MSc Thesis in Finance

# American Options on Commodities Under Stochastic Convenience Yield and Stochastic Volatility

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# Abstract

American and Bermudan options have a wide range of applications in financial markets, e.g. in commodities markets among others. The pricing literature of such contingent claims is broad and many different algorithms and frameworks have been developed. The purpose of this thesis is to investigate how the Least-Squares Method (LSM), [11], can be extended to incorporate stochastic convenience yield and stochastic volatility in the pricing algorithm by using a commodity underlying. Moreover, the thesis aims to investigate the impact of stochastic convenience yield and stochastic volatility on the early exercise premium (EEP) of the American option written on a commodity. The results show that only the convenience yield increases the price of the American option. While, volatility does not add any edge to the algorithm when it is used as regressor. The insertion of the convenience yield increases the EEP especially for deep in the money options and long time span contracts. Lastly, the power polynomial specification shows better performances than the Laguerre one.

**Key words:** American option, Least-Squares Method, commodity, stochastic convenience yield, stochastic volatility, early exercise premium.

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# Chapter 1

# Introduction

## 1.1 Preface

Since Black and Scholes (BS) valuation framework was developed in 1973, a multitude of literature has been written about contingent claims at large. Any kind of contract has been modeled mathematically in order to compute its intrinsic value. Almost surely, options are among the most studied and used contingent claims in the financial world. An extensive effort has been given in the valuation of European and American options under different model specifications. Accordingly, this thesis aims to investigate the pricing of an American option under a stochastic convenience yield and stochastic volatility (SCYH) model of an underlying asset, i.e. a commodity price. Furthermore, the thesis investigates the impact of the SCYH model on the EEP of the American option, in order to underline economics insights behind the pure mathematics.

As it is well known, an American option does not have a closed form solution as the case of an European option in the BS market. Even in the simplest frameworks, American options require numerical approximations of the unknown contingent claim price. These approximations are usually computationally intensive and challenging. Some of the most used methodologies can be recalled in the following lines.

To start with, the most traditional one is the finite difference method which solves the partial difference equation (PDE) of the contingent claims at hand numerically, see [9]. Then, there are simulation based methods for computing American options, such as: random tree methods, state space partitioning, stochastic mesh methods, regression based methods and so on, see [7]. Among the regression based methods, the LSM, see [11], is almost surely the most popular one for its simplicity and robustness. However, the LSM has been already improved by more advanced methodologies such as: parametric and nonparametric different types of regression, see [10], stochastic grid method, see [4], and stochastic grid bundling method, see [5], where the latter two base their improvement on the use of the law of iterated expectations. Moreover, analytical approximations have been suggested, see [3].

Even if the LSM has already been extended in different ways, this thesis uses the plain vanilla LSM, [11], to price American options. Then, the stochastic convenience yield and stochastic volatility state variables are inserted in the pricing algorithm to improve the performance of the LSM. This is done only from a numerical perspective rather than theoretically as well. After that, when an acceptable algorithm setup is reached, an analysis on the EEP is performed.

The thesis is organized as follows: there is a quick review of the general pricing methodology of contingent claims and explanation of the LSM algorithm. Then the commodity model is specified, i.e. SCYH model, and the simulation methodology is explained. After that, a basis function analysis is carried out to find potential improvements to the LSM. Lastly, an EEP analysis is carried out focusing on the best algorithm specifications of the basis function analysis previously performed.

## 1.2 Pricing Contingent Claims

The aim of this section is to recall the main points of derivative pricing and to underline the intuition of them. The entire pricing literature is based on arbitrage theory which is for a large portion developed in continuous time. Therefore, the starting point is to well define what an arbitrage opportunity is.

Assume that it is possible to construct a portfolio $h$ with value $V_h(t)$ at time $t$. An arbitrage opportunity or strategy implies the following conditions:

$$V_h(t_0) = 0 \tag{1.1}$$

$$\mathbb{P}(V_h(t_1) > 0) > 0 \tag{1.2}$$

$$\mathbb{P}(V_h(t_1) \geq 0) = 1 \tag{1.3}$$

Where $t_0 < t_1$.

Conditions 1.1-1.3 summarize the entire concept behind the contingent claim literature. In other words, they mean that given a strategy $h$ such that the initial investment is zero at time $t_0$, it is possible to make profit with a probability grater then zero and there is no possibility of making a loss at time $t_1$. Basically, this is the well-known free lunch that should not exit at wall-street. Given such conditions, the task is to price contingent claims in such a way that there is no possibility of arbitrage, i.e. there is no free lunch. Basically, we want to create a framework that is arbitrage free.

### Black and Scholes Market

Given the aforementioned arbitrage conditions, the presentation of BSs' grate idea is fairly straight forward. Assume the BSs' market:

$$dB(t) = rB(t)\,dt \tag{1.4}$$

$$dS(t) = S(t)\,\mu_S dt + S(t)\,\sigma_S dW(t)^{\mathbb{P}} \tag{1.5}$$

Where $B(t)$ is the riskless bank account and $S(t)$ is the underlying asset price modeled as a Geometric Brownian Motion (GBM) w.r.t. $\mathbb{P}$ measure. Further, assume the existence of a traded T-contingent claim $\Pi(t)$, with payoff function $\Phi(S(T))$, that has to be priced in absence of arbitrage. Through Ito's lemma the dynamic of $\Pi(t)$ is derived as a function of $S(t)$.

$$d\Pi(t) = \Pi(t)\,\mu_\Pi dt + \Pi(t)\,\sigma_\Pi dW(t)^{\mathbb{P}} \tag{1.6}$$

At this point a self-financing portfolio, $V$, made up of the $S(t)$ and $\Pi(t)$ can be constructed with relative weights $\omega_S$ and $\omega_\Pi$, respectively. The dynamic of this portfolio is given as:

$$dV(t) = V(t)\,(\omega_S\mu_s + \omega_\Pi\mu_\Pi)\,dt + V(t)\,(\omega_S\sigma_S + \omega_\Pi\sigma_\Pi)\,dW(t)^{\mathbb{P}} \tag{1.7}$$

At this point the Nobel Prize idea becomes, les't define $\omega_S^*$ and $\omega_\Pi^*$ as the weights that make the self-financing portfolio locally risk free. In other words, this implies $dV(t) = V(t)\,\psi dt$ with $\psi = \omega_S^*\mu_s + \omega_\Pi^*\mu_\Pi$. After that, If for instance $\psi > r$, then a $h$ strategy made up of shorting the bank account at a rate $r$ and going long in the portfolio $V$ at a rate $\psi$ would lead to an arbitrage opportunity. To avoid this type of scenarios the drift of the portfolio $V$ must be equal to the risk free rate to guarantee absence of arbitrage.

$$\omega_S^*\mu_s + \omega_\Pi^*\mu_\Pi = r \tag{1.8}$$

Form equation 1.8 the well-known PDE can be derived and the contingent claim price can be computed by the $Feynman - Ka\check{c}$ representation formula, which exploit the fact that Ito's integral is a random variable with expected value equal to zero.

### Risk neutral valuation formula

The replicating portfolio technic developed by BS is elegant though difficult to use in d-dimensional problems, $d > 1$, or in incomplete markets, e.g. the Heston model, see [8]. A more general setting for pricing contingent claims is the risk neutral valuation formula defined as:

$$\Pi(t) = \mathbb{E}^{\mathbb{Q}}\left[\frac{B(t)}{B(T)}\Phi(S(T))\,|\mathcal{F}_t\right] \tag{1.9}$$

Where, $\Pi(t)$ is a T-contingent claim, $\mathbb{E}^{\mathbb{Q}}[\cdot|\mathcal{F}_t]$ is a conditional expectation w.r.t. the equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ given the information set $\mathcal{F}_t$ at time $t$, and the other objects are defined as before. Recall that the $\mathbb{Q}$ measure is unique in complete markets, e.g. BS market, whereas it is not unique in incomplete markets, e.g. Heston market. However, if there exist at least one equivalent probability measure $\mathbb{Q}$ such that the discounted traded asset is a martingale, then there is absence of arbitrage.

The risk neutral formula can be even extended to a more general setting such as:

$$\Pi(t) = \mathbb{E}^{\mathbb{N}}\left[\frac{N(t)}{N(T)}\Phi(S(T))\,|\mathcal{F}_t\right] \tag{1.10}$$

Where $\mathbb{N}$ is martingale measure such that $\mathbb{N} \sim \mathbb{P}$ and $N$ is the corresponding numeraire. Usually the numeraire is chosen wisely in order to reduce a d-dimensional problem to a smaller dimensional integral, e.g. in the case of a derivative depending on multiple assets.

So far only T-contingent claims have been considered, which means that they can be only exercised at expiration day. In the case of an American style option the problem at hand becomes way more complex. The issue now is to find an optimal stopping time, $\tau^*$ with $t \leq \tau^* \leq T$, such that the value of the option is the supremum among the possible values. *Glasserman* and *Björk* have an extensive discussion about such issues and the thesis sends the reader to see the following references for a deep treatment of the topic, [7] and [1].[1]

However, a general risk neutral valuation formula for an American type option can be defined as:

$$\Pi(t)_A = \sup_{\tau \in \mathcal{Y}} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^\tau r(z)dz}\Phi(S(\tau))\,|\mathcal{F}_t\right] \tag{1.11}$$

Where $\mathcal{Y}$ is the set of possible exercise possibilities in the time interval $[t, T]$. Note that the risk free rate can be even stochastic in the above formula. Basically, the problem is to determine when it is optimal to exercise the option or not, i.e. find $\tau^*$. The LSM algorithm solves such a task by using ordinary least squares (OLS) to compute the decision rule in order to choose if to keep the option alive or to exercise it. The LSM is explained in the following chapter.

From now on, we define $\Pi(t)_E$ as the T-contingent claim evaluated with equation 1.9, e.g. the European option. While, the $\Pi(t)_A$ as the American style option evaluated with equation 1.11, e.g. a typical American option with endless exercise possibilities in the time span $[t, T]$.

[1]Note that the entire Pricing Contingent Claims section is heavily based on these two references, which are milestones of the literature.

# Chapter 2

# Least Squares Method

The previous section pointed out the main features of derivative pricing at large. Anyhow, as mentioned earlier, one of the main focus of the thesis is about pricing American options through the LSM of *Longstaff* and *Schwartz*, [11]. Thus, the aim of this chapter is to present their algorithm form a practical and intuitive perspective and to compare the obtained results with those in the original paper, [11].

The main issue in computing the price of an American option is to define if it is optimal to exercise or if it is more convenient to hold the contingent claim longer, maybe until the final day $T$. Basically, the problem is to define among the in the money (ITM) payoffs which of them are worth exercise and which are not. In other words, we need to know if the future payoffs are expected to be higher or lower than exercising the option immediately. In order to accomplish the aforementioned task, the LSM uses a dynamic programming approach which solves the problem by backward induction.

## 2.1 LSM Algorithm

Assume a general stochastic process of an underlying asset $S$ defined as:

$$dS(t) = \mu(t, S(t))\, dt + \sigma(t, S(t))\, dW(t)^{\mathbb{Q}} \tag{2.1}$$

Generate with Monte Carlo simulation n paths and m steps ahead in time. Furthermore, assume for simplicity that the number of steps is the same as the number of exercise possibilities implied by the American style contract, i.e. a Bermudan option, which has payoff function $\Phi(S)$. Note that equation 2.1 can be simulated by a closed form solution of the SDE as well as by a discretization scheme depending on the model at hand. Nonetheless, in the latter case the number of steps ahead in time must be large even if the number of exercise possibilities is relatively low. This is due to the fact that we want to minimize the discretization error of the numerical approximation of the SDE.

After that, we can start the backward induction procedure.

> Define a variable $\Psi$ as the discounted payoffs from time $T$ to $T-1$.
>
> Define $\mathcal{I}\,(u)$ as the set of paths ITM at time $u$.
>
> for $j = 1$ to $m-1$
>
> > $X = f(\phi(S(T-j))) \in \mathcal{I}\,(T-j)$
> >
> > $Y = \Psi \in \mathcal{I}\,(T-j)$
> >
> > $\hat{\beta} = (X'X)^{-1}X'Y$
> >
> > $\mathbb{E}\,[Y|X] = X\hat{\beta}$
> >
> > if $\Phi\,(S(T-j))_k \geq E\,[Y|X]_k \Rightarrow \Psi_k = \Phi\,(S(T-j))_k$, with $k \in \mathcal{I}\,(T-j)$
> >
> > $\Psi = \Psi$ discounted from $T-j$ to $T-j-1$;
>
> $\Pi\,(t)_A = \frac{1}{n}\sum_{i=1}^{n}\Psi_i$

The matrix $X$ is defined as a function, $f\,(\cdot)$, of a set of regressors $\phi\,(\cdot)$. For instance, $f\,(\cdot)$ can be a constant, $c$, and a set of polynomials (basis functions), e.g. weighted Laguerre polynomial, $l_q\,(S)$, power polynomial, $p_q\,(S)$, or others, see [11].

$$l_q\,(S) = e^{-\frac{S}{2}}\frac{e^S}{q!}\frac{d^q}{dS^q}\left(S^q e^{-S}\right) \tag{2.2}$$

$$p_q\,(S) = S^{q+1} \tag{2.3}$$

The number of basis functions in the regression ranges from 0 to $\mathcal{Q} < \infty$. Actually, the choice of the polynomial and the number of polynomials is fundamental. As *Longstaff* and *Schwartz* proved in their paper, [11], the number of polynomials should be increased as long as the value of the American option increases. This is due to the fact that the LSM approaches the unknown value of the contingent claim form below. Basically, the specification of the conditional expectation of the payoffs, given the underlying asset price, is defined as follows, for the weighted Laguerre polynomial:

$$L_{\mathcal{Q}}\,(S) \equiv \mathbb{E}\,[Y|X]_{L_{\mathcal{Q}}(S)} = c + \sum_{q=0}^{\mathcal{Q}} l_q\,(S)\,\hat{\beta}_q \tag{2.4}$$

While with the power polynomial:

$$P_{\mathcal{Q}}\,(S) \equiv \mathbb{E}\,[Y|X]_{P_{\mathcal{Q}}(S)} = c + \sum_{q=0}^{\mathcal{Q}} p_q\,(S)\,\hat{\beta}_q \tag{2.5}$$

From now on the notation $L_{\mathcal{Q}}(S)$ or $P_{\mathcal{Q}}(S)$ means that a constant and the first $\mathcal{Q}+1$ basis functions of the Laguerre or power polynomials are used to compute $\mathbb{E}[Y|X]$, respectively. Note that $\beta$s are estimated with the closed form solution of OLS. However, the $\beta$s could be estimated with others estimation methods such as generalized method of moments (GMM).

In Appendix A.1, under LSM comparison, there is a comparison between the results of the LSM algorithm reported by *Longstaff* and *Schwartz*, in [11], and the one obtained by the author of the thesis. As it can be seen in the Appendix, the codded thesis algorithm performs really well, compared to the original one. Basically, the obtained results of the contingent claim prices are the same. One the other hand, a small bug in the original paper was detected in the use of antithetic variates.

To conclude, note that all the calculations in the entire thesis were done in MATLAB. All the codes were made by the author of the thesis and no predefined functions in MATLAB were used. Of course, all the thesis codes can be provided upon request.

# Chapter 3

# Commodity model

## 3.1 SCYH model

There are several commodity models that have been suggested to capture the features of the commodity markets. For instance, the *Schwartz-Smith* model, see [13], is one of the most famous, which has two latent processes driving the asset, one for the short run deviation and the other for the long run equilibrium. Other examples are the *Gibson-Schwartz* and the *E. Schwartz* models, which have stochastic convenience yield but constant volatility of the commodity, see [6] and [12]. A more recent commodity model application was made by *James S. Doran et al.*, see [2], who assume a Heston stochastic volatility process for commodities, but not stochastic convenience yield. Nonetheless, as it is well known in the literature, commodities markets present contango as well as backwardation and their volatility is not constant over time. As a result, the thesis considers a stochastic convenience yield and stochastic volatility model (SCYH) defined as follows:

**SCYH model**

$$dS\left(t\right) = S\left(t\right)\left(r - \delta\left(t\right)\right)dt + S\left(t\right)\sqrt{V\left(t\right)}dW\left(t\right)_S^{\mathbb{Q}} \tag{3.1}$$

$$d\delta\left(t\right) = k_\delta\left(\theta_\delta - \delta\left(t\right)\right)dt + \sigma_\delta dW\left(t\right)_\delta^{\mathbb{Q}} \tag{3.2}$$

$$dV\left(t\right) = k_V\left(\theta_V - V\left(t\right)\right)dt + \sigma_V\sqrt{V\left(t\right)}dW\left(t\right)_V^{\mathbb{Q}} \tag{3.3}$$

Where $S\left(t\right)$ is the commodity price, $r$ is the constant risk free rate, $\delta\left(t\right)$ is the stochastic convenience yield and $V\left(t\right)$ is the stochastic variance process. The convenience yield is modeled with a typical Ornstein Uhlenbeck process, with speed of mean reversion $k_\delta$, long term mean $\theta_\delta$ and diffusion $\sigma_\delta$. Basically, the stochastic convenience yield, equation 3.2, is modelled as by *E.*

*Schwartz*, in [12]. The volatility is modeled as the Heston model, [8], or equivalently as *James S. Doran et al.*, [2], with speed of adjustment $k_V$, long term mean $\theta_V$ and volatility of volatility $\sigma_V$.

The three Brownian Motions (BMs) are w.r.t. the $\mathbb{Q}$ measure for pricing purpose. They have correlation matrix defined as:

$$\mathbb{COOR}\left[d\tilde{W}\right] = \begin{bmatrix} 1 & \rho_{S\delta} & \rho_{SV} \\ \rho_{S\delta} & 1 & \rho_{\delta V} \\ \rho_{SV} & \rho_{\delta V} & 1 \end{bmatrix} \tag{3.4}$$

Where $d\tilde{W} = \left[dW\left(t\right)_S^{\mathbb{Q}}, dW\left(t\right)_\delta^{\mathbb{Q}}, dW\left(t\right)_V^{\mathbb{Q}}\right]'$. Therefore, the covariance is defined as:

$$\mathbb{COV}\left[d\tilde{W}\right] = dt \cdot \mathbb{COOR}\left[d\tilde{W}\right] \tag{3.5}$$

The SCYH model allows for contango as well as backwardation. Where, in this thesis, the former is defined as: the current spot price is below its future expected value, w.r.t. the $\mathbb{Q}$ measure. While the latter is defined as: the current spot price is above its future expected value, w.r.t. the $\mathbb{Q}$ measure. These features are commonly seen in commodities markets due to the convenience yield, i.e. the benefit of holding the commodity, see [9]. For instance, if the long term mean of the stochastic convenience yield is below the risk free rate, then the simulated market will present a contango, in its overall distribution. On the other hand, if the long term mean of the convenience yield excides the risk free rate, then the commodity distribution will have expected value below the initial spot price, i.e. the market will be in backwardation.

Furthermore, the SCYH model allows for stochastic volatility as well as for inverse leverage effect. The latter feature is commonly seen in commodities markets, which means that volatility increases while the commodity price rises, i.e. $\rho_{S\delta} > 0$. This feature is exactly the opposite of what is seen in equity markets. For further information on inverse leverage effect see [2].

The SCYH model will be used to price American options on this general commodity price, $S\left(t\right)$, by using the LSM. Hence, the simulation of the stochastic differential equations (SDEs) 3.1-3.3 is a fundamental part of the pricing process.

**Euler and Milstein schemes**

Due to a lack of closed form solutions of general SDEs, the Euler and Milstein schemes are used to discretize SDEs and to obtain accurate numerical approximations of unknown closed solutions. Let's explain how this schemes work.

Assume a general stochastic process $\mathcal{X}\left(t\right)$ that has the following SDE:

$$d\mathcal{X}(t) = a(\mathcal{X}(t))\,dt + b(\mathcal{X}(t))\,dW(t) \tag{3.6}$$

If we want to discretize it in a time span from zero to a final point $T$ with $s$ subintervals, we can do it by defining $\Delta t = \frac{T}{s}$ and $\Delta W(t_j) = \sqrt{\Delta t}\epsilon_j$ with $\epsilon_j \sim N(0,1)$ $i.i.d$ standard normal variable for $j \in [1, s]$. Then by assuming an initial condition $\mathcal{X}(t_0) = x(t_0)$, we can simulate recursively $\mathcal{X}(t_j)$ for $j = 1$ to $s$ as follows:

**Euler scheme**

$$\mathcal{X}(t_{j+1}) = \mathcal{X}(t_j) + a(\mathcal{X}(t_j))\,\Delta t + b(\mathcal{X}(t_j))\,\Delta W(t_{j+1}) \tag{3.7}$$

**Milstein schemes**

$$\begin{aligned}\mathcal{X}(t_{j+1}) = {} & \mathcal{X}(t_j) + a(\mathcal{X}(t_j))\,\Delta t + b(\mathcal{X}(t_j))\,\Delta W(t_{j+1}) \\ & + \frac{1}{2} b(\mathcal{X}(t_j))\, b'(\mathcal{X}(t_j)) \left( (\Delta W(t_{j+1}))^2 - \Delta t \right)\end{aligned} \tag{3.8}$$

Note that $b'(\mathcal{X}(t_j))$ is the first derivative w.r.t. the state variable $\mathcal{X}(t_j)$. As $s \to \infty$, the discretization improves in precision for both of the schemes, till reaching the true solution of the SDE theoretically. The Milstein scheme is usually more precise due to the fact that it is a stochastic second order Taylor expansion.

**SCYH model discretization**

In the case of the SCYH model, the equations 3.1-3.3 can be discretized as follows:

$$\begin{aligned}S(t_{j+1}) = {} & S(t_j) + S(t_j)(r - \delta(t_j))\,\Delta t + S(t_j)\sqrt{V(t_j)}\Delta W(t_{j+1})_S^{\mathbb{Q}} \\ & + \frac{1}{2} S(t_j)\, V(t_j) \left( \left( \Delta W(t_{j+1})_S^{\mathbb{Q}} \right)^2 - \Delta t \right)\end{aligned} \tag{3.9}$$

$$\delta(t_{j+1}) = \delta(t_j) + k_\delta(\theta_\delta - \delta(t_j))\,\Delta t + \sigma_\delta \Delta W(t_{j+1})_\delta^{\mathbb{Q}} \tag{3.10}$$

$$\begin{aligned}V(t_{j+1}) = {} & V(t_j) + k_V(\theta_V - V(t_j))\,\Delta t + \sigma_V \sqrt{V(t_j)}\Delta W(t_{j+1})_V^{\mathbb{Q}} \\ & + \frac{1}{4}\sigma_V^2 \left( \left( \Delta W(t_{j+1})_V^{\mathbb{Q}} \right)^2 - \Delta t \right)\end{aligned} \tag{3.11}$$

Note that only the commodity price and the stochastic volatility processes use the Milstein scheme. Albeit the convenience yield SDE was discretized with the Milstein scheme, the approximation would reduce to the Euler scheme. This is due to the fact that the diffusion term of the stochastic convenience yield process does not depend on the convenience yield itself. Thus, the discretization of the latter cannot exploit the famous quadratic variation feature of the Brownian motion. Nonetheless, note that the convenience yield process could have been simulated explicitly, i.e. with a closed form solution, but in order to be consistent with the other two SDEs, it was chosen to use the Euler scheme.

A Cholesky decomposition of the BMs' covariance matrix, equation 3.5, is carried out in order to make the three *i.i.d.* standard normally distributed random variables of the BMs correlated to each other as specified in equation 3.4. Basically, assume a general covariance matrix $A$ dimension $n$ by $n$, e.g. equation 3.5, and a vector $\tilde{\epsilon} = [\epsilon_1, ..., \epsilon_n]'$ where $\epsilon_i \sim N(0,1)$ and the different $i$ are *i.i.d* with $i \in [1, n]$. In order to make the elements of $\tilde{\epsilon}$ correlated to each other, decompose $A$ as $A = LL'$ where $L$ is the lower triangular matrix of the Cholesky decomposition, then a vector of correlated random variable is defined as $\eta = L \cdot \tilde{\epsilon}$.

*Glasserman* has an extensive explanation of Milstein and Euler scheme as well as cholesky decomposition, i.e. for further information about such issues see [7].

# Chapter 4

# Methodology of Analysis

In this chapter, there is a general explanation of the methodology adopted to investigate the following questions, which are the kernel of the thesis:

- Which type of basis function is more suitable to price American options between the Laguerre and the power polynomial, with and without convenience yield and/or stochastic volatility as regressors?
- Can the LSM yields a higher price of the American option, if a stochastic convenience yield and/or stochastic volatility is inserted in the pricing algorithm?
- Which variable matters the most, in the computation of the conditional expectation of the payoffs in the backward induction of the LSM, between the convenience yield and stochastic volatility?

These questions will be investigated numerically in the following chapter. Nonetheless, a general explanation of the used methodology is presented in the following section. This will become useful as a general picture of the numerical analysis, which will state the used methodology step by step when presented anyway.

To conclude, the following methodology is applied to four different information sets: $\mathcal{A}_\sigma$, $\mathcal{A}_k$, $\mathcal{A}_\theta$ end $\mathcal{A}_\rho$ in chapter 5.

## 4.1 Methodology

**Laguerre-power polynomial analysis**

The starting point of the analysis is to provide a general information set $\mathcal{A}_{\mathcal{H}}$, which represents a vector of SCYH model parameters[1], called $\tilde{\theta}$, and the contract specifications (CS) of the American option. It is worth pointing out that $\tilde{\theta}$ does not need to incorporate all the parameters of the SCYH model, but only a subset of them. Usually the complement of $\tilde{\theta}$ is represented by $\mathcal{H}$, unless stated differently[2]. Likewise, $\mathcal{A}_{\mathcal{H}}$ does not need to incorporate all the necessary CS of the American option. Fundamentally, we can defined the complement of $\mathcal{A}_{\mathcal{H}}$ as $\bar{\mathcal{A}}_{\mathcal{H}}$. Basically, $\bar{\mathcal{A}}_{\mathcal{H}}$ represents all the information needed to price an American option with the SCYH model that are not included in $\mathcal{A}_{\mathcal{H}}$.

