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Contents

M-step Bootstrap Percolation on Z^1

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Abstract

We study an extended bootstrap percolation process in several steps in dimension one, assuming noncyclic boundary conditions. We study the asymptotic behavior of the process as the size of the underlying graph goes to infinity. We find a phase transition with respect to the initial conditions. The phase diagram on the set of initial conditionals of the extended process is described completely. This provides necessary and sufficient conditions for complete percolation of the extended process.

1 Some symbols and notations

A_x the number of elements of one in a vector $x = (x_i)_{i=1}^n$.

$A(\tau) = A_{x(\tau)}$

B_t the union of special intervals.

$g(x) = \prod_{j=1}^m g_j(x)$ a generating function of a certain combinatorial problem and $g_j(x)$ are the factors.

$\bar{0} = (0)_{i=1}^n$ the vector of zeros.

Q_s the percolator operator

R_p the randomizer operator

T_j a subset of $N = \{1, 2, 3.....n\}$

T the stopping time for the process.

$V_{s,x}$ the union of intervals in x with $\mid I_j \mid$ equal or less $s \geq 0$ a positive integer and greater than zero.

$U_{s,x}$ The element of $V_{s,x}$.

ω_n the number of zeros in a vector x.

$x_M = (x_i)_{i \in M}$

$\phi(n) \in [0, 1]$ a function which goes to zero when n goes to ∞.

2 Introduction

We study bootstrap percolation on Z^1 which generalizes our earlier study in [2].

Although the most studied cases are the model in higher dimensions, the dimension one is interesting since the problem has an exact asymptotic solution. We show that this is in a perfect agreement with the high-dimension results of Bollobas, Holmgren, Smith and Uzzel [1].

Consider a graph on a vertex set $V = \{1, 2.., i, .., n\}$ with a set of edges $E = \{12, 23.., (i-1)i, .., (n-1)n\}$. We denote this graph $Z_n = (V, E)$ and call it "a strap". Let us define a bootstrap percolation process $X(t) = (x_i(t), i = 1, 2, .., n)$, $t = 0, 1, ..$, on Z_n. We fix the boundary condition, namely we assume that $x_0(t) = 1$ and $x_{n+1}(t) = 1$ for all $t \geq 0$. Then $x_i(t) \in \{0, 1\}$ is defined as follows. We set initially

$$x_i(0) = \begin{cases} 1 & \text{with probability } p, \\ 0 & \text{with probability } 1-p, \end{cases} \quad (1)$$

independent for different i. Then for all $t \geq 1$ and $i = 1, .., n$, we set

$$x_i(t+1) = \begin{cases} 1, & \text{if } (x_{i-1}(t)) = 1 \text{ and } (x_{i+1}(t)) = 1, \\ x_i(t), & \text{otherwise.} \end{cases} \quad (2)$$

We call this process Bootstrap Percolation on Z_n.

Define $A(t) = \#\{i : x_i(t) = 1\}$. Note that by the definition of the process, we have $A(t+1) \geq A(t)$ for all $t \geq 0$. Since $A(t) \leq n$ we can define the stopping time T for Bootstrap Percolation on Z_n

$$T = min\{t : A(t+1) = A(t)\}.$$

T is also bounded by n. Furthermore it is not difficult to see that $T \leq 1$. We shall study $A(T)$ for different choices of p, as well as of the initial conditions. We study the conditions of percolation and probability of percolation. The asymptotics of $P(A(1) = n)$ was studied in Pehrsson [2]. Here we define further dynamics for the case when $A(1) < n$ given a binary initial vector. We shall study $P(A(1) = n | A(0) = y)$ and $P(A(1) = n)$, which is the probability of percolation at the first step.

3 Preliminaries

Let us recall results from Pehrsson [2] which we use here.

Proposition 3.1 *(Fixed initial conditions.) For all $n \geq 0$ and we have*

$$P(A(1) = n | A(0) = x) = \begin{cases} \frac{\binom{x+1}{n-x}}{\binom{n}{x}}, & \text{if } x \geq \lceil n/2 \rceil, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$P(A(1) = n) = E(P(A(1) = n \mid A(0))$$
$$= \sum_{x=\lceil n/2 \rceil}^{n} \binom{x+1}{n-x} p^x (1-p)^{n-x}.$$

Proposition 3.2 *(Phase transition.)*
(i) If $\omega_n \ll \sqrt{n}$ (i.e., $\omega_n = o(\sqrt{n})$) then

$$P(A(1) = n \mid A(0) = n - \omega_n) \to 1$$

as $n \to \infty$.
(ii) If $\omega_n \gg \sqrt{n}$ (i.e., $\sqrt{n} = o(\omega_n)$) then

$$P(A(1) = n \mid A(0) = n - \omega_n) \to 0$$

as $n \to \infty$.
(iii) If $\omega_n = c\sqrt{n}$ then

$$P(A(1) = n \mid A(0) = n - \omega_n) \to e^{-c^2},$$

as $n \to \infty$.

Theorem 3.1 *([2])*
Assume that the initial conditions $A(0)$ have a $Bin(n, p)$ distribution. Let

$$p = p(n) = 1 - \phi(n), \tag{3}$$

where $\phi(n)$ is a decreasing monotone function, $0 \leq \phi(n) \leq 1$ and

$$\lim_{n\to\infty} \phi(n) = 0 \tag{4}$$

Then

$$\lim_{n\to\infty} P(A(1) = n) = 1, \tag{5}$$

if $\phi(n) < n^{-\alpha}$ and $\alpha > 1/2$, and

$$\lim_{n\to\infty} P(A(1) = n) = 0, \tag{6}$$

if $\phi(n) > n^{-\alpha}$ and $0 < \alpha < 1/2$.

3.1 Proof of Proposition 3.1.

We regard the strap with assigned values to the vertexes as a vector $(x_i)_{i=1}^n$, where each x_i can take value 0 or 1. We can form $\binom{n}{x}$ different vectors with x ones.

Consider the vector of initial conditions. If in the initial vector we have a zero between two ones then this zero will be changed to one at the first step.

We are interested in those vectors of the initial conditions, which yield complete percolation at the first step. We denote the number of such vectors $C(A(1) = n \mid A(0) = x)$ or only $C(n \mid A(0) = x)$. In the case of straps it is a little bit simplified. We have initially $x + 1$ intervals I_k , $1 \leq k \leq x + 1$, where we define an interval as the longest list of the consecutive indices of elements of zero, could be an empty set. Together we have $n - x$ zeros, which gives a simple relation:

$$\sum_{k=1}^{x+1} |I_k| = n - x. \tag{7}$$

We have

$$0 \leq |I_k| \leq 1. \tag{8}$$

We place the first zero in $x+1$-st interval. The second zero is in the xth interval, and so on: we place the $n-x$ zero in $x-(n-x-1)$ intervals. We have $n-x$ different orders. By multiple principle we have $(x + 1)x..\frac{2x-n+1}{(n-x)!}$ different equally probable vectors satisfying (7) and (8). This implies $C(A(1) = \binom{x+1}{n-x}$, $x \geq \lceil n/2 \rceil$. Otherwise, there will be not enough elements with value one. We conclude

$$P(A(1) = n|A(0) = x) = \begin{cases} \frac{\binom{x+1}{n-x}}{\binom{n}{x}}, & \text{if } x \geq \lceil n/2 \rceil, \\ 0, & \text{otherwise,} \end{cases} \tag{9}$$

and then we get

$$P(A(1) = n) = EP(A(1) = n|A(0)) \tag{10}$$

$$= \sum_{x=1}^{n} P(A(1) = n|A(0) = x)P(A(0) = x)$$

$$= \sum_{x=\lceil \frac{n}{2} \rceil}^{n} \binom{x+1}{n-x} p^x (1-p)^{n-x}.$$

This finishes the proof of Proposition 3.1. □

3.2 Proof of Proposition 3.2.

Note that $P(A(1) = n \mid A(0) = x)$ is a growing function of x, and it is zero at $x = \lceil \frac{n}{2} \rceil$, and one at $x = n$. Therefore we shall find a maximal number ω_n such that when $x > n - \omega_n$ we have $P(A(0) = n \mid A(1) = x) \to 1$ as n tends to infinity.

