An Introduction to the Fundamental Group and some Applications

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Fundamentalgruppen

Antag att vi har en bit papper och en tråd. Om vi fixerar en av trådens två ändar någonstans på ytan av pappret så kan vilken slinga som helst på papprets yta bli dragen till den fixerade änden utan att passera någon punkt som inte tillhör pappret. Då säger vi att papprets "fundamentalgrupp" är trivial eller enkelt sammanhängande. Om pappret har ett hål i sig, t.ex. i sitt centrum, så kan slingan runt detta hål inte sjunka in i den fixa punkten utan att passera över hålet, men vi kan snurra tråden medsols eller motsols runt hålet hur många gånger vi vill (teoretiskt sett). Om vi t.ex. snurrar den två gånger medsols och tre gånger motsols så får vi enslingesvindar -1 gånger runt hålet. Minustecknet betyder att riktningen är motsols. Då säger vi att fundamentalgruppen för det enhåliga pappret är heltal, d.v.s. $\{..., -2, -1, 0, 1, 2, ...\}$. Genom att använda samma metod kan vi se att fundamentalgruppen av en fotboll och en enhålig fotboll är trivial eller enkelt sammanhängande, medan fundamentalgruppen för en tvåhålig fotboll är heltalen. Detta är ett approximativt tillvägagångsätt för att försöka att få en uppfattning om fundamentalgruppen. Betänk nu att du har en kopp kaffe (med ett handtag) och en munk, båda gjorda i modellera. Då kan en av dessa former omskapas till att likna den andra utan att man måste plocka isär den ursprungliga formen i smådelar. I denna masteruppsats bevisas det att kaffekoppens fundamentalgrupp är densamma som munkens. Det går däremot inte att omvandla formen av en mugg till en boll, så dessa formers fundamentalgrupper kan vara olika varandra, och det kommer i den här uppsatsen att visas att de verkligen är olika. Denna teori har blivit etablerad av den franske matematikern Henri Poincaré (1854-1912), och den är i hög grad tillämpad inom matemtiken och även inom andra områden, så som fysik, biologi och medicin. Som ett exempel på tillämpningar kan nämnas ham-sandwich-teoremet, vilket säger att: givet en bit bröd, skinka och ost (i vilken form som helst), som placeras hur som helst, så kommer det att existera en flat skiva skuren av en kniv som kommer att bisektera var och en av dem, brödet, skinkan och osten, simultant. Att ta reda på det mänskliga hjärtats fundamentalgrupp är användbart i vissa fall för att avgöra huruvida en patient behöver opereras eller ej. Vidare så spelar fundamentalgruppen en stor roll inom Knot-teorin, som är ganska viktig för att studera egenskaperna hos DNA. I denna uppsats så introducerar vi "fundamentalgruppen" matematiskt, och diskuterar också dess tillämpningar.

Abstract

This thesis provides a self-contained introduction to the Fundamental Group and presents some of its applications, such as Brouwer fixed point Theorem, the Game of Hex, Nielsen-Schreier Theorem, the van-Kampen Theorem and some of its consequences, Borsuk-Ulam Theorem and some of its applications, the Jordan curve Theorem, and some other examples and applications. Some results in Group Theory will be introduced as necessary, in order to study the van-Kampen Theorem, which plays a key role in our proof of the Jordan Curve Theorem.

Acknowledgments

I would like to thank my supervisor professor Arne Meurman for his advice, patient guidance, editorial support, constant encouragement and enthusiasm. I am grateful and proud of being his student in a half of my master program at Lund University.

I would also like to thank my father for supporting me throughout the years practically, financially, and with moral support. Whenever, wherever, and whatever the success I achieve, I must thank my mother.
To my family ... I dedicate this work.

Abdel Rahman
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Introduction

It is known that a homeomorphism between two topological spaces preserves all the topological properties, such as connectedness and compactness. Therefore, the 2-sphere $\mathbb{S}^2$ is not homeomorphic to the plane $\mathbb{R}^2$ because $\mathbb{S}^2$ is compact whilst $\mathbb{R}^2$ is not. But it is not always easy to disprove that a couple of spaces are homeomorphic using only basic notions in general topology. For instance, to decide whether the punctured plane $\mathbb{R}^2 \setminus \{0,0\}$ is homeomorphic to the plane $\mathbb{R}^2$. Or, for another instance, whether the torus is homeomorphic to the 2-sphere $\mathbb{S}^2$, we need to introduce some new invariants. In this thesis the invariant will be the Fundamental Group. The idea of the fundamental group has been defined by the extraordinary French mathematician Henri Poincaré (1854-1912). In addition, the Fundamental Group can be used as a great tool in mathematics to prove many facts.

In Chapter 1, we we introduce the definition of homotopy, homotopic maps, homotopy equivalent spaces, and some important consequences that allow us to find the Fundamental Group of the circle $\mathbb{S}^1$ which has many practical applications. In particular, the Brouwer fixed point Theorem which will also be used in the proof of the Hex Theorem. In Chapter 2, we introduce some necessary Group Theory. In particular, Free Abelian Groups, Free Groups, Group Presentations, and Free Products. These notions play a pivotal role in Chapter 3, where we present the van-Kampen Theorem which is one of the main theorems in Algebraic Topology. This theorem has many important applications and we will present some of them in Chapter 3. In addition, the van-Kampen Theorem will be used in the proof of the Jordan Curve Theorem, Chapter 5. In Chapter 4, we introduce basic notions about Covering Spaces and the Fundamental Group that will be used to prove the Borsuk-Ulam Theorem which is very important and practicable. Furthermore, we will apply the notion of Covering Spaces and the Fundamental Group to prove that a subgroup of a free group is itself free (Nielsen-Schreier Theorem) as an application in Group Theory.

The reader is assumed to be familiar with the basics of General Topology, such as continuity, homeomorphism, connectedness, compactness, quotient topology, one point compactification, etc. Familiarity with basic Abstract

Algebra is also assumed. In particular, definitions, such as group, subgroup, normal subgroup, the commutator, homomorphism, etc. and theorems, like the isomorphism theorems.

"As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection"

Lagrange 1736-1813

Chapter 1

Homotopy

In order to define the fundamental group we shall consider equivalence classes of maps (paths) between topological spaces, these classes are called homotopy classes. The concept of homotopic maps plays pivotal role in our construction of the fundamental group. Results in this chapter are based on [2], [7], [9], and [14].

1.1 Homotopic Maps

Definition 1.1.1 *Let X and Y be topological spaces. Two maps $f, g : X \to Y$ are said to be homotopic if there exists a map $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$, $\forall x \in X$. We call H a homotopy from f to g and denoted by $f \underset{H}{\approx} g$.*
Furthermore, if $f = g$ in $A \subseteq X$ then if there is a homotopy H between f and g s.t $H(x,t) = f(x) = g(x), \forall x \in A$ we say f and g are homotopic relative to A and write $f \underset{H}{\approx} g$ rel A.

Now denote $[0,1]$ by I, we mean by a path in X a continuous map $\alpha : I \to X$. Let $\alpha_0, \alpha_1 : I \to X$ be two paths in X from a to b (meaning that $\alpha(0) = \beta(0) = a$ and $\alpha(1) = \beta(1) = b$). We say α_0, α_1 are homotopic with end points fixed if there exists a homotopy $H : I \times I \to X$ s.t

$$\begin{aligned} H(s,0) &= \alpha_0, \\ H(s,1) &= \alpha_1, \\ H(0,t) &= a \text{ and}, \\ H(1,t) &= b. \end{aligned}$$

This situation is written as $\alpha_0 \underset{H}{\approx} \alpha_1$ rel $\{0,1\}$.
It is sometimes more convenient if we write $H(s,t) = \alpha_t(s)$ for $0 \leq s,t \leq 1$.

Theorem 1.1.2 *If X is a convex subset of $\mathbb{R}^n$, and if $a, b \in X$ then all paths from a to b in X are homotopic.*

Proof: Let α_0 and α_1 be two paths in X from a to b. Define $\alpha_t : I \times I \to X$, $\alpha_t(s) = (1-t)\alpha_0(s) + t\alpha_1(s)$. This defines a homotopy between α_0 and α_1. $\square$

Theorem 1.1.3 *Let X and Y be topological spaces, then the homotopy relation between maps from X to Y is an equivalence relation.*

Proof: Let $f_1, f_2, f_3 : X \to Y$ be maps, then $f_1 \underset{F}{\approx} f_1$, where $F(x,t) = f_1(x)$. If $f_1 \underset{f_t}{\approx} f_2$ then $f_2 \underset{f_{-t}}{\approx} f_1$. Finally, if $f_1 \underset{F}{\approx} f_2$ and $f_2 \underset{G}{\approx} f_3$ then $f_1 \underset{H}{\approx} f_3$ where

$$H(x,t) = \begin{cases} F(x,2t) & : 0 \leq t \leq \frac{1}{2}, \\ G(x,2t-1) & : 1/2 \leq t \leq 1. \end{cases}$$

Hence H is continuous because $H(x, \frac{1}{2}) = F(x,1) = g(x) = G(x,0)$. We denote this equivalence relation by $\sim$. $\square$

Definition 1.1.4 *The equivalence classes of maps from X to Y as in the Theorem 1.1.3 are called the homotopy classes and denoted by $[f]$ as the homotopy class of the map f. While we use the notation $\langle \gamma \rangle$ when γ is a loop in X based at $x_0 \in X$.*

From now on we assume all spaces are topological spaces.

Theorem 1.1.5 *Let X, Y, Z be spaces. Let $f, g : X \to Y$ and $h : Y \to Z$ be continuous maps. If $f \underset{H}{\approx} g$ rel A then $h \circ f \underset{h \circ H}{\approx} h \circ g$ rel A.*

$\square$

1.2 The Fundamental Group

Definition 1.2.1 *Let α, β be two paths in a space X from a to b and from b to c respectively. We define the multiplication of paths*

$$(\alpha \cdot \beta)(t) := \begin{cases} \alpha(2t) & : 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & : \frac{1}{2} \leq t \leq 1. \end{cases}$$

Since $\alpha(1) = \beta(0) = b$, then the multiplication defined above is a continuous path from a to c passing through b. We note that if γ, γ', and γ'' are loops in X based at x_0 then

$$\begin{aligned}(\gamma \cdot (\gamma' \cdot \gamma''))(t) &= \begin{cases} \gamma(2t) & : 0 \leq t \leq \frac{1}{2} \\ \gamma' \cdot \gamma''(2t-1) & : \frac{1}{2} \leq t \leq 1. \end{cases} \\ &= \begin{cases} \gamma(2t) & : 0 \leq t \leq \frac{1}{2} \\ \gamma'(2(2t-1)) & : \frac{1}{2} \leq t \leq \frac{3}{4} \\ \gamma''(2(2t-1)-1) & : \frac{3}{4} \leq t \leq 1. \end{cases}\end{aligned}$$

Which does not necessarily equal $((\gamma \cdot \gamma') \cdot \gamma'')(t)$. Therefore, the space of loops based at x_0 in X under this multiplication does not form a group.

Lemma 1.2.2 *Let $a, b, c \in X$, let α, α' be paths in X from a to b and β, β' be paths in X from b to c. If $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$ then $[\alpha \cdot \beta] = [\alpha' \cdot \beta']$.*

Proof: Let $\{\alpha_t\}_{0 \leq t \leq 1}$ be a homotopy between α and α' so that $\alpha \underset{\alpha_t}{\approx} \alpha'$ rel $\{0, 1\}$ and $\{\beta_t\}_{0 \leq t \leq 1}$ be a homotopy between β and β' so that $\beta \underset{\beta_t}{\approx} \beta'$ rel $\{0, 1\}$. Then $\{\alpha_t \cdot \beta_t\}_{0 \leq t \leq 1}$ is a homotopy between $\alpha \cdot \beta$ and $\alpha' \cdot \beta'$ so that $\alpha \cdot \beta \underset{\alpha_t \cdot \beta_t}{\approx} \alpha' \cdot \beta'$ rel $\{0, 1\}$. $\square$

Definition 1.2.3 *Let α, β be two paths from a to b in X and let $[\alpha]$ and $[\beta]$ be the homotopy classes of α and β respectively. We define the multiplication of the path classes by $[\alpha] \cdot [\beta] := [\alpha \cdot \beta]$.*

This multiplication is well-defined by Lemma 1.2.2.

Lemma 1.2.4 *The multiplication in Definition 1.2.3 is associative.*

Proof: Let α, β and σ be paths in X from a to b, from b to c, and from c to d respectively.

Then $(\alpha \cdot (\beta \cdot \sigma))(t) = \begin{cases} \alpha(2t) & : 0 \leq t \leq \frac{1}{2} \\ \beta(4t-2) & : \frac{1}{2} \leq t \leq \frac{3}{4} \\ \sigma(4t-3) & : \frac{3}{4} \leq t \leq 1. \end{cases}$

Define the map $\rho : I \to I$, $\rho(s) = \begin{cases} 2s & : 0 \leq s \leq \frac{1}{4} \\ s + \frac{1}{4} & : \frac{1}{4} \leq s \leq \frac{1}{2} \\ \frac{s+1}{2} & : \frac{1}{2} \leq s \leq 1. \end{cases}$

Then ρ is continuous with $\rho(0) = 0$ and $\rho(1) = 1$, and

$$(\alpha \cdot (\beta \cdot \sigma) \circ \rho)(s) = \begin{cases} \alpha(4s) & : 0 \leq s \leq \frac{1}{4} \\ \beta(4s-1) & : \frac{1}{4} \leq s \leq \frac{1}{2} \\ \sigma(2s-1) & : \frac{1}{2} \leq s \leq 1. \end{cases} = ((\alpha \cdot \beta) \cdot \sigma)(s).$$

But $[\rho] = [1_I]$ because I is a convex subset of $\mathbb{R}$, then

$$\begin{aligned}[(\alpha \cdot \beta) \cdot \sigma] &= [\alpha \cdot (\beta \cdot \sigma) \circ \rho] \\ &= [\alpha \cdot (\beta \cdot \sigma) \circ 1_I] \\ &= [\alpha \cdot (\beta \cdot \sigma)].\end{aligned}$$

□

Lemma 1.2.5 *Let α be a path from a to b in X and let e_a and e_b be the constant paths at a and b respectively. Then $[e_a][\alpha] = [\alpha] = [\alpha][e_b]$.*

Proof: Define

$$H(s,t) := \begin{cases} \alpha(\frac{2s}{1+t}) & : 0 \leq s \leq \frac{1}{2}(1+t) \\ b & : \frac{1}{2}(1+t) \leq s \leq 1. \end{cases}$$

We have

$$H(s,0) = \begin{cases} \alpha(2s) & : 0 \leq s \leq \frac{1}{2} \\ b & : \frac{1}{2} \leq s \leq 1 \end{cases} = (\alpha \cdot e_b)(s),$$

and

$$H(s,1) = \begin{cases} \alpha(s) & : 0 \leq s \leq 1 \\ b & : s = 1 \end{cases} = \alpha(s).$$

Thus $\alpha \cdot e_b \underset{H}{\approx} \alpha$ rel $\{0,1\}$. Similarly, $e_a \cdot \alpha \simeq \alpha$ rel $\{0,1\}$. □

By a loop γ in X based at $x_0 \in X$ we mean a path $\gamma : I \to X$ such that $\gamma(0) = \gamma(1) = x_0$.

Theorem 1.2.6 *Fix a base point $x_0 \in X$. Let*

$$G := \Big\{ \langle \gamma \rangle ; \gamma \textit{ is a loop in } X \textit{ based at } x_0 \Big\},$$

then G forms a group under the multiplication of the homotopy classes (Definition (1.2.3)).

Proof: Associativity and the existence of the identity follow from Lemmas (1.2.4) and (1.2.5) respectively. It remains to prove the existence of the inverse of each homotopy class $\langle \gamma \rangle$ i.e $\langle \gamma \rangle^{-1}$. Let the loop $\gamma^{-1}(t) := \gamma(1-t)$; $0 \leq t \leq 1$ be the inverse of γ (inverse here means traveling backwards along γ). Define

$$\rho : I \to I,\ \rho(s) = \begin{cases} 2s & : 0 \leq s \leq \frac{1}{2} \\ 2 - 2s & : \frac{1}{2} \leq s \leq 1. \end{cases}$$

Then

$$(\gamma\cdot\gamma^{-1})(s) = \begin{cases} \gamma(2s) & : 0 \le s \le \frac{1}{2} \\ \gamma^{-1}(2s-1) & : \frac{1}{2} \le s \le 1. \end{cases}$$

From the definition of the map ρ and the path multiplication one finds that $\gamma\cdot\gamma^{-1} = \gamma\circ\rho$.
Since $[\rho] = [e_0]$ (where e_0 is the constant path in I at 0) then $\langle\gamma\circ\rho\rangle = \langle\gamma\circ e_0\rangle = \langle e_{x_0}\rangle$ hence $\langle\gamma\cdot\gamma^{-1}\rangle = \langle e_{x_0}\rangle$. Therefore, $\langle\gamma\rangle\langle\gamma^{-1}\rangle = \langle e_{x_0}\rangle$ (where e_{x_0} is the constant path in X at x_0). We can define $\langle\gamma\rangle^{-1}$ to be $\langle\gamma^{-1}\rangle$. □

Definition 1.2.7 *The group G defined above is called the fundamental group of X with base point x_0 and denoted $\pi_1(X, x_0)$.*

Example 1.2.8 *If X is a convex subset of $\mathbb{R}^n$ then the fundamental group of X based at any point $x_0 \in X$ is trivial because all loops in X are homotopic to the constant loop at x_0. So $\pi_1(X, x_0) = 1$.*

Proposition 1.2.9 *Assume that there exists a path from x_0 to x_1 in X. Then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

Proof: Let α be a path in X from x_0 to x_1. We note that if γ is a loop in X based at x_0 then $\alpha^{-1}\gamma\alpha$ is a loop in X based at x_1; $x_1 \xrightarrow{\alpha^{-1}} x_0 \xrightarrow{\gamma} x_0 \xrightarrow{\alpha} x_1$. We define a map $\alpha_* : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by $\langle\gamma\rangle \mapsto \langle\alpha^{-1}\gamma\alpha\rangle$. If γ' is another loop in X based at x_0 then,

$$\begin{aligned}
\alpha_*(\langle\gamma\rangle\langle\gamma'\rangle) &= \alpha_*(\langle\gamma\cdot\gamma'\rangle) \\
&= \langle\alpha^{-1}\cdot\gamma\cdot\gamma'\cdot\alpha\rangle \\
&= \langle\alpha^{-1}\cdot\gamma\cdot(\alpha\cdot\alpha^{-1})\cdot\gamma'\cdot\alpha\rangle \\
&= \langle\alpha^{-1}\gamma\cdot\alpha\rangle\langle\alpha^{-1}\cdot\gamma'\cdot\alpha\rangle \\
&= \alpha_*(\langle\gamma\rangle)\alpha_*(\langle\gamma'\rangle).
\end{aligned}$$

This shows that α_* is a group homomorphism.
Similarly,
$(\alpha^{-1})_* : \pi_1(X, x_1) \to \pi_1(X, x_0)$ is a group homomorphism with

$$\begin{aligned}
((\alpha^{-1})_*\circ\alpha_*)(\langle\gamma\rangle) &= \langle\alpha\alpha^{-1}\gamma\alpha\alpha^{-1}\rangle \\
&= \langle\gamma\rangle \\
\Rightarrow (\alpha^{-1})_*\circ\alpha_* &= 1_{\pi_1(X,x_1)},
\end{aligned}$$

and $\alpha_*\circ(\alpha^{-1})_* = 1_{\pi_1(X,x_0)}$. Thus, α_* is an isomorphism. □

Corollary 1.2.10 *If X is a path-connected space, then the fundamental group is independent of the choice of the base point x_0 up to isomorphism.* □

In this case we may sometimes write $\pi_1(X)$ for $\pi_1(X, x_0)$.

Definition 1.2.11 *A path-connected space X is said to be simply connected if its fundamental group is trivial.*

Definition 1.2.12 *A pointed topological space (X, x_0) is a topological space X together with the base point x_0.*

Proposition 1.2.13 *Homeomorphic spaces have isomorphic fundamental groups.*

Proof: Let $\varphi : X \to Y$ be a homeomorphism, let $x_0 \in X, y_0 = \varphi(x_0)$ then $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, $\langle\gamma\rangle \mapsto \langle\varphi(\gamma)\rangle$ defines a group isomorphism. □

1.3 Induced Homomorphisms

Theorem 1.3.1 *Let (X, x_0) and (Y, y_0) be pointed spaces. Let also the map $f : (X, x_0) \to (Y, y_0)$ be a base point preserving ($f(x_0) = y_0$). Then f induces a homomorphism from $\pi_1(X, x_0)$ to $\pi_1(Y, y_0)$.*

Proof: Define $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, $f_*(\langle\gamma\rangle) = \langle f \circ \gamma\rangle$. First, f_* is well-defined: if $\gamma \approx \gamma' \text{ rel}\{0, 1\}$ then $\langle f\gamma\rangle = \langle f\gamma'\rangle$. Secondly, f_* is a homomorphism: if $\langle\gamma_1\rangle, \langle\gamma_2\rangle \in \pi_1(X, x_0)$ then $f_*(\langle\gamma_1\rangle\langle\gamma_2\rangle) = \langle f \circ (\gamma_1\gamma_2)\rangle = \langle f\gamma_1\rangle\langle f\gamma_2\rangle = f_*(\gamma_1) = f_*(\gamma_2)$.

