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Quantum entanglement in the double slit experiment with qubit which-path detectors

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Abstract

After reviewing basic concepts and tools in order to quantify entanglement, a theoretical set up is proposed, which is based on the double slit experiment and the Mach-Zehnder interferometer. A particle, which is in a superposition of two paths, can interfere with a qubit on each path before it is scattered by a beam splitter and measured by two separate detectors. After a detailed investigation, it can be shown that it is possible to induce entanglement to the qubits via post-selection. This means that the measurement does not induce entanglement, but can be used to select the entangled qubits after a number of iterations. A remarkable result is, that this works independently from the probability of interaction between the particle and the qubits, which could turn out be very beneficial in a possible realization of the set up.

Contents

1 Introduction

Quantum Entanglement describes the condition that the state of various particles are not only correlated, but can also not be described independently from each other. In the context of quantum mechanics this means that a measurement on a particle A does not only determine the result of A, but also causes the *immediate* collapse of the wave function of the entangled particle B, even if that one is light-years apart. This goes far beyond any type of classical correlation. Quantum entanglement is indeed at the very heart of Quantum Information Theory and has many striking applications, of which two very special ones shall be mentioned here, namely superdense coding and quantum teleportation.

A classical bit is a representation of anything that can be in one of two distinct states and can therefore contain one piece of information. A quantum bit however, which is in a superposition of these two states, can even contain two pieces of information by using entanglement. This works the following way. The sender creates a pair of fully entangled qubits and sends one of these qubits directly to the receiver. The sender cannot perform any measurement on the qubit, because that would cause the wave function to collapse, but he can perform an unitary transformation. It turns out that there are exactly four entangled states, the so-called Bell states, for a two qubit system and therefore four possible ways to encode the qubit. The sender sends that encoded qubit to the receiver, who can measure the two qubits jointly and deduce which unitary transformation was performed. Since there are four possibilities, there are also four pieces of information, even though only two qubits were sent. This technique is called superdense coding. [2]

The name quantum teleportation is somewhat misleading, since it has little to do with the teleportation from science fiction. Quantum teleportation describes the process of 'robbing' the state of a particle X and sending it via a quantum channel to another particle Y, so it seems like the particle itself was 'teleported'. Again the key to this process is quantum entanglement. Sender and receiver have each one fully entangled qubit. Now the sender performs a Bell measurement on his shared qubits and on the original particle X. There are four possible outcomes of this measurement and he lets the receiver know which outcome it was by sending two classical bits. After the Bell measurement, X has lost its independent state, but the receiver's qubit Y is now correlated to X. With the obtained information about the Bell measurement, he can adjust his particle Y, so that it is identical to the original state of X. Interestingly, quantum teleportation is the inverse of superdense coding. While in the latter two classical bits were sent by using one qubit, in quantum teleportation one needs two classical bits to send one qubit. [1]

Apparently entanglement can be used for processes that have absolutely no analogue in classical physics. Since entanglement plays such a crucial part in modern quantum physics, it is the aim of this thesis to investigate how entanglement could be created. To do so, the first part of the thesis (Theory & concepts) reviews some important tools that are necessary to quantify entanglement. Then a thought experiment, which is based on the double slit experiment and the Mach-Zehnder Interferometer, is introduced. By applying the acquired concepts such an experiment will be analyzed with the ultimate goal of finding a way to entangle two qubits.

2 Theory & concepts

2.1 Quantum Bits

The consideration of merely two possible states of a system is interesting not only in the context of possible applications in quantum computing, but is also an easy-to-handle system that yet bears all the important features to study quantum entanglement. An ordinary computer operates in two distinct eigenstates, namely $|1\rangle$, which represents a certain voltage, and $|0\rangle$, which represents another distinguishable voltage.

Single Quantum Bits In quantum bits (or just "qubits"), the eigenstates $|0\rangle$ and $|1\rangle$ might be represented by any two distinct eigenstates such as the charge of an electron, the polarization of a photon or the spin of a nucleon. The novelty is, that qubits are not necessarily either in eigenstate $|0\rangle$ or $|1\rangle$, but rather in a superposition of both (Eq. 2.1).

$$|\psi\rangle = a|0\rangle + b|1\rangle \tag{2.1}$$

Only the measurement causes the superposition irretrievably to collapse and the system to be in a definite eigenstate: In $|0\rangle$ with probability $|a|^2$ and in $|1\rangle$ with probability $|b|^2$, where $|a|^2 + |b|^2 = 1$ due to normalization.

Multiple Quantum Bits *"Stepping up from one qubit to two is a bigger leap than you might expect. Much that is weird and wonderful about quantum mechanics can be appreciated by considering the properties of the quantum states of two qubits"* (John Preskill) [11]
Since isolated particles almost don't exist in reality, it is much more interesting to consider the connection of a system to its environment. To keep things simple, the environment is assumed to be a second qubit. The general combined state is then the tensor product of the two qubits with all possible combinations (Eq. 2.2).

$$|\psi\rangle = \alpha|0\rangle_A \otimes |0\rangle_B + \beta|0\rangle_A \otimes |1\rangle_B + \gamma|1\rangle_A \otimes |0\rangle_B + \delta|1\rangle_A \otimes |1\rangle_B \tag{2.2}$$

Instead of talking about systems A and B it is customary to refer to two fictional experimentalists called Alice and Bob.

2.2 Density matrix representation

The formulation of density matrix representation goes back to John von Neumann, who introduced the concept in 1927 to take care of quantum systems in mixed states and to develop a statistical treatment of quantum mechanics. [14] If one knows precisely in what state a system is, that system is said to be in a pure state. Therefore any state that can be written as a state ket is in fact such a pure state. One might come across systems though, about which one doesn't have full knowledge and can therefore not assign a state ket to that system. Such systems are then said to be in a mixed state. This is for example the case if it is unknown in which state the system was prepared and one can only measure the probabilities of the respective eigenstates. Or, and this is the case for the set up in this thesis, the system is entangled to another system, to

which one has no access to.
At first sight one might see no difference between the case where one knows the actual state or only the probabilities of the respective eigenstates, since both could correctly predict the outcome of a measurement. There is seemingly no loss of information for the experimentalists *after* performing the measurement, but it is very important to think about the system *before* the measurement. The fact that the amplitudes can interfere and the probabilities are not simply summed up, is absolutely crucial to quantum mechanics and is the reason that distinguishes it from classical physics. The double slit experiment, which will be explained in detail later, is designed to illustrate this difference.

Definitions Every system consists of an ensemble of pure states $\{p_i, |\psi_i\rangle\}$. [9] The density operator is then defined as

$$\rho \equiv \sum_i p_i |\psi_i\rangle\langle\psi_i|. \tag{2.3}$$

The trace of the density operator must be 1 due to normalization (Eq. 2.4). It might be interesting to know that the trace is independent of the choice of base vectors.

$$\mathrm{tr}(\rho) = 1 \tag{2.4}$$

The expectation value of an observable M in density matrix representation is given by

$$\begin{aligned} \langle M\rangle &= \sum_i p_i \langle\psi_i|M|\psi_i\rangle \\ &= \sum_{i,j} p_i \langle\psi_i|\psi_j\rangle\langle\psi_j|M|\psi_i\rangle \\ &= \sum_i \langle\psi_i| \sum_j p_j |\psi_j\rangle\langle\psi_j|M|\psi_i\rangle \\ &= \mathrm{tr}(\rho M). \end{aligned} \tag{2.5}$$

Pure states If the system is in a pure state the ensemble consists of course only of that state itself with a probability of one, so that Eq. 2.3 simplifies to

$$\rho = |\psi\rangle\langle\psi|. \tag{2.6}$$

The density matrix concept is going to be illustrated with a single qubit state (Eq. 2.7) as described in the previous section.

