

Evaluation of analytical approximations of two-asset basket option price

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Abstract

This thesis applies the decomposition suggested by Alexander and Venkatramanan (2012) to the pay-off of a basket option of two assets with a non-zero strike to derive an approximate price of corresponding basket option. The decomposition yields two sub-baskets and a situation where sub-strikes must be chosen. Approximation errors are studied for different sub-strikes and whenever possible, a rule of thumb is given. Furthermore, the approximation is evaluated against Monte Carlo simulations for a wide range of parameters. Finally, the approximation is benchmarked against the well renowned approximation suggested by Ju (2002) based on Taylor expansion around zero-volatility. It is concluded that the approximation suggested by Alexander and Venkatramanan (2012) is highly dependent on the property of being in the money and is sometimes preferred over the approximation suggested by Ju (2002).

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Contents

1	Introduction	1
2	Theory and background	3
2.1	Option pricing and closed formulas	3
2.2	Basket options	5
3	Analytical approximations	6
3.1	Alexander and Venkatramanan (2012)	6
3.1.1	Two-dimensional decomposition	7
3.1.2	Pricing CEO under weak log-normality	8
3.2	Ju's Taylor expansion (2002)	10
4	Approximation errors	14
5	Comparison	20
6	Summary and discussion	24
	Bibliography	26

1 Introduction

Options are well-known financial contracts, traded in many alternating forms on virtually every financial market. A special, but not uncommon, category of options are called basket options, written on several underlying assets e.g. currencies, commodities or stocks. The structure of the basket option makes it possible to hedge a portfolio of assets with only one contract instead of hedging each individual assets with a corresponding option. The flexibility of basket options gives rise to both usefulness as well as importance in the financial world but makes them also hard to price, in particular due to multiple underlying assets. A sum of assets with log-normal distributed prices does not have log-normal distributed price, implying closed pricing formulas by Black and Scholes (1973) [5] and Margrabe (1978) [10] can not be applied to price a multi-asset basket option. This constraint is encountered in many areas of finance and in the case of basket options often solved by using Monte Carlo simulations combined with variance reducing techniques. Closed form pricing-formulas have the desired property of providing exact and unique prices within negligible time and their absence have encouraged several suggestions of analytical approximations which provide an approximate price within negligible computational time in contrast to Monte Carlo methods which require significant computational power. Several analytical approximations have been suggested such as moment matching methods by Levy (1992) [9], conditional expectation techniques by Beisser (1999) [3] and Taylor expansion around zero volatility by Ju (2002) [7] to mention some. Krekel et al. (2004) [8] compared those methods and three others, concluding Ju's Taylor expansion being overall best with a few exceptions.

Recently, Alexander and Venkatramanan (2012) [2] suggested a new analytical approximation of basket option prices which has not yet been benchmarked against existing approximations. This thesis evaluates the approximation in the special case of two underlying assets and provides deeper understanding of the approximation in this particular case. A desirable property of an analytical approximation is to be able to provide sufficiently good approximations independent of the number of underlying assets. In partic-

ular, an approximation managing to provide good approximations with four underlying assets but not with two underlying assets should be used with care. In fact, when evaluating approximations the basket option tend to be written on at least four assets, e.g. as Krekel et al. (2004) [8]. More underlying assets obviously increase the complexity of the pricing problem but also cause a diversification effect between the underlying assets, lowering the total volatility and making the pricing easier. Therefore, as mentioned, this thesis aims to evaluate this new approximation by Alexander and Venkatramanan and it is done in the case of two underlying assets.

Chapter 2 provides the basics in option theory including asset dynamics and closed pricing-formulas while chapter 3 treats the analytical approximations by Alexander and Venkatramanan and Ju. Especially, the approximation by Alexander and Venkatramanan applied to a basket option with two underlying assets is outlined here. In Chapter 4, the approximation is compared to Monte Carlo simulations for a wide range of parameters to reveal its strengths and drawbacks and secondly, in chapter 5, benchmarked against the approximation from Ju's Taylor expansion which was considered a winner by Krekel et al. (2004) [8]. In total, the purpose of this thesis covers both an analysis of the approximation error as well as a conclusion in chapter 6 whether the approximation suggested by Alexander and Venkatramanan, under certain conditions, may outperform Ju's Taylor expansion.

2 Theory and background

An option is a financial contract whose value depends non-randomly of a set of underlying assets and the price of asset i at time t is denoted $S_{i,t}$ throughout this thesis. The owner of an option does not own any of the underlying assets but the non-negative price of the option is due to non-negative pay-off at the expiration day of the option. Time-point $t = T$ denotes the expiration time of the option and remaining time until expiration, $T - t$, is referred to as time to maturity. The pay-off at the expiration day of the option is a deterministic function of the prices of the underlying assets at one or more time-points and generally written as

$$\hat{\Phi}(t_1, \dots, t_m) = [f(S_1, \dots, S_N, t_1, \dots, t_m) - K]^+ \quad (2.1)$$

where $0 \leq K$ is referred to as strike and f is a real-valued function depending on the price of asset S_1, \dots, S_N , $1 \leq N$ at the time-points $t = t_1, \dots, t_m$, $1 \leq m$ where $0 \leq t_j \leq T$ for each $1 \leq j \leq m$. Contracts whose pay-off function depends only on the asset-values at time $t = T$ are called simple (claims) and have the general pay-off function

$$\Phi(T) = [f(S_{1,T}, \dots, S_{N,T}) - K]^+ \quad (2.2)$$

Throughout this thesis, only simple claims will be considered. The structure of the option makes it suitable for both hedging and leveraging of portfolios and since the holder does not own the underlying assets, options can easily be written on commodities, currencies as well as other derivatives/options. For a more rigorous definition of the option-contract and extended background information, see Björk (2009) [4] and Hull (2008) [6].

2.1 Option pricing and closed formulas

Theoretically, the price $p(t)$ at time t , $0 \leq t \leq T$, of an option is given by the discounted expected value of the pay-off under the risk-neutral measure

\mathbb{Q} , see Björk (2009) [4],

$$p(t) = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}}(\Phi(T)) \quad (2.3)$$

where of course

$$p(T) = \Phi(T) \quad (2.4)$$

In practise however, calculating this quantity may be quite cumbersome or even impossible except in a few simple and well-known special cases. Considering Black-Scholes' framework where $(\Omega, (F_t)_{t \geq 0}, \mathbb{Q})$ is a measure space with Ω set of outcomes for S_t , \mathbb{Q} being a risk-neutral measure and F_t the natural filtration generated by the sigma algebras for all time-points up to t . Furthermore, let the asset dynamics be given by

$$dS_t = rS_t dt + \sigma S_t dW_t \quad (2.5)$$

with risk-free interest rate r , asset volatility σ and standard Brownian motion W_t . A call option (on the single asset) gives the holder the right (option) but not the obligation to, at the expiration day, buy the asset at the price of the strike, K . This yields terminal pay-off for the call-option as

$$\Phi_{call}(T) = [S_T - K]^+ \quad (2.6)$$

Denoting the price of the call at time t as $c(t)$, $0 \leq t \leq T$ we get

$$c(t) = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}}([S_T - K]^+) \quad (2.7)$$

Within Black-Scholes' framework, the call option price $c(t)$ is given by the well-known Black-Scholes formula, see Black and Scholes (1973) [5]:

$$c(t) = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}}([S_T - K]^+) = S_0 N(d_1) - e^{-r(T-t)} K N(d_2) \quad (2.8)$$

$$d_1 = \frac{\log(S_0/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}} \quad (2.9)$$

$$d_2 = d_1 - \sigma \sqrt{T-t} \quad (2.10)$$

where $N(\cdot)$ denotes the cumulative distribution function of a standard normal random variable.