□

Since $\langle\gamma\rangle^{-1} = \langle\gamma^{-1}\rangle$ (see the proof of Theorem 1.2.6), then $(f_*(\langle\gamma\rangle))^{-1} = f_*(\langle\gamma^{-1}\rangle)$.

Proposition 1.3.2 *Let*

$$\begin{aligned} f &: (X, x_0) \to (Y, y_0) \\ g &: (Y, y_0) \to (Z, z_0) \end{aligned}$$

be (base point preserving) maps. Then $(gf)_* = g_*f_*$.

Proof: It follows from the definition that $(gf)_*(\langle\gamma\rangle) = \langle gf\gamma\rangle$,
and that $g_*f_*(\langle\gamma\rangle) = \langle gf\gamma\rangle$. □

1.4 Homotopy of Maps

Definition 1.4.1 *A space X is said to be contractible if the identity map on X is homotopic to a constant map on X.*

Example 1.4.2 *Let X be a convex subset of $\mathbb{R}^n$. If $f, g : X \to X$, $f(x) = x$ and $g(x) = 0\ \forall x \in X$. Let $H : X \times I \to X$ be a map given by $H(x,t) = tx$ then $f \underset{H}{\approx} g$, hence X is a contractible space.*

Proposition 1.4.3 *All maps into a contractible space are homotopic.*

Proof: Let X be contractible and $f, g : Y \to X$ be any maps. Since $1_X \approx x_0 \Rightarrow f = 1_X \circ f \approx x_0 \circ f = x_0 \circ g \approx 1_X \circ g = g$. $\square$

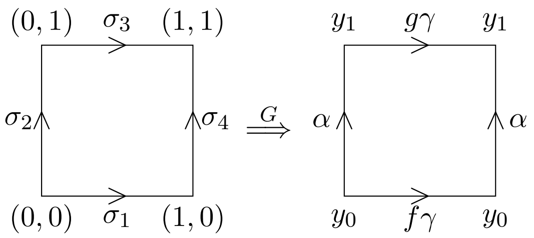
Theorem 1.4.4 *A contractible space is simply connected.*

Proof: Let X be a contractible space. We only need to prove the path-connectedness, if $1_X \underset{H}{\approx} x_0$ then $\alpha(t) = H(x,t)$ joins x_0 with any $x \in X$. Therefore, X is path-connected. $\square$

Theorem 1.4.5 *Let $f, g : X \to Y$ be maps s.t $f \underset{H}{\approx} g$. Fix $x_0 \in X$ and $y_0 = f(x_0) \in Y$, $y_1 = g(x_0)$. Let $\alpha(t) := H(x_0, t)$ be a path in Y from y_0 to y_1. Then the following diagram commutes.*

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "f_*"] \arrow[dr, "g_*"'] & \pi_1(Y, y_0) \arrow[d, "\alpha_*"] \\
 & \pi_1(Y, y_1)
\end{tikzcd}$$

Proof: Let γ be a loop in X based at x_0. Define $G(s,t) := H(\gamma(s), t)$. Let $\sigma_1, \sigma_2, \sigma_3$, and σ_4 be paths on the boundary of the unit square in $\mathbb{R}^2$ s.t $\sigma_1(u) = (u, 0)$, $\sigma_2(v) = (0, v)$, $\sigma_3(u) = (u, 1)$, and $\sigma_4(v) = (1, v)$; $0 \le u, v \le 1$



Since the unit square is convex subset of $\mathbb{R}^2$ then ${\sigma_2}^{-1}\sigma_1\sigma_4 \approx \sigma_3$ rel $\{0,1\}$. Furthermore, $G\circ({\sigma_2}^{-1}\sigma_1\sigma_4) \approx G\circ\sigma_3$ rel $\{0,1\} \Rightarrow (G\circ{\sigma_2}^{-1})(G\circ\sigma_1)(G\circ\sigma_4) \approx G\circ\sigma_3$ rel $\{0,1\}$. From the figure above $\alpha^{-1}(f\gamma)\alpha \approx g\gamma \Rightarrow \alpha_*(\langle f\gamma\rangle) = g_*(\langle\gamma\rangle)$ (by the definitions of α_* and of g_*) i.e $\alpha_* f_*(\langle\gamma\rangle) = g_*(\langle\gamma\rangle)$. Since γ was arbitrary loop in X based at x_0 we conclude that $\alpha_* f_* = g_*$, thus the diagram is commutative. □

Corollary 1.4.6 *In the theorem above f_* is an isomorphism iff g_* is.*

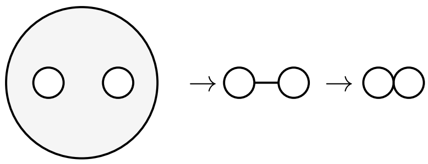
Proof: Because α_* is an isomorphism and $\alpha_* f_* = g_*$. □

1.5 Homotopy Equivalence

Definition 1.5.1 *Let X and Y be two spaces, then we say X and Y are homotopy equivalent if there exists two maps $f : X \to Y$ and $g : Y \to X$ s.t $fg \approx 1_Y$ and $gf \approx 1_X$. We call g a homotopy inverse of f.*

Example 1.5.2 *(1.) If X and Y are homeomorphic then they are homotopy equivalent; $\varphi\varphi^{-1} = 1_Y, \varphi^{-1}\varphi = 1_X$ where φ is a homeomorphism $X \to Y$.*
(2.) $\mathbb{R}^n \setminus \{0\}$ and $\mathbb{S}^{n-1}$ are homotopy equivalent: Let $f : \mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ and $g : \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^{n-1}$, $x \mapsto \frac{x}{\|x\|} \Rightarrow g \circ f = 1_{\mathbb{S}^{n-1}}$ and $f \circ g \underset{H}{\approx} 1_{\mathbb{R}^n\setminus\{0\}}$ where $H(x,t) = \frac{x}{\|x\|^t}$.

Definition 1.5.3 *A subspace A of X is called a deformation retract of X if there is a map $r : X \times I \to X$ s.t $r(x,0) = x$ and $r(x,1) \in A, \forall x \in X$, and $r(a,t) = a, \forall a \in A$, and $\forall t \in I$. The map r is called a retraction.*



Lemma 1.5.4 *If A is a deformation retract of X then A and X are homotopy equivalent.*

Proof: Let r be as in Definition 1.5.3. Define $f : A \to X$ and $g : X \to A$ by $f(a) = r(a,0)$ and $g(x) = r(x,1)$, hence $gf = 1_A$, and if $a = r(x,1)$ then $fg(x) = f(a) = r(a,0) = a = r(x,1)$; Therefore, $fg \underset{r}{\approx} 1_X$. □

Theorem 1.5.5 *If X and Y are homotopy equivalent then their fundamental groups are isomorphic.*

Proof: Let f and g be s.t $gf \underset{F}{\approx} 1_X$ and $fg \underset{H}{\approx} 1_Y$. In order to apply Theorem 1.4.5, we have $gf, 1_X : X \to X$ which are homotopic maps. Suppose that $x_0 \in X$, $x_1 = gf(x_0)$ and $\alpha(t) := F(x_0, t), 0 \leq t \leq 1$, hence we have the following commutative diagram,

$$\begin{tikzcd}
\pi_1(X,x_0) \arrow[r, "(1_X)_*"] \arrow[dr, "g_* \circ f_*"'] & \pi_1(X,x_0) \arrow[d, "\alpha_*"] \\
& \pi_1(X,x_1)
\end{tikzcd}$$

since $(1_X)_*$ is an isomorphism then so is $g_* \circ f_*$. Mimicking the way above to see that $f_* \circ g_*$ is also an isomorphism. Therefore f_* is an isomorphism $\pi_1(X, x_0) \to \pi_1(Y, f(x_0))$. $\square$

Corollary 1.5.6 $\pi_1(\mathbb{R}^n \setminus \{0\}) \cong \pi_1(\mathbb{S}^{n-1})$. $\square$

1.6 The Fundamental Group of $\mathbb{S}^1$

Many applications of the fundamental group depend on the fundamental group of the unit circle $\mathbb{S}^1$, so in this section we will calculate $\pi_1(\mathbb{S}^1)$ and in Chapter 4 we will study the general concept used in this section.
The ideas in this section are similar to the ones in [7] and [9].

Let a map $p : \mathbb{R} \to \mathbb{S}^1$ be given by $x \mapsto e^{2\pi i x}$, then p is a group homomorphism $(\mathbb{R}, +) \to (\mathbb{S}^1, \times)$, because $p(x + y) = e^{2\pi i(x+y)} = e^{2\pi i x}e^{2\pi i y} = p(x)p(y)$.
If $n \in \mathbb{Z}$ then $(n - \frac{1}{2}, n + \frac{1}{2})$ is homeomorphic to $\mathbb{S}^1 \setminus \{-1\}$. Let q be the inverse of p in the set $\mathbb{S}^1 \setminus \{-1\}$.

Lemma 1.6.1 (Lifting of $\mathbb{S}^1$) *Let p be defined as above, and let α be a path in $\mathbb{S}^1$ that starts at 1. Then there exists a unique path $\tilde{\alpha}$ in $\mathbb{R}$ that starts at 0 so that $p\tilde{\alpha} = \alpha$. i.e the following diagram commutes.*

$$\begin{tikzcd}
& (\mathbb{R}, 0) \arrow[d, "p"] \\
I \arrow[ur, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & (\mathbb{S}^1, 1)
\end{tikzcd}$$

Lemma 1.6.2 (Monodromy Theorem of $\mathbb{S}^1$) *Let $F : I \times I \to \mathbb{S}^1$ be a map s.t $F(0,t) = 1, \forall t \in I$. Then there exists a unique $\tilde{F} : I \times I \to \mathbb{R}$ s.t $\tilde{F}(0,t) = 0, \forall t \in \mathbb{R}$. Furthermore, the following diagram commutes.*

$$\begin{tikzcd} & (\mathbb{R},0) \arrow[d, "p"] \\ I \times I \arrow[ur, "\tilde{F}"] \arrow[r, "F"'] & (\mathbb{S}^1,1) \end{tikzcd}$$

Proof: Both lemmas can be proved by the same method, so we prove them together at the same time.
Let X be either I or $I \times I$, $0 \in X$ be either $0 \in I$ or $(0,0) \in I \times I$ and $f : X \to \mathbb{S}^1$ be either α or F. First we prove the existence of $\tilde{f}$ that makes the following diagram commute.

$$\begin{tikzcd} & (\mathbb{R},0) \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & (\mathbb{S}^1,1) \end{tikzcd}$$

Since X is compact subset of $\mathbb{R}^n$ then f is uniformly continuous. So $\exists \delta > 0$ s.t if $\mid x - y \mid < \delta$ then $\mid f(x) - f(y) \mid < 2$, which means that whenever $|x - y| < \delta$ then $(f(x), f(y))$ is not an antipodal pair of points on $\mathbb{S}^1$. Hence $f(x) \neq -f(y) \Rightarrow \frac{f(x)}{f(y)} \neq -1 \Rightarrow q\left(\frac{f(x)}{f(y)}\right)$ is defined. Let $M \in \mathbb{N}$ be large enough s.t $\mid x \mid < \delta M$, $\forall x \in X \Rightarrow \mid \frac{j}{M}x - \frac{j-1}{M}x \mid < \delta$, $\forall j = 1, ..., M \Rightarrow q\left(\frac{f(\frac{j}{M}x)}{f(\frac{j-1}{M}x)}\right)$ is defined $\forall j = 1, ..., M$.
Define $\tilde{f} : X \to \mathbb{R}$ s.t $\tilde{f}(x) = \sum\limits_{j=1}^{M} q\left(\frac{f(\frac{j}{M}x)}{f(\frac{j-1}{M}x)}\right)$. Hence $\tilde{f}$ is continuous,
$\tilde{f}(0) = \sum\limits_{j=1}^{M} q\left(\frac{f(0)}{f(0)}\right) = \sum\limits_{j=1}^{M} q(1) = 0$, and
$p\tilde{f}(x) = p\left(\sum\limits_{j=1}^{M} q\left(\frac{f(\frac{j}{M}x)}{f(\frac{j-1}{M}x)}\right)\right) = \prod\limits_{j=1}^{M} \frac{f(\frac{j}{M}x)}{f(\frac{j-1}{M}x)} = \frac{f(x)}{f(0)} = f(x)$.
It remains to prove the uniqueness of $\tilde{f}$. To this end, let $\tilde{f}$ and $\tilde{g}$ be s.t $\tilde{f}(0) = \tilde{g}(0) = 0$ and $f = p\tilde{f} = p\tilde{g}$ then $p(\tilde{f} - \tilde{g})(x) = 0$, $\forall x \in X$. Thus we have $\tilde{f} - \tilde{g} : X \to \ker p \cong \mathbb{Z}$, but $\tilde{f} - \tilde{g}$ is continuous and X is connected while $\mathbb{Z}$ is not. So $\tilde{f} - \tilde{g}$ must be a constant map.
Since $(\tilde{f} - \tilde{g})(0) = 0 \Rightarrow \tilde{f} - \tilde{g} = 0$ then $\tilde{f} = \tilde{g}$. $\square$

Corollary 1.6.3 *Let α and β be two paths in $\mathbb{S}^1$ starting at 1 so that $\alpha \underset{F}{\approx} \beta$ rel $\{0,1\}$. Then $\exists!\ \tilde{\alpha}, \tilde{\beta}, \tilde{F}$ s.t $p\tilde{\alpha} = \alpha$, $p\tilde{\beta} = \beta$, and $p\tilde{F} = F$ with $\tilde{\alpha} \underset{\tilde{F}}{\approx} \tilde{\beta}$ rel $\{0,1\}$.*

Proof: The existence and uniqueness of each of these lifting maps follow immediately from last two lemmas. We have $p\tilde{F}(s,0) = \alpha(s)$ and $p\tilde{F}(s,1) = \beta(s)$. By the uniqueness of liftings $\tilde{F}(s,0) = \tilde{\alpha}(s)$ and $\tilde{F}(s,1) = \tilde{\beta}(s)$. Therefore, $\tilde{\alpha} \underset{\tilde{F}}{\approx} \tilde{\beta}$ rel $\{0,1\}$. $\square$

Corollary 1.6.4 *If $\alpha, \beta : I \to (\mathbb{S}^1, 1)$ are homotopic with $\alpha(0) = \beta(0) = 1$, then $\tilde{\alpha}$ and $\tilde{\beta}$ have the same terminal point in $\mathbb{R}$.* $\square$

Theorem 1.6.5 $\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$.

Proof: Define $\varphi : \pi_1(\mathbb{S}^1, 1) \to \mathbb{Z}$ by $\langle\gamma\rangle \mapsto \tilde{\gamma}(1)$ (γ is a loop in $\mathbb{S}^1$ iff $\tilde{\gamma}(1) \in \mathbb{Z}$).
• Well-defined: If $\gamma_0 \approx \gamma_1$ rel $\{0,1\}$ then $\tilde{\gamma_0}(1) = \tilde{\gamma_1}(1)$ (by the last corollary).
• Homomorphism: Assume the following;
∗ $\langle\gamma_0\rangle, \langle\gamma_1\rangle \in \pi_1(\mathbb{S}^1, 1)$,
∗ $\tilde{\gamma_0}, \tilde{\gamma_1}$ are the lifting paths of γ_0 and γ_1 to $\mathbb{R}$ respectively,
∗ $\tilde{\gamma_0}(1) = m$ and $\tilde{\gamma_1}(1) = n$, and
∗ $\alpha : I \to \mathbb{R}$ is a path in $\mathbb{R}$ from m to $m+n$ given by $\alpha(t) = m + \tilde{\gamma_1}(t)$.
Now $\tilde{\gamma_0}\alpha$ is a path in $\mathbb{R}$ from 0 to $m+n$, and $p(\tilde{\gamma_0}\alpha) = (p\tilde{\gamma_0})(p\alpha) = \gamma_0\gamma_1 \Rightarrow \tilde{\gamma_0}\alpha = \widetilde{\gamma_0\gamma_1} \Rightarrow \varphi(\langle\gamma_0\gamma_1\rangle) = \widetilde{\gamma_0\gamma_1}(1) = \tilde{\gamma_0}\alpha(1) = m+n = \varphi(\langle\gamma_0\rangle) + \varphi(\langle\gamma_1\rangle)$.
• One-to-one: Let $\varphi(\langle\gamma\rangle) = 0 \Rightarrow \tilde{\gamma}(1) = 0$ so $\tilde{\gamma}$ is a loop in $\mathbb{R}$ based at 0. Since $\mathbb{R}$ is contractible then $\tilde{\gamma} \approx 0$ rel $\{0,1\} \Rightarrow \gamma = p\tilde{\gamma} \approx 1$ rel $\{0,1\} \Rightarrow \langle\gamma\rangle = 1$.
• Onto: Let $m \in \mathbb{Z}$ and let $\gamma(t) = e^{2\pi imt}$ be a loop in $(\mathbb{S}^1, 1)$. Since $\tilde{\gamma}(t) = mt$ then $\varphi(\langle\gamma\rangle) = \tilde{\gamma}(1) = m$. $\square$

Theorem 1.6.6 *If (X, x_0) and (Y, y_0) be pointed spaces. Then,*

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Proof: Let $(p_X)_*$ and $(p_Y)_*$ be the homomorphisms induced from the projections (p_X) and (p_Y).

$$\begin{tikzcd} & \pi_1(X \times Y, (x_0, y_0)) \arrow[dl, "(p_X)_*"'] \arrow[dr, "(p_Y)_*"] & \\ \pi_1(X, x_0) & & \pi_1(Y, y_0) \end{tikzcd}$$

Define $\psi : \pi_1(X\times Y,(x_0,y_0)) \to \pi_1(X,x_0)\times\pi_1(Y,y_0)$ by $\langle\gamma\rangle \mapsto (\langle p_X\gamma\rangle, \langle p_Y\gamma\rangle)$, we claim that ψ is an isomorphism; Indeed, if $p_X\gamma \underset{F}{\approx} e_{x_0}$ rel $\{0,1\}$ and $p_Y\gamma \underset{H}{\approx} e_{y_0}$ rel $\{0,1\}$, then $(p_X, p_Y) \underset{(F,H)}{\approx} (e_{x_0}, e_{y_0})$ rel $\{0,1\}$, so ψ is one-to-one.
Conversely, if $\langle\tau\rangle \in \pi_1(X,x_0)$ and $\langle\sigma\rangle \in \pi_1(Y,y_0)$ then $\psi(\langle(\tau,\sigma)\rangle) = (\langle\tau\rangle,\langle\sigma\rangle)$, so ψ is onto. Homomorphism follows from p_X and p_Y being homomorphisms.
$\square$

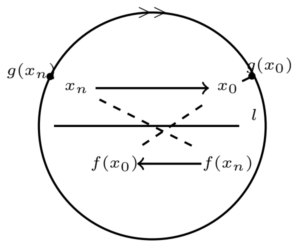
Example 1.6.7 *Consider the torus $T = \mathbb{S}^1 \times \mathbb{S}^1$. Then $\pi_1(T) \cong \mathbb{Z}\times\mathbb{Z}$.*

1.7 Brouwer Fixed Point Theorem

This is a first application of the fundamental group. This theorem also has many subsequent applications, we will introduce some of these applications in this section. The proof of this theorem is somewhat similar in almost every reference, see for example [2], [7] and [9].

Theorem 1.7.1 *Let $f : \mathbb{D}^2 \to \mathbb{D}^2$ be a continuous map. Then there exists an $x \in \mathbb{D}^2$ s.t $f(x) = x$.*

Proof: Assume to the contrary that $f(x) \neq x$, $\forall x \in \mathbb{D}^2$. Define $g : \mathbb{D}^2 \to \mathbb{S}^1$ in the following way, $\forall x \in \mathbb{D}^2$ let $g(x)$ be the point on $\mathbb{S}^1$ obtained from the intersection of the extension of the line segment from $f(x)$ to x with $\mathbb{S}^1$. The continuity of g can be followed by the following **Fact**: If (X, d_1) is a metric space, and if $h : X \to X$ is a continuous map s.t $h(x) \neq x$, $\forall x \in X$. Then $\exists\delta > 0$ s.t $h(B(x,\delta)) \cap (B(x,\delta)) = \varnothing$. **Proof** (Fact): If $\exists x_0 \in X$ s.t $\forall\delta > 0$ $h(B(x_0,\delta)) \cap B(x_0,\delta) \neq \varnothing$ then $h(x_0) \to x_0$ as $\delta \to 0$ because h is continuous. A contradiction.
As a consequence of this fact; If X is a convex subset of $\mathbb{R}^n$ and if $h : X \to \mathbb{R}^n$ is continuous then there exists an $(n-1)$-plane in $\mathbb{R}^n$ separates y's from $f(y)$'s $\forall y \in B(x,\delta)$ for some $\delta > 0$.



In our situation, let $x \in \mathbb{D}^2$ and let $x_n \longrightarrow x$ be a convergent sequence in $\mathbb{D}^2$ $\Rightarrow f(x_n) \longrightarrow f(x)$. Let $\delta > 0$ be s.t there exists a line l separates x_n's from $f(x_n)$'s, $\forall x_n \in B(x_0,\delta) \Rightarrow g(x_n) \longrightarrow g(x_0) \Rightarrow$ g is continuous.
Or alternatively, one can find the map g explicitly; Define

$\gamma(t) = x + t(x - f(x))$, then $||\gamma(t)||^2 = 1$ i.e
$\langle x+t(x-f(x)), x+t(x-f(x))\rangle = ||x||^2+t^2||x-f(x)||^2+2t\langle x, x-f(x)\rangle = 1$.
Let $t_{>0} = \frac{-2\langle x,x-f(x)\rangle+\sqrt{4\langle x,x-f(x)\rangle^2-4||x-f(x)||^2(||x||^2-1)}}{2||x-f(x)||^2}$.

