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Model Predictive Control for Stock Postfolio Selection

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Abstract
The motivation behind the thesis lies in some interesting results in the article 'Portfolio Optimization Applications of Stochastic Receding Horizon Control' written by James Primbs, regarding applying control theory to problems in the financial market. The purpose will be to implement a method called Model Predictive Control, MPC, for selecting optimal portfolio weights in two portfolios with different objectives, a risk adjusted portfolio and an index tracking portfolio. The basic idea behind choosing this method when optimizing portfolios in the financial market is its ability to use feedback and in the same time includes the future when optimizing at each time step. The method also handles multivariable problems and constraints and is therefore suitable for the complex portfolio weight optimization. To see how the method works, experiments with index tracking is performed with real data provided by Danske Bank. This shows promising results for small portfolios but for larger portfolios the setup is needed to be developed further. A new approach for this adjustment is presented. Some further research will be needed to be able to handle problems in reality in a good way and this will be discussed. The thesis holds all needed definitions regarding the modeling technique and terms used.

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Chapter 1

1. Introduction

1.1 Background

During courses taken at the department of Automatic Control it has come to our attention that control theory can be applied to problems relating to the financial market. The interest of applying control theory on a financial problem was the background to the choice of subject of the thesis, which will be an implementation of a method called Model Predictive Control, MPC, for selecting optimal portfolio weights. The method of MPC has the advantages of handling multivariable problems and constraints in a good way which makes it suitable for the complex portfolio weight optimization problems. This project follows the methodology developed in the article 'Portfolio Optimization Applications of Stochastic Receding Horizon Control' [10] written by James Primbs, Assistant Professor of Management Science and Engineering at Stanford University.

In the thesis two portfolios with different objectives were studied, a risk adjusted wealth portfolio and an index tracking portfolio. Portfolio optimization began in the fifties with the work of Markowitz, see [3], who initiated the mathematical problems of portfolio optimization where the goal was to maximize a risk adjusted measure of wealth. Index tracking, on the other hand, aims to follow the performance of an index. This can be of interest for investment managers as they can create a portfolio with a subset of stocks from the index to decrease their transaction costs and simplify the portfolio selection, while still tracking the index. This portfolio will also be easier to rebalance since it can contain fewer stocks than the whole set of stocks in the index. Index tracking is the more suitable of the two portfolio examples when it comes to passive management because index tracking is not in need of that much analysis and interpretation as the previous example.

The ambition with the thesis is to be able to make a good comparison between the optimized portfolios and some historical index tracking portfolios with data provided by Danske Bank. To be able to achieve this comparison a deeper investigation on simpler theoretical examples will be performed initially. The interest in index tracking comes from the fact that over a longer time period active fund managers rarely beat the index, see [1]. In [10] it is mentioned that this tracking can be done with a subset of the stocks in the S&P 500 index, containing 500 stocks. This is then investigated with a tracking of an index of 5 stocks which off course is a huge simplification. By looking at the how the index tracking problem grows with the size of the index it can already here be suspected that some problems can occur when trying to track larger benchmarks, as the S&P 500. This setup and the suspected problems will be further discussed when handling large portfolios.

1.2 Tasks

The tasks of the thesis are the following:

- Implementation and simulation of the method for stock portfolio selection with Model Predictive Control for theoretical examples.
- Testing functionality of the method and adding different constraints.
- Applying the method on portfolios provided by Danske Bank for comparisons and analysis.

1.3 Outline

In Chapter 2 the thesis starts with describing various theories to get a deeper understanding of the considered problem. This section of theory describes some different concepts in finance and the method, MPC, which will be used for portfolio selection. Chapter 2 also involves a description of how the MPC-formulation on this model is formulated to a convex optimization problem, which is tractable to solve. This is the background of the thesis. The following section, Chapter 3, contains of the theoretical examples and a presentation of the numerical data from [10] and the data provided data by Danske Bank. The results and the analysis of the two different types of portfolios will be presented in Chapter 4. This section also contains the case where the method is applied to real historical data. The thesis ends with conclusions and a discussion on further research and improvements, found in Chapter 5.

1.4 Method

The thesis is based on an evaluation of the method stated in [10]. Simpler implementations and problems were extended to more complex and reality based setups. To be able to analyse this for applications in reality, collaboration with Danske Bank gave the opportunity to compare our results with real historical portfolio values. The method and approach used by Danske Bank to achieve these portfolios are not discussed in detail due to confidentiality. Although, the information has helped and been considered in the layout and the analysis in the project.

The resulting optimization problem that is to be solved in each time step, is a semi-definite program. A solver for such optimization problem is SeDuMi. We use Yalmip where it is easy to formulate the optimization problem, Yalmip then translate it to the program language of SeDuMi.

1.5 Data

Besides the data in [10], Danke Bank has provided data for three historical index tracking portfolios. The three portfolios tracks two indexes which are of very different size. The first portfolio "DI Biotechnology" is the smaller of the portfolios with a number of stocks around 20 and tracks the index "Amex Biotechnology", which contains the same amount of stocks. The second and third portfolios "DI Verdensindeks" and "BGI PAL Verden Valutasikret", with a number of around 1000 stocks, track the index "MSCI World". The larger index includes over 1000 stocks. The data provided by Danske Bank includes weekly values of the portfolio and the index, returns, stock prices and weights for two years, from November 9th 2007 to November 9th 2009. Some of the values of the prices were missing, therefore these stocks were removed from the set of stocks, both in the indexes and in the given portfolios. Then the remaining stocks were given a new weight which was proportional to the old but as a percentage of the new total set. The reduced sets and values were then later imported into Matlab.

1.6 Limitations and assumptions

Due to limited data the optimization is later performed on the same time period as the provided data set. This will in a sense provide the method with knowledge of the future which might improve the result.

The problem setup has some financial assumptions, it assumes no transaction costs and the same ask and bid prices. The setup also assumes that money can be invested and borrowed at the same interest rate.

Chapter 2

2. Theory

2.1 Portfolio theory

A portfolio can be described with portfolio weights, which are a fraction of the total investment invested in one stock. Each stock will have a return described by the percentage change of the stock value for every time period. When this portfolio is constructed some risk can be eliminated through diversification, which means that you spread your risk on different securities, [6]. Therefore you need to know the risk and return of each stock as well as the covariance between the different stocks to find the risk for the entire portfolio. Besides diversification there is another way of reducing risk from the portfolio. This can be done by investing some money in risk-free investments, for example by investing in government bonds. The investor can also be given the opportunity to borrow money to invest in the stock market for a more aggressive approach.

A positive amount invested in a stock is called a long position and a negative amount invested in a security is called a short position. A short sale is a transaction in which you sell a stock that you do not own and then buy that stock back in the future. This can be a good strategy if you expect the stock price to decline in the future. An advantage that comes with allowing short sales is that the set of possible portfolios will be extended, for more information see [1].

2.2 Active/Passive portfolio management

Passive management is a financial strategy where the manager himself tries to make as few portfolio decisions as possible and in this way minimizing the labor costs. The portfolio manager then tries to trust the strategy which leaves the human factor out of the decision. This strategy often aims to replicate the performance of a benchmark. Managers with this strategy believe in the efficient market hypothesis that states that all information is reflected and incorporated in the stock prices. Active portfolio managers instead use information to predict the future returns and construct their investment portfolios according to this to try to outperform a benchmark. Managers in this active strategy do not believe in the efficient market hypothesis, [5].

2.3 Capital Asset Pricing Model, CAPM

A prominent researcher in the field of portfolio theory is Markowitz who described the set of all portfolios that will give the highest expected return for each given level of risk. His research was an essential base in development of CAPM, Capital Asset Pricing Model, with Sharp as one of the originators. This is an equilibrium model that states the relationship between the expected return and the risk of the security. The starting point behind CAPM is that if all investors on the market have the same expectations, called *homogeneous* expectations, when it comes to expected return and risk of the stocks and the correlations between the stocks, then they will all choose the same stock portfolio. This means that they will all own shares of the market portfolio, the most diversified portfolio, i.e. all stocks and securities in the market. The risk in the market portfolio, the market risk, is necessary but the specific risk of the individual stocks can be avoided by diversification and this is the reason for choosing this portfolio. Another main assumption underlying CAPM is that investors trade securities at competitive market prices and can borrow and lend at the risk-free rate. When all CAPM assumptions hold, the optimal portfolio is a combination of the market portfolio and the risk-free investment. For more information see [1], [4] and [5].