Assume that $\mathcal{A}_{\mathcal{H}}$ is given and fixed with some values and CS of the American option. Then, we can also define $\bar{\mathcal{A}}_{\mathcal{H}}$ as $\bar{\mathcal{A}}_{\mathcal{H}_1}$ with some chosen values for the parameters and CS. At this point, it is possible to generate a number of paths of the SCYH model by Monte Carlo simulation. Let's define this latter object as $Paths_{\bar{\mathcal{A}}_{\mathcal{H}_1}}$. The same generated paths can be used to price the same American option but with different conditional expectations of the payoffs in the backward induction of the LSM, i.e. $\mathbb{E}\left[Y|X\right]$. In this thesis, $\mathbb{E}\left[Y|X\right]$ can be either defined as $\mathbb{E}\left[Y|X\right]_{L_{\mathcal{Q}}(S)}$ or as $\mathbb{E}\left[Y|X\right]_{P_{\mathcal{Q}}(S)}$ with $\mathcal{Q} = 0, ..., \vartheta$, where $\vartheta < \infty$.

Assume that we have priced $2 \cdot (\vartheta + 1)$ American options. Half of them are priced by using the Laguerre polynomial expectation by increasing $\mathcal{Q}$, i.e. $\mathbb{E}\left[Y|X\right]_{L_{\mathcal{Q}}(S)}$, and the other half is priced by the power polynomial expectation by increasing $\mathcal{Q}$, i.e. $\mathbb{E}\left[Y|X\right]_{P_{\mathcal{Q}}(S)}$. Note that the $2 \cdot (\vartheta + 1)$ options were priced by using the same $Paths_{\bar{\mathcal{A}}_{\mathcal{H}_1}}$. Let's store the $\vartheta + 1$ American options priced by the Laguerre polynomial in the first row of a matrix $\mathcal{D}_L$ and the $\vartheta + 1$ ones priced by the power polynomial in the first row of a matrix $\mathcal{D}_P$. Assume that $\mathcal{D}_L$ and $\mathcal{D}_P$ have dimension $\zeta$ by $\vartheta + 1$.

Fix $\mathcal{A}_{\mathcal{H}}$ as it was for the aforementioned case, but define $\bar{\mathcal{A}}_{\mathcal{H}}$ differently, as $\bar{\mathcal{A}}_{\mathcal{H}_2}$. For example, $\bar{\mathcal{A}}_{\mathcal{H}_2}$ can differ from $\bar{\mathcal{A}}_{\mathcal{H}_1}$ in: the values of some parameters in the SCYH model, the time span of the American contract and the moneyness[3] of the option. Then, generate a new group of paths defined as $Paths_{\bar{\mathcal{A}}_{\mathcal{H}_2}}$. These paths can be used to price other new $2 \cdot (\vartheta + 1)$ American options, which half of them are stored in the second row of $\mathcal{D}_L$ matrix because priced by $\mathbb{E}\left[Y|X\right]_{L_{\mathcal{Q}}(S)}$ and the other half are stored in the second row of $\mathcal{D}_P$ because priced by $\mathbb{E}\left[Y|X\right]_{P_{\mathcal{Q}}(S)}$. Note that the $\vartheta + 1$ American option prices are computed by increasing $\mathcal{Q}$ from 0 to $\vartheta$ for the Laguerre and power polynomial case, respectively.

This procedure can be repeated recursively and generate $\zeta$ of $\bar{\mathcal{A}}_{\mathcal{H}}$ and $Paths_{\bar{\mathcal{A}}_{\mathcal{H}}}$, by keeping

[1]Considering also the initial condition of the SDEs of SCYH model.

[2]$\mathcal{H}$ represents the complement of $\tilde{\theta}$ regarding the parameters in the latent processes in the SCYH model, but not the initial condition of the SDEs.

[3]In this thesis moneyness is defined as the ratio between the strike price and spot price of the underlying.

$\mathcal{A}_{\mathcal{H}}$ fix. Basically, we have generated $\bar{\mathcal{A}}_{\mathcal{H}_i}$ and $Paths_{\bar{\mathcal{A}}_{\mathcal{H}_i}}$ and priced $2 \cdot (\vartheta + 1)$ options for each $i$ in the previously explained way, for $i = 1, ..., \zeta$. In this way the entire matrices $\mathcal{D}_L$ and $\mathcal{D}_P$ are filled up. Element $\mathcal{D}_{L,ij}$ is the option price for the $i$ complement information set $\bar{\mathcal{A}}_{\mathcal{H}_i}$ and the $j-1$ Laguerre specifivation of the conditional expectation, i.e. $L_{j-1}(S)$, for $i = 1, ..., \zeta$ and $j = 1, ..., \vartheta + 1$[4]. The same logic is for element $\mathcal{D}_{P,ij}$ but w.r.t. the power polynomial.

In the $j$ column of $\mathcal{D}_L$, there are American option prices that have been computed from $\zeta$ different $\bar{\mathcal{A}}_{\mathcal{H}}$, given the same $\mathcal{A}_{\mathcal{H}}$. Nevertheless, all of them have been priced with the same $j-1$ Laguerre specification of the conditional expectation, i.e. $L_{j-1}(S)$. The same idea holds for $\mathcal{D}_P$ but w.r.t. the power polynomial.

Now define the average along the column of $\mathcal{D}_L$ and $\mathcal{D}_P$ as $\mathcal{D}_L^{Av}$ and $\mathcal{D}_P^{Av}$, respectively. This means that the $\mathcal{D}_{L,j}^{Av}$ element is the average American option price computed by the $j-1$ Laguerre conditional expectation of the payoffs in the backward induction of the LSM algorithm, i.e. $L_{j-1}(S)$. The same holds for $\mathcal{D}_{P,j}^{Av}$ element but w.r.t. the power polynomial. This implies that if we plot and analyze $\mathcal{D}_L^{Av}$ and $\mathcal{D}_P^{Av}$, we can see which specification maximizes the price of the American option in average sense for the Laguerre and power polynomial, respectively.

Moreover, $\mathcal{D}_L^{Av}$ and $\mathcal{D}_P^{Av}$ can be compared to each other, in order to identify which between Laguerre and power basis function performs better in terms of highest average American option price, stability and behavior[5]. For instance, we could check, not only the average American option price but, also each single case along the rows of $\mathcal{D}_L$ and $\mathcal{D}_P$, i.e. if the "average" result changes in each single case or if it is robust. In other words, we want to see the variability of the results in the latter example.

**Yield-volatility basis function analysis**

Assume that a well-behaved basis function specification, called benchmark, has been detected between the Laguerre and power polynomial case, from the previous analysis. Call this latter specification of the conditional expectation $\mathbb{E}[Y|X]^1_{L_{\mathcal{Q}(\mathcal{A})}(S)}$, if a Laguerre polynomial was the chosen one[6], e.g. $L_{\mathcal{Q}(\mathcal{A})}(S)$. While, call it $\mathbb{E}[Y|X]^1_{P_{\mathcal{Q}(\mathcal{A})}(S)}$, if a power polynomial was taken[7], e.g. $P_{\mathcal{Q}(\mathcal{A})}(S)$. Note that a general information set $\mathcal{A}$ was used, rather than $\mathcal{A}_{\mathcal{H}}$. This is due to the fact that we can analyze different information sets and it is better to generalize the notation[8]. $\mathbb{E}[Y|X]^1_{L_{\mathcal{Q}(\mathcal{A})}(S)}$ or $\mathbb{E}[Y|X]^1_{P_{\mathcal{Q}(\mathcal{A})}(S)}$ is the specification that provides the highest value of the American option in average sense and it has a robust behavior[9], for the information

[4]The numerical indexing of matrices is the usual mathematical convention, i.e. starting from one. Nevertheless, the numerical indexing of the polynomials starts from zero. Thus there is one lag value between the two notations.

[5]This will become clearer in the numerical analysis in the following chapter.

[6]Note that the exponent in $\mathbb{E}[Y|X]^1_{L_{\mathcal{Q}(\mathcal{A})}(S)}$ is referred to the regression specification in table 4.1.

[7]Note that the exponent in $\mathbb{E}[Y|X]^1_{P_{\mathcal{Q}(\mathcal{A})}(S)}$ is referred to the regression specification in table 4.2.

[8]Note that $\mathcal{Q}(\mathcal{A})$ depends on the variable information set $\mathcal{A}$. Basically, different information sets can have different specification of $\mathcal{Q}$.

[9]For instance, the average result holds also in each single case.

set $\mathcal{A}$. After this, we wonder if extending $\mathbb{E}\left[Y|X\right]^{1}_{L_{\mathcal{Q}(\mathcal{A})}(S)}$ or $\mathbb{E}\left[Y|X\right]^{1}_{P_{\mathcal{Q}(\mathcal{A})}(S)}$ by inserting the convenience yield, $\delta$, and/or the stochastic volatility, $\sqrt{V}$, as regressors can improve the price of the American option. In other words, if $\delta$ and/or $\sqrt{V}$ can even further increase the value of the American option by keeping the robustness of the LSM decision rule. The suggested conditional expectation extensions for the Laguerre case are reported in table 4.1, whereas the power extensions are reported in table 4.2. As it can be seen from the two tables, $\mathbb{E}\left[Y|X\right]^{1}_{\mathcal{G}_{\mathcal{Q}(\mathcal{A})}(S)}$ is the benchmark, whereas $\left\{\mathbb{E}\left[Y|X\right]^{\mathcal{K}}_{\mathcal{G}_{\mathcal{Q}(\mathcal{A})}(S)}\right\}_{\mathcal{K}=2}^{9}$ are the extensions, for $\mathcal{G} = L, P$.

Table 4.1: This table shows different regression specifications for the computation of the conditional expectation of the payoffs, i.e. $\mathbb{E}\left[Y|X\right]_{L_{\mathcal{Q}(\mathcal{A})}(S)}$, in the backward induction of the LSM. There is use of the commodity price $S$, stochastic convenience yield $\delta$ and stochastic volatility $\sqrt{V}$.

| $RS$ | $\mathbb{E}\left[Y|X\right]_{L_{\mathcal{Q}(\mathcal{A})}(S)}$ |
|---|---|
| 1 | $L_{\mathcal{Q}(\mathcal{A})}(S)$ |
| 2 | $L_{\mathcal{Q}(\mathcal{A})}(S) + l_0(\delta)\hat{\beta}_{\delta_0}$ |
| 3 | $L_{\mathcal{Q}(\mathcal{A})}(S) + \sum_{j=0}^{1} l_j(\delta)\hat{\beta}_{\delta_j}$ |
| 4 | $L_{\mathcal{Q}(\mathcal{A})}(S) + l_0\left(\sqrt{V}\right)\hat{\beta}_{V_0}$ |
| 5 | $L_{\mathcal{Q}(\mathcal{A})}(S) + \sum_{j=0}^{1} l_j\left(\sqrt{V}\right)\hat{\beta}_{V_j}$ |
| 6 | $L_{\mathcal{Q}(\mathcal{A})}(S) + l_0(\delta)\hat{\beta}_{\delta_0} + l_0\left(\sqrt{V}\right)\hat{\beta}_{V_0}$ |
| 7 | $L_{\mathcal{Q}(\mathcal{A})}(S) + \sum_{j=0}^{1} l_j(\delta)\hat{\beta}_{\delta_j} + \sum_{i=0}^{1} l_i\left(\sqrt{V}\right)\hat{\beta}_{V_i}$ |
| 8 | $L_{\mathcal{Q}(\mathcal{A})}(S) + l_0(\delta)\hat{\beta}_{\delta_0} + l_0\left(\sqrt{V}\right)\hat{\beta}_{V_0} + l_0(\delta)\, l_0\left(\sqrt{V}\right)\hat{\beta}_{\delta_0 V_0}$ |
| 9 | $L_{\mathcal{Q}(\mathcal{A})}(S) + \sum_{j=0}^{1} l_j(\delta)\hat{\beta}_{\delta_j} + \sum_{i=0}^{1} l_i\left(\sqrt{V}\right)\hat{\beta}_{V_i} + l_0(\delta) l_0\left(\sqrt{V}\right)\hat{\beta}_{\delta_0 V_0}$ |

*Notes: RS stands for regression specification and the numerical notation will be used in other tables, plots and in the text too.* $\mathbb{E}\left[Y|X\right]_{L_{\mathcal{Q}(\mathcal{A})}(S)}$ *is the conditional expectation of the payoffs in the backward induction of the LSM. Note that* $\mathcal{Q}(\mathcal{A})$ *depends on the variable information set* $\mathcal{A}$. *Basically, different information sets can have different specification of* $\mathcal{Q}$. $l_q(\mathcal{J})$ *is defined in equation 2.2, for a general variable* $\mathcal{J}$.

The most important issue is that matrix $X$, in the LSM algorithm, can depend on the commodity price, $S$, on the stochastic convenience yield, $\delta$, and on the stochastic volatility, $\sqrt{V}$, for both of the tables. The following methodology for analyzing yield-volatility specifications will be explained for a general $\mathcal{G}$, where $\mathcal{G} = L, P$. This methodology is fairly similar to the one explained for the Laguerre-power polynomial case.

Table 4.2: This table shows different regression specifications for the computation of the conditional expectation of the payoffs, i.e. $\mathbb{E}\left[Y|X\right]_{P_{\mathcal{Q}(\mathcal{A})}(S)}$, in the backward induction of the LSM. There is use of the commodity price $S$, stochastic convenience yield $\delta$ and stochastic volatility $\sqrt{V}$.

| *RS* | $\mathbb{E}\left[Y\vert X\right]_{P_{\mathcal{Q}(\mathcal{A})}(S)}$ |
|---|---|
| 1 | $P_{\mathcal{Q}(\mathcal{A})}(S)$ |
| 2 | $P_{\mathcal{Q}(\mathcal{A})}(S) + p_0(\delta)\hat{\beta}_{\delta_0}$ |
| 3 | $P_{\mathcal{Q}(\mathcal{A})}(S) + \sum_{j=0}^{1} p_j(\delta)\hat{\beta}_{\delta_j}$ |
| 4 | $P_{\mathcal{Q}(\mathcal{A})}(S) + p_0\left(\sqrt{V}\right)\hat{\beta}_{V_0}$ |
| 5 | $P_{\mathcal{Q}(\mathcal{A})}(S) + \sum_{j=0}^{1} p_j\left(\sqrt{V}\right)\hat{\beta}_{V_j}$ |
| 6 | $P_{\mathcal{Q}(\mathcal{A})}(S) + p_0(\delta)\hat{\beta}_{\delta_0} + p_0\left(\sqrt{V}\right)\hat{\beta}_{V_0}$ |
| 7 | $P_{\mathcal{Q}(\mathcal{A})}(S) + \sum_{j=0}^{1} p_j(\delta)\hat{\beta}_{\delta_j} + \sum_{i=0}^{1} p_i\left(\sqrt{V}\right)\hat{\beta}_{V_i}$ |
| 8 | $P_{\mathcal{Q}(\mathcal{A})}(S) + p_0(\delta)\hat{\beta}_{\delta_0} + p_0\left(\sqrt{V}\right)\hat{\beta}_{V_0} + p_0(\delta)\,p_0\left(\sqrt{V}\right)\hat{\beta}_{\delta_0 V_0}$ |
| 9 | $P_{\mathcal{Q}(\mathcal{A})}(S) + \sum_{j=0}^{1} p_j(\delta)\hat{\beta}_{\delta_j} + \sum_{i=0}^{1} p_i\left(\sqrt{V}\right)\hat{\beta}_{V_i} + p_0(\delta)\,p_0\left(\sqrt{V}\right)\hat{\beta}_{\delta_0 V_0}$ |

*Notes: RS stands for regression specification and the numerical notation will be used in other tables, plots and in the text too.* $\mathbb{E}\left[Y|X\right]_{P_{\mathcal{Q}(\mathcal{A})}(S)}$ *is the conditional expectation of the payoffs in the backward induction of the LSM. Note that* $\mathcal{Q}(\mathcal{A})$ *depends on the variable information set* $\mathcal{A}$. *Basically, different information sets can have different specification of* $\mathcal{Q}$. $p_q(\mathcal{J})$ *is defined in equation 2.3, for a general variable* $\mathcal{J}$.

Assume the same fixed information set $\mathcal{A}_{\mathcal{H}}$ and its complements $\bar{\mathcal{A}}_{\mathcal{H}_i}$, for $i = 1, ..., \zeta$, as explained in the Laguerre-power polynomial analysis. Generate new $Paths_{\bar{\mathcal{A}}_{\mathcal{H}_i}}$, for $i = 1, ..., \zeta$, and price 9 different American options with the following conditional expectation specifications

$\left\{\mathbb{E}\left[Y|X\right]^{\mathcal{K}}_{\mathcal{G}_{\mathcal{Q}(\mathcal{A}_{\mathcal{H}})}(S)}\right\}_{\mathcal{K}=1}^{9}$, for each $i$. Then, all these option prices are stored in a matrix $\mathcal{B}_{\mathcal{G}}$, with dimension $\zeta$ by 9.

Now, let's define the average along the column of $\mathcal{B}_{\mathcal{G}}$ as $\mathcal{B}^{Av}_{\mathcal{G}}$. The element $\mathcal{B}^{Av}_{\mathcal{G},\mathcal{K}}$, for $\mathcal{K} = 1, ..., 9$, is the average American option price computed with the $\mathcal{K}$ regression specification (RS) defined in table 4.1 or 4.2, depending on what is $\mathcal{G}$, i.e. $L$ or $P$, respectively[10].

By plotting and analyzing $\mathcal{B}^{Av}_{\mathcal{G}}$, it is possible to find which regression specifications increase the American option price in average sense compared to the benchmark, i.e. $\mathcal{B}^{Av}_{\mathcal{G},1}$. In other words, it is possible to see which variable impacts the option price the most between $\delta$ and $\sqrt{V}$. Moreover, by analyzing the rows of $\mathcal{B}_{\mathcal{G}}$ one by one, it is possible to define the variability and robustness of the "average" result.

**In sample-out of sample test**

Assume that the $\mathcal{K}$ regression specification, i.e. $\mathbb{E}\left[Y|X\right]^{\mathcal{K}}_{\mathcal{G}_{\mathcal{Q}(\mathcal{A}_{\mathcal{H}})}(S)}$, was found particularly relevant for increasing the value of the American option[11]. Call this specification improved benchmark. At this point, it is possible to perform an in sample-out of sample test (IOT)[12] on the benchmark, i.e. $\mathbb{E}\left[Y|X\right]^{1}_{\mathcal{G}_{\mathcal{Q}(\mathcal{A}_{\mathcal{H}})}(S)}$, and on the improved benchmark. Essentially, if the addition of further explanatory variables is strong, then the average difference price between in sample and out of sample options of the improved benchmark should not differ too much from the benchmark one. In other words, there should not be a break down when the improved benchmark is used.

Note that the IOT is supposed to be performed on the same information set of the previous analysis, i.e. $\mathcal{A}_{\mathcal{H}}$. Also the same complements of the information set $\mathcal{A}_{\mathcal{H}}$, i.e. $\bar{\mathcal{A}}_{\mathcal{H}_i}$ for $i = 1, ..., \zeta$, are used to price the in sample and out of sample options. On the other hand, new paths have to be generated, i.e. $Paths_{\bar{\mathcal{A}}_{\mathcal{H}_i}}$ for $i = 1, ..., \zeta$.

**Early exercise premium analysis**

There is also an EEP analysis at the end of the thesis. Nevertheless, it is less theoretical and it concludes the overall picture of the information set analysis for four different sets, i.e. $\mathcal{A}_{\sigma}$, $\mathcal{A}_{k}$, $\mathcal{A}_{\theta}$ end $\mathcal{A}_{\rho}$. As a result, its explanation will be given successively, while presenting the numerical result in chapter 5.

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[10] Theoretically speaking, the benchmark should not be calculated again. Nevertheless, the benchmark was computed again for practical explanation of the topic and implementation of the algorithm in MATLAB.

[11] Of course, $\mathcal{K} \neq 1$.

[12] The IOT is suggested by *Longstaff* and *Schwartz* in their paper, [11], to see the strength of the LSM decision rule. The IOT consists in pricing an American option with some paths and simultaneously storing the estimated $\beta$s in the backward induction of the LSM. After that, a new American option is computed with the same features of the first one. Nonetheless, the second option is priced with new generated paths of the underlying model and no regression is estimated. In the second option the computation of the conditional expectation of the payoffs is performed by using the stored $\beta$s of the first option. In such a way we can investigate if the decision rule, i.e. $\beta$s, of the LSM is robust even out of sample. Fundamentally, if the decision rule is strong, the in sample and out of sample American option prices should not be so different from each other.

# Chapter 5

# Basis Function Analysis

To start with, the entire basis function analysis, for any information set, is done on a plain vanilla American option with payoff function $\Phi(S) = (K - S)^+$. Where, $K$ is the strike price and $S$ is commodity spot price modeled by the SCYH model. The American contract has a time span from zero to time $T$. All the rest of the information about the CS and the parameters values will be given in each information set that is analyzed.

An important note is given on the choice of the parameter values. Due to the fact that the SCYH model, as a whole, has never been tested empirically in the literature[1], the choice of which values assume, in the numerical analysis, has been challenging. As it was explained previously, the SCYH model can be seen as a composition of the *E. Schwartz*, [12], and the *James S. Doran et al.*, [2], models[2]. Accordingly, the author chose to use the estimated oil parameters of the convenience yield process from *E. Schwartz*, [12][3], and the estimated oil parameters for the volatility process from *James S. Doran et al.*, [2][4]. On the one hand, the taken parameters were calibrated on oil commodity and w.r.t. the $\mathbb{Q}$ measure, in both of the papers[5]. On the other hand, the two papers use different sample periods and different methodology for their estimation of the oil parameters[6]. By using the literature as starting point of our analysis, four different information sets will be analyzed in the following sections, i.e. $\mathcal{A}_\sigma$, $\mathcal{A}_k$, $\mathcal{A}_\theta$ and $\mathcal{A}_\rho$. Then, conclusions about the Laguerre-power polynomial analysis and the yield-volatility analysis will be given. After that, an *EEP* analysis will conclude the thesis.

However, note that in each information set there will be a comprehensive explanation of which parameters were assumed and which were taken from the literature, with connected reference.

[1] At least of what the thesis author is aware of.

[2] See chapter 3.

[3] Table VI in [12] for oil parameters.

[4] Table 3 in [2] for crude oil parameters.

[5] In other words, we will simulate oil prices in the numerical application w.r.t. the $\mathbb{Q}$ measure.

[6] Unfortunately, this was the best combination of parameters that could be found in the literature.

## 5.1 Information Set $\mathcal{A}_\sigma$

The starting point of our analysis is to present the Laguerre-power polynomial analysis and to identify the regression specification that better fits the information set at hand. After identifying the benchmark, the yield-volatility basis function analysis is carried out to see if $\delta$ and/or $\sqrt{V}$ can further increase the value of the American option. Lastly, an IOT is implemented to see the different reaction of the decision rule of LSM between the benchmark and the improved benchmark[7].

$\mathcal{A}_\sigma$ is the first information set that is analyzed, table 5.1. As it can be seen, its complement, i.e. $\bar{\mathcal{A}}_\sigma$, includes: the initial condition of the commodity process, $S_0$, the volatility of the stochastic convenience yield, $\sigma_\delta$, the volatility of the volatility process, $\sigma_V$, and the length of the American option contract, $T$.

Table 5.1: Information set $\mathcal{A}_\sigma$

| $\mathcal{A}_\sigma$ | $r$ | $k_\delta$ | $\theta_\delta$ | $\delta_0$ | $k_V$ | $\theta_V$ | $V_0$ | $\rho_{S\delta}$ | $\rho_{SV}$ | $\rho_{\delta V}$ | $K$ | $n$ | $dy$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Value | 0.06 | 1.876 | 0.000456 | $\theta_\delta$ | 27.636 | 0.077 | $\theta_V$ | 0.766 | 0.023 | 0 | 100 | $2 \cdot 10^5$ | 360 |

*Notes: Parameters $r$, $k_\delta$, $\theta_\delta$ and $\rho_{S\delta}$ are taken from [12]. While parameters $k_V$, $\theta_V$, $\rho_{SV}$ are taken from [2]. $\delta_0$ and $V_0$ are the initial conditions of the SDEs for the yield and volatility process, respectively. They are assumed to be equal to the long term mean of their process. $\rho_{\delta V}$ and $K$ are assumed to be equal to 0 and 100, respectively. $n$ is the number of simulated paths, 50% of them are antithetic variates. $dy$ is the number of days per year and it is assumed to be also the number of exercise possibilities per year of the American option. The values assumed by $S_0$, $\sigma_\delta$, $\sigma_V$ and $T$ are reported in the simulation tables in the Appendix, where information set $\mathcal{A}_\sigma$ is used.*

Tables B.1 and B.2, in the Appendix, report the computation of matrix $\mathcal{D}_L$ and $\mathcal{D}_P$ as well as $\bar{\mathcal{A}}_{\sigma_i}$, for $i = 1, ..., \zeta = 20$, for information set $\mathcal{A}_\sigma$, respectively[8]. Figure, 5.1a shows the $\mathcal{D}_L^{Av}$ and $\mathcal{D}_P^{Av}$ plots for information set $\mathcal{A}_\sigma$[9]. As it can be seen from this figure, i.e. 5.1a, the power polynomial increases the average American option price until reaching a maximum. Then, the average American option price starts decreasing when we increase the complexity of the conditional expectation of the payoffs. It is worth pointing out the smoothness of the average American option price function in the power polynomial case. On the other hand, the Laguerre polynomial has a more irregular pattern. The Laguerre polynomial pushes the average American option price to its maximum and then it drops suddenly. In the author's option, this

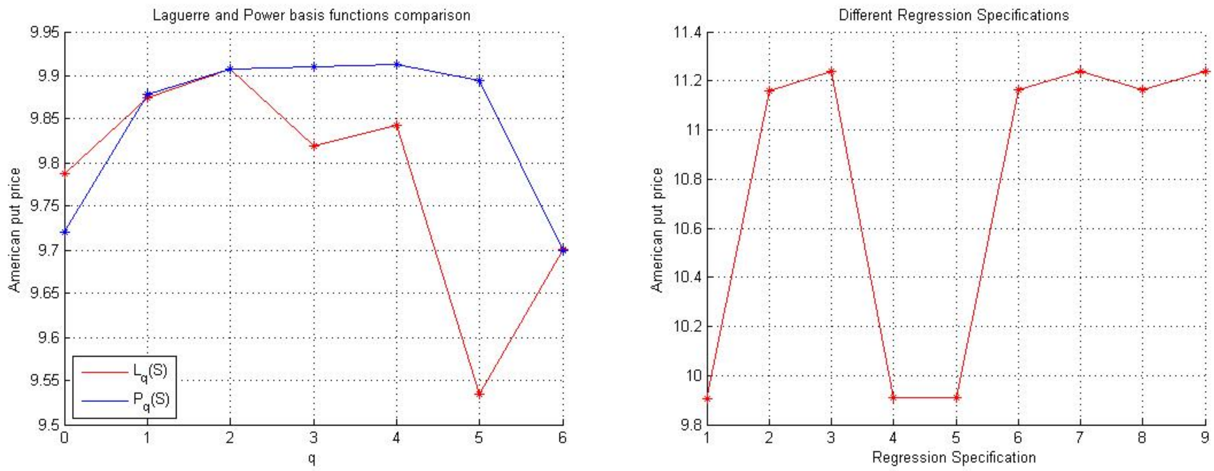
[7] See chapter 4 for the adopted terminology.