Set $x = n - \omega_n$. Assume, that $\omega_n/n \to 0$ as $n \to \infty$. From the Stirling's formula

$$n! = (\frac{n}{e})^n (2\pi n)^{\frac{1}{2}} e^{\alpha_n} \tag{11}$$

set

$$\beta_n = 2\alpha_x - \alpha_{2x-n} - \alpha_n.$$

Then by $P(A(1) = n|A(0) = k) = \binom{k+1}{n-k}/\binom{n}{k}$ we have

$$P(A(1) = n \mid A(0) = x) = \frac{(x+1)x!x!}{(2x-n+1)(2x-n)!n!}$$

$$= (x+1)(2x-n+1)^{-1} e^{\beta_n} (2 - nx^{-1})^{n-2x-1/2} (nx^{-1})^{-n-1/2} + o(1)$$
$$= (1 + 1 - nx^{-1})^{n-2x-1/2} (1 - 1 + nx^{-1})^{-n-1/2} + o(1)$$
$$= (1 + xx^{-1} - nx^{-1})^{n-2x-1/2} (1 - xx^{-1} + nx^{-1})^{-n-1/2} + o(1)$$
$$= (1 - \omega_n x^{-1})^{2\omega_n - n - 1/2} (1 + \omega_n x^{-1})^{-n-1/2} + o(1)$$
$$= (1 - \omega_n n^{-1})^{2\omega_n} (1 - (\omega_n n^{-1})^2)^{-n-1/2} + o(1)$$
$$= (1 - \omega_n n^{-1})^{2n\omega_n n^{-1}} (1 - ({\omega_n}^2 n^{-1}) n^{-1})^{-n-1/2} + o(1)$$
$$= e^{-2{\omega_n}^2 n^{-1} + {\omega_n}^2 n^{-1}} + o(1)$$
$$= e^{-{\omega_n}^2 n^{-1}} + o(1)$$

as $n \to \infty$. Now we see that if $\omega_n n^{-1/2} \to 0$ then the last expression converges to 1, and if $\omega_n n^{-1/2} \to \infty$, the last expression converges to 0. If $\omega_n n^{-1/2} \to c$, where c is a constant then the last expression converges to e^{-c^2} as $n \to \infty$, which is strictly between zero and one.
This proves Proposition 3.2. □

Example 3.1 *In the proof of proposition3.2 we get*

$$P(A(1) = n \mid A(0) = n - \omega_n) = e^{-\omega_n^2 n^{-1}} (1 + o(1)).$$

We let $\omega_n = n^{1-\beta}$, which yields

$$P(A(1) = n \mid A(0) = n - n^{1-\beta}) = e^{-n^{1-2\beta}} (1 + o(1)).$$

(See figure 1.)
We have two values $n = 10^3$ (red) and $n = 10^{12}$ (blue).
We obtain that the critical $\beta_c = 1/2$ seems reasonable already for these finite n.

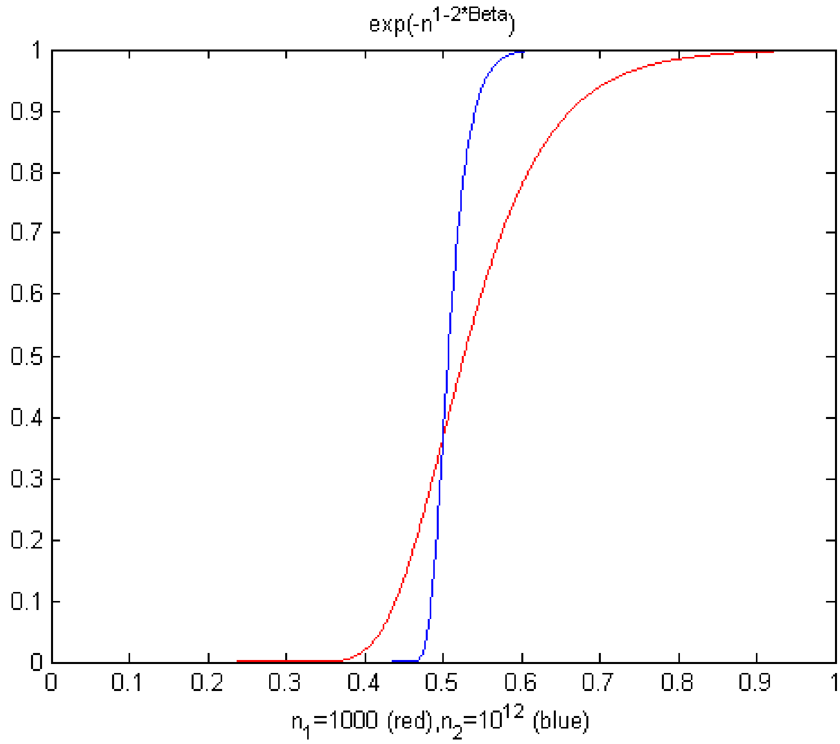

Figure 1: $y(\beta) = e^{-n^{1-2\beta}}(1 + o(1)), n = 10^3(red), n = 10^{12}(blue)$

Example 3.2 *We have*

$$P(A(1) = n) = \sum_{j=0}^{n} P(A(1) = n \mid A(0) = j)P(A(0) = j). \tag{12}$$

We use ideas of the proof of Theorem 3.1 Pehrsson [2].
Assume that

$$P(A(0) \in [n - n^{1-\alpha+\epsilon}, n - n^{1-\alpha-\epsilon}]) = 1 - o(1))$$

for all $\epsilon > 0$. *This leads us to the expectation value* $EA(0) = n - n^{1-\alpha}$ *in (12).*
We get

$$P(A(1) = n) = \sum_{j=n-n^{1-\alpha+\epsilon}}^{n-n^{1-\alpha-\epsilon}} P(A(1) = n \mid A(0) = j)P(A(0) = j) \tag{13}$$

$$= e^{-n^{1-2\alpha}}(1 + o(1)) = e^{-n(1-p)^2}(1 + o(1)).$$

This is the same as the conditional case with β *replaced by* α.

4 An extended process

Here we extend the dynamics of activation beyond time $\tau = 1$, considering a generalization of the above model of Bootstrap Percolation on Z_n. The new process has parameters: m, s, p, where m is the number of steps, s is the length of intervals which can percolate, and p is the probability of an element to be one, or to be "active" (we call it an external activation).

Let us introduce notation I_k for the inactie intervals (i.e., consecutive vertices with value 0) defined as follows. Let $x(\tau) = (x_i(\tau))_{i=1}^n$ be any process defined at time τ with $x_i(\tau) = 0$ or 1 independent i.
Given a set

$$\mathcal{A}(\tau)$$

of active vertices at time τ, we define

$$A(\tau) = \mid \mathcal{A}(\tau) \mid .$$

Consider the set of vertices

$$\{v_0\} \bigcup \mathcal{A}(\tau) \bigcup \{v_{n+1}\}$$

$$= \{v_{j_0}, v_{j_1}, .., v_{j_r}, ..v_{j_{A(\tau)+1}}\},$$

where $0 \leq j_r \leq n+1$
and $j_u > j_v$ whenever $u > v$ and $0 \leq r, v, u \leq A(\tau) + 1$.
Define $I'_r = [0, j_r]$ the interval (i.e., vertices i with value $x_i = 0, 1$) between 0 and j_r. We now define the intervals at time τ as

$$I_r = (I'_{r+1} \setminus I'_r) \setminus \{j_{r+1}\}$$

where $0 \leq r \leq A(\tau)$.
We may let $k = r + 1$ and define I_k, $1 \leq k \leq A(\tau) + 1$.

Then given a set $\mathcal{A}(\tau)$ we define for $\tau = 2h - 2$ whenever $1 \leq h \leq m$:

$$P(x_i(\tau + 1) = 1 \mid x_i(\tau) = 0) = p, \tag{14}$$

$$P(x_i(\tau + 1) = 1 \mid x_i(\tau) = 1) = 1, \tag{15}$$

independent for different i, and for $\tau = 2h - 1$

$$x_i(\tau + 1) = \begin{cases} 1, & \text{if } i \in I_k : \mid I_k \mid \leq s, \\ x_i(\tau), & \text{otherwise,} \end{cases} \tag{16}$$

where $I_k = I_k(\tau)$ as defined above, and $1 \leq i \leq n$. The process starts with the first step at $\tau = 1$ and stops after the m-th step at $\tau = 2m$.

5 Results

We are interested in the probability that at the last step, we have complete percolation. Here we classify the Bootstrap percolation in one dimension on a strap Z_n with respect to the initial conditions and the last step. More precisely we describe conditions for a complete percolation. In one dimensional case it is sufficient to study the process at all steps. We will also show that the percolation steps except the last percolation step can be omitted. We give a rough sketch about the structure of the process.

Observe that we can separate the process in two parts: percolation and randomizing. This leads to the following definition.

Definition 5.1 *(External activation or updating)*
Let $x = (x_i)_{i=1}^n$ be a strap with $x_i \in \{0, 1\}$.
Define a random vector

$$R_p x = R_p (x_i)_{i=1}^n = (R_{p,i} x_i)_{i=1}^n \tag{17}$$

for $1 \leq i \leq n$ by

$$P(R_{p,i} x_i = 1 | x_i = 1) = 1, \tag{18}$$

$$P(R_{p,i} x_i = 1 | x_i = 0) = p, \tag{19}$$

independent for different i.

