Pricing and Hedging using Hedge Monte-Carlo method

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Abstract

In this master's thesis The Hedge Monte-Carlo method (HMC) is evaluated. The HMC method is used to price financial derivatives and at the same time obtain optimal hedge portfolios. The optimal hedge is of great importance as it enables risk management in option trading. The advantage of this method is also that different types of options with features like path-dependent and early exercise can be priced.

The evaluation is based on the quality of the price and hedge estimates of European options. To further evaluate the performance of the method the price process of the underlying asset followed initially a *Geometric Brownian Motion process (GBM)* and then the *Normal Inverse Gaussian process (NIG)*. Several different scenarios are considered in the evaluation of retrieving good prices and hedges, i.e. different times to maturity, initial stock prices and variances. Results shows that the method is very promising when considering the quality of the price and as for the quality of the hedge good levels are obtained for GBM when the option is in the money. A desirable feature as the probability of exercise of an in the money option is very high. For options where the underlying asset follows NIG acceptable levels on the hedging errors were difficult to obtain. As the performance of the method is measured on both good prices and good hedges, the NIG process isn't as suitable as the GBM process when the HMC method i used.

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1 Introduction

1.1 Background

The Hedge Monte-Carlo method (HMC) is a method derived in order to price financial derivatives and at the same time obtain optimal hedge portfolios. The method has been inspired by the Least Square Method (LSM) of Longstaff and Schwartz (2001) -pp 122 and thereof has the advantage of being able to price different types of options with features like path-dependent and early exercise. The HMC method also has the advantages of variance reduction, optimal hedges and that historical data can be used. Potters, Bouchaud and Sestovic(2001) -pp 518.

In this thesis European options are being considered, which gives the holder of a call option the right to buy the underlying asset and respectively if holding a put option to sell the asset, i.e. plain Vanilla options that are traded on exchange markets. In most cases risk in option trading can't be eliminated completely and arbitrage opportunities do not exist, which is the reason that option markets exists. Due to the fact that risk can't be eliminated completely, I have with the article *Hedged Monte-Carlo: low variance derivative pricing with objective probabilities* in mind, in this thesis evaluated the Hedge Monte-Carlo method based on the quality of the price and hedge. Potters, Bouchaud and Sestovic (2001) -pp 519, presents in the article a new Hedged Monte-Carlo method to price financial derivatives and at the same time obtain the optimal hedge. The possibility of obtaining the optimal hedge is of great importance in the banking industry, due to the fact that it enables risk management in option trading.

Initially the price process of the underlying asset followed a *Geometric Brownian Motion process (GBM)* and then the *Normal Inverse Gaussian process (NIG)* which is a Lévy-process. This due to that the NIG-distribution assigns a large amount of probability mass to the tails of the distribution, enabling to better capture the recent historical movement in the financial markets. In a heavy-tailed distribution the probability for outcomes significantly deviating from the mean is much greater than in the case of the normal distribution.

1.2 Method

The HMC method starts out at maturity (when the payoff of the derivative is known) and works backwards in time, to obtain the price and optimal hedge for the derivative. The method is based on minimizing the local quadratic risk of the financial risk which occurs due to the imperfect replication of a derivative by a hedging strategy. The numerical implementation is done with the same approach as used by Longstaff and Schwartz (2001).

The algorithm in Longstaff and Schwartz method is based on finding the conditional expectation of the option value at each time step before the option has reached maturity. This is done by using regression and results in that the optimal stopping times for all trajectories are retrieved. The idea behind the regression method is that the conditional expectation at each time step is approximated by using a set of basis functions.

1.3 Purpose

The purpose of the thesis was to give a thorough description of the theory behind the HMC method, the definitions and theorems in order to have an understanding of the framework, the numerical implementation of the HMC method and to present the results for how well the method preforms based on price and hedge estimates. The evaluation of the method was done for European options where the price process of the underlying asset followed initially a GBM and then a NIG process.

2 Theory - The Hedge Monte-Carlo method

The hedge Monte-Carlo method (HMC), is a method to price options among other financial derivatives and to find the optimal hedge. This method works backwards due to the fact that the exact option price is known at maturity, when it is equivalent to the pay-off and with the local quadratic measure of risk in mind the variance of the wealth change in a time step is minimized.

Before going into how the HMC method is used to price and hedge derivatives there are some definitions and theorems that needs to be stated as well as the models of the underlying asset and Black & Scholes needs to be described. The following terminology will be used:

T : Maturity

δt : Timestep ($\delta t = T/n$)

n : Number of the time steps

$t : 0 < t \leq T$

$P_t(S_t)$: The price of the option

L_t^S, L_t^B : Number of stocks and bonds in the portfolio L at time t with The price S_t respectively B_t

V_t^L : The value of the portfolio L,$V_t^L = L_t^B B_t + L_t^S S_t$

r : Risk free interest rate.

2.1 Definitions

In this thesis European options are being considered, which gives the buyer the right (not the obligation), to buy the underlying asset at maturity if holding a call option or to sell the asset if holding a put option. Options are financial derivatives and the definition of a derivative is,

Definition 1. *(Derivative), an asset that is completely defined in terms of an underlying financial asset is a* ***derivative asset****. Which is also called a* ***contingent claim****. Björk (2004) -pp 9.*

As the Black & Scholes model for option pricing is used as a benchmarking method, the formula will be described and the definition follows below.

Definition 2. *(Black & Scholes), consist of two assets with the dynamics given by*

$$dB(t) = rB(t)dt \tag{1}$$

$$dS(t) = \alpha S(t)dt + \sigma S(t)d\bar{W}(t) \tag{2}$$

where B(t) is the price process of a risk free asset, S(t) the price process of a stock, $\bar{W}(t)$ *is a Standard Brownian motion, and* r, α *and* σ *are deterministic constants, Björk (2004) -pp 89.*

A risk neutral valuation framework is considered, i.e. the price of an asset today is given by the discounted expected asset price of tomorrow. To have a risk neutral valuation the so called $\mathbb{Q}$-probability is used, which is a risk neutral measure with the following property:

Definition 3. *(Risk neutral measure), a probability measure* $\mathbb{Q}$ *is called a* ***risk neutral measure*** *or alternatively* ***a martingale measure*** *if the following condition holds:*

$$\frac{S_0}{B_0} = E^{\mathbb{Q}}\left[\ \tfrac{S_t}{B_t} \mid S_0\ \right] \tag{3}$$

Björk (2004) -pp 9.

A risk neutral framework enables option pricing without estimating the drift of the underlying assets or modeling stochastic discount factors and still receive the same price as with no-arbitrage valuation.