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \tag{2.7}$$

Its relating density matrix according to Eq. 2.6 is simply

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{2.8}$$

Eq. 2.8 might look like a mixed density matrix, but that depends on the choice of base kets. If one chooses to represent the matrix in terms of the eigenstates of ρ, it looks as expected with only one occupied pure state.

$$\rho' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.9}$$

Mixed states Apparently it is not always obvious whether a density matrix refers to a pure or mixed state. This can be easily checked though by calculating the purity γ, which is the trace of the squared matrix. [9]

$$\gamma = \mathrm{tr}\left(\rho^2\right) \leqslant 1 \tag{2.10}$$

γ is always less or equal to one and only equals one in the case of a pure state. Mixed states are a mixture of various pure states with a respectively probability. As an example the case is considered, where an experimentalists measures the state from Eq. 2.7 without knowing how it was prepared. Since he could only measure the probabilities, the related density matrix would equal

$$\rho' = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.11}$$

The purity γ of Eq. 2.11 is indeed smaller than 1 and apparently it differs from the pure state matrix (Eq. 2.8) by the interference term of the amplitudes. That is the loss of knowledge in mixed states.

Reduced density matrix As mentioned before it is much more interesting to consider composite systems with at least two subsystems. The density matrix of that composite system is the tensor product of the subsystems, if those subsystems are independent.

$$\rho_A \otimes \rho_B = \rho_{AB} \tag{2.12}$$

It is of great interest to determine the density matrix ρ_A in cases where the composite matrix ρ_{AB} is known, but ρ_B is inaccessible. When determining the expectation value of an observable M of the system A, it should make no difference to measure M directly on system A or to measure $M \otimes I_B$ on the composite system AB. [9]

$$\langle M_A \rangle = \mathrm{tr}_A\left(\rho_A M_A\right) \overset{!}{=} \mathrm{tr}_{AB}\left(\rho_{AB}\left(M_A \otimes I_B\right)\right) \tag{2.13}$$

This equation is fulfilled if the density matrix ρ_A equals the partial trace over B of the composite matrix ρ_{AB}.

$$\rho_A = \mathrm{tr}_B\left(\rho_{AB}\right) \tag{2.14}$$

To take the partial trace over B means that while tracing out over ρ_B, ρ_A remains untouched.

$$\mathrm{tr}_B\left(\rho_A \otimes \rho_B\right) = \rho_A \mathrm{tr}\left(\rho_B\right) = \rho_A \tag{2.15}$$

The pure state in Eq. 2.16 serves as an example (also a simplified notation for tensor products is used from here on) to illustrate a very strange property of the subsystem.

$$
\begin{aligned}
|\psi\rangle &= \frac{|1\rangle_A \otimes |0\rangle_B + |0\rangle_A \otimes |1\rangle_B}{\sqrt{2}} = \frac{|10\rangle + |01\rangle}{\sqrt{2}} \\
\Rightarrow \rho_{AB} &= \frac{|10\rangle\langle 10| + |10\rangle\langle 01| + |01\rangle\langle 10| + |01\rangle\langle 01|}{2} \\
\Rightarrow \rho_A &= \frac{|1\rangle\langle 1| + |0\rangle\langle 0|}{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned} \tag{2.16}
$$

The composite system ρ_{AB} is in a pure state, while the subsystem ρ_A is in a complete mixed state (as is ρ_B for symmetry reasons). The reason for this is quantum entanglement, which is going to be discussed in the next section.

2.3 Quantum entanglement

"I would not call [entanglement] one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought."(Erwin Schrödinger) [12]

Quantum entanglement is best explained by considering a simple example. A fictional source produces two electrons with opposite spin and sends them in opposite directions to Alice and Bob (Eq. 2.16). Both electrons are in a superposition of spin-up and spin-down with equal probabilities and this superposition collapses only if one experimentalist, say Alice, measures the spin of her particle. In that moment the spin of Bob's electron will be immediately determined to be opposite, although Bob does not know that unless Alice tells him. The striking property of this event is, that as soon one of the superposition collapses the other one collapses *immediately* too. This is much stronger than a classical correlation and Albert Einstein famously coined that behaviour as *"spooky action at a distance"*. [3]

Product states It proves to be quite simple to determine whether two subsystems are entangled, as long as the composite system is in a pure state. A composite system is not entangled if it is possible to write it as a tensor product of its subsystems. [13]

$$
\begin{aligned}
|\psi\rangle &= |\psi\rangle_A \otimes |\psi\rangle_B \\
&= (\alpha|0\rangle_A + \beta|1\rangle_A) \otimes (\gamma|0\rangle_B + \delta|1\rangle_B)
\end{aligned} \tag{2.17}
$$

So a subsystem is entangled if it is not possible to assign an independent state ket to it. This shall be proven by the following considerations:

$$
\begin{aligned}
\langle M_A\rangle &= \langle\psi|M_A \otimes I_B|\psi\rangle \\
&= \langle\psi|_A M_A|\psi\rangle_A \langle\psi|_B I_B|\psi\rangle_B \\
&= |\alpha|^2\langle 1|M_A|1\rangle + \alpha^*\beta\langle 1|M_A|0\rangle + \beta^*\alpha\langle 0|M_A|1\rangle + |\beta|^2\langle 0|M_A|0\rangle \\
&= \mathrm{tr}_A\left(\rho_A M_A\right)
\end{aligned} \tag{2.18}
$$

It follows, that

$$\rho_A = \begin{pmatrix} |\alpha|^2 & \beta^*\alpha \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} \tag{2.19}$$

since

$$\mathrm{tr}_A\left(M_A|i\rangle\langle\, j|\right) = \sum_{k=0}^{1} \langle k|M_A|i\rangle\langle j|k\rangle = \langle i|M_A|j\rangle. \tag{2.20}$$

ρ_A in Eq. 2.19 is idempotent (ρ_A=ρ_A^2) and therefore in a pure state, since its purity equals one. The choosing of ρ_A is of course arbitrary, so the same goes for ρ_B. This shows that every separable state has pure subsystems, which means that they are not entangled since one has full knowledge about them.

$$|\varphi\rangle = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} = \frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}} \otimes \frac{|0\rangle_B + |1\rangle_B}{\sqrt{2}} \tag{2.21}$$

Eq. 2.22 is an example for such a product state. It is easy to see that any measurement of Alice will not influence Bobs, in opposite to the previous example (Eq. 2.17).

2.4 Entanglement measures

2.4.1 Entropy of entanglement

To use the von Neumann entropy as a measure of entanglement is the standard method for pure states. [8]

Entropy in Information Theory The concept of entropy, which comes originally from thermodynamics, is also used in the context of Information Theory. There it is used to quantify the amount of information contained in a message and relates roughly to the uncertainty about that information. [4]

$$S = -\sum_{i=1}^{N} p_i \log_2(p_i) \tag{2.22}$$

Entropy as entanglement measure John von Neumann introduced an entropy, which was named after him, that can be applied to density matrices. [9]

$$S = -\mathrm{tr}\left(\rho \log_2(\rho)\right) = -\sum_{i=1}^{N} \rho_i \log_2(\rho_i) \tag{2.23}$$

The last equality holds, if ρ_i are the eigenvalues of ρ. It is always possible to obtain this result, since the choice of base kets is arbitrary when tracing out. The amount of entanglement E of a system is then defined as the amount of von Neumann entropy of the subsystems.

$$E(\rho_{AB}) = S(\rho_A) = S(\rho_B) \tag{2.24}$$

A system, which subsystems have zero entropy and therefore fully given knowledge, are not entangled at all. If it has totally mixed subsystems it is fully entangled and the entropy is maximized. In the case of qubits the maximal value is actually 1 due to the base of the logarithm. [10]

Pure subsystems If one consider the product state

$$|\psi\rangle = \frac{|11\rangle + |10\rangle + |01\rangle + |00\rangle}{2}, \tag{2.25}$$

as discussed in the previous section, the relating density matrix in the base of its eigenstates equals

$$\rho_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.26}$$

Apparently pure states don't have any entropy (Eq. 2.28), since one has full knowledge about them.

$$S_A = -\log_2(1) = 0 \quad \Rightarrow E(\rho_{AB}) = 0 \tag{2.27}$$

The entanglement of Eq. 2.26 and actually any product state is therefore zero, which is of course exactly the value one would expect.