[8] It is clear that $\vartheta = 6$ from the Appendix tables and from figure 5.1a

[9] In the figure the x-axis is represented by the number of Laguerre or power polynomials included in the regression, i.e. $L_\mathcal{Q}(S)$ or $P_\mathcal{Q}(S)$. Nevertheless, the $\mathcal{Q}$ symbol was not accessible in MATLAB, thus the $q$ one was used. However, its meaning does not change.

sudden drop is due to a loss of precision during the numerical inversion of the $X'X$ matrix in the OLS regression. The Laguerre polynomial reaches a point when its complexity overwhelms the benefit of capturing the non-linarites of the American option payoffs.

Figure 5.1: This figure shows: Laguerre-power polynomial analysis, tables B.1 and B.2 in the Appendix, and yield-volatility basis function analysis, table B.3 in the Appendix, for information set $\mathcal{A}_\sigma$, table 5.1.




(a) Laguerre-power polynomial analysis

(b) yield-volatility basis function analysis

The issue now is to identify the maximum value obtained by the average American option price in figure 5.1a, between the Laguerre and power polynomial cases. First, we define the maximum in the Laguerre case and in the power case, respectively. After that, a comparison of the two maximums will point out the most suitable specification of the conditional expectation of the payoffs in the backward induction of LSM for information set $\mathcal{A}_\sigma$.

At first hand, one could just select the conditional expectation specification that maximizes the average American option price in figure 5.1a, for the Laguerre and power cases, respectively. Nevertheless, this could mean selecting a conditional expectation specification that does not give any sensible improvement from the previous one. This would mean to have a regression that is over specified. In other words, this would increase the multicollinearity and difficulty in the inversion of the $X'X$ matrix in the OLS regression[10]. Note that especially the latter issue is the most relevant, i.e. the loss in precision can substantially impact the American option price.

As a result, we can focus on the increments of the plot, 5.1a, for the Laguerre and the power case, respectively. Basically, if by increasing the conditional expectation specification, the increase in the average American option price is lower than a certain cutoff value, then the latter increment is worthless. For example, Assume that $L_1(S)$ yields an average American option price of $\bar{\Pi}_A^{L_1(S)}$ and $L_2(S)$ yields $\bar{\Pi}_A^{L_2(S)}$; if the difference between $\bar{\Pi}_A^{L_2(S)}$ and $\bar{\Pi}_A^{L_1(S)}$ is greater than a U.S. dollar

[10] Also computational time would increase in such a way, but this is not an issue in this type of analysis.

(USD) cent, then $L_2(S)$ is preferred, while if it is less than a USD cent, $L_1(S)$ is preferred[11]. The USD cent cutoff value is used as decision rule to exclude or include a further regressor in the conditional expectation throughout the entire basis function analysis[12].

Table B.21 and B.22, in the Appendix, report the increments of the average American option price function in plot 5.1a for the Laguerre and power polynomial cases, respectively. As the tables point out the $L_2(S)$ and $P_2(S)$ are the two conditional expectation specifications that maximize the average American option price for the Laguerre and power cases, respectively. This choice is based on the one USD cent rule of thumb. Now, we should choose which between $L_2(S)$ and $P_2(S)$ gives the highest American option price in average. Table B.23, in Appendix, reports the difference between the average American option price function computed with Laguerre polynomial and the one computed with the power polynomial, for information set $\mathcal{A}_\sigma$. In other words, this table shows the difference between the functions in plot 5.1a. It emerges from the table that $P_2(S)$ is preferred to $L_2(S)$, because the former yields a higher average American option price compared to the latter.

Lastly, by analyzing the single rows of matrix $\mathcal{D}_L$ and $\mathcal{D}_P$, which correspond to tables B.1 and B.2 in the Appendix, it is possible to check if the average result, i.e. $P_2(S)$, holds also in the single cases or at least if it is roubst. Basically, we wonder if the average result presents a large variability or if it is stable[13]. After an accurate analysis, it is possible to state that the average result is stable and robust. It means that $P_2(S)$ specification is shown to be fairly consistent in maximizing the American option price also in the single cases. This is not systematic, but it holds in the large majority of the cases, i.e. $\bar{\mathcal{A}}_{\sigma_i}$ for $i = 1, ..., \zeta = 20$. To conclude, we can state that $P_2(S)$ is the conditional expectation specification that yields the highest average American option price in the Laguerre-power polynomial analysis, and it is robust in the single cases as well, for information set $\mathcal{A}_\sigma$.[14]

The yield-volatility basis function analysis starts after identifying the benchmark form the Laguerre-power polynomial analysis, i.e. $\mathbb{E}[Y|X]^1_{P_{\mathbb{Q}(\mathcal{A}_\sigma)}(S)} = P_2(S)$. Therefore, the extensions that will be compared to the benchmark are those specified in table 4.2 in chapter 4, i.e.

[11] Of course this is true if $\bar{\Pi}_A^{L_2(S)} > \bar{\Pi}_A^{L_1(S)}$, while if $\bar{\Pi}_A^{L_2(S)} < \bar{\Pi}_A^{L_1(S)}$ than $L_1(S)$ is without computation whatsoever better than $L_2(S)$.

[12] After extensive numerical simulations and computations, the one USD cent rule of thumb has been seen quite effective. Nonetheless, there might be cases where it is up to the researcher to choose if the USD cent rule is applicable or not. For example, if the average American option increment is 0.0099999 or 0.0100000001.

[13] On the one hand, the author chose not to report this part of analysis, in the Appendix, because it would have overloaded the reader with figures and tables. One the other hand, the overall picture and result is reported and commented in the main text. Nonetheless, this analysis can be extracted by the $\mathcal{D}_L$ and $\mathcal{D}_P$ tables reported in the Appendix, if the reader wants to check.

[14] The Laguerre-power polynomial analysis was carried out also on the GBM example that *Longstaff* and *Schwartz*, [11], have in their paper. The results lead to choose the same specification of the conditional expectation of the payoffs as the one that *Longstaff* and *Schwartz* use in their paper, i.e. $L_2(S)$ with the thesis notation. The summary of the analysis is reported in the Appendix A.1, under performance test.

$\left\{\mathbb{E}\left[Y|X\right]^{\mathcal{K}}_{P_{\mathcal{Q}(\mathcal{A}_\sigma)}(S)}\right\}^{9}_{\mathcal{K}=2}$. Table B.3, in the Appendix, reports the values of matrix $\mathcal{B}_P$, while figure 5.1b the vector, $\mathcal{B}^{Av}_P$. As it can be seen in figure 5.1b[15], the specifications $\left\{\mathbb{E}\left[Y|X\right]^{\mathcal{K}}_{P_{\mathcal{Q}(\mathcal{A}_\sigma)}(S)}\right\}^{3}_{\mathcal{K}=2}$ and $\left\{\mathbb{E}\left[Y|X\right]^{\mathcal{K}}_{P_{\mathcal{Q}(\mathcal{A}_\sigma)}(S)}\right\}^{9}_{\mathcal{K}=6}$ increase the average American option price, whereas, the $\left\{\mathbb{E}\left[Y|X\right]^{\mathcal{K}}_{P_{\mathcal{Q}(\mathcal{A}_\sigma)}(S)}\right\}^{5}_{\mathcal{K}=4}$ ones do not. This means that only the stochastic convenience yield as regressor raises the average American option price, while, the stochastic volatility as regressor does not. This implies that all the specifications that contain the volatility process as regressor only increase the complexity of the conditional expectation of the payoffs w.r.t. the benchmark, but not the price. On the other hand, it is obvious form figure 5.1b that the convenience yield provides a higher American option price in average w.r.t. the benchmark. In fact the difference between the average American option price provided by $\mathbb{E}\left[Y|X\right]^{2}_{P_{\mathcal{Q}(\mathcal{A}_\sigma)}(S)}$, in table 4.2, and the one given by the benchmark is 1.2553 USD. While the difference between the average American option price given by $\mathbb{E}\left[Y|X\right]^{3}_{P_{\mathcal{Q}(\mathcal{A}_\sigma)}(S)}$, in table 4.2, and the one given by the benchmark is 1.3292 USD.

These are remarkable results that point out the strength of adding $\delta$ as regressor in the computation of the conditional expectation of the payoffs in the backward induction of the LSM algorithm. Furthermore, these average results also hold in each single case, i.e. $\bar{\mathcal{A}}_{\sigma_i}$ for $i = 1, ..., \zeta = 20$. This means that $\delta$ gives a edge in the LSM algorithm. It is clear that $\delta$ matters more than $\sqrt{V}$ in the computation of the conditional expectation. This is due to the fact that the drift of the SCYH model impacts more the value of the American option rather than the commodity higher distribution moments modeled by the volatility.

At this point an IOT is carried out on the benchmark, i.e. $\mathbb{E}\left[Y|X\right]^{1}_{P_{\mathcal{Q}(\mathcal{A}_\sigma)}(S)} = P_2\left(S\right)$ table 4.2 or in equation 5.1, and on improved benchmark, i.e. $\mathbb{E}\left[Y|X\right]^{2}_{P_{\mathcal{Q}(\mathcal{A}_\sigma)}(S)}$ table 4.2 or equation 5.2[16].

$$\mathbb{E}\left[Y|X\right]^{1}_{P_{\mathcal{Q}(\mathcal{A}_\sigma)}(S)} = c + \sum_{q=0}^{2} p_q\left(S\right)\hat{\beta}_q \tag{5.1}$$

$$\mathbb{E}\left[Y|X\right]^{2}_{P_{\mathcal{Q}(\mathcal{A}_\sigma)}(S)} = c + \sum_{q=0}^{2} p_q\left(S\right)\hat{\beta}_q + p_0\left(\delta\right)\hat{\beta}_{\delta_0} \tag{5.2}$$

Tables B.4 and B.5, in the Appendix, report the IOT for the benchmark and the improved benchmark, respectively. Moreover, table B.24, in the Appendix, reports the average difference between the in sample and out of sample option prices for the benchmark and the improved benchmark, respectively. The IOT shows that the LSM decision rule is really strong for both the conditional expectation specifications, i.e. equation 5.1 and 5.2. This is due to the fact that the average difference between in sample and out of sample options is lower than a USD cent in both of the cases, i.e. equation 5.1 and 5.2 respectively.

[15] The x-axis represents the different regression specifications.

[16] Note that the improved benchmark could have been $\mathbb{E}\left[Y|X\right]^{2}_{P_{\mathcal{Q}(\mathcal{A}_\sigma)}(S)}$, in table 4.2, as well. Nevertheless, it is better to compare the closest extension to the benchmark as good rule of research.

To conclude, we can say that equation 5.2 is the best possible specification that was identified in the carried out analysis so far, i.e. in information set $\mathcal{A}_\sigma$. This is in term of: average American option price, robustness in the single cases, regression specification parsimonies and in the IOT[17]. In other words, the stochastic convenience yield as regressor improves the precision of the LSM, when an American option is written on a commodity and the SCYH model is used as stochastic process, for information set $\mathcal{A}_\sigma$.

## 5.2 Information Set $\mathcal{A}_k$

The previous analysis pointed out the relevance of $\delta$ in predicting the continuation value of the American option. Even if there is significant evidence of such a founding, the author chose to investigate such issue even further. Basically, we wonder if by changing information set and its complements the previous results will change too. In other words, will the convenience yield be still so relevant in the LSM extension? Is this pattern persistent and reliable? In order to investigate these questions, further numerical analysis is carried out.

To begin with, a new information set $\mathcal{A}_k$ is provided, table 5.2. Its general complement $\bar{\mathcal{A}}_k$ is made up of: $S_0$, $k_\delta$, $k_V$ and $T$. Basically, the main difference between $\mathcal{A}_\sigma$ and $\mathcal{A}_k$ is that in the latter the speed of mean reversion of the yield process and the volatility one will change over the complements sets, $\bar{\mathcal{A}}_{k_i}$ for $i = 1, ..., \zeta = 20$. While in the former the volatilities changed over the difference complements. Fundamentally, we want to see if with different setups of the SCYH model parameters the general previously obtained results hold or not.

Table 5.2: Information set $\mathcal{A}_k$

| $\mathcal{A}_k$ | $r$ | $\sigma_\delta$ | $\theta_\delta$ | $\delta_0$ | $\sigma_V$ | $\theta_V$ | $V_0$ | $\rho_{S\delta}$ | $\rho_{SV}$ | $\rho_{\delta V}$ | $K$ | $n$ | $dy$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Value | 0.06 | 0.527 | 0.000456 | $\theta_\delta$ | 0.443 | 0.077 | $\theta_V$ | 0.766 | 0.023 | 0 | 100 | $2 \cdot 10^5$ | 360 |

*Notes: Parameters $r$, $\sigma_\delta$, $\theta_\delta$ and $\rho_{S\delta}$ are taken from [12]. While parameters $\sigma_V$, $\theta_V$, $\rho_{SV}$ are taken from [2]. $\delta_0$ and $V_0$ are the initial conditions of the SDEs for the yield and volatility process, respectively. They are assumed to be equal to the long term mean of their process. $\rho_{\delta V}$ and $K$ are assumed to be equal to 0 and 100, respectively. $n$ is the number of simulated paths, 50% of them are antithetic variates. $dy$ is the number of days per year and it is assumed to be also the number of exercise possibilities per year of the American option. The values assumed by $S_0$, $k_\delta$, $k_V$ and $T$ are reported in the simulation tables in the Appendix, where information set $\mathcal{A}_k$ is used.*

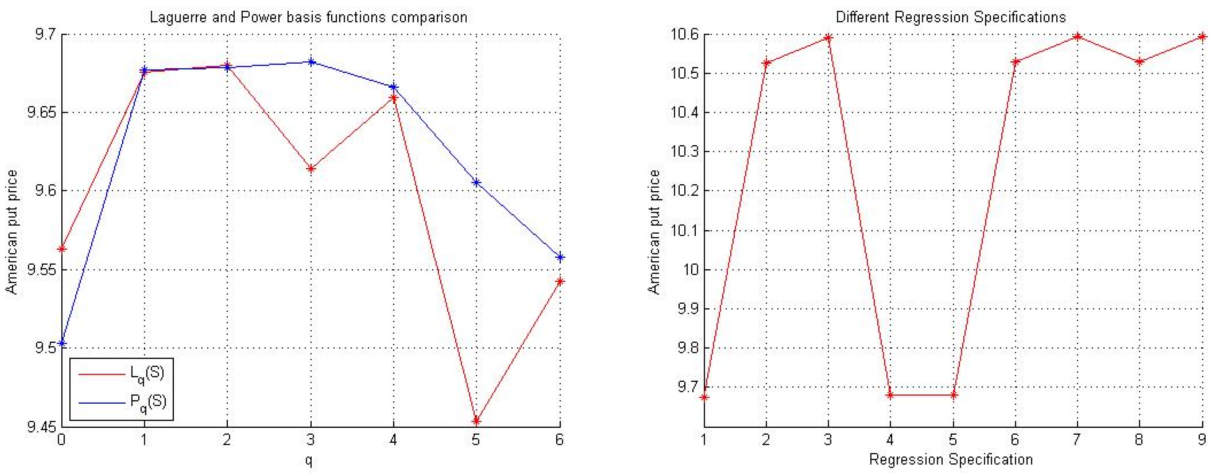
Figure 5.2a shows the average American option price as a function of different Laguerre and power polynomial specifications of the conditional expectation, i.e. $L_\mathcal{Q}(S)$ and $P_\mathcal{Q}(S)$ for $\mathcal{Q} =$

[17] Also $\mathbb{E}[Y|X]^2_{P_{\mathcal{Q}(\mathcal{A}_\sigma)}(S)}$, table 4.2, could have been a really good specification, but no IOT was carried out for it. Furthermore, the author prefers to be parsimonious in the regression specification.

$0, ..., \vartheta = 6$, respectively[18]. As it can be seen from figure 5.2a, the power polynomial case shows a quite smooth pattern after reaching the maximum level of the average American option price, while the Laguerre one still presents an irregular movement. Both of them point out a decrease trend in the average American option price in too complex specifications of the conditional expectation of the payoffs. It is clear from the figure that the average American option reaches its maximum at $L_1(S)$ and $P_1(S)$, for the Laguerre and power polynomial cases, respectively. This is also shown by the one USD cent rule of thumb in table B.21 and B.22, in the Appendix. In addition, table B.23, in the Appendix, points out that $P_1(S)$ yields a higher average American option price compared to the $L_1(S)$ case. Besides, the $P_1(S)$ conditional expectation of the payoffs presents robust features also in the single cases, i.e. $\bar{\mathcal{A}}_{k_i}$ for $i = 1, ..., \zeta = 20$ [19].

It is worth pointing out that in information set $\mathcal{A}_\sigma$ the benchmark was $P_2(S)$, whereas in information $\mathcal{A}_k$ the benchmark becomes $P_1(S)$. This means that by changing some SCYH model parameters, the regression specification that maximizes the American option changes too. On the other hand, as it will be seen in the following lines, $\delta$ still remains a predominant explanatory variable.

Figure 5.2: This figure shows: Laguerre-power polynomial analysis, tables B.6 and B.7 in the Appendix, and yield-volatility basis function analysis, table B.8 in the Appendix, for information set $\mathcal{A}_k$, table 5.2.




(a) Laguerre-power polynomial analysis

(b) yield-volatility basis function analysis

As stated before, the new benchmark for the yield-volatility basis function analysis is $\mathbb{E}[Y|X]^1_{P_{\mathcal{Q}(\mathcal{A}_k)}(S)} = P_1(S)$. As a result, its extensions are $\left\{\mathbb{E}[Y|X]^{\mathcal{K}}_{P_{\mathcal{Q}(\mathcal{A}_k)}(S)}\right\}^9_{\mathcal{K}=2}$ and they are reported in table 4.2. Figure 5.2b shows the average American option price computed with the benchmark, i.e. regression specification one, and with the extensions of table 4.2[20]. As it can be see, the figure

[18]The respective $\mathcal{D}_L$ and $\mathcal{D}_P$ matrices for the Laguerre and power polynomial are reported in the Appendix in table B.6 and B.7, respectively. Moreover, figure 5.2a has on the x-axis the $q$ symbol but it stands for $\mathcal{Q}$ one.

[19]This can be seen by analyzing the rows of table B.6 and B.7, in the Appendix, one by one.

[20]The computation of matrix $\mathcal{B}_P$ is reported in the Appendix in table B.8.

underlines the fact that only those extensions that incorporate $\delta$ increase the average American option price w.r.t. the benchmark. This means that the specifications that contain the volatility as regressor add only complexity and no precision in the computation of the continuation value of the American option. As a result, only the benchmark and $\left\{\mathbb{E}\left[Y|X\right]^{\mathcal{K}}_{P_{\mathcal{Q}(\mathcal{A}_k)}(S)}\right\}^{3}_{\mathcal{K}=2}$ are analyzed successively. In figure 5.2b the increment in the average American option price is 0.8524 USD, when we move from the benchmark, i.e. $\mathbb{E}\left[Y|X\right]^{1}_{P_{\mathcal{Q}(\mathcal{A}_k)}(S)}$, to $\mathbb{E}\left[Y|X\right]^{2}_{P_{\mathcal{Q}(\mathcal{A}_k)}(S)}$. This points out again that $\delta$ improves the LSM algorithm when taken into account. Besides, this result holds also in each single case of matrix $\mathcal{B}_P$, i.e. there is little variation from the average American option price result and the single cases $\bar{\mathcal{A}}_{k_i}$ for $i = 1, ..., \zeta = 20$[21].

Lastly, an IOT was performed on the following two specifications of the conditional expectations of the payoffs:

$$\mathbb{E}\left[Y|X\right]^{1}_{P_{\mathcal{Q}(\mathcal{A}_k)}(S)} = c + \sum_{q=0}^{1} p_q\left(S\right)\hat{\beta}_q \tag{5.3}$$

$$\mathbb{E}\left[Y|X\right]^{2}_{P_{\mathcal{Q}(\mathcal{A}_k)}(S)} = c + \sum_{q=0}^{1} p_q\left(S\right)\hat{\beta}_q + p_0\left(\delta\right)\hat{\beta}_{\delta_0} \tag{5.4}$$

In other words, equation 5.3 is the benchmark, whereas equation 5.4 is the improved benchmark[22]. The average difference of in sample and out of sample prices for the benchmark is $2.82 \cdot 10^{-4}$, whereas the one for the improved benchmark is $-5.97 \cdot 10^{-3}$ [23]. They are both really close to zero, which means that the decision rule of the LSM is till robust when $\delta$ is added as regressor. This is another remarkable results that underlines the consistency of adding $\delta$ in the regression specification.

## 5.3 Information Set $\mathcal{A}_\theta$

The previous two information sets pointed out the better performances of the power polynomial compared to the Laguerre one, and the increasing price of the American option when $\delta$ is used as regressor. This section studies another information set, i.e. $\mathcal{A}_\theta$ table 5.3, which has the long term mean of the stochastic convenience yield and stochastic volatility processes changing in $\bar{\mathcal{A}}_{\theta_i}$ for $i = 1, ..., \zeta = 20$ [24]. A peculiarity of this section is that the values assumed by $\theta_\delta$ and $\theta_V$, in $\bar{\mathcal{A}}_\theta$, are fairly higher than what the literature has empirically presented, i.e. w.r.t. the parameters taken from [12] and [2]. This choice of relatively higher values was made to test

[21] $\mathcal{B}_P$ corresponds to table B.8 in the Appendix for information set $\mathcal{A}_k$.

[22] The IOT for the benchmark is reported in table B.9, while the one of the improved benchmark in table B.10 in the Appendix.

[23] These values are also reported in table B.24 in the Appendix.

[24] Also $S_0$ and $T$ still change in $\bar{\mathcal{A}}_{\theta_i}$ for $i = 1, ..., \zeta = 20$ as usual.

the reaction of our methodology of analysis and investigation questions even in fairly extreme situations, e.g. in a black swan market.

Table 5.3: Information set $\mathcal{A}_\theta$

| $\mathcal{A}_\theta$ | $r$ | $k_\delta$ | $\sigma_\delta$ | $\delta_0$ | $k_V$ | $\sigma_V$ | $V_0$ | $\rho_{S\delta}$ | $\rho_{SV}$ | $\rho_{\delta V}$ | $K$ | $n$ | $dy$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Value | 0.06 | 1.876 | 0.527 | $\theta_\delta$ | 27.636 | 0.443 | $\theta_V$ | 0.766 | 0.023 | 0 | 100 | $2 \cdot 10^5$ | 360 |

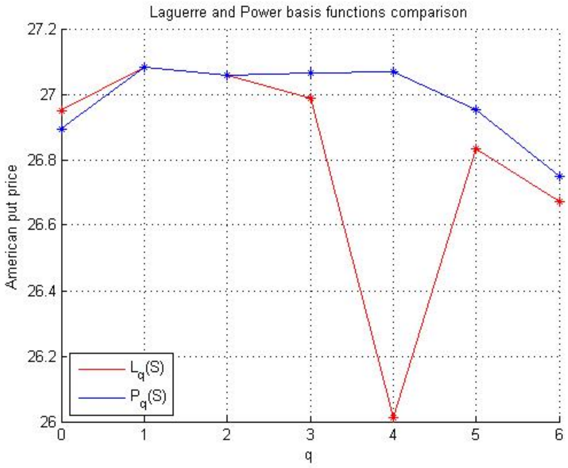
*Notes: Parameters $r$, $k_\delta$, $\sigma_\delta$ and $\rho_{S\delta}$ are taken from [12]. While parameters $k_V$, $\sigma_V$, $\rho_{SV}$ are taken from [2]. $\delta_0$ and $V_0$ are the initial conditions of the SDEs for the yield and volatility process, respectively. They are assumed to be equal to the long term mean of their process. $\rho_{\delta V}$ and $K$ are assumed to be equal to 0 and 100, respectively. $n$ is the number of simulated paths, 50% of them are antithetic variates. $dy$ is the number of days per year and it is assumed to be also the number of exercise possibilities per year of the American option. The values assumed by $S_0$, $\theta_\delta$, $\theta_V$ and $T$ are reported in the simulation tables in the Appendix, where information set $\mathcal{A}_\theta$ is used.*

Figure 5.3a shows the Laguerre-power polynomial analysis for information set $\mathcal{A}_\theta$[25]. As it can be seen from the figure, $L_1(S)$ and $P_1(S)$ maximize the value of the average American option price, for the Laguerre and power cases, respectively[26]. As usual, the Laguerre polynomial case decreases rapidly, whereas the power one slower and smoothly. On the other hand, in this case the Laguerre polynomial specification pushes the average American option price higher than its competitor[27]. Therefore, $L_1(S)$ is preferred to $P_1(S)$ in average sense; as well as in the single cases too, i.e. $\bar{\mathcal{A}}_{\theta_i}$ for $i = 1, ..., \zeta = 20$[28] However, it is fair to point out that the variability is a bit larger than in the previous two analyzed information sets.