In order to derive the price and the hedge strategy the assumptions that the market is arbitrage free and complete are made, these assumptions are defined as,

Definition 4. *(Arbitrage free market), a market is considered arbitrage free if there is no portfolio* L_t, $t \in [0, T]$ *consisting of assets traded in the market, where the value of the portfolio* $V_t^L = 0$ *and* $V_\tau^L > 0$ *with* $P(V_\tau^L > 0) = 1$, $0 \le t \le \tau \le T$, *Hanke (2003)-pp 4, Lüders (2004) -pp 6.*

Definition 5. *(Complete market) An arbitrage-free market is considered complete if every contingent claim is attainable, i.e. a financial market in which the value of any contract can be replicated by selecting an appropriate portfolio of assets in the market and investing an initial amount, Föllmer, Schied (2011) -pp 287.*

2.2 Theorems

Black & Scholes model for option pricing is used as my benchmarking method. In order to have the complete understanding of the model, below follows the theorem of B&S.

Theorem 1. *(Black & Scholes equation), if one assumes that the market is specified by the price processes in the definition of Black & Scholes and a derivative instrument* X *with the contract function* $\Phi(S(t))$ *is to be priced, then the only pricing function fulfilling the requirement of being arbitrage free and with the form* $D(t) = F(t, S(t))$ *where* $F \in C^{1,2}$ *is a smooth function (of the time and the price of the underlying asset S), is when* F *is the solution of the boundary value problem below, Björk (2004) -pp 97.*

$$F_t(t,s) + rsF_s(t,s) + 1/2s^2\sigma^2F_{ss}(t,s) - rF(t,s) = 0 \qquad (4)$$

$$F(T,s) = \Phi(s) \qquad (5)$$

The Black & Scholes model can be expected to be arbitrage free and complete by the meta-theorem:

Theorem 2. *(Meta-theorem), let M be the number of underling traded assets in the model where the risk free asset is not included, and R is the number of random sources. We then have the following relations:*

1. *The model is arbitrage free if and only if* $M \leq R$.
2. *The model is complete if and only if* $M \geq R$.
3. *The model is complete and arbitrage free if and only if* $M = R$.

Björk (2004) -pp 118

As previously mentioned, a risk neutral valuation framework is considered.

Theorem 3. *(Risk Neutral Valuation), the arbitrage free price of the claim* $\Phi(S(t))$ *is given by* $D(t) = F(t, S(t))$ *where* F *given by:*

$$F(t, s) = e^{-r(T-t)} E_{t,s}^{\mathbb{Q}}[\Phi(S(T))], \tag{6}$$

and where $dS(t) = rS(t)dt + S(t)\sigma dW^{\mathbb{Q}}(t)$ *and* W *is a* $\mathbb{Q}$*-Wiener process, Björk (2004) -pp 99.*

2.3 Stochastic Differential Equation, SDE

In this thesis a stock is the underlying asset, which is described as a stochastic differential equation (SDE). This approach is used to capture the behavior of the asset price. How well the movement of the asset price is captured is of great importance because it will affect the derivative pricing. Stochastic differential equations arise from ordinary differential equations, when for example white noise is a part of the equation. In the world of finance we see this when there is uncertainty in the rate of return. If we consider r being the risk free interest rate then $r + \xi(t)$ is the uncertain interest rate, where $\xi(t)$ represents the uncertainty and is referred to as the white noise of the equation.

2.3.1 Geometric Brownian motion

SDE can rarely be solved in a precise way but there are a few cases when the SDE can be solved, one of these are when the equation is the *Geometric Brownian motion* with $X(t)$ as the unknown process,

$$dX(t) = \mu X(t)dt + \sigma X(t)d\bar{W}(t) \tag{7}$$

this equation can be solved in the following way,

$$dX(t) = \mu X(t)dt + \sigma X(t)d\bar{W}(t), \; X(0) = 1 \tag{8}$$

consider $f(x)$ being $\ln x$ then $f'(x) = \frac{1}{x}$ and $f''(x) = -\frac{1}{x^2}$

$$d(lnX(t)) = \frac{1}{X(t)}dX(t) + \frac{1}{2}(-\frac{1}{X^2(t)})\sigma^2 X^2(t)dt \tag{9}$$

$$= \frac{1}{X(t)}(\mu X(t)dt + \sigma X(t)d\bar{W}(t)) - \frac{1}{2}\sigma^2 dt \tag{10}$$

$$= (\mu - \frac{1}{2}\sigma^2)dt + \sigma d\bar{W}(t). \tag{11}$$

If we then set $Y(t) = \ln X(t)$,

$$dY(t) = (\mu - \frac{1}{2}\sigma^2)dt + \sigma d\bar{W}(t), \tag{12}$$

the solution to Y is given by,

$$Y(t) = Y(0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma\bar{W}(t) \tag{13}$$

and we get X to be,

$$X(t) = X(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma\bar{W}(t)} \tag{14}$$

and the expected value is,

$$E[X(t)] = X(0)e^{\mu t} \tag{15}$$

Mörters, Peres, Schramm and Werner (2010) -pp 190, Klebaner (2005) -pp 124.

The properties of the Wiener process in the equation of the GBM, .i.e. in equation 14 enables simulation of the stock price, $S(t)$ at a specific time t by the formula:

$$S(t) \stackrel{d}{=} S(0)e^{((r-1/2\sigma^2)t+\sigma\sqrt{t}Z(t))}.$$

where Z(t) $\sim$ N$(0, 1)$ and d is equality in distribution. In order to get values for all time points this formula is used:

$$S(t_{i+1}) \stackrel{d}{=} S(t_i)e^{((r-1/2\sigma^2)(t_{i+1}-t_i)+\sigma\sqrt{t_{i+1}-t_i}Z(t_{i+1}))}.$$

2.3.2 Normal Inverse Gaussian process

The *Normal Inverse Gaussian* process is a so called Lévy process. Lévy processes are a general class of one-dimensional stochastic processes, with the following definition:

Definition 6. *(Lévy Process), a process $\{X(t)_{t\geq 0}\}$ defined on a probability space (Ω, F, P) is said to be a Lévy process if it has the following properties:*

1. *The paths of X are $P-$almost surely right continuous with left limits.*
2. *$P(X_0 = 0) = 1$, a.s.*
3. *For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s}.*
4. *For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\{X_u : u \leq s\}$.*

Kyprianou (2006) -pp 2.

From the definition of a Lévy process we have that for any $n = 1, 2, ...,$

$$X_t = X_{t/n} + (X_{2t/n} - X_{t/n}) + ... + (X_t - X_{(n-1)t/n}) \tag{16}$$

and that X has stationary independent increments that still are Lévy distributed, these properties result in that for any $t > 0$, X_t is a random variable belonging to the class of *infinite divisible distributions*.

Definition 7. *(Infinite divisible distributions), a real-valued random variable Y has an infinitely divisible distribution if for each $n = 1, 2, ...$ there exists series of i.i.d. random variables $Y_{1,n}, ..., Y_{n,n}$ such that*

$$Y \overset{\mathrm{d}}{=} Y_{1,n} + ... + Y_{n,n} \tag{17}$$

Kyprianou (2006) -pp 3.