Mixed subsystems The entangled state

$$|\psi\rangle = \frac{|10\rangle + |01\rangle}{\sqrt{2}}, \tag{2.28}$$

on the other hand has a mixed reduced density matrix (Eq. 2.30). It is actually as mixed as it can get, because equal probabilities mean that one has absolutely no knowledge about the system whatsoever.

$$\rho_A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.29}$$

The entropy of this matrix is hence 1 and the state Eq. 2.29 is maximally entangled.

$$S_A = -\log_2\left(\frac{1}{2}\right) = 1 \quad \Rightarrow E(\rho_{AB}) = 1 \tag{2.30}$$

Bell states It turns out, that there are actually only four bipartite qubit states that have maximal entanglement (Eq. 2.32 & 2.33). [13] These states are of great importance and are called Bell states after John Stewart Bell, one of the pioneers in the investigation of quantum entanglement.

$$|\psi^{\pm}\rangle = \frac{|10\rangle \pm |01\rangle}{\sqrt{2}} \tag{2.31}$$

$$|\varphi^{\pm}\rangle = \frac{|11\rangle \pm |00\rangle}{\sqrt{2}} \tag{2.32}$$

2.4.2 Entanglement of formation

The concept of entropy of entanglement works unfortunately only for pure states, since the subsystems of mixed states can contain entropy independently of their entanglement. There doesn't exist a standard measure of entanglement for mixed states, but rather several methods depending on the system of interest. The method used in this thesis is the entanglement of formation (Eq. 2.33), which is defined such that only the entropy due to entanglement is taken into account. [7, 10]

$$E(\rho) = \min \sum_i p_i E(\psi_i) \tag{2.33}$$

Precisely the entanglement of formation is defined as the average over the entanglement of the pure ensemble states, minimized over all possible ensembles of ρ. In case of a pure state this definition is equivalent to the entropy of entanglement. In the special case of two qubit systems it can be shown after some further calculations that the entanglement is a function of another easy to calculate function called concurrence, which is itself a function of the density matrix. [7, 15]

$$E(\rho) = f\left(C(\rho)\right) \tag{2.34}$$

Interestingly, the concurrence shows a qualitative behaviour similar to that of the entropy of entanglement. The concurrence is 1 for a fully entangled state and 0 for totally independent subsystems. Hence it is practical to use the concurrence, which is easier to calculate than the final entanglement of formation, as a measure of entanglement instead.

Concurrence The concurrence itself is defined by the eigenvalues λ_i of the matrix $\mathbf{M}$, which is a function of ρ. [16]

$$\mathbf{M}(\rho) = \rho\tilde{\rho}, \quad \text{where} \quad \tilde{\rho} = \left(\sigma_y \otimes \sigma_y\right) \rho^* \left(\sigma_y \otimes \sigma_y\right) \tag{2.35}$$

After taking the square root of each of the eigenvalues, the concurrence can be calculated by taking the highest of these values and subtract all the others. In case that has a negative result, the concurrence is zero.

$$C\left(\rho\right) = \max\left\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right\} \tag{2.36}$$

As mentioned before, the concurrence is a value between 0 and 1, where 0 stands for not entangled at all and 1 for maximally entangled.

3 Double slit experiment

The double slit experiment is probably the most famous example to show the nature of quantum mechanics. It is explained here because the actual set up of the thesis is based on this experiment and also to show with a simple example what a crucial part the obtainable information about a system plays.
As shown in figure 3.1, a source emits a particle which might pass either slit A or B and gets then detected at the screen. Three different scenarios are discussed, namely that only one of the slits is open, both slits are open and that both slits are open but additionally a detector is placed in slit A. To simplify things, only 5 detectors at the screen are taken into account.

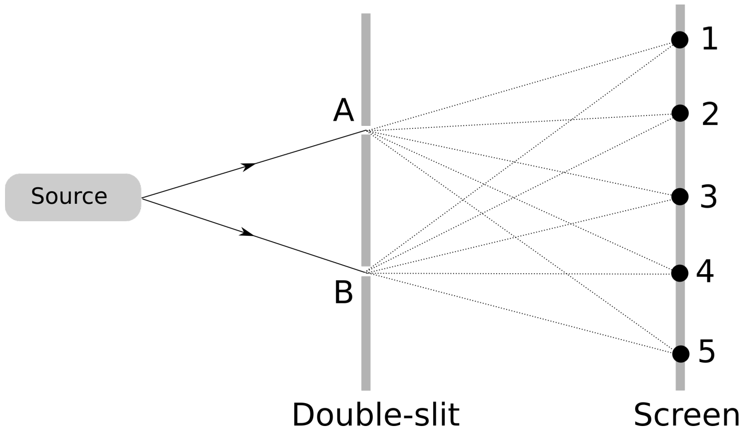


Figure 3.1: The source emits a particle which goes to either slit A or B and gets detected at the screen by one of the detectors

3.1 Only one of the slits is open

Slit A is open If only slit A is open the following evolution holds, where $|\phi\rangle_A$ is the state at the double slit and $|\psi\rangle$ is the state at the screen.

$$|Source\rangle \to |\phi\rangle_A \to |\psi\rangle = \sum_{n=1}^{5} a_n |n\rangle \tag{3.1}$$

The states $|n\rangle$ stand for the different detectors and are orthogonal to each other ($\langle n|m\rangle = \delta_{nm}$). No normalization constants are introduced, since many cases are not taken into account and only the relative probabilities to other scenarios are of interest. The projection on the ith detector $P_A(i)$ in this scenario is:

$$\begin{aligned} P_A(i) &= \langle\psi|i\rangle\langle i|\psi\rangle \\ &= \left(\sum_{n=1}^{5} a_n^* \langle n|i\rangle\right)\left(\sum_{m=1}^{5} a_m \langle i|m\rangle\right) \\ &= |a_i|^2. \end{aligned} \tag{3.2}$$

$P(i)$ is the relative probability of finding the particle at the ith detector, so that one can compare the different scenarios.

Slit B is open The procedure if only slit B is open, is basically the same as before.

$$|Source\rangle \to |\phi\rangle_B \to |\psi\rangle = \sum_{n=1}^{5} b_n |n\rangle \tag{3.3}$$

The projection on the ith detector after the particle passed slit B is again simply the square of the respective amplitude.

$$P_B(i) = \langle\psi|i\rangle\langle i|\psi\rangle = |b_i|^2 \tag{3.4}$$

Consecutive measurements For a huge number of consecutive measurements where alternately only one of the slits is open, the probability of finding the particle at the ith detector (after normalization) is merely the sum of the respective probabilities over two.

$$P_{AB}(i) = \frac{P_A(i) + P_B(i)}{2} = \frac{|a_i|^2 + |b_i|^2}{2} \tag{3.5}$$

3.2 Both slits are open

With both slits open and not knowing which path the particle takes, the final state is a superposition of the respective states.

$$|Source\rangle \rightarrow \frac{|\phi\rangle_A + |\phi\rangle_B}{\sqrt{2}} \rightarrow |\psi\rangle = \frac{1}{\sqrt{2}} \sum_{n=1}^{5} (a_n|n\rangle + b_n|n\rangle) \tag{3.6}$$

The probability of finding the particle at the ith detector (Eq. 3.7) is in this case *not* the sum of the respective probabilities, but the square of the sum of the amplitudes.

$$\begin{aligned} P_{AB}(i) &= \langle\psi|i\rangle\langle i|\psi\rangle \\ &= \frac{|a_i + b_i|^2}{2} \end{aligned} \tag{3.7}$$

If for one of the detectors $a_i = b_i$ holds, the probability according to the first scenario is $|a_i|^2$ while the second scenario would predict $2|a_i|^2$, which is actually twice as high. At another detector, where $a_i = -b_i$, the probability of detecting the particle would even be zero according to the second scenario while the first scenario still predicts $|a_i|^2$.
In experiments where the scenario with both slits open had been realized by sending monochromatic photons through the double slit, one could indeed observe an interference pattern at the screen. If one placed detectors in the slits though, the interference pattern vanished just like in the here considered first scenario. The only difference between the scenarios is that one can say for sure which way the particle took in the first one, while there is no knowledge about the path in the second case.

