Some properties of Bessel functions with applications to Neumann eigenvalues in the unit disc

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Abstract

A study is made of the multiplicities of the eigenvalues of the Laplace operator in the unit disc with Neumann boundary conditions. Our result is that all eigenvalues are simple or double, a result which is well-known if one instead considers Dirichlet boundary conditions. The result is proved by using some properties of Bessel functions. We also give an application to standing water waves in a cylinder.

Sammanfattning

Vi studerar multipliciteten av egenvärdena till Laplace-operatorn i enhetsskivan med Neumann-randvillkor. Vårt resultat är att alla egenvärden är enkla eller dubbla, ett resultat som är välkänt om man istället föreskriver Dirichlet-randvillkor. Beviset bygger på några egenskaper hos Besselfunktioner. Vi ger också en tillämpning på stående vattenvågor i en cylinder.

1 Introduction

In this thesis we investigate the multiplicities of the eigenvalues of the Laplace operator in the unit disc with Neumann boundary conditions. For the corresponding one-dimensional problem it is well-known that all eigenvalues are simple [1]. For certain two-dimensional geometries it is however possible to obtain eigenvalues with arbitrarily large multiplicity (see e.g. [2]). For the disc it has been shown (using the properties of Bessel functions) that all eigenvalues are simple or double if one instead considers Dirichlet boundary conditions [12]. We prove the corresponding result with Neumann boundary conditions. We investigate this by using:

a) the recurrence relations between Bessel functions of different orders;

b) the fact that the non-zero stationary points of the Bessel functions of rational order are transcendental numbers.

As an application of this result we study linear standing water waves in a cylinder. The study of linear standing water waves goes back at least to the beginning of the 19th century, with contributions by Poisson [9] and Cauchy [3] (see [4] for a historical account). A rigorous proof of the existence of standing-wave solutions of the two-dimensional nonlinear water wave problem was obtained only recently by Iooss, Plotnikov and Toland [5, 6, 8]. The problem is notoriously difficult due to the appearance of small divisors. In other words, the linearised operator does not have a bounded inverse in a suitable Banach space due to 'near resonances', and one is forced to resort to Nash-Moser theory. In fact, in the case of infinite depth there are infinitely many true resonances, so that the kernel of the linearised operator is infinite-dimensional. The corresponding cylindrical problem has so far only been considered formally by a few authors. In particular, Mack [7] showed that there are no resonances for all but countably many fluid heights in the radial case, so that the solution space of the linearised problem is one-dimensional if one imposes a symmetry condition. The above result about the multiplicities of the eigenvalues allows us to generalise this conclusion to the non-radial case.

2 Bessel functions

In this section we define the Bessel functions and derive some recurrence formulae which will be used in the next section.

Definition 2.1. Let $v \in \mathbb{R} \setminus \{-1, -2, \ldots\}$. The Bessel function of the first kind of order v, $J_v(x)$, is defined by

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(v+n+1)} \left(\frac{x}{2}\right)^{v+2n}$$

where $\Gamma(x)$ is the gamma function. If v is a negative integer, we define $J_v(x) = (-1)^v J_{-v}(x)$. If v is an integer, then $J_v(x)$ defines an entire analytic function. For non-integer v, it has a branch point at the origin. We then choose the principal branch of the logarithm when defining x^v.

By differentiating under the sum, one finds that J_v solves Bessel's equation

$$x^2 y'' + xy' + (x^2 - v^2)y = 0. \tag{1}$$

When v is not an integer, the functions J_v and J_{-v} are linearly independent solutions and thus form a basis for the solution space. When v is an integer, a basis is given by J_v and Y_v, where Y_v is the Bessel function of the second kind of order v. For non-integer v it is defined by

$$Y_v(x) = \frac{J_v(x)\cos(v\pi) - J_{-v}(x)}{\sin(v\pi)}$$

and for integer values as the limit

$$Y_n(x) = \lim_{\substack{v \to n \\ v \in \mathbb{R}\setminus\mathbb{Z}}} Y_v(x).$$

Y_n is singular at the origin with $Y_n(x) \sim x^{-|n|}$ for $|n| \geq 1$ and $Y_0(x) \sim \ln|x|$ as $x \to 0$. In what follows we will only investigate the Bessel functions of the first kind.