2.4 Index

The principle behind CAPM says that you should hold a value-weighted portfolio of all risky securities in the market, e.g. the market portfolio. The question is then how to achieve this in practice. In reality there are several popular market indexes that try to match the performance of a particular stock market. The most famous and oldest stock index in the United States is the Dow Jones Industrial Average, DJIA, a portfolio of 30 large industrial stocks. DJIA is a price-weighted portfolio which holds an equal number of shares of each stock, independent of their size. Even if this portfolio represents different sectors of the economy it is obviously not a perfect imitation of the entire market.

The most used portfolios today are value-weighted, also called capitalization-weighted, where every component is weighted according to the total market value of their outstanding shares. During the late nineties most companies providing indexes started to adjust their indexes for stocks where a significant portion of the shares outstanding is not available to investors. These indexes are sometimes called free floated adjusted portfolios, which reflects an index with shares available, [4].

A better replication of the U.S. stock market compared to the DJIA is the familiar S&P 500, a value-weighted and free floated adjusted portfolio containing of the 500 largest U.S stocks. The S&P 500 has become a benchmark for professional investors and is the most used index when representing the U.S stock market. Even if the S&P 500 only contains of 500 stocks out of more than 7000 U.S stocks it holds the largest stocks which makes it a good representation, [14]. This is the reason for this portfolio to be presented as the standard representation of the market when it comes to make practice of the CAPM. There is of course indexes with a larger amount of stocks that is more representative for the market but they do not always share the popularity of the smaller ones because they often accurse to perform with similar returns.

These indexes are of course not known as the market portfolio rather as a market proxy, which investors do believe tracks the true market portfolio, [1].

Another method to select indexes is called equally-weighted which means that share quantities for each of the stocks in the index are determined as if one were buying an equal amount of each stock in the index, [4].

2.5 Index Tracking

There are two types of methods to replicate these indexes, the first is full replication, which invest in the same set and proportion of stocks as the index. The second one is called partial replication and is created with a subset of the stocks of the index. These two types of passive portfolio management are called index tracking. The partial replication method, does not give a perfect match of the index, but has the advantages of reducing the transaction costs and simplifying the rebalance of the portfolio. The transaction cost is the difference between bid and ask price on the stock, which is what the investor most pay in order to trade. The customer buy at the ask price and sell at the bid price and the ask price always exceed the bid price, for more information see [1].

To find the optimal set of portfolio weights in the partial replication portfolio it is common to minimize the squared tracking error between the index and the portfolio, [11].

2.6 Model Predictive Control, MPC

From the principle of optimality, Bellman, 1957, E.B Lee and L. Markus, 1967, came to the conclusion that *"One technique for obtaining a feedback controller synthesis from knowledge of open-loop controllers is to measure the current control process state and then compute very rapidly for the open-loop control function. The first portion of this function is then used during a short time interval, after which a new measurement of the process state is made and a new open-loop control function is computed for this new measurement. The procedure is then repeated."* This was the beginning of the idea behind Model Predictive Control. A few years later after the conclusion of Lee and Markus MPC was developed for process control and since then it has been widely applied in petro-chemical and related industries. From the beginning, the developed MPC did not always ensure stability, but today a lot of research on the subject has been made. Nowadays many stabilized settings of MPC can be seen, only differing in their choice of objective function, states and control actions, which almost always can be found in the problem setup.

Model Predictive Control, also called Receding Horizon Control (RHC), is a control method where the current control actions are obtained by solving an optimization problem on-line, at each time step. The optimization problem is a finite horizon open-loop optimal control problem, where the current state of the problem is used as the initial state, and this optimization method gives a sequence of control actions along the chosen horizon. The idea behind MPC is then to only use the first control action in this sequence and to repeat the procedure for the next time step. An important advantage with MPC is its ability to handle multivariable control problems with many constraints on the control actions and the states which make it attractive in industrial applications, information can be found in [7] and [8].

The model to be controlled is described by an ordinary difference equation which describes how the states are updated in time:

$$x(t+1) = f\big(x(t), u(t)\big)$$

where x is the state of the system and u is the control actions. In addition, the state and the control actions are required to satisfy:

$$x(t) \epsilon X$$

$$u(t) \epsilon U$$

where X and U are convex, closed subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

To formulate an optimization problem a stage cost needs to be introduced, which measure the cost in each time step:

$$\ell(x(t), u(t))$$

The cost function can then be stated as the sum of all stage costs. The optimization problem which is to be solved in each time instance is:

$$\min \sum_{t=t_0}^{t_0+N-1} \ell\big(x(t), u(t)\big)$$

subject to

$$x(t+1) = f\big(x(t), u(t)\big) \qquad (1)$$

$$x(t) \epsilon X$$

$$u(t) \epsilon U$$

If $f\big(x(t), u(t)\big)$ is linear, $\ell(x(t),u(t))$ is a convex function then (1) will be a convex optimization problem, which can be solved efficiently.

2.7 Method description and model setup according to Primbs

In [10] a discrete time linear systems with multiplicative noise is considered. Hence, the dynamic equation that is considered in this thesis is given by:

$$x(t+1) = Ax(t) + Bu(t) + \sum_{j=1}^{q} \left(C_j x(t) + D_j u(t)\right) w_j(t) \quad (2)$$

where $x(t)$ is the state at time t, $u(t)$ is the control action at time t, and the iid random variables $w_j(t)$, $j = 1, \dots q$, at time t with $E\left(w_j(t)\right) = 0$, $E\left(w_j(t) w_l(t)\right) = 0$, $j \neq l$, and $E\left(w_j(t) w_j(t)\right) = 1$. This model is suitable to describe how different stocks changes over time and fits the purpose of the paper which is to select the control actions $u(k)$, the weights, to optimize some performance criterion while satisfying certain constraints. Below, an outline of how to go form (2) to a convex optimization problem is given, details can be found in [10].

Having noise in the model does not exactly fit the general framework of MPC, where a deterministic dynamic system is considered. To have a deterministic model, instead the expected value and control actions of the system (2) will be considered.

$$\bar{x}(t+1) = f\left(\bar{x}(t), \bar{u}(t)\right) = A\bar{x}(t) + B\bar{u}(t)$$

where $\bar{x} = E(x)$ and $\bar{u} = E(u)$.

To be able to punish variance of states and control actions, the variance dynamics is also introduced. The variance is defined as $\Sigma(\mathrm{t}) = E\left(\hat{x}(t)\hat{x}^T(t)\right)$ where $\hat{x} = x - \bar{x}$ and $\hat{u} = u - \bar{u}$. Assuming that $\hat{u} = K\hat{x}$:

$$u(t) = \bar{u}(t) + K(t)(x(t) - \bar{x}(t))$$

$$\hat{x}(t+1) = \left(A + BK(t)\right)\hat{x}(t) + \sum_{j=1}^{q} \left(\left(C_j + D_j K(t)\right)\hat{x}(t) + C_j \bar{x}(t) + D_j \bar{u}(t)\right) w_j(t)$$

which gives the dynamics of the covariance matrix:

$$\begin{aligned} \Sigma(t+1) = {} & \left(A + BK(t)\right)\Sigma(t)\left(A + BK(t)\right)^T \\ & + \sum_{j=1}^{q} \left(C_j + D_j K(t)\right)\Sigma(t)\left(C_j + D_j K(t)\right)^T \\ & + \sum_{j=1}^{q} (C_j \bar{x}(t) + D_j \bar{u}(t))(C_j \bar{x}(t) + D_j \bar{u}(t))^T \end{aligned}$$

Introducing $U(t) = K(t)\Sigma(t)$ and using Schur complements, this dynamic equation can be formulated as an LMI (Linear Matrix Inequality).

Using Σ(t), following matrices can be introduced. These matrices will prove to be essential when formulated both the Risk adjusted wealth and the Index tracking:

$$S(t) = E\left(\begin{bmatrix}\hat{x}(t)\\ \hat{u}(t)\end{bmatrix}\begin{bmatrix}\hat{x}(t)\\ \hat{u}(t)\end{bmatrix}^T\right)$$

$$P(t) = E\left(\begin{bmatrix}x(t)\\ u(t)\end{bmatrix}\begin{bmatrix}x(t)\\ u(t)\end{bmatrix}^T\right)$$

Finally the cost function is defined as:

$$\ell\big(\bar{x}(t), \bar{u}(t), S(t), P(t)\big) = g^T\begin{bmatrix}\bar{x}(t)\\ \bar{u}(t)\end{bmatrix} - Tr\big(MP(t)\big) - Tr\left(\widehat{M}S(t)\right)$$

$$+\phi^T\bar{x}(N) - Tr\left(\Phi^{'}\mathrm{P(N)}\right) - \mathrm{Tr}\left(\widehat{\Phi}^{'}\mathrm{S(N)}\right) \quad (3)$$

with $M,\ \widehat{M},\ \Phi,\ \widehat{\Phi} \succcurlyeq 0$, where $\succcurlyeq$ means that the matrix is positive-semidefinite. The notations $\Phi^{'}$ and $\widehat{\Phi}^{'}$ are Φ and $\widehat{\Phi}$ augmented by zeros to the size $(n+m)\times(n+m)$.