Definition 5.2 *(intervals of a strap)*
Let $x = (x_i)_{i=1}^n$,where $x_i = 0$ or $x_i = 1$
Give a set

$$\mathcal{A}_x$$

of active vertices in x define

$$A_x = |\mathcal{A}_x|$$

We define $I_r \subseteq V$:
Let us consider the set of vertices

$$\{x_{j_0} = x_0\} \bigcup \mathcal{A}_x \bigcup \{x_{j_{A_x+1}} = x_{n+1}\}$$

$$= \{v_{j_0} = x_0, v_{j_1}, .., v_{j_r}, ..v_{j_{A_x+1}} = x_{n+1}\}$$

where $0 \leq j_r \leq n + 1$ and $j_u > j_v$ whenever $u > v$ and $0 \leq u, v, r \leq A_x + 1$
Define $I'_r = [0, j_r]$ the interval between 0 and j_r of the whole strap. We now define the intervals as

$$I_r = (I'_{r+1} \setminus I'_r) \setminus \{j_{r+1}\}$$

where $0 \leq r \leq A_x$.
We may let $k = r + 1$ and define I_k ,$1 \leq k \leq A_x + 1$.

Definition 5.3 *(Percolator)*
Let $x = (x_i)_{i=1}^n$ be a vector with $x_i \in \{0, 1\}$.
Then define an operator on x

$$Q_s x = Q_s(x_i)_{i=1}^n = (Q_{s,i}x_i)_{i=1}^n, \tag{20}$$

where

$$Q_{s,i}x_i = \begin{cases} 1, \text{if } i \in I_j \text{ such that } |I_j| \leq s \\ x_i, \text{otherwise} \end{cases} \tag{21}$$

for $1 \leq i \leq n$.

Definition 5.4 *We define*

$$[\prod_{j=1}^{m}(Q_s^{k_j}\prod_{i=0}^{l_j}R_p^{(i)})]x := [\prod_{j=1}^{m}(Q_s R_p^{l_j})]x,$$

where $R_p^{(0)} = 1$ and $R_p^{(i)}, i > 0$ are independent and defined from definition 5.1 Q_s percolators and $0 \leq k_j, l_j < \infty$

Let us derive now a useful representation of the process $x(\tau)$ defined above using operators R_p and Q_s.

Proposition 5.1 *Let $\bar{0} = (0)_{i=1}^n$*
We have the following equalities in distribution

$$x(2m) = (Q_s R_p)^m \bar{0}, \tag{22}$$

where

$$x(0) = \bar{0}, \tag{23}$$

$$x(1) = R_p\bar{0}, \tag{24}$$

$$x(2) = Q_s R_p \bar{0}, \tag{25}$$

and

$$x(2h-1) = R_p x(2h-2), \tag{26}$$

$$x(2h) = Q_s x(2h-1), \tag{27}$$

and at time $\tau = 2m$

$$x(2m) = Q_s R_p x(2m-2) \tag{28}$$

for all $1 \leq h \leq m$.

The follwoing statement tells us that the distribution of our process is same as the distribution of its following simplified version.

Proposition 5.2 *Let $x = (x_i)_{i=1}^n$ and $x_i \in \{0, 1\}$ for all i, $1 \leq i \leq n$ Then*

$$Q_s R_p Q_s x \stackrel{d}{=} Q_s R_p x \tag{29}$$

Here we get the information about how to rationalize the process.

Corollary 5.1 *We have the following equality in distribution:*

$$Q_s R_p Q_s R_p x \stackrel{d}{=} Q_s R_p R_p x.$$

This statement can be generalized, which explanes further simplifications.

Proposition 5.3 *We have the following equality in distribution:*

$$(Q_s R_p)^m x \stackrel{d}{=} Q_s R_p^m x. \tag{30}$$

This separates The Operators and collects the Randomizing parts on the right.

We shall use the following definition and results on generating functions.

Consider $\mid I_j \mid$ as unknown variables, then we have the equation system

$$\sum_{j=1}^{m} \mid I_j \mid = r. \tag{31}$$

We define

$$g(x) = \sum_{r=0}^{\infty} a_r x^r$$

where a_r is the number of solutions of (31) to be a generating function of (31). We define

$$g_j(x) = \sum_{i \in M_j \subseteq N} x^i,$$

where $N = \{0, 1, 2, 3......\}$ as a generating factor. We regard the exponents of $g_j(x)$ as values of the variables $\mid I_j \mid$ in (31), which build up the set M_j. We shall use the following result on generating functions for boxes.

Theorem 5.1 *Function*

$$g(x) = \prod_{j=1}^{m} g_j(x)$$

is a generating function for all factors $g_j(x)$.

Lemma 5.1 *For all $\omega_n < n$*

$$P(A(1) = n \mid A(0) = n - \omega_n)$$

$$= \sum_{j=0}^{\omega_n/(s+1)} (-1)^j \binom{n - \omega_n + 1}{j} \frac{\binom{n-j(s+1)}{\omega_n - j(s+1)}}{\binom{n}{\omega_n}}. \tag{32}$$

Lemma 5.2 *If $\omega_n = o(n)$ we have*

$$\frac{\binom{n-\omega_n+1}{j}\binom{n-j(s+1)}{\omega_n-j(s+1)}}{\binom{n}{\omega_n}} = \frac{a_j}{j!},$$

Then

$$w_j \le a_j^{\frac{1}{j}} = z \le n(\frac{\omega_n}{n-\omega_n})^{s+1} = z_2, \tag{33}$$

where

$$w_j = \begin{cases} z_1 & \text{if } 0 \le j \le \frac{\omega_n^\gamma}{s+1} \\ z_{1,j} & \text{if } \frac{\omega_n^\gamma}{s+1} < j \le \frac{\omega_n}{s+1} \\ 0 & \text{otherwise} \end{cases} \tag{34}$$

with

$$z_1 = (n-2\omega_n)[\frac{\omega_n - \omega_n^\gamma}{n}]^{s+1}$$

and

$$z_{1,j} = (n-2\omega_n)[\frac{\omega_n - j(s+1)}{n}]^{s+1}$$

for $0 < \gamma < 1$ and n large.

Proposition 5.4 *Let*

$$y(0) = R_p\bar{0}$$

and

$$y(1) = Q_s y(0).$$

Let $A(t)$ be the number of ones in vector $y(t)$, $t = 0, 1$. Then

$$P(A(1) = n \mid A(0) = n - \omega_n) = e^{-n(\frac{\omega_n}{n})^{s+1}} + o(1).$$

The following Theorem is the main result here.

Theorem 5.2 *Let $p = 1 - n^{-\alpha}$. Then for any $m \ge 1$*
if $\alpha > \frac{1}{(s+1)m}$

$$P(A(2m) = n) \to 1 \quad \text{as} \quad n \to \infty,$$

while if $\alpha < \frac{1}{(s+1)m}$

$$P(A(2m) = n) \to 0 \quad \text{as} \quad n \to \infty.$$

This tells us that our process have very similar behavior like the one step process with the same initial conditions. Based on these results one can determine conditions for complete percolation for an m-step process. Corollary 5.1 and Proposition 5.2 tells us that percolation process in the first step can be omitted. Then Proposition 5.1 describes our given process in terms of operators. The Proposition 5.3 is a generalization of Proposition 5.2 in several steps. Theorem 5.1 contains a short description of the method of generating function which is used in the first Lemma. Lemma 5.1, Lemma 5.2 and Proposition 5.4 give us the corresponding expression

$$P(A(1) = n \mid A(0) = n - \omega_n) = e^{-n(\frac{\omega_n}{n})^{s+1}}$$

which generalizes the results in [2].

