault 01] summarize the three properties as follows, where $\Pi(\cdot)$ denotes the capital allocation method, N the set of portfolios and $\rho(\cdot)$ is a risk measure, thus defining the allocation problem as $\Pi(N,\rho)$.

No undercut:

$$\forall M \subseteq N, \sum_{i \in M} c_i \leq \rho\left(\sum_{i \in M} X_i\right)$$

Symmetry: If by joining any subset $M \subseteq N \backslash \{i,j\}$, portfolios i and j both make the same contribution to the economic capital, then $c_i = c_j$

Risk-less allocation:

$$c_n = \rho(\alpha B) = -\alpha$$

Recall that the n^{th} portfolio is a risk-less instrument.

[Denault 01, p.5 Definition 3].

The no undercut property ensures that the sum of the allocations c_i is less or equal to the risk, X_i, of the sum of the portfolios. This results in an allocation where each portfolio is allocated a smaller amount of economic capital than it would have a separate portfolio. By sub-additivity of the risk measure, this is ensured since no portfolio can add more risk than the portfolio has on its own. The symmetry property ensure that all portfolios with equal risk will be allocated the same amount of economic capital. Each allocation is thus only concerned with that portfolios risk contribution. The \ in the definition, means the set relative complement, the set of all elements of N except those that intersect with $\{i,j\}$. The risk-less allocation property ensures that each portfolio is allocated capital according to its risk contributions. By adding cash or another type of risk-less portfolio the economic capital should be allocated according to the risk measure, which will be negative.

2.4 Allocation Methods

The following section will provide the theoretical characteristics and assumptions of the three chosen capital allocation methods.

2.4.1 Game theory

The mathematical theory of games was first developed in the 1930's [Neumann 28] and then popularised in the 1950's by [Neumann 44]. In 1953, Shapley [Shapley 53], produced a paper which developed a method for dividing costs between players in a game, called the Shapley-value. That contribution developed the theory concerning capital allocation. Since then many authors have made contributions to the theory e.g. Aubin [Aubin 81] who applies theory from co-operative games to fuzzy co-operative games and presents a more intuitive approach to values of fuzzy games;
Billera and Raanan [Billera 81] evaluates a linear non-atomic production game and its cores. Billera and Heath [Billera 82] derive a procedure for sharing costs in a process, based on four desirable properties.
Mirman and Tauman [Mirman 82] evaluates an economic cost sharing approach based on the Shapley-value and justify their proposed price mechanism by a set of axioms and cost functions and not game theory.
Tsanakas and Barnett [Tsanakas 03] discuss the allocation of insurance liabilities according to Aumann-Shapley values and use a distortion risk measure.
Tsanakas [Tsanakas 04] discuss the use of distortion risk measures for determining the economic capital that is allocated.
Homburg and Scherpereel [Homburg 08] provide a comparison between different allocation methods, when using VaR as a risk measure, and evaluate how well the allocation methods allocate capital through a simulation.
Tsanakas [Tsanakas 09] discuss the use of distortion exponential risk measures for determining economic capital applied on Aumann-Shapley values as allocation method.
De Waegenaere, Boonen and Norde [Boonen 12] discuss the use of game theory when the and Aumann-Shapley values are not defined, an asymptotic approach to a solution that corresponds with Mertens value is proposed.

With the most significant contributions by Aumann and Shapley [Aumann 74] who develops the theory of Aumann-Shapley-values from the Shapley-value. Aubin [Aubin 79] introduces the concept of fuzzy games, which are games that allow fractions of players. Denault [Denault 01] summarize main points from, Shapley [Shapley 53], Aumann-Shapley [Aumann 74] and Aubin [Aubin 79] and [Aubin 81] and compare their different approaches in a numerical example.

The game theory assumes that each of the players, $i = 1,...,n \in N$, in a game, $(X(N),F)$, wants the best outcome for themselves. It further assumes that the players can co-operate with each other, and also that all players think that it is something good to make someone better off, provided that no other player is worse off. These set of multi-strategies is called Pareto strategies. $X(N)$ is the set of multi-strategies of the n players and F is the multi-loss operator, where $F(x) = K$. The stochastic variable K denotes the multi-loss for the game, and thus the cost for the game. Each player, i, is assumed to have a loss function f_i, which determine player i's loss distribution. The strategy were the n players co-operate to form the best outcome for all players is called a multi-loss strategy and denoted $\bar{x} \in X(N)$. A fundamental assumption of the theory is that in order to reach the best strategy, losses can be transferred from one player to another. The vector λ defines 'rates of transfer of loss' between the players. $\lambda = (\lambda_1,...,\lambda_n) \in M^n$ is used by players to fuse their loss functions f_i. Any weak Pareto strategy $\bar{x} \in X(N)$ minimizes over $X(N)$ a collective loss $\langle\lambda,F(\cdot)\rangle$ for an adequate rate of transfer $\lambda \in M^n$. To every multi-loss $K \in \mathbb{R}^n$ a number of units of account $\langle\lambda,K\rangle$ called a 'side payment', ω, is associated with the vector λ. Lets say there are two multi-losses, x and y. They are transferable, under the rate λ, if and only if $\langle\lambda,x\rangle = \langle\lambda,y\rangle$.

> [...] any strategy which minimizes $\sum_{i=1}^{n} \lambda^i f_i(\cdot)$ is a Pareto minimum, we prove that the converse statement holds under convexity assumptions. We generalize this theorem in the case of an economic game. Under convenient assumptions, an allocation $\bar{x} = \{\bar{x}^1,...,\bar{x}^n\} \in X(N)$ is a weak Pareto minimum if and only if there exist a rate of transfer $\bar{\lambda} \in \mathbb{R}^{n*}$ and a price $\bar{p} \in \mathbb{R}^{n*}$ such that
>
> $$\begin{cases} \text{(i)} \quad \forall i \in N, \bar{\lambda}^i f_i(\bar{x}^i) + \langle\bar{p},\bar{x}^i\rangle = \min_{y\in\mathbb{R}}[\bar{\lambda}^i f_i(y) + \langle\bar{p},y\rangle], \\ \text{(ii)} \quad \sum_{i=1}^{n}\langle\bar{p},\bar{x}^i\rangle = \sup_{y\in Y}\langle\bar{p},y\rangle \end{cases} \tag{2.4}$$
>
> [Aubin 79, p.295]

The set of these solutions, Pareto minima, are too large and therefore it is desired to be reduced. In order to decrease the possible solutions, restrictions are added to the game. If the cost for participation in the game is too large, player i threat to leave the game. The level of tolerated cost for participation is determined by the players threat function t. For player i, t_i determines the maximum amount of loss that player i accepts and still stays in the game. Thereby a possible outcome of the game, a multi-loss strategy $\bar{x}$, needs to fulfil two conditions in order to become a solution.

i) Not being rejected by the set N of players.

ii) For all players i, an allocation $f(\bar{x}) \leq t_i(f_i)$ can be found.

If these conditions are met, the solution is part of a set called imputation. Imputations are a subset of the set of solutions. One should realize that the concept of imputations and Pareto minima are simply types of selection functions, denoted s. The concept will be explained by the following example from [Aubin 79]. The choice of Pareto minima can be determined by minimizing $s[F_T(x)]$ on $X(N)$ where F_T is defined as

$$f_{t_i}(x) = \frac{f_i(x) - \alpha(f_i)}{t_i(f_i) - \alpha(f_i)},\ F_T(x) = \{f_{t_i}(x)\}_{i\in N}$$

With the following restrictions.

$$\begin{cases} \text{(i)} \quad \alpha(f_i) = \inf_{x\in X(N)} f_i \\ \text{(ii)} \quad t_i(f_i) > \alpha(f_i)\ \forall i \in N \end{cases} \tag{2.5}$$

Lets say that $s(K) = \sup_{i \in N} K_i$, then the multi-loss strategy, $\bar{x}$, called 'best compromise' can be computed through the following calculations.

$$\forall i \in N, f_i(\bar{x}) \leq (1 - \bar{d})\alpha(f_i) - \bar{d}t_i(f_i)$$

$$\bar{d} = \inf_{x \in X(N)} \max f_{t_i}(x) \in [0,1]$$

In order to eliminate the possibility that the selection function depend on positive affine transformations of the loss function, the threat function must fulfil the following conditions.

$$t_i : f_i \rightarrow t_i(f_i) \text{ must satisfy } t_i(\alpha f + \beta) = \alpha t_i(f_i) + \beta$$

$$\alpha \in \mathbb{R}_+ \text{ and } \beta \in \mathbb{R}$$

which implies the existence of an $\bar{x}$ minimizing

$$x \mapsto \sum_{i=1}^{n} \frac{|f_i(x) - \alpha_i|^2}{|t_i(f_i) - \alpha_i|^2}, \text{ where } \alpha_i = \alpha(f_i)$$

The example present an alternative of how selection functions can be used to solve the allocation problem and present a solution, which is part of the set of imputations. The problem with the set of imputations are unfortunately the same as with the Pareto strategies, the set of solutions is too large. The remedy to this problem is too impose more restrictions, a concept called coalitions are introduced.

