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Abstract

Timer options are relatively new exotic options with the feature that they expire as soon as the accumulated realized variance exceeds a predefined level. This construction leads to a random time to maturity instead of having a fixed exercise day. As shown by Bernard an Cui [7], Timer options can be priced by solving a partial differential equation or by time-changing the stock price process and then using Monte-Carlo methods when assuming a diffusion process for the stock price and the variance. The purpose of this thesis is to show the results of [7] and then to extend their pricing techniques to jump-diffusion processes for the stock price. The jumps are assumed to follow a compound Cox process with independent and identically distributed jumps. Due to the jumps, the partial differential equation extends to a partial integro-differential equation. Furthermore, one can time-change the stock price process like in [7] and then use an adapted Monte-Carlo method with control variates to efficiently simulate the price of a Timer option. As an example, results for Timer Calls are shown when using Monte-Carlo methods. Finally, the pricing error for Timer Calls is studied when assuming a stock price process with continuous paths although it jumps.

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1 Introduction

Timer options are relatively new exotic options which have the feature to allow the buyer to choose the variance level instead of having to accept the volatility implied by the market. A special type of a Timer option is a Timer Call which is similar to a European Call, but as mentioned above, it expires when the accumulated realized variance hits a predefined threshold. Due to this construction, the time of maturity is random.

In April 2007, a Timer Call was issued the first time by Société Générale Corporate and Investment Banking. As Sawyer [31] explains, Timer Calls or in general Timer options protect investors from overpaying which is caused by the difference between the implied volatility and the realized volatility. One can observe, he continues, that the volatility implied by the market, which the price of Call options with fixed time to maturity depends on, is often higher than the actually realized volatility.

Since Timer options depend on the realized volatility and not on the volatility implied by the market, this contract can be used to trade volatility like variance or volatility swaps [9]. Having this application, Timer options are interesting derivatives considering the growing demand for derivatives on the realized variance. However, the success of this option of course depends on a sufficient theory which contributes to compute prices efficiently and precisely. Thus, this thesis hopefully adds some helpful theory to this topic.

During the recent years, some research in this area has been done whereas before the first issue of a Timer Call in 2007, there was only little research going on. A first mention of Timer options was in a working paper of Neuberger [26] who called them "mileage" options. Five years later, Bick [8] considered Timer options in connection with portfolio insurance. In 2010, Li [21] showed how Timer options can be hedged and priced under the Heston model [15], especially by means of solving a partial differential equation or Monte-Carlo methods. One year later, these ideas were generalized to other stochastic volatility models by Bernard and Cui [7] who adapted the idea of time-change considered by Li. Another way of pricing Timer options was shown by Saunders [30] when the stochastic volatility process has a fast mean-reverting property in 2010. A path integral approach to price Timer options with the Duru-Kleinert time transformation was considered by Liang, Lemmens and Tempere [22] in 2011. Besides the study of perpetual Timer options, the authors considered finite time-horizon Timer options, too. A finite time-horizon Timer option expires either if the accumulated realized variance hits a certain level like in the perpetual case or if a fixed time to maturity is reached. It avoids facing the risk of having extreme long or maybe even infinite stopping times, which reduces in general the price of a Timer option compared to the perpetual setting.

In the articles mentioned above, the stock price process was always assumed to be continuous but this might not be sufficient sometimes. In 1975, Merton [24] considered a stock price process which allowed for jumps modelled by a compound Poisson process with lognormally distributed jumps. Since that, a lot of research has been done to investigate the question if stock prices jump. Clifford and Walter [4], for instance, compared prices of Call options under the Merton and the Black-Scholes model but could not find "operationally significant differences". But in 1988, Jorion [18] provided evidence that there are jumps in stocks as well as exchange rates. In 2002, Aït–Sahalia [2] investigates if one can detect jumps in discrete data of stock markets. He finally concluded that jumps should be included. This coincides with Barndorff and Nielsen [5] who provided nonparametric tests in order to test the hypothesis of having continuous paths. As they found out, this hypothesis was sometimes rejected.

This thesis is basically divided into two parts:
The first part picks up the work of Li [21] and especially of Bernard and Cui [7] who consider general stochastic volatility models where the stock price and variance process are continuous, i.e. driven by Brownian motions. Readers who are familiar with the article of Bernard and Cui can skim over chapter 3 because the main theorems are similar but sometimes slightly extended and one can also find some additional material. For instance, a brief discussion of different stock price processes like the SABR model [14] is added. Furthermore, it includes a suggestion to take into account the interest yield curve when using Monte-Carlo methods for pricing Timer options. This seems to be reasonable because of the random time to maturity.
In the second part, chapter 4, this theory is adapted to general stochastic volatility models with jumps in the stock price process. The jump process is assumed to be independent of the diffusion part and to follow a compound Cox process with the integrated variance times a constant as intensity. Interestingly, the time-changed stock price process conditioned on the integrated variance reduces to the stock price process considered by Merton [24] with $\sigma = 1$ if the interest rate is zero. If the interest rate is greater than zero, this does not hold because the time-changed stock price process also depends on the variance process due to the random time to maturity. However, in the simulation part the jumps are assumed to be lognormally distributed like Merton suggested. But in contrast to the pure diffusion model assumed in the first part, the random time to maturity also depends on the jump process because the budget is not only consumed by the integrated variance but also by the squared jumps. This dependence leads to an astonishing behaviour of the price of a Timer Call when changing the number of jumps or the standard deviation of the jump size. In fact, increasing these parameters can lead to decreasing prices.
In chapter 5, finally, the model without jumps assumed in the first part and the model with jumps assumed in the second part are compared. This is done by studying the price difference of a Timer Call when assuming the Heston model although the stock price has lognormally distributed jumps.

2 What is a Timer option?

Essentially, a Timer Option is an exotic option whose feature is that it expires as soon as a variance budget is consumed which also leads to a random time to maturity. For instance, a Timer Call has the same payoff function as a European Call but the expiration date is in contrast random. As mentioned, this randomness is due to the fact that a Timer option expires once the realized variance of the price process of the underlying hits a predefined threshold.
In particular, for a given equidistant partition $0 = t_0 < t_1 < \cdots < t_n = T$ of a time interval $[0, T]$ the estimator for the realized variance of this interval is given by

$$\boldsymbol{RV} = \sum_{k=0}^{n-1} \log \left(\frac{S(t_{k+1})}{S(t_k)} \right)^2 . \tag{2.1}$$

where $S(t_k)$ is the stock price at the time point t_k. Note, that this estimator is in general only asymptotically unbiased. Making the partition finer and finer, the limit of $\boldsymbol{RV}$ is the quadratic variation of the stock price process which will be used to develop the theory in the following chapters.
Now, a Timer option is constructed as follows: In principal, a Timer option has the same conditions like options with a fixed time to maturity. The difference is that a variance budget $\boldsymbol{b}$ is set instead of fixing a suitable time to maturity. This is done by setting a target time to maturity T and a target volatility $\hat{\sigma}$. Having 252 trading days a year and N_T trading days until T the variance budget amounts to

$$\boldsymbol{b} = \hat{\sigma}^2 \frac{N_T}{252}.$$

In case of a Timer Call, for instance, the investor gets the payoff of a European Call option $\max\{S_{\mathcal{T}} - K\}$ at the stopping time

$$\mathcal{T} := \inf\{t_k : \boldsymbol{RV} \geq \boldsymbol{b}\}.$$

An example of a price process of a Timer Call is shown in figure 2.1. The strike price was $K = 7778.80$ (Closing Price on 02.01.2013) and the variance budget was $\boldsymbol{b} = 0.02$. The interest rate was assumed to be $r = 0$. On 11.12.2013, the realized variance was above the variance budget of 0.02 the first time, or differently said, the variance budget had been fully consumed. Thus, the Timer Call expired and the holder of the Timer Call option got a cash settlement of 1298.30 Euros. In the lower right corner the relative price difference between the above mentioned Timer Call and two European Calls with a fixed time to maturity is plotted with the same strike price K and the expiry date 11.12.2013. These European Call options are priced with the Black-Scholes formula. The European Call with an implied volatility equal to the realized volatility, which was approximately 14%, had the same initial price as the Timer Call. In contrast, the initial price of the Timer Call was about 15% lower than the initial price of the European Call with a slightly higher implied volatility of 17%. As one can see, the Timer Call outperformed the European Calls in periods in which the Dax increased and it underperformed the European Calls in periods in which the Dax decreased. The negative correlation between the Dax and its variance is the main reason for this behaviour. In other words, the stochastic clock, which could also be called business clock, runs faster than the real time clock when the Dax decreases and vice versa. This observation is crucial because it points out when a Timer Call should be preferred to a European Call option. Roughly speaking, a Timer Call or Put option is the better choice if one wants to be hedged against the occurrence of an extreme event which is expected to happen during the next weeks or months and until the event happens the market is expected to move only little. On the contrary, a European Call or Put option is better if the occurrence of such an event is good to guess or if the time until it happens one expects a higher volatility in the market than the implied volatility of the option assumes. This also shows another application of Timer options. A Timer option can also be used to trade volatility. If one thinks that the implied volatility of a European Call option is too high, one can sell this Call and buy a Timer Call option with a target volatility that is equal to the implied volatility and a target time to maturity that is equal to the time to maturity of the European Call option. If then the realized volatility is actually lower than the implied volatility, the Timer Call option is worth more than the European Call option and by reversing selling and buying one gets the difference as profit. As an illustration, one can look again at the lower right plot in figure 2.1 which shows a comparison of a Timer Call option and a European Call option (blue line). They had the same initial price but at the end of January the Timer Call option was about 3% more expensive than the European Call option.

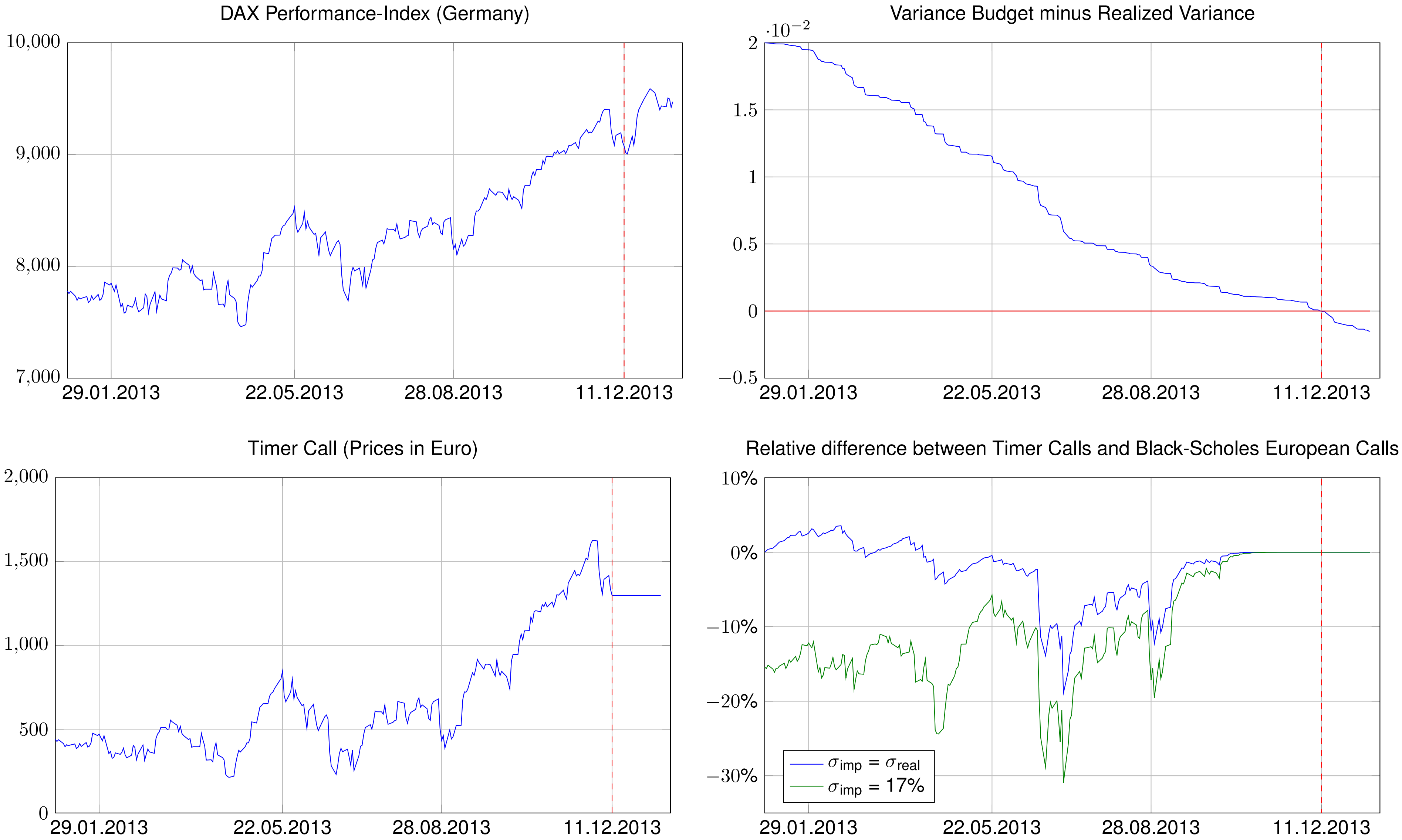

Figure 2.1: Example of a price process of a Timer Call based on the german DAX Performance-Index, which is plotted in the upper left corner from 02.01.2013 until 10.01.2014. In the lower left corner the price of a Timer Call is shown with $K = 7778.80$ and variance budget $\boldsymbol{b} = 0.02$. On the right hand side the consumption of the variance budget is plotted and below the relative difference (Timer Call/European Call-1) of a Timer Call and two European Calls with different implied volatilities. The interest rate is assumed to be zero.

3 Pricing Timer options under pure diffusion processes

This chapter picks up the ideas of Bernard and Cui [7] but some statements are slightly extended and some proofs are different. Before stating the main results of this chapter, a mathematical description of the financial market as well as of the Timer option is needed.

Financial Market

To model the financial market, a general stochastic volatility model like in [7] is considered. Let S_t and V_t be the stock price process and the variance process respectively defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q})$. The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is assumed to be generated by the two Brownian motions W^1 and W^2 defined below. Then, the dynamics under the risk-neutral measure $\mathbb{Q}$ are defined as

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{V_t} S_t d(\sqrt{1-\rho^2} W_t^1 + \rho W_t^2) \\ dV_t &= \alpha(t, V_t)dt + \beta(V_t) dW_t^2 \end{aligned} \tag{3.1}$$

where r is the constant risk-free interest rate, W^1 and W^2 are two independent Brownian motions, $\rho \in [0, 1]$ is the correlation between the changes in the stock and the variance and $\alpha(t, V_t)$ and $\beta(V_t)$ are some measurable functions w.r.t. the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that there exists a global solution to the variance process. Furthermore, it is assumed that for any $t \geq 0$ it holds $V_t > 0$ almost surely and that the integrated variance converges to infinity as $T \to \infty$, i.e.

$$\begin{aligned} \xi_T^c &:= [\log S]_T \\ &= \int_0^T V_t dt \to \infty \ a.s.,\ T \to \infty. \end{aligned}$$

where $[\log S]_T$ is the quadratic variation of the stock price process. The stochastic process ξ_t^c is called stochastic clock and is the continuous version of the realized variance (2.1).

Timer option

As already said, the feature of a Timer option is the random time to maturity τ as a result of the automatic expiration when the predefined variance budget b is fully consumed. In particular, the

Timer option is exercised at the stopping time

$$
\begin{aligned}
\tau_{\boldsymbol{b}} &:= \inf\left\{t \geq 0 : \xi_t^c \geq \boldsymbol{b}\right\} \\
&= \inf\left\{t \geq 0 : \int_0^t V_s ds \geq \boldsymbol{b}\right\}.
\end{aligned} \tag{3.2}
$$

Since the integral over the variance process is in this setting continuous, the stopping time τ can be written as

$$
\tau_{\boldsymbol{b}} = \inf\left\{t \geq 0 : \int_0^t V_s ds = \boldsymbol{b}\right\}.
$$

At τ a Timer option gives the investor a payoff according to the payoff function $f(S_\tau)$ where f is a measurable function w.r.t. the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and S_τ is the value of the stock price process at the stopping time τ. In case of a Timer Call, for instance, the payoff function is $f(S_\tau) = \max\{S_\tau - K, 0\}$. Note, that with τ, used for keeping notations legible, the stopping time $\tau_{\boldsymbol{b}}$ as defined here is meant. During this chapter, it is sometimes also reasonable to write $\tau_{\boldsymbol{b}}$ with maybe other values than $\boldsymbol{b}$.

3.1 Theoretical results for pure diffusion processes

In this section the main theorems are stated which are used for the numerical analysis in the following section. Due to the construction of the Timer option, the main idea for pricing it is to use the feature that the integrated variance can be seen as a deterministic variable because of the stopping time τ (3.2). This is done by transferring the model (3.1) to a filtered probability space where the integrated variance, so to speak, plays the role of the time t. To achieve this goal, a proper time change is used by applying the Dambis-Dubins-Schwarz theorem [19]. Then, one can use the resulting representation of the financial market in terms of the integrated variance to calculate the expectation of the discounted payoff function under the risk neutral measure. The following theorem clarifies this approach.

Theorem 3.1.1. *Let S_t and V_t be like in* (3.1)*, r the risk-free interest rate, f the payoff function and $\boldsymbol{b}$ the variance budget with corresponding stopping time τ, which is defined as in* (3.2)*. Then, the initial price of a Timer option is given by*

$$
\mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau} f\left(S_0 e^{r\tau + \boldsymbol{B}_{\boldsymbol{b}} - \frac{1}{2}\boldsymbol{b}}\right)\right]
$$

where $\boldsymbol{B}$ is a $\mathbb{Q}$-standard Brownian motion.

Before continuing with the proof, it is worth pointing out that a Timer option can easily be priced according to the above theorem when the risk-free interest rate r is zero. In the case of pricing a Timer Call, for instance, the expectation can easily be computed as shown later. But also in cases where one has to use numerical methods to compute the expectation of the payoff function

f one can either use Monte-Carlo methods by drawing N normally distributed random variables with variance $\boldsymbol{b}$ or by means of numerical integration methods since the density function of the normal distribution is well known. In contrast, pricing a Call under a general stochastic variance model is usually more complicated.

Proof. As mentioned, the main idea is to apply the Dambis-Dubins-Schwarz theorem. But firstly, use Itô's lemma with the function $f(x) = \log(x)$ to obtain

$$d\log(S_t) = rdt + \sqrt{V_t}d(\sqrt{1-\rho^2}W_t^1 + \rho W_t^2) - \frac{1}{2}V_t dt$$

and define

$$M_t^i := \int_0^t \sqrt{V_s}dW_s^i,\ i = 1, 2.$$

The two processes M_t^1 and M_t^2 are continuous local martingales with $M_0^i = 0$ and $\lim_{t\to\infty}\left[M^i\right]_t = \lim_{t\to\infty}\int_0^t V_s d(s) = \infty,\ a.s.$, for $i = 1, 2$. Applying the Dambis-Dubins-Schwarz theorem as stated in [19, p. 174] , M_t^1 and M_t^2 can be represented as time-changed Brownian motions, i.e. it holds in distribution

$$M_t^i = B^i_{\int_0^t V_s ds},\ i = 1, 2.$$

B_s^1 and B_s^2 are two $\mathbb{Q}$-standard Brownian motion w.r.t. the filtration generated by the stopping time τ. Furthermore, the two Brownian motions are independent since W_t^1 and W_t^2 are independent. Having this, S_t can be represented as

$$\begin{aligned}\log(S_t) &\overset{\mathrm{d}}{=} rt + \sqrt{1-\rho^2}B^1_{\int_0^t V_s ds} + \rho B^2_{\int_0^t V_s ds} - \frac{1}{2}\int_0^t V_s ds\\ &= rt + \boldsymbol{B}_{\int_0^t V_s ds} - \frac{1}{2}\int_0^t V_s ds\end{aligned}$$

where $\boldsymbol{B}_s = \sqrt{1-\rho^2}B_s^1 + \rho B_s^2$ is a $\mathbb{Q}$-standard Brownian motion since B_s^1 and B_s^2 are independent $\mathbb{Q}$-standard Brownian motions. Hence, the initial price of a Timer option is given by

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}f(S_\tau)\right] &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}f\left(S_0 e^{r\tau + \boldsymbol{B}_{\int_0^\tau V_s ds} - \frac{1}{2}\int_0^\tau V_s ds}\right)\right]\\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}f\left(S_0 e^{r\tau + \boldsymbol{B}_{\boldsymbol{b}} - \frac{1}{2}\boldsymbol{b}}\right)\right]\end{aligned}$$

where $\int_0^\tau V_s ds = \boldsymbol{b}$ by definition of τ. $\square$

Theorem 3.1.1 and its proof can also be found in [7] if f is the payoff function of a Call option, i.e. $f(x) = \max\{x - K, 0\}$ with some strike price K. The proof of theorem 3.1.1 shows the dynamics of the stock price process on the probability space with the time-changed filtration $\{\mathcal{F}_{\tau(v)}\}_{v\geq 0}$. In contrast to the filtered probability space of the financial market (3.1), which is $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q})$, the filtered probability space with the time-changed filtration is given by $\left(\Omega, \mathcal{F}, \{\mathcal{F}_{\tau(v)}\}_{v\geq 0}, \mathbb{Q}\right)$. In order to distinguish the two closely related, filtered probability spaces the first one is called time space and the latter one is called variance space. Like the stock price process, the variance process can also be transferred to the variance space. A sketch how to do so and how the financial market looks on the probability space with the time-changed filtration is stated at the end of subsection 3.1.1.

Furthermore, theorem 3.1.1 can be extended to compute Timer options at any given time $0 \leq t \leq \tau$ by updating the variance budget as suggested in [7]. The new variance budget $\tilde{\boldsymbol{b}}$ is computed as

$$\tilde{\boldsymbol{b}} = \boldsymbol{b} - \int_0^t V_s ds$$

and accordingly the new stopping time $\tilde{\tau}$ is given by

$$\tilde{\tau} = \inf \left\{ s \geq t : \int_t^s V_u du = \tilde{\boldsymbol{b}} \right\}.$$

Furthermore, theorem 3.1.1 can be used with the updated parameters $S_0 = S_t$ and $\tilde{\boldsymbol{b}}$ and $\tilde{\tau}$ as stated above. The next Corollary follows directly from theorem 3.1.1. But firstly, denote by $C_{BS}(S_0, K, r, \sigma, T)$ the Black-Scholes price of a European Call option, i.e.

$$C_{BS}(S_0, K, r, \sigma, T) = S_0 \mathcal{N}(d_1) - K e^{-rT} \mathcal{N}(d_2)$$

where

$$d_1 = \frac{\frac{\log S_0}{K} + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} \qquad \text{and} \qquad d_2 = d_1 - \sigma \sqrt{T}.$$

Accordingly, the price of a Timer Call for $r = 0$ is, as also shown in [7], as follows:

Corollary 3.1.2. *Let $C_{TC}(S_0, K, r, \boldsymbol{b})$ be the initial price of a Timer Call with strike price K, interest rate $r = 0$ and payoff function $f(S_\tau) = \max\{S_\tau - K, 0\}$. Then,*

$$\begin{aligned} C_{TC}(S_0, K, 0, \boldsymbol{b}) &= C_{BS}(S_0, K, 0, \sqrt{\frac{\boldsymbol{b}}{T}}, T) \\ &= S_0 \mathcal{N}(d_1) - K \mathcal{N}(d_2) \end{aligned}$$

where

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \frac{1}{2}\boldsymbol{b}}{\sqrt{\boldsymbol{b}}} \qquad \textit{and} \qquad d_2 = d_1 - \sqrt{\boldsymbol{b}}.$$

Proof. By theorem 3.1.1 the price of a Timer Call can be written as

$$\begin{aligned} C_{TC}(S_0, K, 0, \boldsymbol{b}) &= \mathbb{E}^{\mathbb{Q}}[f(S_\tau)] \\ &= \mathbb{E}^{\mathbb{Q}}\left[f\left(S_0 e^{B_{\boldsymbol{b}} - \frac{1}{2}\boldsymbol{b}}\right)\right] \\ &= \mathbb{E}^{\mathbb{Q}}\Big[\max\{S_0 e^{B_{\boldsymbol{b}} - \frac{1}{2}\boldsymbol{b}} - K, 0\}\Big]. \end{aligned}$$

Since B is a Brownian motion, it can easily be checked that the solution of the above expectation is indeed the Black-Scholes formula with $\sigma^2 T = \boldsymbol{b}$. $\square$

Remark 3.1.3. As stated in Bernard and Cui [7], having the price of a Timer Call, the price of a Timer Put can be calculated via the put-call parity for Timer options which is given by

$$C_{TC}^t - P_{TC}^t = S_t - K\mathbb{E}^{\mathbb{Q}}[e^{-r(\tau - t)}].$$

Alternatively, all the calculations done for a Timer Call can be likewise done for a Timer Put by just replacing the payoff function $f(S_\tau) = \max\{S_\tau - K, 0\}$ by the payoff function $f(S_\tau) = \max\{K - S_\tau, 0\}$.