3.3 A detector is placed at slit A

An interesting scenario as suggested by Richard Feynman occurs when a detector detects the particle at the slit only with a certain probability. Would there still be an interference pattern? [6]
In this scenario one places a detector with state $|D\rangle$ at slit A, which detects the wave function $|\phi\rangle_A$ with a probability of $|p|^2$. The probability that the particle passes the slit undetected is $|q|^2$, so that $|p|^2 + |q|^2 = 1$. An interaction changes both the wave function $|\phi\rangle_A$ and the detector state $|D\rangle$, which are therefore denoted with a prime in the case of interaction (see Eq. 3.8).

$$\begin{aligned} |Source\rangle &\rightarrow |\phi\rangle_A|D\rangle + |\phi\rangle_B|D\rangle \\ &\rightarrow p|\phi'\rangle_A|D'\rangle + q|\phi\rangle_A|D\rangle + |\phi\rangle_B|D\rangle \\ &\rightarrow \sum_{n=1}^{5} (pa'_n|n'\rangle|D'\rangle + qa_n|n\rangle|D\rangle + b_n|n\rangle|D\rangle) = |\psi\rangle \end{aligned} \tag{3.8}$$

For simplicity we assume that the states which were changed due to the interaction are orthogonal to their former selves, so $\langle n'|n\rangle = \langle D'|D\rangle = 0$.

$$\begin{aligned} P(i) &= \langle\psi| \left(|i\,D\rangle\langle i\,D| + |i\,D'\rangle\langle i\,D'| + |i'\,D\rangle\langle i'\,D| + |i'\,D'\rangle\langle i'\,D'|\right)|\psi\rangle \\ &= |p|^2|a_i'|^2 + |qa_i + b_i|^2 \end{aligned} \tag{3.9}$$

If the particle at slit A is always detected ($p = 1$) or never detected ($q = 1$), the projection is either the sum of the squared amplitudes as in scenario one or the sum of the amplitudes squared as in scenario two. It is of course much more interesting to consider cases in which the detector works only partially. As can be seen in Eq. 3.9, the interference has a monotonically decreasing influence for increasing probabilities of detection.

4 Quantum double slit experiment

The following set up is based both on the usual double slit experiment and on the Mach-Zehnder interferometer. A source emits a particle which travels towards a beam splitter and is sent from there on either path A or B (see figure 4.1). As in the double slit experiment, there are only two possible paths but the amplitudes are not necessarily equal, but depend on the setting of the beam splitter. On each path the particle can interact with system A or B respectively, which would change both the particle's and the system's state. It is assumed that there is only one specific kind of interaction, so that the systems' states can be expressed as qubits. Then the particle travels to a second beam splitter and is sent to one of two detectors, here called '+ detector' and '- detector'.

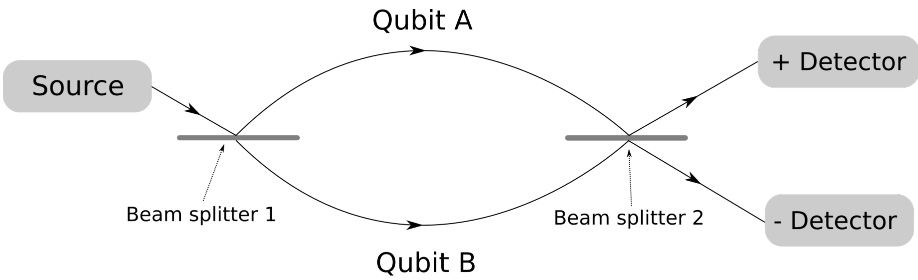


Figure 4.1: The source emits a particle which travels either on path A or B, where it can interact with one of the qubits respectively. The particle is then sent from a beam splitter to one of the two detectors

Initial state The state of the particle is expressed in terms of its energy eigenstates $|E\rangle$. The state is in a superposition $|E_0\rangle$ (Eq. 4.1) of these eigenstates. Without loss in generality, it is assumed that $|E_0\rangle$ has the shape of a Gaussian Wave Packet centred around the average energy E_0. ω is the normalization constant and σ the width of the wave packet.

$$|E_0\rangle = \omega \int_{-\infty}^{\infty} dE\, e^{-\frac{(E-E_0)^2}{2\sigma^2}} |E\rangle \tag{4.1}$$

Apart from the energy kets it is necessary to introduce another quantum number for the position, namely $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for the upper and $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for the lower path.

$$|\psi\rangle = |\phi\rangle_+ |+\rangle + |\phi\rangle_- |-\rangle = \begin{pmatrix} |\phi\rangle_+ \\ |\phi\rangle_- \end{pmatrix} \tag{4.2}$$

$|\phi\rangle_\pm$ represents the combined state of the particle and the two qubits. The qubits A and B are initially in their ground states $|0\rangle_A$ and $|0\rangle_B$ respectively.

$$|\psi\rangle = |0\rangle_A |0\rangle_B |E_0\rangle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.3}$$

It is further arbitrarily chosen that the source is placed at the upper path, so the complete initial state (Eq. 4.3) is a composite system of all these states.

First beam splitter The beam splitter is assumed to be lossless, so it merely rotates the state in position space. Since the position is expressed in vector notation, one can assign a scattering matrix to the beam splitter.

$$\begin{pmatrix} |\phi\rangle_+ \\ |\phi\rangle_- \end{pmatrix} \rightarrow \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \begin{pmatrix} |\phi\rangle_+ \\ |\phi\rangle_- \end{pmatrix} \tag{4.4}$$

$|r|^2$ and $|t|^2$ are the probabilities that $|\phi\rangle_+$ is getting reflected or transmitted by the beam splitter. $|r'|^2$ and $|t'|^2$ have the same effect for $|\phi\rangle_-$. To guarantee normalization is it necessary, that the transformation is unitary and $UU^\dagger = UU^{-1} = 1$ holds.

$$\begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \begin{pmatrix} r^* & t^* \\ t'^* & r'^* \end{pmatrix} = \begin{pmatrix} |r|^2 + |t'|^2 & rt^* + t'r'^* \\ r^*t + t'^*r' & |t|^2 + |r'|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.5}$$

It can be shown by further calculation, that Eq. 4.5 implies that $|r| = |r'|$ and $|t| = |t'|$. They differ in their phases though and are not necessarily equal. The initial state after passing the first beam splitter transforms to:

$$|0\rangle_A |0\rangle_B |E_0\rangle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow |0\rangle_A |0\rangle_B |E_0\rangle \begin{pmatrix} r_1 \\ t_1 \end{pmatrix} \tag{4.6}$$

Interaction with systems A and B On the upper path exist a probability $|p_A|^2$ that the particle interacts with system A, which would change both the particle's and the system's state, namely to $|1\rangle_A$ and $|E_1\rangle$ respectively. The latter is assumed to represent still a Gaussian Wave Packet with a shifted average energy. The same process could happen on the lower path where system B would take on $|1\rangle_B$ with a probability of $|p_B|^2$. The kind of interaction between the particle and system B is assumed to be equivalent to that with system A, so the particle's state would change to $|E_1\rangle$ too.

$$|0\rangle_A |0\rangle_B |E_0\rangle \begin{pmatrix} r_1 \\ t_1 \end{pmatrix} \rightarrow \begin{pmatrix} r_1 p_A |1\rangle_A |0\rangle_B |E_1\rangle + r_1 q_A |0\rangle_A |0\rangle_B |E_0\rangle \\ t_1 p_B |0\rangle_A |1\rangle_B |E_1\rangle + t_1 q_B |0\rangle_A |0\rangle_B |E_0\rangle \end{pmatrix} \tag{4.7}$$

Second beam splitter The final state is due to its complexity not written in vector notation, but as in Eq. 4.2.

$$|\psi\rangle = |\psi\rangle_+ + |\psi\rangle_- = |\phi\rangle_+|+\rangle + |\phi\rangle_-|-\rangle \tag{4.8}$$

After passing the second beam splitter the upper detector state equals

$$\begin{aligned}|\phi\rangle_+ = {} & r_1 r_2 q_A |0\rangle_A|0\rangle_B|E_0\rangle + t_1 t_2' q_B |0\rangle_A|0\rangle_B|E_0\rangle \\ & + r_1 r_2 p_A |1\rangle_A|0\rangle_B|E_1\rangle + t_1 t_2' p_B |0\rangle_A|1\rangle_B|E_1\rangle\end{aligned} \tag{4.9}$$

and the lower detector state equals

$$\begin{aligned}|\phi\rangle_- = {} & r_1 t_2 q_A |0\rangle_A|0\rangle_B|E_0\rangle + t_1 r_2' q_B |0\rangle_A|0\rangle_B|E_0\rangle \\ & + r_1 t_2 p_A |1\rangle_A|0\rangle_B|E_1\rangle + t_1 r_2' p_B |0\rangle_A|1\rangle_B|E_1\rangle.\end{aligned} \tag{4.10}$$