The receding horizon on-line optimization problem is then formulated in its final form (where * denotes the symmetrical reflection of the opposite matrix element):

$$\max\sum_{t=0}^{N-1}\ell\big(\bar{x}(t), \bar{u}(t), S(t), P(t)\big)$$

subject to:

$$\bar{x}(t+1) = A\bar{x}(t) + B\bar{u}(t),\ \text{for t} = 1, \dots, \mathrm{N}-1 \text{ and } \bar{x}(0) = x_0$$

$$\begin{bmatrix}\Sigma(t+1) & * & * & *\\ (A\Sigma(t)+BU(t))^T & \Sigma(t) & * & *\\ (C\Sigma(t)+DU(t))^T & 0 & \Sigma(t) & *\\ (C\bar{x}(t)+D\bar{u}(t))^T & 0 & 0 & 1\end{bmatrix} \succcurlyeq 0, \qquad \text{for t} = 1, .., \mathrm{N}-1$$

$$\begin{bmatrix}\Sigma(1) & *\\ (Cx(0)+D\bar{u}(0))^T & 1\end{bmatrix} \succcurlyeq 0$$

$$\begin{bmatrix} P(t) & * & * \\ [\Sigma(t) \quad U^T(t)] & \Sigma(t) & * \\ [\bar{x}^T(t) \quad \bar{u}^T(t)] & 0 & 1 \end{bmatrix} \succcurlyeq 0, \quad \text{for t} = 0,..,\text{N} - 1$$

$$\begin{bmatrix} S(t) & * \\ [\Sigma(t) \quad U(t)^T] & \Sigma(t) \end{bmatrix} \succcurlyeq 0, \quad \text{for t} = 0,..,\text{N} - 1$$

$$Tr\big(H_r P(t)\big) + Tr\left(\widehat{H}_r S(t)\right) + h_r^T \begin{bmatrix} \bar{x}(t) \\ \bar{u}(t) \end{bmatrix} \leq \beta_r \quad (4)$$

where $H_r, \widehat{H}_r \succcurlyeq 0, r = 1, \dots R,$ R is the number of added constraints. The constraints that only involve the state are imposed from $t = 1, \dots N$ and the other constraints are imposed from $t = 0, \dots N-1$. The constraints (4) can be used to form a range of different specific constraints, one example of this is found in next section 2.7.1.

2.7.1 Probabilistic constraints

Constraints of the type $P(X \leq c) \geq 1 - d,$ where $X = a^T x + b^T u$, will prove to be valuable when doing the portfolio optimization. These constraints will be referred to as probabilistic constraints. Next it will be shown that these constraints can with some approximation, be reduced to constraints in the form of (4).

First to be able to use this type of constraints the probability distribution of *X* has to be known. The constraint is rewritten as:

$P\left(\frac{X-m}{\sigma} \leq \frac{c-m}{\sigma}\right) \geq 1 - d$ where $m = E(X)$ and $\sigma^2 = E((X-m)^2)$.

By approximating *X* to be Gaussian, the constraint can be formulated as $N\left(\frac{c-m}{\sigma}\right) \geq 1 - d$ which is equivalent to the expression $(c-m)^2 \geq (N^{-1}(1-d))^2 \sigma^2$. Here $N(\cdot)$ is the cumulative distribution function of the Gaussian distribution, for simplicity, denote $\gamma = (N^{-1}(1-d))^2$.

This constraint is not convex in *m* and σ^2 and hence a linearization is performed around a selected point m_0. This point can be selected as previous predicted mean of *X*. After the linearization is performed the constraint is convex in *m* and σ^2 and can be formulated as:

$$(c - m_0)^2 - 2(c - m_0)(m - m_0) \geq \gamma \sigma^2$$

As $m = a^T \bar{x} + b^T \bar{u}$ and $\sigma^2 = E\left(\begin{bmatrix} \hat{x} \\ \hat{u} \end{bmatrix}^T \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} \hat{x} \\ \hat{u} \end{bmatrix}\right) = Tr\left(\begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}^T S(t)\right)$ the constraint above can now be included in the Receding Horizon Control Approach since it fits in the for of equation (4).

2.7.2 Portfolio optimization problems

Many portfolio optimization problems can be formulated as the general linear system with state and control multiplicative noises as the one described in previous section. To be able to formulate the two portfolio optimization problems that is later evaluated the following section involves important variables and their setups.

A risk free asset exists, with a risk free rate of r_f per period. The prices of l stocks can be described as:

$$S_i(t+1) = \big(1 + \mu_i + w_i(t)\big)S_i(t) \quad i = 1, \dots l$$

where μ_i is the expected return per period, $w = \begin{bmatrix} w_1 \\ \vdots \\ w_l \end{bmatrix}$ and $w_i(t)$ is the iid driving noise term with $E(w_i) = 0$ and covariance matrix $E(\overline{w}\overline{w}^T) = \mathrm{V}$ per period.

We formulate the wealth dynamics of the portfolio as follows:

$$W(t+1) = \big(1 + r_f\big)W(t) + \sum_{i=1}^{l} \Big(\mu_i - r_f + w_i(t)\Big)\, u_i(t) =$$

$$\big(1 + r_f\big)(W(t) - \sum_{i=1}^{l} u_i(t)) + \sum_{i=1}^{l} \big(1 + \mu_i + w_i(t)\big)u_i(t)$$

where u_i is the amount invested in asset *i* at time *t*. The wealth dynamics can be separated into two parts, one risk free investment and one risky investment:

$$\text{risk free investment: } \big(1 + r_f\big)(W(t) - \sum_{i=1}^{l} u_i(t))$$

$$\text{risky investment: } \sum_{i=1}^{l} \big(1 + \mu_i + w_i(t)\big)u_i(t)$$

Chapter 3

3. Experiment

3.1 Numerical data from the article by Primbs

The numerical examples in [10] is based on five stocks, $S_1 = IBM$, $S_2 = 3M$, $S_3 = Altria$, $S_4 = Boeing$ and $S_5 = AIG$. The expected return and covariance matrix for these five stocks has been calculated out of fifteen years of weekly data from Yahoo! finance and is presented below as a yearly mean and yearly covariance matrix:

$$\mu = \begin{bmatrix} 0.0916 \\ 0.1182 \\ 0.1462 \\ 0.0924 \\ 0.1486 \end{bmatrix}$$

$$V = \begin{bmatrix} 0.09401 & 0.01374 & 0.01452 & 0.01237 & 0.01838 \\ 0.01374 & 0.05062 & 0.01475 & 0.02734 & 0.02200 \\ 0.01452 & 0.01475 & 0.09017 & 0.01029 & 0.01286 \\ 0.01237 & 0.02734 & 0.01029 & 0.09922 & 0.02674 \\ 0.01838 & 0.02200 & 0.01286 & 0.02674 & 0.07318 \end{bmatrix}$$

The following examples, presented in the article, are using a risk free rate of $r_f = 0.05$, initial wealth of $W(0) = \$100$, initial prices of all stocks of $S_i(0) = \$100$ and with a period of 1 month, $\Delta t = 1/12$, with all parameters adjusted accordingly.

3.2 Risk adjusted wealth maximization problem

The first portfolio problem in the article is based on the Markowitz theory, where maximization is performed on the risk adjusted wealth of a portfolio under given risk and allocation constraints. This problem has one state, the portfolio wealth, and it is allowed to invest in all stocks. The receding horizon performance objective is then given by:

$$\text{Risk adjusted wealth maximization problem: } \max \sum_{t=1}^{N} E\big(W(t)\big) - \lambda Var\big(W(t)\big) \quad (5)$$

The first numerical result is solved for a single instance of the receding horizon on-line optimization problem. Therefore, the initial control action from this optimization gives the receding horizon control action at the current state. When a horizon of $N = 5$ and a relatively small λ is used as in the article, the problem gets extremely aggressive control actions because the risk has a small affect on the objective function and the problem turns towards an expected wealth maximization. The effects of choosing different λ will be investigated further later on.