Theorem 2.2 (Recurrence formulae).
We have that

a) $J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x} J_v(x)$,

b) $J_{v-1}(x) - J_{v+1}(x) = 2J_v'(x)$,

c) $xJ_v'(x) + vJ_v(x) = xJ_{v-1}(x)$,

d) $xJ_v'(x) - vJ_v(x) = -xJ_{v+1}(x)$.

Proof. To derive these formulae, we observe first that

$$\frac{d}{dx}(x^v J_v(x)) = x^v J_{v-1}(x) \tag{2}$$

when $v - 1$ is not a negative integer. Indeed, by the definition of $J_v(x)$ we have that

$$\begin{aligned} x^v J_v(x) &= \sum_{n=0}^{\infty} \frac{x^v (-1)^n}{n!\Gamma(v+n+1)} \left(\frac{x}{2}\right)^{v+2n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2v+2n}}{n! 2^{v+2n}\Gamma(v+n+1)} \end{aligned}$$

Differentiating both sides yields

$$\begin{aligned}\frac{d}{dx}(x^v J_v(x)) &= \sum_{n=0}^{\infty} \frac{2(v+n)(-1)^n x^{2v+2n-1}}{n! 2^{v+2n}\Gamma(v+n+1)} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2v+2n-1}}{n! 2^{v+2n-1}\Gamma(v+n)} \\ &= \sum_{n=0}^{\infty} \frac{x^v (-1)^n x^{(v-1)+2n}}{n! 2^{(v-1)+2n}\Gamma((v-1)+n+1)} \\ &= x^v J_{v-1}(x).\end{aligned}$$

Here we have used the fact that $\Gamma(v+1) = v\Gamma(v)$. Similarly,

$$\frac{d}{dx}(x^{-v} J_v(x)) = -x^{-v} J_{v+1}(x) \tag{3}$$

when v is not a negative integer. If now $v = -n$ is a negative integer, then

$$\frac{d}{dx}(x^v J_v(x)) = (-1)^n \frac{d}{dx}(x^{-n} J_n(x)) = (-1)^{n+1} x^{-n} J_{n+1}(x) = x^v J_{v-1}(x)$$

by (3), so that (2) still holds. In a similar way, one finds that (3) remains true if v is a negative integer. By expanding the left-hand side of (2), we get

$$x^v J_v'(x) + v x^{v-1} J_v(x) = x^v J_{v-1}(x).$$

Dividing both sides by x^{v-1} yields

$$x J_v'(x) + v J_v(x) = x J_{v-1}(x) \tag{4}$$

Hence we obtain (c). Similarly, expanding (3) gives

$$x^{-v} J_v'(x) - v x^{-v-1} J_v(x) = -x^{-v} J_v(x)$$

and multiplying by x^{v+1} gives (d). Adding (c) and (d) gives

$$2x J_v'(x) = x J_{v-1}(x) - x J_{v+1}(x)$$

which reduces to (b) after dividing by x. On the other hand, subtracting (d) from (c) gives

$$2v J_v(x) = x J_{v-1}(x) + x J_{v+1}(x)$$

which reduces to (a) after dividing by x. $\square$

3 Bourget's hypothesis

Bourget's hypothesis states that $J_{v+k}(x)$ and $J_v(x)$ have no common zeros other than the origin if v is a non-negative integer and k a positive integer. Siegel proved the following more general result in 1929 [11].

Theorem 3.1. *$J_{v+k}(x)$ and $J_v(x)$ have no common zeros other than the origin if v is a rational number and k a positive integer with $k \neq -2v$ if v is a negative integer.*

We recall the proof below for completeness.

It is natural to hypothesise that the same result is true for the stationary points of the Bessel functions. This is the main new result of this thesis[1].

Theorem 3.2. *$J'_{v+k}(x)$ and $J'_v(x)$ have no common zeros other than the origin if v is a rational number and k a positive integer with $k \neq -2v$ if v is a negative integer.*

The proofs of Theorems 3.1 and 3.2 rely on the recurrence formulae in Theorem 2.2 and some results on transcendental numbers. We begin by recalling the definition of transcendental and algebraic numbers.

Definition 3.3. A complex number z is said to be *algebraic* if it is a root of a non-zero polynomial with rational coefficients. A complex number which is not algebraic is said to be *transcendental*.