5 Results & Analysis

5.1 Single detectors

5.1.1 Determining the density matrix

Measuring the position A measurement of the position at the single detectors leads to a collapse of the wave function and a projection on either the upper or lower path.

$$P_\pm|\psi\rangle = |\pm\rangle\langle\pm|\psi\rangle = |\psi\rangle_\pm = |\phi\rangle_\pm|\pm\rangle \tag{5.1}$$

The position is now well known, so the density matrix is just

$$\rho_\pm = \frac{|\phi\rangle_\pm\langle\phi|_\pm}{\langle\phi|\phi\rangle_\pm}, \tag{5.2}$$

where the inner product $\langle\phi|\phi\rangle_\pm$ guarantees normalization.

$$\begin{aligned}\frac{|\phi\rangle}{\sqrt{\langle\phi|\phi\rangle}} &= \frac{a|0\rangle_A|0\rangle_B|E_0\rangle + b|0\rangle_A|1\rangle_B|E_1\rangle + c|1\rangle_A|0\rangle_B|E_1\rangle}{\sqrt{|a|^2+|b|^2+|c|^2}} \\ &= \alpha|00\rangle|E_0\rangle + \beta|01\rangle|E_1\rangle + \gamma|10\rangle|E_1\rangle\end{aligned} \tag{5.3}$$

Since the following procedure is the same for $|\phi\rangle_+$ and $|\phi\rangle_-$ it is useful to go on with a generalized state $|\phi\rangle$ and a simplified notation (Eq. 5.3).

$$\begin{aligned}\rho = {} & \frac{|\phi\rangle\langle\phi|}{\sqrt{\langle\phi|\phi\rangle}} \\ = {} & |\alpha|^2|00\rangle\langle00||E_0\rangle\langle E_0| + \left(\alpha\beta^*|00\rangle\langle01| + \alpha\gamma^*|00\rangle\langle10|\right)|E_0\rangle\langle E_1| \\ & + \beta\alpha^*|01\rangle\langle00||E_1\rangle\langle E_0| + \left(|\beta|^2|01\rangle\langle01| + \beta\gamma^*|01\rangle\langle10|\right)|E_1\rangle\langle E_1| \\ & + \gamma\alpha^*|10\rangle\langle00||E_1\rangle\langle E_0| + \left(\gamma\beta^*|10\rangle\langle01| + |\gamma|^2|10\rangle\langle10|\right)|E_1\rangle\langle E_1|\end{aligned} \tag{5.4}$$

The related density matrix (Eq. 5.4) represents the system after the measurement of position. It is still in a pure state, so one has full knowledge about the system.

Tracing out over the energy It is assumed that the detectors are not capable of measuring the energy of the particle and it is therefore necessary to trace out over the energy. Before doing so one needs to calculate the inner product of two Wave Packets. The inner products of the eigenkets $|E\rangle$ are orthogonal, so $\langle E|E'\rangle = \delta(E-E')$.

$$\begin{aligned}\langle E_n|E_m\rangle &= |\omega|^2 \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} dE'\, e^{-\frac{(E-E_n)^2}{2\sigma^2}} e^{-\frac{(E'-E_m)^2}{2\sigma^2}} \langle E|E'\rangle \\ &= |\omega|^2 \int_{-\infty}^{\infty} dE\, e^{-\frac{1}{2\sigma^2}\left(2E^2-2E(E_n+E_m)+E_n^2+E_m^2\right)} \\ &= |\omega|^2 \sigma\sqrt{\pi} e^{-\frac{(E_n-E_m)^2}{4\sigma^2}} = \epsilon_{nm}\end{aligned} \tag{5.5}$$

The inner product of two identical Wave Packets equals one ($\langle E_n|E_n\rangle$=1), so $|\omega|^2 = \frac{1}{\sigma\sqrt{\pi}}$. ϵ_{nm} is then a real number between 0 and 1, depending on the difference of the two average energies and the width of the Wave Packet, and represents the overlap of the two Wave Packets. To keep the calculations simple, the Wave Packets were chosen to be Gaussian but could have had any shape. The shape itself is not of importance, since we are interested only in the overlap ϵ_{nm} of two different Wave Packets.
Since we know the result of the inner product of two Wave Packets now, it is possible to determine the reduced density matrix by taking the partial trace over the energy.

$$\begin{aligned}\rho' &= \mathrm{tr}_E(\rho) \\ &= |\alpha|^2|00\rangle\langle 00| + \epsilon\alpha\beta^*|00\rangle\langle 01| + \epsilon\alpha\gamma^*|00\rangle\langle 10| \\ &+ \epsilon\beta\alpha^*|01\rangle\langle 00| + |\beta|^2|01\rangle\langle 01| + \beta\gamma^*|01\rangle\langle 10| \\ &+ \epsilon\gamma\alpha^*|10\rangle\langle 00| + \gamma\beta^*|10\rangle\langle 01| + |\gamma|^2|10\rangle\langle 10|\end{aligned} \tag{5.6}$$

ρ' can be written in usual matrix notation:

$$\rho' = \begin{pmatrix} |\alpha|^2 & \epsilon_{01}\alpha\beta^* & \epsilon_{01}\alpha\gamma^* & 0 \\ \epsilon_{01}\beta\alpha^* & |\beta|^2 & \beta\gamma^* & 0 \\ \epsilon_{01}\gamma\alpha^* & \gamma\beta^* & |\gamma|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.7}$$

So finally we have found the reduced density matrix for the two qubit system. ρ' is in a mixed state except for the case, that the two energy Wave Packets are indistinguishable and ϵ_{nm}=1. Not measuring the energy does not lead to a loss of information then, since the energy is uniform anyway. It would imply though, that an interaction wouldn't change the particle's energy whatsoever which is unlikely if not impossible, so the system is in a mixed state. In order to quantify its entanglement one needs to use the concept of concurrence.

5.1.2 Determining the concurrence

The concurrence of the general density matrix as in Eq. 5.7 shall be calculated here.

$$\tilde{\rho} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |\gamma|^2 & \gamma^*\beta & -\epsilon\gamma^*\alpha \\ 0 & \beta^*\gamma & |\beta|^2 & -\epsilon\beta^*\alpha \\ 0 & -\epsilon\alpha^*\gamma & -\epsilon\alpha^*\beta & |\alpha|^2 \end{pmatrix} \tag{5.8}$$

After calculating $\tilde{\rho}$ which equals $(\sigma_y \otimes \sigma_y)\, \rho^* \,(\sigma_y \otimes \sigma_y)$ it is fairly easy to determine $\mathbf{M}$.

$$\mathbf{M}(\rho) = \rho\tilde{\rho} = \begin{pmatrix} 0 & 2\epsilon\alpha\beta^*|\gamma|^2 & 2\epsilon\alpha|\beta|^2\gamma^* & -2\epsilon^2\alpha^2\beta^*\gamma^* \\ 0 & 2|\beta|^2|\gamma|^2 & 2|\beta|^2\beta\gamma^* & -2\epsilon\alpha|\beta|^2\gamma^* \\ 0 & 2\beta^*\gamma|\gamma|^2 & 2|\beta|^2|\gamma|^2 & -2\epsilon\alpha\beta^*|\gamma|^2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.9}$$

The eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $\lambda_4 = 4|\beta|^2|\gamma|^2$. Since there is only one non-zero eigenvalue the concurrence is simply

$$C(\rho) = 2|\beta||\gamma| \tag{5.10}$$

or, if the related density matrix hadn't been normalized,

$$C(\rho) = \frac{2|b||c|}{|a|^2 + |b|^2 + |c|^2}. \tag{5.11}$$

5.1.3 Assigning specific parameters

After obtaining the results for the general case, we can now also assign specific parameters to $|\phi\rangle_+$ and $|\phi\rangle_-$. We assume that the beam splitters operate just like a common rotation matrix:

$$\begin{pmatrix} r_i & t_i' \\ t_i & r_i' \end{pmatrix} = \begin{pmatrix} \cos(\theta_i) & \sin(\theta_i) \\ -\sin(\theta_i) & \cos(\theta_i) \end{pmatrix} \tag{5.12}$$

The matrix depends only on the scattering angle θ_i. The interaction between the particle and qubit A or B was assumed to be equivalent, so there is no reason why the probabilities of the interaction should differ, which means that $p_A = p_B = p$.