An essential property of the Lévy process is the connection to *infinite divisible distributions*, because from the definition above follows that a Lévy process $\{X(t)_{t\geq 0}\}$ has a unique *characteristic exponent*, which is a continuous function $\lambda(u)$, for all $u \in \mathbb{R}$ and the characteristic function of $X(t)$ is given by,

$$\phi(u) = e^{t\lambda(u)}, \text{for u} \in \mathbb{R}^{\mathrm{d}} \text{ and t} \geq 0 \tag{18}$$

where,

$$\lambda(u) = ibu - \frac{1}{2}uau + \int_{\mathbb{R}\setminus\{0\}} [e^{iuy} - 1 - iuy\mathbf{1}_{\{0<|y|<1\}}(y)]\nu(dy), \tag{19}$$

for for some real b, real a > 0 and $\nu(\cdot)$ a Lévy measure on $\mathbb{R} \setminus \{0\}$ so that $\int_{R\setminus\{0\}} \min(1, | y |^2)\nu(dy) < \infty$.

In Section 2.3.1 the solution of *Geometric Brownian motion* was described, worth noting is that *Brownian motion* is a Lévy process $\{X(t)_{t\geq 0}\}$ that has Gaussian increments. The Lévy measure $\nu(\cdot)$ controls the jumps of the Lévy process so if one sets $\nu = 0$ then X(t) is a Brownian motion with drift b and volatility a.

The NIG process is a subordinated Brownian motion, as earlier stated also a Lévy process $\{X(t)_{t\geq 0}\}$ but with normal inverse Gaussian distributed increments and the random variable $X(t)$ has a $NIG(\alpha, \beta, \delta, \mu t)$-distribution, where $\alpha > 0, \Big| \beta \Big| < \alpha, \delta > 0$ and $\mu \in \mathbb{R}$. For real values x, the probability density function is given by,

$$F_{NIG}(x;\alpha,\beta,\delta,\mu) = \frac{\alpha\delta}{\pi}\frac{K_1(\alpha\sqrt{\delta^2-(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}}e^{\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)}, \quad (20)$$

where

$$K_v(z) = \frac{1}{2}\int_0^\infty u^{v-1}e^{-\frac{z}{2}(u+\frac{1}{u})}du \quad (21)$$

is the modified Bessel function of the third kind and the characteristic function is given by,

$$\phi_{NIG(u;\alpha,\beta,\delta,\mu)} = e^{-\delta(\sqrt{\alpha^2-(\beta+iu)^2}-\sqrt{\alpha^2-\beta^2})}e^{i\mu u} \quad (22)$$

The distribution function is determined by numerical methods and the parameters has the following interpretation:

α controls the shape of the density

β controls skewness

μ manages the position of the density function and

δ is the scaling parameter.

The random variable X of a NIG-distribution has the mean,

$$E(X) = \mu + \frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}} \quad (23)$$

and the variance,

$$Var(X) = \delta\frac{\alpha^2}{(\alpha^2-\beta^2)^{\frac{3}{2}}} \quad (24)$$

Rachev, Hoechstoetter, Fabozzi and Focardi (2010) -pp 285.

2.4 The Black and Scholes formula

Based on the definition and the theorem of Black & Scholes described in the two previous sections one can see that the SDE;

$$dS(t) = rS(t)dt + S(t)\sigma dW(t) \quad (25)$$

can be described by the Geometric Brownian Motion model. By applying the solution of the GBM equation to the S-process one gets $S(T)$:

$$S(T) = S(t)e^{((r-1/2\sigma^2)(T-t)+\sigma(W(T)-W(t)))} \quad (26)$$

and the pricing formula is

$$F(t,s) = e^{-r(T-t)} \int_{-\infty}^{\infty} \Phi(se^Z) f(z) dz \tag{27}$$

where f is the density function and Z is a stochastic variable with the distribution

$$N[(r - \frac{1}{2}\sigma^2)(T-t), \sigma\sqrt{T-t}]$$

The pricing formula is an integral that requires numerical evaluation except for some contract functions Φ like an European call option where $\Phi(x) = \max[x - K, 0]$ which gives

$$E_{t,s}^{\mathbb{Q}}[\max[se^Z - K, 0]] = 0 * \mathbb{Q}(se^Z \leq K) + \int_{\ln(K/s)}^{\infty} (se^z - K) f(z) dz \tag{28}$$

by solving the equation, one is left with the Black & Scholes formula: *The price of an European call option with strike price K and time of maturity T is given by the pricing formula F:*

$$F(t,s) = sN[d_1(t,s)] - e^{-r(T-t)} K N[d_2(t,s)] \tag{29}$$

N is the cumulative distribution function for the N[0,1] distribution and d_1, d_2 *equals to:*

$$d_1(t,s) = \frac{1}{\sigma\sqrt{T-t}} \{\ln(\frac{s}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)\}, \tag{30}$$

$$d_2(t,s) = d_1(t,s) - \sigma\sqrt{T-t} \tag{31}$$

Björk (2004) -pp 101.

2.5 The price and the hedge strategy

In practice the most cost-conscious model is determined by the model associated with minimum risk, so with quadratic measure of risk in mind the variance of the wealth change between t and $t+1$ is minimized. If we assume that the market is arbitrage free and complete then the price process of the option D_t, can be replicated by a so called hedging portfolio that consist of two assets, a bond B (with a deterministic price process) and a stock S (with a stochastic price process). With time the value process V_t^L will correspond to:

$$V_t^L = L_t^B B_t + L_t^S S_t \tag{32}$$

at time t,

$$V_{t+1}^L = L_t^B B_t e^{r(t+1-t)} + L_t^S S_{t+1} \tag{33}$$

at time $t+1$.

Then for every time point t the option price process is:

$$V_t^L = D_t \to V_t^L = P_t(S_t) \tag{34}$$

From one time point to another the portfolio has the following local value change:

$$e^{-r(t+1-t)}V_{t+1}^L - V_t^L = \tag{35}$$

$$e^{-r(t+1-t)}(L_t^B B_t e^{r(t+1-t)} + L_t^S S_{t+1}) - (L_t^B B_t + L_t^S S_t) = \tag{36}$$

$$e^{-r(t+1-t)}(L_t^S S_{t+1}) - L_t^S S_t = L_t^S(e^{-r(t+1-t)}S_{t+1} - S_t) \tag{37}$$

Which in theory should be equal to the change of the option price at the same time interval:

$$e^{-r(t+1-t)}P_{t+1}(S_{t+1}) - P_t(S_t) \tag{38}$$

The equality does not hold, due to the fact that one assumes an arbitrage free and complete market in theory, which is not the case in reality. So we have a financial risk:

$$e^{-r(t+1-t)}P_{t+1}(S_{t+1}) - P_t(S_t) - L_t^S(e^{-r(t+1-t)}S_{t+1} - S_t) \neq 0 \tag{39}$$