We observe that if $x \neq 0$ is algebraic then so is x^{-1}, since

$$a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0 = 0$$

implies

$$a_m + a_{m-1} x^{-1} + \cdots + a_1 (x^{-1})^{m-1} + a_0 (x^{-1})^m = 0.$$

Lemma 3.4 ([10, 11])**.** *Let v be rational. Except for the origin, the zeros of $J_v(x)$ and $J'_v(x)$ are transcendental numbers.*

The proof of this lemma is quite long and we therefore leave it out. The proof for the zeros of $J_v(x)$ is given in [11, p. 231] and for the zeros of $J'_v(x)$ in [10, p. 217].

Repeated use of formula (a) in Theorem 2.2 shows that J_{v+k} can be expressed in terms of J_v and J_{v-1}. The exact relation is given below. For the proof we refer to [12].

[1] We have not been able to find this result in the literature.

Lemma 3.5.

$$J_{v+k}(x) = J_v(x)R_{k,v}(x) - J_{v-1}(x)R_{k-1,v+1}(x),$$

for $x \neq 0$*, where*

$$R_{k,v}(x) = \sum_{n=0}^{\lfloor k/2 \rfloor} (-1)^n \binom{k-n}{n} (v+n)_{k-2n} \left(\frac{2}{x}\right)^{k-2n}, \tag{5}$$

in which

$$(a)_m = a(a+1)\cdots(a+m-1)$$

denotes the Pochhammer symbol[2]*.*

Remark 3.6. *The functions* $R_{k,v}(x)$ *are called* Lommel polynomials*; they are polynomials in* x^{-1}*. Note that* $(v+n)_{k-2n}$ *is rational when* v *is rational, so that* $R_{k,v}(x)$ *is then a polynomial in* x^{-1} *with rational coefficients. The coefficient* $(v+n)_{k-2n}$ *is often written in the form* $\Gamma(v+k-n)/\Gamma(v+n)$ *when* $v+n$ *is not a non-positive integer.*

The following result is a direct corollary of Lemma 3.5 and Theorem 2.2 (c).

Lemma 3.7.

$$J_{v+k}(x) = J_v(x)\tilde{R}_{k,v}(x) - J'_v(x)R_{k-1,v+1}(x), \tag{6}$$

for $x \neq 0$*, where*

$$\tilde{R}_{k,v}(x) = R_{k,v}(x) - \frac{v}{x}R_{k-1,v+1}(x).$$

We are now ready to prove Theorems 3.1 and 3.2.

Proof of Theorem 3.1. Suppose that $x_0 \neq 0$ is a common zero of J_v and J_{v+k}. From Lemma 3.7 we find that $J'_v(x_0)R_{k-1,v+1}(x_0) = 0$. By Lemma 3.4 we know that $R_{k-1,v+1}(x_0) \neq 0$ since $R_{k-1,v+1}(x)$ is a non-zero polynomial in x^{-1} with rational coefficients, and x_0 is transcendental. This implies that $J'_v(x_0) = 0$, so that $J_v(x_0) = J'_v(x_0) = 0$. This contradicts the fact that J_v is a non-trivial solution of a second-order linear homogeneous ordinary differential equation. □

Proof of Theorem 3.2. We begin by assuming that $v \neq 0, -1, \ldots, -k$. Differentiating both sides of equation (6) gives

$$\begin{aligned} J'_{v+k}(x) = J'_v(x)\tilde{R}_{k,v}(x) + \tilde{R}'_{k,v}(x)J_v(x) \\ - J''_v(x)R_{k-1,v+1}(x) - J'_v(x)R'_{k-1,v+1}(x) \end{aligned}$$

[2] With the convention that $(a)_0 = 1$.