2.4.1.1 Coalitions

In order to reduce the number of solutions in the set of imputations, the players are allowed to form coalitions. If player i joins a coalition then the behaviour of that player alters, thereby the set of solutions decrease. Lets say player i joins coalition A, the behaviour of the coalition can be described by the pair $\{X(A),F^A\}$. A is naturally a subset of N and denotes the set of all players in coalition A. $X(A)$ describes the set of multi-loss strategies for coalition A and $F^A : X(A) \mapsto \mathbb{R}^A$ is the multi-loss operator of coalition A, which is defined as $F^A(x) = \{f_i^A(x)\}_{i \in A}$. The co-operative game $\{X(A),F^A\}_{A \in \mathscr{A}}$ describes the behaviour of each player in A and the coalition A; $\mathscr{A}$ denotes a family of coalitions. Coalition A can be denoted by the vector $\tau^A : N \mapsto \{0,1\}$, this vector join each player i with his 'rate of participation' τ_i^A in the coalition.

$$\tau_i^A = \begin{cases} 1 & \text{if player } i \text{ participates in coalition } A \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

All n players $\in N$ are either in or out of coalition A. Lets say a player $m \in N$ does not join coalition A, then per default player m join the adverse coalition, denoted coalition $\bar{A}$ of the set $\bar{A}$. The players in the adverse coalition co-operate to form the best solution for coalition $\bar{A}$ and thereby not co-operating with players i in coalition A. In order to adapt to the new situation player i need to alter i's loss function. Thereby each player $i \in A$ join into a multi-strategy $x^A \in X(A)$ with the worst losses, for player i, defined by the loss function in equation 2.7 below.

$$f_i^{A\sharp}(x^A) = \sup_{\substack{y \in X(N) \\ \pi^A y = x^A}} f_i(y) = \sup_{\{x^A, x^{\hat{A}}\} \in X(N)} f_i(x^A, x^{\hat{A}}) \tag{2.7}$$

$f_i^{A\sharp}(x^A)$ is the loss function for player i of coalition A opposed to the adverse coalition $\hat{A}$. Coalition $A \in \mathscr{A}$ is said to reject any multi-loss strategy $x \in X(N)$ if the coalition can improve on the strategy by adopting a multi-loss strategy x^A of the set of multi-loss strategies $X(A)$. x^A would yield to each player $i \in A$ a loss $f_i^A(x^A)$, which is smaller than the loss $f_i(x)$. The set of solutions which have not been rejected by any coalition A is called the core of the game, denoted $\mathscr{O}(\{X(A), F^A\}_{A \in \mathscr{A}})$. The core of the game can sometimes include more than one solution and thereby being too large. The remedy for this problem is the same as previously; introduce more restrictions.

2.4.1.2 Fuzzy Coalitions

In order to reduce the number of solutions in the core, Aubin [Aubin 79] introduces the concept called fuzzy coalitions. It can be defined as follows

> We shall say that any $\tau = [\tau_1, ..., \tau_n] \in [0,1]^n$ is a "fuzzy coalition" and that its i-th component, τ_i, is the rate of participation of player i in this fuzzy coalition. A subset $\mathscr{T}$ of $[0,1]^n$ which contain $\mathscr{A}$ is called a family of fuzzy coalitions.
> [Aubin 79, p.317 Definition 1]

The main difference to the coalitions is that fuzzy coalitions allow for fractions of players to participate in the game. The λ-vector needs to be adjusted in order to administer fuzzy coalitions. λ enables players, i, in a fuzzy coalition, τ, to share a given side payments, ω, according to the following rule

$$K \in \mathbb{R}^n \text{ such that } \sum_{i=1}^{n} \lambda^i \tau_i K_i = \langle \lambda, \tau K \rangle = \omega$$

under the condition

$$\forall i \in N, \tau_i = \sum_{\substack{A \in \mathscr{A} \\ i \in A}} m(A)$$

where $m(A)$ is the probability that coalition A forms. The rate of participation of player i is the sum of the probabilities of the formation of coalitions A to which i belongs. With the introduction of fuzzy coalitions and fractional players, the analogy to a game with players is not intuitive any more. The concept of the game situation is therefore instead applied to the more relevant allocation of capital. The game, $\{X(A), F^A\}$ represents the allocation of economic capital C_E. The multi-loss operator F^A determines the multi-loss K, which for financial notations are known as, the risk measure $\rho(\cdot)$ and economic capital C_E. The n players in the game, divided into fractions, symbolize the n portfolios of the bank with their corresponding assets. Thereby the concept of pooling fractions of portfolios is more intuitive compared with the player notation. Fractions are given as decimals e.g. $10\% = 0.1$, which indicates that 10% of all assets from that portfolio participates in the coalition. When using the concept of fractional players fuzzy coalition can be approximated by a Debrau-Scarf coalition. The approximation gives a simple and intuitive expression to the 'rate of participation' τ-vector.

> We shall say that a fuzzy coalition $\tau \in \mathscr{A}$ is a Debrau-Scarf coalition if its rate of participation $\tau_i = p_i/q \in Q$ are rational. This implies that $q \geq p_i \forall i = 1,...,n$.
> [Aubin 79, p.318]

i is regarded as a 'type of player' and thus p_i is the number of players of type i, q denotes total number of players of all types in that coalition. Then the 'rate of participation', τ_i, of 'type' i is the proportion of players of type i involved.

2.4.1.3 Application to the allocation problem

A fuzzy game ($\mathscr{A}$,$\rho(\cdot)$) with side-payments is denoted as the game $(\mathscr{T}, J)$ where

$$\forall \tau \in \mathscr{T},\ J(\tau) = \{C_E \in \mathbb{R}^\tau \text{ such that } \sum_{i=1}^{n} c_i = \rho(\tau)\}$$

The function $\rho(\cdot)$ is in the game theory literature called a 'loss function' but is essentially is a risk measure, which join each fuzzy coalition τ with its loss $\rho(\tau)$. The initial setting proposed by the allocation method require that the loss $\rho(\tau)$ is shared by portfolios that are part of τ, proportionally to their 'rates of participation'. By assuming that the loss function $\rho(\cdot)$ is positively homogeneous, the set of Pareto minima is $J(\tau^N)$. Thereby the core $\mathscr{O}(\mathscr{T},\rho)$ of the allocation can be described as the sub-differential of ρ at τ^N. If the set of multi-losses $C_E \in \mathbb{R}^n$ fulfil the following conditions

$$\begin{cases} \text{(i)} & \sum_{i=1}^{n} c_i = \rho(\tau^N) \\ \text{(ii)} & \forall \tau \in \mathscr{T}, \sum_{i=1}^{n} \tau_i c_i \leq \rho(\tau) \end{cases} \tag{2.8}$$

then $\mathscr{O}(\mathscr{T},\rho) = \partial\rho(\tau^N)$. The solution which is in the core, is the optimal solution to the allocation. According to Aumann and Shapley [Aumann 74], the optimal allocation is called the 'Aumann-Shapley-prices' or 'Aumann-Shapley per unit allocation'. The theory assumes that there exist a cost function r, which have been defined as the risk measure $\rho(\cdot)$. By solving the following equation the allocation can be found.

$$\phi_i^{AS}(N,\Lambda,\rho) = q_i^{AS} = \int_0^1 \frac{\partial r}{\partial \Lambda_i}(\gamma\Lambda)\,\mathrm{d}\gamma$$

for portfolio i of the set N portfolios. The vector Λ is the 'size of business' for each portfolio, in an arbitrary reference unit. The per unit cost q_i^{AS} is an average of the i-th portfolio, as the level of activity increases uniformly for all portfolios from 0 to Λ. The value of the equation has a simpler expression if ρ is a coherent and thereby a 1-homogeneous risk measure

$$\phi_i^{AS}(N,\Lambda,\rho) = \frac{\partial r}{\partial \Lambda_i(\Lambda)}$$

and the per unit allocation vector is the gradient of the mapping ρ evaluated at the full-presence level Λ.

$$\phi_i^{AS}(N,\Lambda,\rho) = \nabla \rho(\Lambda)$$

The amount of risk allocated capital to each portfolio is then given by the components of the vector Q, which is derived from the component-wise product of the vectors q^{AS} and Λ.

$$Q^{AS} = q^{AS}. * \Lambda$$

Denault neatly summarize the game theory method as "given that the allocation process starts with a coherent risk measure, this coherent allocation simply corresponds to the gradient of the risk measure with respect to the presence level of the constituents of the firm." [Denault 01, p.26] Denault also come to the conclusion that the Aumann-Shapley-value is the same as a coherent fuzzy value, given that the cost function r is coherent and differentiable at Λ.

2.4.2 Finance

The initial theory provided by Markowitz [Markowitz 52] pose some inconveniences due to the notation of risk, which is defined as 'measure of uncertainty in return'. Thus the standard deviation does not take positive fluctuations about the return in account. Later, authors have adopted their methods to a more convenient notion. Tasche [Tache 07] investigates a method with kernels and Monte Carlo simulation for determining the risk contributions for portfolios and sub-portfolios, using the risk measure VaR. Kalkbrenner [Kalkbrener 05] applies an axiomatic approach, which depends on a chosen risk measure. By using a set of three axioms and a coherent risk measure the capital allocation is determined. The same conclusion is reached by Tasche [Tasche 99] who instead uses the theory for calculation of optimal performance measurements based on the risk measures VaR, standard deviation and ES.

The finance based allocation method assumes that the allocation, $\Pi(\cdot)$ to portfolio X_i, only depends on X_i and the bank X. The decomposition of the rest of the portfolios does not affect the allocation to X_i. For this section, let a probability space be defined by $(\Omega,A,\mathbb{P})$. Let L^0 be a vector space of all the equivalence classes of real valued random variables on Ω. Lets also assume that the vector space L^0 contains a subspace V, with $X_i \in V$. This allocation method propose a set of three axioms, which will define the optimal allocation method $\Pi(\cdot)$. The axioms have been proposed by Kalkbrenner [Kalkbrener 05] and are presented below.

Linear aggregation: The economic capital of the bank equals the sum of the economic capital contributions from the portfolios. Let $X_1,...,X_n \in V$ and $a_1,...,a_n \in \mathbb{R}$ and define $X = a_1X_1 + ... + a_nX_n$. Then

$$\rho(X) = \Pi(X,X) = \sum_{i=1}^{n} a_i \Pi(X_i,X)$$

Diversification: The economic capital $\Pi(X_i,X)$ of $X_i \in V$ considered a portfolio of $X \in V$ does not exceed the economic capital $\rho(X_i) = \Pi(X_i,X_i)$ of X_i considered as a stand-alone portfolio.

Continuity: Small changes to the bank only have a limited effect on the economic capital of its portfolios. The economic capital $\Pi(X_i,X + \epsilon X_i)$ of X_i in $X + \epsilon X_i$ converges to the economic capital $\Pi(X_i,X)$ of X_i in X if $\epsilon \in \mathbb{R}$ converges to 0.

The axioms could be explained as follows. Linear aggregation ensures that the sum of the economic capital of the portfolios are the same as the economic capital of the bank. Diversification, ensures that no extra risk can be added by grouping portfolios together. Finally, the continuity ensures that small changes of the bank X, will have limited effects on the allocation to its portfolios. Linear aggregation and diversification is given by the risk measure being sub-additive and positive homogeneous. Continuity at X is given by the existence of a directional derivative.