Then the function $g(x) = x + t_{>0}(x - f(x))$ is continuous since f is continuous and $f(x) - x \neq 0$ by the assumption.
We note that $g_{\big|_{\mathbb{S}^1}} = 1_{\mathbb{S}^1}$ hence the maps $\mathbb{S}^1 \overset{i}{\hookrightarrow} \mathbb{D}^2 \overset{g}{\to} \mathbb{S}^1$ induce the homomorphisms $\pi_1(\mathbb{S}^1) \overset{i_*}{\to} \pi_1(\mathbb{D}^2) \overset{g_*}{\to} \pi_1(\mathbb{S}^1)$ i.e $\mathbb{Z} \overset{i_*}{\to} 1 \overset{g_*}{\to} \mathbb{Z}$. Since $gi = 1_{\mathbb{S}^1}$ then $(gi)_* = g_*i_* = (1_{\mathbb{S}^1})_*$ which is one-to-one $\Rightarrow g_*$ is onto. But $g_* : 1 \to \mathbb{Z}$ can't be onto, a contradiction. Thus $\exists x \in \mathbb{D}^2$ s.t $f(x) = x$. $\square$

Example 1.7.2 (The Fundamental Theorem of Algebra) *Any polynomial with complex coefficients with degree ≥ 1 has a root in $\mathbb{C}$.*

Proof: (*This proof is due to B. H. Arnold [3]*)
Let $p(z) = z^n+a_{n-1}z^{n-1}+...+a_0$ be an arbitrary monic complex polynomial. Set $z = re^{i\theta}, 0 \leq \theta < 2\pi$, and let $R := 2 + |a_{n-1}| + |a_{n-2}| + ... + |a_0|$. Define the map

$$f(z) = \begin{cases} z - \frac{p(z)}{Re^{i(n-1)\theta r}} & ; |z| \leq 1 \\ z - \frac{p(z)}{Rz^{n-1}} & ; |z| \geq 1. \end{cases}$$

Then f is continuous since the two expressions are identical when $|z| = 1$.
Consider $|z| \leq R$, then we have two cases; First, for $|z| \leq 1$,
$|f(z)| \leq |z| + \frac{|p(z)|}{R} \leq 1 + \frac{1}{R} + \frac{|a_{n-1}|+...+|a_0|}{R} \leq 1 + 1 \leq R$.
While when $1 \leq |z| \leq R$,

$$\begin{aligned} |f(z)| =& |\, z - \frac{z}{R} - \frac{1 + z^{-1} + ... + z^{-n+1}}{R} \,| \\ &\leq |\, \frac{(R-1)z}{R} \,| + \frac{|a_{n-1}|}{R} + ... + \frac{|a_0|}{R} \\ &\leq R - 1 + \frac{R-2}{R} = \frac{R^2-2}{R} \\ &\leq R. \end{aligned}$$

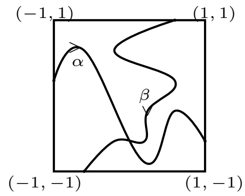
Hence f defines a continuous map $B(0, R) \to B(0, R)$. Therefore, by the Brouwer fixed point theorem there exists $z_0 \in B(0, R)$ s.t $f(z_0) = z_0$, and hence the definition of f implies that $p(z_0) = 0$.

$\square$

Example 1.7.3 *Any 3×3 matrix with positive entries has an eigenvector with positive eigenvalue.*

Proof: Let $U = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1, x_2, x_3 \geq 0, \sum_{i=1}^{3} x_i \neq 0\right\}$,
and $V = \left\{(x_1, x_2, x_3) \in U \mid \sum_{i=1}^{3} x_i = 1\right\}$.
Define $f : U \to V$, $f((x_1, x_2, x_3)) = \frac{1}{\sum_{i=1}^{3} x_i}(x_1, x_2, x_3)$. Now let A be a 3×3 matrix with positive real entries. $A(U) \subseteq U$ (matrix multiplication $A\vec{u}^T \in U$), hence $(fA) : V \to V$ is continuous (linear transformation). But $V \subseteq \mathbb{R}^3$ is homeomorphic to $\mathbb{D}^2$. Therefore, by Brouwer fixed point theorem (fA) has a fixed point. i.e $\exists \vec{v} \in V$ s.t $(fA)(\vec{v}) = \vec{v}$, but since $\vec{v} \in V$ then $(fA)(\vec{v}) = \frac{1}{\sum_{a_{ij} \in A} a_{ij}} A\vec{v}$. Let $\lambda = \sum_{a_{ij} \in A} a_{ij} > 0 \Rightarrow A\vec{v} = \lambda\vec{v}$. $\square$

The following Lemma is an application of the Brouwer fixed point Theorem, and we will use it to prove the following theorem (Hex Theorem). This Lemma is due to [13].



Lemma 1.7.4 *Let* $\alpha(t) = (r_1(t), r_2(t))$ *and* $\beta(t) = (u_1(t), u_2(t))$ *be two paths in the square* $[-1, 1] \times [-1, 1]$, *where* $r_1, r_2, u_1, u_2 : [-1, 1] \to [-1, 1]$ *satisfying the following;* $r_1(-1) = -1$, $r_1(1) = 1$, $u_2(-1) = -1$, *and* $u_2(1) = 1$. *Then* α *and* β *intersect each other inside the square.*

Proof: Assume not, and define $M : [-1, 1] \times [-1, 1] \to \mathbb{R}^+$, by

$M((s, t)) = \max \{\mid r_1(s) - u_1(t) \mid, \mid r_2(s) - u_2(t) \mid\}$. Then $M((s, t)) \neq 0$, for all $-1 \leq s, t \leq 1$.

Define $F : [-1, 1] \times [-1, 1] \to [-1, 1] \times [-1, 1]$, by

$$F((s, t)) = \left(\frac{u_1(t) - r_1(s)}{M((s,t))}, \frac{r_2(s) - u_2(t)}{M((s,t))}\right).$$

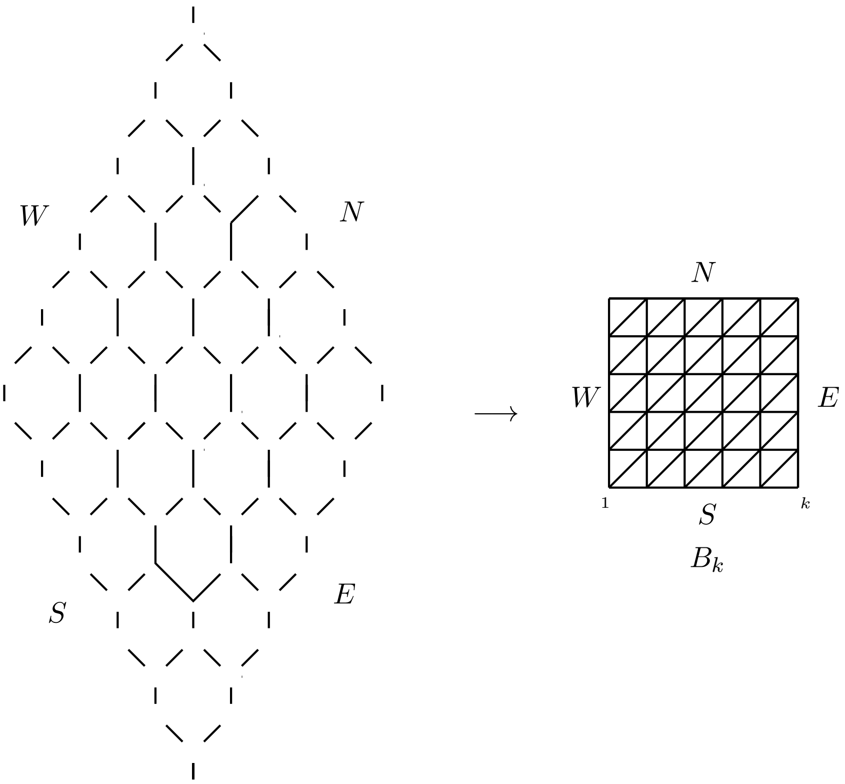
We notice that the image of F is on the boundary of the square. By Brouwer fixed point theorem, $\exists (s_0, t_0) \in [1, -1] \times [1, -1]$ s.t $F((s_0, t_0)) = (s_0, t_0)$, the only possible choices are $(1, t_0)$, $(-1, t_0)$, $(s_0, 1)$, and $(s_0, -1)$, for some $-1 \leq t_0, s_0 \leq 1$. If for example the fixed point is $(-1, t_0)$, then by the definition of the first coordinate of F, $u_1(t_0) - (-1) < 0$. A contradiction. Similarly for the other possibilities. $\square$

For another application of the Brouwer fixed theorem, David Gale [6] proved that the Brouwer fixed point theorem and the Hex theorem are equivalent. Here we will see one direction of his proof; In particular, Brouwer fixed point theorem implies the Hex theorem.

Definition 1.7.5 *The hex game is a 2-player board game, the board consists of (usually* 11*) hexagons. Player 1 aims to connect the 'west' and the 'east'* *sides of the board by a connected coloring path, while Player 2 aims to connect the 'north' and the 'south' sides of the board by a connected path by a different color.*

Theorem 1.7.6 (Hex Theorem) *After filling all cells in the hex board by Player 1 and Player 2 'colors', there is exactly one winner.*

Proof: Uniqueness of the winning follows directly from the last lemma. To prove the existence, for simplicity we replace the board by a graph in the plane with edges and vertices (lattices) as in the figure.

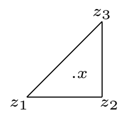


Each corner vertex can be reached in three different ways (edges), and each vertex on the border which is not a corner can be reached in four different ways, while other vertices (inside the graph) in six different ways. So we can see that both the board and the graph are mathematically identical. Assume that it is possible for this game to end without a winner, i.e no connected path from W to E nor from N to S. So the vertices of B_k can be partitioned into two sets X and Y, where X is Player 1 vertices and Y is Player 2 vertcies. Define the four sets
$W' = \{$vertex connected to W by an $X-$path (a path lies in X)$\}$,
$E' = X \setminus W'$,
$N' = \{$vertex connected to N by a $Y-$path (a path lies in Y)$\}$, and

$S' = Y \setminus N'$.
Hence W' and E' (N' and S') are not contiguous, that is no edge connecting a vertex in W' (N') and another one in E' (S'). Define $f : B_k \to B_k$ by

$$f(z) = \begin{cases} z + e_1 & : z \in W' \\ z - e_1 & : z \in E' \\ z + e_2 & : z \in S' \\ z - e_2 & : z \in N'. \end{cases}$$

Where e_1 and e_2 are the unit orthonormal vectors (standard basis) of $\mathbb{R}^2$, f is well-defined. Since any $x \in B_k$ lies in some triangle



then $x = c_1 z_1 + c_2 z_2 + c_3 z_3$ where $\sum_{i=1}^{3} c_i = 1$, and hence we can extend f piecewise linearly to get a continuous $\hat{f} : B_k \to B_k$, $\hat{f}(x) = \sum_{i=1}^{3} c_i f(z_i)$.
By Brouwer fixed point theorem $\hat{f}$ has a fixed point x_0. Say $x_0 = c_1 z_1 + c_2 z_2 + c_3 z_3$ then $x_0 = \hat{f}(x_0) = c_1 f(z_1) + c_2 f(z_2) + c_3 f(z_3) = c_1(z_1 + v_1) + c_2(z_2 + v_2) + c_3(z_3 + v_3)$, where v_i is e_1 or e_2. But this happens iff $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \Rightarrow$ one of the c_i's must be zero, say $c_3 = 0$ and v_1 must equal $-v_2$, if $v_1 = e_1 \Rightarrow f(z_1) = z_1 + e_1$ and $f(z_2) = z_2 - e_1 \Rightarrow z_1 \in W'$ and $z_2 \in E'$. This is a contradiction, because z_1 and z_2 belong to same triangle and by assumption W' and E' are not contiguous. $\square$

Chapter 2

Some Group Theory

To study the fundamental group of more complicated spaces we need some theoretical concepts in group theory, specifically; free groups, free products and the group presentations. The Definitions and results in this Chapter are similar to those in [11], [15], and [17]. There one can also find more theory about free groups and free products.

2.1 Free Abelian Groups

Definition 2.1.1 *A free abelian group G generated by a set X ($\subseteq G$) is an abelian group satisfying the following universal mapping property (UMP): For any abelian group A and a function $f : X \to A$ there exists a unique homomorphism $\varphi : G \to A$ s.t $\varphi|_X = f$, i.e the following diagram commutes,*

$$\begin{tikzcd}
& G \arrow[d, dotted, "\varphi(unique)"] \\
X \arrow[ur, hook] \arrow[r, "f"'] & A
\end{tikzcd}$$

Theorem 2.1.2 *Let G and G' be two free abelian groups generated by the subset X. Then there exists a unique isomorphism $\varphi : G \to G'$ s.t $\varphi|_X$ is the inclusion map $X \hookrightarrow G'$.*

Proof: We have these two commutative diagrams.

$$\begin{tikzcd}
& G \arrow[d, "\varphi(unique)"] \\
X \arrow[ur, hook, "i"] \arrow[r, hook, "j"'] & G'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& G' \arrow[d, "\varphi'(unique)"] \\
X \arrow[ur, hook, "j"] \arrow[r, hook, "i"'] & G
\end{tikzcd}$$

From the right diagram we have $\varphi' j = i$, substituting in the left one we get $(\varphi'\varphi)i = i$. But also $1_G \circ i = i$. Therefore, $\varphi'\varphi = 1_G$ by the UMP of free abelain groups. Similarly, $\varphi\varphi' = 1'_G \Rightarrow \varphi$ is a bijective unique homomorphism. $\square$

Theorem 2.1.3 *Let X be an arbitrary set. The free abelian group generated by X exists.*

Proof: Let X be any set. Define $G = \bigoplus\limits_{x\in X} \langle x\rangle$ ($\cong \sum_{x\in X}\mathbb{Z}$). Then for any abelian group A and any function $f : X \to A$, if $y \in G$ then y has a unique expression $y = \bigoplus\limits_{x\in X} m_x x$, where $m_x \in \mathbb{Z}$. Define $\varphi : G \to A$ by $y \mapsto \bigoplus\limits_{x\in X} m_x f(x)$ (for instance, if $y = (m_1x_1, ..., m_nx_n)$ then $\varphi(y) = (m_1f(x_1), ..., m_nf(x_n)))$. Thus φ is well-defined because the unique expression of y, hence φ is a homomorphism extending f and it is unique because for any other homomorphism $\varphi' : G \to A$ must agree with φ in X which generates G, thus $\varphi = \varphi'$.

$\square$

So we can redefine free abelian groups to be a direct sum of infinite cyclic groups.

Corollary 2.1.4 *If $|X| = |Y|$ then their free abelian groups are isomorphic.* $\square$

Theorem 2.1.5 *Every abelian group A is a quotient of a free abelian group G.*

Proof: Let G be the free abelian group generated by A, then $\exists!\varphi$ s.t the following diagram commutes;

$$\begin{tikzcd} & G \arrow[d, "\varphi"] \\ A \arrow[ur, hook] \arrow[r, "1_A"'] & A \end{tikzcd}$$

Which implies that $\varphi|_A = 1_A \Rightarrow \varphi$ is surjective $\Rightarrow A \cong G/\ker\varphi$. $\square$

Theorem 2.1.6 *A subgroup of a free abelian group is a free abelian.*

Proof: Let G be a free abelian group with a free generating set X, and let $H \leq G$. Since any set can somehow be well-ordered, then we may write $X = \{x_i : i < j\}$ where j is an order type. Define $G_i := \langle x_k : k < i \rangle$, so $G_{i+1} = \langle x_k : k \leq i \rangle = \langle x_k : k < i \rangle \bigoplus \langle x_i \rangle = G_i \bigoplus \langle x_i \rangle$, and $G_j = G$. Define $H_i := H \cap G_i$, then $H_j = H$. Applying the second isomorphism theorem we get $\frac{H_{i+1}}{H_i} = \frac{H_{i+1}}{H_{i+1} \cap H_i} \cong \frac{H_{i+1}+G_i}{G_i} \leq \frac{G_{i+1}}{G_i} \cong \langle x_i \rangle$. Thus either $H_{i+1} = H_i$ or $\frac{H_{i+1}}{H_i}$ is infinite cyclic. In both cases we can write $H_{i+1} = H_i \bigoplus \langle y_i \rangle$ where y_i can be 0. Therefore, $H = \bigoplus\limits_{i<j} \langle y_i \rangle$, hence H is a free abelian group generated by the set $Y := \{y_i \neq 0 : i < j\}$. $\square$

2.2 Free Groups

Definition 2.2.1 *A free group F generated by a set X ($\subseteq F$) is a group satisfying the following UMP: For any group G and any function $f : X \to G$ there exists a unique homomorphism $\varphi : F \longrightarrow G$ s.t $\varphi|_X = f$, i.e the following diagram is commutative.*

$$\begin{tikzcd}
& F \arrow[d, dotted, "\varphi(unique)"] \\
X \arrow[ur, hook] \arrow[r, "f"'] & G
\end{tikzcd}$$

Theorem 2.2.2 *Let F and F' be two free groups generated by the same subset X. Then there is a unique isomorphism $\varphi : F \to F'$ s.t $\varphi|_X$ is the inclusion map $X \hookrightarrow F'$.*

Proof: Similar to the proof of Theorem 2.1.2. $\square$

Theorem 2.2.3 *Given an arbitrary set X, the free group generated by X exists.*

Proof: We construct a free group generated by X.
Define X^{-1} to be a new set disjoint from X s.t both X and X^{-1} have the same cardinality. Fix a bijection between X and X^{-1}, let $x \in X$, we denote by x^{-1} the image of x. Define a word w on X to be a finite string of elements from $X \cup X^{-1}$, and we write $w = x_1^{\lambda_1} x_2^{\lambda_2} \ ... \ x_m^{\lambda_m}$, where $\lambda_i = \pm 1$, $m \geq 0$, where if $m = 0$ then we have the empty word e. Define the inverse w^{-1} of the word w to be the word $w^{-1} := x_m^{-\lambda_m} \ ... \ x_2^{-\lambda_2} x_1^{-\lambda_1}$, and a subword of w to be the word $x_k^{\lambda_k} \ ... \ x_l^{\lambda_l}$ for some $1 \leq k \leq l \leq m$.
Let us define a product on $W(X)$ (:= the set of all words on X) to be a

simple juxtaposition of elements, for example if $w = x_1^{\lambda_1} x_2^{\lambda_2} \ ... \ x_m^{\lambda_m}$ and $\nu = y_1^{\sigma_1} y_2^{\sigma_2} \ ... \ y_n^{\sigma_n}$, then $w.\nu = x_1^{\lambda_1} x_2^{\lambda_2} \ ... \ x_m^{\lambda_m} y_1^{\sigma_1} y_2^{\sigma_2} \ ... \ y_n^{\sigma_n}$, and let's agree that $w \cdot e = e \cdot w = w$.

We see that the above definitions of the identity e, the inverse, and the product operation do not make $W(X)$ a group, for instance, $ww^{-1} \neq e$ unless $w = e$.

So in order to construct our group we define an equivalence relation on $W(X)$.

Define a reduced word $\bar{w}$ of w to be the subword obtained by deleting all the subwords of the form xx^{-1} and $x^{-1}x$, for example if $w = x_1 x_2^{-1} x_2 x_1^{-1} x_3$ then $\bar{w} = x_3$, and if $\nu = x_1^{-1} x_2 x_3 x_3^{-1} x_2^{-1} x_1$ then $\bar{\nu} = e$.

Now we define an equivalence relation $\sim$ on $W(X)$ as the following: $w \sim \nu$ iff $\bar{w} = \bar{\nu}$. Let $[w]$ denote the equivalence class of w, so $\bar{w} \in [w]$ and each equivalence class contains exactly one reduced word, namely $\bar{w}$.

Let $F(X)$ be the set of all equivalence classes, let $[w]^{-1}$ be $[w^{-1}]$. Since if $w \sim w'$ and $\nu \sim \nu'$ then $w\nu \sim w'\nu'$, we define a product ?·? on $F(X)$ by $[w] \cdot [\nu] := [w \cdot \nu]$.

* The product ?·? is associative; $[w_1] \cdot ([w_2] \cdot [w_3]) = [w_1 w_2 w_3] = [w_1 w_2] \cdot [w_3] = ([w_1] \cdot [w_2]) \cdot [w_3]$,
* $[e] \cdot [w] = [w] = [w] \cdot [e]$, $\forall \ [w] \in F(X)$, and
* $[w] \cdot [w]^{-1} = [ww^{-1}] = [e] = [w] \cdot [w^{-1}]$.

Therefore $(F(X), \cdot)$ is a group.

It remains to show that $F(X)$ is a free group generated by X. To this end, assume that G is any group and $f : X \to G$ is any function, then we need a unique homomorphism $\varphi : F(X) \to G$ so that the diagram below commutes.

$$\begin{tikzcd} & F \arrow[d, dotted, "\varphi"] \\ X \arrow[ur, hook] \arrow[r, "f"'] & G \end{tikzcd}$$

Define $\varphi([w]) = (f(x_k))^{\lambda_k} \ ... \ (f(x_l))^{\lambda_l}$, where $x_k^{\lambda_k} \ ... \ x_l^{\lambda_l} = \bar{w}$, then it is well-defined as we noted earlier in the proof that every equivalence class contains exactly one reduced word, it is obviously homomorphism, and it is unique because it is the only way to extend f on $F(X)$ (all homomorphisms agree in the generator set must be equal). Thus, $F(X)$ is the free group generated by X. $\square$

Theorem 2.2.4 *If* $|X| = |Y|$ *then* $F(X) \cong F(Y)$.