The local risk is defined as the variance of the wealth change between t and $t+1$, a risk neutral measure:

$$R_t = E^{\mathbb{Q}}[(e^{-r(t+1-t)}P_{t+1} - P_t - L_t^S(e^{-r(t+1-t)}S_{t+1} - S_t))^2 \mid \mathcal{F}_t] =$$

$$E^{\mathbb{Q}}[(e^{-r(t+1-t)}P_{t+1} - P_t)^2 - 2L_t^S(e^{-r(t+1-t)}P_{t+1} - P_t)(e^{-r(t+1-t)}S_{t+1} - S_t) + ...$$

$$...(L_t^S)^2(e^{-r(t+1-t)}S_{t+1} - S_t)^2 \mid \mathcal{F}_t] =$$

$$= \left\{ \begin{array}{l} B = (e^{-r(t+1-t)}P_{t+1} - P_t)(e^{-r(t+1-t)}S_{t+1} - S_t) \\ A = (e^{-r(t+1-t)}S_{t+1} - S_t)^2 \end{array} \right\} =$$

$$= E^{\mathbb{Q}}[(e^{-r(t+1-t)}P_{t+1} - P_t)^2 - 2L_t^S B + (L_t^S)^2 A \mid \mathcal{F}_t] \tag{40}$$

Consider the function within the square brackets:

$$= \left\{ \begin{array}{l} (e^{-r(t+1-t)}P_{t+1} - P_t)^2 - 2L_t^S B + (L_t^S)^2 A \\ \text{Take the derivative with respect to } L_t^S : 2B + 2L_t^S A \\ \text{The minimum of the function: } 2B + 2L_t^S A = 0 \Longrightarrow L_t^S = \frac{B}{A} \end{array} \right\} =$$

The minimization of the function R_t is obtained when the hedge L_t^S is:

$$L_t^S = \frac{E^{\mathbb{Q}}[(e^{-r(t+1-t)}P_{t+1} - P_t)(e^{-r(t+1-t)}S_{t+1} - S_t)]}{E^{\mathbb{Q}}[(e^{-r(t+1-t)}S_{t+1} - S_t)^2]} \tag{41}$$

The numerator is a covariance and the denominator is conditional expectations, thereby a numerical solution to the problem is rather difficult to obtain, instead the problem is solved by linear regression:

$$Y = \alpha 1 + XL + \epsilon, \tag{42}$$

In matrix notation,

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_t \end{bmatrix} = \begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_t \end{bmatrix} \begin{bmatrix} \alpha \\ L \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_t \end{bmatrix}$$

$$\mathbf{Y} = \mathbf{X\Gamma} + \epsilon \tag{43}$$

Minimizing the residual $\epsilon = \mathbf{Y} - \mathbf{X\Gamma}$ by least Squares, i.e. $\epsilon^T \epsilon$ with respect to Γ one retrieves the normal equation,

$$\mathbf{X^T Y} = \mathbf{X^T \Gamma} \tag{44}$$

Solving this equation for Γ gives the following least squares solution,

$$\mathbf{\Gamma} = (\mathbf{X^T X})^{\mathbf{-1}}\mathbf{X^T Y} \tag{45}$$

where,

$$\mathbf{Y} = \begin{pmatrix} e^{-r(t+1-t)}P_{1+1} - P_1 \\ e^{-r(t+1-t)}P_{2+1} - P_2 \\ \vdots \\ e^{-r(t+1-t)}P_{t+1} - P_t \end{pmatrix} \mathbf{X} = \begin{pmatrix} e^{-r(t+1-t)}S_{1+1} - S_1 \\ e^{-r(t+1-t)}S_{2+1} - S_2 \\ \vdots \\ e^{-r(t+1-t)}S_{t+1} - S_t \end{pmatrix}$$

Which equals to equation 41. Now one can retrieve the optimal hedge by simulations.

The numerical implementation is done in the same manners as in Longstaff and Schwartz (2001), i.e. with the least squares Monte Carlo (LSM).

2.6 Numerical implementation

The value of an option is implied to equal the maximized value of the discounted cash flows from the option. $C(q, s; t_k, T)$ are the cash flows generated by the option for every trajectory q, conditional on the option not being exercised at or before time t and on that the option-holder is following the optimal stopping strategy for all s, $t < s \leq T$. $\mathcal{F}$ is the σ-field of distinguishable events at time T and $\mathbf{F} = \{\mathcal{F}_t; t \in [0, T]\}$ is the so called augmented filtration produced by the price processes for the derivatives in the economy. The value of function $\mathbf{F}(q; t_k)$ if continuing at time t_k can be stated as

$$\mathbf{F}(q; t_k) = E^{\mathbb{Q}}[\sum_{j=k+1}^{n} e^{-r\delta} C(q, t_j; t_k, T) \mid \mathcal{F}_{t_k}], \tag{46}$$

where n is the number of time steps.

The assumption is made that at time t_{n-1} the unknown function $\mathbf{F}(q; t_{n-1})$, i.e. the function from equation 46 can be stated as a linear combination of a countable set of $\mathcal{F}_{t_{n-1}}$-measurable basis functions. This due to that the method is given in terms of Hilbert Spaces, with the property that any function belonging to this space can be represented as a countable linear combination of bases for this vector space. There are several limitations on the basis functions but in this case it is enough for them to be complete and linearly independent, Longstaff and Schwartz (2001) -pp 122.

$$\mathbf{F}(q; t_{n-1}) = \sum_{j=0}^{\infty} a_j f_j(t_{n-1}, S_{t_{n-1}}) \tag{47}$$

The approximation $\hat{\mathbf{F}}(q; t_{n-1})$ of $\mathbf{F}(q; t_{n-1})$ is then done by first specifying the basis functions followed by regressing the discounted values $C(q, t_n; t_{n-1}, T)$ on to the basis functions for the paths where the option is in the money at time t_{n-1}. At t_{n-2} the cash flows for each path are once again approximated and regressed on to the basis functions for the paths where the option is in the money. This is done for all the exercise times for each path until starting time is reached. Then for every path the cash flows on the optimal stopping time are discounted to starting time and the average over all paths is estimated.

In this thesis only European options are considered and as the exercise time for European options equals to time of maturity T the cash flow for each path won't change with time, i.e. no need for recalculating the cash flows at each time step. A difficulty in derivative pricing is that in many cases payoff functions can be discontinuous at certain points but this not being the case for the payoff function of European options one can use it as a basis function and also due to the fact that the closer to maturity the more the value function resembles the payoff function. For the call option one has:

$$E^{\mathbb{Q}}[P(S_{t_{n+1}}) \mid S_{t_n}] \approx \sum_{k=1}^{N} a_{n,k} f_k(t_n, S_{t_n}) = \sum_{k=1}^{N} a_{n,k}(S_{t_n} - K_k)_+ \tag{48}$$

and for the put option,

$$E^{\mathbb{Q}}[P(S_{t_{n+1}}) \mid S_{t_n}] \approx \sum_{k=1}^{N} a_{n,k} f_k(t_n, S_{t_n}) = \sum_{k=1}^{N} a_{n,k}(K_k - S_{t_n})_+ \tag{49}$$

2.6.1 The numerical implementation stepwise

1. Simulating m trajectories for the price of the stock, where the price process:

   $$S_m(t) = S(0)e^{X_m(t)}$$

   is from the exponential Lévy process model and $\{X(t)\}_{t\geq 0}$ is a Lévy process. In this thesis the Lévy processes $X(t)$ being considered are Brownian motion and normal inverse Gaussian process.