Another well-known contract, the exchange option, gives the holder the option but not the obligation to exchange one asset for another. With pay-off function

$$\Phi_{exchange}(T) = [S_{1,T} - S_{2,T}]^+ \quad (2.11)$$

and denoting the price of the exchange option at time t as $e(t)$, $0 \leq t \leq T$ we get

$$e(t) = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}}([S_{1,T} - S_{2,T}]^+) \quad (2.12)$$

To price the contract we need the two-dimensional Black-Scholes' framework consisting of a measure space $(\Omega, (F_t)_{t \geq 0}, \mathbb{Q})$ with Ω as the set of outcomes for the pair $(S_{1,t}, S_{2,t})$, \mathbb{Q} being a bivariate risk-neutral measure and F_t the natural filtration generated by the sigma algebras for all time-points up to t . Corresponding market has two assets S_1 and S_2 with dynamics given by

$$dS_{i,t} = rS_{i,t}dt + \sigma_i S_{i,t}dW_{i,t}, \quad i = 1, 2 \quad (2.13)$$

$$dW_{1,t}dW_{2,t} = \rho dt \quad (2.14)$$

with risk-free interest rate r , individual asset volatilities σ_1 and σ_2 and $W_{1,t}$ and $W_{2,t}$ are correlated standard Brownian motions with correlation $\rho \in [-1, 1]$. Within the two-dimensional framework, Margrabe (1978) [10] derived a closed pricing-formula:

$$e(t) = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}}([S_{1,T} - S_{2,T}]^+) = S_{1,0}N(d_1) - S_{2,0}N(d_2) \quad (2.15)$$

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_1^2\sigma_2^2\rho} \quad (2.16)$$

$$d_1 = \frac{\log(S_{1,0}/S_{2,0}) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \quad (2.17)$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \quad (2.18)$$

2.2 Basket options

A special case of a multi-asset option is a basket option, characterized by its pay-off function. The pay-off of a general basket option (at time $t=T$) with strike K is given by

$$\Phi_{basket}(T) = [\theta_1 S_{1,T} + \theta_2 S_{2,T} + \dots + \theta_N S_{N,T} - K]^+ \quad (2.19)$$

or equivalently $[\Theta \mathbf{S}_T - K]^+$ where $\Theta = (\theta_1, \dots, \theta_N) \in \mathbb{R}^N$ and $\mathbf{S}_T = (S_{1,T}, \dots, S_{N,T})$ is a vector of asset prices, $1 \leq N$. Pricing of a basket option is in practice done via Monte-Carlo simulation combined with variance reducing techniques since there is no closed form pricing-formula for a general basket option. In particular, a linear combination/sum of log-normal-distributed asset prices/stochastic variables is not log-normal implying a different pay-off compared to vanilla options. However, letting $n = 1, \theta_1 = \pm 1$ yields a plain put/call and letting $n = 2, \Theta = (1, -1)$ yields an exchange option, both priced above under the principle of no arbitrage.

3 Analytical approximations

The intention of this chapter is to provide the main ideas of the derivation of the approximations suggested by Alexander and Venkatramanan (2012) [2] and Ju (2002) [7], hence providing the understanding later needed for implementation of a pricing algorithm. In particular, there is an absence of proofs and further outlines/derivations included in the original articles, but instead, the two asset case is treated more comprehensively.

3.1 Alexander and Venkatramanan (2012)

The method suggested by Alexander and Venkatramanan (2012) [2] is derived from decomposition of the pay-off function using the formulas:

$$[f + g]^+ = [f^+ - g^-]^+ + [g^+ - f^-]^+ \quad (3.1)$$

$$[f - g]^+ = [f^+ - g^+]^+ + [g^+ - f^+]^+ \quad (3.2)$$

where f and g are two real-valued functions and

$$f^+ = \max(f, 0) \quad (3.3)$$

$$f^- = \max(-f, 0) \quad (3.4)$$

Denoting the price at time $t, 0 \leq t \leq T$, of a N-asset basket call option $\mathbf{U}_{N,t}$ it is clear that

$$\mathbf{U}_{N,T} = [\Theta_N \mathbf{S}_{N,T} - K]^+ \quad (3.5)$$

and corresponding basket put option price $\mathbf{V}_{N,t}$ follows uniquely from put-call-parity and the concept of no-arbitrage in the Black-Scholes' framework. Applying decomposition (3.2) to corresponding pay-off function yields

$$\begin{aligned} [\Theta_N \mathbf{S}_{N,T} - K]^+ &= [(\Theta_m \mathbf{S}_{m,T} - K_1) + (\Theta_n \mathbf{S}_{n,T} - K_2)]^+ = \\ &= [(\Theta_m \mathbf{S}_{m,T} - K_1)^+ - (\Theta_n \mathbf{S}_{n,T} - K_2)^-]^+ + [(\Theta_n \mathbf{S}_{n,T} - K_2)^+ - (\Theta_m \mathbf{S}_{m,T} - K_1)^-]^+ = \\ &= [\mathbf{U}_{m,T} - \mathbf{V}_{n,T}]^+ + [\mathbf{U}_{n,T} - \mathbf{V}_{m,T}]^+ \quad (3.6) \end{aligned}$$

where Θ_m and Θ_n are chosen arbitrarily but such that $\Theta_m \mathbf{S}_{m,t} + \Theta_n \mathbf{S}_{n,t} = \Theta_N \mathbf{S}_{N,t}$ with $m + n = N$ and $K = K_1 + K_2$. The quantities $\Theta_m \mathbf{S}_{m,t}$ and $\Theta_n \mathbf{S}_{n,t}$ are referred to as sub-baskets and the contract having pay-off

$$\Phi_{CEO}(T) = [\mathbf{U}_{m,T} - \mathbf{V}_{n,T}]^+ \quad (3.7)$$

for some m and n is an exchange option of a sub-basket put option for a sub-basket call option, by Alexander and Venkatramanan (2012) [2] named Compound Exchange Option (CEO). A CEO can be written on two calls, two puts or one of each giving a total of four classes of CEO:s. A CEO is an exchange option since one asset is exchanged for another while it is compound since the assets exchanged are basket options. This set-up reduces the price of a basket option to the equivalent sum of two CEO prices. Denoting the true basket option price at time t as $BP(t)$, $0 \leq t \leq T$, we get

$$BP(t) = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}}([\Theta_N \mathbf{S}_{N,T} - K]^+) = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}}([\mathbf{U}_{m,T} - \mathbf{V}_{n,T}]^+) + e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}}([\mathbf{U}_{n,T} - \mathbf{V}_{m,T}]^+) \quad (3.8)$$

Furthermore, a single asset being log-normally distributed does not imply a linear combination of assets being log-normally distributed which is a well-known constraint in option-pricing. However, *if* the *prices* of the underlying sub-baskets being exchanged would follow a log-normal distribution, Margrabe's formula (2.15) could be used for pricing. Generally, only assets have a log-normal distribution but not option-prices. However, a vanilla call/put deep in the money (ITM) depends linearly on a single underlying asset making the price distribution of the option approximately log-normal as well, an observation playing a key-role in the paper by Alexander and Venkatramanan (2012) [2]. Below, the two-dimensional case ($N = 2$) is outlined and evaluated.