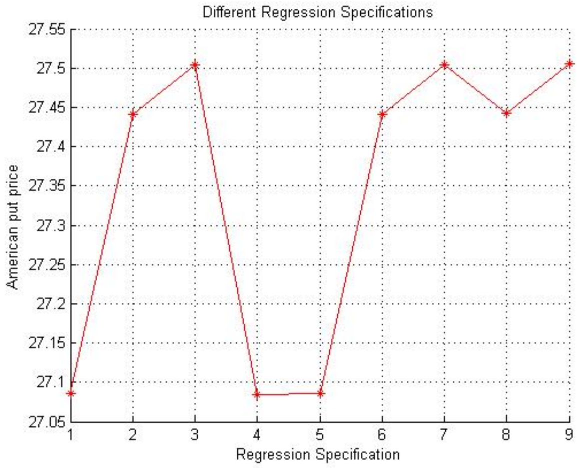
The new benchmark for the yield-volatility basis function analysis becomes $\mathbb{E}[Y|X]^1_{L_{\mathcal{Q}(\mathcal{A}_\theta)}(S)} = L_1(S)$ and its extensions $\left\{\mathbb{E}[Y|X]^{\mathcal{K}}_{L_{\mathcal{Q}(\mathcal{A}_\theta)}(S)}\right\}^9_{\mathcal{K}=2}$, which are presented in table 4.1[29]. Figure 5.3b points out the same fact as the other information sets, i.e. $\delta$ is the variable that matters and not $\sqrt{V}$. Even if the Laguerre regression specification type is used in the yield-volatility basis function analysis, $\delta$ is still the variable that can increase the value of the average American option w.r.t. benchmark[30]. This latter result is also confirmed in the single cases, i.e. $\bar{\mathcal{A}}_{\theta_i}$ for $i = 1, ..., \zeta = 20$. Nonetheless, the variability of the results is a bit larger than the previous information sets, but still really good[31].

[25] The computations of matrix $\mathcal{D}_L$ and $\mathcal{D}_P$ are reported in the Appendix in table B.11 and B.12, respectively.
[26] The numerical values for the USD cent rule of thumb are reported in table B.21 and B.22 in the Appendix.
[27] This is showed in table B.23 in the Appendix.
[28] Each single row of tables B.11 and B.12, in the Appendix, show this latter feature.
[29] This is a different table than in the previous information sets.
[30] The computation of $\mathcal{B}_L$ is reported in table B.13 in the Appendix.
[31] This latter statement can be extracted from table B.13 in the Appendix.

Figure 5.3: This figure shows: Laguerre-power polynomial analysis, tables B.11 and B.12 in the Appendix, and yield-volatility basis function analysis, table B.13 in the Appendix, for information set $\mathcal{A}_\theta$, table 5.3.




(a) Laguerre-power polynomial analysis




(b) yield-volatility basis function analysis

To conclude, an IOT on the benchmark, equation 5.5, and on the improved benchmark, equation 5.6, was performed[32].

$$\mathbb{E}\left[Y|X\right]^1_{L_{\mathcal{Q}(\mathcal{A}_\theta)}(S)} = c + \sum_{q=0}^{1} l_q\left(S\right)\hat{\beta}_q \tag{5.5}$$

$$\mathbb{E}\left[Y|X\right]^2_{L_{\mathcal{Q}(\mathcal{A}_\theta)}(S)} = c + \sum_{q=0}^{1} l_q\left(S\right)\hat{\beta}_q + l_0\left(\delta\right)\hat{\beta}_{\delta_0} \tag{5.6}$$

The average difference between the in sample and out of sample prices for the benchmark case is $1.75 \cdot 10^{-4}$, whereas for the improved benchmark $1.039 \cdot 10^{-2}$ [33]. This is the first time that the improved benchmark shows an IOT average difference that substantially differs from the benchmark. In the author's opinion this is due to the use of fairly high values of $\theta_\delta$ and $\theta_V$ as well as for the use of the Laguerre polynomial regression specification, rather than the power one. Even though the result is not as good as in the previous information sets, $\delta$ still presents strong properties to be used as a regressor in the LSM, even in a black swan market.

[32]The IOT for the benchmark is in table B.14, whereas the one for the improved benchmark is in table B.15 in the Appendix.

[33]This results are also reported in table B.24 in the Appendix.

## 5.4 Information Set $\mathcal{A}_\rho$

Throughout the basis function analysis different information sets have been analyzed to see the reaction of the LSM algorithm by inserting the convenience yield and the volatility as regressors in the computation of the conditional expectation of the payoffs. As the reader notices the choice of the information sets, i.e. $\mathcal{A}_\sigma$, $\mathcal{A}_k$ and $\mathcal{A}_\theta$, and their complements, i.e. $\bar{\mathcal{A}}_\sigma$, $\bar{\mathcal{A}}_k$ and $\bar{\mathcal{A}}_\theta$, was not random. For instance, in $\mathcal{A}_\sigma$ the SCYH model parameters $\sigma_\delta$ and $\sigma_V$ were allowed to change. As it can be seen, in each information set different focus was given to different features of the SCYH model. For example, in $\mathcal{A}_k$ the focus was on the speed of mean reversion of the two state processes, whereas, in $\mathcal{A}_\theta$ the focus was on the long term mean of the processes. Fundamentally, the only pair of parameters that are left to be analyzed are those related to the correlation matrix of the Brownian motions, i.e. $\rho_{S\delta}$ and $\rho_{SV}$ [34].

The information set $\mathcal{A}_\rho$ is presented in table 5.4 and its general complement $\bar{\mathcal{A}}_\rho$ is made up of $S_0$, $\rho_{S\delta}$, $\rho_{SV}$ and $T$. The analysis is carried out as usual and figure 5.4a shows the Laguerre-power polynomial analysis. It is easy to see that $P_3(S)$ is the specification that maximizes the average American option price and it shows a well-defined quadratic function shape. $P_3(S)$ has also a strong and robust feature in each single case, i.e. $\bar{\mathcal{A}}_{\rho_i}$ for $i = 1, ..., \zeta = 20$ [35].

Table 5.4: Information set $\mathcal{A}_\rho$

| $\mathcal{A}_\rho$ | $r$ | $k_\delta$ | $\theta_\delta$ | $\delta_0$ | $k_V$ | $\theta_V$ | $V_0$ | $\sigma_\delta$ | $\sigma_V$ | $\rho_{\delta V}$ | $K$ | $n$ | $dy$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Value | 0.06 | 1.876 | 0.000456 | $\theta_\delta$ | 27.636 | 0.077 | $\theta_V$ | 0.527 | 0.443 | 0 | 100 | $2 \cdot 10^5$ | 360 |

*Notes: Parameters $r$, $k_\delta$, $\theta_\delta$ and $\sigma_\delta$ are taken from [12]. While parameters $k_V$, $\theta_V$, $\sigma_V$ are taken from [2]. $\delta_0$ and $V_0$ are the initial conditions of the SDEs for the yield and volatility process, respectively. They are assumed to be equal to the long term mean of their process. $\rho_{\delta V}$ and $K$ are assumed to be equal to 0 and 100, respectively. $n$ is the number of simulated paths, 50% of them are antithetic variates. $dy$ is the number of days per year and it is assumed to be also the number of exercise possibilities per year of the American option. The values assumed by $S_0$, $\rho_{S\delta}$, $\rho_{SV}$ and $T$ are reported in the simulation tables in the Appendix, where information set $\mathcal{A}_\rho$ is used.*

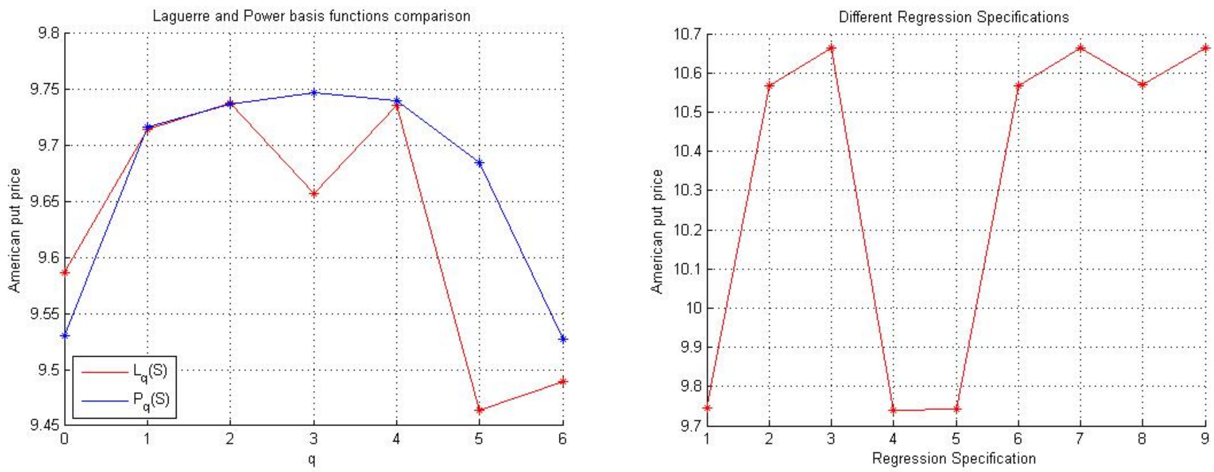
Also information set $\mathcal{A}_\rho$ shows a strong evidence in favor of the $\delta$ as a regressor in the yield-volatility basis function analysis, showed in figure 5.4b. The benchmark $\mathbb{E}[Y|X]^1_{P_{\mathcal{Q}(\mathcal{A}_\rho)}(S)} =$

[34] As it was explained in the previous information sets, $\rho_{\delta V}$ is assumed to be zero. This is due to the fact that no literature has investigated this parameter empirically so far, that the author is aware of. Hence, the author of the thesis chose to leave the yield end volatility processes uncorrelated to each other. Furthermore, the values assumed for $\rho_{S\delta}$ and $\rho_{SV}$ are fairly in line with the literature, i.e. [12] and [2].

[35] As usual the computation of $\mathcal{D}_L$ and $\mathcal{D}_P$ are reported in table B.16 and B.17 in the Appendix. The increments of plot 5.4a are reported in table B.21 and B.22 in the Appendix, for the Laguerre and power cases respectively. Lastly, the value function differences in plot 5.4a are reported in table B.23 in the Appendix.

$P_3(S)$ in table 4.2 is definitively improved by $\mathbb{E}[Y|X]^2_{P_{\mathcal{Q}(\mathcal{A}_\rho)}(S)}$ and $\mathbb{E}[Y|X]^3_{P_{\mathcal{Q}(\mathcal{A}_\rho)}(S)}$ which increase the average American option price by 0.8202 and 0.9165 USD, respectively [36]. These features are also consistent and stable in each single case too, i.e. $\bar{\mathcal{A}}_{\rho_i}$ for $i = 1, ..., \zeta = 20$.

Figure 5.4: This figure shows: Laguerre-power polynomial analysis, tables B.16 and B.17 in the Appendix, and yield-volatility basis function analysis, table B.18 in the Appendix, for information set $\mathcal{A}_\rho$, table 5.4.




(a) Laguerre-power polynomial analysis (b) yield-volatility basis function analysis

Lastly, an IOT was performed between the benchmark, equation 5.7, and the improved benchmark equation 5.8.

$$\mathbb{E}[Y|X]^1_{P_{\mathcal{Q}(\mathcal{A}_\rho)}(S)} = c + \sum_{q=0}^{3} p_q(S)\,\hat{\beta}_q \tag{5.7}$$

$$\mathbb{E}[Y|X]^2_{P_{\mathcal{Q}(\mathcal{A}_\rho)}(S)} = c + \sum_{q=0}^{3} p_q(S)\,\hat{\beta}_q + p_0(\delta)\,\hat{\beta}_{\delta_0} \tag{5.8}$$

The average difference between in sample and out of sample prices for the benchmark is $8.784 \cdot 10^{-3}$, whereas $4.351 \cdot 10^{-3}$ for the improved benchmark[37]. This is the first time that the IOT shows a better result for the improved benchmark compared to the benchmark. This means that the addition of $\delta$ in the LSM decision rule does not increase systematically the $\beta$s imprecision out of sample.

To conclude, all the information sets have showed strong evidence for inserting $\delta$ as regressor in the LSM algorithm while strong rejection of the volatility process. On the other hand, the choice

[36] The computation of matrix $\mathcal{B}_P$ is reported in the Appendix in table B.18. While $\mathbb{E}[Y|X]^2_{P_{\mathcal{Q}(\mathcal{A}_\rho)}(S)}$ and $\mathbb{E}[Y|X]^3_{P_{\mathcal{Q}(\mathcal{A}_\rho)}(S)}$ are reported in table 4.2.

[37] The IOT for the benchmark is reported in table B.19 in the Appendix, while the one for the improved benchmark is in table B.20. Moreover the average price difference between in sample and out of sample options is reported in table B.24 in the Appendix.

of which polynomial specification to use between Laguerre and power one is more uncertain. Four out of three information sets pointed out the preference of the power polynomial for its smoothness and stability in the American option price. In the author's opinion the power polynomial specification is more suitable than the Laguerre one due to less variability in its performances. In a nutshell, the overall picture of the analysis is that a power polynomial specification with $\delta$ as regressor, in addition to the commodity price, is an optimal conditional expectation specification in the LSM with the SCYH model.

## 5.5 Early Exercise Premium Analysis

The EEP analysis closes the thesis investigation questions by seeing the EEP reaction when an American option is priced with and without $\delta$ as regressor. Basically, by using a new information set, which is predominantly based on the literature, surfaces of American option prices are compared to each other, in order to investigate the main insights behind the insertion of $\delta$ as regressor.

The information set $\mathcal{A}_f$ is provided in table 5.5. The $f$ stands for $final$ and the yield parameters $k_\delta$, $\theta_\delta$, $\sigma_\delta$, $\rho_{S\delta}$ and $r$ are taken from *E. Schwartz*, [12]. While the volatility ones, $k_V$, $\theta_V$, $\sigma_V$ and $\rho_{SV}$ are taken from *James S. Doran et al.*, [2]. All the other CS and values are assumed, as usual.

Table 5.5: Information set $\mathcal{A}_f$

| $\mathcal{A}_f$ | Value | $\mathcal{A}_f$ | Value | $\mathcal{A}_f$ | Value |
|---|---|---|---|---|---|
| $S_0$ | 100 | $\sigma_V$ | 0.443 | $\rho_{S\delta}$ | 0.766 |
| $r$ | 0.06 | $k_\delta$ | 1.876 | $\rho_{SV}$ | 0.023 |
| $k_V$ | 27.636 | $\theta_\delta$ | 0.000456 | $\rho_{\delta V}$ | 0 |
| $\theta_V$ | 0.077 | $\delta_0$ | $\theta_\delta$ | $n$ | $2 \cdot 10^5$ |
| $V_0$ | $\theta_V$ | $\sigma_\delta$ | 0.527 | $dy$ | 360 |

*Notes: Yield parameters and $r$ are taken from [12], while the volatility ones from [2]. All the other information is assumed as usual.*

The Laguerre-power polynomial analysis as well as the yield-volatility basis function analysis is carried out in figure B.1, tables B.25 and B.26, in the Appendix, for information set $\mathcal{A}_f$. The results point out that the most suitable specifications for the conditional expectation for the benchmark and the improved benchmark are those in equation 5.9 and 5.10, respectively[38].

[38]Note that in table B.25 the difference between $L_2(S)$ and $P_2(S)$ is basically zero and the author chose to use the power specification due to its well-behaved properties.

$$\mathbb{E}\left[Y|X\right]^{1}_{P_{\mathcal{Q}\left(\mathcal{A}_f\right)}(S)} = P_2\left(S\right) = c + \sum_{q=0}^{2} p_q\left(S\right)\hat{\beta}_q \tag{5.9}$$

$$\mathbb{E}\left[Y|X\right]^{2}_{P_{\mathcal{Q}\left(\mathcal{A}_f\right)}(S)} = c + \sum_{q=0}^{2} p_q\left(S\right)\hat{\beta}_q + p_0\left(\delta\right)\hat{\beta}_{\delta_0} \tag{5.10}$$

Where the notation used in equation 5.9 and 5.10 corresponds to the one reported in table 4.2. These two specifications will be used to price American options surfaces that will be compared to each other to see the insights behind the EEP. Define the American option price generated by equation 5.9, when used in the LSM algorithm, by $\Pi_A^{P_2(S)}$, whereas the one generated by equation 5.10, when used in the LSM algorithm, by $\Pi_A^{\delta}$. We can define the EEP as the difference between a general American option price, $\Pi_A$, and the equivalent European option price, $\Pi_E$.

$$EEP = \Pi_A - \Pi_E \tag{5.11}$$

Note that $\Pi_E$ is computed with Monte Carlo as well, i.e. by using equation 1.9.

Figure 5.5a shows the EEP behavior[39] for different strikes, $K$, end times to maturity, $T$. Basically, the z-axis in computed by the difference between the American option price and the equivalent European option price. The higher surface is computed by the following difference $\Pi_A^{\delta} - \Pi_E$, whereas the lower surface is computed by the following difference $\Pi_A^{P_2(S)} - \Pi_E$. As it can be seen from figure 5.5a, the EEP computed with the improved benchmark always excides the one computed with the benchmark. In both of the cases, the EEP increases with the length of the contract, i.e. $T$, and also with ITM options.

Figure 5.5b shows the EEP premium calculated as in figure 5.5a but it has also been made as a percentage of the equivalent European option. In other words, the z-axis of figure 5.5b shows the increase in value of the American option w.r.t. the equivalent European one in percentage terms. The surface that lies above is the one computed by equation 5.10, while the lower is computed by equation 5.9. As it can be seen, the percentage increase is remarkable in deep OTM options with long term to maturity. This increase can reach up to 50% and 70% for the benchmark and improved benchmark, respectively.

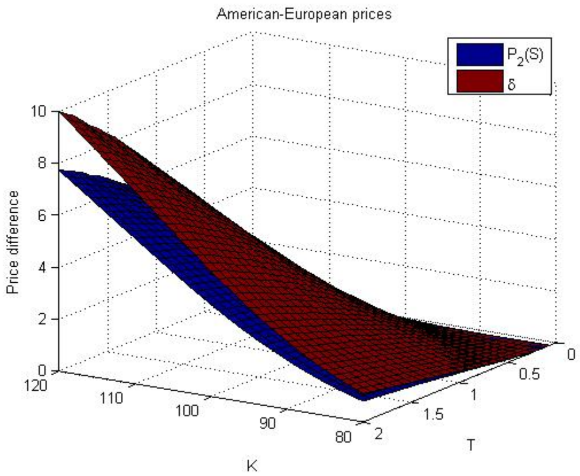
Even if the thesis analysis has been focusing more on the pure price of the contingent claim, few words can be spent on the standard errors of American and European options prices. Figure 5.5c shows the difference of American option standard errors and the equivalent European ones for different $K$ and $T$[40]. In both of the surfaces, the European standard errors are larger for at the money (ATM) options and for OTM options. This is due to the fact that the plot becomes negative in such situations. On the other hand, the American standard errors become larger

[39] The z-axis represents the EEP.

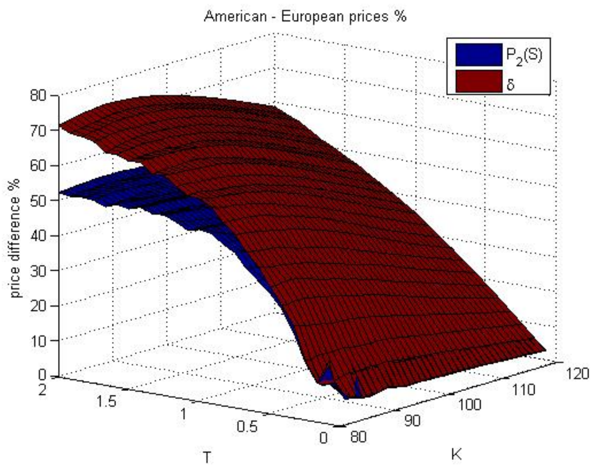
[40] Figure 5.5a, 5.5b and 5.5c all refer to the same computed prices.

in ITM options w.r.t. the equivalent European ones. However, the benchmark surface always lies below the improved benchmark one. This means that the standard errors of the improved benchmark are bigger than the one of the benchmark, i.e. there is less precision and more uncertainty in the contingent claim price in the former case[41].

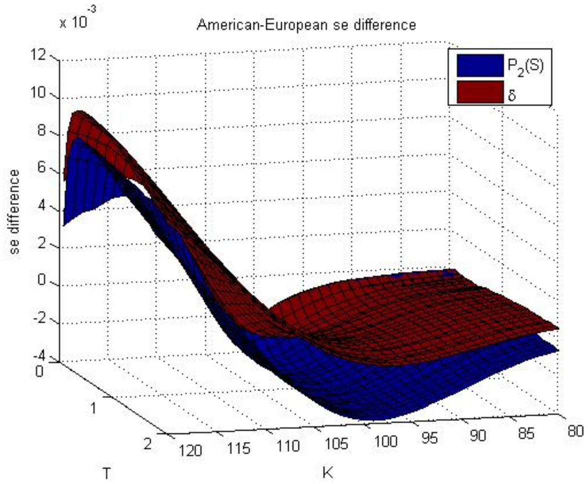
Figure 5.5: These figures show a comparison between American options computed with $P_2(S)$ and with $P_2(S) + p_0(\delta)\hat{\beta}_{\delta_0}$ w.r.t. their equivalent European option. Information set $\mathcal{A}_f$, table 5.5, is used.




(a) $EEP^{A-E}$




(b) $EEP^{A-E}_{\%_E}$




(c) $s.e.^{A-E}$

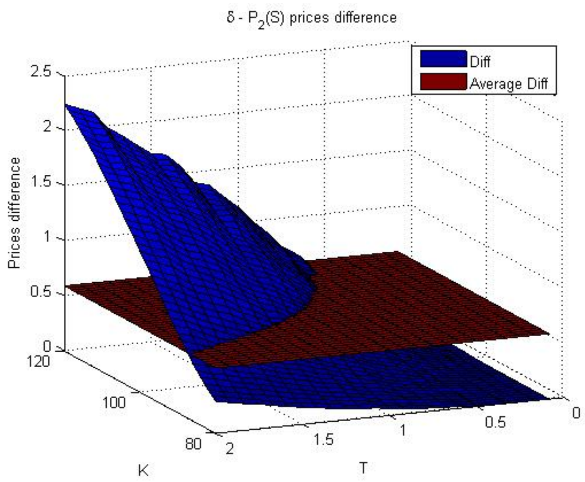
Figure 5.6a shows the difference in price between $\Pi_A^{\delta}$ and $\Pi_A^{P_2(S)}$, which corresponds to the difference in EEP between the improved benchmark and the benchmark. The plot underlines the increase in the difference for long contract terms and deep ITM options. Moreover, the horizontal surface shows the average increase of the EEP of improved benchmark w.r.t. the benchmark one. In other words, the American option price increases by 0.5924 USD when $\delta$ is inserted as regressor, over the entire surface in average.

Figure 5.6b shows the difference in EEP between $\Pi_A^{\delta}$ and $\Pi_A^{P_2(S)}$ in percentage terms, w.r.t.

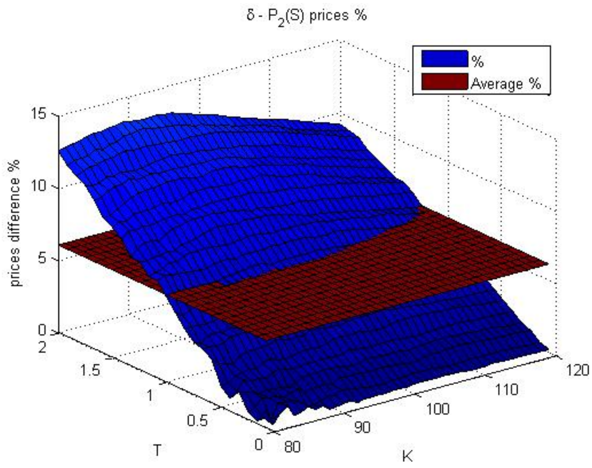
[41] All plots of figure 5.5 and 5.6 are computed from figure B.2, B.3 and B.4 in the Appendix.

$\Pi_A^{P_2(S)}$. Fundamentally, the figure points out how much the American option price computed with equation 5.10 increases w.r.t. the 5.9 specification in percentage terms w.r.t. the latter. Also in this case the biggest improvement is in long term options end especially in deep OTM ones. The horizontal surface implies the average percentage increase over the entire surface, which is 6.017%.

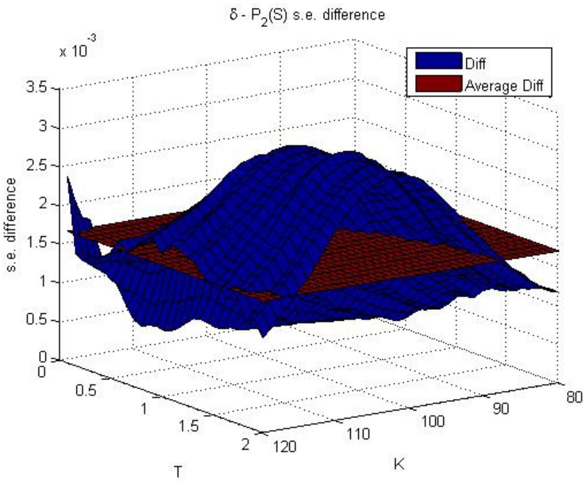
Figure 5.6: These figures show the difference between American options computed with $P_2(S)$ and with $P_2(S) + p_0(\delta)\hat{\beta}_{\delta_0}$ w.r.t. the American option computed with $P_2(S)$. Information set $\mathcal{A}_f$, table 5.5, is used.