6 Discussion

Define as in Bollóbas *et. al.* [1]

$$P_{p_n}(T < t) \tag{35}$$

to be the distribution of the time $T = T(T_n^d)$ of compleate percolation on a torus under the d-neighbor bootstrap percolation model on Z^d given probability sequence p_n which defines the elements of the torus. We will compare this probability with the in one dimension with $P(A(1) = n)$. Define also as in Bollóbas *et. al.* [1]

$$p_\alpha(t) = inf\{p : P_p(T \leq t) \geq \alpha, \}, \tag{36}$$

where $0 < \alpha < 1$ is a constant. We want to compare Theorem 5.2 with a result from Bollóbas *et. al.* [1], which we state here:

Theorem 6.1 *Let $d \geq 2$,let $t = o(\log n / \log\log n)$, let $(p_n)_{n=1}^{\infty}$ be a sequence of probabilities, let $\omega(n) \to \infty$, and let $T = T(T_n^d)$. Under the standard d-neighbor model,*

(i) if, for all n, $q_n \leq (n^{-d}/\omega(n))^{1/m_t}$, then

$$P_{p_n}(T \leq t) \to 1$$

as $n \to \infty$;

(ii) if, for all n, $q_n \geq (n^{-d}\omega(n))^{1/m_t}$, then

$$P_{p_n}(T \leq t) \to 0$$

as $n \to \infty$,

moreover, for any $\alpha \in (0, 1)$,

$$p_\alpha(t) = 1 - (1 + o(1)) \left(\frac{\log(\frac{1}{\alpha})}{d^3 2^{d-1} n^d} \right)^{\frac{1}{m_{t,d}}}.$$

Our conjecture is that the theorem holds for the case $d = 1$. We project the torus on a ring Z_n and adjust the parameters. We set $t = 1$ because it defines our stopping time. Let (see [1] p3.-p4)

$$m_{t,d} = ex(t,d) = min_{A \subset Z^d}\{\mid Z^d \setminus A \mid : 0 \notin A_t\}. \tag{37}$$

We use the result [1] that $m_t = m_{t,d} = ex(t,d)$, and with Z^d replaced by Z_n we have by (37)

$$m_1 = ex(1,1) = \min_{A \in Z_n}\{|Z_n \setminus A| : 0 \notin A_1\}. \tag{38}$$

This gives us $m_1 = 2$. We get then from Theorem 6.1 (i):

$$1 - p_n = q_n \le (n^{-1}/\omega(n))^{\frac{1}{2}} = n^{-\frac{1}{2}} e^{-\frac{1}{2} log(\omega(n))} = n^{-\frac{1}{2}} n^{-\phi(n)},$$

where $\phi(n) > c/\log(n)$ for any positive real constant c.

If $q_n = n^{-2\gamma}$ Theorem 6.1 (i) implies that $\gamma > 1/4$ and $P_{p_n}(T \le 1) \to 1$ as $n \to \infty$.

On the other hand, by Theorem 6.1 (ii) we have

$$q_n \ge (n^{-1}\omega(n))^{\frac{1}{2}} = n^{-\frac{1}{2}} e^{\frac{1}{2} log(\omega(n))} = n^{-1/2} n^{\phi(n)}.$$

Hence, if $q_n = n^{-2\gamma}$, we should have $\gamma < 1/4$, and in this case

$$P_{p_n}(T \le 1) \to 0$$

as $n \to \infty$.

We further estimate $p_\alpha(1)$ from

$$P_{p_n}(T \le 1) = P(A(1) = n) = e^{-(1-p)^2 n}(1 + o(1)), \tag{39}$$

which is derived from example 3.2. From (36) we get

$$p_\alpha(1) = \inf\{p : P(A(1) = n) \ge \alpha\}, \tag{40}$$

where $0 < \alpha < 0$ is a constant. We define $\chi(p) = P(A(1) = n)$ which by (39) is a monotone increasing function of p. We have $\chi(0) = e^{-n} + o(1)$ and $\chi(1) = 1 + o(1)$. Then $0 < \alpha < 1$ cut the function somewhere between $0 < p < 1$ and this must be the lowest limit which preserves the condition $P(A(1) = n) \ge \alpha$ in (40).

So we do only study the case when

$$P(A(1) = n) = \alpha. \tag{41}$$

We solve the equation $P(A(1) = n) = \alpha$ by using

$$P(A(1) = n) = e^{-n(1-p)^2}(1 + o(1))$$

from (39), which yields

$$e^{-n(1-p)^2}(1+o(1)) = \alpha.$$

Hence,

$$-n(1-p)^2 = log(\alpha(1+o(1))^{-1})$$

$$\Leftrightarrow\ (1-p) = [n^{-1}[\log(\frac{1}{\alpha}) + o(1)]]^{\frac{1}{2}}$$

$$\Leftrightarrow\ (1-p) = [n^{-1}\log(\frac{1}{\alpha})[1+\frac{o(1)}{\log(\frac{1}{\alpha})}]]^{\frac{1}{2}},$$

and therefore

$$(1-p) = [1+o(1)]^{\frac{1}{2}}[n^{-1}\log(\frac{1}{\alpha})]^{\frac{1}{2}}.$$

Since $(1+o(1))^{1/2} = 1+o(1)$, we derive

$$p = 1-(1+o(1))[n^{-1}\log(\frac{1}{\alpha})]^{\frac{1}{2}}$$

Then from (40) we get

$$p_\alpha(1) = 1-(1+o(1))(\frac{log(\frac{1}{\alpha})}{n})^{\frac{1}{2}}. \tag{42}$$

Theorem 6.1 agrees with Theorem 5.2 in one dimension in the special case when $m = 1$ as well as the Theorem 3.1.

7 Proofs

7.1 Properties of straps

We suppose that the operators act on strapsm, and we derive some obvious properties.

Let

$$x = (x_i)_{i=1}^n \tag{43}$$

be a strap, let $j \in Z^+$ and

$$w_j = (x_i)_{i \in T_j}, \tag{44}$$

where $T_j \subseteq \{1, 2, 3, \ldots, n\} = N$. Then $w_1 \subseteq w_2$ if and only if $T_1 \subseteq T_2$ and

$$w_2 \setminus w_1 = (x_i)_{i \in T_2 \setminus T_1}, \tag{45}$$

$$w_1 \bigcup w_2 = (x_i)_{i \in T_1 \bigcup T_2}, \tag{46}$$

$$w_1 \bigcap w_2 = (x_i)_{i \in T_1 \bigcap T_2}. \tag{47}$$

7.2 Proof of Proposition 5.1

We study the process at time $\tau = 2h-1$. We define $x(2h-1) = R_p x(2h-2)$ and then

$$P(x_i(2h-1) = 1 \mid x_i(2h-2) = 1) = 1, \tag{48}$$

$$P(x_i(2h-1) = 1 \mid x_i(2h-2) = 0) = p. \tag{49}$$

We define $x(2h) = Q_s x(2h-1)$, and then

$$x_i(2h) = \begin{cases} 1, i \in \{I_j : 0 < |I_j| \leq s\} \\ x_i(2h-1), \text{otherwise} \end{cases} \tag{50}$$

where $I_j = I_j(2h-1)$ and $1 \leq j \leq A(2h-1)$. We start at time $\tau = 0$ and stop at $\tau = 2m$. This completes the proof. □

7.3 The Union of Intervals

We define following sets and vectors.

$$i_x = Index(x) = \{i : x_i \in x\}. \tag{51}$$

In particular, for a vector $x = (x_i)_{i \in S \subseteq N}$, where $N = \{1, 2, 3, .., n\}$, we have

$$i_x = S.$$

Write

$$Ones(x) = (x_i = 1)_{i \in i_x}, \tag{52}$$

$$Zeros(x) = (x_i = 0)_{i \in i_x}. \tag{53}$$

Let $x = (x_i)_{i \in S}$. Then $Ones(x)$ is the subvector of x containing only elements of $x_i = 1$, and $Zeros(x)$ is the subvector of x containing only elements of $x_i = 0$.

Let $x = (x_i)_{i=1}^n$, where $x_i = 0$ or $x_i = 1$, and let I_k be the intervals given by Definition 5.2. Then we set $I_{s,k} = I_k$ if $0 < |I_k| \leq s$. Now define

$$V = V_{s,x} = \bigcup_{k=1}^{A_x+1} I_{s,k}$$

and

$$U_{s,x} = x_V, \tag{54}$$

which will be our fundamental tool in the proof of the Proposition 5.2.

7.4 Proof of Proposition 5.2 and Corollary 5.1

Now we must regard R_p acting identical in the both sequences in (29) in the beginning. Then we let them be independent and they will be equal in distribution, since all R_p have the same distribution from the Definition 5.1.

We start to compare the two sequences in this way.

We start the sequence on the right side of (29) from

$$x = U_{s,x} \bigcup (x \setminus U_{s,x}), \tag{55}$$

and the left side of (29) from

$$Q_s x = Q_s U_{s,x} \bigcup (x \setminus U_{s,x}). \tag{56}$$

The right side of (29) from the first step

$$R_p x = R_p U_{s,x} \bigcup R_p (x \setminus U_{s,x}). \tag{57}$$

The left side of (29) from the first step

$$R_p Q_s x = R_p Q_s U_{s,x} \bigcup R_p (x \setminus U_{s,x}). \tag{58}$$

The right side of (29) from the second step

$$Q_s R_p x = Q_s R_p U_{s,x} \bigcup Q_s R_p (x \setminus U_{s,x}). \tag{59}$$

The left side of (29) from the second step

$$Q_s R_p Q_s x = Q_s R_p Q_s U_{s,x} \bigcup Q_s R_p (x \setminus U_{s,x}). \tag{60}$$

From the definition of Q_s it follows that it only can change the value of the elements $x'_i = Q_s x_i$. That means that $Index(Q_s w) = Index(w)$. The same property has R_p. $Index(R_p w) = Index(w)$. They are both defined elementwise.
Then

$$Index(Q_s U_{s,x}) = Index(U_{s,x}) = U \tag{61}$$

since $U_{s,x}$ is a subset of x.