This example creates a portfolio with the goal to maximize a risk adjusted measure of the wealth of the portfolio as previously stated. Constants for the Risk adjusted wealth maximization example are:

State model constants according to the state dynamic (2), with the wealth as its only state:

$$A = 1 + r_f$$

$$B = [\mu_1 - r_f \quad \mu_2 - r_f \quad \mu_3 - r_f \quad \mu_4 - r_f \quad \mu_5 - r_f]$$

$$C = 0$$

To find the constant *D* for the Risk adjusted wealth maximization example an adjustment is made on the random variable *w* to fit (2). Let $v = Zw$, then $E(v) = 0$ and $E(vv^T) = E(Zww^T Z^T) = \mathrm{ZVZ}$. If $E(vv^T)$ =I this gives $Z = \mathrm{V}^{-1/2}$.

$$\sum_{j=1}^{q} D_j u v_j = \sum_{j=1}^{q} u w_j = u^T w = u^T \mathrm{V}^{1/2} v = (\mathrm{V}^{1/2} u)^T v = \sum_{j=1}^{q} [\mathrm{V}^{1/2}]_j u v_j \Rightarrow$$

$$D_j = [\mathrm{V}^{1/2}]_j, \quad \mathrm{j} = 1,..,5 \qquad (6)$$

where $[\mathrm{V}^{1/2}]_j$ is a row of the matrix $\mathrm{V}^{1/2}$.

The constants of the cost function (3) for the Risk adjusted wealth maximization problem are:

$$g = \begin{bmatrix} 1 \\ 0^{5\times1} \end{bmatrix}, \widehat{\mathrm{M}} = \begin{bmatrix} \lambda & 0^{1\times5} \\ 0^{5\times1} & 0^{5\times5} \end{bmatrix}, \phi = 1, \widehat{\Phi} = \lambda, \mathrm{M} = \Phi = 0$$

3.3. Index tracking example

The second problem that was studied is that of optimally tracking an index of stocks with a subset of those stocks. This example uses an index that is an equally weighted portfolio of the 5 stocks from above and the problem is based on tracking this index with the first three stocks; *IBM*, *3M* and *Altria*. The receding horizon objective in this case will be to minimize the expected squared error between the index and wealth:

$$\text{Index tracking problem: min } E\left(\sum_{t=1}^{N}\big(I(t)-W(t)\big)^2\right) \qquad (7)$$

$$S_i(t+1) = \big(1+\mu_i+w_i(t)\big)S_i(t),\;\; i = 1 \ldots q$$

$$W(t+1) = \big(1+r_f\big)W(t) + \sum_{i=1}^{l}\Big(\mu_i - r_f + w_i(t)\Big)\,u_i(t)$$

$$I(t) = \sum_{i=1}^{q} \alpha_i S_i(t)$$

α_i index weights (in this case the weights are equal)

In this example the model is built on six states, the wealth and the five stock prices. The following constants $q = 5,\; l = 3,\; N = 5$ and $\alpha_i = 0.2$ are being used, and the initial value of the index are, according to the function above, \$100.

State model constants according to the state dynamic (2), with the states defined as the wealth and the stock prices, are presented below:

$$A = \begin{bmatrix} 1+r_f & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+\mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+\mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\mu_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\mu_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\mu_5 \end{bmatrix}$$

$$B = \begin{bmatrix} \mu_1 - r_f & \mu_2 - r_f & \mu_3 - r_f \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To find the constants *C* for the Index tracking example an adjustment is made on the random variable *w* to fit (2). As previously stated in (6) $Z = \mathrm{V}^{-1/2}$. The adjustment below is only considering the stock prices.

$$C_j S_i v_j = S_i w_j = S_i {V^{1/2}}_i v_j = \sum_{j=1}^{q} S_i {V^{1/2}}_{ij} v_j = S_i \sum_{j=1}^{q} {V^{1/2}}_{ij} v_j, \qquad \text{for i} = 1,..,5$$

This adjustment handles one row of element from $V^{1/2}$ in every C_j-matrix and then places these elements diagonally so that each stock price is multiplied with its corresponding element in the row $[V^{1/2}]_j$.

$$C_j = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & [V^{1/2}]_{j1} & 0 & 0 & 0 & 0 \\ 0 & 0 & [V^{1/2}]_{j2} & 0 & 0 & 0 \\ 0 & 0 & 0 & [V^{1/2}]_{j3} & 0 & 0 \\ 0 & 0 & 0 & 0 & [V^{1/2}]_{j4} & 0 \\ 0 & 0 & 0 & 0 & 0 & [V^{1/2}]_{j5} \end{bmatrix}, \text{j} = 1,..,5$$

To find the constants *D* for the Index tracking example an adjustment is made according to (6):

$$D_j = \begin{bmatrix} [V^{1/2}]_{j1} & [V^{1/2}]_{j2} & [V^{1/2}]_{j3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{for j} = 1,..,3 \;\; D_j = 0 \text{ for j} = 4,5$$

Next are the constants of the cost function (3) presented, here is a negative total cost function used to find the constants since this is a minimization problem and the original problem is set to be maximized.

$$M = \begin{bmatrix} -1 \\ \alpha \\ 0^{3\times1} \end{bmatrix} \begin{bmatrix} -1 & \alpha^T & 0^{1\times3} \end{bmatrix} \quad \text{where } \alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_5 \end{bmatrix}$$

$$\Phi = \begin{bmatrix} -1 \\ \alpha \end{bmatrix} \begin{bmatrix} -1 & \alpha^T \end{bmatrix}$$

$$g = \hat{M} = \phi = \hat{\Phi} = 0$$

3.4 Data from Danske Bank

Three historical index tracking portfolios and indexes were studied with data provided by Danske Bank. The first index that was studied was the Amex Biotechnology. The Amex Biotechnology index is designed to measure the performance of a cross section of companies in the biotechnology industry in the U.S. This index is equal-dollar weighted and is rebalanced four times per year to keep stocks from dominating the value of the index. The Amex Biotechnology index only contains 20 firms today and began on October 18, 1991.

The first mutual fund that was evaluated was the DI Biotechnology and it contains almost all the stocks in the index Amex Biotechnology, which is changing over the time period. The historical data from November 9th 2007 until November 9th 2009 was studied where extra cash flows has been added to or subtracted from the portfolio.

The mutual fund, DI Biotechnology, and the index, Amex Biotechnology, do not contain the same amount of money. Therefore, to evaluate the performance of the index tracking portfolio obtained by Danske Bank, the returns of this mutual fund, DI Biotechnology, and the index, Amex Biotechnology, are being compared.

When studying the real portfolios and comparing them to the indexes it can be seen that the tracking is made by minimizing the difference between the return of the portfolio and the return of the benchmark over time. This result in a close tracking that can be seen in Figure 3.1. A better and reality-based way of comparing the historical portfolio data to the historical index data is to calculate the normalized value of the two data sets. Figure 3.2 represent the two values with an initial value of 100 and then respectively added returns over the time period of November 9th 2007 until November 9th 2009.

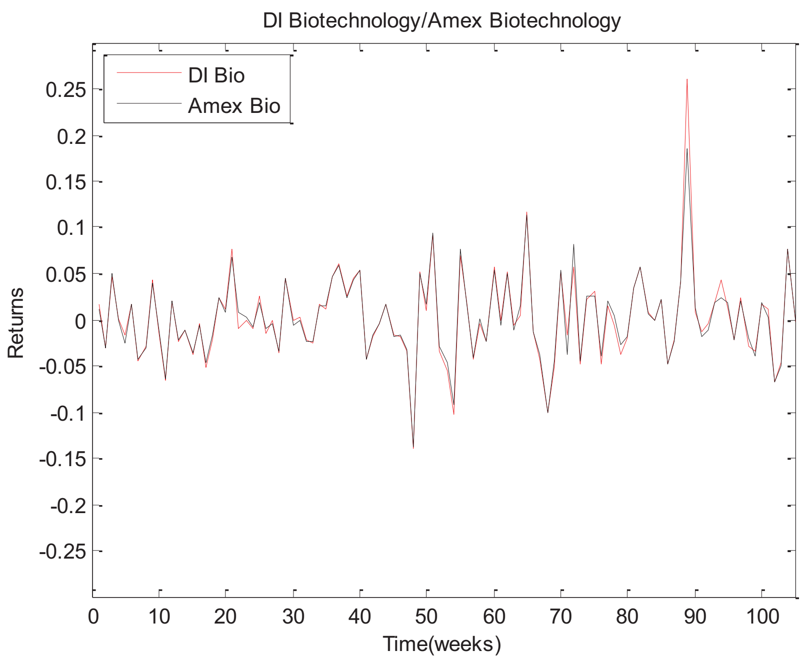

Figure 3.1. The return of the portfolio DI Biotechnology and the index Amex Biotechnology from November 9th 2007 until November 9th 2009.