We know that

$$J_v''(x) = \frac{-1}{x} J_v'(x) - \Big(1 - \frac{v^2}{x^2}\Big) J_v(x)$$

from (1). So

$$\begin{aligned} J_{v+k}'(x) &= J_v'(x)\Big(\tilde{R}_{k,v}(x) + \frac{1}{x} R_{k-1,v+1}(x) - R_{k-1,v+1}'(x)\Big) \\ &\quad + J_v(x) \left(\tilde{R}_{k,v}'(x) + \left(1 - \frac{v^2}{x^2}\right) R_{k-1,v+1}(x)\right) \\ &= J_v(x) P_{k,v}(x) + J_v'(x) Q_{k,v}(x), \end{aligned}$$

where

$$P_{k,v}(x) = \tilde{R}_{k,v}'(x) + \left(1 - \frac{v^2}{x^2}\right) R_{k-1,v+1}(x)$$

and

$$Q_{k,v}(x) = \tilde{R}_{k,v}(x) + \frac{1}{x} R_{k-1,v+1}(x) - R_{k-1,v+1}'(x)$$

are polynomials in x^{-1} with rational coefficients. We now show that $P_{k,v}(x)$ is not the zero polynomial. We have that

$$\begin{aligned} P_{k,v}(x) &= \tilde{R}_{k,v}'(x) + \left(1 - \frac{v^2}{x^2}\right) R_{k-1,v+1}(x) \\ &= \frac{d}{dx}\Big(R_{k,v}(x) - \frac{v}{x} R_{k-1,v+1}(x)\Big) + \left(1 - \frac{v^2}{x^2}\right) R_{k-1,v+1}(x) \\ &= R_{k,v}'(x) + \frac{v}{x^2} R_{k-1,v+1}(x) - \frac{v}{x} R_{k-1,v+1}'(x) + \left(1 - \frac{v^2}{x^2}\right) R_{k-1,v+1}(x) \\ &= R_{k,v}'(x) + \left(1 + \frac{v - v^2}{x^2}\right) R_{k-1,v+1}(x) - \frac{v}{x} R_{k-1,v+1}'(x). \end{aligned}$$

Set

$$R_{k,v}(x) = a_{k,v} x^{-k} + b_{k,v} x^{-k+2} + \dots \tag{7}$$

Then

$$R_{k-1,v+1}(x) = a_{k-1,v+1} x^{-k+1} + \dots \tag{8}$$

Differentiating both sides gives respectively

$$R_{k,v}'(x) = -k a_{k,v} x^{-k-1} + \dots \tag{9}$$

and

$$R_{k-1,v+1}'(x) = (-k+1) a_{k-1,v+1} x^{-k} + \dots \tag{10}$$

Thus

$$\begin{aligned}P_{k,v}(x) &= (-ka_{k,v}x^{-k-1} + \ldots) + \left(1 + \frac{v - v^2}{x^2}\right)(a_{k-1,v+1}x^{-k+1} + \ldots) \\ &\qquad - \frac{v}{x}((-k+1)a_{k-1,v+1}x^{-k} + \ldots) \\ &= \left(-ka_{k,v} + (v - v^2)a_{k-1,v+1} - v(-k+1)a_{k-1,v+1}\right)x^{-k-1} + \ldots \\ &= (-ka_{k,v} + (-v^2 + vk)a_{k-1,v+1})x^{-k-1} + \ldots\end{aligned}$$

From equation (5) we see that

$$a_{k,v} = 2^k(v)_k.$$

We therefore obtain that

$$\begin{aligned}-ka_{k,v} + (-v^2 + vk)a_{k-1,v+1} &= -k2^k(v)_k + (-v^2 + vk)2^{k-1}(v+1)_{k-1} \\ &= -k2 \cdot 2^{k-1}(v)_k + (-v + k)2^{k-1}(v)_k \\ &= 2^{k-1}(v)_k(-2k + k - v) \\ &= -2^{k-1}(v)_k(v + k) \\ &= -2^{k-1}(v)_{k+1},\end{aligned}$$

showing that $P_{k,v}(x)$ is not the zero polynomial if $v \neq 0, -1, \ldots, -k$.

We can now repeat the argument in the proof of Theorem 3.1. If $x_0 \neq 0$ is a common zero of $J'_{v+k}(x)$ and $J'_v(x)$, then $J_v(x_0)P_{k,v}(x_0) = 0$. Since x_0 is transcendental, it follows that $J_v(x_0) = J'_v(x_0) = 0$ and we have a contradiction.