3.1.1 Two-dimensional decomposition

A (non-trivial) call basket option written on two underlying assets has pay-off function

$$\Phi(T; \theta_1; \theta_2) = [\theta_1 S_{1,T} + \theta_2 S_{2,T} - K]^+ \quad (3.9)$$

where at most one of the weights θ_1, θ_2 is negative and $K > 0$. The case of exactly one negative weight is in the literature commonly referred to as a spread-option and the lack of a closed-form pricing-formula (when $K \neq 0$) has encouraged several suggestions of analytical approximations. The presence of a minus-sign allow more flexibility when deriving an approximation and therefore this case is too wide to properly treat here [1]. With this in mind,

both $\theta_i, i = 1, 2$ are assumed to be positive. With out loss of generality, we can assume $\theta_1 = 1$ by rewriting $[\theta_1 S_{1,T} + \theta_2 S_{2,T} - K]^+ = \theta_1 [S_{1,T} + \frac{\theta_2}{\theta_1} S_{2,T} - \frac{K}{\theta_1}]^+$ but restricting $\theta_2 = 1$ as well yields perspicuous derivations/calculations and hence, it is assumed $\theta_1 = \theta_2 = 1$.

For each time $t, 0 \leq t \leq T$, the price of a call and put option written on (a single) asset S_i is denoted by $U_{i,t}$ and $V_{i,t}$ respectively. Then, applying decomposition (3.2) yields

$$\begin{aligned} [S_{1,T} + S_{2,T} - K]^+ &= [(S_{1,T} - K_1) + (S_{2,T} - K_2)]^+ = \\ &= [(S_{1,T} - K_1)^+ - (S_{2,T} - K_2)^-]^+ + [(S_{2,T} - K_2)^+ - (S_{1,T} - K_1)^-]^+ = \\ &= [(S_{1,T} - K_1)^+ - (K_2 - S_{2,T})^+]^+ + [(S_{2,T} - K_2)^+ - (K_1 - S_{1,T})^+]^+ = \\ &= [U_{1,T} - V_{2,T}]^+ + [U_{2,T} - V_{1,T}]^+ = CEO_{1,t} + CEO_{2,t} \quad (3.10) \end{aligned}$$

where $K_1 + K_2 = K$. Note the equality used,

$$(S_{2,T} - K_2)^- = \max(-(S_{2,T} - K_2), 0) = \max(K_2 - S_{2,T}, 0) = (K_2 - S_{2,T})^+ \quad (3.11)$$

Then, equation (3.8) simplifies to:

$$BP(t) = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}}([S_{1,T} + S_{2,T} - K]^+) = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}}(CEO_{1,t} + CEO_{2,t}) \quad (3.12)$$

Hence, the objective is pricing of CEO:s exchanging calls for puts written on different assets S_1 and S_2 .

3.1.2 Pricing CEO under weak log-normality

Alexander and Venkatramanan (2012) [2] follows the two-dimensional Black-Scholes' framework with filtered measure space defined earlier and market given by equation (2.13) and (2.14). Then, for any time $0 \leq t \leq T$, the price of $CEO_{1,t}$ with pay-off $[U_{1,T} - V_{2,T}]^+$ is denoted by f_t and given as a discounted expectation of the pay-off under the bivariate risk-neutral measure

$$f_t = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}([U_{1,T} - V_{2,T}]^+ | F_t) \quad (3.13)$$

If f_t^* denotes the corresponding price of $CEO_{2,t}$ then equation (3.12) can be written

$$BP(t) = f_t + f_t^* \quad (3.14)$$

However, this theoretical construction remains true but can not be solved in practise.

To find an approximation $BP_{AV}(t)$ of $BP(t)$ such that

$$BP_{AV}(t) \approx BP(t) \quad (3.15)$$

Alexander and Venkatramanan (2012) [2] begin by applying Itô's formula to $U_{i,t}$ which yields:

$$dU_{i,t} = rU_{i,t}dt + \xi_{i,t}U_{i,t}dW_{i,t} \quad (3.16)$$

$$\xi_{i,t} = \sigma_i \frac{S_{i,t}}{U_{i,t}} \frac{\partial U_{i,t}}{\partial S_{i,t}} \quad (3.17)$$

Applying Itô's formula to the volatility process yields

$$d\xi_{i,t} = \xi_{i,t}(\sigma_i - \xi_{i,t} + \sigma_i S_{i,t} \frac{\Gamma_{i,t}}{\Delta_{i,t}})[- \xi_{i,t}dt + dW_{i,t}] \quad (3.18)$$

where $\Delta_{i,t} = \frac{\partial U_{i,t}}{\partial S_{i,t}}$ and $\Gamma_{i,t} = \frac{\partial^2 U_{i,t}}{\partial S_{i,t}^2}$. The term $\sigma_i + \sigma_i S_{i,t} \frac{\Gamma_{i,t}}{\Delta_{i,t}} = \sigma_i(1 + S_{i,t} \frac{\Gamma_{i,t}}{\Delta_{i,t}})$ is approximated by $\tilde{\sigma}_i \equiv \sigma_i + c_i \approx \sigma_i + \sigma_i S_{i,t} \frac{\Gamma_{i,t}}{\Delta_{i,t}}$ hence, letting $S_{i,t} \frac{\Gamma_{i,t}}{\Delta_{i,t}}$ be constant by removing the time-dependence. Under this approximation, volatility dynamics is governed by

$$d\xi_{i,t} = \xi_{i,t}(\xi_{i,t} + \tilde{\sigma}_i)[\xi_{i,t}dt - dW_{i,t}] \quad (3.19)$$

Alexander and Venkatramanan (2012) [2] then note that a solution to (3.19) is given by

$$\xi_{i,t} = \tilde{\sigma}_i(1 + k_i e^{\frac{\tilde{\sigma}_i^2 t}{2} - \tilde{\sigma}_i W_{i,t}})^{-1} \quad (3.20)$$

$$k_i = \left(\frac{\tilde{\sigma}_i}{\xi_{i,0}} - 1 \right) \quad (3.21)$$

where $\xi_{i,0} > \tilde{\sigma}_i$ and $k_i < 0$. With volatility process (3.20) the call prices for each time $t, 0 \leq t \leq T$, is given by

$$U_{i,t} = \frac{U_{i,0}e^{rt}}{1 + k_i} (e^{-\frac{\tilde{\sigma}_i^2 t}{2} + \tilde{\sigma}_i W_{i,t}} + k_i) \quad (3.22)$$

where $k_i = \left(\frac{\tilde{\sigma}_i}{\xi_{i,0}} - 1 \right)$.