> Let $\Pi(\cdot)$ be a linear, diversifying capital allocation with respect to $\rho(\cdot)$. If $\Pi(\cdot)$ is continuous at $X \in V$ then for all $X_i \in V$
>
> $$\Pi(X_i,X) = \lim_{\epsilon \to 0} = \frac{\rho(X + \epsilon X_i) - \rho(X)}{\epsilon} \tag{2.9}$$
>
> [Kalkbrener 05, p.4 Theorem 3.1]

If the allocation fulfils the three axioms the uniqueness of the solution is guaranteed; the capital allocation method thus being linear, diversifying and continuous at $X \in V$. The solution to the allocation method is thus the derivative of the risk measure $\rho(\cdot)$ at X in direction of portfolio X_i.

The existence of such allocation method is determined by using the Hanh-Banach theorem. Let $\rho(\cdot)$ be both positive homogeneous and sub-additive it can be shown that $\rho(X_i)$ can be determined by the element h_{X_i} which is only unique if the directional derivative exist, shown by equation 2.9. For a further discussion of Hanh-Banach theorem and the concepts included, the reader is recommended Kulkarni [Kulkarni] or Kalkbrenner [Kalkbrener 05]. The results from the assumptions about Hanh-Banach theorem is summarized in the following useful description.

i) If there exist a linear, diversifying capital allocation $\Pi(\cdot)$ with respect to $\rho(\cdot)$ then $\rho(\cdot)$ is positively homogeneous and sub-additive.

ii) If $\rho(\cdot)$ is positively homogeneous and sub-additive then Π_ρ is a linear, diversifying capital allocation with respect to $\rho(\cdot)$.

[Kalkbrener 05, p.6 Theorem 4.2]

The Hahn-Banach theorem ensures the existence of a capital allocation method that fulfil the three stated axioms. If the risk measure, $\rho(\cdot)$, is both homogeneous and sub-additive and $X \in V$ the following three statements are equivalent.

i) Π_ρ is continuous at X, i.e. for all $X_i \in V$

$$\lim_{\epsilon \to 0} = \Pi_\rho(X_i, X + \epsilon X_i) = \Pi_\rho(X_i, X)$$

ii) The directional derivative

$$\lim_{\epsilon \to 0} = \frac{\rho(X + \epsilon X_i) - \rho(X)}{\epsilon}$$

exist for every $X_i \in V$

iii) There exists a unique $h \in H_\rho$ with $h(X) = \rho(X)$

If these conditions are satisfied then

$$\Pi_\rho(X_i, X) = \lim_{\epsilon \to 0} = \frac{\rho(X + \epsilon X_i) - \rho(X)}{\epsilon} \tag{2.10}$$

for all $X_i \in V$.
[Kalkbrener 05, p.7 Theorem 4.3]

This equation thereby prove the existence of the allocation method as the directional derivative of X in direction X_i. Both sub-additivity and homogeneous are fulfilled by coherent risk measures. Since ES is a coherent risk measure the subsequent calculations will be developed for ES, but can easily be adjusted to represent another coherent risk measure. As stated in the risk measure section, ES determines a quantile, which depends on the significance level $\alpha \in (0,1)$, let $X \in V$, and denote the smallest α-quantile

$$q_\alpha(X) = \inf_{x \in \mathbb{R}} \mathbb{P}(X \leq x) \geq \alpha$$

The risk measure, ES, is the average of this quantile. Defined as

$$ES_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 q_u(X)\, du.$$

ES can also be represented as

$$ES_\alpha(X) = \frac{1}{1-\alpha}(\mathbb{E}[X\mathbf{1}_{\{X>q_\alpha(X)\}}] + q_\alpha(X)(\mathbb{P}(X \le q_\alpha(X)) - \alpha)) \tag{2.11}$$

and since the allocation problem is a continuous problem $\alpha = \mathbb{P}(X \le q_\alpha(X))$, which simplifies the expression a bit. Since ES is coherent and monotonic, there exist a set $\mathscr{Q}$ of probability measures that can be defined as

$$ES_\alpha(X) = \max_{\mathbb{Q}\in\mathscr{Q}} \mathbb{E}_{\mathbb{Q}}(X) \tag{2.12}$$

$\mathscr{Q}$ is the set of probability measures that are absolutely continuous to $\mathbb{P}$, where $\mathbb{P}$ is the probability space. Equation 2.12 can be justified by the following, let $\mathbb{Q}_X$ be defined by

$$\frac{d\mathbb{Q}_X}{d\mathbb{P}} = \frac{E_{\mathbb{Q}_X}(X) + \beta_X \mathbf{1}_{\{X=q_\alpha(X)\}}}{1-\alpha}$$

where

$$\beta_X = \frac{\mathbb{P}(X \le q_\alpha(X)) - \alpha}{\mathbb{P}(X = q_\alpha(X))} \text{ if } \mathbb{P}(X = q_\alpha(X)) > 0 \tag{2.13}$$

By equation 2.15

$$ES_\alpha(X) = E_{\mathbb{Q}_X}(X) = \max_{\mathbb{Q}\in\mathscr{Q}} \mathbb{E}_{\mathbb{Q}}(X)$$

where $\mathscr{Q}$ is defined by

$$\mathscr{Q} = \{\mathbb{Q}_X | X \in V\}$$

Because the Hanh-Banach theorem ensure $\Pi(X_i, X) = h_Y(X_i)$ and subsequently equation 2.10 holds, the allocation of economic capital is determined by the following equation.

$$\Pi_\alpha^{ES}(X_i, X) = E_{\mathbb{Q}_X}(X_i) = \frac{(\int X_i \mathbf{1}_{\{X>q_\alpha(X)\}} d\mathbb{P} + \beta_X \int X_i \mathbf{1}_{\{X=q_\alpha(X)\}} d\mathbb{P})}{1-\alpha} \tag{2.14}$$

The capital allocation method is both linear and diversifying with respect to ES_α, thus chosen as the allocation method for the financial approach. Equation 2.14 propose that if the loss X is larger than the quantile $q_\alpha(X)$ then the expression $\mathbf{1}_{\{X>q_\alpha(X)\}}$ equals 1, otherwise 0. If 1, the loss X_i should be integrated over the probability space $\mathbb{P}$, which is from α to 1. If the loss X is equal to the quantile it is on the gradient of the integral, therefore a β-variable needs to be taken into account defined by equation 2.13.

$$\mathbb{P}(X > q_\alpha(X)) = 1 - \alpha \text{ or } \mathbb{P}(X \ge q_\alpha(X)) = 1 - \alpha \tag{2.15}$$

then

$$\lim_{\epsilon\to 0} = \frac{d\mathbb{Q}_{X+\epsilon X_i}}{d\mathbb{P}} = \frac{d\mathbb{Q}_X}{d\mathbb{P}} \text{ a.s.} \tag{2.16}$$

for every $X_i \in V$ and therefore Π_α^{ES} is continuous for X. Where a.s. stands for almost surely, which has similar meaning as 'in almost all cases'. If equation 2.15 have $\mathbb{P}(X = q_\alpha(X)) = 0$, then equation 2.16 is satisfied.

2.4.3 Optimization

Dhaene, Denuit, Goovaerts, Kaas and Vyncke present two papers on the subject, the first concerning theory [Dhaene 02b] and the second, implementation [Dhaene 02a]. The papers determine approximations of sums of variables, when distributions are known but their stochastic dependence structure is either unknown or too difficult to calculate. The theory and applications are based on claims of an insurance portfolio. Laeven and Goovaerts [Laeven 03] determine an optimal capital allocation based on marginal distributions and a concept called risk residuals. Salam [Salam 10], discuss an insurance companies possibility to maximize expected net after tax income by an optimization problem and the risk measure TVaR is used for determining risk contributions. Dhaene, Goovaerts and Kaas [Dhaene 03] evaluates different risk measures used within the insurance industry and stress the importance of risk measure which can handle all dependence structures. The paper concludes that the companies need to take the difference of allocated capital and risk into account.

This method assumes that the bank hold an amount of economic capital which is exogenously given. How this amount has been calculated is not defined, and thus the capital is assumed to be known in advance. The full amount the bank exogenously holds is denoted C_D. The allocation method further assumes that the capital, C_D have been distributed to the $i = 1,...,n \in N$ portfolios of the bank. The distribution to the portfolios is defined by the vector $C_D = c_{D1},...,c_{Dn}$. One could think that the distribution is made according to the previous years capital allocation or possibly according to regulatory requirements. The following inequality for the capital is assumed to hold

$$\min[X_1] + ... + \min[X_n] < C_D < \infty$$

$X_1,...,X_n$ denote the risk of portfolios of the bank. Further the marginal distributions $(F_{X_1},...,F_{X_n})$ of the portfolios are assumed to be known but the dependence between the portfolios are not needed in advance. The solution is restricted to the compact set A, defined as in equation 2.17.

$$A = \{(c_{D1},...,c_{Dn}) \in \mathbb{R}^n | c_{Di} \in [a_i, b_i], \forall i = 1,...,n, \sum_{i=1}^{n} c_i = C_D\} \tag{2.17}$$

With $a_i,b_i \in \mathbb{R}\ \forall i$ and by setting $a_i = \min[X_i]$ and $b_i = \max[X_i]$ the random variable vector space $(L^\infty(\Omega,F,\mathbb{P}))$ will be considered. The theory will make use of a certain type of vector space called the Fréchet space. In general it is a topological vector space, it is locally convex and complete in the sense of a translation invariant metric. The Fréchet space is a generalisation of the Banach-space, a space which needs the translation invariant metric to arise from a norm. In this case, the Fréchet space will be constructed by the marginal distributions $F_{X_i} = (F_{X_1},...,F_{X_n})$, which allows the dependence between the portfolios to be undefined. For further reading about Fréchet spaces, the reader is recommended Fréchet [Fréchet 67].