Upper detector With the specific parameters from above, one can find the amplitudes according to Eq. 4.9 & 5.3.

$$\begin{aligned} a_+ &= q\cos(\theta_1 + \theta_2) \\ b_+ &= p\cos(\theta_1)\cos(\theta_2) \\ c_+ &= -p\sin(\theta_1)\sin(\theta_2) \end{aligned} \tag{5.13}$$

The normalization constant equals

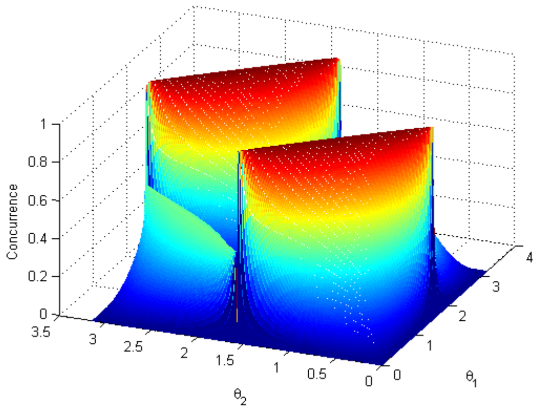
$$\begin{aligned} &|a_+|^2 + |b_+|^2 + |c_+|^2 \\ &= q^2\cos^2(\theta_1 + \theta_2) + p^2\sin^2(\theta_1)\sin^2(\theta_2) + p^2\cos^2(\theta_1)\cos^2(\theta_2) \\ &= \cos^2(\theta_1 + \theta_2) + \frac{1}{2}p^2\sin(2\theta_1)\sin(2\theta_2), \end{aligned} \tag{5.14}$$

so the normalized state at the upper detector is

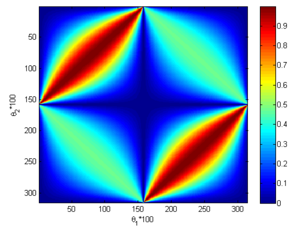
$$\begin{aligned} \frac{|\phi\rangle_+}{\sqrt{\langle\phi|\phi\rangle_+}} &= \frac{q\cos(\theta_1 + \theta_2)|00\rangle|E_0\rangle}{\sqrt{\cos^2(\theta_1 + \theta_2) + \frac{1}{2}p^2\sin(2\theta_1)\sin(2\theta_2)}} \\ &\quad - \frac{p\sin(\theta_1)\sin(\theta_2)|01\rangle|E_1\rangle - p\cos(\theta_1)\cos(\theta_2)|10\rangle|E_1\rangle}{\sqrt{\cos^2(\theta_1 + \theta_2) + \frac{1}{2}p^2\sin(2\theta_1)\sin(2\theta_2)}}. \end{aligned} \tag{5.15}$$

The normalized concurrence of the two qubit density matrix, if the particle is measured at the upper detector, equals

$$C(\rho''_+) = \frac{\frac{1}{2}p^2|sin(2\theta_1)sin(2\theta_2)|}{cos^2(\theta_1+\theta_2)+\frac{1}{2}p^2 sin(2\theta_1)sin(2\theta_2)}. \tag{5.16}$$



(a) 3D-plot



(b) 2D-plot

Figure 5.1: Concurrence of ρ''_+ as a function of θ_1 and θ_2 with p=0.8

Looking at figure 5.1, which shows the concurrence as a function of θ_1 and θ_2 for a fixed value of $p = 0.8$, one can see that the concurrence is maximized for $\theta_2 = \frac{\pi}{2} - \theta_1$ and $\theta_2 = \frac{3\pi}{2} - \theta_1$. Plugging these values in Eq. 4.9 and then normalizing the states gives:

$$\frac{|\phi\rangle_+^{(1)}}{\sqrt{\langle\phi|\phi\rangle_+^{(1)}}} = \frac{|10\rangle - |01\rangle}{\sqrt{2}}|E_1\rangle \tag{5.17}$$

$$\frac{|\phi\rangle_+^{(2)}}{\sqrt{\langle\phi|\phi\rangle_+^{(2)}}} = \frac{|01\rangle - |10\rangle}{\sqrt{2}}|E_1\rangle \tag{5.18}$$

The inherent states Eq. 5.17 and 5.18 are fully entangled Bell states and additionally completely independent of p. That is an astonishing result considering that the concurrence of all other states turns towards zero for small p. Even the slightest probability of interaction is apparently already sufficient to obtain a fully entangled state. This is the case, since the interference is such that the probability of obtaining $|00\rangle$ has completely vanished and the upper detector only detects the events $|01\rangle$ and $|10\rangle$.
One has to keep in mind, that Eq. 5.18 and 5.19 are normalized states though. The probability of actually measuring the particle at the upper detector is

$$\langle\phi|\phi\rangle_+^{(1)} = \frac{1}{2}p^2\sin^2(2\theta_1) \tag{5.19}$$

and of course a function of the interaction probability. And yet, any detected state at the upper state implies that the qubits are perfectly entangled.

Lower detector The very same procedure can be done for the lower detector. The amplitudes of $|\phi\rangle_-$ are:

$$
\begin{aligned}
a_- &= -q\sin(\theta_1+\theta_2) \\
b_- &= -p\cos(\theta_1)\sin(\theta_2) \\
c_- &= -p\sin(\theta_1)\cos(\theta_2)
\end{aligned} \tag{5.20}
$$

After calculating the normalization constant, which is in this case

$$
\begin{aligned}
&|a_-|^2+|b_-|^2+|c_{|}^2 \\
&= q^2\cos^2(\theta_1+\theta_2)+p^2\sin^2(\theta_1)\cos^2(\theta_2)+p^2\cos^2(\theta_1)\sin^2(\theta_2) \\
&= \sin^2(\theta_1+\theta_2)-\frac{1}{2}p^2\sin(2\theta_1)\sin(2\theta_2),
\end{aligned} \tag{5.21}
$$

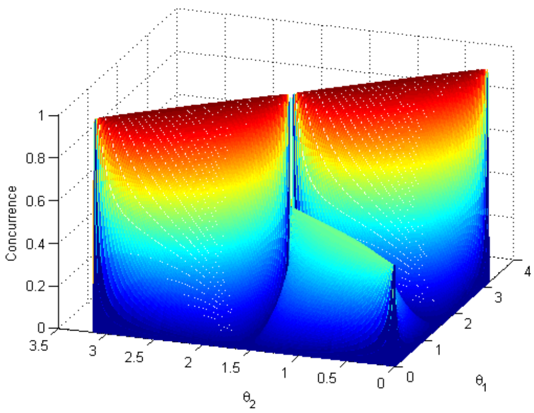
one can compute the normalized wave function, if it collapsed at the lower detector:

$$
\begin{aligned}
\frac{|\phi\rangle_-}{\sqrt{\langle\phi|\phi\rangle_-}} &= \frac{-qsin(\theta_1+\theta_2)|00\rangle|E_0\rangle}{\sqrt{sin^2(\theta_1+\theta_2)-\frac{1}{2}p^2sin(2\theta_1)sin(2\theta_2)}} \\
&\quad - \frac{psin(\theta_1)cos(\theta_2)|01\rangle|E_1\rangle+pcos(\theta_1)sin(\theta_2)|10\rangle|E_1\rangle}{\sqrt{sin^2(\theta_1+\theta_2)-\frac{1}{2}p^2sin(2\theta_1)sin(2\theta_2)}}.
\end{aligned} \tag{5.22}
$$