2. Calculating the initial payoff for all trajectories $P(t_{n+1}, S_{t_{n+1}})$ at time t_{n+1},

3. Calculating the basis functions, $f_k(t_n, S_{t_n})$ which are the payoff functions $(S_{t_n} - K_n)_+$ or $(K_n - S_{t_n})_+$ and calculating the coefficients $a_{n,k}$ at time t_n, by regressing the discounted payoff function from the previous time step on to the basis functions.

4. Now one can retrieve an approximation of $P(t_n, S_{t_n})$ by the coefficient and basis functions calculated in step 3:

$$\hat{P}(t_n, S_{t_n}) = \sum_{k=1}^{N} a_{k,n} f_k(t_n, S_{(t_n)})$$

5. The basis functions are computed for t_{n-1} and the coefficients $a_{n-1,k}$ are calculated by regressing the discounted $\hat{P}(t_n, S_{t_n})$ on to the basis functions.

6. $\hat{P}(t_{n-1}, S_{t_{n-1}})$ can now be approximated.

7. The calculations in step 5 and 6 are done for every time step until starting time.

8. Once all option prices are retrieved and new stock prices are simulated, the optimal hedge is calculated through equation 41.

Note that the basis functions are calculated just once and can then be used for different simulations of stock prices. The price of the option is retrieved for all the trajectories which allow one to use the method on other derivatives with path-dependent and American-exercise features.

3 Numerical results

3.1 Background

I started out with implementing the method and investigated if I could replicate the price of an European option with the underlying asset price process following a GBM. To confirm that the method works and to evaluate how good it works the price and hedge were compared to those retrieved from B&S. I continued with modeling the asset price process as a NIG-process in order to see if other underlying asset price processes can be used in combination with the HMC method and also to see how well it performs, here the estimates were compared to the ones calculated by a Fourier option pricer.

3.2 Numerical results for the GBM-process

Initially the method was implemented for European options with a non-dividend paying stock as an underlying asset, which followed a GBM process and evaluated by comparison with the price and hedge obtained by the B&S formula. As for my analyses I am considered call options with the following values on the parameters:

1. Maturity: 1 month, 6 months and 1 year.
2. Strike price: 100.
3. The initial stock prices: 95, 105 and 115.
4. The volatility of the stock return: 20%, 40% and 80%.
5. The interest rate: 3%.

The evaluation is based on the number of basis functions n and simulations m needed to get good estimates of the price and hedge for the different initial stock prices, times to maturity and volatilities. Studies by Lars Stentoft (2003), express that both the number of simulations and the number of basis functions should tend to infinity in order to approximate the conditional expectation well, and the assumption is made that the same will be for the price estimate. Stentoft (2003) -pp 11. Even if better precision can be obtained by increasing the number of simulations one needs to take into account that it is payed in computational time, and thereof a trade-off between precision and computational time needs to be evaluated when looking at the price and at the hedge. As a higher number of basis functions improves the expected price it also leads to a higher variance of the coefficients in front of the basis functions, which affects the estimate of the hedge. The estimate of the hedge can be improved by increasing the number of simulations but then this will require additional computational time. So one needs to find the right combination of simulations and basis functions that will perform within a reasonable time and at the same time result in acceptable levels of bias and variance.

Initially the evaluation was based on the number of basis functions between 3 and 30 and for each choice simulations between 1 000 and 30 000. The results presented in Appendix A, Table 1, 4 and 7 are the least number of simulations contra basis functions needed in order to obtain good prices with the HMC method where good prices meaning option prices with less than 1% discrepancy from the true value, i.e. the B&S price, the price estimates and the B&S prices are presented in Table 2, 5 and 8. In almost all cases simulations between 10 000 and 20 000 and basis functions between 15 and 30 lead to good prices except for the low variance with 1 month and 6 months to maturity. Consistently the prices got better with the number of simulations increasing as expected but the aim is to find the least number of simulations in order to save time.

However even if it resulted in good price estimates one can't say the same thing about the hedges. Looking at the mean square error, (MSE) of the hedges represented in Table 3, 6 and 9 one can see that the errors are significantly bigger than the ones for B&S delta hedges and that in only one case is the level of the MSE acceptable, this for 20% volatility, 1 month to maturity and strike price 105. The hedging errors are more substantial for one year to maturity and increase as the volatility gets higher. Maybe I was a bit optimistic with the choices of simulations considering that 100 000 paths for the stock price process were used by Longstraff and Schwartz, Longstraff and Schwartz (2001) -pp 127.

Some performance issues where revealed when looking into the hedge for high volatility and 1 month to maturity and it got more distinctive as the volatility got lower, see Figure 1 and 2. The figures show that the hedges are systematically biased, the hedges are consistently underestimated before strike price and overestimated after. Even though the number of simulations and basis functions were enough for calculating good prices they weren't enough to obtain good hedges. To achieve good prices and good hedges one needs to significantly increase the number of simulations and basis functions which is payed in additional computational time.

In the next phase of my evaluation I considered number of basis functions up to 150 and for each choice simulations between 30 000 and 100 000. Results showed that when considering higher number of simulations the systematical bias is reduced considerably in those cases it had appeared during the first phase of the evaluation and the MSEs are significantly reduced in all cases, they are still not as low as the MSEs for B&S delta hedges but acceptable levels were achieved in more than one case. Tables 10, 12 and 14 in Appendix B shows the number of simulations and basis functions needed in

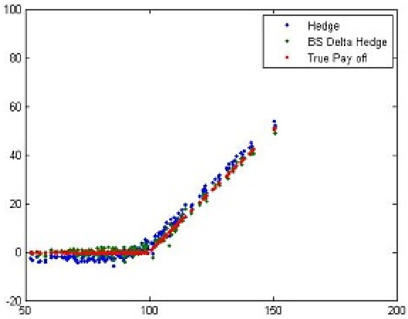

Figure 1: The hedges for HMC and B&S compared to the true pay off in phase 1.