The assumption/scenario $k_i = 0$ is defined by Alexander and Venkatramanan (2012) [2] as the weak-log-normality condition which is equivalent to the call having a log-normal distributed price given by

$$U_{i,t} = U_{i,0}(e^{(r - \frac{\tilde{\sigma}_i^2}{2})t + \tilde{\sigma}_i W_{i,t}}) \quad (3.23)$$

At this point, two concerns/problems raise:

- When does $k_i = 0$ hold ?

- When is this approximation reasonable ?

Considering the relationship $k_i = 0 \Leftrightarrow (\frac{\tilde{\sigma}_i}{\xi_{i,0}} - 1) = 0 \Leftrightarrow \tilde{\sigma}_i = \xi_{i,0} \Leftrightarrow \sigma_i + c_i = \sigma_i \frac{S_{i,0}}{U_{i,0}} \frac{\partial U_{i,0}}{\partial S_{i,0}}$, Alexander and Venkatramanan (2012) [2] points out that the equality approximately holds if $\frac{S_{i,0}}{U_{i,0}} \approx 1$ and $\Delta_{i,0} = \frac{\partial U_{i,0}}{\partial S_{i,0}} \approx 1$ because then $\Gamma_{i,0} = \frac{\partial \Delta_{i,0}}{\partial S_{i,0}} \approx 0$ and hence $c_i \approx 0$. In total $\xi_{i,0} = \tilde{\sigma}_i = \sigma_i$. Moreover, inserting $k_i = 0$ in equation (3.20) gives constant volatility $\xi_{i,t} = \tilde{\sigma}_i$ at all times $0 \leq t \leq T$. Apparently, the weak log-normality condition is reasonable if $\frac{S_{i,t}}{U_{i,t}} \approx 1$ and $\frac{\partial U_{i,t}}{\partial S_{i,t}} \approx 1$ given the economic interpretation of the option being deep in the money (ITM) and having price dynamics approximately equal to the underlying asset. To see this, assume that the probability of an option (with strike K) being out of the money (OTM) before maturity is close to zero making the approximation $U_{i,t} \approx S_{i,t} - K$ valid. Taking derivatives gives $\frac{\partial U_{i,t}}{\partial S_{i,t}} \approx 1$ and dividing with $S_{i,t}$ gives $\frac{U_{i,t}}{S_{i,t}} \approx 1 - \frac{K}{S_{i,t}}$ which is close to 1 only if $S_{i,t} \rightarrow \infty$ or K is close to 0. However, the unlikeliness of these scenarios makes the last approximation questionable. Also, options deep OTM have a value close or equal to zero independent of the volatility which provides another scenario where the suggested approximation is reasonable.

Finally we consider the price of a vanilla put-option on asset S_i denoted as $V_{i,t}$ with dynamics

$$dV_{it} = rV_{it}dt + \eta_{it}V_{it}dW_{it} \quad (3.24)$$

$$\eta_{it} = \sigma_i \frac{S_{it}}{V_{it}} \left| \frac{\partial V_{it}}{\partial S_{it}} \right| \quad (3.25)$$

Since the put has the property $\frac{\partial V_{it}}{\partial S_{it}} \leq 0$, the equality $\eta_{it} = -\sigma_i \frac{S_{it}}{V_{it}} \frac{\partial V_{it}}{\partial S_{it}}$ holds and the volatility process for the put and the call have same dynamics and hence, equal process. Moreover, the minus sign changes sign of the correlation between the put option and the call option, see Alexander and Venkatramanan (2012) [2]. Basically, a put and call written on the same asset have negative correlation while put and call on two negatively correlated assets have positive correlation. Empirically though, assets tend to have positive correlation making the put and call negatively correlated.

3.2 Ju's Taylor expansion (2002)

The approximation suggested by Ju (2002) [7] is based on Taylor expansion around zero-volatility. Since individual assets may have different volatilities, all volatilities are scaled by the same parameter z ,

$$S_{i,t}(z) = S_{i,0}e^{(r-z^2\sigma_i^2/2)t+z\sigma_i w_{i,T}}, \quad i = 1, 2 \quad (3.26)$$

and the average (sum) of assets is defined as

$$A(z) = \sum_{i=1}^2 \chi_i S_{i,T}(z) = S_{1,0} e^{(-z^2 \sigma_1^2/2)T + z \sigma_1 w_{1,T}} + S_{2,0} e^{(-z^2 \sigma_2^2/2)T + z \sigma_2 w_{2,T}} \quad (3.27)$$

where the weights are chosen as $\chi_1 = \chi_2 = 1$. In this notation, the terminal pay-off can be written as $[A(1) - K]^+$. Let $Y(z)$ be a normal random variable with mean $m(z^2)$ and variance $v(z^2)$ such that $e^{Y(z)}$ has the same first two moments as $A(z)$. Furthermore, let $X(z) = \log A(z)$ and consider the characteristic function of $X(z)$

$$\mathbb{E}[e^{i\phi X(z)}] = \mathbb{E}[e^{i\phi Y(z)}] \frac{\mathbb{E}[e^{i\phi X(z)}]}{\mathbb{E}[e^{i\phi Y(z)}]} = \mathbb{E}[e^{i\phi Y(z)}] f(z). \quad (3.28)$$

Finally, $f(z)$ is Taylor expanded around $z = 0$ to include terms of order σ^6 . This approach extends the moment-matching approximation suggested by Levy (1992) [9] which equals the two first terms in the pricing formula and the remaining three terms are derived from the Taylor approximation around zero volatility. As before, let $BP(t)$ denote the exact basket option price and let $BP_{Ju}(t)$ denote the approximate basket price at each time t such that $0 \leq t \leq T$. Then,

$$BP(t) = e^{-r(T-t)} \mathbb{E}[e^{X(1)} - K]^+ \quad (3.29)$$

and

$$BP_{Ju}(t) \approx BP(t) \quad (3.30)$$

where the closed form approximation $BP_{Ju}(t)$ is given by Ju (2002) [7] as

$$BP_{Ju}(t) = e^{-r(T-t)} ([U_1 N(y_1) - K N(y_2)] + K [z_1 p(y) + z_2 \frac{dp(y)}{dy} + z_3 \frac{d^2 p(y)}{dy^2}]) \quad (3.31)$$

where $p(\cdot)$ denotes the standard normal probability density function. Moreover,