Proof: Let $g : X \to Y$ be a bijection, then we have the following commutative diagrams.

$$\begin{tikzcd} & F(X) \arrow[d, "\varphi(unique)"] \\ X \arrow[ur, hook, "i"] \arrow[r, "jg"'] & F(Y) \end{tikzcd} \qquad \begin{tikzcd} & F(Y) \arrow[d, "\varphi'(unique)"] \\ Y \arrow[ur, hook, "j"] \arrow[r, "ig^{-1}"'] & F(X) \end{tikzcd}$$

Since $jg = \varphi i$ and $ig^{-1} = \varphi' j$ then $\varphi'(\varphi i) = (\varphi' j)g = ig^{-1}g = i$, i.e the following diagram is commutative.

$$\begin{tikzcd} & F(X) \arrow[d, "\varphi'\varphi(unique)"] \\ X \arrow[ur, hook, "i"] \arrow[r, hook, "i"'] & F(X) \end{tikzcd}$$

By the UMP of free groups $\varphi'\varphi = 1_{F(X)}$ and similarly $\varphi\varphi' = 1_{F(Y)} \Rightarrow \varphi$ is a unique isomorphism and $F(X) \cong F(Y)$. □

Theorem 2.2.5 *Every group G is a quotient of a free group. That is $G \cong F/R$, where $R \lhd F$.*

Proof: Similar to Theorem 2.1.5. □

Lemma 2.2.6 *Let F be a free group generated by the set $X = \{x_\alpha : \alpha \in \Delta\}$, let F' be the commutator subgroup of F, then F/F' is a free abelian group generated by the set $X' = \{x_\alpha F' : \alpha \in \Delta\}$.*

Proof: Let A be any abelian group, let $\nu : X \to X'$, $x \mapsto xF'$ be the natural map, and let $f : X' \to A$ be any map. Then we have the following diagram, where ψ is unique and a unique φ is to be constructed.

$$\begin{tikzcd} F(X) \arrow[drr, "\psi"'] & X \arrow[l, hook', "i"'] \arrow[r, "\nu"] \arrow[dr, "f\nu"] & X' \arrow[r, hook, "j"] \arrow[d, "f"] & F/F' \arrow[dl, dotted, "\varphi"] \\ & & A & \end{tikzcd}$$

Since A is abelian, then $\psi(w_1 w_2 w_1^{-1} w_2^{-1}) = \psi(w_1)\psi(w_2)\psi(w_1^{-1})\psi(w_2^{-1}) = 1_A \Rightarrow F' \leq \ker \psi$. Hence $\varphi : F/F' \to A$ by $wF' \mapsto \psi(w)$ is well defined (if

$w_1F' = w_2F' \Rightarrow w_1w_2^{-1} \in F' \Rightarrow w_1w_2^{-1} \in \ker\psi \Rightarrow \psi(w_1) = \psi(w_2))$. Also, $\varphi j(xF') = \varphi(xF') = \psi(x) = \psi i(x) = f\nu(x) = f(xF') \Rightarrow \varphi j = f$.
It remains to prove the uniqueness of φ; Let $\varphi' : F/F' \to A$ be a homomorphism satisfying $\varphi j = f \Rightarrow \varphi'(xF') = f(xF') = \varphi(xF')$, $\forall x \in X$. Since all elements in F/F' are on the form wF' where $w = x_1^{\lambda_1} \ ... \ x_m^{\lambda_m}$ and $wF' = x_1^{\lambda_1}F' \ ... \ x_m^{\lambda_m}F'$, then $\varphi(w) = \varphi'(w) \Rightarrow \varphi = \varphi'$. □

Theorem 2.2.7 (Nielsen-Schreier) *Let F be a free group and let $H \leq F$ be a subgroup of F, then H is also a free group.*

The proof will be given in Section 4.7, because we need some theory of covering spaces.

Since any group G is isomorphic to F/N for some free group F generated by X, and some $N \lhd F$, then by the last theorem N is a free group. Let $H \leq N$ so that N is the normal closure of H. So H is also a free group generated by $R \subset H$, say.

Definition 2.2.8 (Generators and Relations) *In the above paragraph, we say X is a generator set and R is a relation set of the group G, and write $G = \langle X|R\rangle$, where the expression $\langle X|R\rangle$ is called a presentation of G. In other words, given a presentation $\langle X|R\rangle$, then $\langle X|R\rangle \cong F(X)/N$, where N is the normal closure of the subgroup generated R (i.e $\langle R\rangle$).*

Example 2.2.9 i. *If $G = \mathbb{Z}_{12}$ then $G \cong \mathbb{Z}/12\mathbb{Z} \cong \langle x\rangle/\langle x^{12}\rangle$ that is, $G = \langle x|x^{12}\rangle$.*
In the Definition 2.2.8, since $N(R)$ is the kernel of some homomorphism $F \to G$ then it is convenient to write G in the example above as $\langle x|x^{12} = e\rangle$. One can express G in another presentation, for example G in the above example can be rewritten as $G = \langle x, y|x^3 = e, y^4 = 1, xyx^{-1}y^{-1} = e\rangle$, where $xyx^{-1}y^{-1} \in R$ guaranties that G is abelian because we have only two generators so $xyx^{-1}y^{-1} = e \Rightarrow xy = yx$. In the next example we discuss the general case.
ii. *Let $G = \langle X|xyx^{-1}y^{-1}, x, y \in X\rangle$. Then G is a presentation of a free abelian group generated by X, i.e $G \cong \bigoplus_{x\in X} \mathbb{Z}$.*
By virtue of this example, the definition of group presentation and Lemma 2.2.6, we see that (the normal closure of) the subgroup generated by the set $\left\{xyx^{-1}y^{-1}; x, y \in X\right\}$ is the commutator subgroup of $F(X)$. Here is a formal proof:
Let F' be the commutator subgroup of F, and let N be the normal closure of the subgroup H generated by the set $A := \{xyx^{-1}y^{-1}; x, y \in X\}$. Since $xyx^{-1}y^{-1} \in N$ for all $x, y \in X$ then $xyN = yxN$ for all $x, y \in X$, and since $N \lhd F$ and $N \subseteq F'$ then $N \lhd F'$. But F' is generated by the set $B = \left\{x_1...x_my_1...y_nx_m^{-1}...x_1^{-1}y_n^{-1}...y_1^{-1}; x_i, y_j \in X, \forall i, j \in \mathbb{N}\right\}$. Therefore,

F'/N is generated by the set $\{bN; b \in B\} = \{N\}$ i.e F'/N is generated by the identity i.e $F'/N \cong \{0\}$, thus $N = F'$ as desired.
iii. A free group F generated by X (may be written as $F/1$) has a presentation $\langle X|\varnothing\rangle$ (where $1 := [e]$ is generated by the empty set $\varnothing$).

2.3 Free Product

Definition 2.3.1 *Let $\{G_\alpha|\alpha \in \Delta\}$ be a family of groups. By a free product of G_α's, $\alpha \in \Delta$ we mean a group P and a collection of homomorphisms $\varphi_\alpha : G_\alpha \to P$ with the following UMP: Given a collection of homomorphisms $f_\alpha : G_\alpha \to G$, then there exists a unique homomorphism $\varphi : P \to G$ satisfying $\varphi\varphi_\alpha = f_\alpha$, $\forall \alpha \in \Delta$. i.e the following diagram is commutative.*

$$\begin{tikzcd} & P \arrow[d, dotted, "\varphi(unique)"] \\ G_\alpha \arrow[ur, "\varphi_\alpha"] \arrow[r, "f_\alpha"'] & G \end{tikzcd}$$

We use the notation $\mathop{\ast}\limits_{\alpha} G_\alpha$ for the free product of G_α's.

Theorem 2.3.2 *If P and P' are both free products of $\{G_\alpha|\alpha \in \Delta\}$, then $P \cong P'$.*

Proof: Similar to the proof of Theorem 2.1.2. $\square$

Lemma 2.3.3 *In the last diagram, φ_α's are monomorphisms.*

Proof: Fix $\alpha_0 \in \Delta$, take $G = G_{\alpha_0}$, then the following diagram commutes,

$$\begin{tikzcd} & P \arrow[d, "\exists!\varphi"] \\ G_\alpha \arrow[ur, "\varphi_\alpha"] \arrow[r, "f_\alpha"'] & G_{\alpha_0} \end{tikzcd}$$

where $f_\alpha = \begin{cases} 1_{G_{\alpha_0}} & ; \alpha = \alpha_0 \\ e \text{ (the identity in } G_{\alpha_0}) & ; \alpha \neq \alpha_0. \end{cases}$
Hence $\varphi\varphi_{\alpha_0} = 1_{G_{\alpha_0}}$ and thus φ_{α_0} is one-to-one. $\square$

Theorem 2.3.4 *For every family $\{G_\alpha : \alpha \in \Delta\}$ of groups, their free product exists.*

Proof: Let $U := \underset{\alpha\in\Delta}{\cup} G_\alpha$, consider $(W(U), \cdot)$ where $W(U)$ is the family of words on U including the empty word e together with the juxtaposition multiplication. We define the reduced word $\bar{w}$ obtained from the word w to be a 'new' word deduced by the following steps:
• If two consecutive elements in w belong to the same group, then we replace them by their product in their group.
• We remove the identity elements from the reduced word obtained from the first step.
We repeat this process (finitely many times) until we have a word which has no consecutive elements that belong to the same group nor identity elements. For example, if $x_1, x_2 \in G_1$, and $y_1, y_2 \in G_2$ s.t $y_1.y_2 = y_3 \in G_2$, then;

$$\begin{tikzcd}
w = x_1y_1x_2^{-1}x_2y_2y_3^{-1}1_{G_1} \arrow[d, Rightarrow, "1"] & x_1y_3y_3^{-1} \arrow[r, Rightarrow, "4"] & x_1 1_{G_2} \arrow[d, Rightarrow, "5"] \\
x_1y_1 1_{G_1} y_2 y_3^{-1} 1_{G_1} \arrow[r, Rightarrow, "2"] & x_1y_1y_2y_3^{-1} \arrow[u, Rightarrow, "3"] & \begin{array}{c} x_1 = \bar{w} \\ \text{(the reduced word)} \end{array}
\end{tikzcd}$$

Let's agree that if $\nu = 1_{G_{\alpha_1}} \ ... \ 1_{G_{\alpha_m}}$, then $\bar{\nu} = e$.
Now we define new equivalence relation $\sim$ on $W(U)$ as the following; $w \sim \nu$ iff $\bar{w} = \bar{\nu}$. Let $[w]$ denote the equivalence class of w, then we notice that every equivalence class contains exactly one reduced word. Hence $\bar{w} \cdot \bar{\nu} = \overline{w \cdot \nu}$ and also $\overline{w}^{-1} = \overline{w^{-1}}$. This permits us to define a multiplication operation on the set of the equivalence classes such as $[w] \cdot [\nu] = [w\nu]$ and an inverse $[w]^{-1}$ to be $[w^{-1}]$.
Let P be the set of all reduced words. Thus $(P, \cdot)$ defines a group (the verification is similar to the one in the proof of Theorem 2.2.3). Now define a monomorphism $\varphi_\alpha : G_\alpha \to P$ by $g_\alpha \mapsto [g_\alpha]$. Let G be a given group and $f_\alpha : G_\alpha \to G$ be a family of homomorphisms. Let $\varphi : P \to G$ be s.t $\varphi([w]) = f_{\alpha_1}(g_{\alpha_1}) \ ... \ f_{\alpha_m}(g_{\alpha_m})$, where $\bar{w} = g_{\alpha_1} \ ... \ g_{\alpha_m}$, hence φ is well-defined and homomorphism. If $\varphi' : P \to G$ is a homomorphism s.t $\varphi'\varphi_\alpha = f_\alpha$ for all α, since $\bar{w}$ has a unique expression of elements from G_α's, hence φ' must equal φ. Therefore, φ is unique. $\square$

Chapter 3

The van-Kampen Theorem and some Applications

In order to calculate the fundamental group of more spaces, we study this important theorem.

3.1 van-Kampen Theorem

Before we state and prove the theorem, we introduce some important definitions and theorems. The references for this section are [16] and [18].

Definition 3.1.1 *Consider the following diagram of groups A, B and C and homomorphisms i and j.*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & B \\ C & \end{tikzcd}$$

A solution (G, f, g) is a data of a group G and homomorphisms f and g which complete the above diagram to be a commutative diagram. That is to have the following commutative diagram.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & G \end{tikzcd}$$

Definition 3.1.2 *A pushout is a solution (P, p, q) satisfying the following UMP: For any other solution (G, f, g) there exists a unique homomorphism $\varphi : P \to G$ s.t the following diagram is commutative.*

$$
\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "f"] \arrow[ddr, "p"] & \\
C \arrow[r, "g"] \arrow[drr, "q"'] & G & \\
 & & P \arrow[ul, dotted, "\varphi"]
\end{tikzcd}
$$

Theorem 3.1.3 *If (P, p, q) and (P', p', q') are both pushouts for the same data, then there exists a unique isomorphism $P \to P'$.*

Proof: In the diagram in Definition 3.1.2 we replace (G, f, g) by (P', p', q') thus $\exists!\ \varphi : P \to P'$ so that $\varphi p = p'$ and $\varphi q = q'$. Similarly, since (P', p', q') is a pushout then $\exists!\ \varphi' : P' \to P$ so that $\varphi' p' = p$ and $\varphi' q' = q$. Since $\varphi'(\varphi p) = \varphi' p' = p$ and $\varphi'(\varphi q) = \varphi' q' = q$ i.e the following diagram is commutative.

$$
\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "p"] \arrow[ddr, "p"] & \\
C \arrow[r, "q"] \arrow[drr, "q"'] & P & \\
 & & P \arrow[ul, dotted, "\varphi'\varphi"]
\end{tikzcd}
$$

By the UMP of free product $\varphi'\varphi$ is unique, and it follows that $\varphi'\varphi = 1_P$. Similarly, $\varphi\varphi' = 1_{P'}$. Thus φ is a unique isomorphism. $\square$

The following theorem is one of two main theorems in this section, see [16].

Theorem 3.1.4 *For any data of groups and homomorphisms,*

$$
\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "j"'] & B \\
C &
\end{tikzcd}
$$

its pushout exists. Specifically, if $P = \frac{B*C}{N}$, $p(b) = bN$ and $q(c) = cN$, where N is the normal closure of the subgroup of $B*C$ generated by $\{i(a)j(a^{-1}); a \in A\}$, then (P, p, q) is the desired pushout. (We call P an amalgamated free product).

Proof: We check first whether (P, p, q) is a solution or not. For $a \in A$ we have $pi(a) = i(a)N = j(a)(i(a^{-1})j(a))^{-1}N = j(a)N = qj(a)$, so $pi = qj$ i.e (P, p, q) is a solution.

Now let (G, f, g) be an arbitrary solution, we need a unique homomorphism $\varphi : P \to G$ s.t the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "f"'] \arrow[ddr, "p"] & \\
C \arrow[r, "g"] \arrow[drr, "q"'] & G & \\
 & & P = \frac{B*C}{N} \arrow[ul, dotted, "\varphi"']
\end{tikzcd}$$

By the definition of free product $\exists! \psi : B * C \to G$ s.t $\psi|_B = f$ and $\psi|_C = g$. So if $b \in B$ and $c \in C$ then bc is a reduced word and hence $\psi(ab) = f(b)g(c)$ (by the unique construction of free product). Moreover, if $a \in A$ then $\psi(i(a)j(a^{-1})) = fi(a)gj(a^{-1}) = 1_G$ which implies that $i(a)j(a^{-1}) \in \ker\psi \Rightarrow N \leq \ker\psi$. Define $\varphi : B * C/N \to G$ by $\varphi(wN) = \psi(w)N$, where $w \in B * C$. Well-defined: if $wN = w'N \Rightarrow ww'^{-1} \in N \Rightarrow ww'^{-1} \in \ker\psi \Rightarrow \psi(w) = \psi(w') \Rightarrow \varphi(wN) = \varphi(w'N)$. Homomorphism: Because so is ψ.
Since $\varphi p(b) = \varphi(bN) = \psi(b)N = f(b)$, and $\varphi q(c) = \varphi(cN) = \psi(c)N = g(c)$ then the last diagram is commutative.
The only thing left is to prove the uniqueness of φ. Suppose that $\varphi' : P \to G$ is like φ, then we have the following diagram.

$$\begin{tikzcd}
 & B \arrow[d, "f"] \arrow[ddr, "p"] \arrow[dddrr, "p'"] & & \\
C \arrow[r, "g"] \arrow[drr, "q"] \arrow[ddrrr, "q'"] & G & & \\
 & & \frac{B*C}{N} \arrow[ul, "\varphi'"] & \\
 & & & B * C \arrow[ul, "\nu"]
\end{tikzcd}$$

Which implies the following commutative diagram.

$$\begin{tikzcd}
 & B * C \arrow[d, "\varphi'\nu(unique)"] \\
\{B, C\} \arrow[ur, "{p',q'}"] \arrow[r, "{f,g}"'] & G
\end{tikzcd}$$

Therefore, $\varphi'\nu = \psi = \varphi\nu$. Since ν is surjective then $\varphi' = \varphi$. $\square$

Let $X = X_1 \cup X_2$ be a path-connected space where X_1 and X_2 are open path-connected subsets of X, suppose that $X_0 = X_1 \cap X_2 \neq \varnothing$ is path-connected. Let $x_0 \in X_0$, then the following diagram of inclusions,

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d, hook] & X_1 \arrow[d, hook] \\ X_2 \arrow[r, hook] & X \end{tikzcd}$$

implies the following commutative diagram of the fundamental groups and induced homomorphisms.

$$\begin{tikzcd} \pi_1(X_0, x_0) \arrow[r, "i"] \arrow[d, "j"'] & \pi_1(X_1, x_0) \arrow[d, "p"] \\ \pi_1(X_2, x_0) \arrow[r, "q"'] & \pi_1(X, x_0) \end{tikzcd}$$

We turn now to the second main theorem in this section, where the proof closely follows the one in [18].

Theorem 3.1.5 *The solution* $(\pi_1(X, x_0), p, q)$ *is a pushout of the last diagram.*

Proof: Let (G, f_1, f_2) be a solution of the data.

$$\begin{tikzcd} \pi_1(X_0, x_0) \arrow[r, "i"] \arrow[d, "j"'] & \pi_1(X_1, x_0) \\ \pi_1(X_2, x_0) & \end{tikzcd}$$

We need to construct a unique homomorphism $\varphi : \pi_1(X, x_0) \to G$ s.t the following diagram becomes commutative.

$$\begin{tikzcd} \pi_1(X_0, x_0) \arrow[r, "i"] \arrow[d, "j"'] & \pi_1(X_1, x_0) \arrow[d, "f_1"] \arrow[dr, bend left, "p"] & \\ \pi_1(X_2, x_0) \arrow[r, "f_2"'] \arrow[rr, bend right, "q"'] & G & \pi_1(X, x_0) \arrow[l, dotted, "\varphi"'] \end{tikzcd}$$

Let γ be a loop in X based at x_0. By Lebesgue lemma, there exists a finite partition $\{y_i\}_{i=0}^m \subset X_0$ along γ s.t the segment τ_{i+1} on γ from y_i to y_{i+1} is contained in either X_1 or X_2. Define β_i to be a path in X_0 from x_0 to y_i. So for $0 \leq i \leq m$ the loop $\gamma_i := \beta_{i-1}\tau_i\beta_i^{-1}$ starts at x_0 and lies entirely within X_1 or X_2. Now for γ in X based at x_0, let $\{\gamma_i\}_{i=0}^m$ be a partition like above. Define $\varphi : \pi_1(X, x_0) \to G$ by $\langle\gamma\rangle \mapsto f_{\lambda_1}(\langle\gamma_1\rangle)...f_{\lambda_m}(\langle\gamma_m\rangle)$, where $\lambda_i = 1$ or 2 depending whether $\gamma_i \subset X_1$ or $\gamma_i \subset X_2$. If $\gamma_i \subset X_0$ then any choice of λ is fine.
In order to prove that φ is the desired homomorphism we have to check the following;

1. Is φ well-defined? That is;
   (a) Does the value of φ depend on the partitions of γ i.e the choice of points y_i and the paths β_i?
   (b) If $\langle\gamma\rangle = \langle\gamma'\rangle$, does $\varphi(\langle\gamma\rangle) = \langle\varphi(\gamma')\rangle$?
2. Is φ a homomorphism? And,
3. Is φ unique?