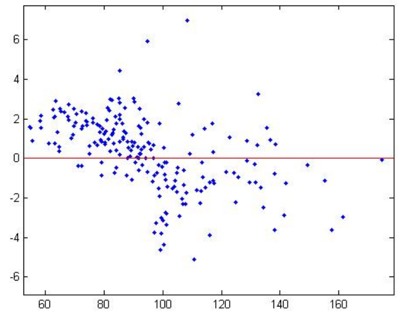

Figure 2: The hedge residuals for HMC in phase 1.

order to achieve significantly lower MSEs, whereas the MSEs are represented in Table 11, 13 and 15. In all cases the choice of number of simulations and basis functions was based on both the price estimate and the MSE for the hedge. The longest CPU time used by MATLAB when running the different cases was 826 seconds which is approximately 14 minutes. Very good price estimates where achieved through out all the different cases but the hedge estimates showed clearly that the method has performance issues when it comes to high variance and out of the money options. In these cases even when the number of simulations was increased to 200 000 a minor improvement of the MSEs were achieved but the levels were still too big.
Taking a closer look when the methods performance very good, i.e. when the option is in the money, close to maturity and the variance is low, meaning volatility at 40% or lower, one can see in Figure 3 and 4 that the HMC hedge follows the true pay off closely and the residuals are mostly within the (-1,1.5) interval except for one at -1.5 and another one at 2. Considering the hedging errors in a histogram as presented in Figure 5 one can see that it is normally distributed and most of the errors are in the interval of -1 to 1. The $VaR_{5\%}$ of - 1.227 shows that with 95% certainty a loss bigger than 1.227 won't occur. Put this against the price of the option at 17.596 one can conclude that this is a rather small loss.

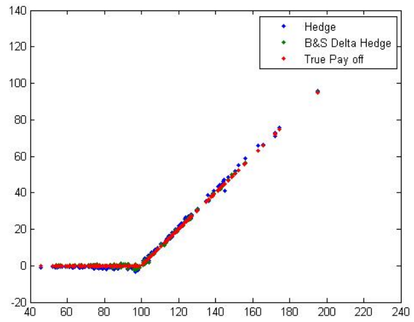

Figure 3: The hedges for HMC and B&S compared to the true pay off in phase 2.

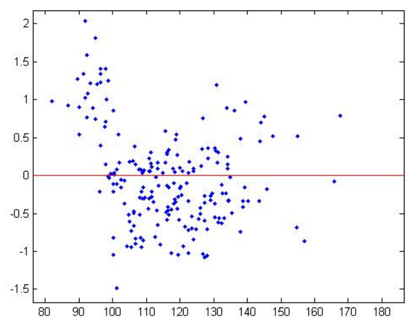

Figure 4: The hedge residuals for HMC in phase 2.

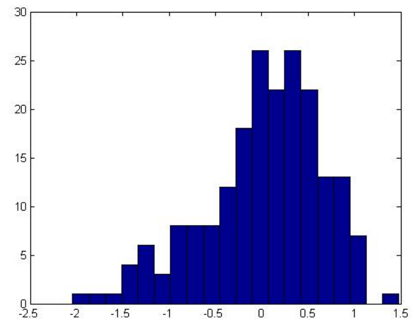

Figure 5: A histogram of the hedging errors in phase 2.

3.3 Numerical results for the NIG-process

By implementing the NIG process to model the asset price I seek to capture the type of movements of the underlying asset that we seen lately in the financial markets, i.e. outcomes considerably differing from the mean and to evaluate how well the HMC method works when a NIG process is being considered. Once again an European option with a non-dividend paying stock as an underlying asset is considered and evaluated by comparison with the price and hedge obtained by a Fourier option pricer, which uses Fourier transform techniques to price options.

For my analyses I considered options with the same values on the parameters as in the previous case:

1. Maturity: 1 month, 6 months and 1 year.
2. Strike price: 100.
3. The initial stock prices: 95, 105 and 115.
4. The interest rate: 3%.

As for the NIG-distribution I used the following estimates on my parameters.

1. The density, α : 37.
2. Skewness, β: -7.
3. The position of the density function, μ: 0,25.
4. Scaling, δ: 2.

The parameter values for the NIG-distribution is from a study by Erik Lindström, Lindström (2012) -pp 23.

The first phase of the evaluation for NIG is done in the same manner as for GBM i.e. I have considered the number of basis functions between 3 and 30 and for each choice simulations between 1 000 and 30 000. The results presented in Appendix C, Table 16 are the least number of simulations contra basis functions needed in order to obtain good option prices i.e. where the discrepancy from the true value is less than 1%. When considering the NIG process the true value of the option price is represented by the option price retrieved from a Fourier option pricer. The price estimates and the Fourier prices are presented in Table 17, where one can clearly see that simulations between 10 000 and 25 000 and basis functions between 5 and 20 leads to good prices except for in one case where the initial stock price is 95 and time to maturity is 1 month. The price estimates got better with the number of simulations increasing as expected but the goal is to find the least number of simulations in order to save computational time.

The conclusion of the results on the MSEs of the hedges in this phase is that the levels are not acceptable, see Table 18, except for in one case when considering a deep in the money option with 1 month to maturity. The size of the errors follows the same pattern as when considering GBM, i.e. the further the maturity is and the more out of the money the option is the bigger the errors are. Even though the levels of the MSEs weren't satisfactory, looking at the hedge estimates in comparison to the ones of Fourier and the true pay off there are no signs of systematic bias, see Figure 6. The hedges follow the shape of the true pay off and the residuals are evenly distributed around the zero level, see Figure 7.

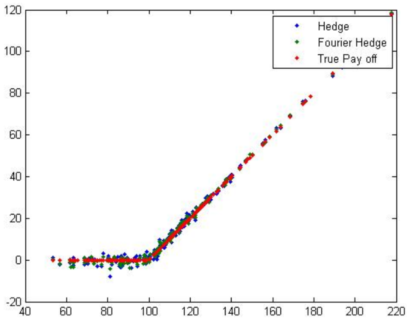

Figure 6: The hedges for HMC and Fourier compared to the true pay off in phase 1.

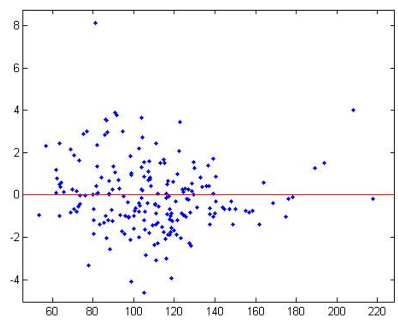

Figure 7: The hedge residuals for HMC in phase 1.

In the next phase the number of basis function goes up to 150 and for each choice simulations between 30 000 and 100 000 are done. The MSEs are reduced significantly and acceptable levels were achieved for the deep in the money options and close to maturity but still they are not as low as the MSEs of the Fourier method. For these computations the longest CPU time used by MATLAB was 394 seconds which is approximately 7 minutes. The results for the MSEs are presented in Appendix D Table 20 and the choices of number of simulations contra basis functions in Table 19.

Just like when the price function followed a GBM, good price estimates where retrieved quite effortlessly in all cases but even though the increase of simulations and basis functions lead to lower MSEs the results for NIG weren't acceptable, except for the deep in the money options with 1 month to maturity. The number of simulations and basis functions where increased in order to see if lower levels could be retrieved but it only resulted in very small improvements.