Finally, we handle the remaining cases $v = 0, -1, \ldots, -k$. The cases $v = -1, \ldots, -k$ can be avoided by replacing $J_v(x)$ by $(-1)^v J_{|v|}(x)$ and exchanging the roles of $J_v(x)$ and $J_{v+k}(x)$ if $2|v| > k$. This argument breaks down trivially if k is even and $v = -k/2$, in which case $J_v(x) = (-1)^v J_{|v|}(x) = (-1)^v J_{v+k}(x)$. This leaves the case $v = 0$, for which the coefficient in front of the term x^{-1-k} in $P_{k,0}(x)$ vanishes. An explicit calculation shows that

$$\begin{aligned}P_{k,0}(x) &= (2^{k-1}(1)_{k-1} + 2^{k-2}(k-2)(k-1)(1)_{k-2})x^{1-k} + \ldots \\ &= 2^{k-2}k!x^{1-k} + \ldots,\end{aligned}$$

when $k \geq 2$, and

$$P_{1,0}(x) = 1.$$

In either case, it's clear that $P_{k,0}(x)$ doesn't vanish identically. Hence we can use the same argument as above to prove that $J'_k(x)$ and $J'_0(x)$ don't have any common zeros except the origin. $\square$

4 Eigenvalues of the Laplacian in the disc

As an application of the preceding theory, we now consider the multiplicities of eigenvalues of the Laplace operator in the unit disc, with Dirichlet or Neumann boundary conditions.

4.1 Dirichlet condition

We want to solve

$$\begin{cases} -\Delta u = \lambda u, & x^2 + y^2 < 1, \\ u = 0, & x^2 + y^2 = 1. \end{cases} \tag{11}$$

In polar coordinates, this can be written

$$\begin{cases} -\left(\partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\theta^2 u\right) = \lambda u, & 0 < r < 1, \quad 0 \leqslant \theta \leqslant 2\pi, \\ u = 0, & r = 1, \quad 0 \leqslant \theta \leqslant 2\pi. \end{cases} \tag{12}$$

The function u also has to satisfy the periodicity conditions $u(r, 2\pi) = u(r, 0)$ and $\partial_\theta u(r, 2\pi) = \partial_\theta u(r, 0)$. The eigenfunctions can now be found by separation of variables. Set

$$u(r, \theta) = R(r)\Theta(\theta).$$

We get

$$\begin{aligned} & -\left(\frac{\Theta}{r}(rR'' + R') + \frac{R}{r^2}\Theta''\right) = \lambda R\theta \\ \Leftrightarrow \quad & \frac{\Theta}{r}(rR'' + R') + \frac{R}{r^2}\Theta'' + \lambda R\Theta = 0 \\ \Leftrightarrow \quad & \frac{r}{R}(rR'' + R') + \lambda r^2 = -\frac{\Theta''}{\Theta}, \end{aligned}$$

assuming that $R, \Theta \neq 0$. Since the left hand side is independent of θ and the right hand side independent of r, they must both be constant. Call this constant m^2. We then obtain

$$\frac{\Theta''}{\Theta} = -m^2,$$

and hence

$$\Theta(\theta) = A\cos(m\theta) + B\sin(m\theta),$$

where m has to be an integer due to the periodicity conditions. Substituting this in the above equation, we find that

$$\begin{aligned} & \frac{1}{rR}(rR'' + R') + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \lambda = 0 \\ \Leftrightarrow \quad & R'' + \frac{1}{r}R' + \left(\lambda - \left(\frac{m}{r}\right)^2\right) R = 0. \end{aligned}$$

Setting $\rho = \sqrt{\lambda} r$ leads to Bessel's equation, and hence

$$R(r) = CJ_m(\sqrt{\lambda} r) + DY_m(\sqrt{\lambda} r).$$

Since we require that R is regular at 0, D must be zero. The boundary condition $R(1) = 0$ now gives $\sqrt{\lambda} = \alpha_{mk}$, where α_{mk} is the kth positive root of $J_m(r)$. Thus

$$R(r) = J_m(\alpha_{mk} r)$$

and so

$$u_{mk}(r, \theta) = J_m(\alpha_{mk} r)(A\cos(m\theta) + B\sin(m\theta)).$$

Moreover, $\lambda = \alpha_{mk}^2$.