By the definition it follows that

$$x \setminus Q_s U_{s,x} \neq x \setminus U_{s,x} = (x_i)_{i \in N \setminus U}. \tag{62}$$

If and only if they have the same value of all elements in the two sets we will regard such two sets equal.
It remains to show that

$$Q_s R_p Q_s U_{s,x} = Q_s R_p U_{s,x} = Q_s U_{s,x}. \tag{63}$$

Since $Q_s U_{s,x}$ only consists of elements $x_i = 1$, it follows from the definition of R_p and Q_s that $Q_s R_p Q_s U_{s,x} = Q_s U_{s,x}$. We claim that

$$U_{s,R_p U_{s,x}} \bigcup Ones(R_p U_{s,x}) = R_p U_{s,x} \tag{64}$$

and

$$U_{s,R_p U_{s,x}} \bigcap Ones(R_p U_{s,x}) = \emptyset \tag{65}$$

since

$$U_{s,R_p U_{s,x}} = Zeros(R_p U_{s,x}). \tag{66}$$

R_p change the value of elements of zero in $U_{s,x}$ to elements of ones, we will still have a union of intervals of length, smaller or equals to s.
This property implies (63).
On the other hand

$$Q_s R_p U_{s,x} = Q_s(U_{s,R_p U_{s,x}} \bigcup Ones(R_p U_{s,x})) \tag{67}$$

$$= Q_s U_{s,R_p U_{s,x}} \bigcup Q_s Ones(R_p U_{s,x}).$$

Then again by their definition R_p and Q_s, are "onto" operators, defined element wise, but Q_s depends on the intervals I_j. This means that it depends on elements outside a given element it acts on. If we let Q_s act on sets of elements of entire intervals, then it acts equal on all such a set. Both sets in the union of (67) consist only of elements $x_i = 1$,since all $\mid I_j \mid \leq s$. The union has the same indexset $Index(Q_s R_p U_{s,x})$, which equals $Index(U_{s,x})$, since the two operators are "onto". This implies

$$Q_s U_{s,R_p U_{s,x}} \bigcup Q_s Ones(R_p U_{s,x}) = Q_s U_{s,x}. \tag{68}$$

Then the statement of the proposition follows immediately. Now the statement is true for identical R_p acting equal in both sequences of (29) in distribution. Let R_p and R'_p be two not identical operators satisfying Definition 5.1, which implies

$$w \stackrel{d}{=} Q_s R_p Q_s x \stackrel{d}{=} Q_s R'_p Q_s x \stackrel{d}{=} Q_s R_p x. \tag{69}$$

This completes the proof. □

7.5 Proof of Corollary 5.1

Put $x = R_p y$ in the Proposition 5.2 and the corollary follows immediately. This completes the proof. □

7.6 Proof of Proposition 5.3

We prove it by induction over m, for $m = 2$ we can apply Corollary 5.1 so

$$y \stackrel{d}{=} Q_s R_p (Q_s R_p)^m x \tag{70}$$

and by the induction assumption

$$y \stackrel{d}{=} Q_s R_p Q_s R_p^m x. \tag{71}$$

This can be written by Corollary 5.2

$$y \stackrel{d}{=} Q_s R_p R_p^m x \tag{72}$$

$$y \stackrel{d}{=} Q_s R_p^{m+1} x \tag{73}$$

and the proposition follows by induction.
This completes the proof. □

7.7 Proof of theorem 5.1

We prove it by induction.
If

$$| I_1 |= r,$$

then by the definition

$$g(x) = g_1(x) = \sum_{i \in M_1 \subseteq N} x^i$$

$a_r = 1$ if and only if $| I_j |= r$ is the only solution. So the statement holds for $m = 1$.

We assume it holds for $m \geq 1$, and derive that it holds for $m+1$. Consider

$$\sum_{j=1}^{m+1} \mid I_j \mid = r \tag{74}$$

We rewrite it as

$$\sum_{j=1}^{m} \mid I_j \mid = r - \mid I_{m+1} \mid .$$

By the induction assumption there exist a generating function $g'(x)$ such that

$$coefficient(g'(x), x^{r-|I_{m+1}|}) = a_{r||I_{m+1}|}.$$

The right side is the number of solutions of (74) given $\mid I_{m+1} \mid$. It can be rewritten as

$$coefficient(g'(x)x^{|I_{m+1}|}, x^r) = a_{r||I_{m+1}|}.$$

Now denote $\mid I_{m+1} \mid = i$ and make following summation and use the definition on the right side.

$$\sum_{i \in M_{m+1}} coefficient(g'(x)x^i, x^r)$$

$$= coefficient(g'(x) \sum_{i \in M_{m+1}} x^i, x^r)$$

$$= coefficient(g'(x)g_{m+1}(x), x^r) = \sum_{i \in M_{m+1}} a_{r|i}$$

$$= a_{r|i \in M_{m+1}}$$

since no solutions are equal for different choice of i. Then $g(x) = g'(x)g_{m+1}(x)$ is a generating function and $g_{m+1}(x)$ is a generating factor for any choice of M_{m+1}. Then the theorem follows by induction.
This completes the proof. □

7.8 Proof of the Lemma 5.1

We consider the intervals:

$$\sum_{j=0}^{A(0)+1} \mid I_j \mid = n - A(0), \tag{75}$$

where

$$\mid I_j \mid \leq s, \tag{76}$$

and

$$A(0) \sim Bin(n, p). \tag{77}$$

First we assume that $A(0) = y = n - \omega_n$, i.e., non-random. Every solution (vector) of (75) has equal probability $p^y(1-p)^{n-y}$. We have totally $\binom{n}{y}$ solutions with this probability. We will define $P(A(1) = n \mid A(0) = n - \omega_n)$ as the quotient between the number of solutions under (76) and the total number of solutions. To estimate the solutions under (76) we use the method of generating function (see [3], p80-p86). From (76) and (75) we obtain the generating function

$$f(z) = (1 + z + ..z^s)^{n-\omega_n+1} = [(1 - z^{s+1})(1-z)^{-1}]^{n-\omega_n+1}$$

$$= \sum_{j=0}^{n-\omega_n+1} \binom{n-\omega_n+1}{j} (-1)^j z^{(s+1)j} \sum_{r=0}^{\infty} \binom{r+n-\omega_n}{r} z^r.$$

The coefficient of z^{ω_n} yields

$$g(z) = \sum_{r+(s+1)j=\omega_n} \binom{n-\omega_n+1}{j} (-1)^j z^{(s+1)j} \binom{r+n-\omega_n}{r} z^r$$

$$= \sum_{r+(s+1)j=\omega_n} \binom{n-\omega_n+1}{j} (-1)^j z^{(s+1)j} \binom{r+n-\omega_n}{r} z^r,$$

which implies

$$= \sum_{j=0}^{\frac{\omega_n}{s+1}} \binom{n-\omega_n+1}{j} (-1)^j z^{(s+1)j} \binom{n-j(s+1)}{\omega_n - j(s+1)} z^{\omega_n - j(s+1)}$$

$$= \sum_{j=0}^{\frac{\omega_n}{s+1}} \binom{n-\omega_n+1}{j} (-1)^j \binom{n-j(s+1)}{\omega_n - j(s+1)} z^{\omega_n}.$$

This gives us

$$P(A(1) = n \mid A(0 = n - \omega_n) = g(1) / \binom{n}{\omega_n}$$

$$= \sum_{j=0}^{\frac{\omega_n}{s+1}} \binom{n-\omega_n+1}{j} (-1)^j \frac{\binom{n-j(s+1)}{\omega_n-j(s+1)}}{\binom{n}{\omega_n}}.$$

This completes the proof □

7.9 Proof of Lemma 5.2

We have by Lemma 5.1

$$\frac{a_j}{j!} = \frac{\binom{n-\omega_n+1}{j}\binom{n-j(s+1)}{\omega_n-j(s+1)}}{\binom{n}{\omega_n}}$$

$$= \frac{(n-\omega_n+1)!(n-j(s+1))!\omega_n!}{j!(n-\omega_n-j+1)!n!(\omega_n-j(s+1))!}$$

$$= \frac{1}{j!}\prod_{i=0}^{j-1}(n-\omega_n-i+1)\prod_{i=0}^{j(s+1)-1}\frac{\omega_n-i}{n-i}.$$

Then

$$z_1 = (n-2\omega_n)[\frac{\omega_n-\omega_n^\gamma}{n}]^{s+1},$$

where $0 \leq j \leq \frac{\omega_n^\gamma}{s+1}$ and large n.

$$z_{1,j} = (n-2\omega_n)[\frac{\omega_n-j(s+1)}{n}]^{s+1},$$

where $\frac{\omega_n^\gamma}{s+1} \leq j \leq \frac{\omega_n}{s+1}$ and

$$z_2 = n[\frac{\omega_n}{n-\omega_n}]^{s+1},$$

where $0 \leq j \leq \frac{\omega_n}{s+1}$ and $0 < \gamma < 1$. We easily see that that (33) and (34) are satisfied.