The result of corollary 3.1.2 shows the strong advantage of Timer options, namely that the price of a Timer option is independent of the choice of the variance process if $r = 0$. In contrast, the price of a European Call option usually heavily depends on the choice of the variance process. Additionally, the sometimes very difficult fitting of the parameters of the variance process is totally avoided. As explained in [7], the benefit for an investor is that, for instance, a Timer Call is usually cheaper than a European Call because she does not have to pay a premium for the uncertainty of the future volatility. In other words, the implied volatility is often higher than the realized volatility, i.e. $\sigma_{implied} \geq \sigma_{realized}$. Comparing a European Call, priced with the Black-Scholes formula, with time to maturity T and volatility $\sigma_{implied}$ and a Timer Call which has a variance budget such that $\boldsymbol{b} = \sigma^2_{realized} T$, it holds

$$\begin{aligned}
C_{BS}(S_0, K, 0, \sigma_{implied}, T) &\geq C_{BS}(S_0, K, 0, \sigma_{realized}, T) \\
&= C_{BS}(S_0, K, 0, \sqrt{\frac{\boldsymbol{b}}{T}}, T) \\
&= C_{TC}(S_0, K, 0, \boldsymbol{b}).
\end{aligned}$$

Unfortunately, the independence from the variance process does not hold if $r > 0$. In this case, the expectation of the discounted payoff function is in general not analytically computable because the time-changed Brownian motion B in theorem 3.1.1 is not independent of the stopping time τ and additionally the distribution of τ is in general unknown. The dependence of τ and B is caused by the dependence of τ on the trajectory of the variance process, in particular on the trajectory of B^2. In fact, one can show that in distribution it holds:

$$\tau_{\boldsymbol{b}} \stackrel{d}{=} \int_0^{\boldsymbol{b}} \frac{1}{X_s} ds$$

where X_s is a solution to an SDE which is driven by B^2. A proof of that is given in theorem 3.1.5. In order to compute the price of a Timer option, numerical methods are needed. One way is to find an alternative representation of the stock price process such that this dependence is no problem and efficient Monte-Carlo methods can be used. Another way is to use the Feynman-Kac formula and compute the expectation by means of solving the corresponding partial differential equation.

3.1.1 Theoretical results for using Monte-Carlo methods

A naive way to use Monte Carlo methods is to just draw samples from (S_τ) but this includes simulating the whole paths oft the stock price process and the variance process. Therefore, this is a very slow algorithm. Another way is to apply theorem 3.1.1 which states that one could just draw samples from the joint distribution $(B_{\boldsymbol{b}}, \tau)$. Unfortunately, this is in general not possible if the stock price process and the variance process are correlated because the joint distribution is unknown and it is only in easy cases of the variance process possible to find the exact density function. However, it is possible to make use of theorem 3.1.1. The first step is to find a way to get rid of the painful dependence between $B_{\boldsymbol{b}}$ and τ. This is the purpose of the following theorem:

Theorem 3.1.4. *Let S_t and V_t be like in* (3.1) *but with β being differentiable and τ given by* (3.2)*. Then, it holds in distribution*

$$S_\tau = S_0 \exp\left\{ r\tau + \sqrt{1-\rho^2} B^1_{\boldsymbol{b}} + \rho(f(V_\tau) - f(V_0) - \int_0^\tau h(s, V_s)ds) - \frac{1}{2}\boldsymbol{b} \right\}$$

where

$$\begin{aligned} f(V_t) &= \int_0^{V_t} \frac{\sqrt{u}}{\beta(u)} du, \\ h(t, V_t) &= \alpha(t, V_t) f'(V_t) + \frac{1}{2}\beta^2(V_t) f''(V_t) \end{aligned} \tag{3.3}$$

and B^1 is a standard Brownian motion.

This theorem and the following proof are not new and can be found in [7] but with α being independent of t, i.e. $\alpha(t, V) = \alpha(V)$.

Proof. Compared to the proof of theorem 3.1.1 the Dambis-Dubins-Schwarz theorem is only used for the first continuous local martingale

$$\begin{aligned} M^1_t &= \int_0^t \sqrt{V_s} dW^1_s \\ &\overset{\mathrm{d}}{=} B^1_{\int_0^t V_s ds} \end{aligned} \tag{3.4}$$

where B^1 is defined like in theorem 3.1.1. Now, one can transform the variance process by applying Itô's lemma to the function $f(x) = \int_0^x \frac{\sqrt{u}}{\beta(u)} du$. This yields

$$\begin{aligned} df(V_t) &= f'(V_t)dV_t + \frac{1}{2} f''(t, V_t) d\left[V\right]_t \\ &= \alpha(t, V_t) f'(t, V_t) + \frac{1}{2}\beta^2(V_t) f''_t(V_t)dt + \sqrt{V_t} dW^2_t \\ &= h(t, V_t)dt + \sqrt{V_t} dW^2_t. \end{aligned}$$

with $h(t,V_t)dt = \alpha(t,V_t)f'(V_t) + \frac{1}{2}\beta^2(V_t)f''(V_t)$. Thus, M_t^2 can be written as

$$\begin{aligned} M_t^2 &= \int_0^t \sqrt{V_t}dW_t^2 \\ &= f(V_t) - f(V_0) - \int_0^t h(s,V_s)ds \end{aligned} \tag{3.5}$$

Using equations (3.4) and (3.5), it holds

$$S_t \stackrel{\mathrm{d}}{=} S_0 \exp\left\{rt + \sqrt{1-\rho^2}B^1_{\int_0^t V_s ds} + \rho(f(V_t) - f(V_0) - \int_0^t h(s,V_s)ds) - \frac{1}{2}\int_0^t V_s ds\right\}$$

Letting $t = \tau$ and recalling that $\int_0^\tau V_s ds = \boldsymbol{b}$ the claim follows. □

Theorem 3.1.4 states that S_τ can be directly drawn by drawing $B^1_{\boldsymbol{b}}$, which is a normally distributed random variable with variance $\boldsymbol{b}$, and jointly simulating τ and the path of the variance process. The next theorem shows how the latter two mentioned random variables can be simulated efficiently as suggested in [7].

Theorem 3.1.5. *Let V_t and τ be like in theorem 3.1.4 and 3.2 respectively. Then, the joint distribution of τ, V_τ and $H(\tau,V_\tau) = \int_0^\tau h(s,V_s)ds$, with h defined as in* (3.3)*, is given by*

$$(\tau, V_\tau, H(\tau,V_\tau)) \sim \left(\tau(\boldsymbol{b}), X_{\boldsymbol{b}}, \int_0^{\boldsymbol{b}} \frac{h(\tau(s),X_s)}{X_s}ds\right) \tag{3.6}$$

with the process X_v solving the following SDE

$$df(X_v) = \frac{h(\tau(v),X_v)}{X_v}dv + dB_v,\ \ X_0 = V_0,$$

where B_v is a standard Brownian motion and

$$\tau(v) = \int_0^v \frac{1}{X_s}ds.$$

The advantage of theorem 3.1.5 is that τ and V_τ can be drawn by simulating the stochastic process X_v on a fixed interval. Instead of simulating the variance process V_t on the stochastic interval $[0,\tau]$, one only needs to simulate X_v on the fixed interval $[0,\boldsymbol{b}]$. In fact, this gives a better control over the error of the simulation. Furthermore, it actually is a reduction of the dimension because instead of simulating the whole path of the stock price process it is sufficient to draw a normally distributed random variable with variance $\boldsymbol{b}$.

Proof. Let $\tau(v)$ be defined as

$$\tau(v) = \inf\{t \geq 0 : \xi^c(t) = v\},\ \ v \geq 0,$$

where $\xi^c(t) = \int_0^t V_s ds$ is the stochastic clock. Since $V_t > 0$ by assumption, $\xi^c(t)$ is strictly increasing. Hence, $\tau(v)$ is the inverse function of $\xi^c(t)$, i.e. $\xi^c(\tau(v)) = v$, and it holds

$$\tau(v) = \int_0^v \tau'(s)ds = \int_0^v \frac{1}{(\xi^c)'(\tau(s))}ds = \int_0^v \frac{1}{V_{\tau(s)}}ds \tag{3.7}$$

because $(\xi^c(\tau(v)))' = (\xi^c)'(\tau(v))\tau'(v) = 1$ and $(\xi^c)'(\tau(v)) = V_{\tau(v)}$. Letting f and h be like in (3.3) and recalling that applying Itô's lemma to the function $f(x) = \int_0^x \frac{\sqrt{u}}{\beta(u)}du$ leads to

$$df(V_t) = h(t, V_t)dt + \sqrt{V_t}dW_t^2 \tag{3.8}$$

one can again apply the Dambis-Dubins-Schwarz theorem to obtain

$$M(\tau(v)) = \int_0^{\tau(v)} \sqrt{V_t}dW_t^2 \overset{\mathrm{d}}{=} B_v$$

where B_v is a standard Brownian motion. Integrating equation (3.8) from 0 to $\tau(v)$ yields

$$\begin{aligned} f(V_{\tau(v)}) &= f(V_0) + \int_0^{\tau(v)} h(t, V_t)dt + \int_0^{\tau(v)} \sqrt{V_t}dW_t^2 \\ &= f(V_0) + \int_0^{\tau(v)} h(t, V_t)dt + B_v \end{aligned}$$

which is in the differential notation

$$\begin{aligned} df(V_{\tau(v)}) &= h(\tau(v), V_{\tau(v)})d\tau(v) + dB_v \\ &\overset{(3.7)}{=} \frac{h(\tau(v), V_{\tau(v)})}{V_{\tau(v)}}dv + dB_v. \end{aligned} \tag{3.9}$$

Recalling that $H(\tau(v), V_{\tau(v)}) = \int_0^{\tau(v)} h(t, V_t)dt$, it holds

$$\begin{aligned} dH(\tau(v), V_\tau) &= h(\tau(v), V_{\tau(v)})d\tau(v) \\ &= \frac{h(\tau(v), V_{\tau(v)})}{V_{\tau(v)}}dv. \end{aligned} \tag{3.10}$$

Letting $X_v = V_{\tau(v)}$, one finally gets when integrating (3.7), (3.9) and 3.10 from 0 to $\boldsymbol{b}$

$$(\tau, V_\tau, H(\tau, V_\tau)) \sim \left(\int_0^{\boldsymbol{b}} \frac{1}{X_s}ds, X_{\boldsymbol{b}}, \int_0^{\boldsymbol{b}} \frac{h(\tau(s), X_s)}{X_s}ds \right)$$

where X_v solves the following SDE

$$df(X_v) = \frac{h(\tau(v), X_v)}{X_v}dv + dB_v, \ X_0 = V_0,$$

and

$$\tau(v) = \int_0^v \frac{1}{X_s}ds.$$

□

Remark 3.1.6. Theorem 3.1.5 and its proof are not new and can be found in [7] but with α being independent of the time variable t, i.e. $\alpha(t,V) = \alpha(V)$. Hence, theorems 3.1.4 and 3.1.5 extend the theory to variance processes which are time dependent in the drift term. For instance, variance processes are include which show a seasonal behaviour in their drift term.

As an example how theorems 3.1.4 and 3.1.5 can be used, consider again the Timer Call.

Corollary 3.1.7. *Denote by $C_{TC}(S_0,K,r,\boldsymbol{b})$ the initial price of a Timer Call with strike price K, interest rate r and variance budget $\boldsymbol{b}$. Then, having the payoff function $f(S_\tau) = \max\{S_\tau - K, 0\}$, it holds*

$$C_{TC}(S_0,K,r,\boldsymbol{b}) = \mathbb{E}^{\mathbb{Q}}\left[C_{BS}(\tilde{S}_0,K,r,\tilde{\sigma},\tau)\right]$$

where C_{BS} is the Black-Scholes price of a European Call option and

$$\begin{aligned}\tilde{S}_0 &= S_0 e^{\rho(f(V_\tau)-f(V_0)-H(\tau,V_\tau)-\frac{\rho}{2}\boldsymbol{b})}\\ \tilde{\sigma} &= \sqrt{\frac{\boldsymbol{b}}{\tau}}.\end{aligned}$$

Proof. Using the representation for S_τ from theorem 3.1.4 and the tower property for iterated expectations one obtains

$$\begin{aligned}C_{TC}(S_0,K,r,\boldsymbol{b}) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}\max\{S_\tau - K, 0\}\right]\\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}\max\left\{S_0 e^{r\tau+\sqrt{1-\rho^2}B_{\boldsymbol{b}}+\rho(f(V_\tau)-f(V_0)-H(\tau,V_\tau))-\frac{1}{2}\boldsymbol{b}} - K, 0\right\}\right]\\ &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}\max\left\{S_0 e^{r\tau+\sqrt{1-\rho^2}B_{\boldsymbol{b}}+\rho(f(V_\tau)-f(V_0)-H(\tau,V_\tau))-\frac{1}{2}\boldsymbol{b}} - K, 0\right\}\middle|(\tau,V_\tau,H(\tau,V_\tau))\right]\right]\\ &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}\max\left\{\tilde{S}_0 e^{r\tau+\sqrt{1-\rho^2}B_{\boldsymbol{b}}-\frac{1-\rho^2}{2}\boldsymbol{b}} - K, 0\right\}\middle|(\tau,V_\tau,H(\tau,V_\tau))\right]\right]\end{aligned}$$

where $\tilde{S}_0 = S_0 e^{\rho(f(V_\tau)-f(V_0)-H(\tau,V_\tau)-\frac{\rho}{2}\boldsymbol{b})}$. Since $\tilde{S}_0$ is just a constant when conditioning on τ, V_τ and $H(\tau,V_\tau)$, the conditional expectation is the Black-Scholes formual, i.e.

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}\max\left\{\tilde{S}_0 e^{r\tau+\sqrt{1-\rho^2}B_{\boldsymbol{b}}-\frac{1-\rho^2}{2}\boldsymbol{b}} - K, 0\right\}\middle|(\tau,V_\tau,H(\tau,V_\tau))\right] &= \tilde{S}_0\mathcal{N}(\tilde{d}_1) - Ke^{-r\tau}\mathcal{N}(\tilde{d}_2)\\ &= C_{BS}(\tilde{S}_0,K,r,\tilde{\sigma},\tau)\end{aligned}$$

where $\tilde{S}_0 = S_0 e^{\rho(f(V_\tau)-f(V_0)-H(\tau,V_\tau)-\frac{\rho}{2}\boldsymbol{b})}$, $\tilde{\sigma} = \sqrt{\frac{\boldsymbol{b}}{\tau}}$ and

$$\tilde{d}_1 = \frac{\log\left(\frac{\tilde{S}_0}{K}\right) + r\tau + \frac{(1-\rho^2)}{2}\boldsymbol{b}}{\sqrt{(1-\rho^2)\boldsymbol{b}}} \qquad \tilde{d}_2 = \tilde{d}_1 - \sqrt{(1-\rho^2)\boldsymbol{b}}.$$

Taking the expectation gives the desired result. □

This result is the basis for the Monte-Carlo simulation done in section 4.2 and can also be found in [7]. Additionally, Bernard and Cui found a very powerful control variate which uses the property that the price of a Timer Call can be computed exactly if the interest rate is zero (cf. corollary 3.1.2).
Another interesting observation is that the Greeks for Timer Calls are similar to those of European Calls priced with the Black-Scholes formula. Some interesting Greeks for Timer Calls are given in table 3.1.

Table 3.1: The Greeks for Timer Calls

	$r > 0$	$r = 0$
Delta $\left(\frac{\partial \mathbf{C}}{\partial \mathbf{S}}\right)$	$\mathbb{E}\left[e^{\rho(f(V_\tau)-f(V_0)-H(\tau,V_\tau)-\frac{\rho}{2}\boldsymbol{b})}\mathcal{N}(\tilde{d}_1)\right]$	$\mathcal{N}(d_1)$
Gamma $\left(\frac{\partial \mathbf{C}}{\partial \mathbf{S^2}}\right)$	$\mathbb{E}\left[\frac{\mathcal{N}'(\tilde{d}_1)}{S_0\sqrt{(1-\rho^2)\boldsymbol{b}}}\right]$	$\frac{\mathcal{N}'(d_1)}{S_0\sqrt{\boldsymbol{b}}}$
Vega $\left(\frac{\partial \mathbf{C}}{\partial \boldsymbol{b}}\right)$	$\mathbb{E}\left[\frac{S_0\mathcal{N}'(\tilde{d}_{1\vert\rho=0})}{2\sqrt{\boldsymbol{b}}} + \frac{r}{V_\tau}Ke^{-r\tau}\mathcal{N}(\tilde{d}_{2\vert\rho=0})\right]^{(1)}$	$\frac{S_0\mathcal{N}'(d_1)}{2\sqrt{\boldsymbol{b}}}$
Rho $\left(\frac{\partial \mathbf{C}}{\partial \mathbf{r}}\right)$	$\mathbb{E}\left[K\tau e^{-r\tau}\mathcal{N}(\tilde{d}_2)\right]$	$\mathbb{E}[\tau\mathcal{N}(\tilde{d}_2^{r=0})]$

(1) The Vega is only exact if $\rho = 0$. In case of $\rho \neq 0$ it is not possible to calculate the derivative because it would include taking the derivative of a Brownian motion. An derivation of Vega is given in the appendix A.1.
(2) It is assumed that all condition necessary for switching derivation and integration are satisfied.
(3) $\tilde{d}_1$ and $\tilde{d}_2$ are as in corollary 3.1.7 and d_1 as in corollary 3.1.2.

Before continuing with the next subsection, the sketch of how to transfer the variance process to the variance space is shown. Using the associativity of integration for the Itô integral [19], the dynamics of the variance process can be written as

$$\begin{aligned} &dV_t = \alpha(t, V_t)dt + \beta(V_t)dW_t^2 \\ \Leftrightarrow\ &dV_t = \frac{\alpha(t, V_t)}{V_t}d\int_0^t V_s ds + \frac{\beta(V_t)}{\sqrt{V_t}}d\int_0^t \sqrt{V_s}dW_s^2 \end{aligned}$$

Applying the Dambis-Dubins-Schwarz theorem, the variance process above is equivalent in distribution to

$$dV_t = \frac{\alpha(t, V_t)}{V_t}d\xi_t^c + \frac{\beta(V_t)}{\sqrt{V_t}}dB^2_{\xi_t^c}$$

where B_v^2 is a standard Brownian motion w.r.t. the filtration generated by the stopping time $\tau(v)$. With $v := \xi_t^c$, $t = \tau(\xi_t^c) = \tau(v)$ and $V_{\tau(v)} = \tilde{V}_v$ it follows:

$$d\tilde{V}_v = \frac{\alpha(\tau(v), \tilde{V}_v)}{\tilde{V}_v}dv + \frac{\beta(\tilde{V}_v)}{\sqrt{\tilde{V}_v}}dB_v^2.$$

Recalling that the time-changed stock price process solves the SDE

$$dS_t = S_t r d\tau(v) + S_t d\left(\sqrt{1-\rho^2} B^1_{\int_0^t V_s ds} + \rho B^2_{\int_0^t V_s ds}\right)$$

and defining $S_{\tau(v)} = \tilde{S}_v$, the financial market on the variance space $(\Omega, \mathcal{F}, \{\mathcal{F}_\tau\}_{v\geq 0}, \mathbb{Q})$ looks as follows:

$$\begin{aligned} d\tilde{S}_v &= r\frac{\tilde{S}_v}{\tilde{V}_v} dv + \tilde{S}_v d\left(\sqrt{1-\rho^2} B^1_v + \rho B^2_v\right) \\ d\tilde{V}_v &= \frac{\alpha(\tau(v), \tilde{V}_v)}{\tilde{V}_v} dv + \frac{\beta(\tilde{V}_v)}{\sqrt{\tilde{V}_v}} dB^2_v \end{aligned} \tag{3.11}$$

3.1.2 Feynman-Kac formula and Timer options

The partial differential equation obtained in the next theorem can also be found in [21] but the proof given there is based on considering a riskless portfolio and using Hestons argument of a market price of volatility risk [15]. Here, the proof is based on the Feynman-Kac formula.

Theorem 3.1.8. *Let S_t and V_t be defined as in* (3.1) *but with $\alpha(t, V) = \alpha(V)$. Furthermore, let τ be given by* (3.2) *and $f(\xi_t, V_t, S_t)$ be some measurable function w.r.t. the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Then, the solution of the partial differential equation*

$$f_{\xi^c} + \frac{\alpha(V)}{V} f_V + \frac{rS}{V} f_S + \frac{1}{2} S^2 f_{SS} + \frac{1}{2}\frac{\beta^2(V)}{V} f_{VV} + \frac{S\rho\beta(V)}{\sqrt{V}} f_{SV} - \frac{r}{V} f = 0 \tag{3.12}$$

with boundary condition $f(\boldsymbol{b}, V, S) = g(S)$, $\forall S > 0$, can be represented as

$$f(\xi_t, V_t, S_t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(\tau(b)-\tau(\xi_t))} g(S_\tau) \mid S_t = x\right]$$

on the domain $(\xi, V, S) = [0, \boldsymbol{b}] \times (0, \infty) \times (0, \infty)$ and with g being the terminal payoff function.

Proof. The proof is basically an application of a multidimensional version of the Feynman-Kac formula [27][p. 25-26]. The only delicate issue is that the time interval $[0, \tau]$ is stochastic. Therefore, consider a fixed time interval $[0, T]$ with $T < \infty$ for now. According to Li [21], the price of a Timer option does not depend on the time t because of its perpetuity property. Thus, it only depends on ξ^c_t, V_t and S_t. Let $f(\xi_t, V_t, S_t)$ be the price function. Using Itô's lemma for $e^{-rt} f(\xi_t, V_t, S_t)$ yields:

$$\begin{aligned} de^{-rt} f(\xi_t, V_t, S_t) = e^{-rt} &\left(V_t f_{\xi^c_t} + \alpha(V_t) f_V + rS_t f_S + \frac{1}{2} S_t^2 V_t f_{SS} + \frac{1}{2}\beta^2(V_t) f_{VV} + S_t\rho\beta(V_t)\sqrt{V_t} f_{SV} - rf\right) dt \\ &+ \underbrace{e^{-rt}\left(f_V \beta(V_t) dW^2_t + f_S S_t \sqrt{V_t} d(\sqrt{1-\rho^2} W^1_t + \rho W^2_t)\right)}_{=:M_t} \end{aligned}$$

Following [27], the process $e^{-rt} f(\xi_t, V_t, S_t)$ is a continuous local martingale. Since M_t is a also a continuous local martingale and by [27][Theorem 1.3.17], the price function $f(\xi_t, V_t, S_t)$ solving the partial differential equation

$$V_t f_{\xi_t^c} + \alpha(V_t) f_V + rS_t f_S + \frac{1}{2} S_t^2 V_t f_{SS} + \frac{1}{2}\beta^2(V_t) f_{VV} + S_t \rho \beta(V_t)\sqrt{V_t} f_{SV} - rf = 0$$

with boundary condition $f(\xi_T, V_T, S_T) = g(S_T)$ can be represented as

$$f(\xi_t, V_t, S_t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} g(S_T) \mid S_t = x\right].$$

This representation holds for any arbitrary $T < \infty$. Now note, that the stopping time τ is non anticipating and it is almost surely finite since the integrated variance is assumed to converge to infinite as $T \to \infty$. Therefore, the representation also holds for $T = \tau$.

Due to the perpetuity of the Timer option, the domain is given by $(\xi, V, S) = [0, \boldsymbol{b}] \times (0, \infty) \times (0, \infty)$ which means it does not include the stochastic time interval $[0, \tau]$. To emphasize the role of the stochastic clock, it might be convenient to divide the partial differential equation by the variance which leads to the partial differential equation

$$f_{\xi^c} + \frac{\alpha(V)}{V} f_V + \frac{rS}{V} f_S + \frac{1}{2} S^2 f_{SS} + \frac{1}{2}\frac{\beta^2(V)}{V} f_{VV} + \frac{S\rho\beta(V)}{\sqrt{V}} f_{SV} - \frac{r}{V} f = 0$$

defined on the domain $(\xi, V, S) = [0, \boldsymbol{b}] \times (0, \infty) \times (0, \infty)$ and with boundary condition $f(\boldsymbol{b}, V_\tau, S_\tau) = g(S_\tau)$. $\square$

Theorem 3.1.8 shows an alternative method to price Timer options by solving the partial differential equation (3.12) which can be numerically done by means of finite differences. Note, that a contract h written on the stock price process S_t from above with a fixed time to maturity T but having the same payoff function $g(S_T)$ leads to the following partial differential equation:

$$h_t + \alpha(V) h_V + rS h_S + \frac{1}{2} V S^2 h_{SS} + \frac{1}{2}\beta^2 h_{VV} + S\rho\beta(V)\sqrt{V} h_{SV} - rh = 0$$

As one can see, the integrated variance ξ seems to play the role of time compared to options with a fixed time to maturity.