It is a classical result that the functions $J_m(\alpha_{mk} r)\cos(m\theta)$, $J_m(\alpha_{mk} r)\sin(m\theta)$, $m = 0, 1, \ldots$ and $k = 1, 2, \ldots$ (with suitable normalisation) form a complete orthonormal basis for $L^2(D)$, where D denotes the unit disc. In other words, they form an orthonormal set and every function $u \in L^2(D)$ can be expressed as a countably infinite linear combination of the basis functions, with convergence in $L^2(D)$. This follows e.g. by using the completeness of the trigonometric functions in $L^2(0, 2\pi)$ combined with the completeness of the Bessel functions $J_m(\alpha_{mk} r)$, $k = 1, 2, \ldots$, for fixed m in the weighted L^2-space on the interval $(0, 1)$ with norm

$$\left(\int_0^1 r|f(r)|^2\, dr\right)^{1/2};$$

see [12, Chapter 18]. Note that $L^2(D)$ can be seen as the tensor product of these two spaces in polar coordinates. It follows as a consequence that all the eigenfunctions can be obtained by separation of variables. In other words, all the eigenvalues of the negative Laplacian in the unit disc with Dirichlet boundary conditions have the form $\lambda = \alpha_{mk}^2$.

The theory in the previous section now allows us to prove the following result.

Theorem 4.1. *Let $\lambda = \alpha_{mk}^2$ be an eigenvalue of equation* (11).

1. *If $m = 0$ then λ has multiplicity 1.*

2. *If $m \neq 0$ then λ has multiplicity 2.*

Proof. By Theorem 3.1 we know that $\alpha_{mk} \neq \alpha_{m'k'}$ if $(m,k) \neq (m',k')$. The first result therefore follows since the corresponding eigenfunctions are simply multiples of $J_0(\alpha_{0k}r)$. In the second case the multiplicity is two since the eigenfunctions have the form $J_m(\alpha_{mk}r)(A\cos(m\theta) + B\sin(m\theta))$, with A and B arbitrary. $\square$

4.2 Neumann condition

Theorem 3.2 allows us to also treat the case of Neumann boundary conditions. Since the proofs are almost exactly the same we simply give a quick summary. The problem is

$$\begin{cases} -\Delta u = \lambda u, & x^2 + y^2 < 1, \\ \partial_n u = 0, & x^2 + y^2 = 1, \end{cases} \tag{13}$$

where ∂_n denotes the normal derivative at the boundary. In polar coordinates, this reads

$$\begin{cases} -\left(\partial_r^2 u + \dfrac{1}{r}\partial_r u + \dfrac{1}{r^2}\partial_\theta^2 u\right) = \lambda u, & 0 < r < 1, \quad 0 \leqslant \theta \leqslant 2\pi, \\ \partial_r u = 0, & r = 1, \quad 0 \leqslant \theta \leqslant 2\pi. \end{cases} \tag{14}$$

By separation of variables, one obtains the eigenvalues $\lambda = \beta_{mk}^2$ and eigenfunctions

$$u_{mk}(r,\theta) = J_m(\beta_{mk}r)(A\cos(m\theta) + B\sin(m\theta)).$$

β_{mk} denotes the kth positive root of $J_m'(r)$. In addition, 0 is an eigenvalue with eigenfunction 1. For notational simplicity, we set $\beta_{00} = 0$. Due to Theorem 3.2 we have the following result.

Theorem 4.2. *Let $\lambda = \beta_{mk}^2$ be an eigenvalue of equation* (13)*.*

1. *If $m = 0$ then λ has multiplicity 1.*
2. *If $m \neq 0$ then λ has multiplicity 2.*

5 Linear standing water waves in a cylinder

In this section we make use of the results developed in the preceding section to study linear standing water waves in a cylinder. Consider an inviscid, incompressible and irrotational fluid in a circular cylinder of radius R under the influence of gravity. We assume that the surface of the fluid is given by the graph of a function, $z = \eta(t,x,y)$, that the fluid has constant density and that the depth of the fluid at rest is $H > 0$. The velocity field then satisfies Laplace's equation with Neumann boundary conditions at the rigid boundaries:

$$\begin{cases} \Delta\phi = 0, & -H < z < \eta, \quad x^2 + y^2 < 1, \\ \partial_z\phi = 0, & z = -H, \quad x^2 + y^2 < 1, \\ \partial_n\phi = 0, & -H < z < \eta \quad x^2 + y^2 = 1. \end{cases} \tag{15}$$