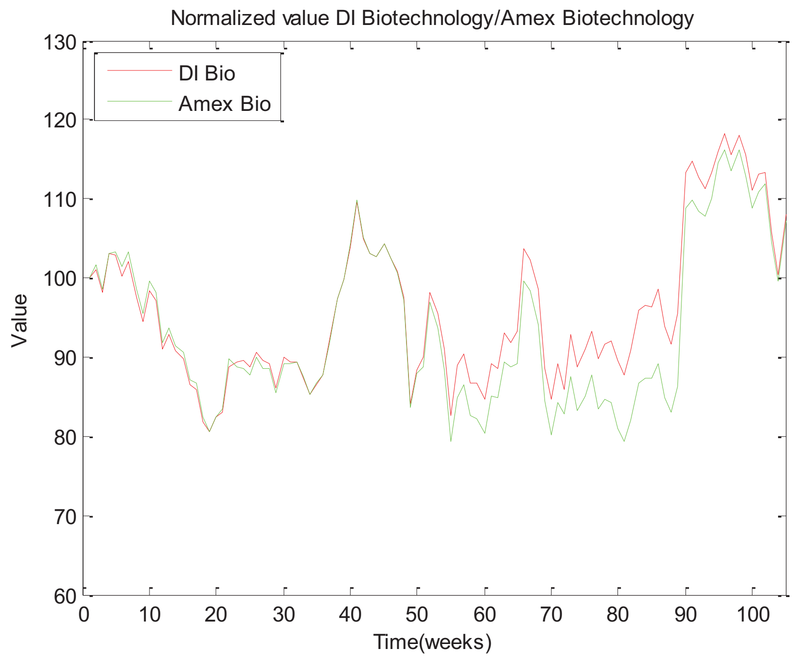

Figure 3.2. The normalized value of the portfolio DI Biotechnology and the index Amex Biotechnology from November 9th 2007 until November 9th 2009.

The second index is the MSCI World index, Morgan Stanley Capital international World index, which is a free-float weighted equity index. The MSCI World index was developed in 1969 and includes 23 developed world markets, without any emerging markets, for more information see [13].

The second mutual fund that was studied was the BGI PAL Verden Valutasikret which is tracking the MSCI World index with a large subset of the approximately 1000 stocks in the index. The historical data for the BGI PAL Verden Valutasikret from November 9th 2007 until November 9th 2009 was studied where extra cash flows has been added to or subtracted from the portfolio.

The mutual fund, BGI PAL Verden Valutasikret, and the index, MSCI World, do not contain the same value. Therefore, to evaluate the performance of this second index tracking portfolio obtained by Danske Bank, the returns of the mutual fund and the index are compared in Figure 3.3 below.

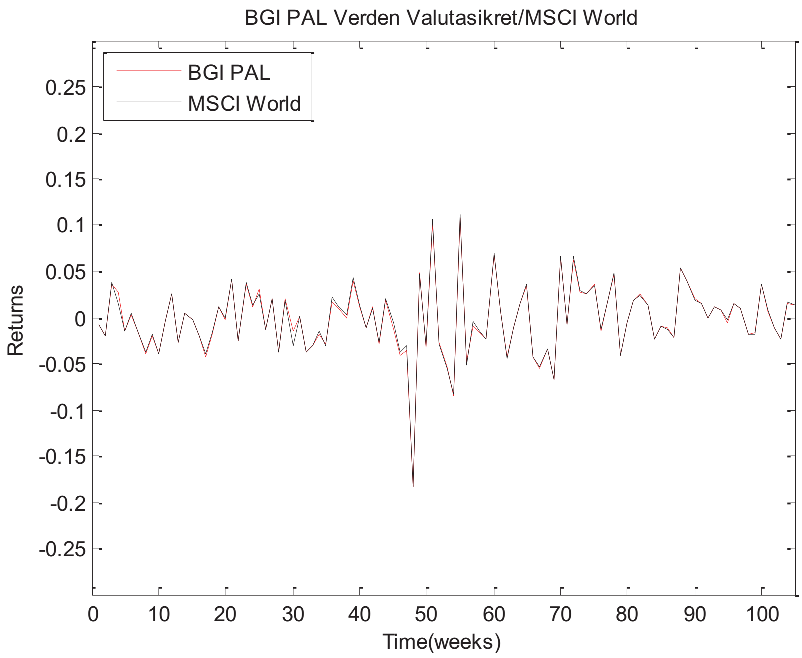

Figure 3.3. The return of the portfolio BGI PAL Verden Valutasikret and the index MSCI World from November 9th 2007 until November 9th 2009.

Just as for previous historical data set a comparison of the historical portfolio value and the historical index data was performed, with a normalized value of the two data sets. Figure 3.4 represent the two values with an initial value of 100 and then respectively added returns over the time period of November 9th 2007 until November 9th 2009.

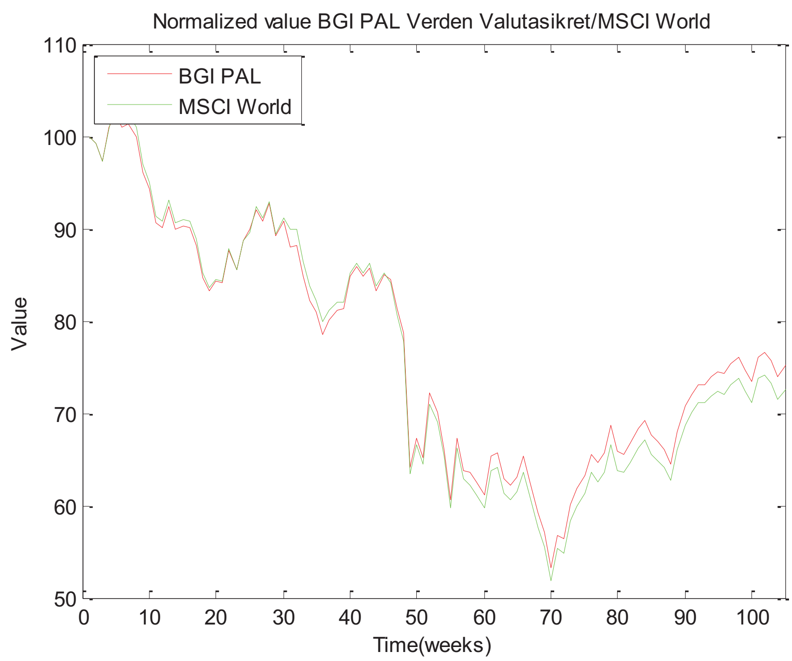

Figure 3.4. The normalized value of the portfolio BGI PAL Verden Valutasikret and the index MSCI World from November 9th 2007 until November 9th 2009.

The third and last historical portfolio, DI Verdensindeks, is tracking the same index as previous mutual fund, the MSCI World index.

Just as for previous historical data sets the mutual fund, DI Verdensindeks, and the index, MSCI World, do not contain the same value. Therefore, to evaluate the performance of this third index tracking portfolio obtained by Danske Bank, the returns of the mutual fund, DI Verdensindeks, and the index, MSCI World, are compared in Figure 3.5 below.

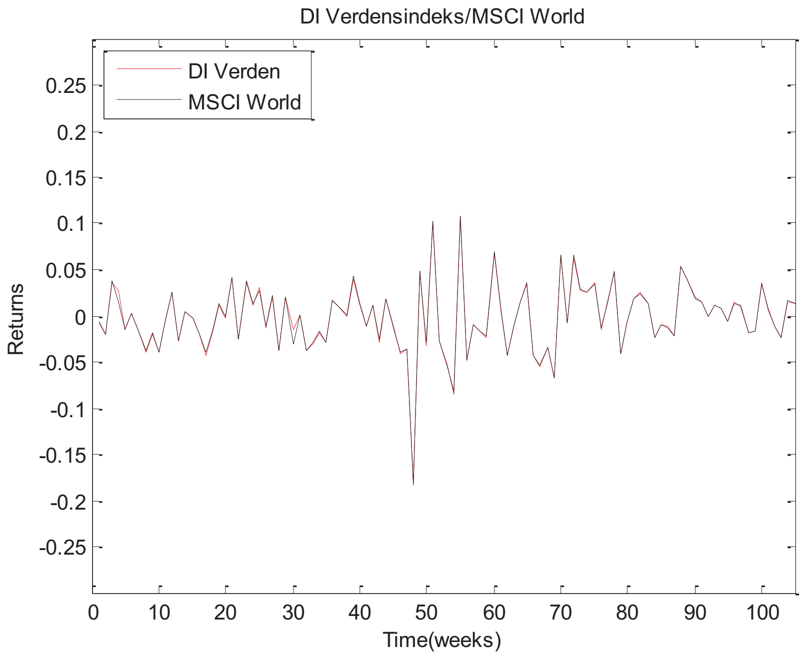

Figure 3.5. The return of the portfolio DI Verdensindeks and the index MSCI World from November 9th 2007 until November 9th 2009.