Considering the case when the method works, i.e. for deep in the money options and 1 month to maturity one gets quite good hedge estimates as represented in Figure 8 where the hedge estimates follows the shape of the true pay off very well and the residuals are mostly within the (-0.5, 0.5) interval, see Figure 9. In figure 10 the hedge errors are presented in a histogram where the errors are normally distributed within the (-0.75, 0.75) interval except for two values deviating out by 1.25 on the scale. The $VaR_{5\%}$ is - 0.482 which indicates that with 95% certainty one won't suffer a loss bigger than 0.482 which in relation to the price of the option at 15.2900 is quite small.

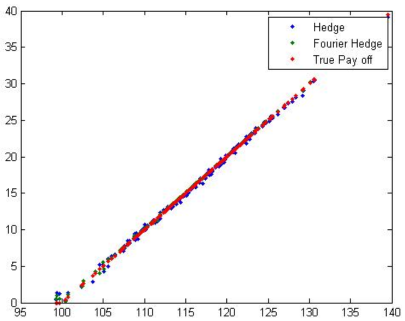

Figure 8: The hedges for HMC and Fourier compared to the true pay off with 1 month to maturity in phase 2.

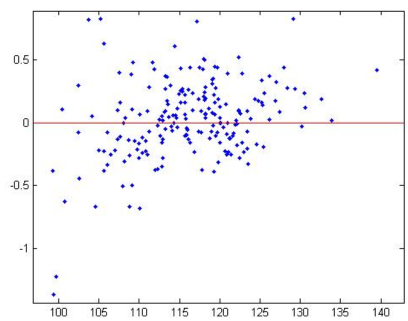

Figure 9: The hedge residuals for HMC with 1 month to maturity in phase 2.

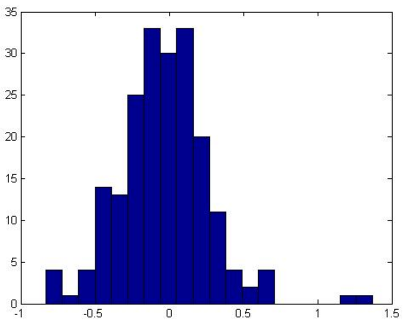

Figure 10: A histogram for the hedge errors with 1 month to maturity in phase 2.

By just increasing the time to maturity to 6 months one still gets good prices but the hedges are not as good, see Figure 11 and Figure 12. Considering the hedge errors in a histogram, Figure 13, one can see that the errors are in a much wider interval than one would have hoped for. The $VaR_{5\%}$ of - 1.941 compared to the price at 18,1940 isn't that high but still one would expect the amount to be a little bit smaller considering that it is an in the money option with 6 months to maturity.

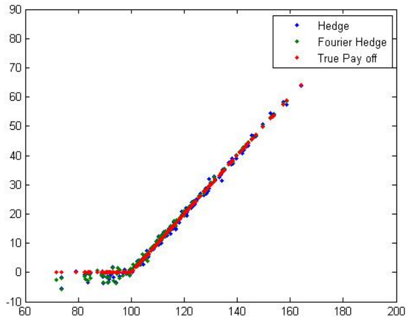

Figure 11: The hedges for HMC and Fourier compared to the true pay off with 6 months to maturity in phase 2.

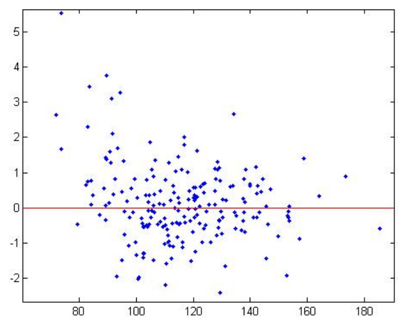

Figure 12: The hedge residuals for HMC with 6 months to maturity in phase 2.

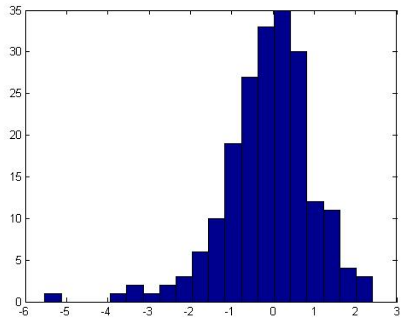

Figure 13: A histogram for the hedge errors with 6 months to maturity in phase 2.

4 Conclusion

Based on my evaluation, a thorough evaluation of the number of simulations contra basis functions needed in order to achieve good price and hedge estimates, I can say with confidence that the method results in very good price estimates and this goes for both when the price process of the underlying asset follows a GBM and a NIG process. The pricing ability of a method is of great importance in the options market where the trade volumes are so big that the values easily exceed millions of Eur's or US dollars, thereof a misprice bigger than 1% isn't acceptable.

From a hedging point of view when considering a GBM process the method performs best for in the money options with variance below 40% and the results gets better as one gets closer to maturity which is not a discouragement as the performance of the HMC method is most important for the options deepest in the money. At this stage the probability of exercise of an option is very high. As for the evaluation of the hedges when the underlying asset price process followed a NIG process, acceptable levels on the MSEs of the hedge estimates were achieved in fewer cases than for GBM. The method performed well from a hedging point of view only when it was a deep in the money option with less the 6 months to maturity. Even though fewer basis functions were needed in order to achieve good prices in the NIG case it didn't result in good hedges. The performance of the method is measured in both good prices and good hedges and thereof is the NIG process not as suitable to use as the GBM process with the HMC method.

When it came to computational time, I only made sure that the calculation was done within a couple of minutes i.e. I made sure that it could be done within reasonable time but a more meticulous evaluation has been done by Lars Stentoft where the results showed that when the number of stochastic factors is increased the method has a better trade-off between computational time and precision, thereof preferable opposed to the Binomial Model. Stentoft (2003) -pp 16.

The method isn't the optimal choice when considering high variance and out of the money options but I strongly believe that the method is of great importance due to it's other properties such as enabling pricing path dependent options and options with underlying assets of higher dimensions. Another desirable property in option trading is the possibility to reduce financial risk which the method enables by the optimal hedging strategy. This is also why I stand by the method even though The B&S delta hedge errors and the Fourier hedge errors where consistently smaller and thereof one can conclude that it's not the optimal choice for pricing European options. The method resulted in acceptable levels on the MSEs and it can be used to price other options when the B&S or Fourier method isn't an option.

5 Future work

It would be of great value to evaluate how well the HMC method performs when considering other type of options especially American options. Pricing American options by simulations is considered difficult due to Monte Carlo simulations going forward in time whereas pricing American options is done by going backwards this due to the probability of early exercise. It would also be interesting to investigate the American options hedge parameters which is considered an unexplored area.

The pricing problem in the HMC method is reduced to selecting a suitable set of basis functions by the assumption that the option value can be described by a linear combination of basis functions. Worth noting is that the precision of the option value one can get with increased computational time is based on the choice of basis functions. I thereof believe that looking into the choice of basis functions would be interesting, especially when other type of contracts, like American style options, are being considered.