<u>Answers</u> **1**.(a) We show first that φ does not depend on the paths β_i's. Indeed, if $\alpha_i \subset X_0$ is another path from x_0 to y_i, then

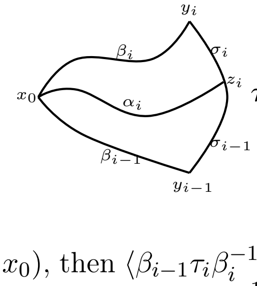
$$\langle\beta_{i-1}\tau_i\beta_i^{-1}\rangle = \langle\beta_{i-1}\tau_i\alpha_i^{-1}\rangle\langle\alpha_i\beta_i^{-1}\rangle \text{ and } \langle\beta_i\tau_{i+1}\beta_{i+1}^{-1}\rangle = \langle\beta_i\alpha_i^{-1}\rangle\langle\alpha_i\tau_{i+1}\beta_{i+1}^{-1}\rangle,$$

thus

$$f_{\lambda_0}(\langle\beta_{i-1}\tau_i\beta_i^{-1}\rangle)f_{\lambda_1}(\langle\beta_i\tau_{i+1}\beta_{i+1}^{-1}\rangle) =$$
$$f_{\lambda_0}(\langle\beta_{i-1}\tau_i\alpha_i^{-1}\rangle)f_{\lambda_0}\cancel{(\langle\alpha_i\beta_i^{-1}\rangle)}f_{\lambda_1}\cancel{(\langle\beta_i\alpha_i^{-1}\rangle)}f_{\lambda_1}(\langle\alpha_i\tau_{i+1}\beta_{i+1}^{-1}\rangle)$$

because $f_{\lambda_0} = f_{\lambda_1}$ in X_0. Hence, changing β_i's by any other paths α_i's $\subset X_0$ does not effect the value of φ.

Similarly, we show that φ is independent from the choice of y_i's.
Let $z_i \in X_0$ be a point on γ that lies between y_{i-1} and y_i as in the figure bellow.



Let $\langle\beta_{i-1}\tau_i\beta_i^{-1}\rangle \in \pi_1(X_{\lambda_0}, x_0)$, then $\langle\beta_{i-1}\tau_i\beta_i^{-1}\rangle = \langle\beta_{i-1}\sigma_{i-1}\alpha_i^{-1}\alpha_i\sigma_i\beta_i^{-1}\rangle = \langle\beta_{i-1}\sigma_{i-1}\alpha_i^{-1}\rangle\langle\alpha_i\sigma_i\beta_i^{-1}\rangle$, because $\langle\beta_{i-1}\sigma_{i-1}\alpha_i^{-1}\rangle$ and $\langle\alpha_i\sigma_i\beta_i^{-1}\rangle$ both belong

to $\pi_1(X_{\lambda_0}, x_0)$. Therefore, $f_{\lambda_0}(\langle\beta_{i-1}\tau_i\beta_i^{-1}\rangle) = f_{\lambda_0}(\langle\beta_{i-1}\sigma_{i-1}\alpha_i^{-1}\rangle)f_{\lambda_0}(\langle\alpha_i\sigma_i\beta_i^{-1}\rangle)$. Hence, adding (removing) proper finite points into (from) $\{y_i\}_{i=0}^m$ can't change the value of φ.
1.(b) Let $\gamma \underset{F}{\approx} \gamma'$ rel$\{0,1\}$, where $F : I \times I \to X$ s.t $F(s,0) = \gamma(s)$ and $F(s,1) = \gamma'(s)$. Since $I \times I$ is a compact subset of $\mathbb{R}^2$, by Lebesgue lemma there exists a finite partition of the square s.t the image of every small rectangle $R_i^j := [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ under F is contained entirely within X_1 or X_2. Hence the partition $\{s_i\}_{i=0}^m$ implies two partitions $\{\tau_i\}_{i=0}^m$ and $\{\tau_i'\}_{i=0}^m$ along γ and γ', respectively, s.t τ_i and τ_i' are contained in X_{λ_0}, $\forall i = 0, ..., m$. Thus $f_{\lambda_0}(\langle\beta_i'(\sigma_i\tau_{i+1}\sigma_{i+1}^{-1})\beta_{i+1}'^{-1}\rangle) = f_{\lambda_0}(\langle\beta_i'\tau_{i+1}'\beta_{i+1}'^{-1}\rangle)$, where σ_i is a path in X_0 from $\gamma'(s_i)$ to $\gamma(s_i)$. Repeat the same process in every single R_i^j we will have $\varphi(\langle\gamma\rangle) = \varphi(\langle\gamma'\rangle)$.
2. φ is a homomorphism:
$\varphi(\langle\gamma\gamma'\rangle) = f_{\lambda_1}(\langle\gamma_1\rangle)...f_{\lambda_m}(\langle\gamma_m\rangle)\cdot f_{\lambda_1'}(\langle\gamma_1'\rangle)...f_{\lambda_n'}(\langle\gamma_n'\rangle) = \varphi(\langle\gamma\rangle)\varphi(\langle\gamma'\rangle)$, where $\{\gamma\}_{i=1}^m$ and $\{\gamma'\}_{j=1}^n$ are two finite sequences of loops corresponding to the partitions of γ and γ', respectively.
3. Let $\varphi' : \pi_1(X, x_0) \to G$ be a homomorphism for which the last diagram commutes, let γ be a loop in X based at x_0 and let $\{y_i\}_{i=0}^m$ be a partition along γ, s.t $\gamma_i \subset X_\lambda \Rightarrow \varphi'(\langle\gamma\rangle) = \varphi'(\langle\gamma_1\rangle)...\varphi'(\langle\gamma_m\rangle)$. The commutative diagram implies that $\varphi'(\langle\gamma\rangle) = f_{\lambda_1}(\langle\gamma_1\rangle)...f_{\lambda_m}(\langle\gamma_m\rangle) = \varphi(\langle\gamma\rangle)$. Therefore, φ is unique. □

Corollary 3.1.6 (The van-Kampen Theorem) *Let X be as in Theorem 3.1.5, then $\pi_1(X) \cong \frac{\pi_1(X_1)*\pi_1(X_2)}{N}$, where N is the normal closure of the subgroup ($\leq \pi_1(X_1)*\pi_1(X_2)$) generated by the set $\{i(\langle\theta\rangle)j(\langle\theta^{-1}\rangle); \langle\theta\rangle \in \pi_1(X_0)\}$.*

Proof: Pushouts of the same data are isomorphic by Theorem 3.1.3. □

3.2 Applications

We turn now to some applications of the van-Kampen theorem. These applications are similar to the applications in [14].

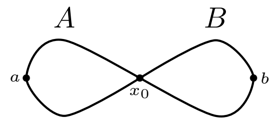
Theorem 3.2.1 *Let $n \geq 2$, then $\pi_1(\mathbb{S}^n) = 1$.*

Proof: Let $x_1 = (1, 0, ..., 0) \in \mathbb{S}^n$ and $x_2 = (-1, 0, ..., 0) \in \mathbb{S}^n$. So we have $X_1 = \mathbb{S}^n \setminus \{x_1\}$ and $X_2 = \mathbb{S}^n \setminus \{x_2\}$ which are path-connected open subsets of $\mathbb{S}^n$, and both are homeomorphic to $\mathbb{R}^n$ and $X_1 \cap X_2 \neq \varnothing$ is path-connected, thus van-Kampen Theorem applies: $\pi_1(X) \cong \frac{\pi_1(X_1)*\pi_1(X_2)}{N} \cong \frac{1*1}{N} = 1$. □

Theorem 3.2.2 *In van Kampen theorem. If X_0 is simply connected, then $\pi_1(X) \cong \pi_1(X_1) * \pi_1(X_2)$* □

Example 3.2.3 *Let X be the figure-eight. Then $\pi_1(X) = \mathbb{Z} * \mathbb{Z}$.*

Proof: Let x_0 be the intersection point of the closed curves A and B. Let $X_1 = X \setminus \{b\}$ and $X_2 = X \setminus \{a\}$, then X_1 and X_2 are path-connected open subset of X.



Thus $X_0 = X_1 \cap X_2 = X \setminus \{a, b\}$ is contractible. Hence we have, $\pi_1(X) = \pi_1(X_1) * \pi_1(X_2) \cong \pi_1(A) * \pi_1(B) \cong \mathbb{Z} * \mathbb{Z}$. $\square$

Example 3.2.4 *The fundamental group of a bouquet of n circles is $\underset{n}{\ast}\mathbb{Z}$.*

Theorem 3.2.5 *In the van-Kampen theorem, assume that $\pi_1(X_2) = 1$, then $p : \pi_1(X_1, x_0) \to \pi_1(X, x_0)$ is surjective; Moreover, $N = \ker p$.*

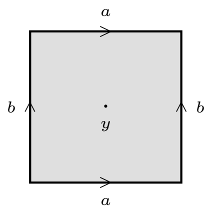
Proof: We have $\pi_1(X, x_0) = \frac{\pi_1(X_1, x_0)}{N}$. As j is trivial then N is the normal closure of Im i. Since $\pi_1(X)$ is a pushout then $\exists! \varphi : \pi_1(X) \to \frac{\pi_1(X_1)}{N}$, and the following diagram is commutative, where ν is the natural map $x \mapsto xN$.

$$\begin{tikzcd}
\pi_1(X_0) \arrow[r, "i"] \arrow[d, "0"'] & \pi_1(X_1) \arrow[d, "\nu"] \arrow[ddr, "p"] & \\
1 \arrow[r, "0"'] \arrow[drr, "q"'] & \frac{\pi_1(X_1)}{N} & \\
& & \pi_1(X) \arrow[ul, "\varphi"']
\end{tikzcd}$$

So $\varphi p = \nu$. Since ν is onto then p must be onto, and $\ker p \leq \ker \nu = N$. Since also $pi = 0 \Rightarrow \text{Im } i \leq \ker p \Rightarrow N \leq \ker p$. Since $\ker p$ is the normal closure of the subgroup generated by Im i (i.e the normal closure of Im i) we conclude that $N = \ker p$. $\square$

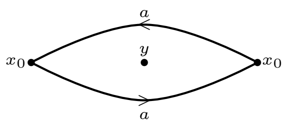
Applications

Example 3.2.6 *We already know that the fundamental group of the torus $T = \mathbb{S}^1 \times \mathbb{S}^1$ is $\mathbb{Z} \times \mathbb{Z}$, yet this result can be derived from the last theorem. First we view the torus as a square with opposite sides identified, as in the figure bellow;*



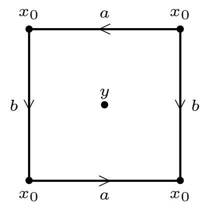
In order to apply Theorem 3.2.5, let X be the torus square, $X_1 = X \setminus \{y\}$, for some $y \in Int(X)$ and $X_2 = Int(X)$. Hence $X_1 \cup X_2 = X$.
X_1, and X_2 are open and path-connected and $X_0 := X_1 \cap X_2 \neq \varnothing$ is path-connected. So,
$\star$ *X_1 is a deformation retract to its boundary.*
$\star$ *X_2 is homeomorphic to the unit disc $\Rightarrow \pi_1(X_2) = 1$.*
$\star$ *$X_1 \cap X_2$ is a deformation retract to $\mathbb{S}^1$.*
Therefore, $\pi_1(X) = \frac{\pi_1(X_1)}{N}$, where N is the normal closure of Im i ($i : \pi_1(\mathbb{S}^1) \to \pi_1(X_1)$ induced from the inclusion). Since the boundary of X_1 is a bouquet of four circles $\{a, b, a^{-1}, b^{-1}\} \Rightarrow \pi_1(X_1) = \langle a, b, a^{-1}, b^{-1} | \varnothing\rangle$ $= \langle a, b|\varnothing\rangle \cong \mathbb{Z} * \mathbb{Z}$.
Since $\pi_1(\mathbb{S}^1)$ is a cyclic group generated by the homotopy class of a loop γ winds one time around 0, then $\pi_1(X_0) = \langle \gamma|\varnothing\rangle \Rightarrow i(\gamma) = aba^{-1}b^{-1}$ because γ will be embedded into the border of X_1, so it becomes the homotopy class of the loop winds once around y. Thus, Im i is the group generated by the set $\{aba^{-1}b^{-1}\}$. Therefore, $\pi_1(X) = \langle a, b|aba^{-1}b^{-1}\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$. (Example 2.2.9 ii).

Example 3.2.7 (The Real Projective Plane P^2) *The real projective n-space P^n is a quotient space obtained by identifying the antipodal points of the unit sphere in $\mathbb{R}^{n+1}$. That is, $P^n \simeq \mathbb{S}^n/\sim$ (homeomorphic), where $x \sim y$ iff $x = \pm y$, $\forall x \in \mathbb{S}^n$.*
So we consider the real projective plane P^2 as a 2-sided polygon with identifying opposite sides.



Let X be the P^2 polygon, $X_1 = X \setminus \{y\}$, and $X_2 = Int(X)$ ($\Rightarrow \pi_1(X_2) = 1$). One can see that $X_1 \cap X_2$ is homotopy equivalent to $\mathbb{S}^1$. We see that X_1 is a deformation retract to its boundary. Thus, $\pi_1(X_1) = \langle a, a|\varnothing\rangle = \langle a|\varnothing\rangle$. If γ is the homotopy class of the loop in $\mathbb{S}^1$ turns once around 0, then $\pi_1(X_0) = \langle \gamma|\varnothing\rangle$ and $i(\gamma) = a^2 \Rightarrow \pi_1(X) = \langle a|a^2 = 1\rangle \cong \mathbb{Z}_2$ (Im $i = 2\mathbb{Z}$ which is a normal subgroup of $\mathbb{Z} \Rightarrow \pi_1(X) = \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2$).

Example 3.2.8 (The Klein Bottle) *The Klein bottle can be obtained by identifying the opposite sides of the square $abab^{-1}$. Let X be Klein bottle square, $X_1 = X \setminus \{y\}$, and $X_2 = Int(X)$ ($\Rightarrow \pi_1(X_2) = 1$).*



Hence $X_1 \cap X_2$ is homotopic equivalent to $\mathbb{S}^1 \Rightarrow \pi_1(X_0) = \langle \gamma | \varnothing \rangle$ and $\pi_1(X_1) = \langle a, b, a, b^{-1} | \varnothing \rangle$ $(= \langle a, b | \varnothing \rangle)$, so $i(\gamma) = abab^{-1}$. Therefore, $\pi_1(X) = \langle a, b | abab^{-1} \rangle$.

3.3 Torus Knot

In this section we study the group of torus knots and we apply the van-Kampen theorem to determine whether two torus knots are equivalent or not. The references for this section are [8],[10] and [12].

Definition 3.3.1 *A knot K is a simple closed curve in $\mathbb{R}^3$. If K can be embedded on the surface of a torus T, then K is called a torus knot, that is $K : \mathbb{S}^1 \to T \subset \mathbb{R}^3$ (K is continuous by the definition).*

So K induces the homomorphism $\pi_1(\mathbb{S}^1) \xrightarrow{K_*} \pi_1(T)$, we know that $\pi_1(\mathbb{S}^1)$ is a free group with one generator a, say, and $\pi_1(T)$ is a free abelian group with two generators b, and c, say. If K winds horizontally on the surface of T m-times and vertically n-times, we write $K(m, n)$, then $K_*(a) = b^m c^n$. In fact, $gcd(m, n) = 1$, that is because $T \simeq \mathbb{R}^2/\mathbb{Z}^2$ and K can be thought of as a line through the origin in $\mathbb{R}^2$ with slope $\frac{m}{n}$, so if $gcd(m, n) \neq 1$ then K can't be a knot (can't be simple).

Definition 3.3.2 *Let K be a knot, then $\pi_1(\mathbb{R}^3 \setminus K)$ is called the group of the knot K and written $G(K)$.*

Definition 3.3.3 *Two knots K_1 and K_2 are said to be similar knots if there exists a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ s.t $h(K_1) = K_2$. If h is an orientation preserving homeomorphism then we say K_1 and K_2 are equivalent.*

Definition 3.3.4 *A knot K is unknotted if it is similar to an euclidean circle in a plane in $\mathbb{R}^3$, that is there exists a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ s.t $h(K) = \{(x, 0, 0); x \in \mathbb{R}\}$.*

Now if K_1 and K_2 are similar then $G(K_1) \cong G(K_2)$, thus if $G(K_1) \ncong G(K_2)$ then K_1 and K_2 are not similar.

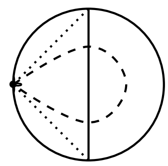
Theorem 3.3.5 *Let C be an euclidean circle in a plane in $\mathbb{R}^3$, then $\pi_1(\mathbb{R}^3 \setminus C) \cong \mathbb{Z}$.*

Proof: Without loss of generality we may choose coordinates s.t

$$\begin{aligned}
\mathbb{S}^2 &= \left\{(x,y,z) \in \mathbb{R}^3; |(x,y,z)| = 1\right\},\\
C &= \left\{(x,y,0); |(x,y,0)| = \frac{1}{2}\right\},\\
d &= \left\{(0,0,z); -1 \leq z \leq 1\right\}.
\end{aligned}$$

Points outside $\mathbb{S}^2$ deformation retract onto the surface $\mathbb{S}^2$ by the retraction $F(x,t) = (1-t)x + t\frac{x}{||x||}$, for $||x|| \geq 1$. We see that the points $\mathbb{S}^2 \setminus C$ can be pushed away from C toward the diameter d or $\mathbb{S}^2$. Now by moving the end points of d toward each other on the surface of $\mathbb{S}^2$ until d becomes a circle, we get the wedge sum $\mathbb{S}^2 \vee \mathbb{S}^1$. Hence $\pi_1(\mathbb{R}^3 \setminus C) \cong \pi_1(\mathbb{S}^2 \cup d) \cong \pi_1(\mathbb{S}^2 \vee \mathbb{S}^1)$.

Thus by the van-Kampen theorem $\pi_1(\mathbb{R}^3 \setminus C) \cong 1 * \mathbb{Z} \cong \mathbb{Z}$. (The generator of $\pi_1(\mathbb{R}^3 \setminus C)$ is the homotopy class of the loop starts somewhere away from the circle and passing through the origin of the circle).



□

Corollary 3.3.6 *If K is unknotted then $G(K) \cong \mathbb{Z}$.* □

Theorem 3.3.7 *Let $K(m,n)$ be a torus knot, then $G(K(m,n)) = \langle a, b | a^m b^{-n}\rangle$.*

Proof: let S be a solid torus in $\mathbb{R}^3$ and T be its surface, so that $K(m,n)$ lives on T. Pick $\varepsilon > 0$, let U be ε-neighborhood of K in $\mathbb{R}^3 \Rightarrow \pi_1(\mathbb{R}^3 - U) = \pi_1(\mathbb{R}^3 - K)$ (K is a deformation retract of U). Now $T - U$ is an annulus $A_{m,n}$ twisting m-times horizontally and n-times vertically, thus $\pi_1(A_{m,n})$ is an infinite cyclic with one generator. (Say, the middle line $l_{m,n}$).
Whereas $S - U$ can be deformation retracted to the meridian circle of T (horizontal circle), hence $\pi_1(S-U) = \langle a | \varnothing\rangle$. In order to apply the van-Kampen theorem, expand $S - U$ and $\mathbb{R}^3 \setminus S - U$ slightly so that they become open sets X_1 and X_2 respectively, and so that $X_0 = X_1 \cap X_2$ is a neighborhood of $A_{m,n}$, so by the van-Kampen theorem we have.

$$\begin{tikzcd}
\pi_1(X_0) \arrow[r, "\langle l_{a,m}\rangle \mapsto a^m"] \arrow[d, "\langle l_{m,n}\rangle \mapsto b^n"'] & \pi_1(X_1) \arrow[d] \\
\pi_1(X_2) \arrow[r] & \pi_1(X_1 \cup X_2) = \pi_1(\mathbb{R}^3 - K)
\end{tikzcd}$$

Therefore, $G(K) = \langle a, b | a^m b^{-n}\rangle$. □

Theorem 3.3.8 *If $K(m,n)$ and $K(m',n')$ are equivalent torus knots then $\{m,n\} = \{m',n'\}$.*

Proof: We have $G := G(K(m,n)) \cong G(K(m',n')) := G'$. By the last theorem $G = \langle a,b|a^m b^{-n}\rangle$, so $a^m = b^n \Rightarrow b^k a^m = b^k b^n = b^n b^k = a^m b^k \Rightarrow a^m \in Z(G)$ (the center of G). Let $N = \langle a^m \rangle$(the subgroup generated by a^m) $\Rightarrow N \leq Z(G) \Rightarrow N$ is a normal subgroup of G. Thus the quotient $G/N = \langle aN, bN|a^m N = b^n N = N\rangle \cong \langle a,b|a^m = b^n = 1\rangle \cong \mathbb{Z}_m * \mathbb{Z}_n$, which has a trivial center (in free groups $xy \neq yx$ unless x or y is the identity), hence $Z(G/N) = \{e\}$.
Claim: $Z(G) = N$. Proof; Let ν be the projection map $G \to G/N \Rightarrow \nu(Z(G)) \leq Z(G/N) = \{e\} \Rightarrow Z(G) \leq \ker\nu = N$, since $N \leq Z(G) \Rightarrow Z(G) = N$, and therefore $N \cong N'$.
Since the abelianization of isomorphic groups gives isomorphic abelian groups, then $G \cong G'$ implies $G/Z \cong G'/Z'$ which implies $\langle a,b|a^m = b^n, ab = ba\rangle \cong \langle a',b'|a'^m = b'^n, a'b' = b'a'\rangle \Rightarrow \mathbb{Z}_m \oplus \mathbb{Z}_n \cong \mathbb{Z}_{m'} \oplus \mathbb{Z}_{n'} \Leftrightarrow \{m,n\} = \{m',n'\}$ because $gcd(m,n) = 1$ and $gcd(m',n') = 1$. $\square$

Chapter 4

Covering Spaces

To continue to study the fundamental group, we study in this chapter some theory of covering spaces which is one of the main subjects in algebraic topology. An extensive introduction to covering spaces and the fundamental group can be found in [14].

4.1 Covering Spaces

We study in this section an essential collection of definitions and theorems of basic covering spaces. Several references are used in this section and the next section, namely; [7], [9], [10], and [12].