$$y = \log(K) \quad (3.32)$$

$$y_1 = \frac{m(1) - y}{\sqrt{v(1)}} - \sqrt{v(1)} \quad (3.33)$$

$$y_2 = y_1 - \sqrt{v(1)} \quad (3.34)$$

and Levy's (1992) [9] matching moments are included as:

$$m(1) = 2 \log(U_1) - \frac{1}{2} \log(U_2(1)) \quad (3.35)$$

$$v(1) = \log(U_2(1)) - 2 \log(U_1) \quad (3.36)$$

Moreover, defining $\bar{\rho}_{ij} = \rho_{ij}\sigma_i\sigma_jT$ and $\bar{S}_i = \chi_i S_{i,0}e^{rT} = S_{i,0}$ where we have used $\chi_1 = \chi_2 = 1$ and $r = 0$ gives:

$$U_1 = \sum_{i=1}^2 \bar{S}_i = \bar{S}_1 + \bar{S}_2 \quad (3.37)$$

$$U_2(z^2) = \sum_{ij=1}^2 \bar{S}_i \bar{S}_j e^{z^2 \bar{\rho}_{ij}} \quad (3.38)$$

$$U_2(0) = \sum_{ij=1}^2 \bar{S}_i \bar{S}_j = \bar{S}_i^2 + 2\bar{S}_i \bar{S}_j + \bar{S}_j^2 \quad (3.39)$$

$$U_2'(0) = \sum_{ij=1}^2 \bar{S}_i \bar{S}_j \bar{\rho}_{ij} \quad (3.40)$$

$$U_2''(0) = \sum_{ij=1}^2 \bar{S}_i \bar{S}_j \bar{\rho}_{ij}^2 \quad (3.41)$$

$$U_2'''(0) = \sum_{ij=1}^2 \bar{S}_i \bar{S}_j \bar{\rho}_{ij}^3 \quad (3.42)$$

The z_i , $i = 1, 2, 3$ are computed as linear combinations:

$$z_1 = d_2(1) - d_3(1) + d_4(1) \quad (3.43)$$

$$z_2 = d_3(1) - d_4(1) \quad (3.44)$$

$$z_3 = d_4(1) \quad (3.45)$$

where the functions d_i are computed as:

$$d_1(1) = \frac{1}{2}(6a_1^2(1) + a_2(1) - 4b_1(1) + 2b_2(1)) \\ - \frac{1}{6}(120a_1^3(1) - a_3(1) + 6(24c_1(1) - 6c_2(1) + 2c_3(1) - c_4(1)))$$

$$d_2(1) = \frac{1}{2}(10a_1^2(1) + a_2(1) - 6b_1(1) + 2b_2(1)) \\ - \left(\frac{128}{3}a_1^3(1) - \frac{1}{6}a_3(1) + 2a_1(1)b_1(1) - a_1(1)b_2(1)\right) \\ + 50c_1(1) - 11c_2(1) + 3c_3(1) - c_4(1)$$

$$d_3(1) = (2a_1^2(1) - b_1(1)) - \frac{1}{3}(88a_1^3(1) + 3a_1(1)(5b_1(1) - 2b_2(1)) \\ + 3(35c_1(1) - 6c_2(1) + c_3(1)))$$

$$d_4(1) = -\frac{20}{3}a_1^3(1) + a_1(1)(-4b_1(1) + b_2(1)) - 10c_1(1) + c_2(1)$$

$$\begin{aligned}
c_1(1) &= -a_1(1)b_1(1) \\
c_2(1) &= \frac{1}{144A(0)}(9[8 \sum_{ijkl=1}^2 \bar{S}_i \bar{S}_j \bar{S}_k \bar{S}_l \bar{\rho}_{il} \bar{\rho}_{jk} \bar{\rho}_{kl} + 2U_2'(0)U_2''(0)] \\
&\quad + 4[6 \sum_{ijkl=1}^2 \bar{S}_i \bar{S}_j \bar{S}_k \bar{S}_l \bar{\rho}_{il} \bar{\rho}_{jl} \bar{\rho}_{kl}]) \\
c_3(1) &= \frac{1}{48A^3(0)}(4[6 \sum_{ijk=1}^2 \bar{S}_i \bar{S}_j \bar{S}_k \bar{\rho}_{ik} \bar{\rho}_{jk}^2] + [8 \sum_{ijk=1}^2 \bar{S}_i \bar{S}_j \bar{S}_k \bar{\rho}_{ij} \bar{\rho}_{ik} \bar{\rho}_{jk}]) \\
c_4(1) &= a_1(1)a_2(1) - \frac{2}{3}a_1^3(1) - \frac{1}{6}a_3(1) \\
b_1(1) &= \frac{1}{4A^3(0)}2 \sum_{ijk=1}^2 \bar{S}_i \bar{S}_j \bar{S}_k \bar{\rho}_{ik} \bar{\rho}_{jk} \\
b_2(1) &= a_1^2(1) - \frac{1}{2}a_2(1) \\
a_1(1) &= -\frac{U_2'(0)}{2U_2(0)} \\
a_2(1) &= 2a_1^2(1) - \frac{U_2''(0)}{2U_2(0)} \\
a_3(1) &= 6a_1(1)a_2(1) - 4a_1^3(1) - \frac{U_2^3(0)}{2U_2(0)}
\end{aligned}$$

4 Approximation errors

The following chapter evaluates the performance of the analytical approximation suggested by Alexander and Venkatramanan (2012) [2] using Monte Carlo simulations with 2000000 independent observations including anti-thetic variables for variance reduction as benchmark. Table 4.1 defines a set of default parameters used throughout this thesis unless otherwise specified. One parameter at the time from table 4.1 is varied while remaining kept fixed making it possible to study the approximation error defined as the absolute value of the difference between the simulation result and the analytical approximation. Considering the concept of forward prices as explained by Krekel et al. (2004) [8], it is justified to let the interest rate equal zero, $r = 0$, as common when benchmarking different methods. Another remark regarding the default parameters in table 4.1 might be the low asset volatilities which is partly motivated by the basket option only having two assets and hence, not much of a diversification effect which usually lower the total option volatility.

Bearing in mind decomposition (3.10), restated as:

$$[S_{1,T} + S_{2,T} - K]^+ = [(S_{1,T} - K_1) + (S_{2,T} - K_2)]^+ = [U_{1,T} - V_{2,T}]^+ + [U_{2,T} - V_{1,T}]^+$$

the choice of $0 < K_1$ and $0 < K_2$ such that $K_1 + K_2 = K$ is still unclear.

Time to maturity	$T = 1$
Interest rate	$r = 0$
Assets prices at time $t = 0$	$S_{1,0} = S_{2,0} = 100$
Strike	$K = 200$
Asset volatilities	$\sigma_1 = \sigma_2 = 0.1$
Asset correlation	$\rho = 0.5$

Table 4.1: Table of default parameters used throughout the thesis if nothing else stated.