Remark 3.1.9. The partial differential equation (3.12) could also have been established by looking at the financial market on the variance space (3.11). Applying Itô's lemma to $e^{-r\tau(v)} f(\xi, V, S)$, but this time with $\xi = v$ and S and V following the dynamics in (3.11), leads to

$$\begin{aligned} de^{-r\tau(v)} f(v, \tilde{V}_v, \tilde{S}_v) = e^{-r\tau(v)} &\left(f_v + \frac{\alpha(\tilde{V}_v)}{\tilde{V}_v} f_{\tilde{V}} + r\frac{\tilde{S}_v}{\tilde{V}_v} f_{\tilde{S}} + \frac{1}{2}\tilde{S}_v^2 f_{\tilde{S}\tilde{S}} + \frac{1}{2}\frac{\beta^2(\tilde{V}_v)}{\tilde{V}_v} f_{\tilde{V}\tilde{V}} + \tilde{S}_v \frac{\rho\beta(\tilde{V}_v)\sqrt{\tilde{V}_v}}{\tilde{V}_v} f_{\tilde{S}\tilde{V}} + \frac{r}{\tilde{V}_v} f \right) dv \\ &+ e^{-r\tau(v)} \left(\tilde{S}_v f_{\tilde{S}} d\left(\rho B_v^1 + \sqrt{1-\rho^2} B_v^2\right) + \frac{\beta(\tilde{V}_v)}{\sqrt{\tilde{V}_v}} f_{\tilde{V}} dB_v^2 \right) \end{aligned}$$

Since $e^{-r\tau(v)} f(v, \tilde{V}_v, \tilde{S}_v)$ is a martingale, the drift term should vanish, i.e.

$$f_v + \frac{\alpha(\tilde{V}_v)}{\tilde{V}_v} f_{\tilde{V}} + r\frac{\tilde{S}_v}{\tilde{V}_v} f_{\tilde{S}} + \frac{1}{2}\tilde{S}_v^2 f_{\tilde{S}\tilde{S}} + \frac{1}{2}\frac{\beta^2(\tilde{V}_v)}{\tilde{V}_v} f_{\tilde{V}\tilde{V}} + \tilde{S}_v \frac{\rho\beta(\tilde{V}_v)}{\sqrt{\tilde{V}_v}} f_{\tilde{S}\tilde{V}} - \frac{r}{\tilde{V}_v} f = 0$$

on the domain $(v, \tilde{V}, \tilde{S}) = [0, \boldsymbol{b}] \times (0, \infty) \times (0, \infty)$ with boundary condition $f(\boldsymbol{b}, \tilde{V}_{\boldsymbol{b}}, \tilde{S}_{\boldsymbol{b}}) = g(\tilde{S}_{\boldsymbol{b}})$. As it turns out, one has to solve the same problem as in 3.1.8. The solution to the partial differential equation above can be represented as

$$f(v, \tilde{V}_v, \tilde{S}_v) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(\tau(\boldsymbol{b})-\tau(v))} g(\tilde{S}_{\boldsymbol{b}} T) \mid \tilde{S}_v = x\right].$$

Since $\tilde{S}_{\boldsymbol{b}} = S_{\tau(\boldsymbol{b})}$ and $\tilde{S}_v = S_{\tau(\xi_t^c)} = S_t$, it has the same value as the expectation in 3.1.8.

3.2 Pricing Timer Calls - Numerical results for the Heston model

In this section the theoretical results of the previous section are used to implement numerical algorithms in order to price Timer Calls under the Heston model [15]. Under the Heston model, the dynamics under the risk neutral measure $\mathbb{Q}$ of the stock price process and the variance process are as follows:

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{V_t} S_t d(\sqrt{1-\rho^2} W_t^1 + \rho W_t^2) \\ dV_t &= \kappa(\theta - V_t)dt + \gamma\sqrt{V_t} dW_t^2 \end{aligned} \tag{3.13}$$

with constants κ, θ and γ fulfilling the relation $2\kappa\theta > \gamma^2$ which means that the variance process is strictly positive according to the Feller test [19]. Furthermore, the Timer Call is characterized by the payoff function $f(S_\tau) = \max\{S_\tau - K, 0\}$ where τ is given by (3.2), K is the strike price and $\boldsymbol{b}$ the variance budget.

3.2.1 Explicit Monte-Carlo algorithm

Now recall the result of corollary 3.1.7 which states that the price of a Timer Call can be computed by taking the expectation of the Black-Scholes formula which can be numerically computed by using Monte-Carlo methods. In particular, one can proceed as follows:

Step 1: Use theorem 3.1.5 to draw a sample of (τ, V_τ). In the Heston case, it is $\alpha(t, v) = \kappa(\theta - v)$ and $\beta(v) = \gamma\sqrt{v}$. Hence, the functions f and h reduce to

$$\begin{aligned} f(v) &= \int_0^v \frac{\sqrt{u}}{\beta(u)} = \frac{v}{\gamma} \\ h(\tau, v) &= \alpha(\tau, v) f'(v) + \frac{1}{2}\beta^2(v) f''(v) = \frac{\kappa(\theta - v)}{\gamma}. \end{aligned}$$

From this it follows that the function $H_\tau = \int_0^{\boldsymbol{b}} \frac{h(\tau(v), X_v)}{X_v} dv = \frac{\kappa(\theta\tau - \boldsymbol{b})}{\gamma}$ is just a linear combination of τ and therefore drawing the above sample is enough which is done by simulating the process

X_v on $[0, \boldsymbol{b}]$ given by the SDE

$$dX_v = \left(\frac{\kappa\theta}{X_v} - \kappa\right) dv + \gamma dB_v,\ X_0 = V_0.$$

This, for instance, can be done by means of Euler's method [13]. Then, the sample of τ and V_τ is given by $\left(\int_0^{\boldsymbol{b}} \frac{1}{X_v} dv, X_{\boldsymbol{b}}\right)$.

Step 2: Having n samples of (τ, V_τ) at hand, one can approximate the desired expectation. In particular, let C_{TC}^{mc} be the Monte-Carlo estimator for the price of a Timer Call. Then according to the theory, the Timer Call can be approximated by

$$\begin{aligned} C_{TC}^{mc} &= \frac{1}{n}\sum_{i=1}^{n} C_{BS}(\tilde{S}_{0,i}, K, r, \tilde{\sigma}_i, \tau_i) \\ &= \frac{1}{n}\sum_{i=1}^{n} \left(\tilde{S}_{0,i}\mathcal{N}(\tilde{d}_{1,i}) - Ke^{-r\tau_i}\mathcal{N}(\tilde{d}_{2,i})\right) \end{aligned}$$

with

$$\begin{aligned} \tilde{S}_{0,i} &= S_0 e^{\frac{\rho}{\gamma}(V_{\tau,i} - V_0 - \kappa(\theta\tau_i - \boldsymbol{b})) - \frac{\rho^2}{2}\boldsymbol{b}} \\ \tilde{d}_{1,i} &= \frac{\log\left(\frac{S_0}{K}\right) + r\tau_i + \frac{\rho}{\gamma}\left(V_{\tau,i} - V_0 - \kappa(\theta\tau_i - \boldsymbol{b})\right) + \left(\frac{1}{2} - \rho^2\right)\boldsymbol{b}}{\sqrt{(1-\rho^2)\boldsymbol{b}}} \\ \tilde{d}_{2,i} &= \tilde{d}_{1,i} - \sqrt{(1-\rho^2)\boldsymbol{b}}. \end{aligned}$$

Step 3: In order to improve the convergence of the Monte-Carlo simulation, one can use corollary 3.1.2 to establish a powerful control variate as suggested in [7]. Corollary 3.1.2 states that the price of a Timer Call can be exactly computed when $r = 0$. Denote by $C_{TC|r=0}$ the price of a Timer Call when the interest rate is zero and define the corresponding control variate CV as

$$CV = \frac{1}{n}\sum_{i=1}^{n}\left(\tilde{S}_{0,i}\mathcal{N}(\hat{d}_{1,i}) - K\mathcal{N}(\hat{d}_{2,i})\right)$$

where

$$\begin{aligned} \hat{d}_{1,i} &= \frac{\log\left(\frac{S_0}{K}\right) + \frac{\rho}{\gamma}\left(V_{\tau,i} - V_0 - \kappa(\theta\tau_i - \boldsymbol{b})\right) + \left(\frac{1}{2} - \rho^2\right)\boldsymbol{b}}{\sqrt{(1-\rho^2)\boldsymbol{b}}} \\ \hat{d}_{2,i} &= \hat{d}_{1,i} - \sqrt{(1-\rho^2)\boldsymbol{b}}. \end{aligned}$$

Note, that the above defined control variate has indeed the expectation $C_{TC|r=0}$. Thus, letting $\lambda := \frac{Cov(C_{TC}^{mc}, CV)}{Var(CV)}$, which is the optimal choice as shown in [25], the variance reduced approximation is given by

$$C_{TC}^{mc} - \lambda(CV - C_{TC|r=0}).$$

3.2.2 Monte-Carlo simulation results and sensitivity analysis

The results shown here are obtained by implementing the previously discussed Monte-Carlo method in Matlab. A routine is shown in A.2.1. As values for the parameters the ones given in [7] are chosen, i.e. :

Table 3.2: Parameters Timer Call

S_0	K	r	$\boldsymbol{b}$	V_0	κ	γ	θ
100	100	0.04	0.0265	0.0625	2	0.1	0.0324

The variance budget seems to be very low but having 0.0265 "variance units" means that the expected time to maturity is one year if the underlying asset has a volatility of around 16% per year.
Before stating the numerical results, note that in this setting the price of a Timer Call is $C_{TC|r=0} = 6.4871$ when the interest rate is zero. In table 3.3 some results are shown. The parameters N and M are the number of simulations and the number of discretization steps used to simulate X_v respectively. As one can observe, the control variate reduces the error massively. In the case when $\rho = 0$, the control variate has no variance reducing effect because it breaks down to a deterministic value which is exactly $C_{TC|r=0} = 6.4871$. Furthermore, it can be seen that a higher interest rate leads to a higher price and that the price is antiproportional to the correlation between the stock price process and its variance process. In order to have a closer look at the influence of different parameters, a sensitivity analysis is added.

Table 3.3: Monte Carlo simulation Timer Calls

Number of simulations N	Number of discretization steps M	Correlation coefficient ρ	Monte Carlo simulation without control variate		Monte Carlo simulation with control variate	
		-0.8	6.495	(0.0143)	6.4871	(exact)
300000	500	0	6.4871	(exact)	6.4871	(exact)
		0.8	6.4973	(0.0143)	6.4871	(exact)
		-0.8	6.4881	(0.0143)	6.4871	(exact)
300000	1000	0	6.4871	(exact)	6.4871	(exact)
		0.8	6.4771	(0.0143)	6.4871	(exact)
		-0.8	6.487	(0.0101)	6.4871	(exact)
600000	500	0	6.4871	(exact)	6.4871	(exact)
		0.8	6.4744	(0.0101)	6.4871	(exact)
		-0.8	6.4703	(0.0101)	6.4871	(exact)
600000	1000	0	6.4871	(exact)	6.4871	(exact)
		0.8	6.4731	(0.0101)	6.4871	(exact)
		-0.8	7.6483	(0.0158)	7.6322	(0.0007)
300000	500	0	7.5337	(0.0003)	7.5337	(0.0003)
		0.8	7.4194	(0.0151)	7.4323	(0.0007)
		-0.8	7.6105	(0.0158)	7.6325	(0.0006)
300000	1000	0	7.5336	(0.0003)	7.5336	(0.0003)
		0.8	7.4197	(0.0151)	7.4321	(0.0007)
		-0.8	7.6462	(0.0112)	7.6320	(0.0005)
600000	500	0	7.5342	(0.0002)	7.5342	(0.0002)
		0.8	7.4336	(0.0107)	7.4319	(0.0005)
		-0.8	7.6422	(0.0112)	7.6316	(0.0005)
600000	1000	0	7.5342	(0.0002)	7.5342	(0.0002)
		0.8	7.4282	(0.0107)	7.4331	(0.0005)

The first four simulation sets are done with $r = 0$ and the second four simulation sets are done with $r = 0.04$.

Sensitivity to the variance budget b

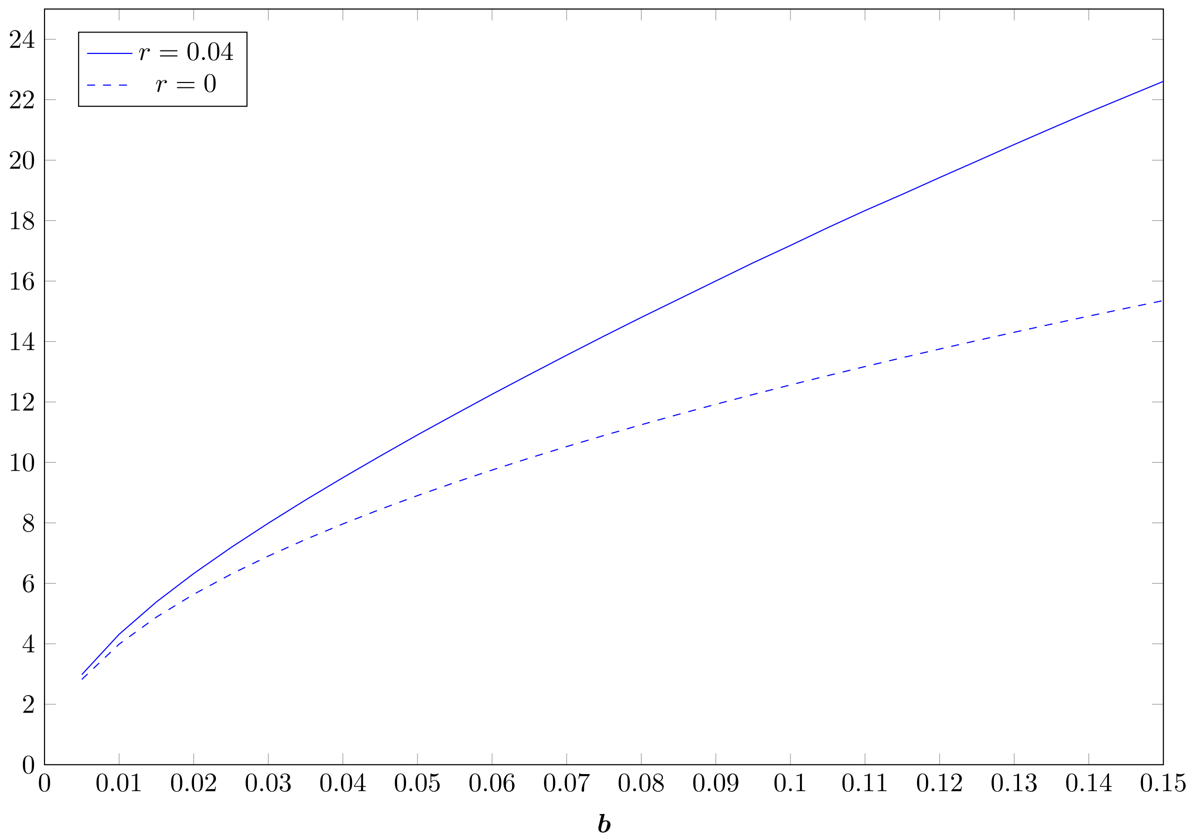

Figure 3.1: Plot of the influence of the variance budget $\boldsymbol{b}$ on the price of a Timer Call (ordinate). Besides the explicitly given parameters r and $\boldsymbol{b}$, all parameters are as in table 3.2.

The Timer Call is increasing in the variance budget variable $\boldsymbol{b}$ which can be seen in figure 3.1. Since a continuous monitoring of the realized variance is assumed, it shows that Timer Calls are slightly underpriced. In fact, monitoring the realized variance daily in practise means that the holder of a Timer Call gets the variance budget $\boldsymbol{b}$ plus an overshoot $\mathbf{b}_{overshoot}$ as mentioned in [7]. This overshoot depends on the volatility of the stock and therefore the higher the variance budget the relatively less underpriced is the Timer Call. For instance, having the parameters of table 3.2 but with $r = 0$ the price of a Timer Call is 6.4871. Letting the stock move 4%, although the variance budget is actually fully consumed, leads to an overshoot of the variance budget of about 0.0015. The price of a Timer Call having a variance budget of $0.0265 + 0.0015 = 0.028$ is 6.6723. Another problem with the daily monitoring is that the estimator $\boldsymbol{RV}$ as stated in 2.1 is only asymptotically unbiased. Indeed, considering the model (3.1) for the stock price process the estimator for a given partition $0 = t_0 < t_1 < \cdots < t_n = T$ is given by

$$\begin{aligned}
\boldsymbol{RV} &= \sum_{k=0}^{n-1} \log\left(\frac{S(t_{k+1})}{S(t_k)}\right)^2 \\
&= \sum_{k=0}^{n-1} \log\left(\frac{S_0 \exp\{rt_{k+1} + \int_0^{t_{k+1}} \sqrt{V_t} d(\sqrt{1-\rho^2}W_t^1 + \rho W_t^2) - \frac{1}{2}\xi^c_{t_{k+1}}\}}{S_0 \exp\{rt_k + \int_0^{t_k} \sqrt{V_t} d(\sqrt{1-\rho^2}W_t^1 + \rho W_t^2) - \frac{1}{2}\xi^c_{t_k}\}}\right)^2 \\
&= \sum_{k=0}^{n-1} \log\left(\exp\{r(t_{k+1}-t_k) + \int_{t_k}^{t_{k+1}} \sqrt{V_t} d(\sqrt{1-\rho^2}W_t^1 + \rho W_t^2) - \frac{1}{2}(\xi^c_{t_{k+1}} - \xi^c_{t_k})\}\right)^2 \\
&= \sum_{k=0}^{n-1} \left(r(t_{k+1}-t_k) - \frac{1}{2}(\xi^c_{t_{k+1}} - \xi^c_{t_k}) + \int_{t_k}^{t_{k+1}} \sqrt{V_t} d(\sqrt{1-\rho^2}W_t^1 + \rho W_t^2)\right)^2 \\
&= \sum_{k=0}^{n-1} \Bigg[(r(t_{k+1}-t_k) - \frac{1}{2}(\xi^c_{t_{k+1}} - \xi^c_{t_k}))^2 + 2(r(t_{k+1}-t_k) - \frac{1}{2}(\xi^c_{t_{k+1}} - \xi^c_{t_k}))\int_{t_k}^{t_{k+1}} \sqrt{V_t} d(\sqrt{1-\rho^2}W_t^1 + \rho W_t^2) \\
&\quad + \left(\int_{t_k}^{t_{k+1}} \sqrt{V_t} d(\sqrt{1-\rho^2}W_t^1 + \rho W_t^2)\right)^2\Bigg].
\end{aligned}$$

Then, taking the expectation yields

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[\boldsymbol{RV} - \int_0^T V_t dt] &= \mathbb{E}^{\mathbb{Q}}\left[\sum_{k=0}^{n-1}\left((r(t_{k+1}-t_k) - \frac{1}{2}(\xi^c_{t_{k+1}} - \xi^c_{t_k}))^2 + \int_{t_k}^{t_{k+1}} V_t dt\right) - \int_0^T V_t dt\right] \\
&= \sum_{k=0}^{n-1} \mathbb{E}^{\mathbb{Q}}(r(t_{k+1}-t_k) - \frac{1}{2}(\xi^c_{t_{k+1}} - \xi^c_{t_k}))^2
\end{aligned}$$

which is the bias of $\boldsymbol{RV}$. If the partition is equidistant with step size $\Delta_t = \frac{T}{n}$, the bias can be written as

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\sum_{k=0}^{n-1}(r(t_{k+1}-t_k) - \frac{1}{2}(\xi^c_{t_{k+1}} - \xi^c_{t_k}))^2\right] &= \mathbb{E}^{\mathbb{Q}}\left[\sum_{k=0}^{n-1}(r(t_{k+1}-t_k))^2 - r(t_{k+1}-t_k)\mathbb{E}^{\mathbb{Q}}(\xi^c_{t_{k+1}} - \xi^c_{t_k}) + \mathbb{E}^{\mathbb{Q}}(\frac{1}{2}(\xi^c_{t_{k+1}} - \xi^c_{t_k}))^2\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[n(r(\Delta_t))^2 - r\Delta_t\mathbb{E}^{\mathbb{Q}}(\xi^c_T) + \sum_{k=0}^{n-1}(\frac{1}{2}(\xi^c_{t_{k+1}} - \xi^c_{t_k}))^2\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\frac{Tr(Tr - (\xi^c_T))}{n} + \frac{1}{4}\sum_{k=0}^{n-1}(\xi^c_{t_{k+1}} - \xi^c_{t_k})^2\right] \\
&=: Bias_{\boldsymbol{RV}}
\end{aligned}$$

Now, letting $\mathcal{T}$ be defined as $\mathcal{T} = \inf\{t_k : \boldsymbol{RV} \geq \boldsymbol{b}\}$, it holds

$$Bias_{\boldsymbol{RV}} = \mathbb{E}^{\mathbb{Q}}\left[\frac{\mathcal{T}r(\mathcal{T}r - (\xi^c_T))}{n} + \frac{1}{4}\sum_{k=0}^{n-1}(\xi^c_{t_{k+1}} - \xi^c_{t_k})^2\right]$$

Since the bias is smaller the more frequent the monitoring is done, one could think of an hourly or minutely monitoring. But one should be aware of the dependent microstructure noise which one observes in real data. Thus, the estimator of the realized variance stated here could fail to converge to the right quantity when increasing the number of steps. As shown in [1] and [16], one can construct robust estimators which converge to the integrated variance when increasing the number of steps although there is a dependent microstructure noise in the data. The disadvantage of these estimators is that they are a lot more complicated than $\sum_{k=0}^{n-1} \log\left(\frac{S(t_{k+1})}{S(t_k)}\right)^2$ and therefore it might be difficult to agree on such an estimator as a measure for the expiry event. However, one could improve the accuracy by using more information such as the intraday high and low like in the Parkinson estimator or the Yang and Zhang estimator [33] which additionally takes into account the opening prices. Especially the Parkinson estimator is widely known and only a little more complex than $\sum_{k=0}^{n-1} \log\left(\frac{S(t_{k+1})}{S(t_k)}\right)^2$ using daily closing prices.

Sensitivity to the correlation coefficient ρ

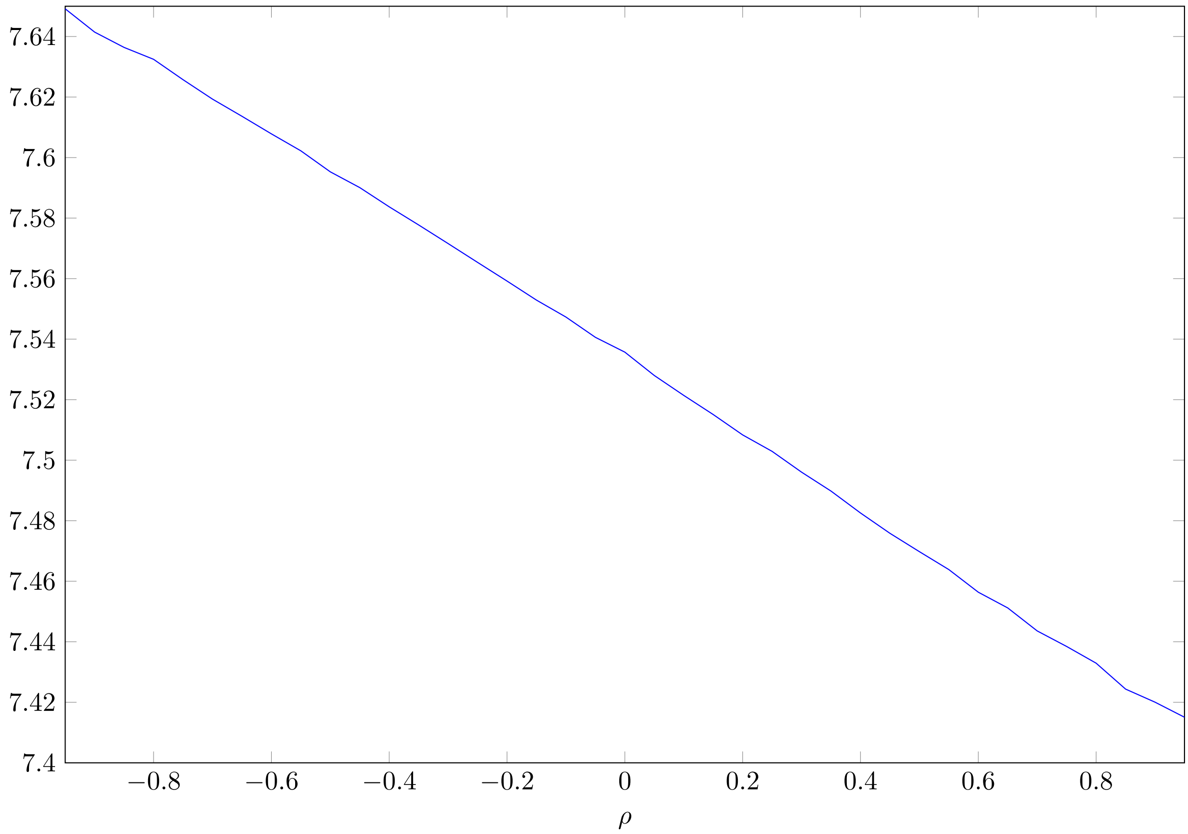

Figure 3.2: Plot of the influence of the correlation coefficient ρ on the price of a Timer Call (ordinate). Besides the explicitly given parameter ρ, all parameters are as in table 3.2.