Definition 4.1.1 *A pair $(\tilde{X}, p)$ consisting of a space $\tilde{X}$ and a map $p : \tilde{X} \to X$ is said to be a covering space of X if each $x \in X$ has a neighborhood U s.t $p^{-1}(U) = \underset{\alpha\in\Delta}{\cup} S_\alpha$, where S_α's are pairwise disjoint open sets so that for all α we have $p|_{S_\alpha}$ is a homeomorphism $S_\alpha \to U$.*
Such U is called an elementary neighborhood (or evenly covered), and each S_α is called a sheet.
The set $p^{-1}(x)$ is called the fiber over $x \in X$ and it is a discrete set as a consequence of the definition (otherwise we can't get the homeomorphism).

Lemma 4.1.2 (Unique Lifting Property) *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space of X (so that $p(\tilde{x}_0) = x_0$), let (Y, y_0) be any connected pointed space and $f : (Y, y_0) \to (X, x_0)$ be any map. If there exists a map $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ s.t $p\tilde{f} = f$, then $\tilde{f}$ is unique.*

$$\begin{tikzcd}
& (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\
(Y, y_0) \arrow[ur, "\tilde{f}(unique)"] \arrow[r, "f"'] & (X, x_0)
\end{tikzcd}$$

Proof: Let $\tilde{\tilde{f}} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ satisfy $p\tilde{\tilde{f}} = f$. Set $A = \{y \in Y; \tilde{f} = \tilde{\tilde{f}}\}$ and $B = \{y \in Y; \tilde{f} \neq \tilde{\tilde{f}}\}$, hence Y is a disjoint union of A and B, since $y_0 \in A$ so if we prove that A and B are both open then by the connectedness of Y we conclude that $B = \varnothing$.
To this end, let $y \in A$ and let U be an elementary neighborhood of $f(y)$ and S be the sheet where $\tilde{f}(y) = \tilde{\tilde{f}}(y) \in S \Rightarrow y \in \underbrace{\tilde{f}^{-1}(S) \cap \tilde{\tilde{f}}^{-1}(S)}_{\text{open}} \subseteq A \Rightarrow A$ is open. While if $y \in B$ then $\tilde{f}(y) \in S_1$ and $\tilde{\tilde{f}}(y) \in S_2$ where $S_1 \cap S_2 = \varnothing$, hence $y \in \underbrace{\tilde{f}^{-1}(S_1) \cap \tilde{\tilde{f}}^{-1}(S_2)}_{\text{open}} \subseteq B \Rightarrow B$ is open. $\square$

Theorem 4.1.3 (Path Lifting Theorem) *If $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering space of X, and if $\alpha : I \to X$ is a path in X starting at x_0 then there exists a unique $\tilde{\alpha} : I \to \tilde{X}$ starting at $\tilde{x}_0$ and s.t $p\tilde{\alpha} = \alpha$.*

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ ([0,1], 0) \arrow[r, "\alpha"'] \arrow[ur, dotted, "\tilde{\alpha}(unique)"] & (X, x_0) \end{tikzcd}$$

Proof: <u>Case 1</u>: If α is contained entirely in an elementary neighborhood U, let S be the sheet contains $\tilde{x}_0$ and define $\tilde{\alpha} : (p|_S)^{-1} \circ \alpha$.
<u>General case</u>: Since I is a compact subset of $\mathbb{R}$, then by Lebesgue lemma we can find a finite partition $0 = t_0 < t_1 < ... < t_n = 1$ of $[0, 1]$ s.t $\alpha([t_{i-1}, t_i]) \subset U_i$, where U_i is an elementary neighborhood of $\alpha(t_i)$, $0 \leq i \leq n$. If $\alpha_1, \alpha_2, ..., \alpha_n$ is a partition along α, s.t α_i starts at $\alpha(t_{i-1})$ and terminates at $\alpha(t_i)$. Then each α_i is contained entirely in U_i. By case 1, $\exists \tilde{\alpha}_i : I \to \tilde{X}$ so that $p\tilde{\alpha}_i = \alpha_i$. Since $\tilde{\alpha}_1(0) = \tilde{x}_0$ and $\tilde{\alpha}_{i+1}$ starts from the point which $\tilde{\alpha}_i$ ends in, then $\tilde{\alpha} := \tilde{\alpha}_1, ...\tilde{\alpha}_n$ is a desired path. It is unique by Lemma 4.1.2.
$\square$

Theorem 4.1.4 (Homotopy Lifting Theorem) *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space of X,and let $F : I \times I \to X$ be a map s.t $F(0, 0) = x_0$. Then there exists a unique $\tilde{F} : I \times I \to \tilde{X}$ so that $\tilde{F}(0, 0) = x_0$ and $p\tilde{F} = F$.*

Proof: The construction of $\tilde{F}$ is similar to the one of $\tilde{\alpha}$ in Theorem 4.1.3. Since $I \times I$ is compact, then there exist two partitions of I,
$0 = s_0 < s_1 < ... < s_m = 1$ and $0 = t_0 < t_1 < ... < t_n = 1$, so that
$F([s_i, s_{i+1}] \times [t_j, t_{j+1}]) \subset U_{ij}$, where U_{ij} is an elementary neighborhood of $F(s_i, t_j)$, $\forall i, j$.

Starting from the rectangle $[0, s_1] \times [0, t_1]$, then we can define $\tilde{F}$ over it, because $\tilde{F}([0, s_1] \times [0, t_1]) \subset U_{00}$, next we extend $\tilde{F}$ consecutively over $[s_i, s_{i+1}] \times [0, t_1]$ for $i = 2, ..., m$ in a similar manner we used in Theorem 4.1.3, which guarantees us that the definitions of $\tilde{F}$ agree on the common edges of consecutive rectangles. Therefore, $\tilde{F}$ is defined over $I \times [0, t_1]$. In the same way we extend it over $I \times [t_1, t_2]$, and so forth.
The uniqueness follows from the uniqueness property Lemma 4.1.2. □

Corollary 4.1.5 (Monodromy Theorem) *Let α_0 and α_1 be two paths in X both start at $x_0 \in X$. If $\alpha_0 \underset{F}{\approx} \alpha_1$ rel $\{0,1\}$, then $\tilde{\alpha}_0 \underset{\tilde{F}}{\approx} \tilde{\alpha}_1$ rel $\{0,1\}$. In particular, $\tilde{\alpha}_0$ and $\tilde{\alpha}_1$ terminate at the same point in $\tilde{X}$.* □

Corollary 4.1.6 *The induced homomorphism $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is one-to-one.*

Proof: Let β be a loop in $\tilde{X}$ at e_{x_0} s.t $p_*(\langle\beta\rangle) = \langle x_0\rangle \Rightarrow p \circ \beta \approx p \circ e_{\tilde{x}_0}$ rel $\{0,1\}$ (where e_{x_0} and $e_{\tilde{x}_0}$ are the constant loops based at x_0 and $\tilde{x}_0$ respectively), hence $\widetilde{p \circ \beta} \approx \widetilde{p \circ e_{\tilde{x}_0}}$ rel $\{0,1\}$. Thus by the unique lifting property we get $\beta \approx e_{\tilde{x}_0}$ rel $\{0,1\}$ hence $\ker p_*$ is trivial, i.e p_* is injective. □

Lemma 4.1.7 *Let $p : \tilde{X} \to X$ be a covering space of X. Suppose that X and $\tilde{X}$ are path-connected. Then $\forall x \in X$ the fibers $p^{-1}(x)$ have the same cardinality.*

Proof: Let $x_1, x_2 \in X$, let α be a path in X from x_1 to x_2. Define a map $\phi_1 : p^{-1}(x_1) \to p^{-1}(x_2)$ by $\tilde{x}_1 \mapsto \tilde{\alpha}(1) := \tilde{x}_2$, where $\tilde{\alpha}$ is the lifting path of α to $\tilde{X}$ starting at $\tilde{x}_1$ and let $\tilde{x}_2 \in p^{-1}(x_2)$ be its terminal point. Hence ϕ_1 is well-defined and one-to-one. Similarly we define $\phi_2 : p^{-1}(x_2) \to p^{-1}(x_1)$ by $\tilde{x}_2 \mapsto \widetilde{\alpha^{-1}}(1) := \tilde{x}_2 \Rightarrow \phi_2 = \phi_1^{-1}$ then also ϕ_2 is well-defined and one-to-one, and $\phi_1 = \phi_2^{-1}$ i.e ϕ_i is a bijection. □

4.2 The Fundamental Group of Covering Spaces

Theorem 4.2.1 (Classification Theorem) *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a path-connected covering space of X. Then the collection*

$$\left\{p_*\left(\pi_1(\tilde{X}, \tilde{x})\right); \tilde{x} \in p^{-1}(x_0)\right\}$$

of subgroups forms a conjugacy class of subgroups of $\pi_1(X, x_0)$.

Proof: Fix $\tilde{x}_0 \in p^{-1}(x_0)$, and let $H := p_*\Big(\pi_1(\tilde{X}, \tilde{x}_0)\Big)$. Let $\tilde{x}_1 \in p^{-1}(x_0)$, and α be a path in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_1$, hence $p\alpha$ is a loop in X based at x_0.
Define $\varphi : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(\tilde{X}, \tilde{x}_1)$ by $\langle\beta\rangle \mapsto \langle\alpha^{-1}\beta\alpha\rangle$,
and $\mu : \pi_1(X, x_0) \to \pi_1(X, x_0)$ by $\langle\gamma\rangle \mapsto \langle p\alpha\rangle^{-1}\langle\gamma\rangle\langle p\alpha\rangle$.
Consider the diagram

$$\begin{tikzcd}
\pi_1(\tilde{X}, \tilde{x}_0) \arrow[r, "p_*"] \arrow[d, "\varphi"'] & \pi_1(X, x_0) \arrow[d, "\mu"] \\
\pi_1(\tilde{X}, \tilde{x}_1) \arrow[r, "p_*"'] & \pi_1(X, x_0)
\end{tikzcd}$$

Since $p_*\varphi(\langle\beta\rangle) = p_*(\langle\alpha^{-1}\beta\alpha\rangle) = \langle p\alpha^{-1} \cdot p\beta \cdot p\alpha\rangle = \langle p\alpha\rangle^{-1}p_*(\langle\beta\rangle)\langle p\alpha\rangle = \mu p_*(\langle\beta\rangle)$ then the diagram above is commutative. Since φ is an isomorphism, then $p_*\Big(\pi_1(\tilde{X}, \tilde{x}_1)\Big) \cong \mu(H) = \langle p\alpha\rangle^{-1}H\langle p\alpha\rangle$ .... $(\star)$
Hence $\Big\{p_*\Big(\pi_1(\tilde{X}, \tilde{x})\Big) ; \tilde{x} \in p^{-1}(x_0)\Big\} \subseteq Cl(H)$.
Now consider the element $\langle\gamma\rangle^{-1}H\langle\gamma\rangle$, where $\langle\gamma\rangle \in \pi_1(X, x_0)$. Let $\tilde{\gamma}$ be the lifting path of γ to $\tilde{X}$ so that $\tilde{\gamma}(0) = \tilde{x}_0$, and let $\tilde{x} = \tilde{\gamma}(1) \Rightarrow \tilde{x} \in p^{-1}(x_0) \overset{\star}{\Rightarrow}$ $\langle\gamma\rangle^{-1}H\langle\gamma\rangle \cong p_*\Big(\pi_1(\tilde{X}, \tilde{x})\Big) \Rightarrow Cl(H) \subseteq \Big\{p_*\Big(\pi_1(\tilde{X}, \tilde{x})\Big) ; \tilde{x} \in p^{-1}(x_0)\Big\}$. i.e $Cl(H) = \Big\{p_*\Big(\pi_1(\tilde{X}, \tilde{x})\Big) ; \tilde{x} \in p^{-1}(x_0)\Big\}$. □

Theorem 4.2.2 *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space of X. Suppose that $\tilde{X}$ is simply connected, then the function $\pi_1(X, x_0) \to p^{-1}(x_0)$ given by $\langle\gamma\rangle \mapsto \tilde{\gamma}(1)$ is a bijection.*

Proof: Well-defined: If $\langle\gamma\rangle = \langle\gamma'\rangle$ then $\gamma \approx \gamma'$ rel $\{0, 1\}$ and by the monodromy theorem $\gamma(1) = \gamma'(1)$.
Onto: Let $\tilde{x} \in p^{-1}(x_0)$ and let α be a path in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}$, then $p\alpha$ is a loop in X based at $x_0 \Rightarrow \langle p\alpha\rangle \mapsto \alpha(1) = \tilde{x}$.
One-to-one: Let γ_1 and γ_2 be two loops in X at x_0 so that $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$ thus $\tilde{\gamma}_1 \cdot \tilde{\gamma}_2^{-1}$ is a loop in $\tilde{X}$ based at $\tilde{x}_0$. Since $\tilde{X}$ is simply connected, so if $\tilde{\gamma}_1 \cdot \tilde{\gamma}_2^{-1} \underset{F}{\approx} e_{\tilde{x}_0}$ rel$\{0, 1\}$ then $\gamma_1 \cdot \gamma_2^{-1} \underset{pF}{\approx} e_{x_0}$ rel$\{0, 1\}$. Thus $\langle\gamma_1 \cdot \gamma_2^{-1}\rangle = \langle e_{x_0}\rangle \Rightarrow \langle\gamma_1\rangle = \langle\gamma_2\rangle$. □

Example 4.2.3 *The projection map $p : \mathbb{S}^n \to P^n$ where P^n is the real projective n-space defines a covering space of P^n (two sheets for every elementary neighborhood. Here any small neighborhood is elementary, but the open set which is about $\frac{3}{4}$ of $\mathbb{S}^n$ is not elementary, for instance).*
However, for $n \geq 2$, $\mathbb{S}^n$ is simply connected. Hence if x is the north pole and y is the south pole of $\mathbb{S}^n$, there is a one to one correspondence between $\pi_1(P^n, p(x))$ and $p^{-1}(p(x)) = \{x, y\} \overset{4.2.2}{\Longrightarrow} \pi_1(P^n) \cong \mathbb{Z}_2$ for all $n > 1$.

4.3 Lifting Criterion

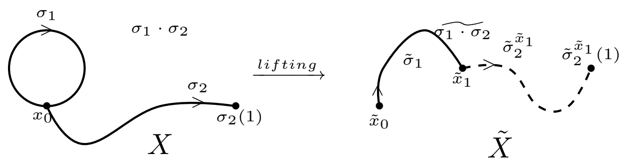
Theorem 4.3.1 *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space of X, and let Y be connected and locally path-connected. If $f : (Y, y_0) \to (X, x_0)$ is an arbitrary map, then there exists a lifting $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ of f iff $f_* \left(\pi_1(Y, y_0)\right) \leq p_* \left(\pi_1(\tilde{X}, \tilde{x}_0)\right)$.*

$$\begin{tikzcd}
 & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\
(Y, y_0) \arrow[ur, dotted, "\tilde{f}"] \arrow[r, "f"'] & (X, x_0)
\end{tikzcd}$$

Proof: ($\Rightarrow$) We have $p_* \tilde{f}_* = f_* \Rightarrow f_*(\pi_1(Y, y_0)) \leq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.
($\Leftarrow$) Let $y \in Y$ and let α be a path in Y from y_0 to y, then $f\alpha$ is a path in X from x_0 to $f(y)$ and it has a unique lifting $\widetilde{f\alpha}$ to $\tilde{X}$ so that $\widetilde{f\alpha}(0) = \tilde{x}_0$. Define $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ by $y \mapsto \widetilde{f\alpha}(1)$.
<u>Claims</u>:

1. $p\tilde{f} = f$.
2. $\tilde{f}$ is well-defined.
3. $\tilde{f}$ is continuous.

<u>Proofs</u>:
1. We have $p\tilde{f}(y) = p\widetilde{f\alpha}(1) = f\alpha(1) = f(y)$.
2. Let α_1 and α_2 be different paths in Y from y_0 to y, so $\alpha_1\alpha_2^{-1}$ is a loop in Y based at y_0, and $f_*(\langle\alpha_1\alpha_2^{-1}\rangle) \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ (by hypotheses), hence $\langle f \circ (\alpha_1\alpha_2^{-1})\rangle = \langle p\beta \rangle$, for some loop β in $\tilde{X}$ based at x_0. Thus $\langle f\alpha_1 \cdot f\alpha_2^{-1}\rangle = \langle p\beta\rangle \Rightarrow f\alpha_1 \cdot f\alpha_2^{-1} \approx p\beta \text{ rel}\{0,1\} \Rightarrow f\alpha_1 \approx p\beta \cdot f\alpha_2 \text{ rel}\{0,1\} \xrightarrow{4.1.5} \widetilde{f\alpha_1}(1) = \widetilde{p\beta \cdot f\alpha_2}(1)$.



By the figure above, we see when $\tilde{\sigma}_1$ is a loop in $\tilde{X}$ then $\tilde{\sigma}_2^{\tilde{x}_1}$ is just $\tilde{\sigma}_2$. In our situation we have $\beta(= \widetilde{p\beta})$ is a loop $\tilde{X}$. Hence $\widetilde{p\beta \cdot f\alpha_2}(1) = \widetilde{f\alpha_2}(1) \Rightarrow \tilde{f}$ is well-defined.
3. Let $y \in Y$ and let $U \subseteq X$ be an elementary neighborhood of $f(y)$. Let $S \subseteq \tilde{X}$ be the sheet containing $\tilde{f}(y)$. Let V be a path-connected open neighborhood of y so that $f(V) \subseteq U$. For any $y' \in V$ define a path from y_0

to y' in Y by $\tau\sigma$, where τ is a path in Y from y_0 to y and σ is in V from y to y'. Thus $f \circ (\tau\sigma)$ is a path in X from x_0 to $f(y') \in U$, so this path has a unique lifting path $\widetilde{f \circ (\tau\sigma)}$ to $\tilde{X}$ from $\tilde{x}_0$ to $\widetilde{f \circ (\tau\sigma)}(1) = \tilde{f}(y')$, but $\widetilde{f \circ (\tau\sigma)}(1) \in S \Rightarrow \tilde{f}(V) \subseteq S$. Since also $\widetilde{f \circ (\tau\sigma)}(1) = (p|_U)^{-1}(f(y'))$, then $\tilde{f}|_V = (p|_U)^{-1} f$ thus $\tilde{f}$ is continuous in V, which implies the continuity at y. □

Corollary 4.3.2 *Suppose that Y in Theorem 4.3.1 is simply connected, then for any map $f : (Y, y_0) \to (X, x_0)$ there exists a unique lifting map $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ so that $p\tilde{f} = f$.* □

Corollary 4.3.3 *Let $p_1 : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and $p_2 : (\tilde{\tilde{X}}, \tilde{\tilde{x}}_0) \to (X, x_0)$ be two simply connected covering spaces of X. Then there is a unique homeomorphism $\varphi : (\tilde{X}, \tilde{x}_0) \to (\tilde{\tilde{X}}, \tilde{\tilde{x}}_0)$ so that $p_1\varphi = p_2$.*

Proof: We have this situation;

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p_1"] \arrow[dl, bend left, "\exists!\varphi'"] \\ (\tilde{\tilde{X}}, \tilde{\tilde{x}}_0) \arrow[r, "p_2"'] \arrow[ur, bend left, "\exists!\varphi"] & (X, x_0) \end{tikzcd}$$

Therefore, $p_1\varphi\varphi' = p_2\varphi' = p_1$ and $p_2\varphi'\varphi = p_1\varphi = p_2 \Rightarrow \varphi = \varphi'^{-1}$ and $\varphi^{-1} = \varphi'$ hence φ is a homeomorphism and it is unique. □

4.4 Universal Covering Space

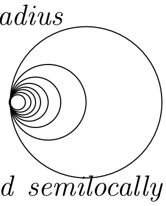
The results in this section are based on [9] and [10].

Definition 4.4.1 *Two covering spaces $p_1 : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and $p_2 : (\tilde{\tilde{X}}, \tilde{\tilde{x}}_0) \to (X, x_0)$ are said to be equivalent if there is a homeomorphism $\varphi : (\tilde{X}, \tilde{x}_0) \to (\tilde{\tilde{X}}, \tilde{\tilde{x}}_0)$ so that $p_1\varphi = p_2$.*

It is a straightforward consequence of Corollary 4.3.3 that if X has a simply connected covering space then it is unique up to homeomorphism. We call this covering space *a universal covering space*.
⋆ Does any space X have a simply connected covering space?
First, assume that X has a simply connected covering space $p : \tilde{X} \to X$, then every $x \in X$ has a neighborhood U which is homeomorphic to an open subset $S \subseteq \tilde{X}$.

Now any loop γ in U has a lifting $\tilde{\gamma}$ to S. But $\tilde{\gamma}$ is nullhomotopic in $\tilde{X}$, implying that $p\tilde{\gamma} = \gamma \subset U$ is nullhomotopic in X. Thus the homomorphism $\pi_1(U, x) \to \pi_1(X, x)$ induced by the inclusion is indeed trivial.
A space X which has the property above is called **semilocally simply connected**.