Just as for previous historical data sets a comparison of the historical portfolio value and the historical index data was performed, with a normalized value of the two data sets. Figure 3.6 represent the two values with an initial value of 100 and then respectively added returns over the time period of November 9th 2007 until November 9th 2009.

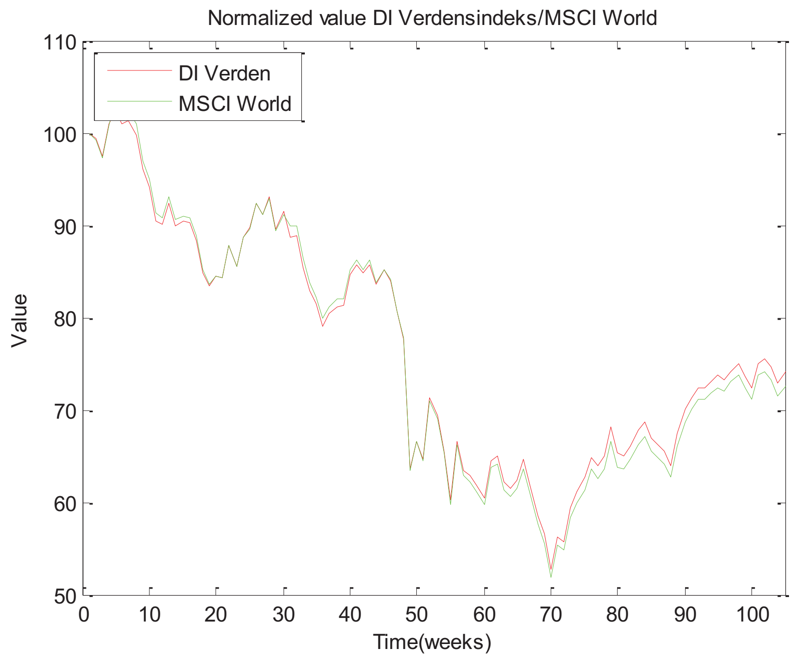

Figure 3.6. The normalized value of the portfolio DI Verdensindeks and the index MSCI World from November 9th 2007 until November 9th 2009.

The risk free rate has been approximated by a mean of the 10 year Danish government bond and the 3 months money market rate over the given time period. The mean was approximately 4 % and this is the rate that is later used in the optimization with the provided data from Danske Bank. Furthermore, Danske Bank is using some constraints to control their index tracking portfolios based on regulatory social restrictions and preferences on the size of the different stock weights.

The setup of the Index tracking problem in the article assumes equally weighted stocks in the index and the wealth of the portfolio, the index value and the stock prices assumes the same initial values. These assumptions work in the artificial example but is not very realistic. The index is rarely a mean of the prices of the containing stocks and the portfolio rarely has the same size as the stocks and should be able to hold any size of amount. To handle this problem a normalization of the portfolio wealth and the index value with its initial values in the objective function (7) is performed below.

$$E\left[\sum_{t=1}^{N}\left(I_{ny}(t)-W_{ny}(t)\right)^2\right]=E\left[\sum_{t=1}^{N}\left(\frac{100}{I(0)}I(t)-\frac{100}{W(0)}W(t)\right)^2\right] \quad (8)$$

The constants of the new cost function (3) are presented below:

$$M=\begin{bmatrix}\frac{100}{W(0)}\\ -\alpha\frac{100}{I(0)}\\ 0^{3\times1}\end{bmatrix}\begin{bmatrix}\frac{100}{W(0)} & -\alpha^T\frac{100}{I(0)} & 0^{1\times3}\end{bmatrix} \quad \text{where } \alpha=\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_q\end{bmatrix}$$

$$\Phi = \begin{bmatrix} \frac{100}{W(0)} \\ -\alpha \frac{100}{I(0)} \end{bmatrix} \begin{bmatrix} \frac{100}{W(0)} & -\alpha^T \frac{100}{I(0)} \end{bmatrix}$$

$$g = \widehat{M} = \phi = \widehat{\Phi} = 0$$

Another change of the previous setup is that the index weights vector, α, are updated with the index weights provided by Danske Bank instead of holding equal values during the whole optimization period. This will lead to a necessary update of the constants at each time step for the optimization problem. The same comes for the states, which instead of being simulated now are being updated with real returns from the provided data set.

Chapter 4

4. Result and analysis

4.1 Risk adjusted wealth maximization example

The first example that was implemented considers the Risk adjusted wealth maximization problem. The objective function (5), where the expected wealth is adjusted with the variance, is easy to understand and well connected to the underlying financial theory of Markowitz. This program may rebalance the portfolio in every time step, but this does not strictly mean that the portfolios need to be changed at every time step. The program is meant to be utilized as a passive portfolio strategy and is therefore in need of interpretation before rebalancing. For example, weights close to zero can be interpreted as no risky investment at that time, and small changes can in some cases depend on the price changes but in either case be ignored. Because transaction costs are not included in this model small changes can therefore lead to profit losses for the portfolio if the predicted profit of the changes is too small to compensate the transaction costs. This is a drawback of this method and could be included in further research.

How the size of λ, the level of risk, is affecting the Risk adjusted wealth maximization problem was first to be investigated. Different sizes of λ were chosen to see which one that gave an appropriate level of risk, seen Figure 4.1. In this first example, when the goal is to maximize the risk adjusted wealth, λ determines how the risk is weighted in the maximization. When λ is set to 10^{-3}, the problem goes strongly towards an expected wealth maximization problem with very little consideration of the variance. The result clearly indicates that when a smaller λ is used the wealth has large fluctuations. This shows that the risk is not highly considered in the choice of portfolio weights, which results in a very aggressive portfolio. In this case the wealth is negative during the time period which means that a negative amount, which in practice means to borrow money, is invested in the portfolio. When λ is chosen to 10^{-2} the behavior of the wealth is much calmer and shows a more risk averse portfolio. The largest λ that was evaluated, 10^{-1}, highly considers the risk in the Risk adjusted wealth maximization problem. In this case the portfolio almost exclusively invests in the risk free asset. Here is the wealth almost exclusively growing with the risk free rate of 5% per year and the weights in the risky investments are close to zero. λ set to 10^{-2} seems like the most appropriate choice because of its modest fluctuations. Some further values around 10^{-2} was therefore evaluated, these evaluations shows the same behavior as the previously discussed so these graphs are excluded from the result and λ is chosen to 10^{-2}. For all of these evaluations the same random vector was used to make the results easier to analyze.

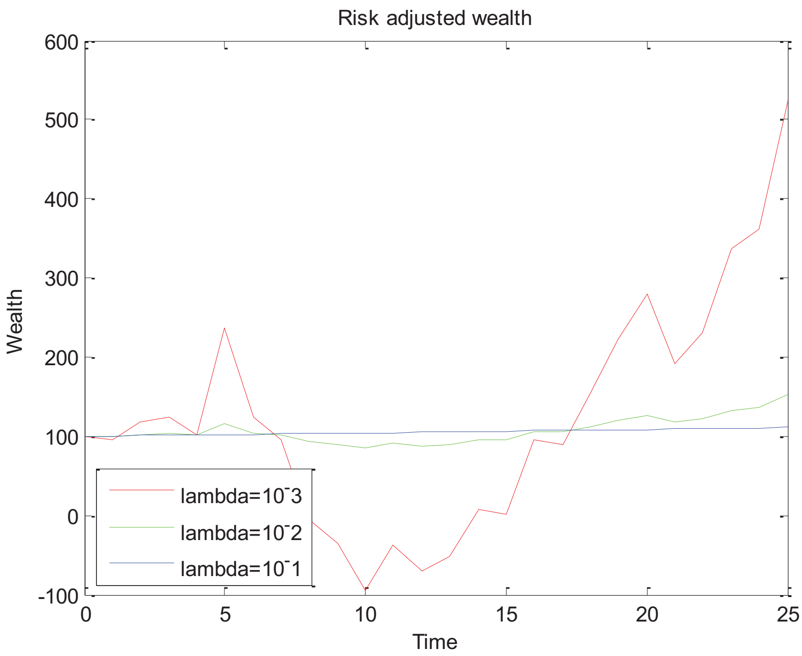

Figure 4.1. The portfolio value of the Risk adjusted wealth with different lambdas.