The higher correlated the stock price process and its variance process are, the lower is the price of the Timer Call. This phenomena can be explained by looking at the hedging costs as mentioned by [7]. A negative correlation between the stock price process and its variance process means that the stock price process is positive correlated with the stopping time τ. In particular, τ tends to be larger if the stock price process expires in the money, i.e. the stock increased. In other words, hedging is probably more expensive when the stock increases and accordingly the Timer Call expires in the money. Conversely, hedging tends to be cheaper if the correlation between the stock price process and its variance process is positive because τ is likely to be smaller if the stock price process increases and accordingly the Timer Call expires in the money.

Sensitivity to the interest rate r

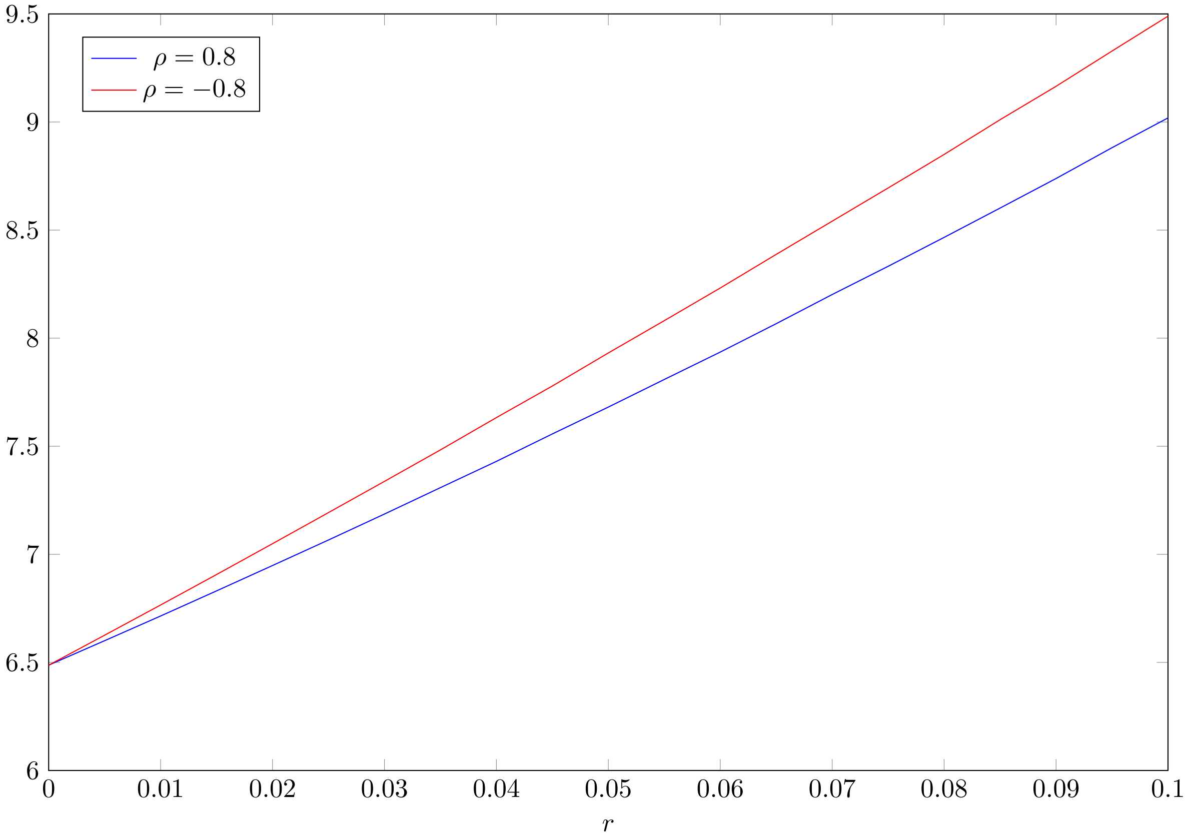

Figure 3.3: Plot of the influence of the interest rate r on the price of a Timer Call (ordinate). Besides the explicitly given parameters ρ and r, all parameters are as in table 3.2.

As figure 3.3 shows, the interest rate r has an increasing effect on the price of Timer Calls. Indeed, to hedge a Timer Call one has to borrow money to buy some fraction of the stock as mentioned in [7]. This borrowing becomes more expensive the higher the interest rate is. Here, a constant interest rate is assumed. When actually having stochastic interest rates, assuming a constant interest rate bears some risk one has to take into account. For instance, if the interest rate process and the variance process are negatively correlated, an increasing interest rate goes more or less hand in hand with an increasing time to maturity because the variance decreases. Especially if the budget is relatively high, assuming a constant interest rate could have expensive consequences.

3.3 A glance on the risk free interest rate

As mentioned in the sensitivity analysis of the interest rate r, assuming a constant interest rate for any period of time bears some risk. One problem is that the time to maturity is random. If the interest rate would be the same for all periods of borrowing, it is no problem at all, but in reality, the interest rate is different for different periods of borrowing. Thus, one reasonable modification is to take the yield curve into account which usually means that for shorter periods the interest rate is lower than for longer periods. Since this yield curve is a deterministic function, it is straightforward to modify the mentioned theorems and the Monte-Carlo simulation. Let $y(\tau)$ be the yield curve function. Then, it holds in distribution:

$$S_\tau = S_0 \exp\left\{ y(\tau)\tau + \sqrt{1-\rho^2} B_{\boldsymbol{b}} + \rho(f(V_\tau) - f(V_0) - \int_0^\tau h(s, V_s)ds) - \frac{1}{2}\boldsymbol{b} \right\}.$$

The price of a Timer Call, for instance, is then given by

$$C_{TC}(S_0, K, y(\tau), \boldsymbol{b}) = \mathbb{E}^{\mathbb{Q}}\left[C_{BS}(\tilde{S}_0, K, y(\tau), \tilde{\sigma}, \tau) \right]$$

where all other parameters are given as in corollary 3.1.7.
However, including a stochastic interest rate model is somewhat more complicated. In fact, one hast to compute the following expectation:

$$\mathbb{E}^{\mathbb{Q}}\left[C_{BS}(\tilde{S}_0, K, r_\tau, \tilde{\sigma}, \tau) \right]$$

If the interest rate process is independent of the Brownian motions W^1 and W^2 (cf. 3.1), one can adjust the Monte-Carlo simulation by just simulating separately r_τ for every simulated τ. This can be done by using the dynamics either on the time space or on the variance space:

$$dr_t = a(t, r_t)dt + b(r_t)dW_t^3 \tag{3.14}$$

$$d\tilde{r}_v = \frac{a(\tau(v), \tilde{r}_v)}{\tilde{V}_v}dv + \frac{b(\tilde{r}_v)}{\sqrt{\tilde{V}_v}}dB_v^3 \tag{3.15}$$

where W^3 and B^3 are standard Brownian motions. The SDE (3.14) shows the dynamics of the interest rate process on the time space with a and b being sufficient functions. This SDE has to be simulated from 0 to τ. The second SDE (3.15) shows the dynamics of the interest rate process on the variance space. This SDE has to be simulated from 0 to $\boldsymbol{b}$. If the variance process and the interest rate process are dependent, the simulation cannot be done straightforward.

3.4 A glance on different models for the stock price process

Until now, the stock price process was assumed to be modelled by

$$dS_t = rS_t dt + \sqrt{V_t} S_t d(\sqrt{1-\rho^2} W_t^1 + \rho W_t^2)$$

but actually one could also assume a more general model. Therefore, let the stock price process S_t follow the SDE

$$dS_t = S_t r dt + b(S_t, V_t)(\sqrt{1-\rho^2} W_t^1 + \rho W_t^2)$$

under the risk neutral measure $\mathbb{Q}$ with a proper function b. For instance, this generalization includes the SABR model [14] with $b(S_t, V_t) = S_t^\beta V_t$ in which $\beta \in [0, 1]$. Next, the sochastic clock ξ_t^c has to be defined. Applying Itô's lemma, the log-price process of the stock is given by

$$d\log(S_t) = rdt + \frac{b(S_t, V_t)}{S_t}(\sqrt{1-\rho^2} W_t^1 + \rho W_t^2) - \frac{b(S_t, V_t)^2}{2S_t^2} V_t dt.$$

Taking the quadratic variation, the stochastic clock ξ_t^c looks as follows:

$$\xi_t^c = [\log(S)]_t = \int_0^t \frac{b(S_u, V_u)^2}{S_u^2} du.$$

Recall, that setting $b(S_t, V_t) = S_t \sqrt{V_t}$ the stochastic clock is as it was up to now. However, assuming the above general stock price process and the adjusted stochastic clock, one can also apply the Dambis-Dubins-Schwarz theorem to obtain the price for a Timer option given by (cf. theorem 3.1.1)

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau} f\left(S_0 e^{r\tau + \boldsymbol{B_b} - \frac{1}{2}\boldsymbol{b}}\right)\right]$$

If the interest rate r is zero, the expectation can be similarly computed like before and it turns out that the price of a Timer option is not only independent of the choice of the variance process but also independent of the choice of the stock price process as long as it is continuous. But as soon as the interest rate is greater than zero, dealing with a general stock price process becomes a difficult task because in general τ is not only dependent on the variance process but also on the stock price process. Indeed, this is always the case when $b(S_t, V_t) \neq S_t c(V_t)$ for some appropriate function c.

4 Pricing Timer options under jump-diffusion processes

In the previous chapter the stock price process was assumed to be continuous but this might be an insufficient assumption. As in the introduction mentioned, a lot of research has been going on during the last years about this question. As [18],[2] and [5] showed, it is reasonable to consider jumps when, for instance, modelling stock prices. Thus, extending the theory of chapter 3 to jump-diffusion processes to model the stock price is the purpose of this chapter. To model the jumps, a compound Cox process is considered. In particular, the financial market is assumed to be as follows:

Financial Market

To model the financial market, a general stochastic volatility model with jumps is considered. Let S_t and V_t be the stock price process and the volatility process respectively defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q})$. The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is assumed to be generated by W^1, W^2 and J which are defined below. Then, under the risk-neutral measure $\mathbb{Q}$ the dynamics are defined as

$$\begin{aligned} d\log(S_t) &= rdt + \sqrt{V_t}d(\sqrt{1-\rho^2}W_t^1 + \rho W_t^2) - \frac{1}{2}d\xi_t^c + d(J(\xi_t^c) - c(\xi_t^c)) \\ dV_t &= \alpha(t, V_t)dt + \beta(V_t)dW_t^2 \end{aligned} \tag{4.1}$$

where r is the constant risk-free interest rate, W^1 and W^2 are two independent Brownian motions, $\rho \in [0,1]$ is the correlation between the changes in the stock and the volatility and $\alpha(t, V_t)$ and $\beta(V_t)$ are some measurable function w.r.t. the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that there exists a global solution for the variance process V_t and such that for any $t \geq 0$ it holds $V_t > 0$ almost surely. Furthermore, it is assumed that the integrated volatility converges to infinity as $T \to \infty$, i.e.

$$\xi_T^c := \int_0^T V_t dt \to \infty \ a.s.,\ T \to \infty.$$

The stochastic process ξ_t^c is the stochastic clock of chapter 3 but in this setting it is only the continuous part of the stochastic clock as shown later. The process J is a compound Cox process, i.e.

$$J(\xi_t^c) := \sum_{i=1}^{N(t)} Y_i \tag{4.2}$$

where N_t is a Cox process, also known as double stochastic Poisson process [20], with intensity $\lambda(t) = \lambda V_t$ which means that $N_t \sim \mathcal{P}\left(\int_0^t \lambda(s)ds\right) = \mathcal{P}\left(\lambda \int_0^t V_s ds\right) = \mathcal{P}\left(\lambda \xi_t^c\right)$. The parameter λ will be referred to as jump intensity parameter. The jump sizes Y_i are assumed to be independent and identically distributed random variables such that $\mathbb{E}[e^{2Y_1}] < \infty$. Furthermore, J is assumed to be independent of W^1 and W^2. The function $c(\xi_t^c)$ is called the compensator of $\tilde{J}$ where

$$\tilde{J}(\xi_t^c) := \sum_{i=1}^{N(t)} (e^{Y_i} - 1).$$

Due to the definition of the jumps, the compensator is given by

$$c(\xi_t^c) = \lambda \xi_t^c \mathbb{E}[e^{Y_1} - 1].$$

Note, that $e^{J(\xi_t^c) - c(\xi_t^c)}$ is thus a martingale.

Remark 4.0.1*.* The dynamics of the stock price process are stated on log-scale because it gives a more intuitive understanding of the jumps and shortens proofs. But for the sake of completeness, the stock price dynamics are given by

$$dS_t = rS_t dt + \sqrt{V_t} S_t d(\sqrt{1-\rho^2} W_t^1 + \rho W_t^2) + S_t d(\tilde{J}(\xi_t^c) - c(\xi_t^c))$$

where

$$\tilde{J}(\xi_t^c) := \sum_{i=1}^{N(t)} (e^{Y_i} - 1)$$

with Y_i defined as above. Since

$$\log(S_{T_i-} + \Delta S_i) - \log(S_{T_i-}) = \log\left(\frac{S_{T_i-} + S_{T_i-}(e^{Y_i} - 1)}{S_{T_i-}}\right) = \log(e^{Y_i}) = Y_i,$$

where T_i is the i-th jump time, one obtains when applying Itô's lemma for semimartingales with jump processes of finite variation, as it is given in [10], to the function $f(x) = \log(x)$

$$d\log(S_t) = rdt + \sqrt{V_t} d(\sqrt{1-\rho^2} W_t^1 + \rho W_t^2) - \frac{1}{2} d\xi_t^c + d(J(\xi_t^c) - c(\xi_t^c))$$

which is the stock price dynamics as stated in (4.1).

Remark 4.0.2*.* If the variance process is defined as an Cox-Ingersoll-Ross process [11] like in the Heston model, the model (4.1) reminds of the Bates model [6]. The basic difference is that in the Bates model the process N_t is a compound Poisson process, i.e. $N_t \sim \mathcal{P}(\lambda t)$.

The Timer option is defined as in chapter 3 but the stopping time τ changes due to the different model for the stock price process. According to chapter 3, the stochastic clock in this chapter ξ_t

is defined as

$$\xi_t := [\log(S)]_t = \int_0^t V_s ds + [J]_{\xi_t^c}$$
$$= \xi_t^c + \xi_t^j \tag{4.3}$$

where $[J]_{\xi_t^c} = \xi_t^j$ is the quadratic variation of the jump process J. Since J is a compound Cox process, the quadratic variation can be written as

$$\xi_t^j = \sum_{i=1}^{N(t)} Y_i^2$$

As one can see, the stochastic clock ξ_t in this chapter is different from the one of the previous chapter. Indeed, jumps in the stock price process cause jumps in the stochastic clock which means that ξ_t is no longer continuous. Thus, the stochastic clock consists of a continuous part ξ_t^c, which is the same as in chapter 3, and a jump part ξ_t^j. Extending the interpretation of the stochastic clock as business time, one could say that the business time is usually moving continuously but external shocks cause an infinitely acceleration of the business time which no human can keep up with. According to the definition of ξ_t, the stopping time τ is defined as

$$\tau = \inf\{t \geq 0 : \xi_t \geq \boldsymbol{b}\} \tag{4.4}$$

Note, that $\xi_t \geq \boldsymbol{b}$ is essential here because of the jumps. In this context, a Timer option expires as soon as ξ_t either hits the variance budget $\boldsymbol{b}$ exactly or jumps over it. The possibility of heavily over-consuming the budget by having a big jump when the variance budget is relatively low is here the main risk when neglecting jumps in the stock price process. Fortunately, having the jumps defined as in (4.1), the basic theory of chapter 3 can be reused.

4.1 Theoretical results for jump-diffusion processes

The main issue is to adapt the theory of chapter 3 to the different financial market (4.1). As a first step, theorem 3.1.1 is adapted to the new situation.

Theorem 4.1.1. *Let S_t and V_t be like in* (4.1)*, r the interest rate, f the payoff function and $\boldsymbol{b}$ the variance budget with corresponding stopping time τ, which is defined as in* (4.4)*. Then, the initial price of a Timer option is given by*

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau} f\left(S_0 e^{r\tau + \boldsymbol{B}_{\xi_\tau^c} - \frac{1}{2}\xi_\tau^c + J(\xi_\tau^c) - c(\xi_\tau^c)}\right)\right]$$

where $\boldsymbol{B}$ is a $\mathbb{Q}$-standard Brownian motion.

Proof. By theorem 3.1.1 it holds for any $t \geq 0$ in distribution

$$S_t = S_0 e^{rt + \int_0^t \sqrt{V_s} d(\sqrt{1-\rho^2} W_t^1 + \rho W_t^2) - \frac{1}{2}\xi_t^c} = S_0 e^{rt + \boldsymbol{B}_{\xi_t^c} - \frac{1}{2}\xi_t^c}$$

where $\boldsymbol{B}$ is a standard Brownian motion. Hence, it holds

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-rt} f\left(S_0 e^{rt + \int_0^t \sqrt{V_s} d(\sqrt{1-\rho^2} W_t^1 + \rho W_t^2) - \frac{1}{2}\xi_t^c + J(\xi_t^c) - c(\xi_t^c)}\right)\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-rt} f\left(S_0 e^{rt + \boldsymbol{B}_{\xi_t^c} - \frac{1}{2}\xi_t^c + J(\xi_t^c) - c(\xi_t^c)}\right)\right].$$

Letting $t = \tau$ the claim follows. $\square$

Remark 4.1.2. According to theorem 4.1.1, the dynamics of the time-changed stock price process are given by

$$S_0 e^{\boldsymbol{B}_{\xi_t^c} - \frac{1}{2}\xi_t^c + J(\xi_t^c) - c(\xi_t^c)}$$

when the interest rate is assumed to be zero. Conditioning on ξ_t^c and denoting $\xi_t^c = v$, the dynamics look as follows:

$$S_0 e^{\boldsymbol{B}_v - \frac{1}{2}v + J(v) - c(v)}.$$

Considering the jumps Y_i being normally distributed, these dynamics are the same as considered by Merton [24] with $\sigma = 1$ and $r = 0$.

Apart from the jump process and its compensator, one crucial difference is that ξ_τ^c is a random variable instead of having the value $\boldsymbol{b}$ at the stopping time τ almost surely. Thus, one has to deal with a stochastic ξ_τ^c in addition to the problem of the dependence of the Brownian motion and the stopping time. As an example how theorem 4.1.1 can be used, the Timer Call is again considered.

Corollary 4.1.3. *Let $C_{TC}(S_0, K, r, \boldsymbol{b})$ be the initial price of a Timer Call with strike price K, interest rate $r = 0$ and payoff function $f(S_\tau) = \max\{S_\tau - K, 0\}$. Then,*

$$\begin{aligned} C_{TC}(S_0, K, 0, \boldsymbol{b}) &= \mathbb{E}^{\mathbb{Q}}\left[C_{BS}(\hat{S}_0, K, 0, \sqrt{\frac{\xi_\tau^c}{T}}, T)\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\hat{S}_0 \mathcal{N}(d_1) - K\mathcal{N}(d_2)\right] \end{aligned}$$

where $\hat{S}_0 = S_0 e^{J(\xi_\tau^c) - c(\xi_\tau^c)}$

$$d_1 = \frac{\log\left(\frac{\hat{S}_0}{K}\right) + \frac{1}{2}\xi_\tau^c}{\sqrt{\xi_\tau^c}} \qquad \textit{and} \qquad d_2 = d_1 - \sqrt{\xi_\tau^c}.$$

Proof. By theorem 4.1.1 and the tower property it holds:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[f(S_\tau)] &= \mathbb{E}^{\mathbb{Q}}[\max\{S_\tau - K, 0\}] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\max\Big\{S_0 e^{\boldsymbol{B}_{\xi_\tau^c} - \frac{1}{2}\xi_\tau^c + J(\xi_\tau^c) - c(\xi_\tau^c)} - K, 0\Big\}\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\mathbb{E}^{\mathbb{Q}}\left[\max\Big\{S_0 e^{\boldsymbol{B}_{\xi_\tau^c} - \frac{1}{2}\xi_\tau^c + J(\xi_\tau^c) - c(\xi_\tau^c)} - K, 0\Big\} \mid (\xi_\tau^c, J(\xi_\tau^c))\right]\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\left[C_{BS}(\hat{S}_0, K, 0, \sqrt{\frac{\xi_\tau^c}{T}}, T)\right]
\end{aligned}$$

where $\hat{S}_0 = S_0 e^{J(\xi_\tau^c) - c(\xi_\tau^c)}$ and

$$d_1 = \frac{\log\left(\frac{\hat{S}_0}{K}\right) + \frac{1}{2}\xi_\tau^c}{\sqrt{\xi_\tau^c}} \qquad \text{and} \qquad d_2 = d_1 - \sqrt{\xi_\tau^c}.$$

□

As it turns out, there is in general no closed expression for the Timer Call even if the interest rate is zero. However, as long as the jumps Y_i can be simulated efficiently the expectation can be computed quickly with Monte-Carlo methods as shown later.

4.1.1 Theoretical results for using Monte Carlo methods

In a first step, it is again taken care of the correlation between the stopping time and the Brownian motion $\mathbf{B}$. This is done by adapting theorem 3.1.4 which can be done similarly to theorem 4.1.1.

Theorem 4.1.4. *Let S_t and V_t be like in* (4.1) *but with β being differentiable and τ given by* (4.4)*. Then, it holds in distribution*

$$S_\tau = S_0 \exp\left\{r\tau + \sqrt{1-\rho^2} B^1_{\xi_\tau^c} + \rho(f(V_\tau) - f(V_0) - \int_0^\tau h(s, V_s) ds) - \frac{1}{2}\xi_\tau^c + J(\xi_\tau^c) - c(\xi_\tau^c)\right\}$$

where

$$\begin{aligned}
f(V_t) &= \int_0^{V_t} \frac{\sqrt{u}}{\beta(u)} du \\
h(t, V_t) &= \alpha(t, V_t) f'(V_t) + \frac{1}{2}\beta^2(V_t) f''(V_t)
\end{aligned} \tag{4.5}$$

and B^1 is a standard Brownian motion.

Proof. Applying theorem 3.1.4 to

$$S_0 \exp\left\{rt + \sqrt{V_t} d(\sqrt{1-\rho^2} W_t^1 + \rho W_t^2) - \frac{1}{2}\xi_t^c\right\},$$

the claim follows immediately by the same argument as in the proof of theorem 4.1.1. The only difference is that $\xi_\tau^c \leq \boldsymbol{b}$ instead of $\xi_\tau^c = \boldsymbol{b}$ almost surely. □

Theorem 4.1.4 can now be used to extend corollary 4.1.3 and to derive a pricing formula for Timer Calls for all $r \geq 0$.

Corollary 4.1.5. *Denote by $C_{TC}(S_0, K, r, \boldsymbol{b})$ the initial price of a Timer Call with strike price K, interest rate r and variance budget $\boldsymbol{b}$. Then, having the payoff function $f(S_\tau) = \max\{S_\tau - K, 0\}$, it holds*

$$C_{TC}(S_0, K, r, \boldsymbol{b}) = \mathbb{E}^{\mathbb{Q}}\left[C_{BS}(\tilde{S}_0, K, r, \tilde{\sigma}, \tau)\right]$$

where C_{BS} is the Black-Scholes price of a European Call option and

$$\begin{aligned}
\tilde{S}_0 &= S_0 e^{\rho(f(V_\tau) - f(V_0) - H(\tau, V_\tau) - \frac{\rho}{2}\xi^c_\tau) + J(\xi^c_\tau) - c(\xi^c_\tau)} \\
\tilde{\sigma} &= \sqrt{\frac{\xi^c_\tau}{\tau}}.
\end{aligned}$$

Proof. Using the representation for S_τ from theorem 4.1.4 and the tower property for iterated expectations, one obtains

$$\begin{aligned}
&C_{TC}(S_0, K, r, \boldsymbol{b}) \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}\max\left\{S_\tau - K, 0\right\}\right] \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}\max\left\{S_0 e^{r\tau + \sqrt{1-\rho^2}B_{\xi^c_\tau} + \rho(f(V_\tau) - f(V_0) - H(\tau, V_\tau)) - \frac{1}{2}\xi^c_\tau + J(\xi^c_\tau) - c(\xi^c_\tau)} - K, 0\right\}\right] \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}\max\left\{\tilde{S}_0 e^{r\tau + \sqrt{1-\rho^2}B_{\xi^c_\tau} - \frac{1-\rho^2}{2}\xi^c_\tau} - K, 0\right\}\middle| \left(\xi^c_\tau, J_{\xi^c_\tau}, \tau, V_\tau, H(\tau, V_\tau)\right)\right]\right]
\end{aligned}$$

where $\tilde{S}_0 = S_0 e^{\rho(f(V_\tau) - f(V_0) - H(\tau, V_\tau) - \frac{\rho}{2}\boldsymbol{b}) + J(\xi^c_\tau) - c(\xi^c_\tau)}$. Since $\tilde{S}_0$ is just a constant when conditioning on ξ^c_τ, $J_{\xi^c_\tau}$, τ, V_τ and $H(\tau, V_\tau)$, the conditional expectation is the Black-Scholes formual, i.e.