For the solution to the allocation problem, the marginal distributions, $(F_{X_1},...,F_{X_n})$, play an important role. In order to develop the theory for allocation of capital according to an optimization, an alternative definition of the marginal distribution needs to be created. It is called the α-mixed inverse distribution function, which was introduced by Dhaene [Dhaene 02b, p.10]. For a random variable X the inverse distribution function F_X^{-1} is defined as

$$F_X^{-1}(p) = \inf_{x \in \mathbb{R}} \{F_X(x) \geq p\},\ p \in [0,1]$$

with $\inf \emptyset = +\infty$ by convention. The following definition is also introduced:

$$F_X^{-1+}(p) = \sup_{x \in \mathbb{R}} \{F_X(x) \leq p\},\ p \in [0,1]$$

The two parts of the α-mixed inverse distribution function is defined and can be combined to the following distribution.

$$F_X^{-1(\alpha)}(p) = \alpha F_X^{-1}(p) + (1-\alpha) F_X^{-1+}(p),\ p \in [0,1], \alpha \in [0,1]$$

The α-mixed distribution will be utilized when solving the allocation. For the distribution the following hold; for any random variable X and for all g with $0 < F_X(g) < 1$, there exist an $\alpha_g \in [0,1]$ such that $F_X^{-1(\alpha_g)}(F_X(g)) = g$. Together with the α-mixed distribution the allocation method make use of a concept called risk residual for solving the allocation problem. The risk residual is defined as follows:

$$\max(X_i - C_{Di},0) = (X_i - C_{Di})_+ \tag{2.18}$$

The risk residual considers the difference between the risk, X_i, of portfolio i and the capital exogenously distributed, C_{Di}. Whenever there is positive risk in the portfolio, the difference is gathered in a vector called the risk residual. The residual ensures that all risk of the portfolios will be covered by the allocation of economic capital, which can be summarized by the following inequality:

$$(\sum_{i=1}^{n} X_i - C_D)_+ \leq \sum_{i=1}^{n} (X_i - C_{Di})_+ \text{ a.s.} \tag{2.19}$$

Where a.s. again is the acronym for almost surely, in this case the exceptions to this can e.g. be when imaginary numbers are concerned. Equation 2.19 holds under the condition of full allocation, which is stated in equation 2.20.

$$C_D = \sum_{i=1}^{n} C_{Di} \tag{2.20}$$

If not otherwise stated, full allocation is assumed to hold for the rest of the section. Full allocation ensures that all the capital distributed to the portfolios is the same as the capital exogenously given, C_D. If $(\sum_{i=1}^{n} C_{Di} \leq C_D)$ holds, the allocation is also super-additive and equation 2.21 holds.

$$\rho(\sum_{i=1}^{n} X_i - C_D)_+ \leq \rho(\sum_{i=1}^{n}(X_i - C_{Di})_+) \tag{2.21}$$

If the allocation is super-additive, the risk measure $\rho(\cdot)$ can be defined as a first-order stochastic dominance preserving risk measure. This is a desired characteristic, because it ensures that the allocation method will function in a situation where the dependence structure of the portfolios are unknown. The risk measure, $\rho(\cdot)$, is thereby additive over all types of dependence structures. A risk measure that fulfil these conditions are $\rho(\cdot) = \mathbb{E}(\cdot)$, thereby equation 2.22 holds.

$$\mathbb{E}[(X_1 +\ldots +X_n - C_D)_+] \leq (\sum_{i=1}^{n} \mathbb{E}[(X_i - C_{Di})_+]) \tag{2.22}$$

The allocation method suggest that the optimal allocation will be reached by minimizing the exposure that the risk residual has. The capital allocation $c_i = (c_1,\ldots,c_n)$ is optimal if it minimizes the risk measure applied to the sum of the risk residuals. Where $c_i = (c_1,\ldots,c_n)$ represents the allocation to the portfolios of the bank performed after the initial distribution, c_{Di}, have been made. The relationship between the exogenously given allocation and the optimal allocation is given by equation:

$$\rho((X_1 - c_1)_+ +\ldots +(X_n - c_n)_+) \leq \rho((X_1 - c_{D1})_+ +\ldots +(X_n - c_{Dn})_+) \tag{2.23}$$

Equation 2.23 state that the sum of the risk residuals for the optimal allocation, c_i is lower than the sum of the risk residuals of the exogenously given allocation of capital, c_{Di}. A lesser sum of the residuals indicate that the capital allocated towards the risks are better adjusted to correspond to the risks X_i, it also indicates that the capital $C_E \geq C_D$. The minimization problem of the risk measure is described in equation 2.24.

$$\min_{\Pi(\cdot)} \rho(\sum_{i=1}^{n}(X_i - \Pi(X_i))_+), \text{ subject to } \sum_{i=1}^{n} \Pi(X_i) = C_D \tag{2.24}$$

In order for the risk measure to be sensible, it is desired that the risk of the portfolios are covered. Therefore the risk measure should be maximized over all possible dependency structures, which can be summarized as:

$$\max_{(X_1,\ldots,X_n)\in\Gamma} \rho((X_1 +\ldots , + X_n - C_D)_+) \tag{2.25}$$

Γ denotes the set of random vectors with corresponding marginal distributions i.e. the Fréchet space. In summary, by maximizing the amount of capital allocated over all dependence structures, a full allocation of the economic capital is ensured. At the same time minimizing the risk measure, thus the exposure of the risk residual, results in an allocation that ensures that the risk exposure of the portfolios are taken into account. By choosing $\rho(\cdot) = \mathbb{E}[\cdot]$, equations 2.24 and 2.25 reach the same result.

$$\sum_{i=1}^{n} \mathbb{E}[(X_i - F_{X_i}^{-1(\alpha_{c_D})}(F_{X_1^c+...+X_n^c})(C_D)))_+] = \mathbb{E}[(X_1^c + ... + X_1^c - C_D)_+]$$

in which $(X_1^c+...+X_1^c)$ denotes the co-monotonic random vector in the Fréchet space. Henceforth, we will often denote $X^c = X_1^c + ... + X_1^c$, and corresponding $(F_{X^c}(C_D) = (F_{X_1^c+...+X_n^c})(C_D))$. We tacitly assume that the solution $(c_1,...,c_n)$ of optimal allocations is internal in the domain G defined in equation 2.17.
[Laeven 03, p.9 Theorem 2.1]

By having $\rho(\cdot) = \mathbb{E}(\cdot)$, the optimal allocation of economic capital to portfolio X_i is determined by $c_i = F_{X_i}^{-1(\alpha_i)}(F_{X^c}(C_D))$, for the vector $(\alpha_1,...,\alpha_n)$ in the set G. Where G is defined in equation 2.26.

$$G = \{(\alpha_1,...,\alpha_n) : \sum_{i=1}^{n} F_{X_i}^{-1(\alpha_i)}(q) = C_D\} \tag{2.26}$$

In equation 2.26, q is a constant $q \in (0,1)$, so that $F_{X_i}(c_i) = q$ holds, which is the same as $c_i = F_{X_i}^{-1(\alpha_i)}(q)$ holds. If $F_X^{-1(\alpha)}(q) = F_X^{-1}(q)$ for all $q \in (0,1)$ and all $\alpha \in [0,1]$ the marginal distributions are strictly increasing and continuous. This ensures that the solution $(c_1,...,c_n)$ is unique, which indicates that the allocation is the optimal allocation.

2.5 Capital allocation used as a performance measure

The following section will explore how banks work with performance measurements and in the method section it will be determined how suitable the three capital allocation methods would be. There are different approaches to performance measurement, normally they are grouped into something called Risk-Adjusted-Return-On-Capital, RAROC. Mark and Bishop [Mark 07] who present an overview of the performance measure, its components and its many uses. James [James 96] describe the RAROC system and how the performance measurement is implemented at Bank of America. Stoughton and Zechner [Stoughton 06] show that optimal capital budgeting and performance can be derived from RAROC and an alternative method, Economic Value Added (EVA). Loebnitz and Roorda [Loebnitz 11] define a framework to make economic capital and RAROC sensitive to illiquidity; a liquidity adjusted risk measure is proposed and the paper show how this can help to determine if combning positions are beneficial. Le Leslé and Avramova [Leslé 12] address the concerns with the many alternatives that exist for calculating RWA, and how this can undermine the Basel III framework. Tasche [Tasche 99] develops a capital allocation method based on the financial approach and determines the optimal vector of values used for performance measurements. Perold [Perold 01] develops and allocation and discuss the use and misuse of RAROC as tool for performance measurement, capital budgeting and manager compensation.

2.5.1 General RAROC

RAROC was developed in the 1980's by Banker's Trust, an alternative performance measurement method that matured from the existing Return-On-Capital (ROC). ROC is essentially the ratio between expected return and capital, problems arise since there is no account taken to the risk. Two alternative investments with different risk exposure cannot be evaluated in a fair way. The difference and improvement RAROC contributes with, compared to ROC, is that all assets are weighted according to risk exposure. Thereby investment opportunities with different risk exposure can be evaluated against each other and one can determine which alternative that gives the highest return based on initial risk. In general, the saying "the higher the risk, the higher the return", makes sense according to RAROC. In the same way as investments alternatives can be ranked, RAROC can be used to determine the performance of portfolios of a bank.

When comparing portfolios of the bank, riskier portfolios must be estimated different from less risky portfolios. The RAROC-method, allow for such risk-adjusted comparison. By discounting portfolio returns with corresponding portfolio risk, the performance of the portfolios can be compared. By comparing the portfolio performance, management will have an overview of how profitable portfolios are. Thereby management of the bank can adjust capital requirements of returns for certain portfolios, which gives an incentive to give a fair performance based compensation to portfolio managers.

The general formula of RAROC is defined as the ratio between expected return and economic capital distributed to the portfolio. The expected return is (Revenue - Expenses - Expected Loss + Income from Capital), which is denoted $\mathbb{E}[X]$ for the bank. Theory state that economic capital should be defined as a function of credit risk, market risk and operational risk. Since this thesis only focus on credit risk, the economic capital will have the same definition as before, and only depend on credit risk.

$$\text{RAROC } = \frac{\mathbb{E}[X]}{C_E}$$

The formula stated above give a RAROC for the whole bank. By dividing the parameters into expected portfolio returns and also economic capital allocated to portfolios, RAROC for the portfolios are determined.

3

Method

In order to determine the difference between the three allocation methods proposed, a simulation evaluated the respective methods. Special focus was put to determine the possibility for the methods to allocate capital below portfolio level, and still be stable, transparent and intuitive. This is of interest, because the allocation of economic capital to each specific customer simultaneously determine the risk exposure of the customer. The bank can thereby adapt their pricing strategy to each customers individual risk exposure. In order to determine differences among the three methods, the simulations were made on two different scenarios. Scenario 1, called the Equal bank, aimed to display differences between the allocation methods, in amount of allocated economic capital. Scenario 2, called the Unbalanced bank, aimed to determine how stable the allocations were by testing the allocation methods on a riskier scenario, with less low- and medium-risk assets in the bank compared to the Equal bank scenario.