The velocity potential and the surface elevation are coupled by the boundary conditions

$$\begin{cases} \eta_t = \phi_z - \eta_x\phi_x - \eta_y\phi_y, \\ \phi_t = -\dfrac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) - g\eta \end{cases} \tag{16}$$

at the free surface $z = \eta$. Here, g is the gravitational constant of acceleration. We non-dimensionalise the equations by setting

$$\begin{cases} x' = \dfrac{x}{R}, \quad y' = \dfrac{x}{R}, \quad z' = \dfrac{z}{R}, \quad t' = \dfrac{t}{\sqrt{\frac{R}{g}}} \\ \eta' = \dfrac{\eta}{R}, \quad \phi' = \dfrac{\phi}{\sqrt{g}R^{\frac{3}{2}}}. \end{cases} \tag{17}$$

The new equations are

$$\begin{cases} \Delta\phi = 0, & -h < z < \eta, \quad x^2 + y^2 < 1 \\ \partial_z\phi = 0, & z = -h, \quad x^2 + y^2 < 1, \\ \partial_n\phi = 0, & -h < z < \eta, \quad x^2 + y^2 = 1 \end{cases} \tag{18}$$

and

$$\begin{cases} \eta_t = \phi_z - \eta_x\phi_x - \eta_y\phi_y \\ \phi_t = -\dfrac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) - \eta \end{cases} \tag{19}$$

at $z = \eta$, where $h = \frac{H}{R}$ and we have removed the primes for notational convenience.

Looking for small-amplitude solutions, we linearise the equations by dropping all the terms which are at least quadratic in ϕ and η. This gives

$$\begin{cases} \Delta\phi = 0, & -h < z < 0, \quad x^2 + y^2 < 1 \\ \partial_z\phi = 0, & z = -h, \quad x^2 + y^2 < 1, \\ \partial_n\phi = 0, & -h < z < 0, \quad x^2 + y^2 = 1 \end{cases} \tag{20}$$

and

$$\begin{cases} \eta_t = \phi_z, \\ \phi_t = -\eta, \end{cases} \tag{21}$$

at $z = 0$. We separate variables by setting

$$\phi = f(x,y)g(z)u(t).$$

This implies

$$(\Delta f)gu + fg''u = 0, \quad -h < z < 0, \quad x^2 + y^2 < 1,$$

which reduces to

$$\frac{\Delta f}{f} = \frac{-g''}{g} = -\lambda.$$

Thus

$$\begin{cases} -\Delta f = \lambda f, & x^2 + y^2 < 1, \\ \partial_n f = 0, & x^2 + y^2 = 1 \end{cases} \tag{22}$$

and

$$\begin{cases} g'' = \lambda g, \\ g'(-h) = 0. \end{cases} \tag{23}$$

Equation (22) represents the Neumann eigenvalue problem for $-\Delta$ in the unit disc. According to previous results each eigenvalue has multiplicity at most 2. Equation (23) gives

$$g(z) = A\cosh(\sqrt{\lambda}(z+h)),$$

so

$$\phi(t,x,y,z) = f_n(x,y)\cosh(\sqrt{\lambda_n}(z+h))u(t),$$

where f_n is an eigenfunction corresponding to λ_n. Substitution in equation (21) gives

$$\eta_t(t,x,y) = f_n(x,y)\sqrt{\lambda_n}\sinh(\sqrt{\lambda_n}h)u(t) \tag{24}$$

and

$$f_n(x,y)\cosh(\sqrt{\lambda_n}h)u'(t) = -\eta(t,x,y). \tag{25}$$

Substitution of equation (25) in equation (24) gives

$$-f_n(x,y)\cosh(\sqrt{\lambda_n}(z))u''(t) = \sqrt{\lambda_n}f_n(x,y)\sinh(\sqrt{\lambda_n}h)u(t),$$

which implies

$$-u''(t) = \frac{\sqrt{\lambda_n}\sinh(\sqrt{\lambda_n}h)}{\cosh(\sqrt{\lambda_n}h)}u(t).$$

This equation has the solution

$$u(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$$

where

$$\omega_n^2 = \sqrt{\lambda_n}\tanh(\sqrt{\lambda_n}h). \tag{26}$$

Hence

$$\phi(t,x,y,z) = f_n(x,y)\cosh(\sqrt{\lambda_n}(z+h))(A\cos(\omega_n t) + B\sin(\omega_n t))$$

and

$$\eta(t,x,y,z) = \omega_n f_n(x,y)\cosh(\sqrt{\lambda_n}h)(A\sin(\omega_n t) - B\cos(\omega_n t)).$$