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}\max\left\{\tilde{S}_0 e^{r\tau + \sqrt{1-\rho^2}B_{\boldsymbol{b}} - \frac{1-\rho^2}{2}\boldsymbol{b}} - K, 0\right\}\middle| \left(\xi^c_\tau, J_{\xi^c_\tau}, \tau, V_\tau, H(\tau, V_\tau)\right)\right] &= \tilde{S}_0\mathcal{N}(\tilde{d}_1) - Ke^{-r\tau}\mathcal{N}(\tilde{d}_2) \\
&= C_{BS}(\tilde{S}_0, K, r, \tilde{\sigma}, \tau)
\end{aligned}$$

where $\tilde{\sigma} = \sqrt{\frac{\xi^c_\tau}{\tau}}$ and

$$\tilde{d}_1 = \frac{\log\left(\frac{\tilde{S}_0}{K}\right) + r\tau + \frac{(1-\rho^2)}{2}\xi^c_\tau}{\sqrt{(1-\rho^2)\xi^c_\tau}} \qquad \tilde{d}_2 = \tilde{d}_1 - \sqrt{(1-\rho^2)\xi^c_\tau}.$$

Taking the expectation gives the desired result. □

In order to compute the price of a Timer Call, one can use corollary 4.1.5 and apply Monte-Carlo methods for which the joint distribution of $(\xi^c_\tau), J(\xi^c_\tau), \tau, V_\tau, H(\tau, V_\tau)$ is needed. This is not straightforward since the stopping time τ is distinctly different from the one in chapter 3. Because of the jumps in the stochastic clock, the techniques shown in the proof of theorem 3.1.5 cannot be used directly. The trick is to split the stopping time τ into two stopping times. One stopping time handles the jumps and determines the value of the continuous part of the stochastic clock, i.e. it shows how to draw ξ_τ^c and $J(\xi_\tau^c)$. The second stopping time can then be somehow similarly defined as in chapter 3 which makes it possible to apply theorem 3.1.5 to draw τ, V_τ and $H(\tau, V_\tau)$. How to define those stopping times is shown in the next lemma:

Lemma 4.1.6. *Let the stopping time τ be given by* (4.4)*, i.e.*

$$\begin{aligned}\tau &= \inf\{t \geq 0 : \xi_t \geq \boldsymbol{b}\} \\ &= \inf\left\{t \geq 0 : \int_0^t V_s ds + [J]_{\xi_t^c} \geq \boldsymbol{b}\right\},\end{aligned}$$

and define

$$\begin{aligned}\tilde{\tau} &:= \inf\{t \geq 0 : \xi_t^c \geq \nu\} \\ \nu &:= \inf\{u \geq 0 : u + [J]_u \geq \boldsymbol{b}\}.\end{aligned} \tag{4.6}$$

Then, the stopping times τ and $\tilde{\tau}$ are almost surely equal.

Proof. Firstly note, that τ and $\tilde{\tau}$ are both defined on the same probability space, which is $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q})$ (cf. (4.1)). It remains to show that for any realisation of $\{J_{\xi_t^c}\}_{t\geq 0}$ and $\{\xi_t^c\}_{t\geq 0}$ the stopping times are equal, i.e. $\tau(\omega) = \tilde{\tau}(\omega),\ \forall \omega \in \Omega$.
Let $\{J_{\xi_t^c}(\omega)\}_{t\geq 0}$ and $\{\xi_t^c(\omega)\}_{t\geq 0}$ be given paths of the jump process and the integrated variance process respectively. Firstly assume, that $\tau(\omega) < \tilde{\tau}(\omega)$ which implies $\xi_\tau^c(\omega) < \xi_{\tilde{\tau}}^c(\omega) = \nu(\omega)$ since ξ^c is strictly increasing and continuous. Furthermore, by definition of ν, one obtains $\xi_{\tilde{\tau}}^c(\omega) + [J]_{\xi_{\tilde{\tau}}^c(\omega)} \geq \boldsymbol{b}$ and simultaneously $\xi_\tau^c(\omega) + [J]_{\xi_\tau^c(\omega)} < \boldsymbol{b}$ because $\xi_\tau^c(\omega) < \xi_{\tilde{\tau}}^c(\omega)$ and the quadratic variation is non decreasing. But this is in contradiction to the definition of τ. Now assume, that $\tau(\omega) > \tilde{\tau}(\omega)$ which implies this time $\xi_\tau^c(\omega) > \xi_{\tilde{\tau}}^c(\omega)$ by the same argument and by definition of ν it holds $\xi_{\tilde{\tau}}^c(\omega) + [J]_{\xi_{\tilde{\tau}}^c(\omega)} \geq \boldsymbol{b}$. But since $\xi_\tau^c(\omega) > \xi_{\tilde{\tau}}^c(\omega)$ and the quadratic variation is non decreasing it also holds $\xi_\tau^c(\omega) + [J]_{\xi_\tau^c(\omega)} > \xi_{\tilde{\tau}}^c(\omega) + [J]_{\xi_{\tilde{\tau}}^c(\omega)} \geq \boldsymbol{b}$ which is in contradiction to the definition of τ. Hence, the two stopping times must be equal, i.e. $\tau(\omega) = \tilde{\tau}(\omega)$. Since this holds for all $\omega \in \Omega$, the claim follows. $\square$

With the stopping times $\tilde{\tau}$ and ν given by (4.6), it is now possible to draw a sample from $(\xi_\tau^c, J(\xi_\tau^c), \tau, V_\tau, H(\tau, V_\tau))$ by on the one hand using the definition of ν and on the other hand applying theorem 3.1.5 with the stopping time $\tilde{\tau}$.

Theorem 4.1.7. *Let V_t and τ be like in theorem 4.1.4 and 4.4 respectively. Then, the joint distribution of ξ_τ^c, $J_{\xi_\tau^c}$, τ, V_τ and $H(\tau, V_\tau) = \int_0^t h(s, V_s)ds$ with h defined as in* (4.5) *is given by*

$$\left(\xi_\tau^c, J_{\xi_\tau^c}, \tau, V_\tau, H(\tau, V_\tau)\right) \sim \left(\nu, J_\nu, \tau(\nu), X_\nu, \int_0^\nu \frac{h(\tau(s), X_s)}{X_s} ds\right) \tag{4.7}$$

with $\nu = \inf\{u \geq 0 : u + [J]_u \geq \boldsymbol{b}\}$ and the process X_v solving the following SDE

$$df(X_v) = \frac{h(\tau(v), X_v)}{X_v} dv + dB_v,\ X_0 = V_0,$$

where B_v is a standard Brownian motion and

$$\tau(v) = \int_0^v \frac{1}{X_s} ds.$$

Proof. Let $\nu = \inf\{u \geq 0 : v + [J]_u \geq \boldsymbol{b}\}$ and $\tilde{\tau} = \inf\{t \geq 0 : \xi_t^c \geq \nu\}$ be like in (4.6). Then, by lemma 4.1.6 the stopping times $\tilde{\tau}$ and τ are almost surely equal. Hence, the distribution of $\left(\xi_\tau^c, J_{\xi_\tau^c}, \tau, V_\tau, H(\tau, V_\tau)\right)$ is equal to the distribution of $\left(\xi_{\tilde{\tau}}^c, J_{\xi_{\tilde{\tau}}^c}, \tilde{\tau}, V_{\tilde{\tau}}, H(\tilde{\tau}, V_{\tilde{\tau}})\right)$. Since ξ_t^c is continuous and by definition of ν, it holds $\xi_{\tilde{\tau}}^c = \nu$ almost surely, i.e. $\left(\xi_{\tilde{\tau}}^c, J_{\xi_{t\tilde{a}u}^c}, \tilde{\tau}, V_{\tilde{\tau}}, H(\tilde{\tau}, V_{\tilde{\tau}})\right) \sim (\nu, J_\nu, \tilde{\tau}, V_{\tilde{\tau}}, H(\tilde{\tau}, V_{\tilde{\tau}}))$. Now let $(x_1, x_2, x_3, x_4, x_5) \in [0, \boldsymbol{b}] \times \mathbb{R} \times [0, \infty)^3$ and $F_\nu(v)$ be the distribution function of ν. Then, it holds by law of the total probability

$$\begin{aligned}
&\mathbb{P}(\nu \leq x_1, J_\nu \leq x_2, \tilde{\tau} \leq x_3, V_{\tilde{\tau}} \leq x_4, H(\tilde{\tau}, V_{\tilde{\tau}}) \leq x_5) \\
&= \int_0^{\boldsymbol{b}} \mathbb{P}(\nu \leq x_1, J_\nu \leq x_2, \tilde{\tau} \leq x_3, V_{\tilde{\tau}} \leq x_4, H(\tilde{\tau}, V_{\tilde{\tau}}) \leq x_5 \mid \nu = v) dF_\nu(v) \\
&= \int_0^{\boldsymbol{b}} \mathbb{P}(v \leq x_1, J_v \leq x_2, \tilde{\tau} \leq x_3, V_{\tilde{\tau}} \leq x_4, H(\tilde{\tau}, V_{\tilde{\tau}}) \leq x_5 \mid \nu = v) dF_\nu(v)
\end{aligned}$$

Note, that the jumps conditioned on ν are independent of the variance process conditioned on ν and that the variance process is independent of ν. Hence,

$$\begin{aligned}
&\int_0^{\boldsymbol{b}} \mathbb{P}(v \leq x_1, J_v \leq x_2, \tilde{\tau} \leq x_3, V_{\tilde{\tau}} \leq x_4, H(\tilde{\tau}, V_{\tilde{\tau}}) \leq x_5 \mid \nu = v) dF_\nu(v) \\
&\quad = \int_0^{\boldsymbol{b}} \mathbb{P}(v \leq x_1, J_v \leq x_2 \mid \nu = v) \mathbb{P}(\tilde{\tau} \leq x_3, V_{\tilde{\tau}} \leq x_4, H(\tilde{\tau}, V_{\tilde{\tau}}) \leq x_5) dF_\nu(v)
\end{aligned}$$

Now, the stopping time $\tilde{\tau}$ conditioned on ν is $\tilde{\tau} = \inf\{t \geq 0 : \xi_t^c \geq v\}$. Thus, by theorem (3.1.5) the common distribution of $\tilde{\tau}$, $V_{\tilde{\tau}}$ and $H(\tilde{\tau}, V_{\tilde{\tau}})$ is given by $\tau(v), X_v$ and $\int_0^v \frac{h(\tau(s), X_s)}{X_s} ds = H_v$ where X_v is a solution to the SDE

$$df(X_v) = \frac{h(\tau(v), X_v)}{X_v} dv + dB_v, \ X_0 = V_0,$$

where B_v is a standard Brownian motion and

$$\tau(v) = \int_0^v \frac{1}{X_s} ds.$$

Then, again by the law of total probability, one gets

$$\begin{aligned}
&\int_0^{\boldsymbol{b}} \mathbb{P}(v \leq x_1, J_v \leq x_2, \tilde{\tau} \leq x_3, V_{\tilde{\tau}} \leq x_4, H(\tilde{\tau}, V_{\tilde{\tau}}) \leq x_5 \mid \nu = v) dF_\nu(v) \\
&\quad = \int_0^{\boldsymbol{b}} \mathbb{P}(v \leq x_1, J_v \leq x_2, \tau(v) \leq x_3, X_v \leq x_4, H(\tau(v), V_v) \leq x_5 \mid \nu = v) dF_\nu(v) \\
&\quad = \mathbb{P}(\nu \leq x_1, J_\nu \leq x_2, \tau(\nu) \leq x_3, X_\nu \leq x_4, H(\tau(\nu), V_\nu) \leq x_5)
\end{aligned}$$

which concludes the proof. $\square$

Theorem 4.1.7 gives a clear instruction how a sample of $\left(\xi^c_\tau, J_{\xi^c_\tau}, \tau, V_\tau, H(\tau, V_\tau)\right)$ can be drawn in order to use Monte-Carlo methods. An explicit Monte-Carlo algorithm for the case that the variance process follows a Cox-Ingersoll-Ross process like in the Heston model is shown in section 4.2.

Remark 4.1.8. If one would like to price other Timer options than Timer Call options, it is probably not possible to get rid of the Brownian Motion B^1 (cf. theorem 4.1.4). But since B^1 is only dependent on ξ^c_τ, one can just draw $\left(\xi^c_\tau, J_{\xi^c_\tau}, \tau, V_\tau, H(\tau, V_\tau), B^1_{\xi^c_\tau}\right)$ which is in distribution equal to $\left(\nu, J_\nu, \tau(\nu), X_\nu, \int_0^\nu \frac{h(\tau(s), X_s)}{X_s} ds, B^1_\nu\right)$ where $B^1_\nu \sim \mathcal{N}(0, \nu)$.

4.1.2 Timer options and partial integro-differential equations

Similar to chapter 3, one can price Timer options also by solving, in this case, a partial integro-differential equation. The integro part is a consequence of the jumps in the stock price process.

Theorem 4.1.9. *Let S_t and V_t be like in* (4.1)*. Then, the initial price of a Timer option with payoff function g is given by $f(\xi, V, S)$ which solves the partial integro-differential equation*

$$\begin{aligned} f_\xi &+ \frac{\alpha(V)}{V} f_V + \frac{S(r - \lambda \mathbb{E}[e^{Y_1} - 1])}{V} f_S + \frac{1}{2} S^2 f_{SS} + \frac{1}{2}\frac{\beta^2(V)}{V} f_{VV} \\ &+ \frac{S\rho\beta(V)}{\sqrt{V}} f_{SV} - \frac{r}{V} f + \lambda \int_{\mathbb{R}} \left[f(\xi + y^2, V, Se^y) - f(\xi, V, S) \right] dF_Y(y) = 0 \end{aligned} \tag{4.8}$$

on the domain $(\xi, V, S) \in [0, \boldsymbol{b}] \times (0, \infty)^2$ with boundary condition

$$f(\xi, V, S) = g(S), \ \forall \xi \geq \boldsymbol{b}, \ S > 0.$$

and with $F_Y(y)$ being the distribution function of the jumps Y_i.

Sketch of Proof. The proof is as in chapter 3 basically an application of Itô's lemma and setting the resulting drift term to zero. Due to the perpetuity property, the price of a Timer option is independent of the time variable t [21]. Thus, let the price be given by

$$f(\xi_t, V_t, S_t) = \mathbb{E}^{\mathbb{Q}} \left[e^{-r(\tau - t)} g(S_\tau) \mid S_t = x \right].$$

Now apply Itô's lemma for semimartingales with jump processes of finite variation to $e^{-rt} f(\xi_t, V_t, S_t)$. Note, that due to the definition of J, the number of jumps is finite on finite intervals. Hence, one obtains

$$\begin{aligned} de^{-rt} f(\xi_t, V_t, S_t) = e^{-rt} \Bigg(& V_t f_\xi + \alpha(V_t) f_V + r S_{t-} f_S + \frac{1}{2} S^2_{t-} V_t f_{SS} + \frac{1}{2} \beta^2(V_t) f_{VV} \\ & + S_{t-} \rho \beta(V_t) \sqrt{V_t} f_{SV} - rf + \lambda V_t \int_{\mathbb{R}} [f(\xi_{t-} + y^2, V_t, S_{t-} e^y) - f(\xi_{t-}, V_t, S_{t-})] dF_Y(y) \Bigg) dt \\ & + \underbrace{e^{-rt} \left(f_V \beta(V_t) dW^2_t + f_S S_{t-} \sqrt{V_t} d(\sqrt{1 - \rho^2} W^1_t + \rho W^2_t) \right)}_{=:M_t} \end{aligned}$$

with S_{t-} being the left-handed limit of S_t and $F_Y(y)$ being the probability distribution function of the jumps Y_i. Assuming M to be a local martingale, the drift term must vanish because $e^{-rt}f(\xi_t, V_t, S_t)$ is a martingale by construction, i.e.

$$\begin{aligned} &Vf_\xi + \alpha(V)f_V + rSf_S + \frac{1}{2}S_{t-}^2 Vf_{SS} + \frac{1}{2}\beta^2(V)f_{VV} \\ &\quad + S\rho\beta(V)\sqrt{V}f_{SV} - rf + \lambda V\int_{\mathbb{R}}[f(\xi+y^2, V, Se^y) - f(\xi, V_t, S)]dF_Y(y) = 0. \end{aligned}$$

Now, one might again divide by V in order to point out the role of ξ, i.e.

$$\begin{aligned} &f_\xi + \frac{\alpha(V)}{V}f_V + \frac{S(r-\lambda\mathbb{E}[e^{Y_1}-1])}{V}f_S + \frac{1}{2}S^2 f_{SS} + \frac{1}{2}\frac{\beta^2(V)}{V}f_{VV} \\ &\quad + \frac{S\rho\beta(V)}{\sqrt{V}}f_{SV} - \frac{r}{V}f + \lambda\int_{\mathbb{R}}\left[f(\xi+y^2, V, Se^y) - f(\xi, V, S)\right]dF_Y(y) = 0 \end{aligned}$$

with the boundary condition

$$f(\xi, V, S) = g(S),\ \forall\xi \geq \boldsymbol{b},\ S > 0$$

because a Timer option expires as soon as $\xi \geq \boldsymbol{b}$. Actually, ξ is not bounded because of the jumps. But due to the boundary condition, one only has to solve the partial integro-differential equation on the domain $(\xi, V, S) \in [0, \boldsymbol{b}] \times (0, \infty)^2$. □

Remark 4.1.10. The assumption that M_t is a continuous local martingale is crucial and actually one has to check that the solution to the partial integro-differential equation fulfils this assumption.

Remark 4.1.11. The boundary condition in theorem 4.1.9 allows to simplify the integral because the value of the payoff function g is known for $\xi + y^2 \geq \boldsymbol{b} \Leftrightarrow |y| \geq \sqrt{\boldsymbol{b}-\xi}$. In particular,

$$\begin{aligned} &\int_{|y|\geq\sqrt{\boldsymbol{b}-\xi}}\left[f(\xi+y^2, V, Se^y) - f(\xi, V, S)\right]dF_Y(y) \\ &\qquad = \int_{|y|\geq\boldsymbol{b}-\xi}[g(Se^y)]\,dF(y) - f(\xi, V, S)\int_{|y|\geq\sqrt{\boldsymbol{b}-\xi}}dF_Y(y). \end{aligned}$$

Since F_Y is a probability distribution function, this leads to the partial integro-differential equation

$$\begin{aligned} &f_\xi + \frac{\alpha(V)}{V}f_V + \frac{S(r-\lambda\mathbb{E}[e^{Y_1}-1])}{V}f_S + \frac{1}{2}S^2 f_{SS} + \frac{1}{2}\frac{\beta^2(V)}{V}f_{VV} + \frac{S\rho\beta(V)}{\sqrt{V}}f_{SV} - \left(\frac{r}{V}+\lambda\right)f \\ &\quad + \lambda\int_{|y|\geq\boldsymbol{b}-\xi}[g(Se^y)]\,dF_Y(y) + \lambda\int_{|y|<\sqrt{\boldsymbol{b}-\xi}}\left[f(\xi+y^2, V, Se^y)\right]dF(y) = 0. \end{aligned}$$

This simplification can be helpful when solving the partial integro-differential equation (4.8) with numerical methods.

4.2 Pricing Timer Calls - Numerical results

As an example, an explicit Monte-Carlo algorithm is stated like in chapter 3. The Timer option of consideration is a Timer Call. The variance process is assumed to follow a Cox-Ingersoll-Ross process like in the Heston model and the jumps are assumed to be normally distributed with mean μ and variance σ^2 like in the Merton model [24]. In particular, the dynamics under the risk-neutral measure $\mathbb{Q}$ of the logarithmic stock price process and the variance process are as follows:

$$\begin{aligned} d\log(S_t) &= rdt + \sqrt{V_t}d(\sqrt{1-\rho^2}W_t^1 + \rho W_t^2) - \frac{1}{2}d\xi_t^c + d(J(\xi_t^c) - c(\xi_t^c)) \\ dV_t &= \kappa(\theta - V_t)dt + \gamma\sqrt{V_t}dW_t^2 \end{aligned} \tag{4.9}$$

with constants κ, θ and γ fulfilling the relation $2\kappa\theta > \gamma^2$ which means that the variance process is strictly positive according to the Feller test. The jump process J is a compound Cox process with $Y_i \sim \mathcal{N}(\mu, \sigma^2)$ (cf. (4.1)) and hence the compensator is given by

$$c(v) = \mathbb{E}\left[e^{J(v)}\right] = \exp\{\lambda v(e^{\mu + \frac{\sigma^2}{2}} - 1)\}$$

Furthermore, the Timer Call is characterized by the variance budget $\boldsymbol{b}$ and the payoff function $f(S_\tau) = \max\{S_\tau - K, 0\}$ where τ is given by (4.4) and K is the strike price.

4.2.1 Explicit Monte-Carlo algorithm

Now recall the result of corollary 4.1.5 which states that the price of a Timer Call can be computed by taking the expectation of the Black-Scholes formula which can be numerically computed by using Monte-Carlo methods. In particular, one can proceed as follows:

Step 1: Use theorem 4.1.7 to draw a sample of $(\xi_\tau^c, J_{\xi_\tau^c}, \tau, V_\tau)$. It is a good idea to start with drawing ξ_τ^c and $J_{\xi_\tau^c}$ because by theorem 4.1.7 this can be done by drawing ν and J_ν where $\nu = \inf\{u \geq 0 : u + [J]_u \geq \boldsymbol{b}\}$. In particular, draw $J(\boldsymbol{b})$ which is a Poisson distributed random variable, i.e. $J(\boldsymbol{b}) \sim P(\lambda\boldsymbol{b})$. Then, use the fact that the jumpingtimes are uniformly distributed on $[0, \boldsymbol{b}]$, i.e. draw $J(\boldsymbol{b})$ many uniformly distributed random variables on $[0, \boldsymbol{b}]$. Now construct the process $u + [J]_u$ and check when it exceeds $\boldsymbol{b}$. This value is then ν. Simultaneously, one gets the random variable $J(\nu)$. Now, one can proceed like in chapter 3. Recalling that $f(v) = \frac{v}{\gamma}$ and $h(\tau, v) = \frac{\kappa(\theta - v)}{\gamma}$ one has to simulate the process X_v on $[0, \nu]$ for every ν which is done by simulating

$$dX_v = \left(\frac{\kappa\theta}{X_v} - \kappa\right)dv + \gamma dB_v, \ X_0 = V_0$$

on $[0, \nu]$. Then, the sample of ξ_τ^c, $J_{\xi_\tau^c}$, τ and V_τ is given by $\left(\nu, J(\nu), \int_0^\nu \frac{1}{X_v}dv, X_\nu\right)$.