3.1 General structure for the simulation

The simulation have been set up and conducted in the software 'R'. It primarily considers a bank which have a large set of portfolios of customer loans and a small part of other business; a retail bank. The simulated bank have, for simplicity reasons, been generalized to only consider customer loans and no other types of business. The bank has a simple structure divided into three levels; company, portfolio and asset level, Figure 3.1 on page 34 demonstrate the structure of the bank. The top portfolio consist of the whole bank, the company level, which is split into three large portfolios; A, B and C. Each portfolio holds a number of assets with similar risk. Portfolio A, consist of low-risk loans, Portfolio B with medium-risk loans and portfolio C of high-risk loans. The simulation consider two different scenarios; the Equal bank, described in table 3.1 on page 34 and the Unbalanced bank described in the tables 3.2 on page 34. The equal bank scenario, aim to determine the methods relative differences in allocation and considered the point

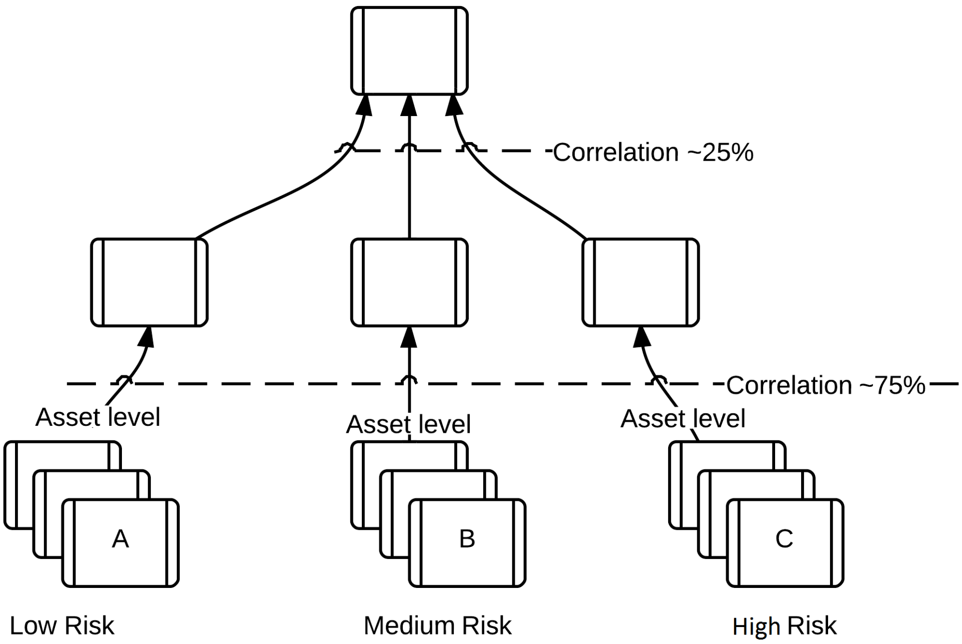

Figure 3.1: Structure of the fictional bank in the simulation

Scenario 1	The Equal bank	
Portfolio A	Portfolio B	Portfolio C
100 low-risk assets	100 medium-risk assets	100 high-risk assets

Table 3.1: The Equal Bank - Scenario 1

of reference. The Equal bank have the same number of assets in all three portfolios, which is considered the 'normal' setting for the simulation. Evaluation of the allocation methods in this scenario determine how the proportions of allocated economic capital differ among them. The scenario also examine the ability of the methods to allocate capital below portfolio level. The Unbalanced bank scenario, aim to establish how the methods handle the capital allocation in a different setting, were the bank is exposed to more risk. The comparison between the allocations of the two scenarios give an indication of how stable the allocation methods are. The goal for the simulation is to determine the overall performance of the allocation methods and especially determine how well the three methods satisfy the desired properties; stability, transparency and

Scenario 2	The Unbalanced bank	
Portfolio A	Portfolio B	Portfolio C
20 low-risk assets	50 medium-risk assets	100 high-risk assets

Table 3.2: The Unbalanced Bank - Scenario 2

intuitivity.

3.1.1 General setting for the simulation

Because the three methods use different input parameters for the allocation, it is desired to have a common setting that can be applied on all three methods. Thereby the results from the allocation can be easily compared and the differences between the methods are distinct. Before the allocation methods are introduced the economic capital of the bank is determined by the use of a risk measure. The proposed measure is ES, because it is coherent and sufficiently fulfils the proposed conditions desired by a risk measure, on page 16 in Table 2.1 a summary of properties held by ES can be found. The following section will explain this setting.

3.1.1.1 Assets

Portfolio A,B and C are formed by a number of assets; on page 34 Table 3.1 and 3.2 define the number of assets of each portfolio for both scenarios. The required input parameters for the risk measure and the different allocation methods differ a bit. By modelling the loss of an asset from a distribution, each asset have a loss function and a distribution of losses; satisfying the characteristics needed by all methods tested. It has been shown by many authors, that financial data does not follow a normal distribution, but rather an extreme distribution e.g. a Pareto distribution [Haas 11, p.7]. The Pareto distribution is a family of distributions with different types of characteristics and parameters.

These types of distributions have fatter tails, thereby the probability for an extreme event is bigger than for other distributions e.g. the normal distribution. The relevant data for this simulation is the extreme events i.e. the data in the tails of the distribution and thus the family of Pareto distributions seem like a wise choice, for a deeper discussion about the family of Pareto distributions [Tajvidi 06] is recommended. The distribution chosen to characterise the asset is a General Pareto Distribution (GPD) type II. This GPD have three parameters and by altering these significantly different characteristics can be achieved. Because there are three different parameters there are many possibilities to set up the distribution so that each asset is given an individual loss distribution, which obviously is desired. In order to determine the optimal number of simulations a convergence test can be made; the test determine at what number of simulations the result of the calculation of C_E does not improve with more simulations. The paper by Gustafsson [Gustafsson 11] address this problem and determine a rule of thumb for the limit of sufficiently many simulations. By using sufficiently many calculation, the confidence level α can be chosen freely and thereby the significance level can be lowered to 1% or 0.1%. Resulting in increased accuracy the of the C_E calculations. The convergence test is outside the the scope of the paper and calculations are limited to 1000, which limit the choice of significance level $1 - \alpha$.

Because the main purpose with the simulations is to compare the differences between the capital allocation methods and not determine an real-life allocation, the number of simulations seem sufficiently large. These losses are centralised, a concept explained in the risk measure section, which gives each loss an expected value of 0. Table 3.3 describe how the three parameters of the distribution affect the losses.

Parameter	Sign	Effect
Location	$\mu \in \mathbb{R}$	The parameter determines the smallest value the distribution can have.
Scale	$\sigma \in \mathbb{R}^+$	A scalar that enhances the value from the distribution
Shape	$\xi \in \mathbb{R}$	Determines the shape of the distribution, a larger value gives a larger tale of the distribution and thereby larger losses in the distribution

Table 3.3: Characteristics of the parameters of the GPD type II

Equation 3.1 below describes the equation for simulating GPD values. Where U is a uniformly distributed parameter between [0,1].

$$X = \mu + \frac{\sigma(U^{-\xi} - 1)}{\xi} \sim GPD(\mu, \sigma, \xi \neq 0) \tag{3.1}$$

The parameters of the distributions for the assets are shown in table 3.4.

Parameter	Portfolio A	Portfolio B	Portfolio C
Location	0	0	0
Scale	0.75	1	1.25
Shape	[0.75/2, 0.75]	[0.5, 1]	[1.25/2, 1.25]

Table 3.4: Parameters of the GPD type II, used to simulate the assets distributions

The location parameter determines the smallest loss that the distribution can generate. Because the allocation methods only are interested in losses, it is for convenient reasons set to 0 and thus not generating any negative losses. The scale parameter is a scalar that resize the GPD-value; a higher value indicates a higher scaled loss. For determining the scale parameter the following approximation is used $\sigma = 0.5 + 0.25 * r$ were r is a risk factor, determined by the risk of the portfolio. It is set to 1,2 or 3, for the portfolio A, B and C respectively. The shape parameter determines the randomness and also the volatility of the losses. The parameter is given by an interval, which is extracted from the scale, by the following formula [σ/2,σ]. A higher interval indicates a higher probability for a high loss.

3.1.1.2 Model points

One of the goals with the simulation is to determine how the capital allocation methods differ in determining the risk exposure of a single customer. The exposure of a customer can be derived as the economic capital allocated to the corresponding customer. In order to determine what a customer of the bank is in the simulation, a concept called model points will be used. Instead of deriving a loss distribution and loss function for each customer, persons with similar characteristics and thus risk profile are grouped together into a model point. With a large enough set of different model points the risk profiles of the customers grouped together are not that different. Thereby the risk of the customers are more or less similar and the loss of accuracy of the model is minimal. The optimal level of model points can be determined by a similar convergence test described in the Assets section. In the simulation each asset of the portfolios is thought of as a model point. The computational benefit of model points is quickly comprehended with the following example:

> Lets say, a bank has 1 million customers and each customers risk is determined by a distribution of a 1000 simulation, that is 1 billion calculations to determine the loss distributions. By increasing the number of simulations the accuracy of the model increase, but the amount of data quickly becomes enormous. What if the bank is slightly bigger and has 10 million customers and simulate each risk profile 10000 times?

The trade-off between accuracy of the method and calculations is hard to determine and depends on what the accuracy is used for. For this simulation the number of simulation of an asset is set to 1000, which limits the choice of significance level $1-\alpha$. For the scope of the simulation, which is to compare how the methods allocated capital differently and not how much they actually allocate, the accuracy of the method is accepted and the calculations is not too time-consuming.