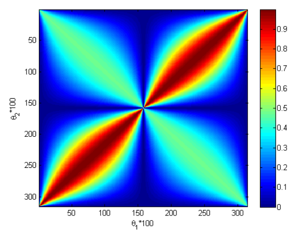
The normalized concurrence of the two qubit density matrix, if the particle is measured at the upper detector, equals

$$
C(\rho''_-) = \frac{\frac{1}{2}p^2|sin(2\theta_1)sin(2\theta_2)|}{sin^2(\theta_1+\theta_2)-\frac{1}{2}p^2sin(2\theta_1)sin(2\theta_2)}. \tag{5.23}
$$

Apparently the concurrence shows the same qualitative behaviour after mea-



(a) 3D-plot



(b) 2D-plot

Figure 5.2: Concurrence of ρ''_- as a function of θ_1 and θ_2 with p=0.8

suring the particle at the lower detector as it did after measuring it at the upper detector. The angles for maximized concurrence are of course different, but for $\theta_2 = \pi - \theta_1$ and $\theta_2 = -\theta_1$ one obtains the very same Bell states as for the upper detector.

$$\frac{|\phi\rangle_-^{(1)}}{\sqrt{\langle\phi|\phi\rangle_-^{(1)}}} = \frac{|10\rangle - |01\rangle}{\sqrt{2}}|E_1\rangle \tag{5.24}$$

$$\frac{|\phi\rangle_-^{(2)}}{\sqrt{\langle\phi|\phi\rangle_-^{(2)}}} = \frac{|01\rangle - |10\rangle}{\sqrt{2}}|E_1\rangle \tag{5.25}$$

The effects and consequences are the same as for the upper detector, so they are not discussed again.

5.2 Combined detector

5.2.1 Determining the density matrix

If I decide to measure the position of the particle, but don't look at the exact position that is the same as not measuring the particle at all since I assume that I do not have any losses on the way. So a combined measurement is simply a projection of the state onto itself.

$$P|\psi\rangle = (|+\rangle\langle+| + |-\rangle\langle-|)|\psi\rangle = |\psi\rangle \tag{5.26}$$

Therefore, the relating density matrix is just the outer product of that state.

$$\begin{aligned} \rho &= |\psi\rangle\langle\psi| \\ &= |\phi\rangle_+\langle\phi|_+|+\rangle\langle+| + |\phi\rangle_+\langle\phi|_-|+\rangle\langle-| \\ &+ |\phi\rangle_-\langle\phi|_+|-\rangle\langle+| + |\phi\rangle_-\langle\phi|_-|-\rangle\langle-| \end{aligned} \tag{5.27}$$

Since the exact position is not measured, one needs to trace out over the position kets.

$$\rho' = \mathrm{tr}_\pm(\rho) = |\phi\rangle_+\langle\phi|_+ + |\phi\rangle_-\langle\phi|_- \tag{5.28}$$

The result is merely the sum of the two single detector density matrices, since the overlap $\langle+|-\rangle$ of the paths is zero. As before the energy is not measured, so one needs tp trace out over the energy here too.

$$\begin{aligned} \rho'' &= \mathrm{tr}_E(\rho') \\ &= \mathrm{tr}_E(|\phi\rangle_+\langle\phi|_+) + \mathrm{tr}_E(|\phi\rangle_-\langle\phi|_-) \\ &= \rho''_+ + \rho''_- \end{aligned} \tag{5.29}$$

ρ'' is apparently the sum of the two single density matrices and therefore:

$$\rho'' = \begin{pmatrix} q^2 & \epsilon pqcos^2(\theta_1) & \epsilon pqsin^2(\theta_1) & 0 \\ \epsilon pqcos^2(\theta_1) & p^2cos^2(\theta_1) & 0 & 0 \\ \epsilon pqsin^2(\theta_1) & 0 & p^2sin^2(\theta_1) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This density matrix has two striking features. It is completely independent of θ_2 and two entries are zero, which were not in case of the single detector matrices.

These two entries refer to the interference terms of $|01\rangle$ and $|10\rangle$. To make sure, that this is not just a odd speciality of the assigned parameters, we consider the most general case and calculate the entries in terms of the initial parameters.

$$\begin{aligned}\langle 10|\rho''_+|01\rangle + \langle 10|\rho''_-|01\rangle &= p_A^* p_B r_1^* r_2^* t_1 t'_2 + p_A^* p_B r_1^* t_2^* t_1 r'_2 \\ &= p_A^* p_B r_1^* t_1 (r_2^* t'_2 + t_2^* r'_2) \\ &= 0\end{aligned} \tag{5.30}$$

This is zero, since $(r_2^* t'_2 + t_2^* r'_2)$ are designed to be zero (see Eq. 4.5) in order to guarantee that the scattering matrix is unitary. Those density matrix are hence always zero regardless of the choice of parameters. Before discussing this we investigate the second odd property, the independence of θ_2. Assuming $|\psi'\rangle$ represents the state after passing the second beam splitter and $|\psi\rangle$ the state before doing so, $|\psi'\rangle = S_2|\psi\rangle$ and therefore

$$\rho = |\psi'\rangle\langle\psi'| = S|\psi\rangle\langle\psi|S^\dagger. \tag{5.31}$$

We then trace out over the position:

$$\rho' = \mathrm{tr}_\pm(\rho) = \mathrm{tr}_\pm(S|\psi\rangle\langle\psi|S^\dagger) = \mathrm{tr}_\pm(SS^\dagger|\psi\rangle\langle\psi|) = \mathrm{tr}_\pm(|\psi\rangle\langle\psi|) \tag{5.32}$$

The order of matrices when taking the trace does not matter ($\mathrm{tr}(AB) = \mathrm{tr}(BA)$) and the scattering matrix is unitary ($SS^\dagger = \mathbb{I}$), so measuring with a combined detector after the second beam splitter is physically the same as placing the detectors before the second beam splitter. The upper and lower state are still mixed, but if one does not distinguish between lower and upper path while measuring, the mixing has no effect whatsoever. Any information on possible mixing of the upper and lower path is destroyed, when one traces out over the position. This explains why θ_2 does not appear in the density matrix and it also explains why those interference terms vanish. They cannot possibly give any information about the entanglement of the qubits.

5.2.2 Determining the concurrence

To calculate the concurrence, we assume a very general case again except that the two interference terms are zero.

$$\rho = \begin{pmatrix} a & b & c & 0 \\ d & e & 0 & 0 \\ f & 0 & g & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \tilde{\rho} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & g & 0 & -f^* \\ 0 & 0 & e & -d^* \\ 0 & -c^* & -b^* & a \end{pmatrix} \tag{5.33}$$

While calculating $\tilde{\rho}$ we use the fact, that diagonal elements of a density matrix are per definition always real regardless of the base kets since they represent probabilities. In order to find the eigenvalues of $\rho\tilde{\rho}$ we calculate the eigenvalues and find that $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \lambda_4 = eg$.

$$\det(\rho\tilde{\rho} - \lambda\mathbb{I}) = \begin{vmatrix} -\lambda & bg & ce & -bf^* - cd^* \\ 0 & eg - \lambda & 0 & -ef^* \\ 0 & 0 & eg - \lambda & -d^* g \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = \lambda^2 (eg - \lambda)^2 \tag{5.34}$$

So the concurrence equals

$$C(\rho) = \sqrt{eg} - \sqrt{eg} = 0. \tag{5.35}$$

It is apparently always zero regardless of the scattering angles and the probability of interaction. If the particle's path after the second beam splitter is not measured, the qubits A and B are never entangled.

6 Discussion

At a first glance this is a very odd result, since it seems that only the knowledge of the exact position of the particle induces entanglement to the qubits A and B. The measurement happens later than the interaction though and can therefore impossibly affect the states of A and B.
To make sense of this we take the perspective of three fictional experimentalists, who shall called Alice, Bob and Chris. So Alice measures the qubit A, Bob the qubit B and Chris the position of the particle after it passed the second beam splitter. Chris measures the position first and only after that will Alice and Bob get green lights to perform their measurements on the system.
In a first scenario, Chris only writes down whether he detected the particle, but not where. Alice and Bob on the other hand write down their exact results and after performing ten consecutive measurements (always with a new particle and new qubits) the three meet and try to make sense of their obtained information. Chris detected the particle in all ten cases, since the set up is assumed to be lossless. Apparently Chris' measurement was pointless, since his results do not obtain any valuable information. Alice and Bob will find the events $|0\rangle_A|0\rangle_B$, $|0\rangle_A|1\rangle_B$ and $|1\rangle_A|0\rangle_B$ in random appearance and order when comparing their results (see table 1). Since they do not have any additional information they cannot see any pattern or have any reason to assume that their qubits are entangled. This scenario refers of course to the previous calculations, where the particle was detected by a combined detector that could not distinguish between upper and lower path. And indeed the result of these calculations, that the entanglement is always zero (Eq. 5.35), makes perfect sense when adopting the perspective of the three experimentalists.