Step 2: Repeating step 1 n times, one can approximate the desired expectation. In particular, let C_{TC}^{mc} be the Monte-Carlo estimate of the price of a Timer Call. Then, according to the theory provided in section 4.1.1, the Timer Call is computed by

$$\begin{aligned} C_{TC}^{mc} &= \frac{1}{n}\sum_{i=1}^{n} C_{BS}(\tilde{S}_{0,i}, K, r, \tilde{\sigma}_i, \tau_i) \\ &= \frac{1}{n}\sum_{i=1}^{n}\left(\tilde{S}_{0,i}\mathcal{N}(\tilde{d}_{1,i}) - Ke^{-r\tau_i}\mathcal{N}(\tilde{d}_{2,i})\right) \end{aligned}$$

with

$$\begin{aligned} \tilde{S}_{0,i} &= S_0 e^{\frac{\rho}{\gamma}(V_{\tau,i}-V_0-\kappa(\theta\tau_i-\xi^c_{\tau,i}))-\frac{\rho^2}{2}\xi^c_{\tau,i}+J_i(\xi^c_{\tau,i})-c(\xi^c_{\tau,i})} \\ \tilde{d}_{1,i} &= \frac{\log\left(\frac{S_0}{K}\right) + r\tau_i + \frac{\rho}{\gamma}\left(V_{\tau,i} - V_0 - \kappa(\theta\tau_i - \xi^c_{\tau,i})\right) + \left(\frac{1}{2} - \rho^2\right)\xi^c_{\tau,i} + J_i(\xi^c_{\tau,i}) - c(\xi^c_{\tau,i})}{\sqrt{(1-\rho^2)\xi^c_{\tau,i}}} \\ \tilde{d}_{2,i} &= \tilde{d}_{1,i} - \sqrt{(1-\rho^2)\xi^c_{\tau,i}}. \end{aligned}$$

Step 3: In order to improve the convergence of the Monte-Carlo simulation, one can again make use of control variates. Unfortunately, it is not possible to use the same idea like in the pure diffusion case because even if the interest rate is zero the exact price of a Timer Call is unknown. However, one can construct one control variate to reduce the variance caused by the compound Cox process and ξ^c_τ and one to reduce the variance caused by τ and V_τ. For the first one the property that the process $e^{J(v)-c(v)}$ is a martingale can be used. Thus, the first control variate is given by

$$CV_1 = \frac{1}{n}\sum_{i=1}^{n} e^{J(\xi^c_{\tau,i})-c(\xi^c_{\tau,i})}$$

The expectation of CV_1 is $\mathbb{E}^{\mathbb{Q}}[e^{J(0)-c(0)}] = 1$ because of the martingale property. The second control variate is a try to still use that a Timer Call can be exactly computed when having no jumps and if the interest rate is zero. Let the second control variate CV_2 be defined as

$$CV_2 = \frac{1}{n}\sum_{i=1}^{n}\left(\hat{S}_{0,i}\mathcal{N}(\hat{d}_{1,i}) - Ke^{r\tau_i}\mathcal{N}(\hat{d}_{2,i})\right)$$

where

$$\begin{aligned} \hat{S}_{0,i} &= S_0 e^{\frac{\rho}{\gamma}(V_{\tau,i}-V_0-\kappa(\theta\tau_i-\xi^c_{\tau,i}))-\frac{\rho^2}{2}\xi^c_{\tau,i}+J_i(\xi^c_{\tau,i})-c(\xi^c_{\tau,i})} \\ \hat{d}_{1,i} &= \frac{\log\left(\frac{S_0}{K}\right) + \frac{\rho}{\gamma}\left(V_{\tau,i} - V_0 - \kappa(\theta\tau_i - \xi^c_{\tau,i})\right) + J_i(\xi^c_{\tau,i}) - c(\xi^c_{\tau,i}) + \left(\frac{1}{2} - \rho^2\right)\xi^c_{\tau,i}}{\sqrt{(1-\rho^2)\xi^c_{\tau,i}}} \\ \hat{d}_{2,i} &= \hat{d}_{1,i} - \sqrt{(1-\rho^2)\xi^c_{\tau,i}}. \end{aligned}$$

The only difficulty is that it is not possible to compute $\mathbb{E}^{\mathbb{Q}}[CV_2]$. But the expectation conditioned on ξ^c_τ and $J(\xi^c_{\tau,i})$ is exactly computable. In other words, one could view the control variate CV_2

as a sum of control variates for every realisation of $(\xi_\tau^c, J(\xi_{\tau,i}^c))$. The conditional expectation of CV_2 is then given by

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[CV_2 \mid (\xi_\tau^c, J(\xi_\tau^c))] &= \frac{1}{n}\sum_{i=0}^{n}\Big(S_0 e^{J_i(\xi_{\tau,i}^c)-c(\xi_{\tau,i}^c)}\mathcal{N}(d_{1,i}) - K\mathcal{N}(d_{2,i})\Big)\\ &=: C_{TC|\xi^c,J}\end{aligned}$$

where

$$d_{1,i} = \frac{\log\left(\frac{S_0}{K}\right) + J_i(\xi_{\tau,i}^c) - c(\xi_{\tau,i}^c) + \frac{1}{2}\xi_{\tau,i}^c}{\sqrt{\xi_{\tau,i}^c}} \qquad \text{and} \qquad d_{2,i} = d_{1,i} - \sqrt{\xi_{\tau,i}^c}.$$

Now, letting $\lambda_1 := \frac{Cov(C_{TC}^{mc}, CV_1)}{Var(CV_1)}$ and $\lambda_2 := \frac{Cov(C_{TC}^{mc}, CV_2)}{Var(CV_2)}$, the variance reduced approximation is then given by

$$C_{TC}^{mc} - \lambda_1(CV_1 - 1) - \lambda_2(CV_2 - C_{TC|\xi^c,J}).$$

It turns out, that the control variates $CV1$ and $CV2$ reduce the variance significantly although the coefficients $\lambda 1$ and $\lambda 2$ are maybe not optimal. For further details on multivariate control variates refer to [28] and [34].

Remark 4.2.1. If the interest rate is zero, it is not necessary to simulate τ and V_τ because one can apply theorem 4.1.1 and corollary 3.1.2. In particular, the Timer Call can be computed by

$$\begin{aligned}C_{TC}^{mc} &= \frac{1}{n}\sum_{i=1}^{n} C_{BS}(\tilde{S}_{0,i}, K, r, \tilde{\sigma}_i, T)\\ &= \frac{1}{n}\sum_{i=1}^{n}\Big(\tilde{S}_{0,i}\mathcal{N}(\tilde{d}_{1,i}) - K\mathcal{N}(\tilde{d}_{2,i})\Big)\end{aligned}$$

with

$$\begin{aligned}\tilde{S}_{0,i} &= S_0 e^{J_i(\xi_{\tau,i}^c)-c(\xi_{\tau,i}^c)}\\ \tilde{d}_{1,i} &= \frac{\log\left(\frac{S_0}{K}\right) + J_i(\xi_{\tau,i}^c) - c(\xi_{\tau,i}^c) + \frac{1}{2}\xi_{\tau,i}^c}{\sqrt{\xi_{\tau,i}^c}}\\ \tilde{d}_{2,i} &= \tilde{d}_{1,i} - \sqrt{\xi_{\tau,i}^c}.\end{aligned}$$

In order to reduce the variance of the Monte-Carlo estimator, one can still use the control variate CV_1.

4.2.2 Monte-Carlo simulation results and sensitivity analysis

Now, the results of the Monte-Carlo algorithm for a Timer Call obtained when assuming the financial market (4.1) are presented. The parameters which are needed are given in table 4.1. Actually, this is the set of parameters of chapter 3 extended by the parameters of the compound

Cox process. The jumps are assumed to be normally distributed, i.e. $Y_i \sim \mathcal{N}(\mu, \sigma^2)$. The jump intensity is denoted by λ.

Table 4.1: Parameters Timer Call

S_0	K	r	$\boldsymbol{b}$	V_0	κ	γ	θ	λ	μ	σ
100	100	0.04	0.0265	0.0625	2	0.1	0.0324	100	0	0.06

Having $\sigma = 0.06$, the jumps on log-scale are assumed to be between -18% and 18% in about 99% of all realizations. A λ of 100 means that if the yearly realized volatility without jumps is about 20% then the underlying jumps on average four times a year. In table 4.2 and 4.3 some simulation results are shown. In table 4.2 the interest rate r is assumed to be zero. In this case, the price of a Timer Call is around 6.72 which is about 0.23 higher than the price in chapter 3. The control variate $CV1$ reduces the error of the Monte-Carlo estimation by a factor of about 4. In table 4.3 the results are stated under the assumption of having an interest rate of 4%. One can observe that the correlation coefficient ρ seems to influence the price of a Timer Call in the same way as in chapter 3. The two control variates are not that powerful as the control variate in chapter 3 but they still increase the accuracy by a factor of about eight if the correlation is different from zero and of about five if the correlation is zero. This difference is due to the fact that the control variate CV_2 is exactly its expectation and has therefore no variance reducing effect like in chapter 3. In order to have a look at the prices of Timer Calls, when changing certain parameters, a sensitivity analysis is shown next. If not stated differently, the needed parameters are given as in table 4.1 and the correlation coefficient is set to $\rho = 0$.

Table 4.2: Monte Carlo simulation of the price of a Timer Call with interest rate $r = 0$.

Number of simulations N	Monte Carlo simulation without control variates		Monte Carlo simulation with control variates	
300000	6.7195	(0.0088)	6.7195	(0.0021)
600000	6.7130	(0.0062)	6.7205	(0.0015)

Table 4.3: Monte Carlo simulation of the price of a Timer Call with interest rate $r = 0.04$.

Number of simulations N	Number of discretization steps M	Correlation coefficient ρ	Monte Carlo simulation without control variates		Monte Carlo simulation with control variates	
		-0.8	7.5534	(0.0169)	7.5509	(0.0021)
300000	500	0	7.4719	(0.0092)	7.4884	(0.0020)
		0.8	7.4009	(0.0165)	7.4264	(0.0021)
		-0.8	7.5401	(0.0169)	7.5490	(0.0021)
300000	1000	0	7.4646	(0.0092)	7.4827	(0.0020)
		0.8	7.4565	(0.0165)	7.4333	(0.0021)
		-0.8	7.5605	(0.0120)	7.5497	(0.0015)
600000	500	0	7.4952	(0.0065)	7.4904	(0.0014)
		0.8	7.4324	(0.0117)	7.4277	(0.0015)
		-0.8	7.5574	(0.0120)	7.5482	(0.0015)
600000	1000	0	7.5018	(0.0065)	7.4906	(0.0014)
		0.8	7.4482	(0.0117)	7.4292	(0.0015)

Sensitivity to the jump intensity parameter λ

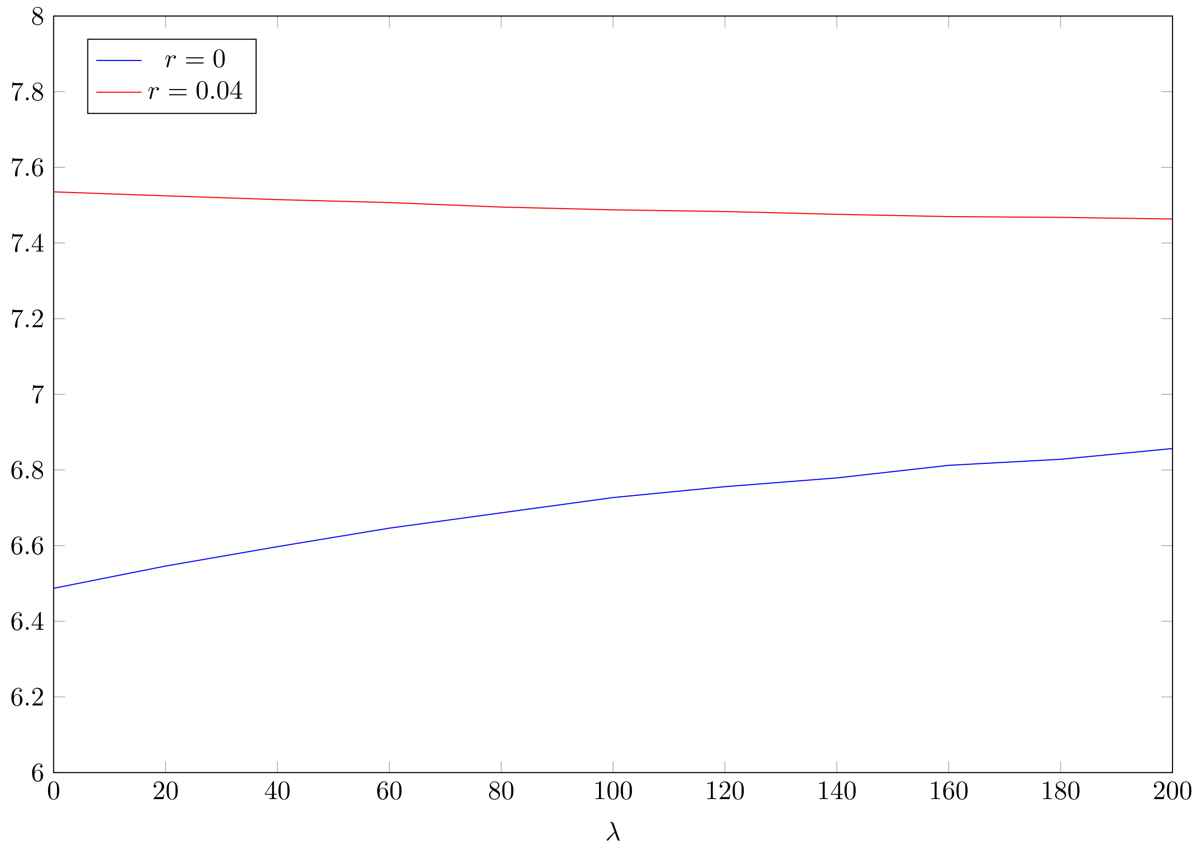

Figure 4.1: Plot of the influence of the jump intensity parameter λ on the price of a Timer Call (ordinate). Besides the explicitly given parameters λ and r, all parameters are as in table 4.1.

The jump intensity parameter λ has a very interesting influence on the price of a Timer Call. If the interest rate r is zero, a higher jump intensity λ leads to higher prices. This seems to be natural since a higher λ means that there are more jumps and thus overshooting the variance budget $\boldsymbol{b}$ becomes more probable. But if the interest rate is greater than zero, another influence direction shows up. It is still true that overshooting the budget becomes more probable and it still has an increasing effect. But more jumps also cause that ξ_τ^c becomes smaller on average and thus τ becomes smaller on average. But a shorter time to maturity means that hedging becomes less expensive and hence this has a decreasing effect on the price. Having an interest rate of 4% like in figure 4.1 these two effects seem to be almost equally strong and astonishingly the price of a Timer Call stays almost constant when changing the jump intensity parameter λ.

Sensitivity to the jump size parameter σ

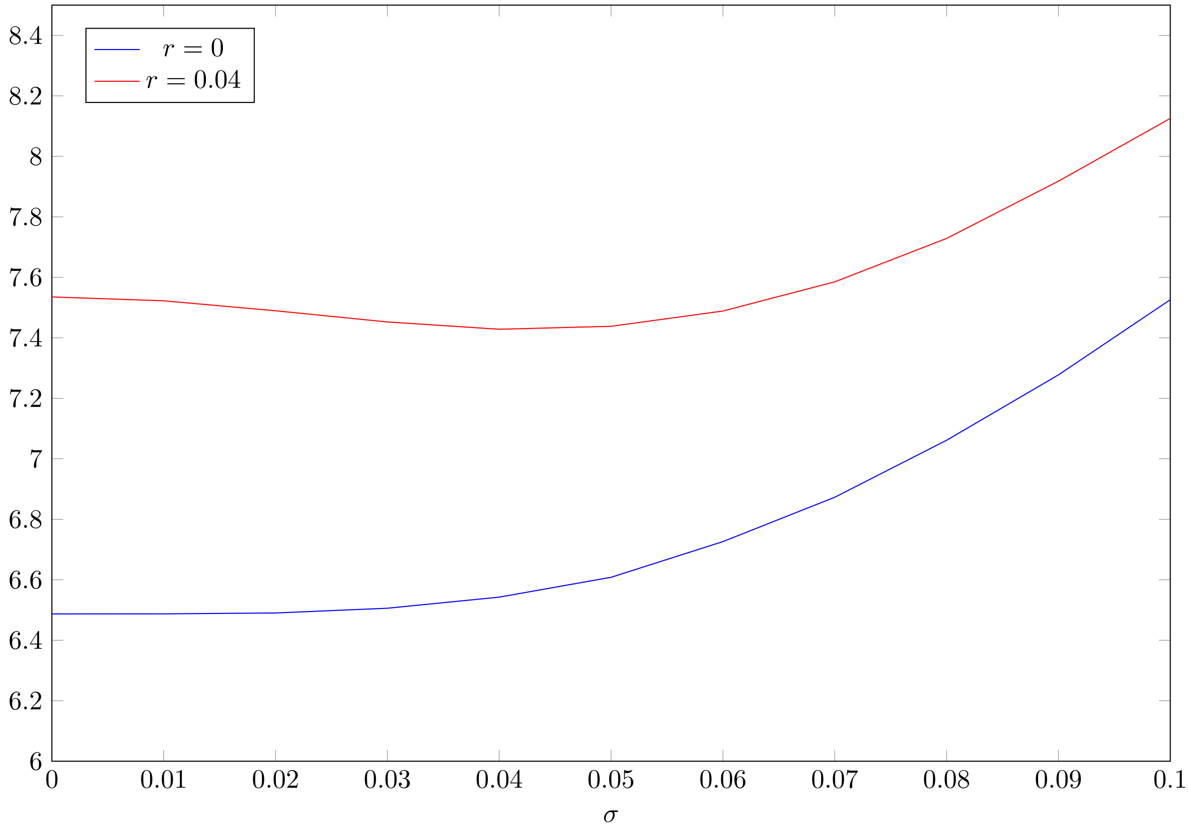

Figure 4.2: Plot of the influence of the jump size parameter σ on the price of a Timer Call (ordinate). Besides the explicitly given parameters σ and r, all parameters are as in table 4.1.

The influence of the jump size parameter σ is somehow similar to the jump intensity parameter λ. When the interest rate is zero, a higher σ leads to higher prices because larger jumps become more probable and accordingly the probability of overshooting the variance budget massively increases. For small σ, this effect is relatively weak but the higher σ the higher is the price. If the interest rate is greater than zero, the price slightly decreases for small σ and then starts increasing like in the zero interest rate case. This behaviour has the same reason as the behaviour of λ. Indeed, larger jumps still lead to larger overshoots which has an upward influence but it also leads overall to shorter times to maturity like a greater λ. Having an interest rate of 4% the downward effect seems to dominate the upward effect until $\sigma \approx 0.04$ and then the upward effect becomes stronger. This turning point is probably dependent on the interest rate because the higher the interest rate the stronger is the influence of an on average shorter time to maturity.

Sensitivity to the mean jump size μ

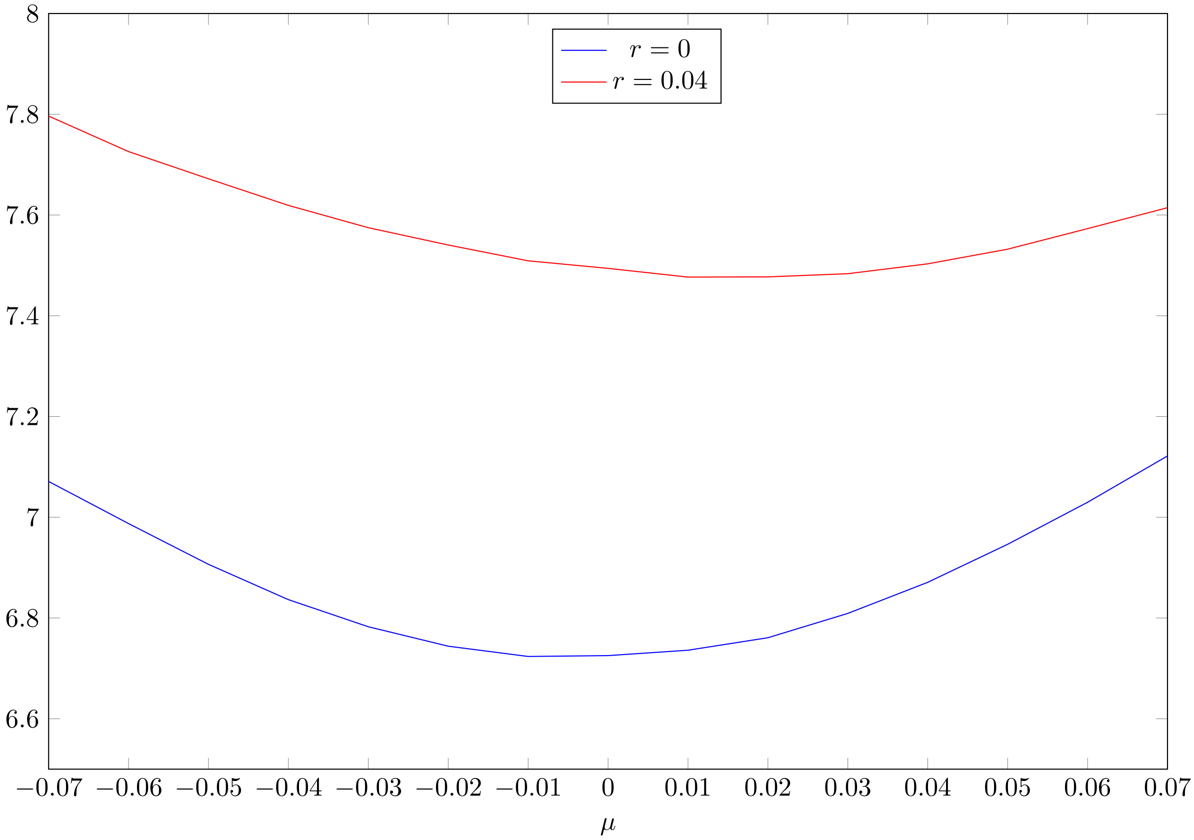

Figure 4.3: Plot of the influence of the mean jump size parameter μ on the price of a Timer Call (ordinate). Besides the explicitly given parameters μ and r, all parameters are as in table 4.1.

If the interest rate is zero, changing the mean size jump parameter μ leads to a smile where the price is a bit higher when μ is positive compared to negative μ. The minimum is around zero. If the interest rate is greater than zero, this smile is slightly tilted to the right and is flatter. The minimum is moved to the right. On the one hand a μ distinctly different form zero leads to larger jumps and thus similar to σ increases the probability of overshooting the budget massively. Furthermore, a large positive jump at the end is in case of a Timer Call better for the price than large negative jumps which explains the minor asymmetry. If the interest rate is greater than zero, another effect shows up. As for σ, larger jumps lead to overall shorter times to maturity which has a lowering effect on the price. This explains why the smile is a bit flatter. In order to explain the tilt to the right, one has to have a look at the dynamics of the stock price process and, in particular, at the compensator of the jump process. Here, normally distributed jumps are assumed with mean μ and σ which has a compensator of the following form as result:

$$c(\xi_t^c) = \lambda \xi_t^c \left(e^{\mu + \frac{\sigma^2}{2}} - 1 \right)$$

As one can see, the compensator is an increasing function in μ and therefore if μ is positive, the compensator is positive which means it has a lowering effect on the price. If μ is negative such that $\frac{\sigma^2}{2} < -\mu$ the compensator is negative and has an increasing effect on the price. The tilted smile is now a consequence of the correlation of the compensator and the stopping time τ. If there is no jump until τ, for instance, the stock price process is higher the lower μ is because of the compensator. This means, long times to maturity are connected to high stock prices if μ is negative and connected to low stock prices if μ is positive. On the other hand if there are jumps, the time to maturity tends to be shorter and overall the jumps, so to speak, overcompensate the compensator. Mathematically, this means that $J(\xi^c_\tau) - c(\xi^c_\tau)$ tends to be greater than zero if μ is positive and smaller than zero if μ is negative. Thus, short times to maturity are connected to high stock prices if μ is positive and connected to low prices if μ is negative. In a nutshell, if μ is negative, the stock price tends to be high when time to maturity is long and thus hedging becomes more expensive. On the contrary, if μ is positive, the stock price tends to be high when time to maturity is short and thus hedging becomes cheaper. It is somehow a very similar effect like the effect of the correlation coefficient ρ.

Sensitivity to the variance budget b, the correlation coefficient ρ and the interest rate r

The price of a Timer Call is influenced the same way by the variance budget b, the correlation coefficient ρ and the interest rate r as in the model without jumps. It is increasing in b and in r as can be seen in figure 4.4 and 4.6 respectively. The correlation coefficient is again antiproportional to the price. A negative correlation leads to higher and a positive correlation to lower prices as shown in figure 4.5. The rough behaviour is caused by the error of the Monte-Carlo simulation. The reasons for the behaviour in b, r and ρ are the same as in chapter 3. Thus, the discussion can be shortened here by referring to chapter 3.

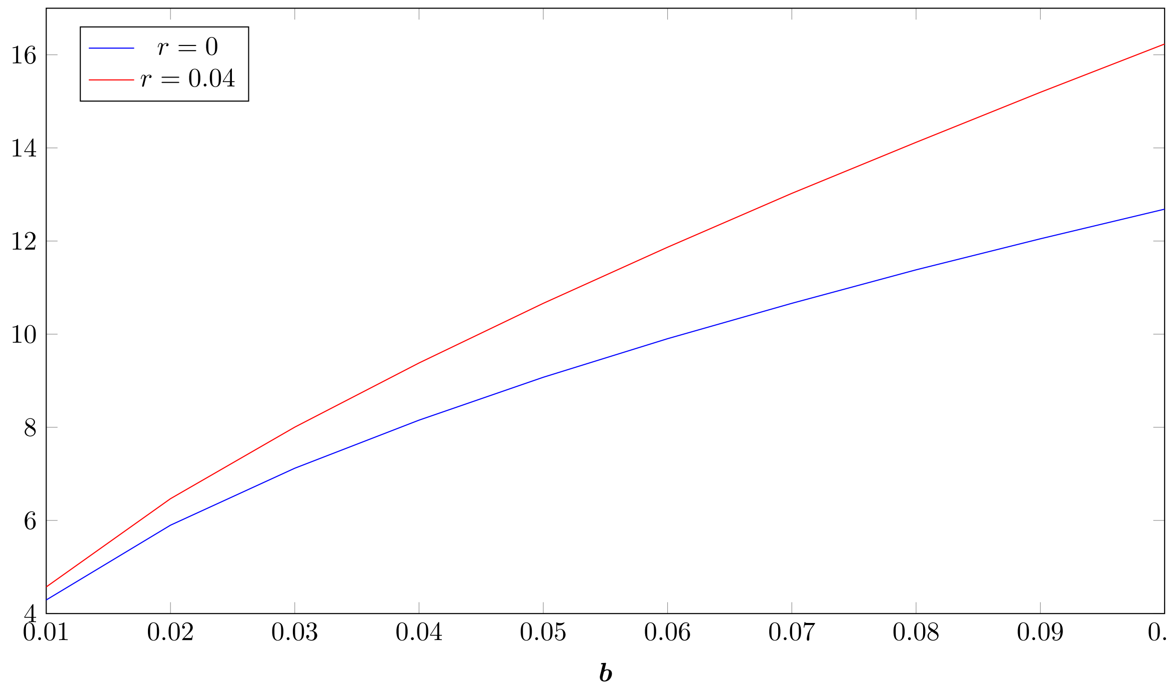

Figure 4.4: Plot of the influence of the variance budget $\boldsymbol{b}$ on the price of a Timer Call (ordinate). Besides the explicitly given parameters $\boldsymbol{b}$ and r, all parameters are as in table 4.1.