This completes the proof. □

7.10 Proof of Proposition 5.4

We conclude from Lemma 5.2 that

$$z_1 = (n-2\omega_n)[\frac{\omega_n-\omega_n^\gamma}{n}]^{s+1}$$

for $0 \leq j \leq \frac{\omega_n^\gamma}{s+1}$, and

$$z_{1,j} = (n-2\omega_n)[\frac{\omega_n-j(s+1)}{n}]^{s+1}$$

for $\frac{\omega_n^\gamma}{s+1} \leq j \leq \frac{\omega_n}{s+1}$ is the lowest bound of $a_j^{\frac{1}{j}}$, while

$$z_2 = n[\frac{\omega_n}{n-\omega_n}]^{s+1}$$

is the highest bound of $a_j^{\frac{1}{j}}$.

We shall use the Taylor-Maclaurins theorem, which states that

$$\sum_{j=0}^{n}\frac{f^{(j)}(0)x^j}{j!} + \frac{f^{(n+1)}(\theta x)x^{n+1}}{(n+1)!} = f(x), \tag{78}$$

where $0 < \theta < 1$.

We study first the upper bound. By Lemma 5.2 and Maclaurins theorem (78)

$$P(A(1)=n \mid A(0)=n-\omega_n) \geq \sum_{j=0}^{m_n=\frac{\omega_n}{s+1}} \frac{(-1)^j z_2^j}{j!} + R_{m_n}(z_2).$$

We claim that the all rest terms are $o(1)$. Therefore

$$P(A(1)=n \mid A(0)=n-\omega_n) \geq e^{-z_2} + o(1)$$

Then similar for the lower bound.

$$P(A(1)=n \mid A(0)=n-\omega_n) \tag{79}$$

$$\leq \sum_{j=0}^{m'_n=\frac{\omega_n^\gamma}{s+1}} (-1)^j \frac{z_1^j}{j!} + \sum_{j=m'_n+1}^{m_n=\frac{\omega_n}{s+1}} (-1)^j \frac{z_{1,j}^j}{j!}.$$

Consider the right term in right side of (79)

$$\left| \sum_{j=m'_n+1}^{m_n} \frac{(-1)^j z_{1,j}^j}{j!} \right| \leq \sum_{j=m'_n+1}^{m_n} \frac{z_2^j}{j!} = \tilde{R}_{m'_n}(z_2) - \tilde{R}_{m_n}(z_2).$$

We claim that all the restt erms vanish under a given condition. Our conclusion is

$$P(A(1)=n \mid A(0)=n-\omega_n) \leq e^{-z_1} + o(1).$$

We are left to show that all the restterms vanish when n goes to infinity. All the restterms is of the form $R_m(z) = e^{\theta z} z^m/m!$, where $0 < \theta < 1$ and z positive or negative. It is quite easy to see that if the quotient $z/m \to 0$ when $m \to \infty$ then the rest term vanishes. We use that $m_n \to \infty$ as $n \to \infty$. It is enough to study $n(\omega_n/n)^{s+1}/m'_n$ in the case $0 < \gamma \leq 1$. We substitute $\omega_n = n^{1-\beta}$ and $m'_n = \omega_n^\gamma$. We study the exponents and when they are negative. We get

$$1 - \beta(s+1) - (1-\beta)\gamma < 0$$

which give us

$$\frac{1-\gamma}{s+1-\gamma} < \beta$$

so we can choose γ arbitrary near one and one and all of our restterms will vanish for $\beta > 0$. Since

$$z_1 = (n - 2\omega_n)\left(\frac{\omega_n - \omega_n^\gamma}{n}\right)^{s+1}$$

$$= n\left(\frac{\omega_n}{n}\right)^{s+1}(1 - 2\omega_n n^{-1})(1 - \omega_n^{\gamma-1})^{s+1}$$

$$= n(\frac{\omega_n}{n})^{s+1} - o(n(\frac{\omega_n}{n})^{s+1})$$

and

$$z_2 = n(\frac{\omega_n}{n-\omega_n})^{s+1} = n(\frac{\omega_n}{n})^{s+1}(1-\frac{\omega_n}{n})^{-(s+1)}$$
$$= n(\frac{\omega_n}{n})^{s+1} + o(n(\frac{\omega_n}{n})^{s+1}),$$

the Proposition 5.4 follows. □

Example 7.1 *As in Example 3.1 from Proposition 5.4 , we get from*

$$P(A(1) = n \mid A(0) = n - \omega_n) = e^{-n(\omega_n/n)^{s+1}}(1 + o(1))$$

$$P(A(1) = n \mid A(0) = n - n^{1-\beta}) = e^{-n^{1-(s+1)\beta}}(1 + o(1)) \tag{80}$$

shown in figure 2.
We can easily see the critical $\beta_c = 1/(s+1)$ for $s = 1, 3, 7, 15$.

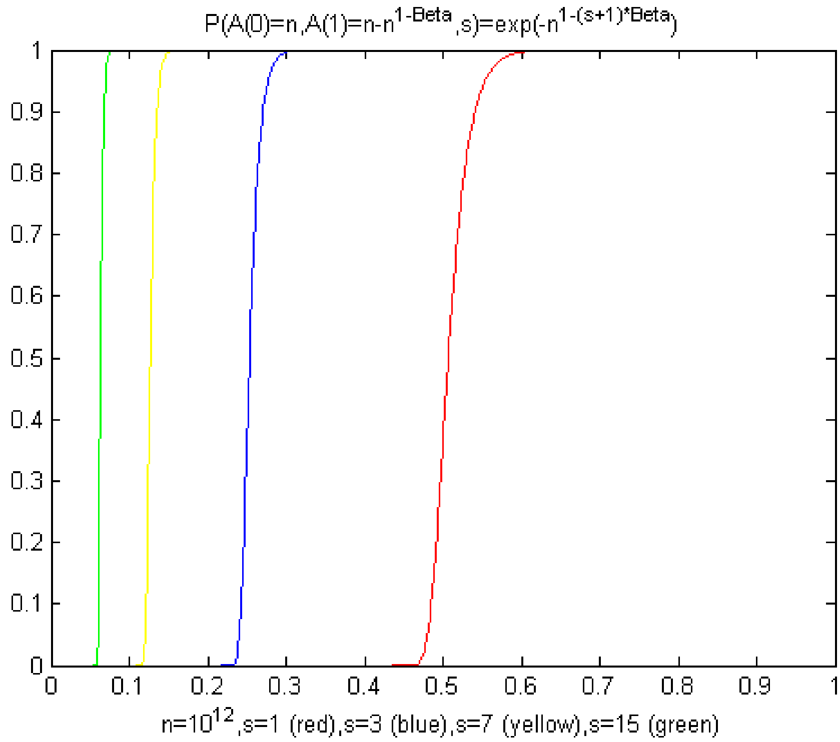

Figure 2: $y(\beta) = e^{-n^{1-(s+1)\beta}}(1 + o(1)), s = 1(red), s = 3(blue), s = 7(yellow), s = 15(green)$