3.1.1.3 Aggregation with consideration of correlation

In order to determine the economic capital of the bank the contents of the portfolios needs to be aggregated to the top level, where a risk measure can be used. When risky assets are grouped together the overall risk of the aggregated portfolio decrease, called diversification effect. How diversification affect the losses are determined by a correlation matrix. There are a couple of different methods available for determining the dependence between uncorrelated assets. A popular and fast method is the Cholesky decomposition, which decomposes a real valued positive definite matrix, C_U, into a lower triangular matrix L and L^* the conjugate transpose of L.

$$C_U = L \times L^*$$

The level of correlation is assumed to change on different levels of the simulation. Because all assets in a portfolio have the same level of risk the correlation between them is assumed to be higher than between the portfolios. On the asset level of the bank C_U^i, for portfolio i, is determined by a matrix of ones on the diagonal and normally distributed numbers, between $[-1,1]$ with mean 0.75 in the lower triangle. The lower triangle is mirrored so that the matrix becomes symmetric, figure 3.2 on page 38 give an indication of what the matrix C_U would look like. On the portfolio level a new correlation matrix needs to be

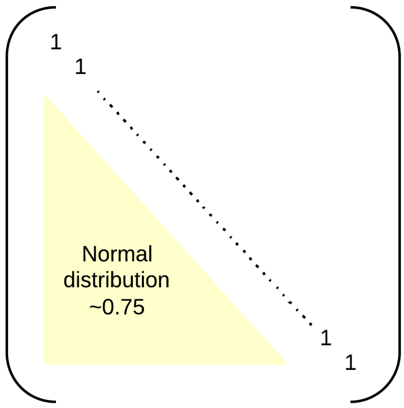

Figure 3.2: A model of the matrix C_U

formed since the correlation is assumed to be lower. Therefore the numbers in the lower triangle of the new matrix C_U^P, for the portfolio level of the bank, instead is assumed to have a mean around 0.25. Because the 1000 simulations of each asset is derived from the same stochastic variable and correlation occur between two different stochastic variables, the whole set of assets and simulations are kept for calculation of correlation dependence. This matrix is called A^i with size $j^i \times 1000$, for portfolio i. The matrix of correlated values is derived from the multiplication $L \times A$ which results in a matrix with the size $j^i \times 1000$, called C_C^i. The same calculation is performed for all portfolios A, B and C, creating matrices C_C^A, C_C^B and C_C^C. In order to aggregate the losses to company level, the three matrices are each rearranged into a vector of $j^i * 1000 \times 1$ values. The vectors from each portfolio are positioned into one matrix, which will have the size $j^i * 1000 \times 3$. For different lengths of the j^i-vector, the longest is used and shorter vectors are filled with zeros so all vectors are of equal length, j. The new matrix is then run through the portfolio level correlation matrix, C_C^P, which result in a matrix consisting of values of losses from all assets of the bank with the correlation taken into account.

3.1.1.4 Calculation of Economic Capital

The proposed method for calculating the economic capital is ES, which is the average loss in the $1-\alpha$ quantile of worst losses. For calculation of the economic capital the correlated losses of the bank are sorted into a vector, according to size. For this simulation $\alpha = 95\%$ have been chosen, which results in a quantile that take into account the $1-\alpha = 5\%$ worst losses. $1/0.05 = 20$ means that the 5% significance level determine losses that occur every

20 years. As discussed in the Asset section previously, the number of simulations limits the accuracy of α. In a real-life scenario one might consider a higher confidence level for the quantile, such as 99% or 99.9%, a higher accuracy of the method would require a larger set of data. $ES_{0.95}$ is calculated by taking the average of the 5% worst losses of the vector with $3 * j * 1000$ correlated losses. $ES_{0.95}$ represents the amount of economic capital that the bank wants to allocate towards the unexpected losses of the assets. The $ES_{0.95}$ is also calculated for each portfolio and for each asset in all three portfolios.

3.2 Specific set up for the allocations methods

The specific methods and parameters used for calculating the capital allocation will be given below.

3.2.1 Game theory

For the game theory approach, it is assumed that the whole bank join into one coalition. This coalition is called D and contain all assets of all portfolios of the bank. The parameters needed are thus the τ vector which represents each assets part of participation and a coherent and differentiable risk measure, $\widehat{\rho}$. The rate of participation, can either be approximated by a Debreu-Scarf coalition, as suggested in the theory, or defined as the relative sizes of the portfolios, in some reference unit. The later is chosen for the simulation and the reference unit is defined as economic capital. The τ-vector for the assets is thus determined by the ratio of the losses for that asset and the total losses of the bank.

The allocation to assets and portfolios are determined by the gradient of the risk measure evaluated at the corresponding value of the τ-vector. The gradient of the risk measure is considered to be the same as the marginal of ES, which can be abbreviated to MES. All the risks X_i of a sub-portfolio that is larger than the $q_{0.95}(X)$ of portfolio X, is averaged and called MES of sub-portfolio X_i. The MES gives the vector q, which determines the % allocation to each portfolio. When C_E for the bank have been divided into C_i for the portfolios, the assets are allocated the ratio of MES for the asset divided by total MES for the portfolio $\frac{MES_{asset}}{MES_{portfolio}}$. The MES for the assets are given as a % and therefore multiplied with C_i of the portfolio to determine the allocated capital.

3.2.2 Finance

The financial approach to portfolio X_i is conditional on the parent portfolio, X having a loss and determines the allocation thereby. By first calculating economic capital for the bank and then for the portfolios, the initial setting is given. $ES_{0.95}$ of the bank determine the level of risk that the underlying portfolio is conditional on. Given the level of risk, the quantile of the 95% confidence level of the portfolio is determined, and calculated as the average of losses over that level. If there is no value larger than the $ES_{0.95}$ of the parent portfolio, X, the largest value of the portfolio, X_i is chosen as $ES_{0.95}$ for X_i.

MES is calculated the same way as for the game theory approach, by dividing each asset MES with the portfolio MES.

3.2.3 Optimization

The optimization method takes a different approach and begins at the lowest level, asset level. It calculates for one portfolio at the time the optimal allocation. In order to calculate the optimal allocation for the assets the α-vector of the α-inverse distribution needs to be evaluated. This is made according to the formula

$$F_X^{-1(\alpha)}(p) = \alpha F_X^{-1}(p) + (1-\alpha)F_X^{-1+}(p),\ p \in [0,1], \alpha \in [0,1]$$

where

$$F_X^{-1}(p) = \inf_{x\in\mathbb{R}}\{F_X(x) \geq p\},\ p \in [0,1],$$

and

$$F_X^{-1+}(p) = \sup_{x\in\mathbb{R}}\{F_X(x) \leq p\}\ p \in [0,1]$$

Since the 'R' software only handle finite series of data, the values of inf and sup needs to be estimated. A vector [0,20000], which consist of 200000 values is set to represent all real variables. The limit 20000 is set since the probability of a loss being higher than that is minimal. Thus it represents an upper limit of real numbers, sensible for this simulation. This creates a one dimensional optimization problem of $F_X^{-1(\alpha)}(p)$ to find the optimal α-value for all assets. Given $F_X^{-1(\alpha)}(p)$ and α the relationship of $F_X^{-1(\alpha_g)}(F_X(g)) = g$ can be optimized. The software makes use of a standardized solution for the one dimensional optimization called the Brent-method. The parameters from every asset's original pareto distribution are saved and used to form individual probability distributions. By using a moment matching estimation (MME), the losses of the top portfolio are fitted to a normal distribution, resulting in $F_{X^c}(\dot{})$. Given all the assets, with corresponding α-vector, the optimal allocation is determined by the equation $c_i = F_{X_i}^{-1(\alpha_i)}(F_{X^c}(C_D))$.

3.3 Performance measurement calculations

A general bank would determine their expected profit of a loan as a percentage return on the invested loan. The estimated returns for the bank for portfolios A, B and C are assumed to be $3\%, 6\%$ and 9%, thus assumed that the bank does not have any further evaluation of return than on portfolio level. The RAROC is the ratio of estimated return of the asset and economic capital allocated to that asset. This is the general conclusion, here stated by Tasche.

> Let $\emptyset \neq U \subset \mathbb{R}^n$ be an open set and $\rho : U \to \mathbb{R}$ be a function that is partially differentiable in U with continuous derivatives. Let $a = (a_1,...,a_n)$:

$U \to \mathbb{R}^n$ be a continuous vector field. Then a is suitable for performance measurement with ρ if and only if

$$a_i(X) = \frac{\partial \rho}{\partial X_i}(X), i = 1,\ldots,n, X \in U$$

[Tasche 99, p.11 Theorem 4.4]

The vector, a, is exactly the vector used to determine economic capital and thus the relation of estimated return and economic capital seems reasonable.

4

Results

THE FOLLOWING CHAPTER summarize the results from the simulations in 'R' [R Core Team 13]. The capital allocation methods use three different concepts to approach the calculation of optimal allocation. Interestingly, the game theory and finance method come to exactly the same conclusion, the best allocation is achieved by differentiation of the risk measure. The optimization method does not take the risk measure in account and instead develop an allocation that depend on the marginal distribution of the asset. The results first present the Equal Bank scenario followed by the Unbalanced Bank scenario. Results are displayed by bar-plotted graphs of the allocations on portfolio level followed by diagrams over the allocation to each asset of the portfolios. The optimization method allocates more capital to the low and medium risk portfolios compared to the two other methods.

4.1 General results

The results from the two scenarios are displayed below

4.1.1 The Equal bank - Scenario 1

The Figure 4.1 show how the three capital allocation methods allocated economic capital, to portfolios A, B and C. Where economic capital is determined by $ES_{0.95}$ and is calculated on asset level, which means that the diversification effects between the assets have been taken into account but not between the portfolios. By looking at Figure 4.1, one quickly realize that the allocated capital from the game theory and finance method allocate the same amount of capital to the three portfolios. This is no surprise since the two methods use the same rule to allocate by. It is interesting that two such different concepts manage to come to the same conclusion about optimal allocation. The optimization method allocates capital more evenly distributed between the portfolios. The optimization method has a higher allocation to portfolio A and B, which represents the low- and medium-risk portfolios. To the low-risk portfolio the method even allocates more economic capital than the portfolio risk, determined by ES is. The next set of figures show how the allocation methods allocate capital to the respective asset. Figure 4.2, show the allocation to Portfolio A, Figure 4.3, show the allocation to Portfolio B and Figure 4.4, show the allocation to Portfolio C.