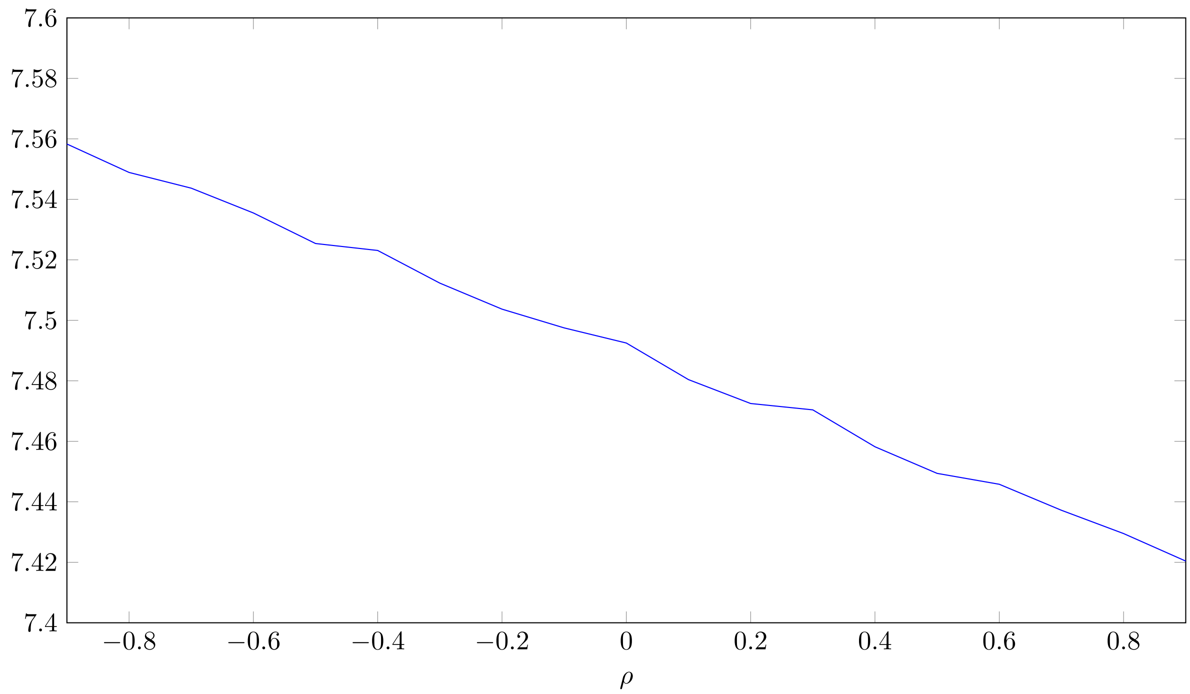

Figure 4.5: Plot of the influence of the correlation coefficient ρ on the price of a Timer Call (ordinate). Besides the explicitly given parameter ρ, all parameters are as in table 4.1.

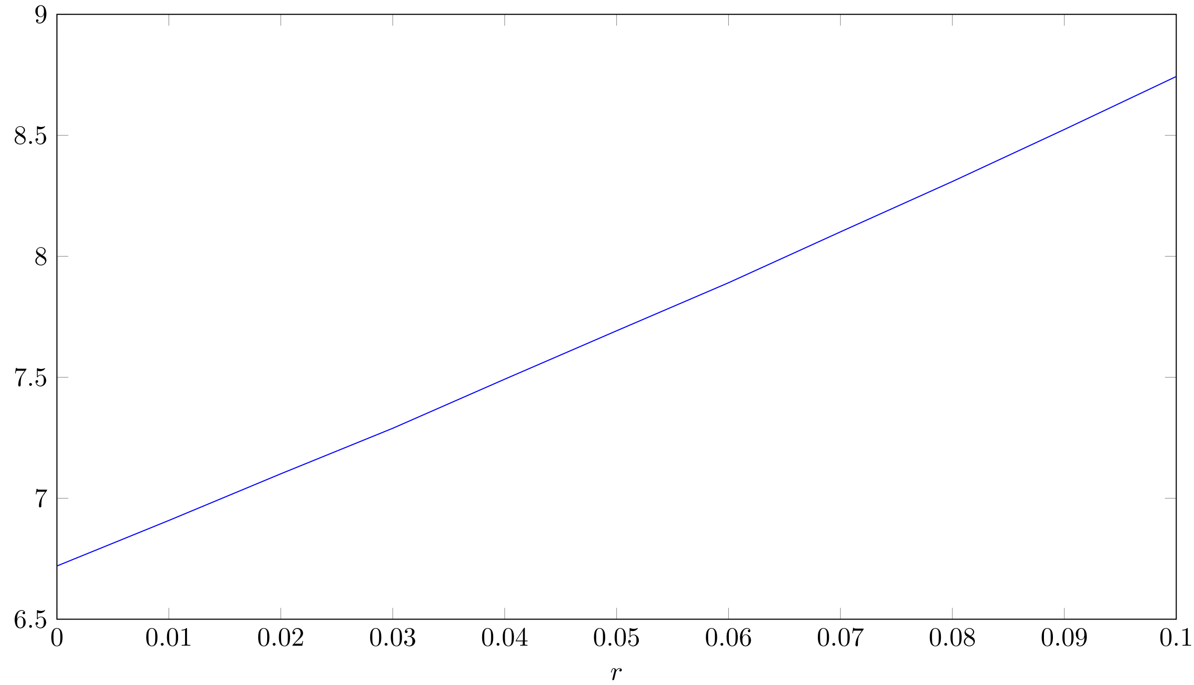

Figure 4.6: Plot of the influence of the interest rate r on the price of a Timer Call (ordinate). Besides the explicitly given parameter r, all parameters are as in table 4.1.

5 Comparison of the results of chapter 3 and chapter 4

In chapter 3 the stock price process was assumed to be driven by a Brownian motion only whereas in chapter 4 the stock price process was additionally allowed to jump. The question, which arises, is, what is the difference between the price of a Timer Call priced under (4.1) and the price of a Timer Call priced under (3.1) but assuming that the model (4.1) is the true model. Note, that the model for the stock price process in (3.1) is nested in the model of the stock price process in (4.1). If λ is not significantly greater than zero when fitting the model (4.1) to the market data, it just reduces to the model (3.1). Thus, the interesting case is when λ is significantly greater than zero. If the interest rate is zero, it is easy to do the comparison because the price does not depend on the variance process at all. If the interest rate is greater than zero, one can use synthetic data to study the pricing error when using the model 3.1 although the stock price has jumps 4.1. The variance process is assumed to be like in the Heston model. In particular, the data is generated by simulating

$$\begin{aligned} d\log(S_t) &= rdt + \sqrt{V_t}d(\sqrt{1-\rho^2}W_t^1 + \rho W_t^2) - \frac{1}{2}d\xi_t^c + d(J(\xi_t^c) - c(\xi_t^c)) \\ dV_t &= \kappa(\theta - V_t)dt + \gamma\sqrt{V_t}dW_t^2 \end{aligned} \tag{5.1}$$

where J is a compound Cox process with normally distributed jumps 4.1 and ξ_t^c being the continuous part of the stochastic clock. The parameters κ, θ, γ, ρ, μ and σ and the starting parameters S_0 and V_0 have the values given in table 5.1.

Table 5.1: Parameters for (5.1)

S_0	V_0	κ	γ	θ	λ	μ	σ
100	0.0625	2	0.1	0.0324	100	0	0.06

The procedure is then to price standard European Call options with different strike prices and times to maturity under 5.1 using Monte-Carlo methods. Having done this, a non-linear least square solver is used to fit the parameters of the Heston model which is defined by

$$\begin{aligned} dS_t &= rS_tdt + \sqrt{V_t}S_td(\sqrt{1-\rho_h^2}W_t^1 + \rho_h W_t^2) \\ dV_t &= \kappa_h(\theta_h - V_t)dt + \gamma_h\sqrt{V_t}dW_t^2 \end{aligned} \tag{5.2}$$

and with initial values S_0 and V_0^h to the prices of the European Call options obtained under 5.1. For more details, one can have a look at [3].
If not stated differently, the parameters for the comparison are as in table 5.1 and the interest rate is assumed to be $r = 0.04$. Since the price is independent of the choice of the variance process if $r = 0$, the price of a Timer Call is a constant if priced under the jump free model (3.1). Thus, the comparison when $r = 0$ is not explicitly shown. But the behaviour of the price of a Timer Call priced under (4.1) with $r = 0$ is presented in chapter 4 which can easily been taken to figure out the pricing error. Recall, that the price of a Timer Call in chapter 3 was 6.4871 when the interest rate was zero.

5.1 Comparison under different jump intensities λ

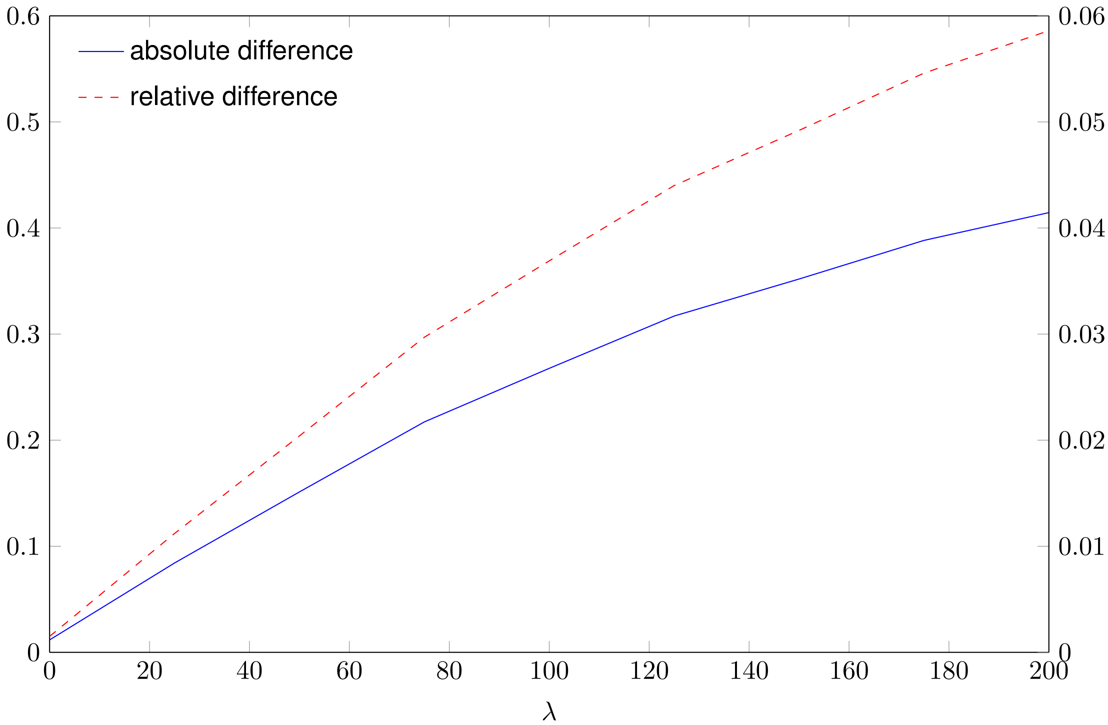

Figure 5.1: Comparison of a Timer Call priced under the Heston model (5.2) and under the jump model (5.1) for different jump intensity parameters λ. The absolute difference (Timer Call with jumps - Timer Call without jumps) is plotted against the left ordinate and the relative difference (Timer Call with jumps/Timer Call without jumps-1) is plotted against the right ordinate.

The absolute difference as well as the relative difference between the model with jumps (5.1) and the Heston model (5.2) with respect to the jump intensity parameter λ becomes larger when λ is increased. Indeed, the greater λ the more likely it is that the variance budget is over-consumed whereas overshoots are impossible in the Heston model. When $\lambda = 200$, the relative difference is about 6%. Note, that $\lambda = 200$ means that there are only two jumps in expectation if the volatility of the jumps adjusted stock price is 10% per year and there are eight jumps in expectation if this volatility is 20% per year. This shows that only a few jumps per year lead to significant pricing errors when the Heston model is assumed although there are even only few jumps. The results for the estimates of the parameters of the Heston model (5.2) as well as of the Monte-Carlo estimates for the prices of the Timer Call priced under (5.2) and (5.1) can be found in table A.2 in the appendix.

5.2 Comparison under different jump sizes σ

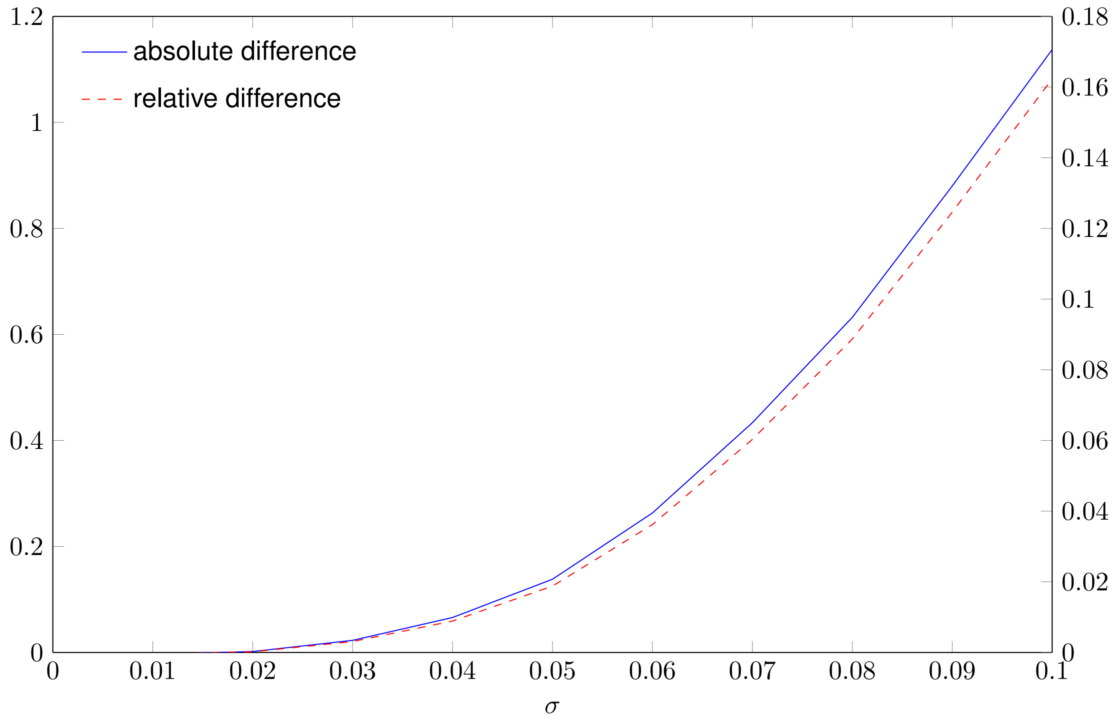

Figure 5.2: Comparison of a Timer Call priced under the Heston model (5.2) and under the jump model (5.1) for different jump sizes σ. The absolute difference (Timer Call with jumps - Timer Call without jumps) is plotted against the left ordinate and the relative difference (Timer Call with jumps/Timer Call without jumps-1) is plotted against the right ordinate.

As figure 5.2 shows, the difference between the price of a Timer call priced under (5.2) although (5.1) is true is almost insignificant if $\sigma \leq 0.02$. But then, the relative and absolute difference seems to grow exponentially fast. Thus, small jumps can be almost ignored but underestimating very large jumps on the contrary leads to clearly too low prices. This was to be expected because large jumps lead to movements which are not covered by the variance budget $\boldsymbol{b}$. For instance, if the remaining variance budget exactly before the jump occurs is $\boldsymbol{b} = 0.02$ a jump of 15% leads to an overshoot of 0.0025 of the variance budget which is not anticipated by the jump free model and therefore it is not priced in. In this setting, the jumps of the log stock price process are assumed to be normally distributed like in the Merton model [24]. But maybe one might use other distributions which lead to different prices. A distribution, for instance, which has heavier tails than the normal distribution will probably lead to larger prices than distributions which are more concentrated around zero and hence imply only small jumps. The results for the estimates of the parameters of the Heston model (5.2) as well as the Monte-Carlo estimates for the prices of the Timer Call priced under (5.2) and (5.1) can be found in table A.3 in the appendix.

5.3 Comparison under different mean jump sizes μ

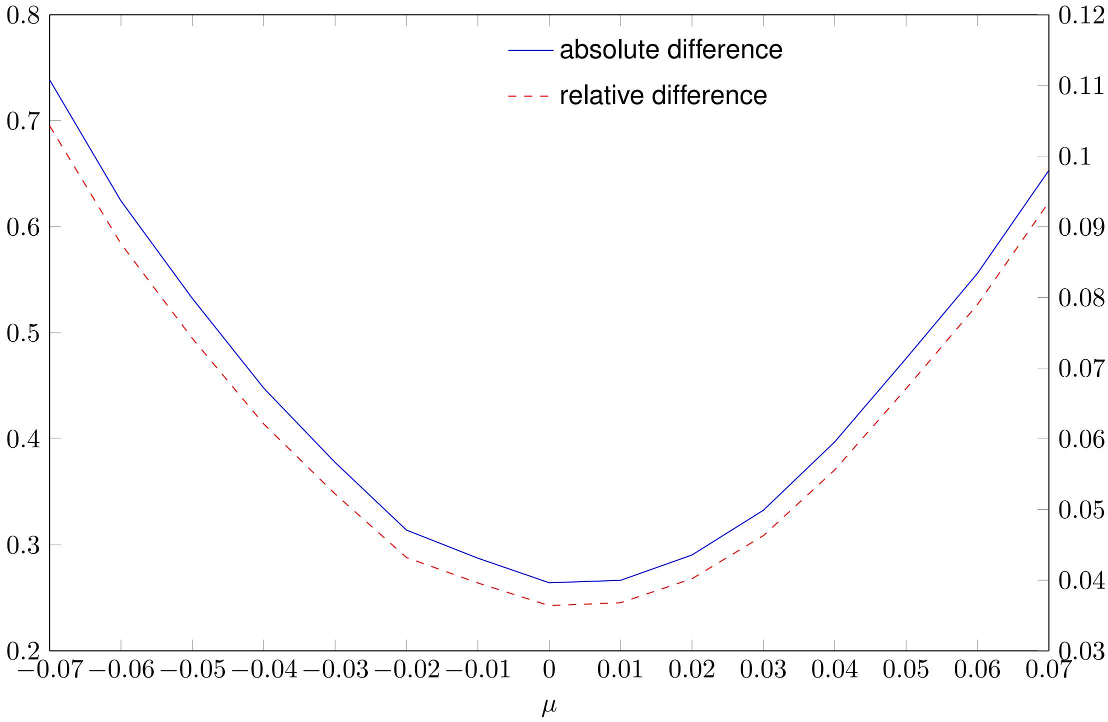

Figure 5.3: Comparison of a Timer Call priced under the Heston model (5.2) and under the jump model (5.1) for different mean jump sizes μ. The absolute difference (Timer Call with jumps - Timer Call without jumps) is plotted against the left ordinate and the relative difference (Timer Call with jumps/Timer Call without jumps-1) is plotted against the right ordinate.

The pricing error for different mean jump sizes is shown in figure 5.3. As it turns out, the smile obtained in the sensitivity analysis of μ shows up again. A μ around zero leads to a relative error of about 4% whereas a very high or a very low μ leads to a relative difference of up to 11%. One reason of course is that a high or low μ implies large (positive or negative) jumps and thus overshooting the budget massively becomes more probable like in the case of σ. The results for the estimates of the parameters of the Heston model (5.2) as well as the Monte-Carlo estimates for the prices of the Timer Call priced under (5.2) and (5.1) can be found in table A.4 in the appendix.

5.4 Comparison under different strike prices K

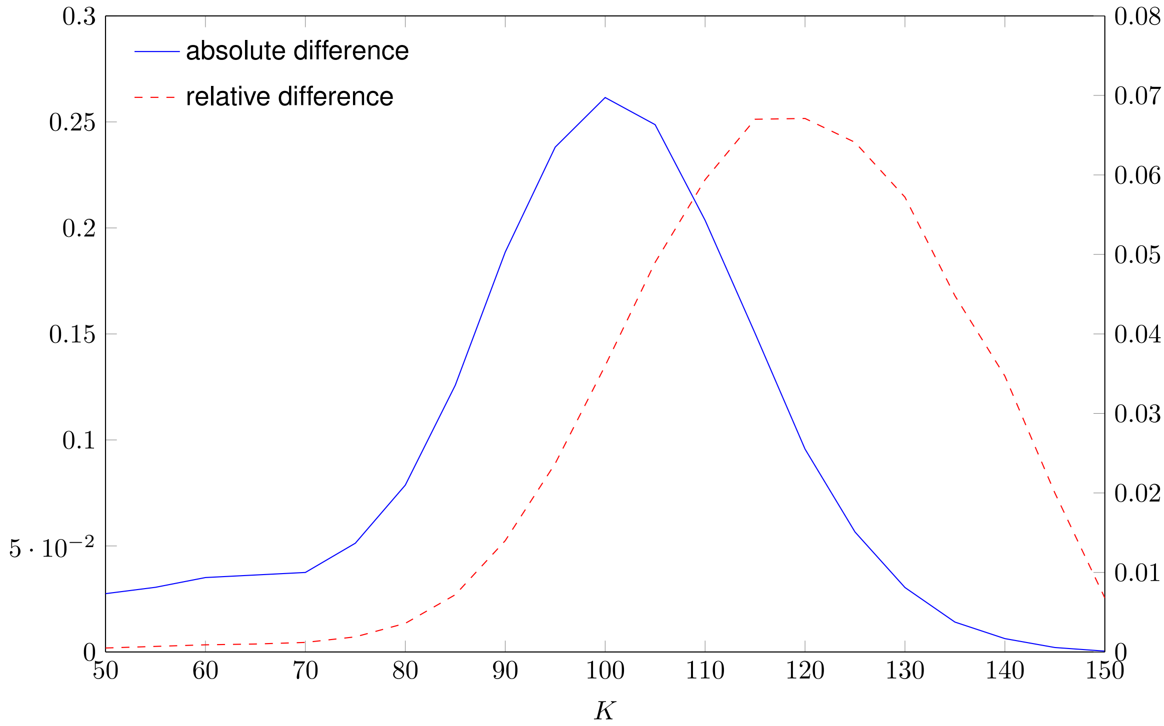

Figure 5.4: Comparison of a Timer Call priced under the Heston model (5.2) and under the jump model (5.1) for different strike prices K. The absolute difference (Timer Call with jumps - Timer Call without jumps) is plotted against the left ordinate and the relative difference (Timer Call with jumps/Timer Call without jumps-1) is plotted against the right ordinate.

Letting the pricing function of a Timer Call being a variable of the strike price K, leads to an absolute error having a maximum of about 0.26 at $K = 100$ and then being almost symmetrically decreasing. The relative error looks similar but is moved to the right. The maximum is here approximately 6.7% at $K = 120$. This indicates that Timer Calls at and out of the money are relatively underpriced when assuming (5.2) although (5.1) is true. Expiring in the money for Calls which are at or out of the money is more likely when having jumps than without. The results for the estimates of the parameters of the Heston model (5.2) as well as the Monte-Carlo estimates for the prices of the Timer Call priced under (5.2) and (5.1) can be found in table A.5 in the appendix.

5.5 Comparison under different interest rates r

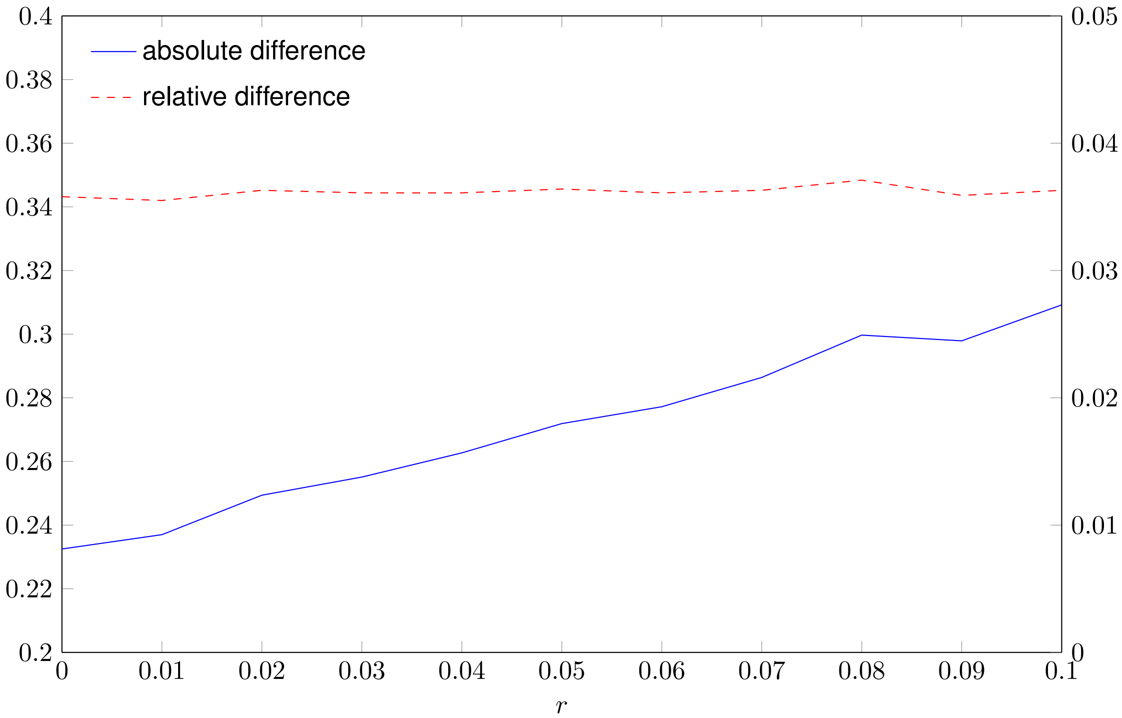

Figure 5.5: Comparison of a Timer Call priced under the Heston model (5.2) and under the jump model (5.1) for different interest rates r. The absolute difference (Timer Call with jumps - Timer Call without jumps) is plotted against the left ordinate and the relative difference (Timer Call with jumps/Timer Call without jumps-1) is plotted against the right ordinate.