The first constraint to be evaluated puts a limit on a weight as a percentage of the wealth. This type of constraint can be used to diversify the portfolio and divide the risk on different stocks. Restrictions on the different weights were tried based on the historical variances and returns trying to keep down the weights on the stocks with low return and high variance. The linearization of the constraints is performed, as previously mentioned in section 2.7.1, around a selected point m_0. These points can either be selected as a constant, for example as $m_0 = 0$, or by a more sophisticated method such as the previous predicted mean of *X*. The implementation of this constraint turns out to, in some cases, not manage to handle the whole simulation period when the linearization is done with previous predicted mean of the states. The problems of handling this constraint occurs when the linearization point come close to the limit and thereby ignoring these limitations after a while. In section 2.7.1. the linearization in performed with the following setup $(c - m_0)^2 - 2(c - m_0)(m - m_0) \geq \gamma\sigma^2$. It is here obvious to see that if m_0 is close to the limit, c, the constraint is then being ignored. This problem is solved in these cases by keeping the linearization, on a small value away from the limit, to maintain these constraints. The program automatically chooses small weights of the stocks with low return and high variance. So therefore this constraint is more important when it comes to regulatory reasons where there is a need to keep a weight of a stock on a certain level. One of these regulatory reasons can be that no shortselling is allowed, which is the same as keeping the weights over or equal to zero, this can be seen in Figure 4.2. This was implemented as a non-probabilistic constraint. This graph shows that when the no shortselling restriction is added the weights changes to adjust to the constraint but since there was almost no shortselling before this gives a small difference in the wealth. The setup of constraints in this method allows all types of limits on the weights except for having a lower positive limit and at the same time allowing the weights to be zero. This is because a positive lower limit, with allowing the weights to be zero, will not lead to a convex constraint since the set is not convex which the method demands. In a convex set every pair of points in the set must be able to be connected in a straight line within the set. This is a disadvantage of the method because in many cases portfolio managers wants to exclude small weighs because of the transactions cost, in practice not to buy small quantities of a stock.

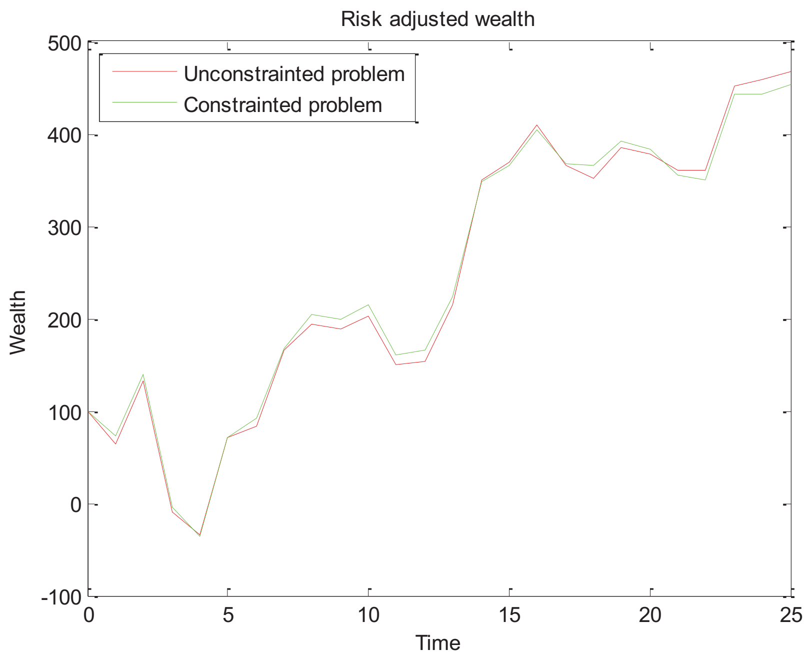

Figure 4.2. The portfolio value of the Risk adjusted wealth with and without allowing shortselling.

The second constraint that was implemented contains a limit on the wealth according to a percentage, delta, of the initial wealth, $P(W(t) \geq \delta(\$100)) \geq 0.95,\ \delta < 1$. This type of constraint sets a lower limit on the wealth which can be used to avoid big losses. To make the constraint reasonable delta has an upper limit of one. To evaluate this constraint a simulation over a time period of 25 months is performed. Delta of 0.7 gives a more aggressive portfolio since it allows the wealth to fluctuate more while delta of 0.9 is a safer approach and the delta of 0.8 in between. In Figure 4.3 it can clearly be seen that the bigger delta gives smaller losses. It is also seen that when the wealth is located further away from the limit the performance of both portfolios starts to look very similar to the risky unconstrained portfolio. In the opposite way, when the wealth is close to the limit the portfolio has less fluctuations. This constraint was implemented as a probabilistic constraint as seen in the description of the setup in the article, in section 2.7.1. Another way of implementing a constraint is in a non-probabilistic setup which says that the constraint should hold at all time. This might lead to an infeasible optimization problem in the next step if the wealth happens to become below the constraint.

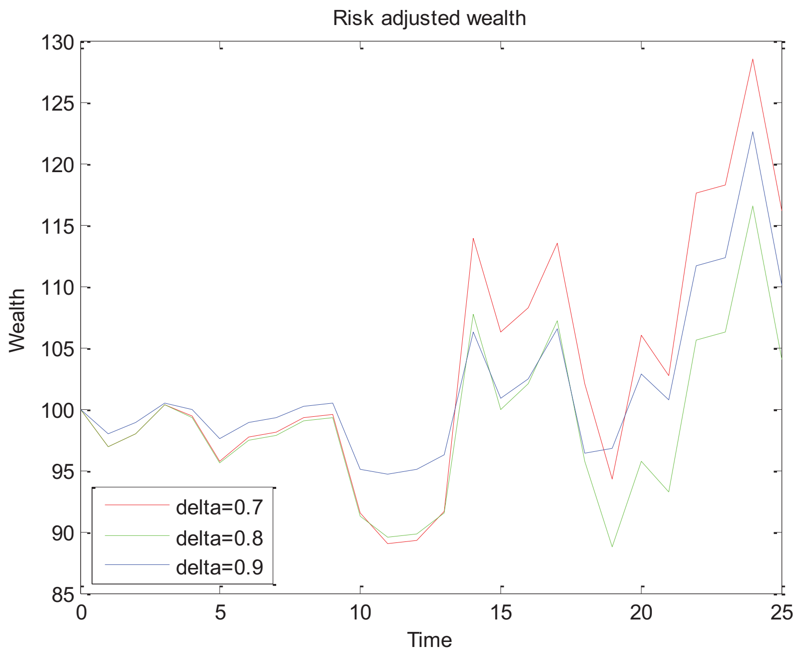

Figure 4.3. The portfolio value of the Risk adjusted wealth with different values of delta in the added constraint $P(W(t) \geq \delta(\$100)) \geq 0.95$.

To summarize the first example, it is calculating the receding horizon allocations successfully for most added constraints. In reality it is common to apply active management on this type of portfolios and this is the reason why the index tracking problem is further investigated.

4.2 Index tracking example

The second example, the Index tracking problem, where the objective is to minimize the squared tracking error (7), was evaluated next. This is a strategy that is used because of the background theory which is statistically proved, that over longer time periods actively managed portfolios rarely beats a well diversified index, see [1]. This makes the strategy very interesting in a passive portfolio management angle. The setup of the problem described in the article assumes equally weighted stocks in the index and that the initial wealth of the portfolio, the index and the stock prices are the same. This works in the artificial example, with the numerical data from the article, but is not very realistic because the portfolio is rarely the same size as the stocks. This problem will be discussed further in the case with real data.

The program solves the problem of Index tracking without constraints in a satisfying manner with a relative small tracking error, see Figure 4.4. To be able to diversify your portfolio or to meet with your preferences or regulations it is important to manage to control the stocks in the portfolio. To try this out a constraint that restricts the second control action is added to the problem. The constraint puts a limit on the second control action as a percentage of the wealth over time, $P(u_2(t) \leq 0.15W(t)) \geq 0.95$, see Figure 4.4 and Figure 4.5. This is more useful than to place a constant limit in reality since the weights then will be weighted in contrast to the wealth which is more relevant. As the figure shows, limits on the control actions increases the tracking error a little bit, and this is the cost you pay to control your assets.

Figure 4.4. The unconstrained portfolio wealth and the portfolio wealth and index value for the Index tracking example with the constraint $P(u_2(t) \leq 0.15W(t)) \geq 0.95$.

Figure 4.5. The portfolio weights of the Index tracking example with the constraint $P(u_2(t) \leq 0.15W(t)) \geq 0.95$.

Sometimes there are restrictions on not allowing shortselling in portfolios, which is equal to setting a constraint that do not allow negative allocations. It is structured as a strict nonprobablistic constraint for all three stocks in the portfolio. The stocks will be controlled to exclude negative values but, also in this case, you need to pay the price of getting larger tracking errors. In the unrestricted case there was a limited amount of shortselling so therefore the restricted portfolio value will not differ much from its unrestricted portfolio value.