Approximation (3.23) is motivated if both the put and call is deep ITM or equivalently K_1 close to zero and K_2 close to K . Another concern is how different choices of K_1 yield different approximations and hence different errors without any rule for finding (if it exists) an optimal K_1 . The existence is unclear since even though the error is a continuous function of K_1 on the interval $(0, K)$, the interval itself is not compact. This uncertainty makes it suitable to consider the approximation error as a function of K_1 as done in the remainder of this chapter.

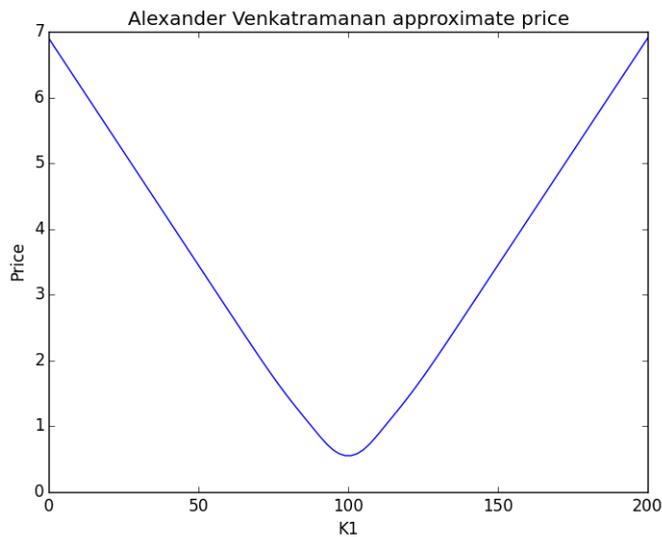


Figure 4.1: Two-asset basket option price-approximation with default parameters as in table 4.1 for different values of K_1 . Monte Carlo simulations provide an estimated price of 6.91.

Figure 4.1 shows the approximate price under default parameters for different choices of K_1 where Monte Carlo simulations give an approximate price of 6.91. Moreover, the approximation is worst off for $K \approx 100$ making all four options OTM while choosing K_1 close to either zero or K yield a suitable approximation since one CEO exchanges two deep ITM options while the value of the other is negligible.

Figure 4.2 shows how the approximation is worse off for longer maturities but to its defence, contracts with time to maturity longer than three years could without doubt be considered exotic. Considering strikes, we find the approximation worst off when $K = 200$ and $K_1 = K_2 = 100$ as in the situation described above. As expected, and seen in Figure 4.3, higher strike makes the basket option out of the money and worth less, making it easier

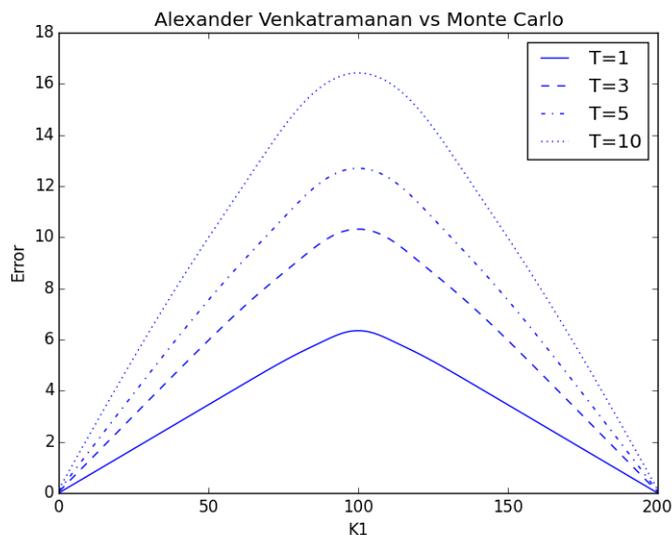


Figure 4.2: Error as the absolute value between the approximated price and corresponding Monte-Carlo simulated price for different times to maturity and different K_1 .

to approximate as the price goes to zero as K tends to infinity. Interesting though, is how lower strike makes the approximation in Figure 4.4 quite acceptable and more important, almost independent of the decomposition $K_1 + K_2 = K$.

Sadly though, Figure 4.6 indicates how the approximation gets better with negative correlation which empirically does not fit market data. Finally, in Figure 4.7 σ_1 is varied as labelled in the figure while $\sigma_2 = 0.1$ is constant, making the volatility ratio vary between one and 7 but also induces asymmetry. As usual, higher asset volatility makes pricing models more uncertain as in Figure 4.7.

Finally, we draw three conclusions:

- A sufficiently low strike makes the basket option deep ITM yielding an acceptable approximation independent of K_1 .
- Choosing $K_1 \approx 0$ and $K_2 \approx K$ gives (as a rule of thumb) a good approximation unless the basket option is OTM.
- The peaking error at $K_1 = K_2 = 100$ appears for all parameters except the strike and can not be avoided by choosing certain subsets of the set of parameters not including K . In total, unless the two asset basket option is OTM, the best we can do is to choose K_1 close to zero.

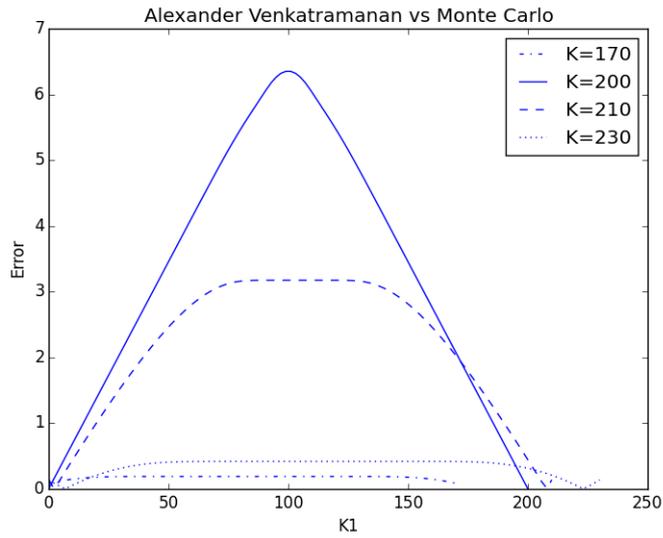


Figure 4.3: Error as the absolute value between the approximated price and corresponding Monte-Carlo simulated price for different strikes and different K_1 .

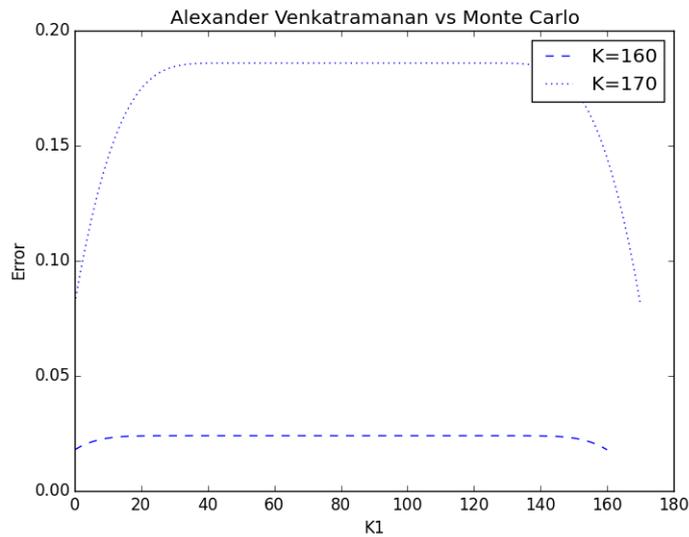


Figure 4.4: Error as the absolute value between the approximated price and corresponding Monte-Carlo simulated price for different strikes and different K_1 .