7.11 Proof of Theorem 5.2

Let $p = 1 - n^{-\alpha}$. Proposition 5.3 implies

$$x(2m) = Q_s R_p^m \bar{0} \tag{81}$$

for $m \geq 1$, and

$$x(2m-1) = R_p^m \bar{0}. \tag{82}$$

Consider

$$P(x_i(2m-1) = 0) = P({R_{p,i}}^m 0 = 0) \tag{83}$$

$$= P({R_{p,i}}^0 0 = 0) \prod_{k=1}^{m} P({R_{p,i}}^k 0 = 0 | {R_{p,i}}^{k-1} 0 = 0)$$

$$= (1-p)^m.$$

Hence,

$$P(x_i(2m-1) = 1) = 1 - (1-p)^m. \tag{84}$$

Then we reduce the process to a one step process

$$y(1) = Q_1 R_{1-(1-p)^m} \bar{0} = x(2m) \tag{85}$$

$$y(0) = R_{1-(1-p)^m} \bar{0} = x(2m-1). \tag{86}$$

Let $p = 1 - n^{-\alpha}$ and regard a one step process $y(\tau)$ for $m = 1$. We have following model for percolated combinations

$$\sum_{j=1}^{A(0)+1} | I_j |= n - A(0) \tag{87}$$

$$| I_j | \leq s \tag{88}$$

$$A(0) \sim Bin(n, p) \tag{89}$$

By Lemma 5.1, we get

$$f(\omega_n) = \sum_{j=0}^{\frac{\omega_n}{s+1}} (-1)^j \binom{n - \omega_n + 1}{j} \frac{\binom{n-j(s+1}{\omega_n - j(s+1)}}{\binom{n}{n-\omega_n}} \tag{90}$$

where $f(\omega_n) = P(A(1) = n \mid A(0) = n - \omega_n)$. By Lemma 5.2, we get

$$f(\omega_n) = \sum_{j=0}^{\omega_n/(s+1)} (-1)^j \frac{a_j}{j!}$$

with $z_{1,j} \leq a_j^{1/j} \leq z_2$, where

$$z_1 = (n - 2\omega_n)(\frac{\omega_n - \omega_n^\gamma}{n})^{s+1}$$

for $0 \le j \le \omega_n^\gamma/(s+1)$ and

$$z_{1,j} = (n - 2\omega_n)(\frac{\omega_n - j(s+1)}{n})^{s+1}$$

for $\frac{\omega_n^\gamma}{s+1} \le j \le \frac{\omega_n}{s+1}$ and $z_1 \le z_{1,j}$ in the defined region for z_1. This is the lower bound.

$$z_2 = n(\frac{\omega_n}{n - \omega_n})^{s+1}$$

for $0 \le j \le \omega_n/(s+1)$ and this is the upper bound. We use Proposition 5.4 and get

$$P(A(1) = n \mid A(0) = n - \omega_n) = e^{-n(\frac{\omega_n}{n})^{s+1}}(1 + o(1)) \tag{91}$$

We insert $\omega_n = n^{1-\beta}$ in (91).
This impies

$$P = P(A(1) = n \mid A(0) = n - \omega_n) = e^{-n^{1-\beta(s+1)}}(1 + o(1)))$$

$\beta < \frac{1}{s+1}$ implies $P \to 0$ as $n \to \infty$
and $\beta > \frac{1}{s+1}$ implies $P \to 1$ as $n \to \infty$. Now we replace ω_n in the proposition 3.2 in [2] by $\omega_n = n^{1-\frac{1}{s+1}}$ and change Theorem 3.1 in [2] by replacing the critical α by $\alpha_c = \frac{1}{s+1}$, our responsibility to Theorem 5.2 and apply the proof of theorem 3.1 with this changes. This will prove Theorem 5.2 for $m = 1$.
More in details. The proof of the theorem 3.1 implies

$$P(A(0) < n - \omega_n) \le e^{n(1-p)\alpha - \omega_n \alpha}(1 + o(1)) \tag{92}$$

and

$$P(A(0) > n - \omega_n) \le e^{\omega_n \alpha - n(1-p)\alpha}(1 + o(1)) \tag{93}$$

where $\alpha > 0$ is small.
From proposition 5.4 let $\omega_n = n^{1-\frac{1}{s+1}}$. Then by (92) and (93), we get

$$\alpha > \frac{1}{s+1} + \epsilon \Rightarrow P(A(1) = n) = \sum_{i=0}^{n} P(A(1) = n \mid A(0) = i)P(A(0) = i)$$

$$= \sum_{i=n-n^{1-\frac{1}{s+1}-\epsilon'}}^{n} P(A(1) = n \mid A(0) = i)P(A(0) = i) + o(1) = 1 - o(1)$$

and

$$\alpha < \frac{1}{s+1} - \epsilon \Rightarrow P(A(1) = n) = \sum_{i=0}^{n} P(A(1) = n \mid A(0) = i)P(A(0) = i)$$

$$= \sum_{i=0}^{n-n^{1-\frac{1}{s+1}+\epsilon'}} P(A(1)=n \mid A(0)=i)P(A(0)=i) + o(1) = o(1)$$

for arbitrary small $\epsilon > \epsilon' > 0$
This Prove Theorem 5.2 for $m = 1$.
We can also reason like this.
By study $\omega_n = n^{1-\alpha}$ (instead of $\omega_n = n^{1-\frac{1}{s+1}}$) This proof of the theorem 3.1 ,by the limits of(92) and(93) also provides the information

$$P(A(0) \in [n - n^{1-\alpha+\epsilon}, n - n^{1-\alpha-\epsilon}]) = 1 - o(1)$$

for all $\epsilon > 0$. Combining this with Proposition 5.4 , we get

$$P(A(1)=n) = EP(A(1)=n \mid A(0)) = \sum_{i=0}^{n} P(A(1)=n \mid A(0)=i)P(A(0)=i)$$

$$= \sum_{i=n-n^{1-\alpha+\epsilon}}^{n-n^{1-\alpha-\epsilon}} P(A(1)=n \mid A(0)=i)P(A(0)=i)(1-o(1))$$

$$= P(A(1)=n \mid A(0)=np)(1-o(1)) = e^{-n(1-p)^{s+1}}(1-o(1)),$$

which also gives a critical $\alpha_c = \frac{1}{s+1}$ also for $m = 1$.

By (84) this result is generalized to $\alpha_c = \frac{1}{(s+1)m}$ for the extended process. This completes the proof. □

Example 7.2 *As in Example 3.2 we can do the same procedure with*

$$P(A(1)=n \mid A(0)=EA(0)) = e^{-n^{1-(s+1)\alpha}}(1+o(1))$$

and get

$$P(A(1)=n) = e^{-n^{1-(s+1)\alpha}}(1+o(1))$$
$$= e^{-n(1-p)^{s+1}}(1+o(1))$$

7.12 Discussion

Here we discuss some heuristic proof of Theorem 5.2, which uses the fact that the length of the (inactive) intervals are independent asymptotically distributed. Locally it can be shown that each interval I followed by an element of one is $Ge(p)$-distributed, $P(\mid I \mid = k) = (1-p)^k p$, where $I = I_j$ is one of the $A(0)+1$ intervals. If we denote $N_j - 1 = \mid I_j \mid$, where $1 < j < A(0)+1$ then we use

$$\sum_{j=1}^{A(0)+1} (N_j - 1) = n - A(0), \tag{94}$$

where $A(0$ the number of elements of one, and $N_j - 1$ is the number of zeros in one interval, both are depending random variables. If we estimate $A(0) = np$ taken from example 3.2 we loose the random property on the right side of (94). Therefore we can only estimate the left side

$$\sum_{j=1}^{A(0)+1} (N_j - 1)$$

by

$$\sum_{j=1}^{np+1} (N_j - 1)$$

and claim it is asymptotic correct. Now

$$\phi_{n-A(0)}(s) = Ee^{is(n-A(0))} = (p + (1-p)e^{is})^n = e^{n(1-p)(e^{is}-1)}(1+o(1))$$

when $p = 1 - n^{-\alpha}$ and

$$\phi_{\sum_{j=1}^{np+1}(N_j-1)}(s) = Ee^{is\sum_{j=1}^{np+1}(N_j-1)} = Ee^{is\sum_{j=1}^{np+1}N_j-(np+1)is}$$

$$= e^{-(np+1)is}(pe^{is}(1-(1-p)e^{is})^{-1})^{np+1} = (p(1-(1-p)e^{is})^{-1})^{np+1}$$
$$= e^{n(1-p)(e^{is}-1)}(1+o(1)).$$

However, we do not know if the solution is unique. Let us see if we can use the result of Theorem 5.2. By example 7.2, we construct following.