(a) $EEP^{\delta - P_2(S)}$




(b) $EEP^{\delta - P_2(S)}_{\%_{P_2(S)}}$




(c) $s.e.^{\delta - P_2(S)}$

Lastly, figure 5.6c represents the difference of American options standard errors between the one corresponding to $\Pi_A^{\delta}$ price and the one corresponding to $\Pi_A^{P_2(S)}$ price. As it can be seen, the standard errors of the improved benchmark are always greater than the one of the benchmark. However, the difference increases for ATM options with long maturity, while it decreases for deep OTM options with short maturity. The horizontal surface shows the average difference, which is 0.0017. Stated differently, the improved benchmark has higher imprecision especially in ATM options compared to the benchmark.

To conclude, we can state that the EEP increases with the improved benchmark especially with moneyness. Moreover, the EEP increases also for deep OTM options with long contract term, when EEP is looked in percentage terms. On the other hand, also the imprecision of the Monte Carlo method increases with the improved benchmark. Nonetheless, the magnitude of the improved price seems to fully offset the increment of the standard error.

# Chapter 6

# Conclusion

The thesis has presented and discussed a general contingent claim pricing framework as well as a practical implementation of the LSM for pricing American options. Then, a commodity process has been introduced, i.e. SCYH model. Then, the used methodology of analysis introduced the reader to the basis function analysis. Laguerre-power polynomial analysis as well as yield-volatility basis function analysis have been extensively analyzed for different information sets. Then an EEP behavior of the benchmark and improved benchmark has been showed graphically.

The main result of the Laguerre-power polynomial analysis is that the power polynomial shows better properties than the Laguerre one. The average American option price as function of the power polynomial shows a smooth trend and good stability as well. As a result, the power polynomial is suggested to be used in real world applications.

The main result of the yield-volatility analysis is that the convenience yield, when added as regressor, further increases the American option price while the volatility process does not. This is due to the fact that the first moment of the distribution of the underlying asset matters more than higher moments, when the contingent claim is priced. As a result, this leads to suggest the insertion of convenience yield as regressor in the LSM algorithm.

The EEP analysis showed the relevance of the convenience yield especially for deep OTM options with long contract term, when the EEP is looked in percentage terms. Moreover, the benefits of the convenience yields increases with moenyness. As a result, the convenience yield impacts prominently over the entire option term structure as well as contract specifications.

The entire thesis analysis was based on numerical computations. Nevertheless, analytical proofs should be carried out in order to investigate the thesis questions from a more theoretical perspective. Unfortunately, this goes beyond the purpose of this thesis and it is left to future research.

# Appendices

# Appendix A

# Geometeric Browmian Motion Test

## A.1 Numerical Example

### LSM comparison

In order to guarantee an error free code, the thesis reproduces the pricing of an American put option with underlying asset modelled as GBM with no dividends, e.g. a stock. Basically, there is a comparison between the results obtained by *Longstaff* and *Schwartz*, [11], and the one obtained by the author of the thesis. The setup is as the one in their paper and it is as follows.

The payoff function of the American put is defined as $\Phi(S) = (K - S)^+$, where $K$ is the strike price and $S$ is the underlying asset. The dynamic of the underlying financial security is defined as:

$$dS(t) = S(t)\,rdt + S(t)\,\sigma dW(t)^{\mathbb{Q}} \tag{A.1}$$

The simulation of the SDE in equation A.1 was done with its well-known closed form solution. The number of paths is 100.000 (50.000 plus 50.000 antithetic). The risk free rate $r$ and the volatility $\sigma$ are constant. The risk free rate is assumed to be 0.06, while the strike price to be 40 and the number of exercise possibilities is 50 per year. The first three Laguerre basis functions were used to compute the conditional expectation of the payoff, i.e. $L_2(S)$ was used. All the other features are defined in table A.1 in the Appendinx.

As it can be seen in table A.1 the difference between the American option computed in the reference paper, $\Pi_A^+$, and the one computed in the thesis, $\Pi_A^*$, is minimal. In other words, the average mean difference between the two prices is -0.00713, which is less than a USD cent. This points out that the algorithm is fairly good in computing the price of the contingent claim at hand.

On the other hand, the author noticed that the difference between the standard error reported in the reference paper and the one computed in the thesis, i.e. $se^+ - se^*$, is systematically positive. This means that the performance of the algorithm in the thesis outperforms the one in the original paper. This is due to a small bug or imperfection in the original paper in the use of the antithetic variates. The original paper computes the price of the American option as a mere average of the discounted payoffs after the implementation of the core of the LSM algorithm. Nonetheless, this is not fully correct in the case of the use of antithetic variates, when we want to compute the price standard error as well. In such a latter case the computation of the contingent claim price should be as the expectation over the independent generated payoffs. Basically, the first $n/2$ generated payoffs should be added to the corresponding $n/2$ antithetic ones and divided by 2. Then, the average of this new random vector should be used as numerical approximation of the contingent claim price and this new random vector should be used to compute the price standard error. Note that the price of the contingent claim does not change between the two different procedures, whereas the standard error does change, and it becomes smaller. This is due to the fact that in the latter way, the negative covariance of the antithetic variates is exploited while in the former way is not, see *Glasserman* for further explanation on antithetic variates [7].

Anyhow, the algorithm coded in the thesis seems to perform really well and it will be used for the rest of the thesis too.

### Test procedure

Figure A.1: Laguerre and power polynomial comparison of the GBM Numerical Exmaple that correspond to the setting presented by *Longstaff* and *Schwartz*, see [11] or section A.1 under LSM comparison.

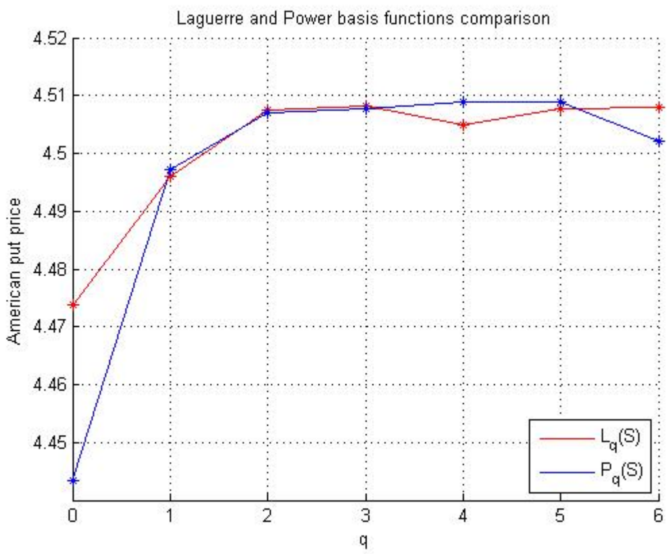


Table A.1: Comparison between the prices of the American put option computed in the paper of *Longstaff* and *Schwartz* and in this thesis.

| $S_0$ | $\sigma$ | $T$ | $\Pi_A^+$ | $se^+$ | $EEP^+$ | $\Pi_A^*$ | $se^*$ | $EEP^*$ | $\Pi_A^+ - \Pi_A^*$ | $se^+ - se^*$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 36 | 0.2 | 1 | 4.472 | 0.01 | 0.628 | 4.477 | 0.006 | 0.633 | -0.005 | 0.004 |
| 36 | 0.2 | 2 | 4.821 | 0.012 | 1.058 | 4.826 | 0.007 | 1.063 | -0.005 | 0.005 |
| 36 | 0.4 | 1 | 7.091 | 0.02 | 0.380 | 7.100 | 0.008 | 0.389 | -0.009 | 0.012 |
| 36 | 0.4 | 2 | 8.488 | 0.024 | 0.788 | 8.520 | 0.011 | 0.820 | -0.032 | 0.013 |
| 38 | 0.2 | 1 | 3.244 | 0.009 | 0.392 | 3.250 | 0.005 | 0.398 | -0.006 | 0.004 |
| 38 | 0.2 | 2 | 3.735 | 0.011 | 0.744 | 3.752 | 0.006 | 0.761 | -0.017 | 0.005 |
| 38 | 0.4 | 1 | 6.139 | 0.019 | 0.305 | 6.141 | 0.008 | 0.306 | -0.002 | 0.011 |
| 38 | 0.4 | 2 | 7.669 | 0.022 | 0.690 | 7.651 | 0.010 | 0.672 | 0.018 | 0.012 |
| 40 | 0.2 | 1 | 2.313 | 0.009 | 0.247 | 2.317 | 0.005 | 0.250 | -0.004 | 0.004 |
| 40 | 0.2 | 2 | 2.879 | 0.01 | 0.523 | 2.890 | 0.006 | 0.534 | -0.011 | 0.004 |
| 40 | 0.4 | 1 | 5.308 | 0.018 | 0.248 | 5.322 | 0.009 | 0.262 | -0.014 | 0.009 |
| 40 | 0.4 | 2 | 6.921 | 0.022 | 0.595 | 6.933 | 0.010 | 0.607 | -0.012 | 0.012 |
| 42 | 0.2 | 1 | 1.617 | 0.007 | 0.152 | 1.621 | 0.006 | 0.157 | -0.004 | 0.001 |
| 42 | 0.2 | 2 | 2.206 | 0.01 | 0.365 | 2.205 | 0.007 | 0.364 | 0.001 | 0.003 |
| 42 | 0.4 | 1 | 4.588 | 0.017 | 0.209 | 4.614 | 0.010 | 0.236 | -0.026 | 0.007 |
| 42 | 0.4 | 2 | 6.243 | 0.021 | 0.507 | 6.239 | 0.011 | 0.503 | 0.004 | 0.010 |
| 44 | 0.2 | 1 | 1.118 | 0.007 | 0.101 | 1.105 | 0.005 | 0.088 | 0.013 | 0.002 |
| 44 | 0.2 | 2 | 1.675 | 0.009 | 0.246 | 1.687 | 0.007 | 0.258 | -0.012 | 0.002 |
| 44 | 0.4 | 1 | 3.957 | 0.017 | 0.174 | 3.963 | 0.011 | 0.180 | -0.006 | 0.006 |
| 44 | 0.4 | 2 | 5.622 | 0.021 | 0.420 | 5.635 | 0.012 | 0.433 | -0.013 | 0.009 |

*Notes: $\Pi_A^+$ and $se^+$ represent the American option price and standard error as reported in, [11], $EEP^+$ is computed as $\Pi_A^+$ minus BS put price, $\Pi_A^*$ and $se^*$ represent the American option price and standard error computed in the thesis, $EEP^*$ is computed as $\Pi_A^*$ minus BS put price.*

As it can be seen in figure A.1 both the Laguerre and the power function show a quite stable asymptotic behavior when reaching the maximum average American option price. However, the Laguerre cases still shows a small drop. By looking at table A.2 and A.3 it is obvious that the $L_2(S)$ specification is the most suitable one. It is not a coincidence that this is the same specification used by *Longstaff* and *Schwartz*, in their paper, see [11]. This points out the strength of the adopted methodology in the thesis for selecting the regression specification.

Table A.2: This table shows the increments of the average American option price plotted in figure A.1, which correspond to the GBM numerical example, in the Laguerre and power polynomial specification.

| $\Delta_{\mathcal{G}}$ | $I_{i=0}$ | $I_{i=1}$ | $I_{i=2}$ | $I_{i=3}$ | $I_{i=4}$ | $I_{i=5}$ |
|---|---|---|---|---|---|---|
| $\Delta_L$ | -0.02224 | -0.01138 | -0.00092 | 0.003288 | -0.00275 | -0.00015 |
| $\Delta_P$ | -0.05377 | -0.00994 | -0.00069 | -0.00119 | 9.91E-05 | 0.006669 |

*Notes: Where* $I_i = \bar{\Pi}_{A,\mathcal{G}_i(S)} - \bar{\Pi}_{A,\mathcal{G}_{i+1}(S)}$ *and* $\bar{\Pi}_{A,\mathcal{G}_i(S)}$ *is the average American option price for a chosen specification* $\mathcal{G}$*, which can be either Laguerre (L) or power (P) polynomial, e.g. the i Laguerre specification.*

Table A.3: This table shows the difference between the value function of the Laguerre and power polynomial plotted in figure A.1, which corresponds to the GBM numerical example.

| $q$ in figure | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| GBM | 0.030367 | -0.00116 | 0.000273 | 0.000502 | -0.00398 | -0.00113 | 0.005682 |

*Notes: Any numerical value is computed by subtracting the power polynomial value function from the Laguerre one. In other words, a negative value implies that the power polynomial specification gives a higher price to the American option in average.*

# Appendix B

# Basis Function Tables and Figures

In the following pages the computation of the $\mathcal{D}_L$, $\mathcal{D}_P$, $\mathcal{B}_L$ and $\mathcal{B}_P$ matrices will be presented as well as the in sample and out of sample tests for the benchmark and improved benchmark. These tables are reported for each information set analyzed in the basis function analysis. Then, surfaces of American options prices and standard errors will be presented as fundamental underlying of the EEP analysis.

## B.1 Tables

Table B.1: This table shows the price of an American put option for different parameters and basis function specifications. The basis function used is the Laguerre polynomial and the information set is $\mathcal{A}_\sigma$, table 5.1.

| $S_0$ | $\sigma_\delta$ | $\sigma_V$ | $T$ | $L_0(S)$ | $L_1(S)$ | $L_2(S)$ | $L_3(S)$ | $L_4(S)$ | $L_5(S)$ | $L_6(S)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 90 | 0.3 | 0.2 | 1 | 13.604 | 13.762 | 13.812 | 13.816 | 13.803 | 13.347 | 13.822 |
| 90 | 0.7 | 0.2 | 2 | 16.173 | 16.280 | 16.320 | 16.313 | 15.716 | 15.145 | 15.312 |
| 90 | 0.3 | 0.6 | 1 | 13.644 | 13.772 | 13.818 | 13.827 | 13.653 | 13.357 | 13.829 |
| 90 | 0.7 | 0.6 | 2 | 16.147 | 16.253 | 16.294 | 16.290 | 16.200 | 15.397 | 16.275 |
| 95 | 0.3 | 0.2 | 1 | 10.669 | 10.786 | 10.822 | 10.832 | 10.802 | 9.861 | 10.643 |
| 95 | 0.7 | 0.2 | 2 | 13.138 | 13.241 | 13.267 | 13.272 | 13.268 | 12.529 | 13.269 |
| 95 | 0.3 | 0.6 | 1 | 10.625 | 10.745 | 10.767 | 10.780 | 10.780 | 10.708 | 10.778 |
| 95 | 0.7 | 0.6 | 2 | 13.121 | 13.209 | 13.246 | 13.240 | 13.029 | 13.072 | 13.238 |
| 100 | 0.3 | 0.2 | 1 | 8.134 | 8.235 | 8.271 | 8.275 | 8.278 | 8.111 | 8.208 |
| 100 | 0.7 | 0.2 | 2 | 10.582 | 10.652 | 10.686 | 10.674 | 10.659 | 10.559 | 9.721 |
| 100 | 0.3 | 0.6 | 1 | 8.115 | 8.212 | 8.245 | 8.247 | 8.235 | 8.083 | 8.218 |
| 100 | 0.7 | 0.6 | 2 | 10.598 | 10.653 | 10.687 | 10.028 | 10.624 | 9.625 | 10.683 |
| 105 | 0.3 | 0.2 | 1 | 6.089 | 6.173 | 6.205 | 6.202 | 6.210 | 6.190 | 6.194 |
| 105 | 0.7 | 0.2 | 2 | 8.395 | 8.461 | 8.491 | 8.134 | 8.491 | 8.223 | 8.439 |
| 105 | 0.3 | 0.6 | 1 | 6.089 | 6.165 | 6.196 | 6.198 | 6.205 | 6.090 | 5.778 |
| 105 | 0.7 | 0.6 | 2 | 8.414 | 8.466 | 8.496 | 7.961 | 8.377 | 8.147 | 8.145 |
| 110 | 0.3 | 0.2 | 1 | 4.483 | 4.541 | 4.560 | 4.557 | 4.566 | 4.516 | 4.371 |
| 110 | 0.7 | 0.2 | 2 | 6.610 | 6.668 | 6.690 | 6.643 | 6.695 | 6.665 | 6.095 |
| 110 | 0.3 | 0.6 | 1 | 4.486 | 4.546 | 4.576 | 4.545 | 4.577 | 4.461 | 4.577 |
| 110 | 0.7 | 0.6 | 2 | 6.626 | 6.674 | 6.697 | 6.553 | 6.697 | 6.618 | 6.406 |

*Notes: $L_Q(S)$ is defined in equation 2.4.*

Table B.2: This table shows the price of an American put option for different parameters and basis function specifications. The basis function used is the power polynomial and the information set is $\mathcal{A}_\sigma$, table 5.1.

| $S_0$ | $\sigma_\delta$ | $\sigma_V$ | $T$ | $P_0(S)$ | $P_1(S)$ | $P_2(S)$ | $P_3(S)$ | $P_4(S)$ | $P_5(S)$ | $P_6(S)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 90 | 0.3 | 0.2 | 1 | 13.505 | 13.774 | 13.809 | 13.822 | 13.824 | 13.822 | 13.801 |
| 90 | 0.7 | 0.2 | 2 | 16.090 | 16.287 | 16.323 | 16.313 | 16.321 | 16.314 | 15.408 |
| 90 | 0.3 | 0.6 | 1 | 13.551 | 13.778 | 13.818 | 13.830 | 13.830 | 13.833 | 13.830 |
| 90 | 0.7 | 0.6 | 2 | 16.062 | 16.256 | 16.300 | 16.287 | 16.296 | 16.291 | 15.449 |
| 95 | 0.3 | 0.2 | 1 | 10.577 | 10.791 | 10.820 | 10.831 | 10.826 | 10.832 | 10.827 |
| 95 | 0.7 | 0.2 | 2 | 13.062 | 13.242 | 13.266 | 13.271 | 13.274 | 13.256 | 13.232 |
| 95 | 0.3 | 0.6 | 1 | 10.539 | 10.753 | 10.767 | 10.782 | 10.784 | 10.786 | 10.272 |
| 95 | 0.7 | 0.6 | 2 | 13.046 | 13.211 | 13.248 | 13.240 | 13.246 | 13.244 | 13.185 |
| 100 | 0.3 | 0.2 | 1 | 8.071 | 8.237 | 8.270 | 8.276 | 8.283 | 8.029 | 8.257 |
| 100 | 0.7 | 0.2 | 2 | 10.519 | 10.655 | 10.682 | 10.679 | 10.681 | 10.640 | 10.648 |
| 100 | 0.3 | 0.6 | 1 | 8.048 | 8.220 | 8.245 | 8.245 | 8.248 | 8.246 | 8.251 |
| 100 | 0.7 | 0.6 | 2 | 10.534 | 10.655 | 10.687 | 10.681 | 10.685 | 10.678 | 10.499 |
| 105 | 0.3 | 0.2 | 1 | 6.033 | 6.175 | 6.204 | 6.210 | 6.215 | 6.212 | 5.807 |
| 105 | 0.7 | 0.2 | 2 | 8.339 | 8.460 | 8.493 | 8.491 | 8.494 | 8.478 | 8.372 |
| 105 | 0.3 | 0.6 | 1 | 6.038 | 6.169 | 6.195 | 6.203 | 6.203 | 6.201 | 6.047 |
| 105 | 0.7 | 0.6 | 2 | 8.360 | 8.466 | 8.495 | 8.490 | 8.495 | 8.465 | 8.029 |
| 110 | 0.3 | 0.2 | 1 | 4.446 | 4.542 | 4.558 | 4.567 | 4.568 | 4.570 | 4.564 |
| 110 | 0.7 | 0.2 | 2 | 6.565 | 6.669 | 6.692 | 6.697 | 6.697 | 6.694 | 6.252 |
| 110 | 0.3 | 0.6 | 1 | 4.443 | 4.549 | 4.576 | 4.576 | 4.583 | 4.582 | 4.579 |
| 110 | 0.7 | 0.6 | 2 | 6.581 | 6.675 | 6.699 | 6.702 | 6.699 | 6.701 | 6.687 |

*Notes: $P_{\mathcal{Q}}(S)$ is defined in equation 2.5.*

Table B.3: This table shows the price of American put options w.r.t. the information set $\mathcal{A}_\sigma$, table 5.1. The conditional expectation of the payoffs, i.e. $\mathbb{E}\left[Y|X\right]_{P_{\mathbb{Q}(\mathcal{A}_\sigma)}(S)}$, is defined as in table 4.2.

| $S_0$ | $\sigma_\delta$ | $\sigma_V$ | $T$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 90 | 0.3 | 0.2 | 1 | 13.838 | 14.067 | 14.120 | 13.841 | 13.841 | 14.066 | 14.119 | 14.068 | 14.119 |
| 90 | 0.7 | 0.2 | 2 | 16.318 | 19.363 | 19.486 | 16.319 | 16.318 | 19.365 | 19.489 | 19.363 | 19.489 |
| 90 | 0.3 | 0.6 | 1 | 13.803 | 14.039 | 14.090 | 13.808 | 13.810 | 14.045 | 14.093 | 14.046 | 14.091 |
| 90 | 0.7 | 0.6 | 2 | 16.287 | 19.404 | 19.515 | 16.300 | 16.303 | 19.405 | 19.511 | 19.405 | 19.513 |
| | | | | | | | | | | | | |
| 95 | 0.3 | 0.2 | 1 | 10.789 | 10.980 | 11.019 | 10.790 | 10.793 | 10.978 | 11.019 | 10.980 | 11.018 |
| 95 | 0.7 | 0.2 | 2 | 13.311 | 16.015 | 16.150 | 13.311 | 13.312 | 16.015 | 16.151 | 16.017 | 16.151 |
| 95 | 0.3 | 0.6 | 1 | 10.762 | 10.952 | 10.994 | 10.766 | 10.768 | 10.958 | 10.993 | 10.961 | 10.994 |
| 95 | 0.7 | 0.6 | 2 | 13.245 | 16.048 | 16.176 | 13.255 | 13.257 | 16.052 | 16.178 | 16.056 | 16.179 |
| | | | | | | | | | | | | |
| 100 | 0.3 | 0.2 | 1 | 8.239 | 8.407 | 8.429 | 8.241 | 8.241 | 8.406 | 8.431 | 8.408 | 8.431 |
| 100 | 0.7 | 0.2 | 2 | 10.670 | 13.019 | 13.156 | 10.672 | 10.670 | 13.019 | 13.157 | 13.019 | 13.158 |
| 100 | 0.3 | 0.6 | 1 | 8.265 | 8.426 | 8.445 | 8.272 | 8.271 | 8.428 | 8.455 | 8.430 | 8.454 |
| 100 | 0.7 | 0.6 | 2 | 10.677 | 13.095 | 13.226 | 10.686 | 10.686 | 13.097 | 13.223 | 13.098 | 13.223 |
| | | | | | | | | | | | | |
| 105 | 0.3 | 0.2 | 1 | 6.210 | 6.331 | 6.346 | 6.211 | 6.209 | 6.334 | 6.345 | 6.333 | 6.347 |
| 105 | 0.7 | 0.2 | 2 | 8.488 | 10.462 | 10.574 | 8.491 | 8.493 | 10.464 | 10.572 | 10.466 | 10.572 |
| 105 | 0.3 | 0.6 | 1 | 6.185 | 6.302 | 6.324 | 6.191 | 6.192 | 6.302 | 6.328 | 6.301 | 6.327 |
| 105 | 0.7 | 0.6 | 2 | 8.506 | 10.447 | 10.571 | 8.508 | 8.508 | 10.448 | 10.573 | 10.451 | 10.575 |
| | | | | | | | | | | | | |
| 110 | 0.3 | 0.2 | 1 | 4.580 | 4.663 | 4.676 | 4.579 | 4.580 | 4.663 | 4.674 | 4.664 | 4.676 |
| 110 | 0.7 | 0.2 | 2 | 6.680 | 8.234 | 8.344 | 6.682 | 6.680 | 8.233 | 8.345 | 8.234 | 8.345 |
| 110 | 0.3 | 0.6 | 1 | 4.584 | 4.673 | 4.682 | 4.585 | 4.586 | 4.673 | 4.683 | 4.672 | 4.684 |
| 110 | 0.7 | 0.6 | 2 | 6.710 | 8.293 | 8.411 | 6.713 | 6.716 | 8.292 | 8.412 | 8.293 | 8.412 |

*Notes: the first regression specification, i.e. 1, is defined as $P_2(S)$. All the other regression specifications follow as descibed in table 4.2*

Table B.4: This table shows the comparison between the computed prices of American put options in sample and out of sample, for the information set $\mathcal{A}_\sigma$, table 5.1. The specification of the conditional expectation of the payoff is $P_2(S)$.

| $S_0$ | $\sigma_\delta$ | $\sigma_V$ | $T$ | $\Pi_A^{in}$ | $se^{in}$ | $\Pi_A^{out}$ | $se^{out}$ | $\Pi_A^{in-out}$ | $se^{in-out}$ |
|---|---|---|---|---|---|---|---|---|---|
| 90 | 0.3 | 0.2 | 1 | 13.801 | 0.013 | 13.819 | 0.013 | -0.018 | 0.000 |
| 90 | 0.7 | 0.2 | 2 | 16.315 | 0.016 | 16.320 | 0.016 | -0.005 | 0.000 |
| 90 | 0.3 | 0.6 | 1 | 13.826 | 0.014 | 13.821 | 0.014 | 0.005 | 0.000 |
| 90 | 0.7 | 0.6 | 2 | 16.280 | 0.017 | 16.278 | 0.017 | 0.001 | 0.000 |
| | | | | | | | | | |
| 95 | 0.3 | 0.2 | 1 | 10.809 | 0.011 | 10.791 | 0.011 | 0.018 | 0.000 |
| 95 | 0.7 | 0.2 | 2 | 13.295 | 0.015 | 13.289 | 0.015 | 0.007 | 0.000 |
| 95 | 0.3 | 0.6 | 1 | 10.779 | 0.012 | 10.758 | 0.012 | 0.021 | 0.000 |
| 95 | 0.7 | 0.6 | 2 | 13.266 | 0.016 | 13.240 | 0.016 | 0.026 | 0.000 |
| | | | | | | | | | |
| 100 | 0.3 | 0.2 | 1 | 8.248 | 0.011 | 8.272 | 0.011 | -0.024 | 0.000 |
| 100 | 0.7 | 0.2 | 2 | 10.685 | 0.014 | 10.655 | 0.014 | 0.030 | 0.000 |
| 100 | 0.3 | 0.6 | 1 | 8.245 | 0.011 | 8.244 | 0.011 | 0.001 | 0.000 |
| 100 | 0.7 | 0.6 | 2 | 10.682 | 0.015 | 10.680 | 0.015 | 0.002 | 0.000 |
| | | | | | | | | | |
| 105 | 0.3 | 0.2 | 1 | 6.175 | 0.011 | 6.203 | 0.011 | -0.028 | 0.000 |
| 105 | 0.7 | 0.2 | 2 | 8.450 | 0.014 | 8.511 | 0.013 | -0.061 | 0.000 |
| 105 | 0.3 | 0.6 | 1 | 6.185 | 0.012 | 6.186 | 0.012 | 0.000 | 0.000 |
| 105 | 0.7 | 0.6 | 2 | 8.521 | 0.014 | 8.509 | 0.014 | 0.011 | 0.000 |
| | | | | | | | | | |
| 110 | 0.3 | 0.2 | 1 | 4.581 | 0.011 | 4.551 | 0.011 | 0.030 | 0.000 |
| 110 | 0.7 | 0.2 | 2 | 6.663 | 0.013 | 6.674 | 0.013 | -0.011 | 0.000 |
| 110 | 0.3 | 0.6 | 1 | 4.576 | 0.011 | 4.559 | 0.011 | 0.017 | 0.000 |
| 110 | 0.7 | 0.6 | 2 | 6.716 | 0.014 | 6.697 | 0.014 | 0.019 | 0.000 |

*Notes:* $\Pi_A^{in}$ *and* $se^{in}$ *represent the price and standard error of the American option computed in sample, respectively, whereas* $\Pi_A^{out}$ *and* $se^{out}$ *represent the price and standard error of the American option computed out of sample, respectively. Note that* $\Pi_A^{in-out} = \Pi_A^{in} - \Pi_A^{out}$ *and* $se^{in-out} = se^{in} - se^{out}$.