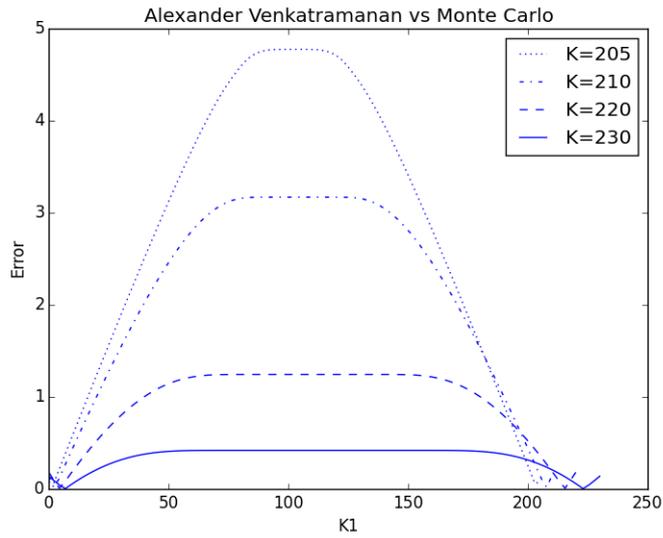


Figure 4.5: Error as the absolute value between the approximated price and corresponding Monte-Carlo simulated price for different strikes and different K_1 .

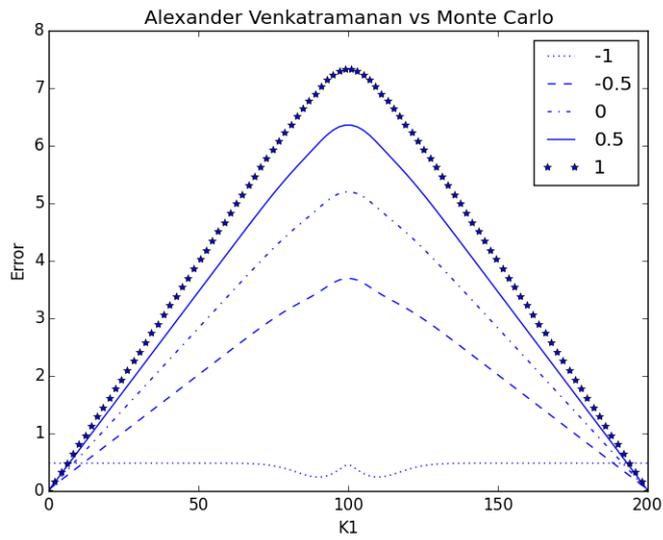


Figure 4.6: Error as the absolute value between the approximated price and corresponding Monte-Carlo simulated price for different correlations and different K_1 .

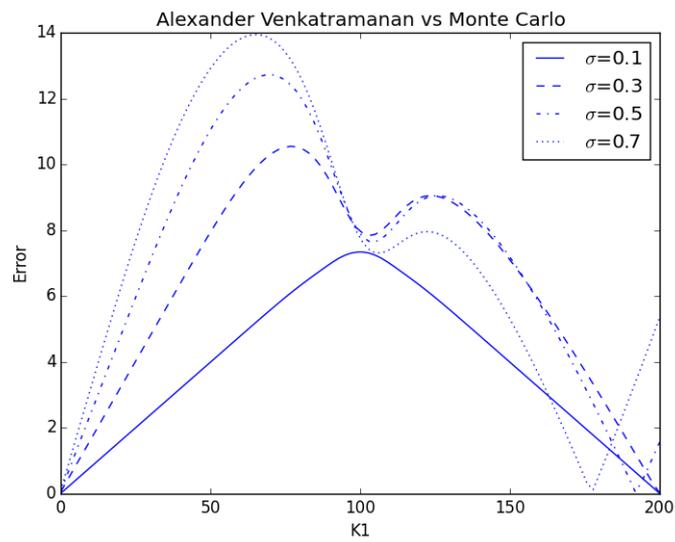


Figure 4.7: Error as the absolute value between the approximated price and corresponding Monte-Carlo simulated price for different volatilities σ_1 while keeping $\sigma_2 = 0.1$ fixed and different K_1 .

5 Comparison

This section covers a comparison between the analytical approximation suggested by Alexander and Venkatramanan (2012) [2] and the one derived by Ju (2002) [7]. As before, the basket consists of two assets and the parameters in table 4.1 are considered default. Bearing in mind Ju being the "winner" of approximations according to Krekel et al. (2004) [8] and is expected to produce error of order 10^{-1} or even 10^{-2} (as in Krekel et al. (2004) [8]) while the maximum error of Alexander and Venkatramanan (2012) [2] as seen in previous chapter might have 100 times the magnitude for bad choices of K_1 such as $K_1 = K/2$. This makes it impossible to plot the errors against each other for all choices of K_1 . Considering the error-plots from previous section we conclude that Alexander and Venkatramanan's approximation would not stand a chance against Ju's Taylor expansion for arbitrary values of K_1 . To overcome this problem and derive a meaningful comparison we note that the error of Alexander and Venkatramanan's approximation shrinks in magnitude as $K_1 \rightarrow 0$ as discussed earlier. Using the two rules of thumb $K_1 = 1, K_2 = K - K_1$ and $K_1 = 0.01, K_2 = K - K_1$ gives the approximations collected below. One might argue that $K_1 < 0.01$ would give even better approximations but numerical errors (zero-division-error) is already an issue which is rather avoided if possible. However, the approximations are satisfactory but the numerical issue should be considered a con. Note the equal bold rows in each table indicating the approximations under the set of default parameters as in table 4.1.

Table 5.1 shows how Ju manage to provide an approximation within the confidence-bound for each time to maturity T . Alexander and Venkatramanan provides equally good approximations if $K_1 = 0.01$ and $T \leq 3$. Furthermore, table 5.1 tells us the approximation fails to be contained within the confidence bound for $K_1 = 1$.

Spectacularly Ju fails to provide a sufficient approximation if the option is too deep ITM as seen in table 5.2 in contrast to the approximation by Alexander and Venkatramanan which is heavily dependent on this property. From Figure 4.4 we recall the nice property of the approximation being almost

T	AV, $K_1 = 1$	AV, $K_1 = 0.01$	Ju	Monte Carlo (StdDev)
0.5	4.83	4.88	4.88	4.88 (0.01)
1	6.83	6.90	6.90	6.91 (0.01)
3	11.80	11.92	11.96	11.94 (0.02)
5	15.20	15.36	15.43	15.44 (0.03)
10	21.50	21.71	21.80	21.85 (0.05)

Table 5.1: Approximated basket option prices by Alexander and Venkatramanan (AV) and by Ju benchmarked against Monte Carlo simulations for different times to maturities. Remaining parameters are default as given in table 4.1.