No.	Alice	Bob	Chris
1	$\|0\rangle$	$\|0\rangle$	detected
2	$\|0\rangle$	$\|1\rangle$	detected
3	$\|0\rangle$	$\|0\rangle$	detected
4	$\|0\rangle$	$\|0\rangle$	detected
5	$\|0\rangle$	$\|1\rangle$	detected
6	$\|1\rangle$	$\|0\rangle$	detected
7	$\|0\rangle$	$\|0\rangle$	detected
8	$\|1\rangle$	$\|0\rangle$	detected
9	$\|0\rangle$	$\|1\rangle$	detected
10	$\|0\rangle$	$\|0\rangle$	detected

Table 1: The results of Alice, Bob and Chris after ten consecutive measurements, when Chris does not measure where he detected the particle

In the second scenario, Chris measures the exact position of the particle and writes it down. Furthermore he sets the scattering angles to $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = \frac{\pi}{4}$, since his calculations suggested that the qubits are maximally entangled at the upper detector for these angles. In case the lower detector detects the particle, A and B are not or only to a very small amount entangled. Otherwise the procedure is as before, so after a number of measurements Alice and Bob have again a list of seemingly random events they cannot make sense of. Only this time Chris can check his list, which says him that for example the events 2,5,8 and 9 had been fully entangled (see table 2). And indeed in these cases Alice and Bob would only find $|01\rangle$ and $|10\rangle$ events. So Chris can use his knowledge of the position to single out the entangled cases.

No.	Alice	Bob	Chris
1	$\|0\rangle$	$\|0\rangle$	detected at -
2	$\|0\rangle$	$\|1\rangle$	detected at +
3	$\|0\rangle$	$\|0\rangle$	detected at -
4	$\|0\rangle$	$\|0\rangle$	detected at -
5	$\|0\rangle$	$\|1\rangle$	detected at +
6	$\|1\rangle$	$\|0\rangle$	detected at -
7	$\|0\rangle$	$\|0\rangle$	detected at -
8	$\|1\rangle$	$\|0\rangle$	detected at +
9	$\|0\rangle$	$\|1\rangle$	detected at +
10	$\|0\rangle$	$\|0\rangle$	detected at -

Table 2: The results of Alice, Bob and Chris after ten consecutive measurements, when Chris does measure where he detected the particle

So it turns out that the measurement of position does indeed not create entanglement, but it can be used to check whether A and B are entangled. One could even take the perspective, that the detectors are able to detect entanglement itself. With that knowledge it is fairly easy to create entanglement between A and B. One only needs to prepare a number of pairs, put them in the experiment, measure whether they are entangled and later simply select the entangled ones.

The truly striking property of the experiment though, is that for $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = \frac{\pi}{4}$ a detection at the upper detector implies that the qubits are fully entangled regardless of how unlikely an interaction between qubit and particle might be. For low probabilities it might take a number of times of course, to detect the particle in the upper detector, but nevertheless it would allow one to use particles that seemed utterly impractical before.

7 Outlook

Summary By exactly measuring the particle's position, it is possible to check whether the systems A and B are entangled. As in the normal double slit experiment, the particle's wave function interferes with itself due to its ambiguity, so that the probabilities for certain events might even completely vanish. By setting the scattering angles of the beam splitters in a suitable way, it is therefore possible that one detector detects the particle only when A and B perfectly cor-

relate and the other detector detects it, when that is not the case. So I cannot just magically introduce entanglement to A and B by measuring the particle's position, but I can use the information contained in the particle's position to post-select the entangled qubits.
This outcome also seems to be in agreement with current research. A very recent paper of di Lorenzo from August 2014 considered a very similar set up as the one presented in this thesis and came also to the conclusion, "that a single particle in a superposition of different paths can entangle two objects located on each path". [5]

Applications The set up could be used to induce entanglement between two objects by post-selection. This means that the measurement does not induce entanglement, but can be used to select the entangled qubits after a number of iterations. Extremely interesting is that the upper detector can detect fully entangled pairs independently from the probability of interaction between particle and object. A low probability would of course lead to fewer cases of detection (see Eq. 5.19), but those that are detected would tell us that the objects are fully entangled. This is especially interesting for particles, which interact only weakly with the objects. Experimentalist could consider to use particles or objects that they neglected before due to the low probability of interaction.

Improvements Many of the assumptions are of course strongly idealized. An actual set up will most likely not be lossless, but the particle might get lost on the way or cannot be detected by the detector. It is also definitely a restriction to assign a qubit to the objects A and B, since there are many cases where more than one distinct kind of interaction with a particle is possible. These deficiencies could be possibly incorporated in the thesis, but it would make the calculations a lot more difficult.
Another thing that could be very nicely incorporated though, is that system A and B are not alike, so their interactions with the particle would have different probabilities and would lead to different energy changes. A and B are still qubits, but an interaction would lead to three possible energy levels of the particle, $|E_0\rangle$, $|E_1\rangle$ and $|E_2\rangle$. Tracing out over the energy would than lead to another density matrix and the concurrence could possibly be a function of the overlaps.
Another thing that might be worth to look into, is to use detectors that do not measure the position, but the energy. The goal would be the same, namely to single out the entangled qubit pairs.

References

[1] Charles H Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K Wootters. Teleporting an unknown quantum state via dual classical and einstein-podolsky-rosen channels. *Physical review letters*, 70(13):1895, 1993.

[2] Charles H Bennett and Stephen J Wiesner. Communication via one-and two-particle operators on einstein-podolsky-rosen states. *Physical review letters*, 69(20):2881, 1992.

[3] Max Born, Albert Einstein, and Irene Born. *The Born Einstein Letters: correspondence between Albert Einstein and Max and Hedwig Born from 1916 to 1955 with commentaries by Max Born. Translated by Irene Born.* Basingstoke, Macmillan Press, 1971.

[4] Emmanuel Desurvire. *Classical and quantum Information theory: an introduction for the telecom scientist*. Cambridge University Press, 2009.

[5] Antonio Di Lorenzo. Postselection-induced entanglement swapping from a vacuum-excitation entangled state to separate quantum systems. *Phys. Rev. A*, 90:022121, Aug 2014.

[6] Richard P Feynman, Robert B Leighton, and Matthew Sands. *The feynman lectures on physics. 3: Quantum mechanics: Lectures on physics*. Addison-Wesley, 1965.

[7] Scott Hill and William K Wootters. Entanglement of a pair of quantum bits. *Physical review letters*, 78(26):5022, 1997.

[8] Michal Horodecki. Entanglement measures. *Quantum Information and Computation*, 1, 2001.

[9] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010.

[10] Martin B Plenio and Shashank Virmani. An introduction to entanglement measures. *arXiv preprint quant-ph/0504163*, 2005.

[11] John Preskill. Lecture notes for ph219/cs219: Quantum information and computation, 2013.

[12] Erwin Schrödinger. Discussions of probability relations between separated systems. *Proceedings of the Cambridge Philosophical Society*, 1935.

[13] Joachim Stolze and Dieter Suter. *Quantum computing: a short course from theory to experiment*. John Wiley & Sons, 2008.

[14] John von Neumann. Wahrscheinlichkeitstheoretischer aufbau der quantenmechanik. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1927:245–272, 1927.

[15] William K Wootters. Entanglement of formation of an arbitrary state of two qubits. *Physical Review Letters*, 80(10):2245, 1998.

[16] William K Wootters. Entanglement of formation and concurrence. *Quantum Information & Computation*, 1(1):27–44, 2001.