Example 4.4.2 *Let $X = \bigcup_{n=1}^{\infty} S_n$, where S_n is a circle with radius $\frac{1}{n}$ and center $(\frac{1}{n}, 0)$. Then any neighborhood of $(0,0)$ must contain a circle with some radius which can't be shrunken into a point, hence X has no universal covering space.*



Theorem 4.4.3 *If X is connected, locally path-connected and semilocally simply connected, then X has a universal covering space.*

Proof: Fix $x_0 \in X$ and define $\tilde{X} := \{[\alpha]; \alpha \text{ is a path in } X \text{ s.t } \alpha(0) = x_0\}$, let $\tilde{x}_0 = [x_0]$. Define $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ by $[\alpha] \mapsto \alpha(1)$ (well-defined by the definition of the equivalence relation of paths with end points fixed). Let U a neighborhood of $\alpha(1)$ and define a subset $[U]_\alpha \subseteq \tilde{X}$ by,

$$[U]_\alpha := \{[\alpha \cdot \beta]; \beta \text{ is a path in } U s.t \beta(0) = \alpha(1)\}.$$

Let α' be a another path in X that starts at x_0 and let V be a neighborhood of $\alpha'(1)$. Suppose that $[\sigma] \in [U]_\alpha \cap [V]_{\alpha'}$, then $[\sigma] = [\alpha \cdot \beta] = [\alpha' \cdot \beta']$ for some $\beta \subset U$ and $\beta' \subset V$ which implies that $\beta(1) = \beta'(1)$. Therefore, $U \cap V$ is a neighborhood of $\sigma(1)$ and

$$[U \cap V]_\sigma = \{[\sigma \cdot \tau]; \tau \subset U \cap V, \tau(0) = \sigma(a)\}. \text{ Since}$$
$$[\sigma \cdot \tau] = [(\alpha \cdot \beta) \cdot \tau] = [(\alpha' \cdot \beta') \cdot \tau], \text{ then}$$
$$[\sigma \cdot \tau] = [\alpha \cdot (\beta \cdot \tau)] = [\alpha' \cdot (\beta' \cdot \tau)]. \text{ Thus}$$
$$[U \cap V]_\sigma \subseteq [U]_\alpha \cap [V]_{\alpha'},$$

so the $\{[U]_\alpha\}$'s form a basis for a topology on $\tilde{X}$. Furthermore, $p([U]_\alpha)$ is the path component of U containing $\alpha(1)$, which is open (a space is locally path-connected iff the path component of each open subset is open), hence p is open and hence continuous. Given $x \in X$, let U_α be a path-connected open neighborhood of x s.t any loop in U is contractible to x. Since $p^{-1}(U) = \{[U]_\alpha; \alpha(1) \in U\}$, if $[U]_\alpha \cap [U]_{\alpha'} \neq \varnothing$ we have $[U]_\alpha = [U]_{\alpha'}$. In addition, $p([U_\sigma])$ is a path component of U i.e U. If $[\alpha \cdot \beta] = [\alpha \cdot \beta']$ then $\beta(1) = \beta'(1)$ thus by the choice of U, $[\beta] = [\beta']$ and $[\alpha \cdot \beta] = [\alpha \cdot \beta']$. Hence, $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering space. For α a path in X s.t $\alpha(0) = x_0$ let $\alpha_t(s) = \alpha(st)$, $s, t \in I \Rightarrow (\star)$ $\tilde{\alpha} : t \mapsto [\alpha_t]$ is a path from $\tilde{x}_0$ to $[\alpha]$ in $\tilde{X} \Rightarrow \tilde{X}$ is path-connected. Moreover, $p\tilde{\alpha} = \alpha$.
To show that $\tilde{X}$ is simply connected it suffices to show that $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = 1$ (because p_* is injective). Let γ be a loop in $\tilde{X}$ based at $\tilde{x}_0$, and let $\alpha = p\gamma$, hence α is a loop in X based at x_0 and $\tilde{\alpha} = \gamma$ (by the unique lifting property). By $(\star)$ $\tilde{\alpha}$ is a path from $[e_{x_0}]$ to $[\alpha]$, so $[e_{x_0}] = [\alpha] = \langle p\gamma \rangle$. $\square$

4.5 Covering Transformations and Group Actions

The main references for this section are [10] and [12].

Definition 4.5.1 *Let $p : \tilde{X} \to X$ be a covering space. A homomorphism $\phi : \tilde{X} \to \tilde{X}$ is a deck transformation if $p\phi = p$,*

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\phi"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

i.e ϕ is a homeomorphism preserving the fiber.

These homeomorphisms form a group under composition operation, and we denote this group by $\tilde{G}$.
If $\tilde{X}$ is connected, then by the unique lifting property, the only map preserves the base point is the identity.

Definition 4.5.2 *Let $p : \tilde{X} \to X$ be a covering space. If for all $x \in X$ and $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x)$ exists a deck transformation ϕ s.t $\phi(\tilde{x}_1) = \tilde{x}_2$. Then the covering space is called normal (or regular or Galois).*

Theorem 4.5.3 *Let X be connected and locally path-connected space, and let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be path-connected covering space. Let $G := p_*(\pi_1(\tilde{X}, \tilde{x}_0))$, and N be the normalizer of G in $\pi_1(X, x_0)$. Then:*

1. *The covering space is normal iff $G \triangleleft \pi_1(X, x_0)$.*
2. *$\tilde{G} \cong N/G$.*

Proof: 1. Let $\langle\gamma\rangle \in \pi_1(X, x_0)$. Since $\langle\gamma\rangle^{-1} p_*\Big(\pi_1(\tilde{X}, \tilde{x}_0)\Big)\langle\gamma\rangle = p_*\Big(\pi_1(\tilde{X}, \tilde{\gamma}(1))\Big)$. Here $\tilde{\gamma}$ is a path in $\tilde{X}$ starts at $\tilde{x}_0$ where $\gamma = p\tilde{\gamma}$. Thus $\langle\gamma\rangle \in N$ iff $p_*\Big(\pi_1(\tilde{X}, \tilde{x}_0)\Big) = p_*\Big(\pi_1(\tilde{X}, \tilde{\gamma}(1))\Big)$ iff (by the lifting criterion) there exists a deck transformation $\phi_\gamma : (\tilde{X}, \tilde{x}_0) \to (\tilde{X}, \tilde{\gamma}(1))$. Thus, the covering space is normal iff $N = \pi_1(X, x_0)$ iff $G \triangleleft \pi_1(X, x_0)$.
2. Define $\varphi : N \to \tilde{G}$, $\varphi(\langle\gamma\rangle) = \phi_\gamma$ (the deck transformation s.t $\phi_\gamma(\tilde{x}_0) = \tilde{\gamma}(1)$). Hence $\varphi(\langle\gamma \cdot \beta\rangle) = \phi_{\gamma\cdot\beta}$ (the deck transformation s.t $\phi_{\gamma\cdot\beta}(\tilde{x}_0) = \widetilde{\gamma \cdot \beta}(1)$), but $\widetilde{\gamma \cdot \beta} = \tilde{\gamma} \cdot (\phi_\gamma \tilde{\beta})$ (because $\widetilde{\gamma \cdot \beta} = \tilde{\gamma} \cdot \tilde{\beta}^{\tilde{\gamma}(1)}$, and $\tilde{\beta}^{\tilde{\gamma}(1)} = \phi_\gamma \tilde{\beta}$, by the uniqueness of ϕ_γ) and $\tilde{\gamma} \cdot (\phi_\gamma \tilde{\beta})$ is a path in $\tilde{X}$ from $\tilde{x}_0$ to $\phi_\gamma(\tilde{\beta}(1)) = \phi_\gamma \phi_\beta(x_0) \Rightarrow \phi_{\gamma\beta} = \phi_\gamma \phi_\beta \in \tilde{G}$, hence φ is a homomorphism. By the proof of part 1 in this theorem, $\phi_\gamma \in \tilde{G}$ iff $p_*\Big(\pi_1(\tilde{X}, \tilde{x}_0)\Big) = p_*\Big(\pi_1(\tilde{X}, \tilde{\gamma}(1))\Big)$ iff $\langle\gamma\rangle \in N$. Thus, φ is surjective. Now, $\varphi(\langle\sigma\rangle) = 1_{\tilde{G}}$ iff $\tilde{\sigma}$ is a loop in $\tilde{X}$ based at $\tilde{x}_0$ iff $p_*(\langle\sigma\rangle) \in G$, hence $\ker\varphi = G$. Therefore, by the first isomorphism theorem $\tilde{G} = N/G$. $\square$

Corollary 4.5.4 *If $\tilde{X}$ is a normal covering space of X, then $\tilde{G} \cong \pi_1(X, x_0)/G$.*

Proof: By the proof of part 1 of the last theorem, $\tilde{X}$ is normal iff $N = \pi_1(X, x_0)$, and then by part 2, $\tilde{G} \cong \pi_1(X, x_0)/G$. $\square$

Corollary 4.5.5 *If $(\tilde{X}, \tilde{x}_0)$ in Theorem 4.5.3 is simply connected, then the normalizer $N = \pi_1(X, x_0)$ (i.e $(\tilde{X}, \tilde{x}_0)$ is a normal covering space of (X, x_0)), also $\tilde{G} \cong \pi_1(X, x_0)$.* $\square$

Definition 4.5.6 *A left action of a group G on a set X is a function $G \times X \to X$ by $(g, x) \mapsto gx$ so that, $1 \cdot x = x$ and $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ for all $x \in X$ and $g_1, g_2 \in G$. We say X is a left G-set.*

Theorem 4.5.7 *Fix $g \in G$ then the function $f_g : X \to X$, $x \mapsto gx$ is a bijection.*

Proof: The function $f_{g^{-1}}$ is its inverse. $\square$

Definition 4.5.8 *Let G be a group which acts to the left on a space X. Define an equivalence relation $\sim$ on X by $x \sim y$ iff $\exists g \in G$ s.t $gx = y$. Then The quotient space $X/\sim$ is called the orbit space of X and denoted by X/G. Each element $Gx = \{gx; g \in G\}$ is called an orbit.*

Definition 4.5.9 *Let a space X be a left G-set, we say that G acts properly discontinuously on X, if each $x \in X$ has an open neighborhood U so that $gU \cap U = \varnothing$, $\forall g \in G, g \neq 1$.*

Theorem 4.5.10 *Let $p : \tilde{X} \to X$ be a covering space. Then $\tilde{G}$ acts properly discontinuously on $\tilde{X}$.*

Proof: Let $\tilde{x} \in \tilde{X}$ and $x = p(\tilde{x})$, let U be an elementary neighborhood of x, and let S be the sheet containing $\tilde{x}$. Assume that $\phi S \cap S \neq \varnothing$ for some $\phi \in \tilde{G}$, which means that $\exists\ \tilde{x}_0, \tilde{x}_1 \in S$ so that $\phi(\tilde{x}_0) = \tilde{x}_1$ which is possible only when $\tilde{x}_0 = \tilde{x}_1$ and $\phi = 1$ because only the identity homeomorphism can preserve the base point. $\square$

Theorem 4.5.11 *If a group G acts properly discontinuously on a space X, then:*

1. *The projection map $p : X \to X/G$, $p(x) = Gx$ is a normal covering space of X/G.*
2. *If X is path-connected, then $G \cong \tilde{G}$.*
3. *If X is connected and locally path-connected then $G \cong \pi_1(X/G)/p_*(\pi_1(X))$.*

Proof: 1. Let $x \in X$ and let U be a neighborhood of x which satisfies the properly discontinuous property. Then Gx is an element of X/G and $G \cdot U$ is a neighborhood of Gx in X/G. Then $p^{-1}(G \cdot U) = \{g \cdot U; g \in G\}$ where the sets in the last collection are pairwise disjoint and homeomorphic. Furthermore, $p|_{g \cdot U} : g \cdot U \to G \cdot U$ is a homeomorphism. Thus, $p : X \to X/G$ is a covering space.
To prove that $p : X \to X/G$ is a normal covering space, let $x_1, x_2 \in p^{-1}(Gx)$ for some $x \in X$, then $x_1 = g_1 x$ and $x_2 = g_2 x$ for some $g_1, g_2 \in G$, i.e $x_2 = g_2(x) = g_2 g_1^{-1}(x_1)$. Clearly all $g \in G$, in particular $g_2 g_1^{-1}$ acts by deck transformation (the group $\tilde{G}$). Hence the covering space $p : X \to X/G$ is normal.
2. We have $G \leq \tilde{G}$, $\forall g \in G$. Now if X is path-connected, let $\phi \in \tilde{G}$.

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "p"'] & & X \arrow[dl, "p"] \\ & X/G & \end{tikzcd} \qquad \begin{tikzcd} (X,x) \arrow[rr, "\phi"] \arrow[dr, "p"'] & & (X,\phi(x)) \arrow[dl, "p"] \\ & (X/G, Gx) & \end{tikzcd}$$

Then $p(x) = p\phi(x) = Gx$ i.e $\{gx; g \in G\} = \{g\phi(x); g \in G\}$ implies that $\phi(x) = g_1 x$ for some $g_1 \in G$. By unique lifting property $\phi = g_1$.
3. We have $G \cong \tilde{G} \cong \pi_1(X/G)/p_*(\pi_1(X))$. (The first $\cong$ is by 2 of this theorem and the second $\cong$ is by Corollary 4.5.4). $\square$

Corollary 4.5.12 *Let X be connected, locally path-connected and simelocally simply connected. Let $H \leq \pi_1(X)$, then there exists a unique (up to equivalence) covering space $p_H : X_H \to X$ so that $H = p_{H*}(\pi_1(X_H))$. (With a suitable base point).*

Proof: Let $p : \tilde{X} \to X$ be the universal covering space of X, by Corollary 4.5.5 $\tilde{G} \cong \pi_1(X)$. Let $\tilde{H} \leq \tilde{G}$ s.t $\tilde{H} \cong H$ and let $\tilde{p} : \tilde{X} \to \tilde{X}/\tilde{H}$ be the projection space which is a covering space by part 1 of Theorem 4.5.11. Moreover, $\tilde{H} \cong \pi_1(\tilde{X}/\tilde{H})$ by part 3 of Theorem 4.5.11. Define the map $p_H : \tilde{X}/\tilde{H} \to X$; $\tilde{H}\tilde{x} \mapsto p(\tilde{x})$.

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{p}"] \arrow[d, "p"'] & \tilde{X}/\tilde{H} \arrow[dl, "p_H"] \\ X & \end{tikzcd}$$

Claim 1: The map p_H is well-defined. To prove this, assume that $\tilde{H}\tilde{x}_1 = \tilde{H}\tilde{x}_2$ for some $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}$, then $\tilde{x}_1 = \tilde{h}\tilde{x}_2$ for some $\tilde{h} \in \tilde{H}$. Since $\tilde{H} \leq \tilde{G}$ then $p(\tilde{x}_1) = p(\tilde{h}x_2)$. Therefore, p_H is well-defined.
Claim 2: The space $p_H : \tilde{X}/\tilde{H} \to X$ is a covering space of X. To this

end, let $x \in X$ and let U be an elementary neighborhood of x with respect to the map p. We check first the commutativity of the above diagram, $p_{H}(\tilde{p}(\tilde{x})) = p_{H}(\tilde{H}x) = p(x)$ hence the diagram is commutative. Now $p_{H}^{-1}(U) = \tilde{p}(p^{-1}(U)) = \tilde{p}(\{S_i\}) = \{\tilde{H}S_i\}$, where $\tilde{H}S_i$ are pairwise same or disjoint for if $\tilde{H}S_1 \cap \tilde{H}S_2 \neq \varnothing$ then $\tilde{H}s_1 = \tilde{H}s_2$ for some $s_1 \in S_1$ and $s_2 \in S_2$, so $s_1 = \tilde{h}s_2$ for some $\tilde{h} \in \tilde{H}$, but this happens iff $s_1 = s_2$ and $\tilde{h} = 1$. Furthermore, $\tilde{H}S_i$'s are homeomorphic to U by the following diagram;

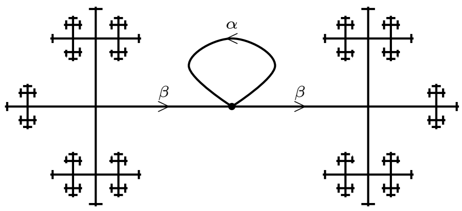
$$\begin{tikzcd}
\tilde{S}_i \arrow[r, "\tilde{p}|_{S_i}"] \arrow[d, "p|_{S_i}"'] & \tilde{H}S_i \arrow[dl, "p_H|_{\tilde{H}S_i}"] \\
U &
\end{tikzcd}$$

Since $\tilde{p}|_{S_i}$ and $p|_{S_i}$ in the diagram above are homeomorphisms then $p_H|_{\tilde{H}S_i}$ is also a homeomorphism. Therefore $p_H : \tilde{X}/\tilde{H} \to X$ is a covering space of X with $p_{H_*}(\pi_1(\tilde{X}/\tilde{H})) \cong p_{H*}(\tilde{H})$ since the homomorphism p_{H*} is one-to-one then $p_{H*}(\tilde{H}) \cong \tilde{H} \cong H$, as desired.

□

Example 4.5.13 *Since $p : \mathbb{S}^n \to \mathbb{S}^n/\mathbb{Z}_2, n \geq 1$ is a covering space of the real projective n-space P^n. Thence $\mathbb{Z}_2 \cong \pi_1(P^n)/p_*(\pi_1(\mathbb{S}^n))$.*
If $n = 1$, then $p_*(\mathbb{Z}) = 2\mathbb{Z} \Rightarrow \pi_1(P^1) \cong \mathbb{Z}$ and if $n \geq 2$, then $\pi_1(P^n) = \mathbb{Z}_2$.

Example 4.5.14 *Since the fundamental group of the figure-eight space $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) \cong \mathbb{Z} * \mathbb{Z} \cong \langle\alpha\rangle * \langle\beta\rangle$, we can construct an irregular covering space corresponding to the subgroup $\langle\alpha\rangle$ which is not normal subgroup of $\langle\alpha\rangle * \langle\beta\rangle$. The following figure is a covering space of the figure-eight and its fundamental group is $\langle\alpha\rangle$. By Theorem 4.5.3 (1) this covering space is not normal.*



4.6 Borsuk-Ulam Theorem and Applications

To make use of the theory presented in sections 4.1-4.5 we will study in this section the Borsuk-Ulam Theorem and some consequences and applications. The main theorems and corollaries in this section are based on [14], where the proof of the Borsuk-Ulam theorem (Theorem 4.6.2) is similar to the one in [12].

Definition 4.6.1 *A map $f : \mathbb{S}^m \to \mathbb{S}^n$ is antipode preserving if for all $x \in \mathbb{S}^m$, $f(-x) = -f(x)$.*

Theorem 4.6.2 (Borsuk-Ulam) *There does not exist any continuous antipode preserving map from $\mathbb{S}^n$ into $\mathbb{S}^{n-1}$.*

Proof: We give the proof when $n = 1, 2$.
(Case n=1) Let $f : \mathbb{S}^1 \to \{-1, 1\}$ be continuous so that $f(-x) = -f(x)$, for all $x \in \mathbb{S}^1$. Hence f is surjective. But this contradicts $\mathbb{S}^1$ being connected.
(Case n=2) Assume there exists a continuous map $f : \mathbb{S}^2 \to \mathbb{S}^1$ so that $f(x) = -f(-x), \forall x \in \mathbb{S}^2$. Consider the following diagram,

$$\begin{tikzcd} (\mathbb{S}^2, a) \arrow[r, "f"] \arrow[d, "p_2"'] & (\mathbb{S}^1, f(a)) \arrow[d, "p_1"] \\ (\mathbb{S}^2/\mathbb{Z}_2, b) \arrow[r, "g"'] & (\mathbb{S}^1/\mathbb{Z}_2, g(b)) \end{tikzcd}$$

where $a = (1, 0, 0) \in \mathbb{S}^2$, $b = p_2(a)$, and $g : \mathbb{S}^2/\mathbb{Z}_2 \to \mathbb{S}^1/\mathbb{Z}_2$; $\mathbb{Z}_2 x \mapsto \mathbb{Z}_2 f(x)$, which is well-defined because

$$g(\mathbb{Z}_2(-x)) = \mathbb{Z}_2 f(-x) = \mathbb{Z}_2(-f(-x)) = \mathbb{Z}_2 f(x) = g(\mathbb{Z}_2 x).$$

So if $x \in \mathbb{S}^2$ then $p_1 f(x) = \mathbb{Z}_2 f(x) = g p_2(x)$. Thus, the diagram is commutative.
Let β be any path in $\mathbb{S}^2$ from a to $-a$ hence $p^2(\beta)$ is a loop in $\mathbb{S}^2/\mathbb{Z}_2$ based at b. Let $g_* : \pi_1(\mathbb{S}^2/\mathbb{Z}_2, b) \to \pi_1(\mathbb{S}^1/\mathbb{Z}_2, g(b))$ be the induced homomorphism from g. Since $\pi_1\left(\mathbb{S}^1/\mathbb{Z}_2, g(b)\right) \cong \mathbb{Z}$ and $\pi_1(\mathbb{S}^2/\mathbb{Z}_2, b) \cong \mathbb{Z}_2$, then $g_* : \mathbb{Z}_2 \to \mathbb{Z}$ must be trivial. Which means that $g_*(\langle p^2\beta\rangle) = \langle e_{g(b)}\rangle$ (the constant loop at $g(b)$) i.e $\langle gp^2\beta\rangle = \langle e_{g(b)}\rangle$. Since p_1 is a covering map then by the unique lifting property $f\beta$ and $e_{f(a)}$ are the unique liftings of $gp^2\beta$ and $gp^2(a) = e_{g(b)}$ to $(\mathbb{S}^1/\mathbb{Z}_2, g(b))$ respectively.

$$\begin{tikzcd} & (\mathbb{S}^1, f(a)) \arrow[d, "p_1"] \\ I \arrow[ur, "e_{f(a)}"] \arrow[r, "e_{g(b)}"'] & (\mathbb{S}^1/\mathbb{Z}_2, g(b)) \end{tikzcd} \qquad \begin{tikzcd} & (\mathbb{S}^1, f(a)) \arrow[d, "p_1"] \\ I \arrow[ur, "f\beta"] \arrow[r, "gp^2\beta"'] & (\mathbb{S}^1/\mathbb{Z}_2, g(b)) \end{tikzcd}$$

Since $gp_2\beta \approx e_{g(b)}$ rel $\{0, 1\}$, then by the monodromy theorem
$e_{f(a)} \approx f\beta$ rel $\{0, 1\}$ and $e_{f(a)}(1) = f\beta(1)$ which implies that $f(a) = f(-a)$.
A contradiction. $\square$

Applications

See [1], [5], [7] and [14].