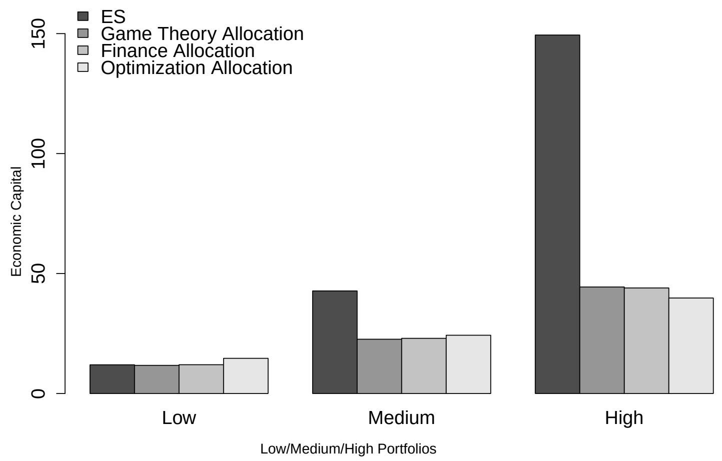

Figure 4.1: The Equal Bank - Optimal capital allocation on portfolio level

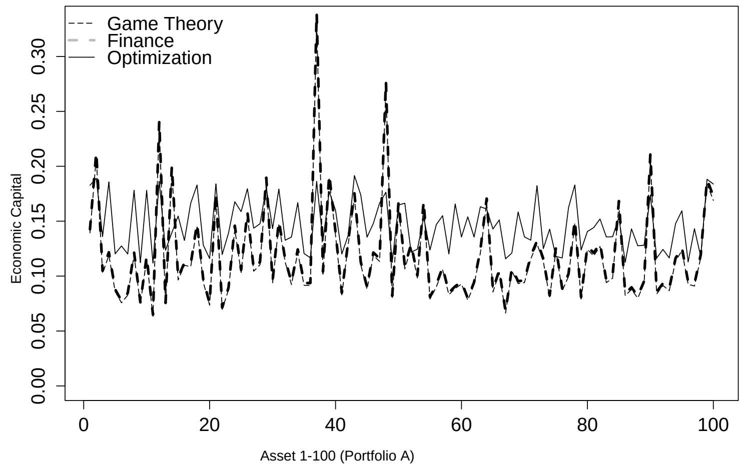

Figure 4.2: The Equal Bank - Optimal allocation to portfolio A

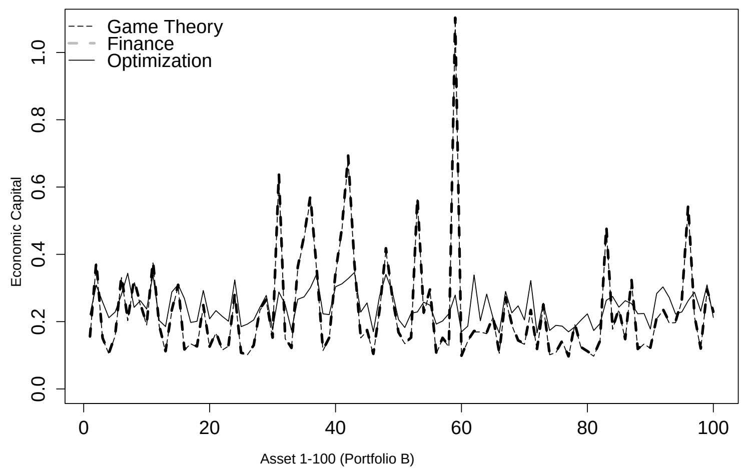

Figure 4.3: The Equal Bank - Optimal allocation to portfolio B

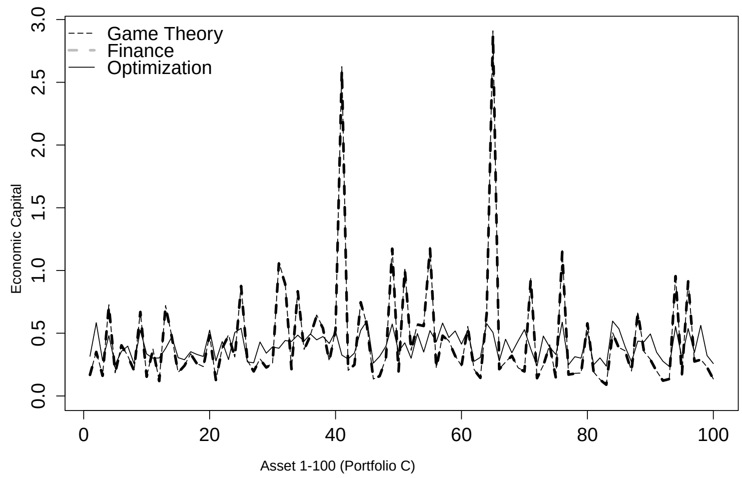

Figure 4.4: The Equal Bank - Optimal allocation to portfolio C

It is again apparent that the game theory and finance method allocate economic capital exactly the same, also on asset level. But generally, all three methods value the risks of the assets in a similar pattern. When looking at the asset allocation curves the methods roughly follow the same pattern for allocation. There are some great differences, but some similarities too. Both the game theory and finance method allocate significantly more capital to some assets, compared to the other assets. The optimization method does not value these 'peak' assets in the same way, instead it value all assets in a portfolio on a more even level. On the other hand the optimization method seems to allocate a generally higher amount of capital to all assets compared to the game theory and finance method. Because the game theory and finance method allocate much capital to the few 'peak' assets, especially apparent for portfolio C, they have a higher level of allocated capital to that portfolio, compared to the optimization method. These results will be evaluated against the second simulation, the Unbalanced bank.

4.1.2 The Unbalanced bank - Scenario 2

This scenario have a similar approach to the simulation and Figure 4.5 begin by showing how the three different capital allocation methods allocate economic capital, to the portfolios A, B and C in a scenario prone to high risk. As for the previous scenario, the economic capital is determined by $ES_{0.95}$ and are calculated on asset level, which means that the correlation between the assets have been taken into account but not on the portfolio level.

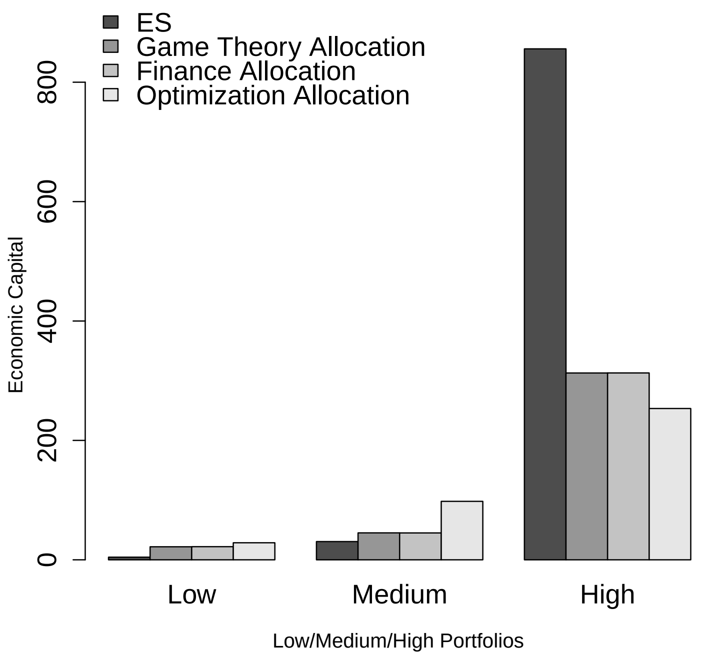

Figure 4.5: The Unbalanced Bank - Optimal capital allocation on portfolio level

Figure 4.5 show results that are consistent with the Equal Bank scenario; the optimization method allocate more capital to the low- and medium-risk portfolios compared to the other methods. The next set of figures show how the allocation methods allocate capital to the respective asset of the Unbalanced Bank scenario. Figure 4.6, show the allocation to Portfolio A, Figure 4.7, show the allocation to Portfolio B and Figure 4.8, show the allocation to Portfolio C.

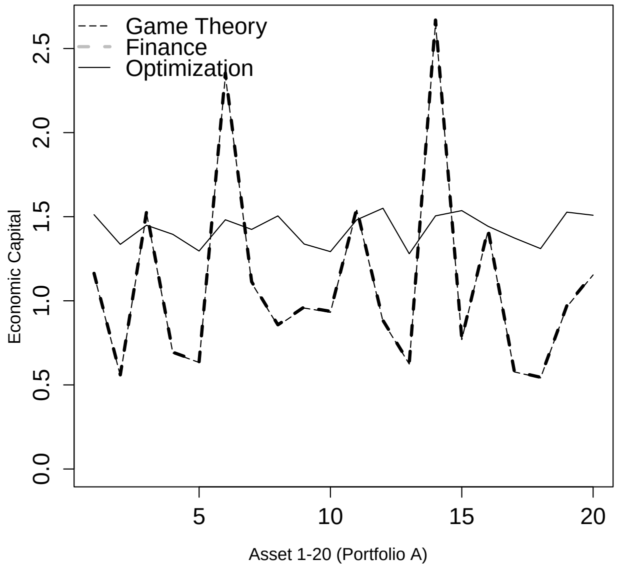

Figure 4.6: The Unbalanced Bank - Optimal allocation to portfolio A

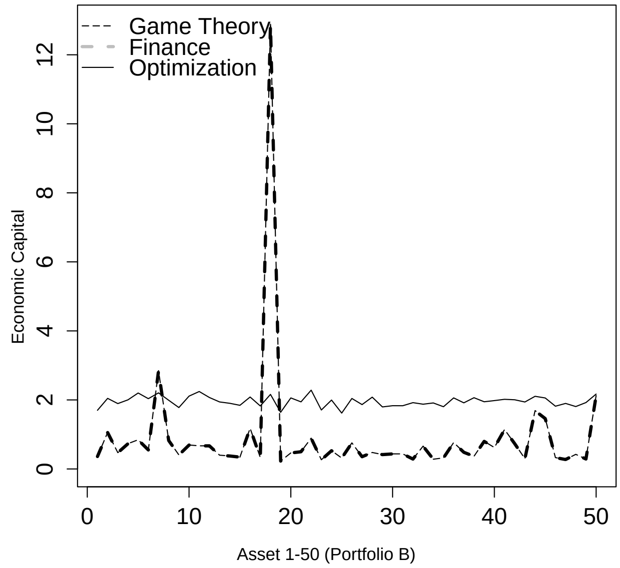

Figure 4.7: The Unbalanced Bank - Optimal allocation to portfolio B

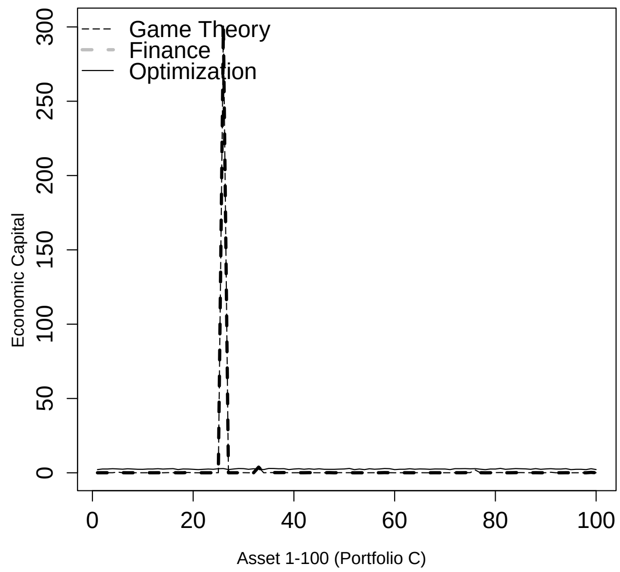

Figure 4.8: The Unbalanced Bank - Optimal allocation to portfolio C

Figure 4.6, 4.7 and 4.8 show consistent results with the Equal Bank scenario; the optimization method generally allocate more capital to the assets, but the game theory and finance method allocate significantly more capital to certain 'peak' assets.