Figure 5.5 shows the absolute and relative error for different interest rates r. The absolute error is slightly increasing whereas the relative error can be assumed to be constant at about 3.6%. This indicates that the interest rate r has no effect on the relative price difference. Again, the detailed results for the estimates of the parameters of the Heston model (5.2) as well as the Monte-Carlo estimates for the prices of the Timer Call priced under (5.2) and (5.1) can be found in table A.1 in the appendix.

6 Conclusion

The purpose of this thesis was to research into the effects of jumps on the price of Timer options. This was done by adding a compound Cox process to the continuous general stochastic volatility model (cf. chapter 3). The intensity of the compound Cox process was assumed to be the variance multiplied by a constant, called intensity parameter. For the simulation in chapter 4, a normal distribution was considered for the jumps of the log stock price process and a Cox-Ingersoll-Ross process for the variance process. Furthermore, a Timer Call option at the money was the Timer option of consideration. If there is no interest rate, the results were as expected when changing the jump size influencing parameters and the intensity parameter of the compound Cox process. Increasing the jump size and increasing the number of jumps separately results in higher prices of the Timer Call. But assuming an interest rate which is strictly greater than zero, the results were not as expected. In fact, it turned out that, for instance, having an interest rate of 4% the price slightly decreased when increasing the number of jumps. As mentioned, the jumps were assumed being normally distributed with mean zero and standard deviation 0.06 for the log stock price process. Furthermore, increasing the jump size also leads to decreasing prices in the beginning and only then starts increasing. Letting the interest rate being 4%, the price slightly decreased of about 0.6% in a range from 0 to 0.06 of the standard deviation of the normally distributed jumps with mean zero for the log stock price process. This behaviour was quite astonishing and is due to the construction of Timer options. A jump leads to an instantaneous and significant consumption of the variance budget which, on the other hand, leads to a shorter time to maturity and thus to lower hedging costs. In contrast, jumps may cause significant overshoots in the budget which, indeed, only depends on the jump size and number of jumps. But this effect increases the price because jumps allow for movements in the stock price which are not covered by the variance budget.
However, as chapter 5 shows, neglecting jumps at all can result in considerably too low prices. For instance, consider a Heston model for the jump free model compared to the jump model with a compound Cox process with normally distributed jumps for the log price process of the stock and a variance process following a Cox-Ingersoll-Ross process as mentioned above. Then, pricing Timer Calls under the Heston model, although the jump model is assumed to be true, results in possibly significantly wrong prices. Especially when jumps are large or if there are many, the price is too low. Furthermore, Timer Calls out of the money and at the money are considerably more expensive than implied by the Heston model, although the underlying asset jumps.

For further research: The theory for Timer options stated in [7] and in this thesis is only applicable to Timer options which are path independent. For instance, the theory does not include

Timer Barrier options or American Timer Call and Put options because one time-changes the stock price and variance process which implies equality in distribution only.
Furthermore, it might be interesting to consider correlated jumps in the stock price process and the variance process.

A Appendix

A.1 Derivation of Vega

Proof. Taking the derivative w.r.t. the variance budget is somewhat more complicated. By corollary 3.1.7 the price of a Timer Call is given by taking the expectation of

$$\tilde{S_0}\mathcal{N}(\tilde{d_1}) - Ke^{-r\tau}\mathcal{N}(\tilde{d_2}) \tag{A.1}$$

with

$$\tilde{d_1} = \frac{\log\left(\frac{S_0}{K}\right) + r\tau + \frac{1}{2}\boldsymbol{b}}{\sqrt{\boldsymbol{b}}} \qquad\qquad \tilde{d_2} = \tilde{d_1} - \sqrt{(1-\rho^2)\boldsymbol{b}}.$$

when $\rho = 0$. Thus, switching derivation and integration and then taking the derivative w.r.t. the variance budget $\boldsymbol{b}$ of (A.1), one obtains

$$\frac{\partial}{\partial \boldsymbol{b}}\tilde{S_0}\mathcal{N}(\tilde{d_1}) - Ke^{-r\tau}\mathcal{N}(\tilde{d_2}) = S_0\mathcal{N}'(\tilde{d_1})\frac{\partial}{\partial \boldsymbol{b}}\tilde{d_1} + \left(\frac{r}{V_\tau}Ke^{-r\tau}\mathcal{N}(\tilde{d_2}) - Ke^{-r\tau}\mathcal{N}'(\tilde{d_2})\frac{\partial}{\partial \boldsymbol{b}}\tilde{d_2}\right)$$

Note, that since $\tau \stackrel{d}{=} \int_0^{\boldsymbol{b}} \frac{1}{X_v}$ and $\frac{1}{X_{\boldsymbol{b}}} \stackrel{d}{=} \frac{1}{V_\tau}$, taking the derivative of τ w.r.t. the variance budget is given by

$$\begin{aligned}\frac{\partial}{\partial \boldsymbol{b}}\tau &\stackrel{d}{=} \frac{\partial}{\partial \boldsymbol{b}}\int_0^{\boldsymbol{b}} \frac{1}{X_v} \\ &= \frac{1}{X_{\boldsymbol{b}}} \stackrel{d}{=} \frac{1}{V_\tau}.\end{aligned}$$

Now, using that $\frac{\partial}{\partial \boldsymbol{b}}\tilde{d_2} = \frac{\partial}{\partial \boldsymbol{b}}\tilde{d_1} - \frac{1}{2\sqrt{\boldsymbol{b}}}$ and that

$$\mathcal{N}'(\tilde{d_2}) = \mathcal{N}'(\tilde{d_1})\frac{S_0}{K}e^{r\tau}$$

one finally obtains the claim. □

A.2 Matlab Code

A.2.1 Routine to simulate Timer Call prices with variance reduced Monte-Carlo methods under the Heston model

```
% Monte-Carlo simulation to price a Timer Call with control variate under the Heston model

% Parameters Timer Call
r = 0.04;
budget = 0.0265;
K = 100;

% Parameters stock price process
S0 = 100;
cor = 0.8;

% Number of samples
N = 1000;
% Discretization steps
M = 100;


% Parameters variance process
V0 = 0.0625;
kappa = 2;
theta = 0.0324;
gamma = 0.1;
h = @(x,y) kappa/gamma*(theta-y);
f = @(x) x/gamma;
invf = @(x) gamma*x;


[C_mc,std_mc] = TimerCallCV(r,budget,K,[S0 cor],[V0 kappa theta gamma],h,f,invf,N,M);


%Function for computing the price of a Timer Call
function [TC_mc,std_mc] = TimerCallCV(r,budget,K,parameterStock,parameterVol,h,f,invf,N,M)

    V0 = parameterVol(1);
    kappa = parameterVol(2);
    theta = parameterVol(3);
    gamma = parameterVol(4);
    S0 = parameterStock(1);
    cor = parameterStock(2);


    V_tau = zeros(1,N);
    tau = zeros(1,N);
    H_tau = zeros(1,N);
    delta_v = budget/M;
```

```
    %Drawing (tau,V_tau,H_tau)
    for l = 1:N
        temp_tau = 0;
        temp_H = 0;
        X = V0;
        for i = 1:M
            temp_H = temp_H + h(temp_tau,X)/X*delta_v;
            X_new = X + invf(h(temp_tau,X)/X*delta_v + normrnd(0,1)*sqrt(delta_v));
            temp_tau = temp_tau + (1/X_new + 1/X)/2*delta_v;
            X = X_new;

        end

        V_tau(l) = X;
        tau(l) = temp_tau;
        H_tau(l) = temp_H;
    end

        %Computing the price of Timer Call (C_mc) and the MC error (std_mc)
    TC = Timer_Call_Price(V_tau,tau,H_tau,f,S0,V0,K,r,cor,budget);
    CV = Timer_Call_Price(V_tau,tau,H_tau,f,S0,V0,K,0,cor,budget);
    lambda = cov(TC,CV)/var(CV);
    TC_vr = TC - lambda(1,2)*(CV-Timer_Call_Price(V_tau,tau,H_tau,f,S0,V0,K,0,0,budget));
    TC_mc = mean(TC_vr);
    std_mc = std(TC_vr)/sqrt(N);
end

%Function for computing the price for a Timer Call (TC) given a sample (tau,V_tau,H_tau)
function [ TC ] = Timer_Call_Price(V,tau,H,f,S0,V0,K,r,cor,budget)

    d1 = (log(S0/K) + r*tau + cor*(f(V)-f(V0)-H) +
                ...(0.5 - cor*cor)*budget)/sqrt((1-cor*cor)*budget);
    d2 = d1 - sqrt((1-cor*cor)*budget);
    TC = S0*exp(cor*(f(V)-f(V0)-H-0.5*cor*budget)).*normcdf(d1)-K*exp(-r*tau).*normcdf(d2);

end
```

A.2.2 Routine to simulate Timer Call prices with variance reduced Monte-Carlo methods under the jump model of chapter 4

```
% Variance reduced Monte-Carlo simulation to price a Timer Call with control variance  under


% Parameters Timer Call
r = 0.04;
budget = 0.0265;
K = 100;

% Parameters stock price process
S0 = 100;
cor = -0.8;
lambda = 100;
mu = 0;
sigma = 0.06;

% Number of samples
N = 100;
% Discretization steps
M = 100;


% Parameters variance process
V0 = 0.0625;
kappa = 2;
theta = 0.0324;
gamma = 0.1;
h = @(x,y) kappa/gamma*(theta-y);
f = @(x) x/gamma;
invf = @(x) gamma*x;


%Function for computing Timer Calls
tic
[C_mc std_mc] = TimerCallCV_Jumps(r,budget,K,[S0 cor lambda mu sigma],
...[V0 kappa theta gamma],h,f,invf,N,M);

%function for simulating the samples needed for Monte-Carlo and returning\\
%the Monte-Carlo estimate and its standard deviation
function [C_mc,std_mc] = TimerCallCV_Jumps(r,budget,K,parameterStock,parameterVol,h,f,invf,N,M)

    V0 = parameterVol(1);
    kappa = parameterVol(2);
    theta = parameterVol(3);
    gamma = parameterVol(4);
    S0 = parameterStock(1);
    cor = parameterStock(2);
    lambda = parameterStock(3);
    mu = parameterStock(4);
```

```
    sigma = parameterStock(5);

    J= zeros(1,N);
    b = zeros(1,N);
    C = zeros(1,N);
    CV = zeros(1,N);
    V_tau = zeros(1,N);
    tau = zeros(1,N);
    H_tau = zeros(1,N);
    delta_v = budget/M;


%Drawing the Jumps and the value of the continuous part of the stochastic clock

    for j= 1:N
        p = poissrnd(budget*lambda);
        Jump = normrnd(mu,sigma,1,p);
        T = [sort(rand(1,p)*budget) budget];

        if (T(1)>=budget || p == 0)
            b(j) = budget;
        else
            b(j) = T(1);
            for i = 1:p
                J(j) = J(j) + Jump(i);

                if(b(j)+sum(Jump(1:i).^2)>=budget)
                    break;
                end
                b(j) = T(i+1);
                if(b(j)+sum(Jump(1:i).^2)>=budget)
                    b(j) = budget - sum(Jump(1:i).^2);
                    break;
                end
            end
        end
    end

%Drawing (tau,V_tau,H(tau,V_tau)) if r>0

    if r>0
        for j= 1:N
            M_temp = ceil(b(j)/budget*M);
            delta_v = b(j)/M_temp;
            temp_tau = 0;
            temp_H = 0;
            X = V0;
            for i = 1:M_temp
                temp_H = temp_H + h(temp_tau,X)/X*delta_v;
                X_new = X + invf(h(temp_tau,X)/X*delta_v + normrnd(0,1)*sqrt(delta_v));
                temp_tau = temp_tau + (1/X_new + 1/X)/2*delta_v;
```

```
                X = X_new;

            end

            V_tau(j) = X;
            tau(j) = temp_tau;
            H_tau(j) = temp_H;
        end
    end

%Computing the price of Timer Call (C_mc) and the MC error (std_mc)

    if r==0
        c = lambda*b*(exp(mu+sigma^2/2)-1);
        C = Timer_Call_Price_Jumps(J,c,V_tau,tau,H_tau,f,S0,V0,K,r,cor,b);
        CV = exp(J-c);
        lambda1 = cov(C,CV)/var(CV);
        if lambda>0 && sigma>0
            C = C - lambda1(1,2)*(CV-1);
        end
        C_mc = mean(C);
        std_mc = std(C)/sqrt(N);
    else
        c = lambda*b*(exp(mu+sigma^2/2)-1);
        C = Timer_Call_Price_Jumps(J,c,V_tau,tau,H_tau,f,S0,V0,K,r,cor,b);
        d1 = (log(S0/K)  + cor*(f(V_tau)-f(V0)-H_tau) +J - c +
                          ...(0.5 - cor*cor)*b)./sqrt((1-cor*cor)*b);
        d2 = d1 - sqrt((1-cor*cor)*b);
        CV1 = S0*exp(cor*(f(V_tau)-f(V0)-H_tau-0.5*cor*b) + J - c).*normcdf(d1)-K*normcdf(d2);

        ECV1 = Timer_Call_Price_Jumps(J,c,1,1,1,f,S0,V0,K,0,0,b);
        CV2 = exp(J-c);
        lambda1 = cov(C,CV1)/var(CV1);
        lambda2 = cov(C,CV2)/var(CV2);
        if lambda>0 && sigma>0
            C = C - lambda1(1,2)*(CV1-ECV1)-lambda2(1,2)*(CV2-1);
        else
            C = C - lambda1(1,2)*(CV1-ECV1);
        end
        C_mc = mean(C);
        std_mc = std(C)/sqrt(N);
    end
end

%function for computing the price of a Timer Call given a sample (\xi_tau,J_tau,tau,V_tau,H_tau)
function [ C ] = Timer_Call_Price_Jumps(J,c,V,tau,H,f,S0,V0,K,r,cor,budget)
    if r > 0
        d1 = (log(S0/K) + r*tau + cor*(f(V)-f(V0)-H) + J - c +
                          ...(0.5 - cor*cor)*budget)./sqrt((1-cor*cor)*budget);
        d2 = d1 - sqrt((1-cor*cor)*budget);
        C = S0*exp(cor*(f(V)-f(V0)-H-0.5*cor*budget)
```

```
                        ...+ J - c).*normcdf(d1)-K*exp(-r*tau).*normcdf(d2);
    else
        d1 = (log(S0/K)  +J - c + 0.5*budget)./sqrt(budget);
        d2 = d1 - sqrt(budget);
        C = S0*exp(J - c).*normcdf(d1)-K*normcdf(d2);
    end
end
```

A.3 Tables of chapter 5

Table A.1: Comparison under different interest rates r

r	$\hat{V}_0^h$	$\hat{\kappa}_h$	$\hat{\theta}_h$	$\hat{\gamma}_h$	$\hat{\rho}_h$	Timer Call with jumps		Timer Call without jumps		absolute difference	relative difference
0	0.0834	1.7054	0.0415	0.1589	-0.2905	6.4871	(0.0000)	6.7196	(0.0019)	0.2325	0.0358
0.01	0.0834	1.7084	0.0416	0.1590	-0.2903	6.6776	(0.0007)	6.9146	(0.0019)	0.237	0.0355
0.02	0.0834	1.7085	0.0416	0.1592	-0.2899	6.8687	(0.0014)	7.1181	(0.0019)	0.2494	0.0363
0.03	0.0834	1.7102	0.0416	0.1595	-0.2895	7.0656	(0.0022)	7.3207	(0.0019)	0.2551	0.0361
0.04	0.0834	1.7121	0.0416	0.1598	-0.2889	7.2686	(0.0029)	7.5313	(0.0019)	0.2627	0.0361
0.05	0.0834	1.7121	0.0416	0.1598	-0.2889	7.4668	(0.0038)	7.7387	(0.0019)	0.2719	0.0364
0.06	0.0834	1.7181	0.0417	0.1610	-0.2870	7.6745	(0.0046)	7.9517	(0.0019)	0.2772	0.0361
0.07	0.0834	1.7218	0.0418	0.1615	-0.2861	7.8807	(0.0056)	8.1671	(0.0020)	0.2864	0.0363
0.08	0.0834	1.7267	0.0418	0.1621	-0.2852	8.0850	(0.0068)	8.3847	(0.0020)	0.2997	0.0371
0.09	0.0834	1.7321	0.0419	0.1628	-0.2842	8.3073	(0.0073)	8.6053	(0.0021)	0.2979	0.0359
0.1	0.0834	1.7366	0.0420	0.1634	-0.2832	8.5209	(0.0084)	8.8302	(0.0022)	0.3092	0.0363

...

Table A.2: Comparison under different jump intensities λ

λ	$\hat{V}_0^h$	$\hat{\kappa}_h$	$\hat{\theta}_h$	$\hat{\gamma}_h$	$\hat{\rho}_h$	Timer Call without jumps		Timer Call with jumps		absolute difference	relative difference
0	0.0623	1.9103	0.0317	0.0932	-0.5228	7.5858	(0.0039)	7.5975	(0.0002)	0.0117	0.0015
25	0.0675	1.8260	0.0341	0.1149	-0.4146	7.4923	(0.003 0)	7.5765	(0.0007)	0.0842	0.0112
50	0.0729	1.8381	0.0372	0.1341	-0.3549	7.4065	(0.0031)	7.5577	(0.0012)	0.1512	0.0204
75	0.0783	1.7394	0.0389	0.1432	-0.3268	7.3238	(0.0029)	7.5410	(0.0015)	0.2172	0.0297
100	0.0837	1.7839	0.0428	0.1666	-0.2724	7.2618	(0.0031)	7.5294	(0.0019)	0.2677	0.0369
125	0.0892	1.7584	0.0451	0.1617	-0.2814	7.2021	(0.0026)	7.5191	(0.0022)	0.3170	0.0440
150	0.0948	1.7856	0.0481	0.1732	-0.2660	7.1542	(0.0026)	7.5061	(0.0025)	0.3519	0.0492
175	0.1001	1.6354	0.0489	0.1742	-0.2493	7.1073	(0.0025)	7.4956	(0.0027)	0.3883	0.0546
200	0.1061	1.9163	0.0556	0.1972	-0.2385	7.0746	(0.0023)	7.4891	(0.003 0)	0.4145	0.0586

...

Table A.3: Comparison under different jump sizes σ

σ	$\hat{V}_0^h$	$\hat{\kappa}_h$	$\hat{\theta}_h$	$\hat{\gamma}_h$	$\hat{\rho}_h$	Timer Call without jumps		Timer Call with jumps		absolute difference	relative difference
0	0.0623	1.9103	0.0317	0.0932	-0.5228	7.6001	(0.0040)	7.5972	(0.0002)	-0.0030	-0.0004
0.01	0.0628	1.9013	0.0320	0.0922	-0.5265	7.5865	(0.0036)	7.5833	(0.0002)	-0.0032	-0.0004
0.02	0.0647	1.9361	0.0332	0.0986	-0.5018	7.5454	(0.0034)	7.5472	(0.0004)	0.0018	0.0002
0.03	0.0679	1.9053	0.0345	0.1028	-0.4715	7.4815	(0.0031)	7.5044	(0.0007)	0.0229	0.0031
0.04	0.0719	1.8458	0.0363	0.1155	-0.4094	7.4113	(0.0026)	7.4773	(0.0011)	0.0660	0.0089
0.05	0.0775	1.9106	0.0401	0.1407	-0.3323	7.3412	(0.0029)	7.4794	(0.0015)	0.1382	0.0188
0.06	0.0839	1.8301	0.0430	0.1625	-0.2869	7.2637	(0.0030)	7.5268	(0.0019)	0.2632	0.0362
0.07	0.0912	1.7661	0.0469	0.1888	-0.2349	7.1940	(0.0035)	7.6275	(0.0023)	0.4335	0.0603
0.08	0.1001	1.6725	0.0507	0.2245	-0.1877	7.1290	(0.0037)	7.7613	(0.0027)	0.6322	0.0887
0.09	0.1098	1.5550	0.0543	0.2446	-0.1661	7.0626	(0.0044)	7.9426	(0.0031)	0.8800	0.1246
0.1	0.1214	1.7096	0.0639	0.3079	-0.1200	7.0093	(0.0047)	8.1467	(0.0035)	1.1374	0.1623

...

Table A.4: Comparison under different mean jump sizes μ

μ	$\hat{V}_0^h$	$\hat{\kappa}_h$	$\hat{\theta}_h$	$\hat{\gamma}_h$	$\hat{\rho}_h$	Timer Call without jumps		Timer Call with jumps		absolute difference	relative difference
-0.07	0.1152	3.4306	0.0708	0.2559	-0.6802	7.0793	(0.0037)	7.8178	(0.0033)	0.7386	0.1043
-0.06	0.1071	3.1222	0.0644	0.2376	-0.6218	7.1307	(0.0031)	7.7551	(0.0030)	0.6244	0.0876
-0.05	0.1000	2.9179	0.0595	0.2170	-0.5915	7.1718	(0.0034)	7.7042	(0.0027)	0.5324	0.0742
-0.04	0.0942	2.6346	0.0548	0.1984	-0.5423	7.2051	(0.0029)	7.6528	(0.0024)	0.4478	0.0621
-0.03	0.0895	2.2820	0.0499	0.1819	-0.4775	7.2369	(0.0026)	7.6146	(0.0022)	0.3777	0.0522
-0.02	0.0864	2.1179	0.0472	0.1760	-0.4061	7.2601	(0.0027)	7.5740	(0.0020)	0.3139	0.0432
-0.01	0.0843	1.8679	0.0439	0.1558	-0.3701	7.2620	(0.0026)	7.5494	(0.0019)	0.2874	0.0396
0	0.0836	1.6823	0.0416	0.1622	-0.2783	7.2642	(0.0032)	7.5284	(0.0019)	0.2642	0.0364
0.01	0.0843	1.6103	0.0405	0.1582	-0.2227	7.2492	(0.0032)	7.5157	(0.0019)	0.2665	0.0368
0.02	0.0869	1.7690	0.0436	0.1711	-0.1453	7.2207	(0.0033)	7.5111	(0.0020)	0.2904	0.0402
0.03	0.0906	1.7378	0.0441	0.1823	-0.0740	7.1876	(0.0034)	7.5202	(0.0022)	0.3326	0.0463
0.04	0.0959	1.8380	0.0473	0.2007	-0.0121	7.1387	(0.0030)	7.5359	(0.0025)	0.3971	0.0556
0.05	0.1027	2.0532	0.0528	0.2242	0.0452	7.0880	(0.0025)	7.5638	(0.0028)	0.4758	0.0671
0.06	0.1126	2.7055	0.0619	0.2784	0.1075	7.0410	(0.0024)	7.5971	(0.0031)	0.5561	0.0790
0.07	0.1231	2.8367	0.0674	0.3221	0.1458	6.9861	(0.0022)	7.6392	(0.0034)	0.6532	0.0935

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Table A.5: Comparison under different strike prices K

K	$\hat{V}_0^h$	$\hat{\kappa}_h$	$\hat{\theta}_h$	$\hat{\gamma}_h$	$\hat{\rho}_h$	Timer Call without jumps		Timer Call with jumps		absolute difference	relative difference
50						50.7621	(0.0026)	50.7912	(0.0008)	0.0291	0.0006
60						40.9166	(0.0032)	40.9515	(0.0009)	0.0349	0.0009
70						31.1241	(0.0036)	31.1625	(0.0011)	0.0384	0.0012
80						21.6850	(0.0038)	21.7661	(0.0016)	0.0810	0.0037
90						13.4352	(0.0039)	13.6204	(0.0027)	0.1853	0.0138
100	0.0839	1.8301	0.0430	0.1625	-0.2869	7.2633	(0.0029)	7.5261	(0.0036)	0.2628	0.0362
110						3.4286	(0.0021)	3.6320	(0.0033)	0.2034	0.0593
120						1.4249	(0.0013)	1.5230	(0.0023)	0.0981	0.0689
130						0.5310	(0.0006)	0.5627	(0.0013)	0.0317	0.0598
140						0.1816	(0.0004)	0.1877	(0.0006)	0.0061	0.0337
150						0.0573	(0.0002)	0.0578	(0.0003)	0.0005	0.0095

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