K	AV, $K_1 = 1$	AV, $K_1 = 0.01$	Ju	Monte Carlo (StdDev)
150	50.00	50.00	64.56	50.00 (0.02)
170	30.09	30.10	30.38	30.18 (0.02)
200	6.83	6.90	6.90	6.91 (0.01)
205	4.78	4.85	4.77	4.77 (0.01)
210	3.25	3.31	3.16	3.17 (0.01)
215	2.15	2.20	2.00	2.03 (0.008)
220	1.38	1.42	1.21	1.24 (0.006)
225	0.87	0.90	0.68	0.73 (0.005)
230	0.53	0.55	0.32	0.41 (0.003)

Table 5.2: Approximated basket option prices by Alexander and Venkatramanan (AV) and by Ju benchmarked against Monte Carlo simulations for different strikes. Remaining parameters are default as given in table 4.1.

ρ	AV, $K_1 = 1$	AV, $K_1 = 0.01$	Ju	Monte Carlo (StdDev)
-1	0.00	0	2.67	0.48 (0.001)
-0.5	3.94	3.98	4.03	4.01 (0.008)
0	5.58	5.63	5.65	5.64 (0.01)
0.5	6.83	6.90	6.90	6.91 (0.01)
1	7.88	7.96	7.67	7.97 (0.02)

Table 5.3: Approximated basket option prices by Alexander and Venkatramanan (AV) and by Ju benchmarked against Monte Carlo simulations for different asset correlations. Remaining parameters are default as given in table 4.1.

σ_1	AV, $K_1 = 1$	AV, $K_1 = 0.01$	Ju	Monte Carlo (StdDev)
0.1	6.83	6.90	6.90	6.91 (0.01)
0.3	14.16	14.30	14.35	14.35 (0.03)
0.5	21.71	21.92	21.96	21.98 (0.06)
0.7	29.12	29.41	29.07	29.53 (0.09)

Table 5.4: Approximated basket option prices by Alexander and Venkatramanan (AV) and by Ju benchmarked against Monte Carlo simulations for different asset volatilities σ_1 . Remaining parameters are default as given in table 4.1.

independent of the choice of K_1 if the basket option is deep ITM as supported by table 5.2. Sadly though, for $K > 200$ or equivalently, the basket being OTM, the approximation suggested by Alexander and Venkatramanan is worse of than Ju's Taylor expansion as well as worse of for $K_1 = 0.01$ than $K_1 = 1$. This property is visualized in Figure 4.5 and quite alarming since it implies the existence of an optimal $K_1 \neq 0$ without an algorithm for how to find such a K_1 .

Table 5.3 shows how both approximations collapses for perfectly negative correlation between the assets which can be considered a theoretical problem rather than a practical. Interesting is how Alexander and Venkatramanan's approximation succeeds in providing good approximations for all correlations while Ju's Taylor expansion fails at correlations above $\rho = 0.5$.

Bering in mind the derivation of Ju's Taylor expansion around zero volatility it is not unexpected how this approximation performs slightly better than Alexander and Venkatramanan's for $\sigma_1 \leq 0.5$, as seen in table 5.4. However, for $\sigma_1 = 0.7$, Alexander and Venkatramanan's approximation is still fairly

close without being inside the confidence interval. Also, Krekel et al. (2004) [8] points out that other approximations than Ju's Taylor Expansion is to be preferred if the ratio of the individual asset volatilities grows too large.

6 Summary and discussion

In summary, the approximation suggested by Alexander and Venkatramanan (2012) [2] (applied to the two asset basket) is highly dependent on the property of being ATM or ITM since only then $K_1 \rightarrow 0$ is a suitable rule of thumb for the decomposition. Due to the symmetry of S_1 and S_2 (at least in this thesis) this could of course be formulated as letting $K_2 \rightarrow K$ as well. From previous chapter we concluded that a correct choice of K_1 is crucial for a sufficiently good approximation and only when the basket is deep ITM, the approximation provides good approximations independent of the splitting of K into K_1 and K_2 . In this case, the approximation by Alexander and Venkatramanan is not only good and better than Ju's Taylor expansion, but also the approximation of Ju fails if the basket is too deep ITM. If, on the other hand, the basket is OTM, the optimal choice of K_1 is for some unknown $K_1 > 0$ as seen in Figure 4.5. In this situation, the non-zero price of the option is due to the non-zero probability of a non-zero pay-off in the future, often referred to as volatility part of the price. In other words, there is no real-value part of the price, making the pricing in general a lot harder. The lack of such an algorithm for finding the optimal K_1 when the option is OTM is a huge drawback for the method, in particular since as mentioned above, correctly pricing of an option slightly OTM is a great challenge for most pricing formulas. As seen in table 5.2, even the approximation by Ju can not provide prices within the confidence-interval in this case. However, one should bear in mind the very strict confidence intervals used in this thesis based on two million observations (including anti-thetic variables) which might be too computational expensive in practise.

From a practical point of view, the choice of a small K_1 induces numerical issues since expressions as $\log(\frac{S_{i,0}}{K_1})$ appears in the closed pricing formulas as part of the approximation. The choices of K_1 in this thesis provides reasonable approximations to be verified against both Ju's Taylor expansion and Monte Carlo methods. In practice however, the existence of analytical approximations is motivated by the lack of such verification.

For times to maturity longer than three years the approximation by

Alexander and Venkatramanan loses precision but in practice, those contracts are to be considered exotic and the drawback could be considered a theoretical issue only. With positive correlation between the assets, as often in empirical findings, Alexander and Venkatramanan provides at least as good approximations and when the ratio between the individual volatilities grows, the approximation by Alexander and Venkatramanan is to be preferred as well.

Another remark being that empirical studies as this often creates a basket consisting of the average of at least four assets implying a non-negligible reduction of portfolio variance due to the diversification effect. An option with lower volatility is often easier to price, partly due to a lower price, influencing the performance of the approximations. Therefore, an approximation treated here may be well-performing when benchmarked using several underlying assets even though it is not considered satisfactory here. Also, an approximation derived under the restriction of only two underlying assets is obviously as least as good as one derived for an arbitrarily number of underlying assets.

As usual in comparisons/benchmarks like this there is no overall winner. Even Krekel (2004) [8] concluded Ju as a winner under certain restrictions where under a few circumstances the approximation by Beisser (1999) [3] were to be preferred. Similarly, this thesis holds Ju as an over-all well performing approximation and if the basket is OTM, the approximation by Alexander and Venkatramanan is not recommended without an algorithm for finding the optimal decomposition of the strike. If, however, the basket option is ATM or even better ITM, the decomposition treated here may be desirable.

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