4.2 Comments of results

There are a few important remarks to be done about the results.
First, all the data is fictional and simulated from an extreme value distribution. This implicates that the asset distributions might not follow a realistic pattern that can be found in empirical data. This is mostly because the simulation of assets vary significantly from one run to the next e.g. the highest allocation to Portfolio C in the two simulation are for the Equal bank around 3 and for the Unbalanced bank around 300. This could be the explanation for the 'peak' allocations found by the game theory and finance methods in all simulations. In the low- and medium-risk portfolios these peak assets could represent loans that have been wrongly estimated and would belong in the high-risk portfolio. In the high-risk portfolio the peaks could represent loans to customers with low credit rating or speculative loans with no security. This problem would probably not be solved with an increased amount of simulations and thereby accuracy of the ES_α estimation. The solutions to receiving more consistent distributions lie in the set up of the distribution parameters. For this simulation the varying values of the distributions is of little significance, since the goal is to compare the characteristics of the methods and not calculate values for allocations.

Secondly, the optimization method does not value these peak assets as risky. This could be explained by the set up of the method, the optimization method is extremely sensitive to the optimal α-vector; in equation $c_i = F_{X_i}^{-1(\alpha_i)}(F_{X^c}(C_D))$. When the vector have been determined, the rest of the allocation is a linear transformation with no randomness. Because the software 'R' only handle finite series of numbers, and not sup and inf which are not finite, the space of real values need to be limited, which could be a source of error in the method. The choice of accuracy depends on how the sup and inf are limited, with a less restricted limitation than the one used the accuracy of the α-vector would increase. With the implication that the accuracy of the model also would increase. With this set up the optimization method is by far the slowest to compute, and further accuracy of the α is a trade-off.

Thirdly, the allocation by the optimization method does not have an upper restriction of possible amount of C_E to allocate. Without scaling the results from the method with C_E the method would allocate more capital than the other methods.
In summary, the simulation provide relevant results for comparing differences and similarities between the allocation methods since it builds on an equal setting for all three methods. Because it is based on fictional data the actual values derived from the simulation, should not be given much attention.

4.3 Results from RAROC simulation

The simulation of the RAROC values have posed some difficulties. The set of data used so far have only concerned losses and loss distributions, thus there is no information about the sizes of the loans in the portfolios. The size of the loan would be a convenient input parameter for determining the sizes of the returns of the loans and thereby the portfolios. The expected % return for the portfolios have been estimated to $3\%, 6\%$ and 9% for the portfolios A, B and C. In order to solve problem, the RAROC of the portfolios are calculated as the ratio of estimated % return and the % of economic capital allocated to each portfolio. Table 4.1 show the results from the RAROC calculations of the Equal Bank scenario and Table 4.1 the same for the Unbalanced Bank scenario. As seen in

	Game Theory	Finance	Optimization
Portfolio A (Low Risk)	6.79%	6.74%	4.93%
Portfolio B (Medium Risk)	4.13%	4.16%	3.28%
Portfolio C (High Risk)	1.84%	1.84%	2.32%

Table 4.1: RAROC for the The Equal Bank Scenario

	Game Theory	Finance	Optimization
Portfolio A (Low Risk)	9.16%	9.16%	7.96%
Portfolio B (Medium Risk)	5.27%	5.20%	4.32%
Portfolio C (High Risk)	1.58%	1.58%	2.58%

Table 4.2: RAROC for the The Unbalanced Bank Scenario

the tables, all three methods work well for performance measures. Since the values are fictional and the returns simple estimations, the numbers are not the important, but the tables still can be used to distinguish characteristics of the methods. The methods show quite similar results but there are some differences. Because the optimization method allocates a larger % of economic capital to the low- and medium-risk portfolios, the RAROC for these portfolios are lower than the corresponding portfolios for the other methods. The RAROC of the riskier portfolio C is thus higher for the optimization method, compared to the RAROC of the other methods for portfolio C. This implies that the optimization method slightly favours risk taking to a greater extent than the other methods; since it gives a higher RAROC to the risky portfolio compared to the other methods. The converse conclusion is that the other methods are slightly more risk averse than the optimization method.

From a portfolio manager's perspective, the result indicate that the more capital the portfolio is allocated the higher return requirement is needed in order to get the same RAROC. This means that the manager of portfolio C needs to take a higher interest on the loans in order to get the same RAROC, and thus compensation, as the manager of portfolio A. For the simulation the expected return are selected freely and thus might not represent realistic values of a real retail bank, which could explain the variations of the RAROC values of portfolios A, B and C.

The RAROC calculation takes the risk of the loans into account, which gives a fair comparison of the three portfolios. Each portfolio is evaluated after the portfolios ability to gain a return. If a ROC calculation would be performed, no account to the risk of the portfolios would be taken into account. Each portfolio would then receive a return requirement based on the capital held by the portfolio, and not the economic capital allocated to the portfolio. A ROC measure would give an unfair requirement to the portfolios with lower risk exposure, if compared to each other. Instead of comparing the portfolios against each other the portfolios would have to be evaluated against some exogenously determined hurdle values. By reaching the hurdle value, or above, the performance of the portfolios would be determined.

With the RAROC measure the hurdle value is not needed, all portfolios can be compared equally against each other. The RAROC is thus more competitive and pose as a simpler to way to determine the performance of the portfolios.

5

Conclusion

This section summarize the results and conclusions made from the thesis into Table 5.1 on page 53. The problem questions are answered through a discussion and the chapter concludes with suggestions for future investigation.

The finance method is considered to be the most intuitive of the proposed capital allocation methods. It is simple and quick to compute and the concept of the model can be easily understood. The method is based on intuitive and accepted financial concepts such as expected value and standard deviation. The game theory approach also introduce an easy concept for allocation of capital. The analogy to a game of players and costs for participation is intuitive and easily translated to a bank with portfolios and an allocation problem. When looking closer at the theory applying the game situation on an allocation problem, the method is less intuitive and propose some complex concepts in order to reach the final computation. The optimization approach is the least intuitive method, the concept of minimizing the risk residual is quite easy to understand but when the computation starts, the method quickly looses simplicity and is demanding to follow.

In order to fulfil the stability condition, the method needs to be insensitive to an extra simulation and not too restrictive on the underlying assumptions. The game theory and finance methods depend on the differentiation of a coherent risk measure. Such method have been presented, ES, and therefore cannot be considered as an overly restrictive underlying condition. Because the allocation depends on the gradient of the risk measure and the exposure by the individual asset, the allocation does not change dramatically by more assets introduced into one or more portfolio of the bank. The optimization method reach a unique solution when $F_X^{-1(\alpha)}(p) = F_X^{-1}(p)$ otherwise the method might reach a local optimal allocation and not the global. The method also depends on where in the corporate tree the allocation is performed, a new α-vector needs to be calculated if e.g. the allocation level changes from portfolio to, for example, business unit level. When these conditions are fulfilled the method does not change much for an introduction of a new asset and could thus be considered stable.

The finance approach is regarded as the most straight-forward and transparent approach. The concept of calculating the marginal risk contribution of assets and portfolios are considered logical. The notion of determining the contributions from the differentiation of the same risk measure that determined the risk of the portfolio, is transparent. The game theory method can be ranked as second among the suggested methods, the analogy to a game makes the method transparent but assumptions following the theory is less revise. The theory leads to a simple transparent calculation, which is the same as the one for the finance approach. The optimization approach is the least transparent, the allocation method can easily be thought of as a black box method, which is hard to follow for anyone that is not part of building the model. Assumptions about the method are complex and not always easy to follow.

As shown by the simulations all of the proposed capital allocation methods are able to allocate capital on a more granular level than portfolio level. In the simulation the most granular level is a model point, which represents a group of customers with similar risk profile to their loan. Both the game theory and finance method can allocate to a single loss, since the two methods depend on the relative size of the risk exposures from the customers. The optimization method is limited by the marginal distribution function of a customer, if that function can be determined for a single customer then optimization is as granular as the two other methods. The granularity increase the accuracy of the method but with increased granularity the amount of calculations is increased. When it comes to the speed of the allocation methods the optimization method is the slowest to compute and the accuracy can be increased by a more accurate calculation of the α-vector. The accuracy of the other two methods is less obvious, it is determined by the accuracy of the C_E calculation. For these simulations the C_E calculation is limited by the number of simulations of each asset, a higher number of simulations increases the accuracy of the calculation. The trade-off between accuracy and speed of the allocation is determined by the application and the computational capacity.

All three methods are good as performance metrics and can be used to determine the RAROC of a portfolio. When comparing the results of the methods, the game theory and finance method give a higher RAROC to the low- and medium-risk portfolios compared to the optimization method, which gives the high-risk portfolios a higher RAROC.

	Stable	Intuitive	Transparent	Allocation below portfolio level	Suitability as Performance Measurement
Game Theory	Yes	Simple concept	The final computation is as transparent as the financial approach but the assumptions leading there are sometimes unclear	Yes, to loss level	Suitable
Finance	Yes	Most intuitive	Most transparent	Yes, to loss level	Suitable
Optimization	Yes	Simple concept but calculations quickly get difficult to understand	Black-box method	Yes, needs marginal distribution of loss thereby most granular allocation is to a model point	Suitable

Table 5.1: Results for the three allocation methods

5.1 Suggestions for future investigations

It would be interesting to test the allocation methods on real data instead of the fictional, in order to be able to compare the specific amount of capital allocated to the portfolios and not only their conceptual differences.
One could also determine the value of the α-vector with higher accuracy and test if the optimization method also finds the 'peak' assets in the simulations.

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