Table B.5: This table shows the comparison between the computed prices of American put options in sample and out of sample, for the information set $\mathcal{A}_\sigma$, table 5.1. The specification of the conditional expectation of the payoff is $P_2(S) + p_0(\delta)\hat{\beta}_{\delta_0}$.

| $S_0$ | $\sigma_\delta$ | $\sigma_V$ | $T$ | $\Pi_A^{in}$ | $se^{in}$ | $\Pi_A^{out}$ | $se^{out}$ | $\Pi_A^{in-out}$ | $se^{in-out}$ |
|---|---|---|---|---|---|---|---|---|---|
| 90 | 0.3 | 0.2 | 1 | 14.055 | 0.014 | 14.059 | 0.014 | -0.004 | 0.000 |
| 90 | 0.7 | 0.2 | 2 | 19.372 | 0.019 | 19.387 | 0.019 | -0.015 | 0.000 |
| 90 | 0.3 | 0.6 | 1 | 14.055 | 0.015 | 14.044 | 0.015 | 0.011 | 0.000 |
| 90 | 0.7 | 0.6 | 2 | 19.401 | 0.020 | 19.409 | 0.020 | -0.008 | 0.000 |
| 95 | 0.3 | 0.2 | 1 | 11.006 | 0.013 | 10.982 | 0.013 | 0.024 | 0.000 |
| 95 | 0.7 | 0.2 | 2 | 16.005 | 0.019 | 16.002 | 0.019 | 0.003 | 0.000 |
| 95 | 0.3 | 0.6 | 1 | 10.993 | 0.013 | 10.962 | 0.013 | 0.031 | 0.000 |
| 95 | 0.7 | 0.6 | 2 | 16.052 | 0.020 | 16.043 | 0.020 | 0.009 | 0.000 |
| 100 | 0.3 | 0.2 | 1 | 8.405 | 0.012 | 8.420 | 0.012 | -0.015 | 0.000 |
| 100 | 0.7 | 0.2 | 2 | 13.046 | 0.019 | 13.011 | 0.018 | 0.035 | 0.000 |
| 100 | 0.3 | 0.6 | 1 | 8.408 | 0.013 | 8.401 | 0.012 | 0.008 | 0.000 |
| 100 | 0.7 | 0.6 | 2 | 13.109 | 0.019 | 13.049 | 0.019 | 0.059 | 0.000 |
| 105 | 0.3 | 0.2 | 1 | 6.283 | 0.012 | 6.316 | 0.012 | -0.034 | 0.000 |
| 105 | 0.7 | 0.2 | 2 | 10.404 | 0.018 | 10.431 | 0.018 | -0.027 | 0.000 |
| 105 | 0.3 | 0.6 | 1 | 6.315 | 0.012 | 6.296 | 0.012 | 0.019 | 0.000 |
| 105 | 0.7 | 0.6 | 2 | 10.489 | 0.019 | 10.484 | 0.019 | 0.005 | 0.000 |
| 110 | 0.3 | 0.2 | 1 | 4.674 | 0.012 | 4.644 | 0.012 | 0.030 | 0.000 |
| 110 | 0.7 | 0.2 | 2 | 8.227 | 0.017 | 8.223 | 0.017 | 0.004 | 0.000 |
| 110 | 0.3 | 0.6 | 1 | 4.659 | 0.012 | 4.649 | 0.012 | 0.010 | 0.000 |
| 110 | 0.7 | 0.6 | 2 | 8.304 | 0.018 | 8.303 | 0.018 | 0.001 | 0.000 |

*Notes:* $\Pi_A^{in}$ *and* $se^{in}$ *represent the price and standard error of the American option computed in sample, respectively, whereas* $\Pi_A^{out}$ *and* $se^{out}$ *represent the price and standard error of the American option computed out of sample, respectively. Note that* $\Pi_A^{in-out} = \Pi_A^{in} - \Pi_A^{out}$ *and* $se^{in-out} = se^{in} - se^{out}$.

Table B.6: This table shows the price of an American put option for different parameters and basis function specifications. The basis function used is the Laguerre polynomial and the information set is $\mathcal{A}_k$, table 5.2.

| $S_0$ | $k_\delta$ | $k_V$ | $T$ | $L_0(S)$ | $L_1(S)$ | $L_2(S)$ | $L_3(S)$ | $L_4(S)$ | $L_5(S)$ | $L_6(S)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 90 | 0.876 | 20 | 1 | 14.476 | 14.551 | 14.533 | 14.529 | 14.521 | 13.970 | 14.109 |
| 90 | 2.876 | 20 | 2 | 15.112 | 15.335 | 15.356 | 15.373 | 15.383 | 15.390 | 15.374 |
| 90 | 0.876 | 34 | 1 | 14.524 | 14.592 | 14.570 | 14.561 | 14.556 | 13.634 | 14.546 |
| 90 | 2.876 | 34 | 2 | 15.100 | 15.328 | 15.337 | 15.359 | 15.320 | 15.365 | 15.365 |
| 95 | 0.876 | 20 | 1 | 11.266 | 11.338 | 11.308 | 11.307 | 11.070 | 10.909 | 11.299 |
| 95 | 2.876 | 20 | 2 | 12.181 | 12.386 | 12.404 | 12.419 | 12.416 | 11.898 | 11.652 |
| 95 | 0.876 | 34 | 1 | 11.265 | 11.332 | 11.304 | 11.305 | 11.304 | 11.182 | 11.227 |
| 95 | 2.876 | 34 | 2 | 12.182 | 12.376 | 12.417 | 12.429 | 12.288 | 11.941 | 12.372 |
| 100 | 0.876 | 20 | 1 | 8.510 | 8.552 | 8.533 | 8.535 | 8.505 | 8.473 | 8.507 |
| 100 | 2.876 | 20 | 2 | 9.714 | 9.880 | 9.906 | 9.924 | 9.932 | 9.201 | 9.924 |
| 100 | 0.876 | 34 | 1 | 8.493 | 8.539 | 8.522 | 8.005 | 8.524 | 8.339 | 8.324 |
| 100 | 2.876 | 34 | 2 | 9.751 | 9.916 | 9.941 | 9.955 | 9.848 | 9.694 | 9.960 |
| 105 | 0.876 | 20 | 1 | 6.231 | 6.269 | 6.258 | 6.017 | 6.259 | 6.167 | 5.879 |
| 105 | 2.876 | 20 | 2 | 7.673 | 7.813 | 7.832 | 7.246 | 7.860 | 7.571 | 7.672 |
| 105 | 0.876 | 34 | 1 | 6.236 | 6.279 | 6.271 | 6.267 | 6.269 | 6.256 | 6.129 |
| 105 | 2.876 | 34 | 2 | 7.668 | 7.815 | 7.843 | 7.836 | 7.858 | 7.867 | 7.855 |
| 110 | 0.876 | 20 | 1 | 4.454 | 4.499 | 4.500 | 4.481 | 4.497 | 4.493 | 4.453 |
| 110 | 2.876 | 20 | 2 | 5.988 | 6.110 | 6.136 | 6.128 | 6.146 | 6.127 | 6.126 |
| 110 | 0.876 | 34 | 1 | 4.452 | 4.491 | 4.488 | 4.490 | 4.490 | 4.487 | 4.183 |
| 110 | 2.876 | 34 | 2 | 5.986 | 6.109 | 6.135 | 6.120 | 6.144 | 6.098 | 5.896 |

*Notes: $L_Q(S)$ is defined in equation 2.4.*

Table B.7: This table shows the price of an American put option for different parameters and basis function specifications. The basis function used is the power polynomial and the information set is $\mathcal{A}_k$, table 5.2.

| $S_0$ | $k_\delta$ | $k_V$ | $T$ | $P_0(S)$ | $P_1(S)$ | $P_2(S)$ | $P_3(S)$ | $P_4(S)$ | $P_5(S)$ | $P_6(S)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 90 | 0.876 | 20 | 1 | 14.445 | 14.552 | 14.535 | 14.530 | 14.521 | 14.397 | 14.412 |
| 90 | 2.876 | 20 | 2 | 14.955 | 15.335 | 15.347 | 15.371 | 15.371 | 15.378 | 15.391 |
| 90 | 0.876 | 34 | 1 | 14.506 | 14.590 | 14.575 | 14.561 | 14.550 | 14.290 | 14.458 |
| 90 | 2.876 | 34 | 2 | 14.957 | 15.331 | 15.331 | 15.366 | 15.363 | 15.372 | 15.366 |
| 95 | 0.876 | 20 | 1 | 11.245 | 11.340 | 11.309 | 11.313 | 11.302 | 11.232 | 11.285 |
| 95 | 2.876 | 20 | 2 | 12.066 | 12.386 | 12.398 | 12.420 | 12.417 | 12.423 | 12.089 |
| 95 | 0.876 | 34 | 1 | 11.243 | 11.329 | 11.305 | 11.310 | 11.295 | 11.291 | 11.102 |
| 95 | 2.876 | 34 | 2 | 12.070 | 12.376 | 12.408 | 12.430 | 12.438 | 12.444 | 11.818 |
| 100 | 0.876 | 20 | 1 | 8.484 | 8.554 | 8.535 | 8.536 | 8.528 | 8.518 | 8.440 |
| 100 | 2.876 | 20 | 2 | 9.612 | 9.887 | 9.904 | 9.925 | 9.920 | 9.921 | 8.972 |
| 100 | 0.876 | 34 | 1 | 8.475 | 8.539 | 8.521 | 8.526 | 8.522 | 8.515 | 8.483 |
| 100 | 2.876 | 34 | 2 | 9.659 | 9.922 | 9.938 | 9.955 | 9.957 | 9.960 | 9.980 |
| 105 | 0.876 | 20 | 1 | 6.215 | 6.268 | 6.259 | 6.260 | 6.258 | 6.046 | 6.202 |
| 105 | 2.876 | 20 | 2 | 7.600 | 7.816 | 7.831 | 7.730 | 7.853 | 7.364 | 7.874 |
| 105 | 0.876 | 34 | 1 | 6.215 | 6.282 | 6.272 | 6.275 | 6.266 | 6.260 | 6.260 |
| 105 | 2.876 | 34 | 2 | 7.588 | 7.817 | 7.838 | 7.852 | 7.492 | 7.732 | 7.881 |
| 110 | 0.876 | 20 | 1 | 4.441 | 4.501 | 4.499 | 4.497 | 4.494 | 4.218 | 4.468 |
| 110 | 2.876 | 20 | 2 | 5.927 | 6.115 | 6.137 | 6.146 | 6.151 | 6.119 | 6.161 |
| 110 | 0.876 | 34 | 1 | 4.436 | 4.490 | 4.489 | 4.490 | 4.487 | 4.483 | 4.477 |
| 110 | 2.876 | 34 | 2 | 5.916 | 6.110 | 6.132 | 6.144 | 6.140 | 6.139 | 6.036 |

*Notes: $P_Q(S)$ is defined in equation 2.5.*

Table B.8: This table shows the price of American put options w.r.t. the information set $\mathcal{A}_k$, table 5.2. The conditional expectation of the payoffs, i.e. $\mathbb{E}\left[Y|X\right]_{P_{\mathcal{Q}(\mathcal{A}_k)}(S)}$, is defined as in table 4.2.

| $S_0$ | $k_\delta$ | $k_V$ | $T$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 90 | 0.876 | 20 | 1 | 14.572 | 15.966 | 16.196 | 14.581 | 14.582 | 15.968 | 16.195 | 15.968 | 16.194 |
| 90 | 2.876 | 20 | 2 | 15.327 | 16.345 | 16.330 | 15.332 | 15.332 | 16.352 | 16.333 | 16.352 | 16.335 |
| 90 | 0.876 | 34 | 1 | 14.586 | 15.979 | 16.199 | 14.592 | 14.591 | 15.976 | 16.199 | 15.973 | 16.199 |
| 90 | 2.876 | 34 | 2 | 15.316 | 16.352 | 16.325 | 15.315 | 15.315 | 16.349 | 16.328 | 16.351 | 16.330 |
| | | | | | | | | | | | | |
| 95 | 0.876 | 20 | 1 | 11.319 | 12.498 | 12.657 | 11.320 | 11.318 | 12.501 | 12.663 | 12.503 | 12.664 |
| 95 | 2.876 | 20 | 2 | 12.391 | 13.305 | 13.288 | 12.402 | 12.397 | 13.307 | 13.298 | 13.307 | 13.296 |
| 95 | 0.876 | 34 | 1 | 11.335 | 12.475 | 12.664 | 11.335 | 11.334 | 12.477 | 12.663 | 12.477 | 12.663 |
| 95 | 2.876 | 34 | 2 | 12.407 | 13.299 | 13.262 | 12.410 | 12.412 | 13.302 | 13.265 | 13.302 | 13.266 |
| | | | | | | | | | | | | |
| 100 | 0.876 | 20 | 1 | 8.541 | 9.461 | 9.610 | 8.550 | 8.550 | 9.461 | 9.617 | 9.461 | 9.618 |
| 100 | 2.876 | 20 | 2 | 9.864 | 10.659 | 10.636 | 9.874 | 9.876 | 10.664 | 10.637 | 10.665 | 10.636 |
| 100 | 0.876 | 34 | 1 | 8.537 | 9.457 | 9.609 | 8.537 | 8.535 | 9.460 | 9.608 | 9.459 | 9.608 |
| 100 | 2.876 | 34 | 2 | 9.901 | 10.681 | 10.659 | 9.904 | 9.904 | 10.679 | 10.660 | 10.677 | 10.660 |
| | | | | | | | | | | | | |
| 105 | 0.876 | 20 | 1 | 6.242 | 6.930 | 7.046 | 6.248 | 6.251 | 6.935 | 7.049 | 6.937 | 7.048 |
| 105 | 2.876 | 20 | 2 | 7.824 | 8.430 | 8.428 | 7.833 | 7.832 | 8.435 | 8.431 | 8.436 | 8.432 |
| 105 | 0.876 | 34 | 1 | 6.266 | 6.952 | 7.065 | 6.271 | 6.274 | 6.953 | 7.065 | 6.954 | 7.065 |
| 105 | 2.876 | 34 | 2 | 7.819 | 8.456 | 8.434 | 7.820 | 7.822 | 8.459 | 8.435 | 8.459 | 8.435 |
| | | | | | | | | | | | | |
| 110 | 0.876 | 20 | 1 | 4.509 | 5.013 | 5.091 | 4.517 | 4.517 | 5.013 | 5.098 | 5.012 | 5.099 |
| 110 | 2.876 | 20 | 2 | 6.119 | 6.639 | 6.622 | 6.122 | 6.122 | 6.642 | 6.621 | 6.644 | 6.622 |
| 110 | 0.876 | 34 | 1 | 4.505 | 5.008 | 5.087 | 4.509 | 4.509 | 5.009 | 5.089 | 5.008 | 5.089 |
| 110 | 2.876 | 34 | 2 | 6.108 | 6.632 | 6.609 | 6.109 | 6.110 | 6.635 | 6.613 | 6.634 | 6.616 |

*Notes: the first regression specification, i.e. 1, is defined as $P_1(S)$. All the other regression specifications follow as descibed in table 4.2*

Table B.9: This table shows the comparison between the computed prices of American put options in sample and out of sample, for the information set $\mathcal{A}_k$, table 5.2. The specification of the conditional expectation of the payoff is $P_1(S)$.

| $S_0$ | $k_\delta$ | $k_V$ | $T$ | $\Pi_A^{in}$ | $se^{in}$ | $\Pi_A^{out}$ | $se^{out}$ | $\Pi_A^{in-out}$ | $se^{in-out}$ |
|---|---|---|---|---|---|---|---|---|---|
| 90 | 0.876 | 20 | 1 | 14.581 | 0.014 | 14.590 | 0.014 | -0.009 | 0.000 |
| 90 | 2.876 | 20 | 2 | 15.276 | 0.017 | 15.307 | 0.017 | -0.031 | 0.000 |
| 90 | 0.876 | 34 | 1 | 14.604 | 0.014 | 14.614 | 0.014 | -0.010 | 0.000 |
| 90 | 2.876 | 34 | 2 | 15.332 | 0.016 | 15.321 | 0.016 | 0.011 | 0.000 |
| | | | | | | | | | |
| 95 | 0.876 | 20 | 1 | 11.307 | 0.013 | 11.280 | 0.013 | 0.027 | 0.000 |
| 95 | 2.876 | 20 | 2 | 12.362 | 0.015 | 12.397 | 0.015 | -0.035 | 0.000 |
| 95 | 0.876 | 34 | 1 | 11.326 | 0.012 | 11.313 | 0.013 | 0.012 | 0.000 |
| 95 | 2.876 | 34 | 2 | 12.405 | 0.015 | 12.386 | 0.015 | 0.019 | 0.000 |
| | | | | | | | | | |
| 100 | 0.876 | 20 | 1 | 8.544 | 0.012 | 8.527 | 0.012 | 0.018 | 0.000 |
| 100 | 2.876 | 20 | 2 | 9.874 | 0.014 | 9.904 | 0.014 | -0.030 | 0.000 |
| 100 | 0.876 | 34 | 1 | 8.531 | 0.011 | 8.533 | 0.011 | -0.002 | 0.000 |
| 100 | 2.876 | 34 | 2 | 9.888 | 0.014 | 9.897 | 0.014 | -0.009 | 0.000 |
| | | | | | | | | | |
| 105 | 0.876 | 20 | 1 | 6.260 | 0.011 | 6.256 | 0.011 | 0.004 | 0.000 |
| 105 | 2.876 | 20 | 2 | 7.822 | 0.014 | 7.805 | 0.014 | 0.017 | 0.000 |
| 105 | 0.876 | 34 | 1 | 6.283 | 0.011 | 6.272 | 0.011 | 0.011 | 0.000 |
| 105 | 2.876 | 34 | 2 | 7.819 | 0.014 | 7.843 | 0.014 | -0.024 | 0.000 |
| | | | | | | | | | |
| 110 | 0.876 | 20 | 1 | 4.496 | 0.010 | 4.453 | 0.010 | 0.043 | 0.000 |
| 110 | 2.876 | 20 | 2 | 6.118 | 0.014 | 6.108 | 0.014 | 0.010 | 0.000 |
| 110 | 0.876 | 34 | 1 | 4.470 | 0.010 | 4.480 | 0.010 | -0.009 | 0.000 |
| 110 | 2.876 | 34 | 2 | 6.106 | 0.014 | 6.113 | 0.014 | -0.007 | 0.000 |

*Notes:* $\Pi_A^{in}$ *and* $se^{in}$ *represent the price and standard error of the American option computed in sample, respectively, whereas* $\Pi_A^{out}$ *and* $se^{out}$ *represent the price and standard error of the American option computed out of sample, respectively. Note that* $\Pi_A^{in-out} = \Pi_A^{in} - \Pi_A^{out}$ *and* $se^{in-out} = se^{in} - se^{out}$.

Table B.10: This table shows the comparison between the computed prices of American put options in sample and out of sample, for the information set $\mathcal{A}_k$, table 5.2. The specification of the conditional expectation of the payoff is $P_1(S) + p_0(\delta)\hat{\beta}_{\delta_0}$.

| $S_0$ | $k_\delta$ | $k_V$ | $T$ | $\Pi_A^{in}$ | $se^{in}$ | $\Pi_A^{out}$ | $se^{out}$ | $\Pi_A^{in-out}$ | $se^{in-out}$ |
|---|---|---|---|---|---|---|---|---|---|
| 90 | 0.876 | 20 | 1 | 15.986 | 0.017 | 15.970 | 0.017 | 0.016 | 0.000 |
| 90 | 2.876 | 20 | 2 | 16.345 | 0.017 | 16.358 | 0.017 | -0.013 | 0.000 |
| 90 | 0.876 | 34 | 1 | 15.961 | 0.017 | 16.006 | 0.017 | -0.046 | 0.000 |
| 90 | 2.876 | 34 | 2 | 16.327 | 0.017 | 16.364 | 0.017 | -0.037 | 0.000 |
| | | | | | | | | | |
| 95 | 0.876 | 20 | 1 | 12.457 | 0.016 | 12.478 | 0.016 | -0.021 | 0.000 |
| 95 | 2.876 | 20 | 2 | 13.284 | 0.016 | 13.312 | 0.016 | -0.028 | 0.000 |
| 95 | 0.876 | 34 | 1 | 12.439 | 0.016 | 12.449 | 0.016 | -0.010 | 0.000 |
| 95 | 2.876 | 34 | 2 | 13.321 | 0.016 | 13.301 | 0.016 | 0.019 | 0.000 |
| | | | | | | | | | |
| 100 | 0.876 | 20 | 1 | 9.482 | 0.015 | 9.459 | 0.015 | 0.023 | 0.000 |
| 100 | 2.876 | 20 | 2 | 10.660 | 0.015 | 10.674 | 0.015 | -0.014 | 0.000 |
| 100 | 0.876 | 34 | 1 | 9.455 | 0.015 | 9.471 | 0.015 | -0.016 | 0.000 |
| 100 | 2.876 | 34 | 2 | 10.684 | 0.015 | 10.653 | 0.015 | 0.031 | 0.000 |
| | | | | | | | | | |
| 105 | 0.876 | 20 | 1 | 6.955 | 0.014 | 6.969 | 0.014 | -0.014 | 0.000 |
| 105 | 2.876 | 20 | 2 | 8.451 | 0.014 | 8.456 | 0.014 | -0.005 | 0.000 |
| 105 | 0.876 | 34 | 1 | 6.959 | 0.014 | 6.960 | 0.014 | -0.002 | 0.000 |
| 105 | 2.876 | 34 | 2 | 8.449 | 0.014 | 8.469 | 0.014 | -0.021 | 0.000 |
| | | | | | | | | | |
| 110 | 0.876 | 20 | 1 | 4.998 | 0.013 | 4.974 | 0.013 | 0.024 | 0.000 |
| 110 | 2.876 | 20 | 2 | 6.644 | 0.014 | 6.614 | 0.014 | 0.030 | 0.000 |
| 110 | 0.876 | 34 | 1 | 4.958 | 0.013 | 4.973 | 0.013 | -0.015 | 0.000 |
| 110 | 2.876 | 34 | 2 | 6.621 | 0.014 | 6.644 | 0.014 | -0.023 | 0.000 |

*Notes:* $\Pi_A^{in}$ *and* $se^{in}$ *represent the price and standard error of the American option computed in sample, respectively, whereas* $\Pi_A^{out}$ *and* $se^{out}$ *represent the price and standard error of the American option computed out of sample, respectively. Note that* $\Pi_A^{in-out} = \Pi_A^{in} - \Pi_A^{out}$ *and* $se^{in-out} = se^{in} - se^{out}$.