Appendices

Appendix A: Results for phase 1 of the evaluation for GBM

Table 1: Number of simulations and basis functions for 20% volatility

Initial stock price	115	105	95
Time to maturity	m\n	m\n	m\n
1 month	20 000\20	80 000\50	80 000\150
6 months	15 000\30	15 000\30	20 000\30
1 year	15 000\30	15 000\30	15 000\20

Table 2: Prices for 20% volatility

Initial stock price	115	105	95
Time to maturity	Price\True Price	Price\True Price	Price\True Price
1 month	15.33\15.26	5.89\5.86	0.61\0.62
6 months	17.51\17.53	9.56\9.55	3.84\3.87
1 year	20.23\20.27	12.57\12.63	6.73\6.67

Table 3: MSE for 20% volatility

Initial stock price	115	105	95
Time to maturity	MSE\B&S MSE	MSE\B&S MSE	MSE\B&S MSE
1 month	0.07\0.01	0.79\0.06	0.34\0.07
6 months	0.92\0.09	1.22\0.10	1.26\0.11
1 year	1.42\0.12	1.69\0.14	2.46\0.13

Table 4: Number of simulations and basis functions for 40% volatility

Initial stock price	115	105	95
Time to maturity	m\n	m\n	m\n
1 month	20 000\25	15 000\25	20 000\25
6 months	15 000\20	20 000\25	20 000\25
1 year	15 000\25	15 000\25	15 000\20

Table 5: Prices for 40% volatility

Initial stock price	115	105	95
Time to maturity	Price\True Price	Price\True Price	Price\True Price
1 month	15.84\15.89	7.83\7.80	2.50\2.50
6 months	22.01\21.99	14.86\14.98	9.19\9.21
1 year	27.23\27.25	20.25\20.29	14.25\14.22

Table 6: MSE for 40% volatility

Initial stock price	115	105	95
Time to maturity	MSE\B&S MSE	MSE\B&S MSE	MSE\B&S MSE
1 month	0.76\0.17	1.83\0.50	2.36\0.47
6 months	2.78\0.44	2.99\0.43	3.44\0.60
1 year	4.37\0.61	3.84\0.62	3.95\0.48

Table 7: Number of simulations and basis functions for 80% volatility

Initial stock price	115	105	95
Time to maturity	m\n	m\n	m\n
1 month	20 000\20	10 000\20	20 000\15
6 months	10 000\15	15 000\20	15 000\25
1 year	15 000\25	20 000\15	10 000\15

Table 8: Prices for 80% volatility

Initial stock price	115	105	95
Time to maturity	Price\True Price	Price\True Price	Price\True Price
1 month	19.33\19.30	12.23\12.27	6.71\6.77
6 months	33.05\32.88	26.20\26.05	19.70\19.84
1 year	42.31\42.63	35.34\35.52	28.68\28.83

Table 9: MSE for 80% volatility

Initial stock price	115	105	95
Time to maturity	MSE\B&S MSE	MSE\B&S MSE	MSE\B&S MSE
1 month	5.07\1.89	4.26\2.07	5.04\1.78
6 months	16.00\2.01	8.44\1.98	9.11\2.09
1 year	26.99\1.92	15.10\2.00	15.12\1.97

Appendix B: Results for phase 2 of the evaluation for GBM

Table 10: Number of simulations and basis functions for 20% volatility

Initial stock price	115	105	95
Time to maturity	m\n	m\n	m\n
1 month	90 000\45	100 000\100	100 000\120
6 months	90 000\60	100 000\90	90 000\90
1 year	90000\30	80 000\65	100 000\65

Table 11: MSE for 20% volatility

Initial stock price	115	105	95
Time to maturity	MSE\B&S MSE	MSE\B&S MSE	MSE\B&S MSE
1 month	0.014\0.006	0.21\0.08	0.14\0.08
6 months	0.36\0.07	0.44\0.11	0.29\0.11
1 year	0.58\0.09	0.58\0.10	0.45\0.16

Table 12: Number of simulations and basis functions for 40% volatility

Initial stock price	115	105	95
Time to maturity	m\n	m\n	m\n
1 month	100 000\45	100 000\65	100 000\80
6 months	90 000\50	100 000\55	100 000\50
1 year	100 000\45	80 000\40	100 000\50

Table 13: MSE for 40% volatility

Initial stock price	115	105	95
Time to maturity	MSE\B&S MSE	MSE\B&S MSE	MSE\B&S MSE
1 month	0,56\0.29	0.79\0.44	0.56\0.35
6 months	1.47\0.49	1.26\0.59	1.15\0.50
1 year	1.75\0.58	1.83\0.55	1.18\0.54

Table 14: Number of simulations and basis functions for 80% volatility

Initial stock price	115	105	95
Time to maturity	m\n	m\n	m\n
1 month	100 000\45	100 000\45	100 000\50
6 months	100 000\40	100 000\45	90 000\40
1 year	100 000\45	100 000\45	90 000\45

Table 15: MSE for 80% volatility

Initial stock price	115	105	95
Time to maturity	MSE\B&S MSE	MSE\B&S MSE	MSE\B&S MSE
1 month	2.92\1.78	3.02\2.27	2.36\1.66
6 months	5.88\2.35	4.79\1.91	3.84\1.68
1 year	8.77\1.99	6.68\1.95	7.33\2.15

Appendix C: Results for phase 1 of the evaluation for the NIG process

Table 16: Number of simulations and basis functions

Initial stock price	115	105	95
Time to maturity	m\n	m\n	m\n
1 month	15 000\15	15 000\20	60 000\60
6 months	20 000\15	20 000\15	25 000\20
1 year	10 000\5	25 000\20	25 000\20

Table 17: Price estimates

Initial stock price	115	105	95
Time to maturity	Price\True Price	Price\True Price	Price\True Price
1 month	15.43\15.35	6.19\6.21	0.88\0.87
6 months	18.13\18.28	10.54\10.56	4.85\4.84
1 year	21.49\21.50	13.95\14.08	8.03\8.09

Table 18: MSE

Initial stock price	115	105	95
Time to maturity	MSE\Fourier MSE	MSE\Fourier MSE	MSE\Fourier MSE
1 month	0.63\0.18	1.92\1.61	0.78\0.60
6 months	2.64\0.71	4.85\2.09	2.55\1.55
1 year	19.79\1.54	3.16\1.47	3.16\1.93

Appendix D: Results for phase 2 of the evaluation for the NIG process

Table 19: Number of simulations and basis functions

Initial stock price	115	105	95
Time to maturity	m\n	m\n	m\n
1 month	60 000\30	60 000\60	40 000\100
6 months	100 000\60	100 000\60	100 000\60
1 year	60 000\30	100 000\60	100 000\60

Table 20: MSE

Initial stock price	115	105	95
Time to maturity	MSE\Fourier MSE	MSE\Fourier MSE	MSE\Fourier MSE
1 month	0.12\0.06	1.03\0.96	0.52\0.52
6 months	1.43\1.05	1.84\1.47	1.80\1.56
1 year	2.17\1.25	1.63\0.89	1.73\1.22

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