Example 7.3 *Let* $\mid I_{dj_t+t} \mid \leq s_t$ *for a fix integer* $d > 1$, $0 \leq t \leq d-1$ *and* $0 \leq j_t \leq \lceil \frac{n-t}{d} \rceil$ *be our percolation conditions. Define*

$$A_t(\tau) = \mid \{x_j(\tau) = 1 : j \in \bigcup_{k=0}^{\lceil \frac{A(0)-t}{d} \rceil} (I_{dk+t} \bigcup \{\max_{i \in I_{dk+t}} i + 1\})\} \mid .$$

We define

$$B_t = \{i : i \in \bigcup_{k=0}^{\lceil (A(0)-t)/d \rceil} (I_{dk+t} \bigcup \{ \max_{i \in I_{dk+t}} i+1\})\} \tag{95}$$

and

$$n = \mid \bigcup_{t=0}^{d-1} B_t \mid = \sum_{t=0}^{d-1} \mid B_t \mid, \tag{96}$$

where all sets B_t *are disjointed. If the set consists of the last interval, we set* $\max_{i\in I} i+1 = \emptyset$ *for that interval. Suppose,*

$$P(A(1) = n) = \prod_{t=0}^{d-1} P(A_t(1) = \mid B_t \mid) + o(1).$$

We assume from (96) that all $\mid B_t \mid$ *are asymptotic equiprobable and therefore*

$$\mid B_t \mid \in Bin(n, 1/d).$$

Then $E \mid B_t \mid = \frac{n}{d}$ *and* $Var(\mid B_t \mid) = \frac{n(1-\frac{1}{d})}{d}$*. By the Chebyshev's inequality*

$$P(\mid\mid B_t \mid - E \mid B_t \mid\mid > n^{\epsilon+1/2}) \leq Var\frac{\mid B_t \mid}{n^{1+2\epsilon}} \leq \frac{n}{n^{1+2\epsilon}} \leq n^{-2\epsilon} \to 0$$

as $n \to \infty$ *for all* $\epsilon > 0$*. Hence*

$$\mid B_t \mid = \frac{n(1+o(1))}{d}.$$

We have $P(A_t(1) = \frac{n}{d} + o(n)) = P(A_t(1) = \frac{n}{d}) + o(1)$ *by example 7.2, where* $o(n) \approx n^{1/2}$*. We expect the same contribution of all elements of zero for all the intervals, so the approximation*

$$P(A(1) = n) = \prod_{t=0}^{d-1} P(A_t(1) = \frac{n}{d}) + o(1) = \prod_{t=0}^{d-1} e^{\frac{n}{d}(1-p)^{s_t+1}} + o(1) \tag{97}$$

is supposed to hold by symmetry of the intervals. Then

$$\prod_{t=0}^{d-1} (1-(1-p)^{s_t+1})^{\frac{n}{d}} + o(1) = \prod_{t=0}^{d-1} P(\mid I \mid \leq s_t)^{\frac{n}{d}} + o(1)$$

which indicates asymptotically independence of the probability of the length of the intervals.

This is of course not a complete argument but it tells us about the nature of the intervals.

Another interesting aspect is whether the result holds with cyclic condition. It is enough to study the case when $m = 1$. In a ring we have

$$P(A(1) = n \mid A(0) = n - \omega_n) = P(A(1) = n \mid A(0) = n - \omega_n, x_1 = 0)P(x_1 = 0)$$

$$+P(A(1) = n \mid A(0) = n - \omega_n, x_1 = 1)P(x_1 = 1)$$

$$= P(A(1) = n \mid A(0) = n - \omega_n, x_1 = 1) + o(1)$$

$$= P^*(A(1) = n - 1 \mid A(0) = n - 1 - \omega_{n-1}) + o(1),$$

where P^* is a probability of a strap with noncyclic conditions. Therefore the asymptotically behavior is the same for cyclic and noncyclic conditions.

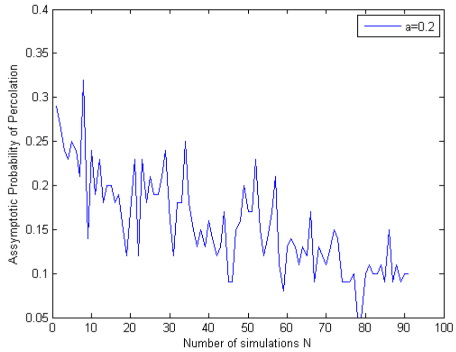

Figure 3: $\alpha = 0.20$

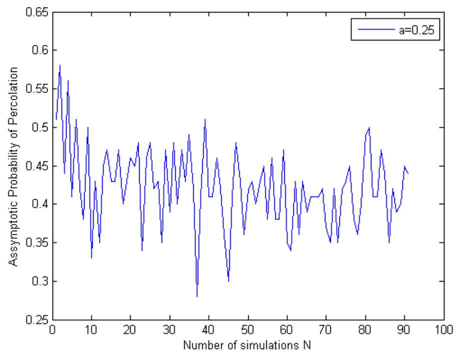

Figure 4: $\alpha = 0.25$

8 Simulations

8.1 Numerical support for Theorem 5.2

We simulated the case s=1, t=2,the same as the Theorem 5.2 by using the following algoritm:
$y = Q_s(R_p)^2 x_0$ 1.We start with $x_0 = (0)_{i=1}^n$ then we randomize all elements of zero.
2.We randomize all the elements of zero again
3.We percolate the elements of zero.
and by $Q_s R_p Q_s R_p x_0$
1.We start with $x_0 = (0)_{i=1}^n$ then we randomize all elements of zero.
2.We percolate the elements of zero.
3.We randomize all the elements of zero again.
4.We percolate the elements of zero.
Both algorithms implies the same results.

8.2 Results

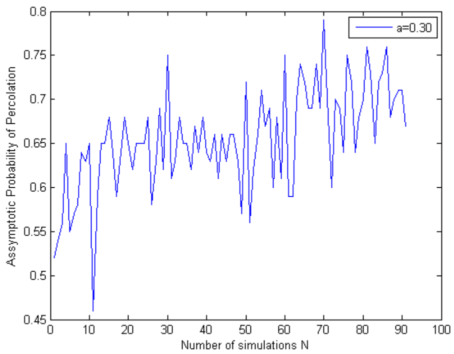

Figure 5: $\alpha = 0.30$

We used n=100 and made 100 realizations for $\alpha = 0.20, 0.25, 0.30$. We plot the number of complete percolated realizations quoted with the number of realizations against the number of realizations and get the "asymptotic probability of percolation". This frequency converges quite slowly but the trend is visible for $n = 100$.
If we start with $\alpha = 0.2$ (Figure 3), we see a vanishing trend and conclude that the critical α is higher than 0.2.

At $\alpha = 0.3$ the last picture (Figure 5), we see that the trend is rising and can conclude that the critical α is lower than 0.3.
At $\alpha = 0.25$ the middle picture (Figure 4), the trend is almost horizontal. We therefor conclude that this is nearby the critical α. The convergence

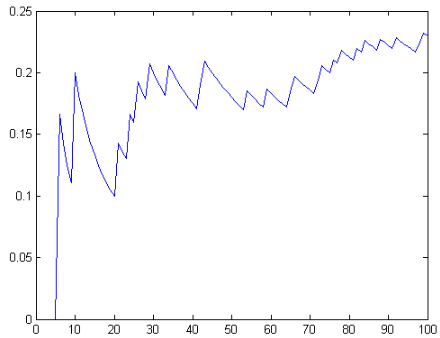

Figure 6: $\alpha = 0.19$

speed in our simulator is quite low to run on small computers. We construct a simulator for a $s, 1-processes$ and ran it for $s = 4$, where we expected $\alpha_c = 0.2$ by theorem 5.2.

In figure 6 the frequency seems low at 0.25 for $\alpha = 0.19$ in the critical zone.

In figure 7 the frequency is about 0.5 for $\alpha = 0.20$ also in the critical zone.

In figure 8 the frequency is about 0.6 for $\alpha = 0.21$ also in the critical zone.We used $N = 10000$.

It is a quite slow convergence speed in both simulations but indicate critical α to be the middle α in both simulations.

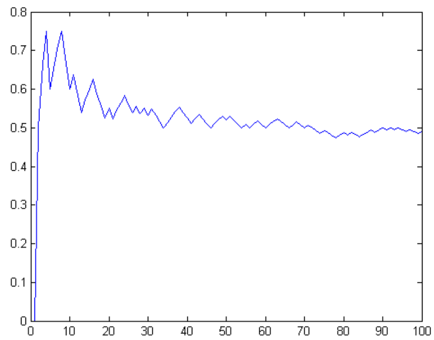

Figure 7: $\alpha = 0.20$

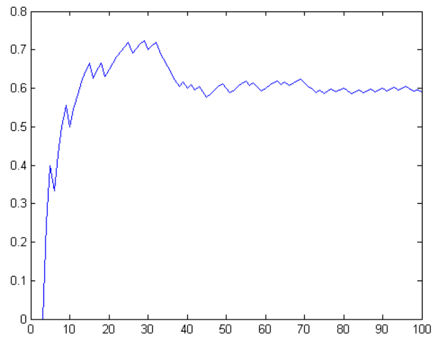

Figure 8: $\alpha = 0.21$

9 References

References

[1] B. Bollobás, C. Holmgren, P. Smith and A. J. Uzzel: THE TIME OF BOOTSTRAP PERCOLATION WITH DENSE INITIAL SETS, *Annals of Probabillity*, 2012

[2] P. Pehrsson "Bootstrap Percolation on Z_1". *Lund University* 2012:K2

[3] V.K. Balakrishnan "Introductory Discrete Mathematics", *Dover*