Table B.11: This table shows the price of an American put option for different parameters and basis function specifications. The basis function used is the Laguerre polynomial and the information set is $\mathcal{A}_\theta$, table 5.3.

| $S_0$ | $\theta_\delta$ | $\theta_V$ | $T$ | $L_0(S)$ | $L_1(S)$ | $L_2(S)$ | $L_3(S)$ | $L_4(S)$ | $L_5(S)$ | $L_6(S)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 90 | 0.1 | 0.2 | 1 | 22.807 | 22.923 | 22.920 | 22.913 | 22.914 | 22.892 | 22.917 |
| 90 | 0.2 | 0.2 | 2 | 36.609 | 36.834 | 36.779 | 36.811 | 35.901 | 36.820 | 36.722 |
| 90 | 0.1 | 0.3 | 1 | 26.085 | 26.225 | 26.243 | 26.234 | 25.984 | 26.185 | 26.167 |
| 90 | 0.2 | 0.3 | 2 | 40.227 | 40.452 | 40.398 | 40.420 | 35.399 | 40.422 | 37.247 |
| | | | | | | | | | | |
| 95 | 0.1 | 0.2 | 1 | 20.090 | 20.182 | 20.185 | 20.182 | 19.985 | 19.982 | 20.161 |
| 95 | 0.2 | 0.2 | 2 | 34.251 | 34.449 | 34.384 | 34.412 | 34.411 | 34.430 | 34.433 |
| 95 | 0.1 | 0.3 | 1 | 23.669 | 23.752 | 23.770 | 23.754 | 23.193 | 23.764 | 23.768 |
| 95 | 0.2 | 0.3 | 2 | 38.152 | 38.355 | 38.291 | 38.299 | 37.843 | 38.310 | 35.616 |
| | | | | | | | | | | |
| 100 | 0.1 | 0.2 | 1 | 17.647 | 17.727 | 17.736 | 17.733 | 16.467 | 17.700 | 17.732 |
| 100 | 0.2 | 0.2 | 2 | 31.965 | 32.128 | 32.073 | 32.093 | 30.470 | 32.104 | 32.098 |
| 100 | 0.1 | 0.3 | 1 | 21.391 | 21.481 | 21.481 | 21.484 | 20.254 | 18.825 | 21.483 |
| 100 | 0.2 | 0.3 | 2 | 36.217 | 36.370 | 36.322 | 36.326 | 36.319 | 36.341 | 36.233 |
| | | | | | | | | | | |
| 105 | 0.1 | 0.2 | 1 | 15.376 | 15.462 | 15.473 | 15.479 | 15.040 | 15.472 | 15.470 |
| 105 | 0.2 | 0.2 | 2 | 29.834 | 29.982 | 29.918 | 29.925 | 29.362 | 29.362 | 29.938 |
| 105 | 0.1 | 0.3 | 1 | 19.297 | 19.380 | 19.381 | 19.370 | 17.773 | 19.363 | 19.380 |
| 105 | 0.2 | 0.3 | 2 | 34.291 | 34.447 | 34.382 | 34.390 | 32.196 | 34.390 | 32.607 |
| | | | | | | | | | | |
| 110 | 0.1 | 0.2 | 1 | 13.394 | 13.455 | 13.469 | 13.470 | 12.711 | 12.643 | 13.456 |
| 110 | 0.2 | 0.2 | 2 | 27.833 | 27.978 | 27.925 | 27.914 | 27.675 | 27.919 | 27.925 |
| 110 | 0.1 | 0.3 | 1 | 17.410 | 17.475 | 17.487 | 15.977 | 17.458 | 17.477 | 17.481 |
| 110 | 0.2 | 0.3 | 2 | 32.457 | 32.622 | 32.565 | 32.544 | 28.889 | 32.239 | 32.584 |

*Notes:* $L_Q(S)$ *is defined in equation 2.4.*

Table B.12: This table shows the price of an American put option for different parameters and basis function specifications. The basis function used is the power polynomial and the information set is $\mathcal{A}_\theta$, table 5.3.

| $S_0$ | $\theta_\delta$ | $\theta_V$ | $T$ | $P_0(S)$ | $P_1(S)$ | $P_2(S)$ | $P_3(S)$ | $P_4(S)$ | $P_5(S)$ | $P_6(S)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 90 | 0.1 | 0.2 | 1 | 22.675 | 22.922 | 22.914 | 22.915 | 22.913 | 20.518 | 22.893 |
| 90 | 0.2 | 0.2 | 2 | 36.643 | 36.836 | 36.782 | 36.815 | 36.820 | 36.818 | 36.820 |
| 90 | 0.1 | 0.3 | 1 | 25.954 | 26.234 | 26.245 | 26.231 | 26.229 | 26.220 | 26.224 |
| 90 | 0.2 | 0.3 | 2 | 40.199 | 40.444 | 40.391 | 40.423 | 40.424 | 40.432 | 40.427 |
| 95 | 0.1 | 0.2 | 1 | 19.999 | 20.191 | 20.188 | 20.181 | 20.173 | 20.185 | 20.171 |
| 95 | 0.2 | 0.2 | 2 | 34.268 | 34.444 | 34.387 | 34.417 | 34.429 | 34.421 | 34.422 |
| 95 | 0.1 | 0.3 | 1 | 23.557 | 23.761 | 23.768 | 23.754 | 23.766 | 23.770 | 17.432 |
| 95 | 0.2 | 0.3 | 2 | 38.115 | 38.335 | 38.286 | 38.301 | 38.307 | 38.323 | 38.316 |
| 100 | 0.1 | 0.2 | 1 | 17.553 | 17.732 | 17.731 | 17.733 | 17.734 | 17.730 | 17.727 |
| 100 | 0.2 | 0.2 | 2 | 31.982 | 32.126 | 32.067 | 32.095 | 32.103 | 32.113 | 32.107 |
| 100 | 0.1 | 0.3 | 1 | 21.287 | 21.487 | 21.482 | 21.483 | 21.490 | 21.486 | 21.483 |
| 100 | 0.2 | 0.3 | 2 | 36.178 | 36.371 | 36.323 | 36.332 | 36.340 | 36.354 | 36.353 |
| 105 | 0.1 | 0.2 | 1 | 15.289 | 15.468 | 15.470 | 15.479 | 15.470 | 15.472 | 15.468 |
| 105 | 0.2 | 0.2 | 2 | 29.823 | 29.977 | 29.901 | 29.926 | 29.942 | 29.951 | 29.948 |
| 105 | 0.1 | 0.3 | 1 | 19.191 | 19.380 | 19.384 | 19.376 | 19.379 | 19.381 | 19.374 |
| 105 | 0.2 | 0.3 | 2 | 34.258 | 34.446 | 34.380 | 34.393 | 34.391 | 34.401 | 34.383 |
| 110 | 0.1 | 0.2 | 1 | 13.321 | 13.458 | 13.468 | 13.467 | 13.468 | 13.464 | 13.460 |
| 110 | 0.2 | 0.2 | 2 | 27.819 | 27.976 | 27.922 | 27.914 | 27.927 | 27.932 | 27.934 |
| 110 | 0.1 | 0.3 | 1 | 17.336 | 17.477 | 17.484 | 17.493 | 17.481 | 17.487 | 17.482 |
| 110 | 0.2 | 0.3 | 2 | 32.423 | 32.612 | 32.563 | 32.554 | 32.575 | 32.581 | 32.580 |

*Notes: $P_{\mathcal{Q}}(S)$ is defined in equation 2.5.*

Table B.13: This table shows the price of American put options w.r.t. the information set $\mathcal{A}_\theta$, table 5.3. The conditional expectation of the payoffs, i.e. $\mathbb{E}\left[Y|X\right]_{L_{\mathcal{Q}(\mathcal{A}_\theta)}(S)}$, is defined as in table 4.1.

| $S_0$ | $\theta_\delta$ | $\theta_V$ | $T$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 90 | 0.1 | 0.2 | 1 | 22.943 | 23.200 | 23.374 | 22.948 | 22.947 | 23.200 | 23.380 | 23.200 | 23.384 |
| 90 | 0.2 | 0.2 | 2 | 36.826 | 37.425 | 37.405 | 36.826 | 36.824 | 37.429 | 37.412 | 37.429 | 37.412 |
| 90 | 0.1 | 0.3 | 1 | 26.247 | 26.448 | 26.593 | 26.243 | 26.248 | 26.451 | 26.588 | 26.450 | 26.586 |
| 90 | 0.2 | 0.3 | 2 | 40.442 | 40.905 | 40.872 | 40.441 | 40.440 | 40.901 | 40.869 | 40.902 | 40.873 |
| 95 | 0.1 | 0.2 | 1 | 20.180 | 20.462 | 20.621 | 20.178 | 20.180 | 20.458 | 20.623 | 20.459 | 20.624 |
| 95 | 0.2 | 0.2 | 2 | 34.435 | 35.001 | 35.008 | 34.431 | 34.431 | 35.004 | 35.011 | 35.003 | 35.010 |
| 95 | 0.1 | 0.3 | 1 | 23.756 | 23.928 | 24.033 | 23.754 | 23.754 | 23.927 | 24.039 | 23.928 | 24.041 |
| 95 | 0.2 | 0.3 | 2 | 38.368 | 38.761 | 38.757 | 38.367 | 38.365 | 38.767 | 38.758 | 38.768 | 38.761 |
| 100 | 0.1 | 0.2 | 1 | 17.723 | 17.958 | 18.073 | 17.725 | 17.726 | 17.959 | 18.071 | 17.962 | 18.075 |
| 100 | 0.2 | 0.2 | 2 | 32.119 | 32.726 | 32.760 | 32.120 | 32.121 | 32.728 | 32.756 | 32.727 | 32.758 |
| 100 | 0.1 | 0.3 | 1 | 21.485 | 21.624 | 21.732 | 21.489 | 21.487 | 21.625 | 21.728 | 21.627 | 21.730 |
| 100 | 0.2 | 0.3 | 2 | 36.317 | 36.757 | 36.777 | 36.315 | 36.320 | 36.755 | 36.770 | 36.754 | 36.774 |
| 105 | 0.1 | 0.2 | 1 | 15.472 | 15.701 | 15.841 | 15.470 | 15.470 | 15.703 | 15.844 | 15.707 | 15.844 |
| 105 | 0.2 | 0.2 | 2 | 29.992 | 30.622 | 30.621 | 29.985 | 29.983 | 30.621 | 30.619 | 30.621 | 30.620 |
| 105 | 0.1 | 0.3 | 1 | 19.424 | 19.555 | 19.645 | 19.426 | 19.428 | 19.560 | 19.641 | 19.560 | 19.644 |
| 105 | 0.2 | 0.3 | 2 | 34.445 | 34.886 | 34.870 | 34.437 | 34.441 | 34.887 | 34.870 | 34.888 | 34.872 |
| 110 | 0.1 | 0.2 | 1 | 13.518 | 13.695 | 13.806 | 13.516 | 13.519 | 13.691 | 13.805 | 13.696 | 13.807 |
| 110 | 0.2 | 0.2 | 2 | 27.957 | 28.537 | 28.547 | 27.952 | 27.952 | 28.533 | 28.550 | 28.536 | 28.553 |
| 110 | 0.1 | 0.3 | 1 | 17.430 | 17.577 | 17.676 | 17.431 | 17.431 | 17.576 | 17.677 | 17.581 | 17.676 |
| 110 | 0.2 | 0.3 | 2 | 32.649 | 33.055 | 33.069 | 32.644 | 32.644 | 33.057 | 33.062 | 33.050 | 33.067 |

*Notes: the first regression specification, i.e. 1, is defined as* $L_1(S)$. *All the other regression specifications follow as descibed in table 4.1*

Table B.14: This table shows the comparison between the computed prices of American put options in sample and out of sample, for the information set $\mathcal{A}_{\theta}$, table 5.3. The specification of the conditional expectation of the payoff is $L_1(S)$.

| $S_0$ | $\theta_\delta$ | $\theta_V$ | $T$ | $\Pi_A^{in}$ | $se^{in}$ | $\Pi_A^{out}$ | $se^{out}$ | $\Pi_A^{in-out}$ | $se^{in-out}$ |
|---|---|---|---|---|---|---|---|---|---|
| 90 | 0.1 | 0.2 | 1 | 22.944 | 0.019 | 22.949 | 0.019 | -0.005 | 0.000 |
| 90 | 0.2 | 0.2 | 2 | 36.829 | 0.021 | 36.809 | 0.020 | 0.021 | 0.000 |
| 90 | 0.1 | 0.3 | 1 | 26.259 | 0.021 | 26.235 | 0.021 | 0.025 | 0.000 |
| 90 | 0.2 | 0.3 | 2 | 40.463 | 0.025 | 40.449 | 0.024 | 0.013 | 0.000 |
| | | | | | | | | | |
| 95 | 0.1 | 0.2 | 1 | 20.157 | 0.017 | 20.200 | 0.017 | -0.043 | 0.000 |
| 95 | 0.2 | 0.2 | 2 | 34.403 | 0.019 | 34.417 | 0.019 | -0.014 | 0.000 |
| 95 | 0.1 | 0.3 | 1 | 23.715 | 0.019 | 23.764 | 0.019 | -0.048 | 0.000 |
| 95 | 0.2 | 0.3 | 2 | 38.350 | 0.024 | 38.335 | 0.024 | 0.015 | 0.000 |
| | | | | | | | | | |
| 100 | 0.1 | 0.2 | 1 | 17.720 | 0.015 | 17.712 | 0.015 | 0.008 | 0.000 |
| 100 | 0.2 | 0.2 | 2 | 32.177 | 0.018 | 32.133 | 0.018 | 0.044 | 0.000 |
| 100 | 0.1 | 0.3 | 1 | 21.485 | 0.017 | 21.484 | 0.017 | 0.001 | 0.000 |
| 100 | 0.2 | 0.3 | 2 | 36.347 | 0.023 | 36.399 | 0.023 | -0.051 | 0.000 |
| | | | | | | | | | |
| 105 | 0.1 | 0.2 | 1 | 15.502 | 0.015 | 15.498 | 0.015 | 0.004 | 0.000 |
| 105 | 0.2 | 0.2 | 2 | 29.986 | 0.017 | 29.972 | 0.017 | 0.014 | 0.000 |
| 105 | 0.1 | 0.3 | 1 | 19.377 | 0.016 | 19.370 | 0.016 | 0.008 | 0.000 |
| 105 | 0.2 | 0.3 | 2 | 34.404 | 0.021 | 34.443 | 0.021 | -0.039 | 0.000 |
| | | | | | | | | | |
| 110 | 0.1 | 0.2 | 1 | 13.481 | 0.015 | 13.500 | 0.015 | -0.019 | 0.000 |
| 110 | 0.2 | 0.2 | 2 | 27.938 | 0.016 | 27.918 | 0.016 | 0.020 | 0.000 |
| 110 | 0.1 | 0.3 | 1 | 17.474 | 0.017 | 17.455 | 0.017 | 0.019 | 0.000 |
| 110 | 0.2 | 0.3 | 2 | 32.646 | 0.020 | 32.614 | 0.020 | 0.032 | 0.000 |

*Notes:* $\Pi_A^{in}$ *and* $se^{in}$ *represent the price and standard error of the American option computed in sample, respectively, whereas* $\Pi_A^{out}$ *and* $se^{out}$ *represent the price and standard error of the American option computed out of sample, respectively. Note that* $\Pi_A^{in-out} = \Pi_A^{in} - \Pi_A^{out}$ *and* $se^{in-out} = se^{in} - se^{out}$.

Table B.15: This table shows the comparison between the computed prices of American put options in sample and out of sample, for the information set $\mathcal{A}_\theta$, table 5.3. The specification of the conditional expectation of the payoff is $L_1(S) + l_0(\delta)\hat{\beta}_{\delta_0}$.

| $S_0$ | $\theta_\delta$ | $\theta_V$ | $T$ | $\Pi_A^{in}$ | $se^{in}$ | $\Pi_A^{out}$ | $se^{out}$ | $\Pi_A^{in-out}$ | $se^{in-out}$ |
|---|---|---|---|---|---|---|---|---|---|
| 90 | 0.1 | 0.2 | 1 | 23.202 | 0.019 | 23.206 | 0.019 | -0.004 | 0.000 |
| 90 | 0.2 | 0.2 | 2 | 37.440 | 0.024 | 37.434 | 0.024 | 0.006 | 0.000 |
| 90 | 0.1 | 0.3 | 1 | 26.423 | 0.022 | 26.430 | 0.022 | -0.007 | 0.000 |
| 90 | 0.2 | 0.3 | 2 | 40.834 | 0.028 | 40.893 | 0.027 | -0.059 | 0.000 |
| 95 | 0.1 | 0.2 | 1 | 20.434 | 0.017 | 20.438 | 0.017 | -0.005 | 0.000 |
| 95 | 0.2 | 0.2 | 2 | 34.997 | 0.024 | 35.050 | 0.023 | -0.052 | 0.000 |
| 95 | 0.1 | 0.3 | 1 | 23.905 | 0.020 | 23.920 | 0.020 | -0.016 | 0.000 |
| 95 | 0.2 | 0.3 | 2 | 38.767 | 0.027 | 38.741 | 0.027 | 0.026 | 0.000 |
| 100 | 0.1 | 0.2 | 1 | 17.970 | 0.016 | 17.971 | 0.016 | -0.001 | 0.000 |
| 100 | 0.2 | 0.2 | 2 | 32.754 | 0.023 | 32.749 | 0.023 | 0.005 | 0.000 |
| 100 | 0.1 | 0.3 | 1 | 21.634 | 0.018 | 21.631 | 0.018 | 0.003 | 0.000 |
| 100 | 0.2 | 0.3 | 2 | 36.771 | 0.026 | 36.744 | 0.026 | 0.026 | 0.000 |
| 105 | 0.1 | 0.2 | 1 | 15.713 | 0.016 | 15.722 | 0.016 | -0.008 | 0.000 |
| 105 | 0.2 | 0.2 | 2 | 30.557 | 0.022 | 30.576 | 0.022 | -0.019 | 0.000 |
| 105 | 0.1 | 0.3 | 1 | 19.520 | 0.017 | 19.524 | 0.017 | -0.004 | 0.000 |
| 105 | 0.2 | 0.3 | 2 | 34.854 | 0.025 | 34.891 | 0.025 | -0.037 | 0.000 |
| 110 | 0.1 | 0.2 | 1 | 13.690 | 0.017 | 13.703 | 0.017 | -0.013 | 0.000 |
| 110 | 0.2 | 0.2 | 2 | 28.534 | 0.021 | 28.577 | 0.021 | -0.043 | 0.000 |
| 110 | 0.1 | 0.3 | 1 | 17.625 | 0.018 | 17.598 | 0.018 | 0.027 | 0.000 |
| 110 | 0.2 | 0.3 | 2 | 33.061 | 0.023 | 33.092 | 0.023 | -0.031 | 0.000 |

*Notes:* $\Pi_A^{in}$ *and* $se^{in}$ *represent the price and standard error of the American option computed in sample, respectively, whereas* $\Pi_A^{out}$ *and* $se^{out}$ *represent the price and standard error of the American option computed out of sample, respectively. Note that* $\Pi_A^{in-out} = \Pi_A^{in} - \Pi_A^{out}$ *and* $se^{in-out} = se^{in} - se^{out}$.

Table B.16: This table shows the price of an American put option for different parameters and basis function specifications. The basis function used is the Laguerre polynomial and the information set is $\mathcal{A}_{\rho}$, table 5.4.

| $S_0$ | $\rho_{S\delta}$ | $\rho_{SV}$ | $T$ | $L_0(S)$ | $L_1(S)$ | $L_2(S)$ | $L_3(S)$ | $L_4(S)$ | $L_5(S)$ | $L_6(S)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 90 | 0.65 | 0 | 1 | 13.858 | 13.967 | 13.999 | 14.005 | 14.003 | 12.892 | 13.194 |
| 90 | 0.85 | 0 | 2 | 15.929 | 16.147 | 16.170 | 16.147 | 16.161 | 15.994 | 16.191 |
| 90 | 0.65 | 0.2 | 1 | 13.896 | 14.000 | 14.029 | 14.027 | 14.016 | 13.379 | 14.038 |
| 90 | 0.85 | 0.2 | 2 | 16.007 | 16.185 | 16.222 | 16.238 | 16.197 | 15.891 | 15.059 |
| 95 | 0.65 | 0 | 1 | 10.744 | 10.856 | 10.874 | 10.376 | 10.877 | 10.573 | 10.882 |
| 95 | 0.85 | 0 | 2 | 12.778 | 12.972 | 13.003 | 13.020 | 13.028 | 12.401 | 13.033 |
| 95 | 0.65 | 0.2 | 1 | 10.758 | 10.861 | 10.880 | 10.883 | 10.882 | 10.367 | 10.657 |
| 95 | 0.85 | 0.2 | 2 | 12.813 | 13.011 | 13.027 | 13.048 | 12.951 | 12.539 | 13.044 |
| 100 | 0.65 | 0 | 1 | 8.187 | 8.274 | 8.303 | 8.268 | 8.303 | 8.266 | 8.295 |
| 100 | 0.85 | 0 | 2 | 10.088 | 10.273 | 10.300 | 10.310 | 10.315 | 10.182 | 10.197 |
| 100 | 0.65 | 0.2 | 1 | 8.200 | 8.289 | 8.304 | 8.301 | 8.307 | 8.280 | 7.739 |
| 100 | 0.85 | 0.2 | 2 | 10.055 | 10.231 | 10.249 | 10.263 | 10.266 | 10.071 | 10.239 |
| 105 | 0.65 | 0 | 1 | 6.137 | 6.211 | 6.226 | 6.230 | 6.230 | 6.064 | 5.893 |
| 105 | 0.85 | 0 | 2 | 7.766 | 7.919 | 7.951 | 7.167 | 7.957 | 7.731 | 7.776 |
| 105 | 0.65 | 0.2 | 1 | 6.116 | 6.186 | 6.210 | 6.163 | 6.210 | 6.204 | 5.719 |
| 105 | 0.85 | 0.2 | 2 | 7.750 | 7.880 | 7.910 | 7.917 | 7.920 | 7.461 | 7.545 |
| 110 | 0.65 | 0 | 1 | 4.519 | 4.567 | 4.582 | 4.576 | 4.588 | 4.577 | 4.578 |
| 110 | 0.85 | 0 | 2 | 5.851 | 5.988 | 6.016 | 6.016 | 6.020 | 5.987 | 5.802 |
| 110 | 0.65 | 0.2 | 1 | 4.471 | 4.518 | 4.536 | 4.281 | 4.516 | 4.533 | 4.356 |
| 110 | 0.85 | 0.2 | 2 | 5.816 | 5.948 | 5.973 | 5.912 | 5.984 | 5.889 | 5.555 |

*Notes: $L_Q(S)$ is defined in equation 2.4.*

Table B.17: This table shows the price of an American put option for different parameters and basis function specifications. The basis function used is the power polynomial and the information set is $\mathcal{A}_\rho$, table 5.4.

| $S_0$ | $\rho_{S\delta}$ | $\rho_{SV}$ | $T$ | $P_0(S)$ | $P_1(S)$ | $P_2(S)$ | $P_3(S)$ | $P_4(S)$ | $P_5(S)$ | $P_6(S)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 90 | 0.65 | 0 | 1 | 13.782 | 13.975 | 13.999 | 14.005 | 14.000 | 13.971 | 13.980 |
| 90 | 0.85 | 0 | 2 | 15.823 | 16.150 | 16.166 | 16.194 | 16.205 | 16.204 | 15.869 |
| 90 | 0.65 | 0.2 | 1 | 13.830 | 14.009 | 14.029 | 14.034 | 14.041 | 14.035 | 13.814 |
| 90 | 0.85 | 0.2 | 2 | 15.922 | 16.189 | 16.220 | 16.239 | 16.246 | 16.097 | 15.514 |
| 95 | 0.65 | 0 | 1 | 10.693 | 10.856 | 10.876 | 10.875 | 10.872 | 9.850 | 10.873 |
| 95 | 0.85 | 0 | 2 | 12.700 | 12.975 | 13.003 | 13.022 | 13.035 | 13.033 | 13.029 |
| 95 | 0.65 | 0.2 | 1 | 10.709 | 10.865 | 10.877 | 10.884 | 10.889 | 10.888 | 10.750 |
| 95 | 0.85 | 0.2 | 2 | 12.736 | 13.009 | 13.028 | 13.051 | 13.058 | 13.062 | 13.058 |
| 100 | 0.65 | 0 | 1 | 8.139 | 8.277 | 8.302 | 8.303 | 8.279 | 8.302 | 7.697 |
| 100 | 0.85 | 0 | 2 | 10.026 | 10.278 | 10.296 | 10.313 | 10.323 | 10.321 | 10.023 |
| 100 | 0.65 | 0.2 | 1 | 8.157 | 8.290 | 8.301 | 8.305 | 8.311 | 8.290 | 8.294 |
| 100 | 0.85 | 0.2 | 2 | 9.988 | 10.230 | 10.247 | 10.265 | 10.277 | 10.166 | 10.125 |
| 105 | 0.65 | 0 | 1 | 6.098 | 6.212 | 6.225 | 6.231 | 6.020 | 6.226 | 6.234 |
| 105 | 0.85 | 0 | 2 | 7.706 | 7.924 | 7.949 | 7.961 | 7.972 | 7.969 | 7.436 |
| 105 | 0.65 | 0.2 | 1 | 6.083 | 6.188 | 6.209 | 6.213 | 6.216 | 6.216 | 6.187 |
| 105 | 0.85 | 0.2 | 2 | 7.697 | 7.884 | 7.910 | 7.920 | 7.921 | 7.928 | 7.100 |
| 110 | 0.65 | 0 | 1 | 4.488 | 4.569 | 4.583 | 4.588 | 4.585 | 4.588 | 4.499 |
| 110 | 0.85 | 0 | 2 | 5.804 | 5.990 | 6.014 | 6.013 | 6.029 | 6.033 | 5.827 |
| 110 | 0.65 | 0.2 | 1 | 4.445 | 4.521 | 4.534 | 4.537 | 4.534 | 4.536 | 4.254 |
| 110 | 0.85 | 0.2 | 2 | 5.777 | 5.949 | 5.972 | 5.985 | 5.989 | 5.987 | 5.987 |

*Notes: $P_Q(S)$ is defined in equation 2.5.*

Table B.18: This table shows the price of American put options w.r.t. the information set $\mathcal{A}_\rho$, table 5.4. The conditional expectation of the payoffs, i.e. $\mathbb{E}\left[Y|X\right]_{P_{\mathcal{Q}(\mathcal{A}_\rho)}(S)}$, is defined as in table 4.2.