Corollary 4.6.3 (n=1,2) *Assume that the map $f : \mathbb{S}^n \to \mathbb{R}^n$ is continuous so that $f(-x) = -f(x), \forall x \in \mathbb{S}^n$, then $\exists x \in \mathbb{S}^n$ with $f(x) = 0$.*

Proof: Assume not, that is $f(x) \neq 0, \forall x \in \mathbb{S}^n$. Define $g : \mathbb{S}^n \to \mathbb{S}^{n-1}$, $g(x) = \frac{f(x)}{|f(x)|}$. Then g is continuous and $g(-x) = -g(x), \forall x \in \mathbb{S}^n$. A contradiction. $\square$

Corollary 4.6.4 (n=1,2) *Assume that $f : \mathbb{S}^n \to \mathbb{R}^n$ is a continuous map. Then there exists an $x \in \mathbb{S}^n$ so that $f(x) = f(-x)$.*

Proof: Assume to the contrary, that $\forall\ x \in \mathbb{S}^n$, $f(x) \neq f(-x)$. Define $g : \mathbb{S}^n \to \mathbb{R}^n$, $g(x) = f(x) - f(-x)$, hence g is continuous and $g(x) \neq 0, \forall x \in \mathbb{S}^n$, while $g(-x) = f(-x) - f(x) = -g(x)$, $\forall x \in \mathbb{S}^n$. A contradiction. $\square$

Corollary 4.6.5 *The sphere $\mathbb{S}^2$ is not homeomorphic to any subset of $\mathbb{R}^2$.*

Proof: By the last corollary, any continuous map $\mathbb{S}^n \to \mathbb{R}^n$ is not injective. $\square$

Example 4.6.6 *Let $x, y \in \mathbb{R}$. The graphs of the following functions*

$$f(x,y) = xy \sin y - (x^2 + y^2) - \sin^2(\frac{\pi}{2} x^4 \sqrt{x^2 + y^2 + 3}) + 3, \text{ and}$$
$$g(x,y) = \cos x \sin y + \sin \pi(x + y^3) + \cos^2(\pi x^4) + 1,$$

intersect each other in the unit circle $\mathbb{S}^1$.

Proof: We have,

$$f|_{\mathbb{S}^1}(x,y) = xy \sin y - \sin^2 \pi x^4 + 2,$$
$$g|_{\mathbb{S}^1}(x,y) = \cos x \sin y + \sin \pi(x + y^3) + \cos^2(\pi x^4) + 1.$$

Define,

$$h(x,y) = (f|_{\mathbb{S}^1} - g|_{\mathbb{S}^1})(x,y) \Rightarrow$$
$$h(x,y) = xy \sin y - \cos x \sin y - \sin \pi(x + y^3) \Rightarrow$$
$$h(-x,-y) = -xy \sin y + \cos x \sin y + \sin \pi(x + y^3) = -h(x,y).$$

Hence $\exists (x_0, y_0) \in \mathbb{S}^1$ with $h(x_0, y_0) = 0$. $\square$

Theorem 4.6.7 *Let S_1, S_2 and S_3 be closed sets in $\mathbb{R}^3$ so that $\{S_i\}_i$ is a covering of $\mathbb{S}^2$. Then one of S_i's must contain a pair $\{a, -a\}$, for some $a \in \mathbb{S}^2$.*

Proof: Define $g : \mathbb{S}^2 \to \mathbb{R}^2$, by $g(x) = (d(x, S_2), d(x, S_3))$, hence g is continuous and therefore $\exists a \in \mathbb{S}^2$ satisfying $g(a) = g(-a)$. But $g(-a) = (d(-a, S_2), d(-a, S_3))$. Thus if $a \in S_2$ then $0 = d(a, S_2) = d(-a, S_2)$, since S_2 is closed, then $-a \in S_2$. Similarly when $a \in S_3$. If $a \notin S_2$, $a \notin S_3$ then $\{a, , -a\} \subseteq S_1$ because the S_i's cover $\mathbb{S}^2$. $\square$

Theorem 4.6.8 (Ham-Sandwich Theorem) *Let B_1, B_2 and B_3 be bounded measurable subsets of $\mathbb{R}^3$. Then there exits a hyperplane in $\mathbb{R}^3$ which divides each of these subsets into two parts with equal volume.*

Proof: Since the sets are bounded, then we can assume them inside the unit sphere $\mathbb{S}^2$. For all $x \in \mathbb{S}^2$ let H^x be the tangent plane at x, and P^x be the hyperplane parallel to H^x which divides B_1 into two parts with equal volume. Let V_i^x the volume of the part of B_i between H^x and P^x. Define $g : \mathbb{S}^2 \to \mathbb{R}^2$, $g(x) = (V_2^x, V_3^x)$, hence g is continuous and so there exists a point $x_0 \in \mathbb{S}^2$ s.t $g(x_0) = g(-x_0)$. Hence $V_2^x = V_2^{-x_0}$ and $V_3^x = V_3^{-x_0}$. But $V_i^{-x} = volume(B_i) - V_i^x$. $\square$

Theorem 4.6.9 (Pancake Theorem) *Let A_1 and A_2 be bounded measurable subsets of $\mathbb{R}^2$. Then there exits a line in $\mathbb{R}^2$ which bisects each of these subsets into two parts with equal area.*

Proof: Similar to the Ham-Sandwich theorem proof. $\square$

Theorem 4.6.10 (Meteorology Theorem) *At any time, there are two antipodal places on the earth having the same temperatures and the same barometric pressures.*

Proof: Temperatures and pressures are assumed to be continuous on the earth. Then $\mathbb{S}^2 \to \mathbb{R}^2 : x \mapsto$(Temperature, Pressure) is continuous. $\square$

4.7 Covering spaces of graphs and the Nielsen-Schreier Theorem

We study in this section an application of the fundamental group in group theory, in particular the the Nielsen-Schreier Theorem.

Definition 4.7.1 *[10] A graph X is an identification space $X^0 \sqcup I_\alpha / \sim$, where X^0 is a discrete set called the set of vertices and $I_\alpha \subset \mathbb{R}$ are the unit intervals, so that the ends points of each interval I_α are identified by $\sim$ with elements from X^0.*
• *I_α's after identification are called edges and denoted by e_α.*
• *A connected graph is a graph so that one can connect two vertices by a sequence of edges. (connectedness of graphs implies path-connectedness).*
• *A subgraph $Y \subseteq X$ is a closed subspace of X, i.e if $e_\alpha \in Y$ then $\bar{e}_\alpha \in Y$.*
• *A tree is a contractible connected graph.*
• *A spanning tree is a tree $T \subseteq X$ containing X^0.*

The following result can be found in [4].

Proposition 4.7.2 *Any connected graph X contains a spanning tree.*

Proof: Consider the set $\Lambda := \{T; T \subseteq X, T \text{is a tree}\}$ which is partially ordered by the subgraph relation. If Λ has a maximal tree then it will be a spanning tree since X is connected. We use Zorn's lemma to prove the existance of a maximal tree. Let T_j be a chain of trees in X. Claim; $T_J := \cup T_j$ is an upper bound tree of the chain T_j. Proof; 1 T_J is connected, if $x_1^0, x_2^0 \in X^0 \cap T_J$ then $\exists T_{j_1}$ and T_{j_2} s.t $x_1^0 \in T_{j1}$ and $x_2^0 \in T_{j2}$, say $T_{j_1} \subseteq T_{j_2}$, then there is a sequence of edges in T_{j_2} connecting x_1^0 and x_2^0 hence this sequence is also in T_J. 2 T_J is a tree, assume there exists a simple loop γ $(\gamma(t_1) \neq \gamma(t_2); 0 \leq t_1, t_2 \leq 1, t_1 \neq t_2)$ in T_j based at a vertex $x_1^0 \Rightarrow \gamma$ has only finitely many edges $\Rightarrow \gamma \subseteq T_{j'}$ for some j', impossible. Therefore, Λ has a maximal element which is a spanning tree in X. □

The next theorem can be found in [14].

Theorem 4.7.3 *Let X be a connected graph and T be a spanning tree in X, then $\pi_1(X)$ is a free group generated by the edges in $X \setminus T$.*

Proof: If the graph X has only one vertex then it is a bouquet of circles, where its fundamental group is a free product of free groups i.e a free group generated by the circles in the bouquet.
For the general case, let $\{e_\beta\}$ be the set of edges in $X \setminus T$. Let x_β be a point that lies at e_β. Therefore, $\{x_\beta\}$ is discrete and closed subset of X (since $X = X^0 \sqcup I_\alpha / \sim$ then it has the weak topology with respect to $\bar{e}_\beta$). Let V be the complement of of this set in X. Then V deformation retracts to T, hence V is contractible. Set $U_\beta = V \cup \{x_\beta\}$, then $U_{\beta_1} \cap U_{\beta_2} = V$, $\beta_1 \neq \beta_2$. In addition, U_β deformation retracts to $T \cup \{e_\beta\}$, where U_β is a homotopic equivalent to a circle based at v_0, say. By the van-Kampen theorem $\pi_1(X, v_0) = \pi_1(\bigcup_\beta U_\beta) = \mathop{\ast}_\beta \mathbb{Z}$. □

Theorem 4.7.4 *[10] Let $p : \tilde{X} \to X$ be a covering space of a graph X. Then $\tilde{X}$ is also a graph.*

Proof: For the vertices $\tilde{X}^0$ take the discrete set $p^{-1}(X^0)$. By the definition $X = X^0 \sqcup_\alpha I_\alpha / \sim$ for all e_α there exists a unique $\tilde{e}_\alpha$ s.t $p\tilde{e}_\alpha = e_\alpha$. Hence $\tilde{e}_\alpha$'s construct the edges of $\tilde{X}$.

$$\begin{tikzcd}
 & \tilde{X} \arrow[d, "p"] \\
I_\alpha \arrow[ur, "\tilde{e}_\alpha"] \arrow[r, "e_\alpha"'] & X^0 \sqcup_\alpha I_\alpha / \sim
\end{tikzcd}$$

□

Theorem 4.7.5 (Nielsen Schreier) *A subgroup of a free groups is a free group.*

Proof: Let F be a free group generated by a set X, and let $H \leq F$. Let X be a graph s.t $\pi_1(X) \cong F$ (for example, a bouquet of circles corresponding to the elements of X). Then there exists a covering space $p_H : X_H \to X$ s.t $p_{H*}(\pi_1(X_H)) \cong H$, since p_{H*} is injective, then $\pi_1(X_H) \cong H$. But X_H is a graph. So $\pi_1(X_H)$ is a free group. $\square$

Chapter 5

The Jordan Curve Theorem

This is the last application of the fundamental group in this thesis. We use an idea that is similar to one in [7] but by making use of the van-Kampen Theorem.

5.1 The Jordan Curve Theorem

Definition 5.1.1 *A Jordan curve (or simple closed curve) in a space X is an injective map $\gamma : \mathbb{S}^1 \to X$. i.e a non-self-intersecting closed curve.*

Since γ is one-to-one then $\mathbb{S}^1 \simeq \gamma(\mathbb{S}^1) := \Gamma$ iff γ is open (closed). As $\mathbb{S}^1$ is compact then if X is Hausdorff then γ must be open (closed). (If $A \subseteq \mathbb{S}^1$ is closed then it is compact and hence $\gamma(A)$ is compact subset of a Hausdorff space which implies the closedness).

Theorem 5.1.2 (Jordan Curve Theorem) *Suppose that γ is a Jordan curve in $\mathbb{R}^2$. Then $\mathbb{R}^2 - \Gamma$ consists of two disjoint connected components.*

Theorem 5.1.3 *Suppose that γ is a Jordan curve in $\mathbb{S}^2$. Then $\mathbb{S}^2 - \Gamma$ consists of two disjoint connected components.*

Theorem 5.1.4 *Let $f : \mathbb{R} \to \mathbb{R}^2$ be a one-to-one map, s.t $|f(t)| \to \infty$ as $|t| \to \infty$. Then $\mathbb{R}^2 - f(\mathbb{R})$ consists of two disjoint connected components.*

These three theorems are equivalent, using two facts; $\mathbb{S}^2 - \{x_0\} \simeq \mathbb{R}^2$ and the one-point compactification $\mathbb{R}^2 \cup \{\infty\} \simeq \mathbb{S}^2$.

Lemma 5.1.5 *Let $f : \mathbb{R} \to \mathbb{R}^2$ be continuous and one-to-one map s.t $|f(t)| \to \infty$ as $|t| \to \infty$. Then there exist a homeomorphism $F : \mathbb{R}^3 \to \mathbb{R}^3$ s.t*

$$(F \circ \iota \circ f)(t) = (0, 0, t),\ t \in \mathbb{R}.$$

where $\iota : \mathbb{R}^2 \to \mathbb{R}^3$ is the canonical embedding $(x, y) \mapsto (x, y, 0)$.

Proof: Let f_1 and f_2 be the coordinate functions of f, s.t $f(t) = (f_1(t), f_2(t))$, $t \in \mathbb{R}$. Define $g : f[\mathbb{R}] \to \mathbb{R}$ by $g(f(t)) = t$, $t \in \mathbb{R}$. Since f is one-to-one then g is well defined, we postpone the verifying of the continuity of g until the end of this proof. By Tietze Extension Theorem, there is a continuous map $G : \mathbb{R}^2 \to \mathbb{R}$ s.t $G|_{f[\mathbb{R}]} = g$ i.e

$$G(f_1(t), f_2(t)) = t,\ t \in \mathbb{R}.$$

Define

$$\begin{aligned} F_1 &: \mathbb{R}^3 \to \mathbb{R}^3; (x, y, z) \mapsto (x, y, z + G(x, y)), \\ F_2 &: \mathbb{R}^3 \to \mathbb{R}^3; (x, y, z) \mapsto (x - f_1(z), y - f_2(z), z). \end{aligned}$$

The maps F_1 and F_2 are homeomorphisms since they are continuous and they have continuous inverses, namely $(x, y, z) \mapsto (x, y, z - G(x, y))$ and $(x, y, z) \mapsto (x + f_1(z), y + f_2(z), z)$; respectively.
Thus the map $F = F_1 \circ F_2$ is a homeomorphism $\mathbb{R}^3 \to \mathbb{R}^3$ and

$$\begin{aligned} (F \circ \iota \circ f)(t) &= F_2(F_1(f_1(t), f_2(t), 0)) \\ &= F_2(f_1(t), f_2(t), t) \\ &= (0, 0, t). \end{aligned}$$

To show that g is continuous, we note that $\forall M \in \mathbb{N}$, $\exists N \in \mathbb{N}$ s.t if $|t| > N$ then $|f_j| > M$ for $j = 1, 2$. Accordingly, the inverse of a bounded subset $K \subset \mathbb{R}^2$ is bounded, since f is continuous then the inverse of a compact subset is also compact i.e f is a proper map. Now since a proper map into a locally compact Hausdorff space is closed, then f is closed i.e g is continuous. $\square$

Lemma 5.1.6 *Let $B \subset \mathbb{R}^2$ be closed. If $W := \mathbb{R}^3 - \{(x, y, 0); (x, y) \in B\}$. Then,*

$$\pi_1(W) \cong \begin{cases} 1 & : \mathbb{R}^2 - B \text{ is path-connected,} \\ \mathop{\ast}\limits_{n-1} \mathbb{Z} & : \mathbb{R}^2 - B \text{ has } n \text{ components.} \end{cases}$$

Proof: Let $n \in \mathbb{N}$ and let $A_1, ..., A_n$ be the n open components in $\mathbb{R}^2 \setminus B$. Therefore,

$$\mathbb{R}^2 \setminus B = \bigcup_{i=1}^{n} A_i.$$

Fix an open disc D_i in A_i for each $i = 1, ..., n$, and define the following sets,

$$\begin{aligned} H^+ &= \{(x, y, z) \in \mathbb{R}^3; z > 0\}, \\ H^- &= \{(x, y, z) \in \mathbb{R}^3; z < 0\}. \end{aligned}$$

Let $\iota : \mathbb{R}^2 \to \mathbb{R}^3$; $(x, y) \mapsto (x, y, 0)$ be the canonical embedding map. So $W = H^+ \cup H^- \cup \bigcup_{i=1}^{n} \iota(A_i)$. We continue the proof by an induction on n.
When $n = 1$, then $W = H^+ \cup H^- \cup \iota(A_1) := U \cup V$, where $U = A_1 \times (-\frac{1}{2}, 1) \cup H^-$ and $V = A_1 \times (-1, \frac{1}{2}) \cup H^+$. Thus, U and V are open and path-connected subsets of W and $U \cap V = A_1 \times (-1, 1) \neq \varnothing$ is path-connected, then the van-Kampen Theorem is applicable. Since $A_1 \times (-\frac{1}{2}, 1)$ is homotopy equivalent to A_1 via an explicit homotopy $F_t(x, y, z) = (x, y, (1 - t)z)$, then U is homotopy equivalent to $H^- \cup A_1$ which is path-connected and any loop γ based at $x_0 \in H^-$ can be shrunken to x_0 by an explicit homotopy $G_t(s) = (1-t)\gamma(s)+tx_0$. Therefore, U and similarly V are simply connected and the van-Kampen Theorem implies that $\pi_1(W) = 1$.
For $n \geq 2$, we proceed in two steps. First, One may without loss of generality replace each A_i by D_i. To see this, let

$$\begin{aligned} U &= H^+ \cup H^- \cup \iota(A_1), \\ V &= H^+ \cup H^- \cup \iota(D_1) \cup \bigcup_{i=2}^{n} \iota(A_i). \end{aligned}$$

Hence,

$$W = H^+ \cup H^- \cup \bigcup_{i=1}^{n} \iota(A_i) = U \cup V.$$

Since $U \cap V = H^+ \cup H^- \cup \iota(D_1)$ is path-connected and contractible and U is simply connected (by the first part of the proof), then by the van-Kampen Theorem,

$$\pi_1(W) \cong \pi_1(U) * \pi_1(V) \cong 1 * \pi_1(V) \cong \pi_1(V).$$

Analogously, replace A_i by D_i, for $i = 2, ..., n$.
Secondly, we resume the proof on D_i-version. For $n = 2$,

$$W = H^+ \cup H^- \cup \iota(D_1) \cup \iota(D_2).$$

We assume without loss of generality that $\iota(D_1)$ and $\iota(D_2)$ have radius $\frac{1}{2}$ and centers $(-1, 0, 0)$ and $(1, 0, 0)$, respectively. Now W is homotopy equivalent with its projection in the (x, z)-plane via an explicit homotopy: $H_t(x, y, z) = (x, (1 - t)y, z)$. Hence W after the projection in the (x, z)-plane becomes; $W' = \{(x, z); z > 0\} \cup \{(x, z); z < 0\} \cup (-\frac{3}{2}, -\frac{1}{2}) \times \{0\} \cup (\frac{1}{2}, \frac{3}{2}) \times \{0\}$ which is path-connected and any loop in W' based at $(1, 1)$ can be shrunken into $(1, 1)$ except those loops passing through the interval $(-\frac{3}{2}, -\frac{1}{2}) \times \{0\}$ (x-axis). Thus, W' is homotopy equivalent to $\mathbb{S}^1$, i.e $\pi_1(W) \cong \pi_1(W') \cong \pi_1(\mathbb{S}^1) \cong \mathbb{Z}$.
We continue the induction on n (D_i-version); Assume that if $\mathbb{R}^2 \setminus B$ has $n-1$

components then $\pi_1(W) \cong \mathop{\ast}\limits_{n-2} \mathbb{Z}$. Now we prove the lemma for any n. Let

$$U = H^+ \cup H^- \cup \iota(D_1) \cup \iota(D_2),$$
$$V = H^+ \cup H^- \cup \bigcup_{i=2}^{n} \iota(D_i).$$

Hence,

$$W = H^+ \cup H^- \cup \bigcup_{i=1}^{n} \iota(D_i) = U \cup V.$$

Since $U \cap V = H^+ \cup H^- \cup \iota(D_2)$ is path-connected and contractible and $\pi_1(U) \cong \mathbb{Z}$ (case n=2), then by the van-Kampen Theorem,

$$\pi_1(W) = \pi_1(U \cup V) \cong \pi_1(U) * \pi_1(V) \cong \mathbb{Z} * \left(\mathop{\ast}\limits_{n-2} \mathbb{Z} \right) \cong \mathop{\ast}\limits_{n-1} \mathbb{Z}.$$

□

Proof:[Theorem 5.1.4] Let $W = \mathbb{R}^3 - \{(x, y, 0); (x, y) \in f[\mathbb{R}]\}$. Then by Lemma 5.1.5 W is homeomorphic to $\{\mathbb{R}^3 - \{(0, 0, z); z \in \mathbb{R}\}$ which is homeomorphic to $(\mathbb{R}^2 - \{0\}) \times \mathbb{R}$, and $\pi_1((\mathbb{R}^2 - \{0\}) \times \mathbb{R}) \cong \mathbb{Z}$. So $\mathbb{R}^2 - f[\mathbb{R}]$ has two connected components. □

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