This represents a standing wave; it is time-periodic with minimal period

$$T_n = \frac{2\pi}{\omega_n}.$$

Equation (26), which gives the permissible values of the angular velocity ω, is called the *dispersion relation.*

When studying the corresponding nonlinear problem, it is of great importance to know the dimension of the space of time-periodic solutions for a fixed period $T = \frac{2\pi}{\omega}$. If we just consider solutions of minimal period $T = T_n$, then the answer can be obtained using the results in the previous section. If the corresponding eigenvalue λ_n is simple,

$$\phi = f_n(x,y)\cosh(\sqrt{\lambda_n}(z+h))A\cos(\omega_n t) + B\sin(\omega_n t)$$

and

$$\eta_n(t,x,y,z) = f_n(x,y)\cosh(\sqrt{\lambda_n}h)(A\sin(\omega_n t) - B\cos(\omega_n t))$$

where for each solution,

$$f_n = CJ_0(\beta_{0k}r).$$

for some C, and

$$\lambda_n = \beta_{0k}^2.$$

We thus get a two-dimensional space of solutions. On the other hand, if λ_n is double, we wll get a four-dimensional space of solutions since

$$f_n = J_j(\beta_{jk}r)(C\cos(j\theta) + D\sin(j\theta)),$$

for some $j \geq 1$ Note however that a solution of minimal period T_m is also periodic of period kT_m for any integer $k \geq 2$. This means that if

$$\omega_m = k\omega_n, \tag{27}$$

$k \geq 2$ integer, then (η_m, ϕ_m) and (η_n, ϕ_n) are both periodic with period $T_n = 2\pi/\omega_n$. This kind of resonance complicates the nonlinear analysis [5, 6]. In the radial case, Mack [7] showed that it is possible to choose the height h so that no resonance occurs. We now generalise his result to non-radial solutions. Using the dispersion relation, condition (27) can be rewritten in the form

$$\frac{\sqrt{\lambda_m}\tanh(\sqrt{\lambda_m}h)}{\sqrt{\lambda_n}\tanh(\sqrt{\lambda_n}h)} = k^2. \tag{28}$$

Lemma 5.1. *Let $m > n \geq 1$ be integers. The function*

$$h \mapsto \frac{\sqrt{\lambda_m}\tanh(\sqrt{\lambda_m}h)}{\sqrt{\lambda_n}\tanh(\sqrt{\lambda_n}h)}$$

is strictly decreasing from λ_m/λ_n to $\sqrt{\lambda_m}/\sqrt{\lambda_n}$ as h goes from 0 to ∞.

Proof. It suffices to show that the function

$$f(x) = \frac{\tanh(ax)}{\tanh(bx)}$$

is strictly decreasing for $x > 0$ if $a > b$ (the limits at 0 and ∞ are obtained from standard properties of hyperbolic functions). A straightforward computation reveals that

$$f'(x) = \frac{abx}{\cosh^2(ax)\sinh^2(bx)}\left(\frac{\sinh(2bx)}{2bx} - \frac{\sinh(2ax)}{2ax}\right).$$

The result now follows from the expansion

$$\frac{\sinh y}{y} = \sum_{k=0}^{\infty} \frac{y^{2k}}{(2k+1)!},$$

which shows that $\sinh y/y$ is strictly increasing for $y \geq 0$. □

Let $n \geq 1$ be a given integer. The lemma implies that for given $m > n$ and $k \geq 2$, equation (28) has at most one solution $h = h_{n,m,k}$. Letting m and k vary, we obtain a countable number of 'resonant' heights. If $h \neq h_{n,m,k}$, there is no resonance and any $T_n = 2\pi/\omega_n$-periodic solution has minimal period T_n. We summarise our findings in a theorem.

Theorem 5.2. *Let $n \geq 1$. There exists a countable set of heights $h_{n,m,k}$, such that for $h \neq h_{n,m,k}$, the space of T_n-periodic solutions is either two- or four-dimensional, depending on whether the eigenfunctions of problem (22) corresponding to λ_n are radial or not.*

The above theorem is a first step towards a rigorous existence proof for standing waves for the nonlinear problem. In such an analysis, one could impose evenness in t and θ to reduce the dimension of the solution space to one.

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