| $S_0$ | $\rho_{S\delta}$ | $\rho_{SV}$ | $T$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 90 | 0.65 | 0 | 1 | 13.991 | 15.092 | 15.188 | 14.001 | 14.002 | 15.094 | 15.193 | 15.094 | 15.194 |
| 90 | 0.85 | 0 | 2 | 16.197 | 17.454 | 17.598 | 16.200 | 16.202 | 17.458 | 17.600 | 17.458 | 17.601 |
| 90 | 0.65 | 0.2 | 1 | 14.002 | 15.088 | 15.180 | 13.977 | 13.976 | 15.092 | 15.178 | 15.091 | 15.180 |
| 90 | 0.85 | 0.2 | 2 | 16.236 | 17.478 | 17.643 | 16.195 | 16.195 | 17.480 | 17.641 | 17.481 | 17.641 |
| 95 | 0.65 | 0 | 1 | 10.913 | 11.827 | 11.923 | 10.922 | 10.924 | 11.832 | 11.924 | 11.831 | 11.924 |
| 95 | 0.85 | 0 | 2 | 13.035 | 14.153 | 14.280 | 13.043 | 13.044 | 14.154 | 14.279 | 14.153 | 14.279 |
| 95 | 0.65 | 0.2 | 1 | 10.882 | 11.785 | 11.879 | 10.869 | 10.869 | 11.784 | 11.875 | 11.782 | 11.875 |
| 95 | 0.85 | 0.2 | 2 | 13.033 | 14.113 | 14.264 | 13.008 | 13.008 | 14.115 | 14.265 | 14.117 | 14.264 |
| 100 | 0.65 | 0 | 1 | 8.305 | 9.018 | 9.095 | 8.316 | 8.313 | 9.020 | 9.099 | 9.021 | 9.101 |
| 100 | 0.85 | 0 | 2 | 10.274 | 11.200 | 11.316 | 10.274 | 10.280 | 11.203 | 11.319 | 11.203 | 11.318 |
| 100 | 0.65 | 0.2 | 1 | 8.307 | 9.040 | 9.102 | 8.302 | 8.301 | 9.039 | 9.109 | 9.037 | 9.107 |
| 100 | 0.85 | 0.2 | 2 | 10.274 | 11.174 | 11.301 | 10.247 | 10.247 | 11.173 | 11.305 | 11.180 | 11.303 |
| 105 | 0.65 | 0 | 1 | 6.218 | 6.757 | 6.817 | 6.222 | 6.221 | 6.758 | 6.821 | 6.761 | 6.821 |
| 105 | 0.85 | 0 | 2 | 7.950 | 8.705 | 8.792 | 7.956 | 7.953 | 8.704 | 8.795 | 8.705 | 8.794 |
| 105 | 0.65 | 0.2 | 1 | 6.199 | 6.739 | 6.785 | 6.195 | 6.196 | 6.741 | 6.790 | 6.744 | 6.789 |
| 105 | 0.85 | 0.2 | 2 | 7.926 | 8.658 | 8.753 | 7.915 | 7.913 | 8.661 | 8.753 | 8.660 | 8.754 |
| 110 | 0.65 | 0 | 1 | 4.586 | 4.921 | 5.038 | 4.581 | 4.601 | 4.921 | 5.041 | 4.936 | 5.042 |
| 110 | 0.85 | 0 | 2 | 6.057 | 6.643 | 6.707 | 6.065 | 6.064 | 6.646 | 6.704 | 6.646 | 6.706 |
| 110 | 0.65 | 0.2 | 1 | 4.527 | 4.914 | 4.968 | 4.526 | 4.527 | 4.917 | 4.966 | 4.917 | 4.964 |
| 110 | 0.85 | 0.2 | 2 | 6.000 | 6.557 | 6.620 | 5.998 | 5.998 | 6.558 | 6.623 | 6.561 | 6.622 |

*Notes: the first regression specification, i.e. 1, is defined as $P_3(S)$. All the other regression specifications follow as descibed in table 4.2*

Table B.19: This table shows the comparison between the computed prices of American put options in sample and out of sample, for the information set $\mathcal{A}_\rho$, table 5.4. The specification of the conditional expectation of the payoff is $P_3(S)$.

| $S_0$ | $\rho_{S\delta}$ | $\rho_{SV}$ | $T$ | $\Pi_A^{in}$ | $se^{in}$ | $\Pi_A^{out}$ | $se^{out}$ | $\Pi_A^{in-out}$ | $se^{in-out}$ |
|---|---|---|---|---|---|---|---|---|---|
| 90 | 0.65 | 0 | 1 | 14.015 | 0.014 | 13.995 | 0.014 | 0.020 | 0.000 |
| 90 | 0.85 | 0 | 2 | 16.233 | 0.016 | 16.184 | 0.016 | 0.049 | 0.000 |
| 90 | 0.65 | 0.2 | 1 | 14.019 | 0.014 | 14.003 | 0.014 | 0.016 | 0.000 |
| 90 | 0.85 | 0.2 | 2 | 16.262 | 0.016 | 16.243 | 0.016 | 0.019 | 0.000 |
| | | | | | | | | | |
| 95 | 0.65 | 0 | 1 | 10.920 | 0.012 | 10.915 | 0.012 | 0.005 | 0.000 |
| 95 | 0.85 | 0 | 2 | 13.013 | 0.015 | 13.062 | 0.015 | -0.049 | 0.000 |
| 95 | 0.65 | 0.2 | 1 | 10.931 | 0.012 | 10.908 | 0.012 | 0.023 | 0.000 |
| 95 | 0.85 | 0.2 | 2 | 13.044 | 0.015 | 13.011 | 0.015 | 0.033 | 0.000 |
| | | | | | | | | | |
| 100 | 0.65 | 0 | 1 | 8.325 | 0.011 | 8.327 | 0.011 | -0.001 | 0.000 |
| 100 | 0.85 | 0 | 2 | 10.290 | 0.014 | 10.295 | 0.014 | -0.005 | 0.000 |
| 100 | 0.65 | 0.2 | 1 | 8.302 | 0.011 | 8.326 | 0.011 | -0.024 | 0.000 |
| 100 | 0.85 | 0.2 | 2 | 10.258 | 0.013 | 10.254 | 0.013 | 0.003 | 0.000 |
| | | | | | | | | | |
| 105 | 0.65 | 0 | 1 | 6.225 | 0.011 | 6.219 | 0.011 | 0.005 | 0.000 |
| 105 | 0.85 | 0 | 2 | 7.955 | 0.013 | 7.960 | 0.013 | -0.005 | 0.000 |
| 105 | 0.65 | 0.2 | 1 | 6.192 | 0.011 | 6.175 | 0.011 | 0.016 | 0.000 |
| 105 | 0.85 | 0.2 | 2 | 7.913 | 0.012 | 7.898 | 0.012 | 0.015 | 0.000 |
| | | | | | | | | | |
| 110 | 0.65 | 0 | 1 | 4.595 | 0.011 | 4.588 | 0.011 | 0.007 | 0.000 |
| 110 | 0.85 | 0 | 2 | 6.047 | 0.012 | 6.035 | 0.012 | 0.011 | 0.000 |
| 110 | 0.65 | 0.2 | 1 | 4.553 | 0.011 | 4.533 | 0.011 | 0.019 | 0.000 |
| 110 | 0.85 | 0.2 | 2 | 5.999 | 0.012 | 5.984 | 0.012 | 0.016 | 0.000 |

*Notes:* $\Pi_A^{in}$ *and* $se^{in}$ *represent the price and standard error of the American option computed in sample, respectively, whereas* $\Pi_A^{out}$ *and* $se^{out}$ *represent the price and standard error of the American option computed out of sample, respectively. Note that* $\Pi_A^{in-out} = \Pi_A^{in} - \Pi_A^{out}$ *and* $se^{in-out} = se^{in} - se^{out}$.

Table B.20: This table shows the comparison between the computed prices of American put options in sample and out of sample, for the information set $\mathcal{A}_\rho$, table 5.4. The specification of the conditional expectation of the payoff is $P_3(S)+p_0(\delta)\hat{\beta}_{\delta_0}$.

| $S_0$ | $\rho_{S\delta}$ | $\rho_{SV}$ | $T$ | $\Pi_A^{in}$ | $se^{in}$ | $\Pi_A^{out}$ | $se^{out}$ | $\Pi_A^{in-out}$ | $se^{in-out}$ |
|---|---|---|---|---|---|---|---|---|---|
| 90 | 0.65 | 0 | 1 | 15.084 | 0.017 | 15.071 | 0.017 | 0.013 | 0.000 |
| 90 | 0.85 | 0 | 2 | 17.475 | 0.018 | 17.445 | 0.018 | 0.030 | 0.000 |
| 90 | 0.65 | 0.2 | 1 | 15.081 | 0.017 | 15.101 | 0.017 | -0.020 | 0.000 |
| 90 | 0.85 | 0.2 | 2 | 17.470 | 0.017 | 17.475 | 0.017 | -0.004 | 0.000 |
| 95 | 0.65 | 0 | 1 | 11.856 | 0.016 | 11.795 | 0.016 | 0.061 | 0.000 |
| 95 | 0.85 | 0 | 2 | 14.134 | 0.017 | 14.156 | 0.017 | -0.022 | 0.000 |
| 95 | 0.65 | 0.2 | 1 | 11.810 | 0.015 | 11.816 | 0.015 | -0.006 | 0.000 |
| 95 | 0.85 | 0.2 | 2 | 14.128 | 0.017 | 14.132 | 0.016 | -0.005 | 0.000 |
| 100 | 0.65 | 0 | 1 | 9.047 | 0.015 | 9.045 | 0.015 | 0.003 | 0.000 |
| 100 | 0.85 | 0 | 2 | 11.204 | 0.016 | 11.226 | 0.016 | -0.022 | 0.000 |
| 100 | 0.65 | 0.2 | 1 | 9.034 | 0.014 | 9.052 | 0.014 | -0.017 | 0.000 |
| 100 | 0.85 | 0.2 | 2 | 11.187 | 0.016 | 11.154 | 0.016 | 0.032 | 0.000 |
| 105 | 0.65 | 0 | 1 | 6.763 | 0.014 | 6.760 | 0.014 | 0.003 | 0.000 |
| 105 | 0.85 | 0 | 2 | 8.699 | 0.015 | 8.698 | 0.015 | 0.001 | 0.000 |
| 105 | 0.65 | 0.2 | 1 | 6.741 | 0.014 | 6.722 | 0.014 | 0.019 | 0.000 |
| 105 | 0.85 | 0.2 | 2 | 8.647 | 0.015 | 8.659 | 0.015 | -0.012 | 0.000 |
| 110 | 0.65 | 0 | 1 | 4.977 | 0.013 | 4.982 | 0.013 | -0.006 | 0.000 |
| 110 | 0.85 | 0 | 2 | 6.645 | 0.014 | 6.629 | 0.014 | 0.016 | 0.000 |
| 110 | 0.65 | 0.2 | 1 | 4.960 | 0.013 | 4.942 | 0.013 | 0.019 | 0.000 |
| 110 | 0.85 | 0.2 | 2 | 6.565 | 0.014 | 6.561 | 0.014 | 0.005 | 0.000 |

*Notes:* $\Pi_A^{in}$ *and* $se^{in}$ *represent the price and standard error of the American option computed in sample, respectively, whereas* $\Pi_A^{out}$ *and* $se^{out}$ *represent the price and standard error of the American option computed out of sample, respectively. Note that* $\Pi_A^{in-out} = \Pi_A^{in} - \Pi_A^{out}$ *and* $se^{in-out} = se^{in} - se^{out}$.

Table B.21: This table shows the increments of the average American option price plotted in figure 5.1a, 5.2a, 5.3a and 5.4a, which correspond to the information sets $\mathcal{A}_\sigma$, $\mathcal{A}_k$, $\mathcal{A}_\theta$ end $\mathcal{A}_\rho$, respectively, in the Laguerre polynomial specification.

| $\Delta_L$ | $I_{i=0}$ | $I_{i=1}$ | $I_{i=2}$ | $I_{i=3}$ | $I_{i=4}$ | $I_{i=5}$ |
|---|---|---|---|---|---|---|
| $\mathcal{A}_\sigma$ | -0.087630399 | -0.03269 | 0.088032 | -0.024 | 0.308038 | -0.16486 |
| $\mathcal{A}_k$ | -0.112248605 | -0.00424 | 0.065475 | -0.04527 | 0.206431 | -0.08953 |
| $\mathcal{A}_\theta$ | -0.133917542 | 0.024885 | 0.072726 | 0.974247 | -0.8198 | 0.161036 |
| $\mathcal{A}_\rho$ | -0.127171582 | -0.02398 | 0.080857 | -0.07916 | 0.27245 | -0.02548 |

*Notes: Where* $I_i = \bar{\Pi}_{A,L_i(S)} - \bar{\Pi}_{A,L_{i+1}(S)}$ *and* $\bar{\Pi}_{A,L_i(S)}$ *is the average American option price for a chosen specification of the Laguerre polynomial, i.e. the i specification.*

Table B.22: This table shows the increments of the average American option price plotted in figure 5.1a, 5.2a, 5.3a and 5.4a, which correspond to the information sets $\mathcal{A}_\sigma$, $\mathcal{A}_k$, $\mathcal{A}_\theta$ end $\mathcal{A}_\rho$, respectively, in the power polynomial specification.

| $\Delta_P$ | $I_{i=0}$ | $I_{i=1}$ | $I_{i=2}$ | $I_{i=3}$ | $I_{i=4}$ | $I_{i=5}$ |
|---|---|---|---|---|---|---|
| $\mathcal{A}_\sigma$ | -1.5784E-01 | -0.02904 | -0.00226 | -0.00288 | 0.018928 | 0.193868 |
| $\mathcal{A}_k$ | -0.174218752 | -0.00111 | -0.00376 | 0.015573 | 0.061151 | 0.04741 |
| $\mathcal{A}_\theta$ | -0.190386104 | 0.027123 | -0.00727 | -0.004 | 0.115967 | 0.201843 |
| $\mathcal{A}_\rho$ | -0.18683052 | -0.01995 | -0.00997 | 0.006903 | 0.054903 | 0.157596 |

*Notes: Where* $I_i = \bar{\Pi}_{A,P_i(S)} - \bar{\Pi}_{A,P_{i+1}(S)}$ *and* $\bar{\Pi}_{A,P_i(S)}$ *is the average American option price for a chosen specification of the power polynomial, i.e. the i specification.*

Table B.23: This table shows the difference between the value function of the Laguerre and power polynomial plotted in figure 5.1a, 5.2a, 5.3a and 5.4a, which correspond to the information sets $\mathcal{A}_\sigma$, $\mathcal{A}_k$, $\mathcal{A}_\theta$ end $\mathcal{A}_\rho$, respectively.

| $q$ in figure | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $\mathcal{A}_\sigma$ | 0.066485681 | -0.00373 | -7.75E-05 | -0.09037 | -0.06925 | -0.35836 | 0.000361 |
| $\mathcal{A}_k$ | 0.060417489 | -0.00155 | 0.001576 | -0.06766 | -0.00681 | -0.15209 | -0.01515 |
| $\mathcal{A}_\theta$ | 0.056627147 | 0.000159 | 0.002397 | -0.0776 | -1.05585 | -0.12009 | -0.07928 |
| $\mathcal{A}_\rho$ | 0.056870864 | -0.00279 | 0.001238 | -0.08959 | -0.00353 | -0.22107 | -0.038 |

*Notes: Any numerical value is computed by subtracting the power polynomial value function from the Laguerre one. In other words, a negative value implies that the power polynomial specification gives a higher price to the American option in average.*

Table B.24: This table shows the average difference value between in sample and out of sample option prices as defined in tables B.4-B.5, B.9-B.10, B.14-B.15 and B.19-B.20 in the Appendix.

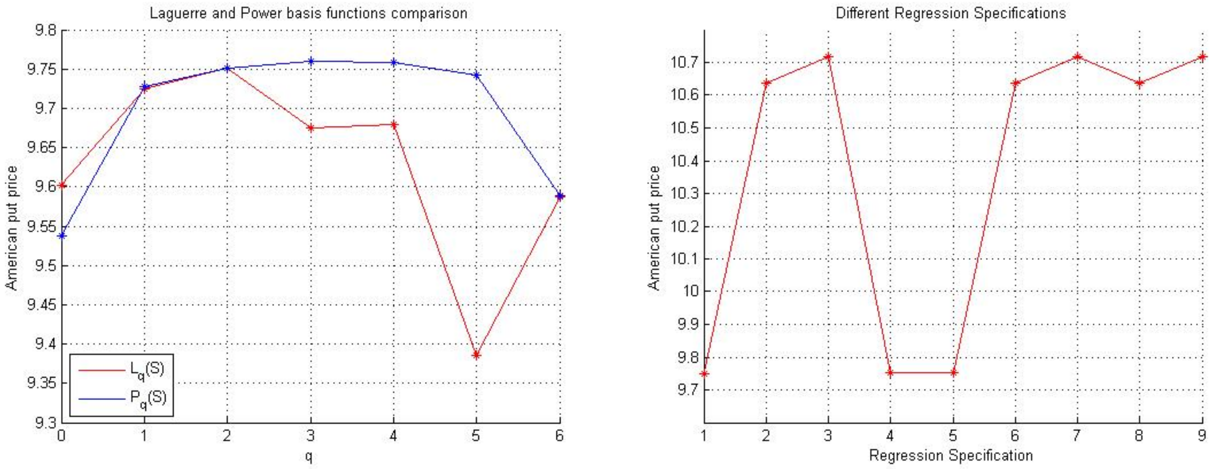
| | $\mathcal{A}_\sigma$ | $\mathcal{A}_k$ | $\mathcal{A}_\theta$ | $\mathcal{A}_\rho$ |
|---|---|---|---|---|
| *Benchmark* | 0.002 | 2.82E-04 | 0.000175 | 0.008784 |
| $Benchmark + \delta$ | 0.007 | -0.00597 | -0.01039 | 0.004351 |

*Notes:* $Benchmark_{\mathcal{A}_\sigma} = P_2(S)$, $Benchmark_{\mathcal{A}_k} = P_1(S)$, $Benchmark_{\mathcal{A}_\theta} = L_1(S)$ *and* $Benchmark_{\mathcal{A}_\rho} = P_3(S)$. $Benchmark + \delta \equiv$ *adding* $\delta$ *as regressor.*

# B.2 Figures

### $\mathcal{A}_f$ Analysis

Figure B.1: This figure shows: Laguerre-power polynomial analysis and yield-volatility basis function analysis for information set $\mathcal{A}_f$, table 5.5.




(a) Laguerre-power polynomial analysis (b) yield-volatility basis function analysis

Table B.25: This table shows the difference between the value function of the Laguerre and power polynomial plotted in figure B.1a, which corresponds to information set $\mathcal{A}_f$, table 5.5.

| $q$ in figure | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $\mathcal{A}_f$ | 0.065215 | -0.00377 | 4.30E-05 | -0.0851 | -0.07857 | -0.35689 | -0.00147 |

*Notes: Any numerical value is computed by subtracting the power polynomial value function from the Laguerre one. In other words, a negative value implies that the power polynomial specification gives a higher price to the American option in average.*

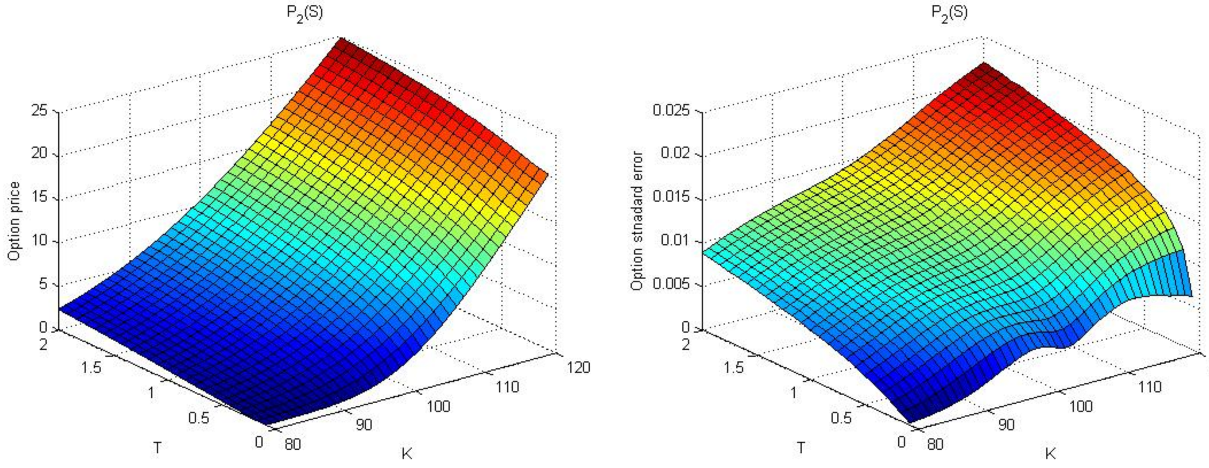
Table B.26: This table shows the increments of the average American option price plotted in figure B.1a, which correspond to information set $\mathcal{A}_f$, table 5.5, in the Laguerre and power polynomial specification.

| $\Delta_{\mathcal{G}}$ | $I_{i=0}$ | $I_{i=1}$ | $I_{i=2}$ | $I_{i=3}$ | $I_{i=4}$ | $I_{i=5}$ |
|---|---|---|---|---|---|---|
| $\Delta_L$ | -0.12198 | -0.02654 | 0.076108 | -0.00487 | 0.294538 | -0.2017 |
| $\Delta_P$ | -0.19096 | -0.02272 | -0.00904 | 0.001663 | 0.016218 | 0.153715 |

*Notes: Where $I_i = \bar{\Pi}_{A,\mathcal{G}_i(S)} - \bar{\Pi}_{A,\mathcal{G}_{i+1}(S)}$ and $\bar{\Pi}_{A,\mathcal{G}_i(S)}$ is the average American option price for a chosen specification $\mathcal{G}$, which can be either Laguerre (L) or power(P) polynomial, e.g. the i Laguerre specification.*

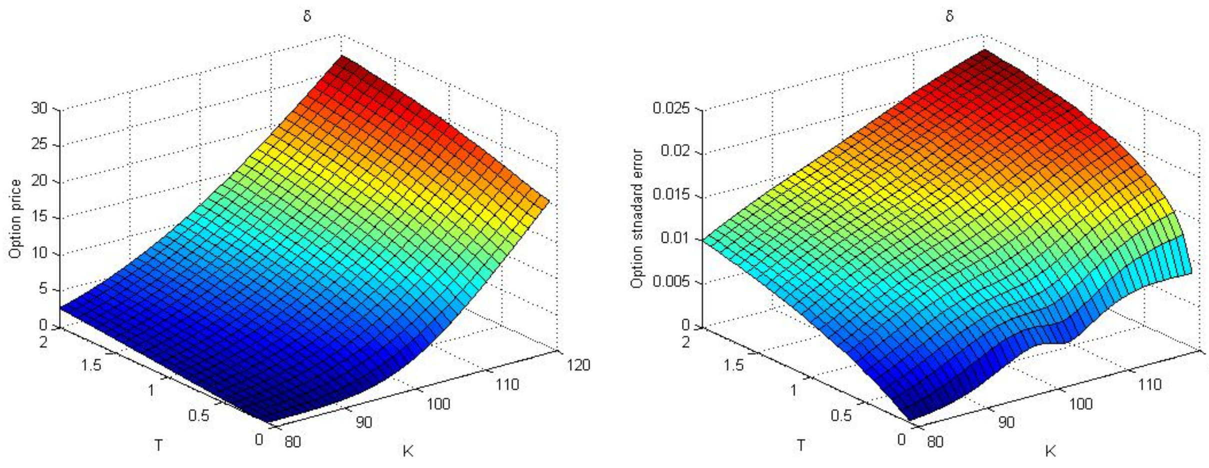
## $\mathcal{A}_f$ Surfaces

Figure B.2: This figure shows the surfaces of American option prices and standard errors computed with $P_2(S)$. Information set $\mathcal{A}_f$, table 5.5.



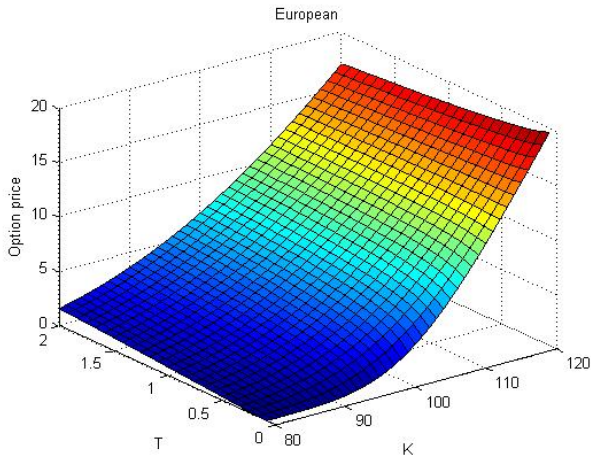

(a) $\Pi_A^{P_2(S)}$ (b) $s.e._A^{P_2(S)}$

Figure B.3: This figure shows the surfaces of American option prices and standard errors computed with $P_2(S) + p_0(\delta)\hat{\beta}_{\delta_0}$. Information set $\mathcal{A}_f$, table 5.5.

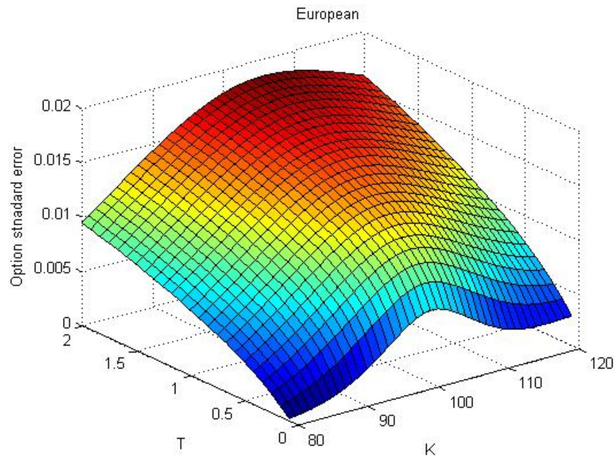



(a) $\Pi_A^{P_2(S)+p_0(\delta)\hat{\beta}_{\delta_0}}$ (b) $s.e._A^{P_2(S)+p_0(\delta)\hat{\beta}_{\delta_0}}$

Figure B.4: This figure shows the surfaces of European option prices and standard errors computed with Monte Carlo by using equation 1.9. Information set $\mathcal{A}_f$, table 5.5.




(a) $\Pi_E$




(b) $s.e._E$

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