To summarize the Index tracking example, to be able to use this in later analysis of real data, the results show that it is possible to track index with a relatively small tracking error. The problem can handle weight constraints with successful results. It is thereby not saying that these constraints produces better index tracking then the unconstrained problem. The tracking of the index is being countered by controlling the value of the portfolio because the movements of the wealth are then being prevented. Therefore the focus should lie on controlling the different assets and not restricting the wealth for best possible index tracking result.

4.3 Case with data provided by Danske Bank

The last and most important part of this thesis involves a case with real data from Danske Bank. Here the focus has been on the Index tracking example since this can be seen as a passive strategy and the program is suitable for that. When implementing the case with real data the index and wealth must be studied carefully. In the previous experiment with an equally weighted index, and stock prices and portfolio values in the same size this was not a problem. These assumptions are not very realistic since the index rarely is a mean of the stock prices as in that example. In this case the index is of a different size than the stock prices and the portfolio value should be able to assume any initial value. To handle this problem a normalization of the wealth and the index is performed in the objective function (8). The setup constants are also being updated during the optimization problem and the details of this can be seen in the end of the experiment section, Chapter 3. The provided data set had some missing values for about 10% of the index and this lack of information will lead to a little bit of a different appearance of the "new" reconstructed index.

When studying the real historical index tracking portfolios and their indexes in section 3.4 it is clearly seen that a larger selection of stocks gives more opportunities and thereby better tracking results. But because of the suspected problem with large systems, the smallest index given to us, Amex Biotechnology, is being investigated first. With this data the objective is to track this index with a new portfolio. The Amex Biotechnology index is being tracked with a subset of 5 stocks. The first result of this with a subset of 5 stocks can be seen in Figure 4.6.

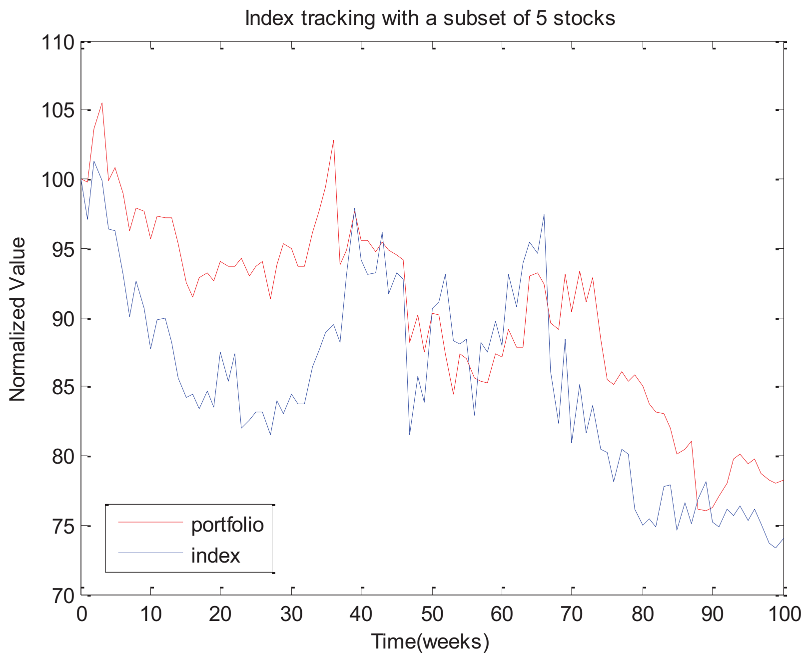

Figure 4.6. The portfolio value of the Index tracking problem, with real data and a subset of 5 stocks.

The tracking error is quite large in this example so a larger subset of 10 stocks is therefore evaluated in Figure 4.7.

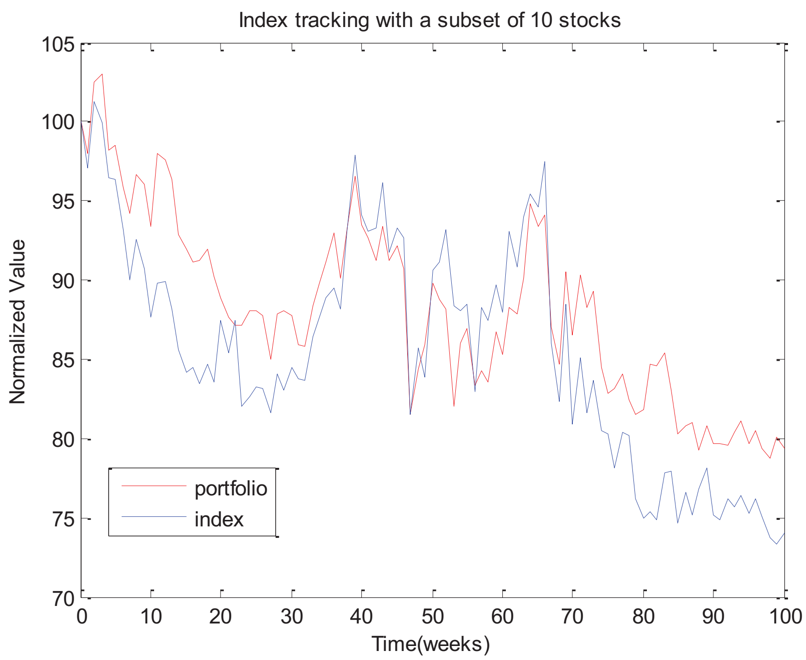

Figure 4.7. The portfolio value of the Index tracking problem, with real data and a subset of 10 stocks.

When using this larger subset to track the index an obvious decrease of the tracking error is seen. This result motivates an even larger subset, so a tracking using full replication is used, in other words all 17 stocks are included in the portfolio. The result of this can be seen in Figure 4.8.

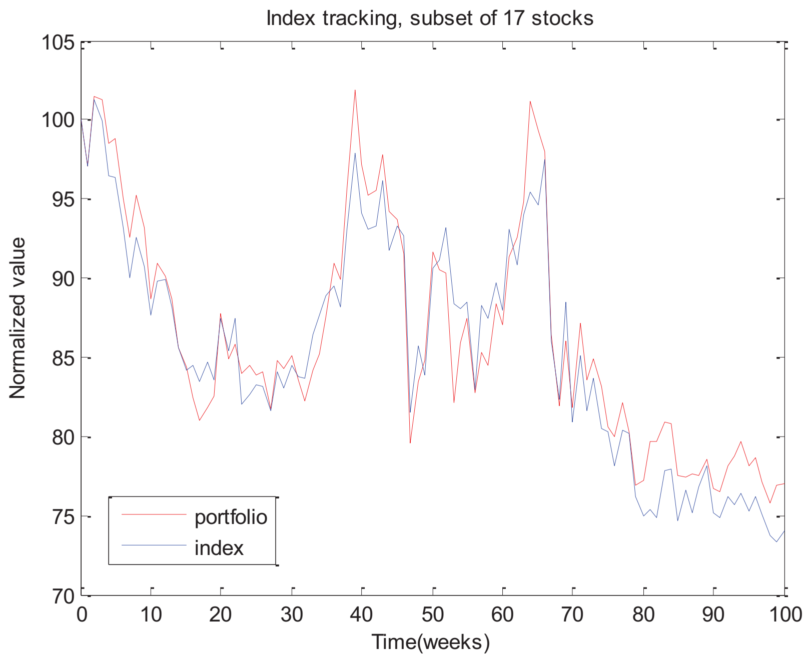

Figure 4.8. The portfolio value of the Index tracking problem, with real data and a subset of 17 stocks.

This, full replication, index tracking gives a satisfying result, but there are still some small tracking errors that can depend on the optimization constraints in the problem setup. This result is being compared with the graphic result of the tracking performed by Danske Bank, in Figure 3.2, which is using full tracking as well. The results are comparable with consideration of the simplicity of the passive management program that is evaluated. The risk free rate of 4 % seems reasonable in the figure above. Since the risk free rate change over time it could have given better results to change the rate in the model as well. This can especially be seen in the last 20 weeks were the real risk free rate is significantly smaller than 4 %.

To analyze the method further constraints are added to the problem. As discussed in the result and analysis of the previous experiment, in section 4.2, restrictions for the Index tracking problem should focus on restricting the control actions. A common restriction is to not allow shortselling in the portfolio, so therefore this non-probabilistic constraint was added to the problem. The result of adding this constraint to a subset of 10 stocks can be seen in Figure 4.9. This clearly shows that the tracking error increases when adding constraints. This increase in the tracking error depends on the restricted options of selecting the weights.

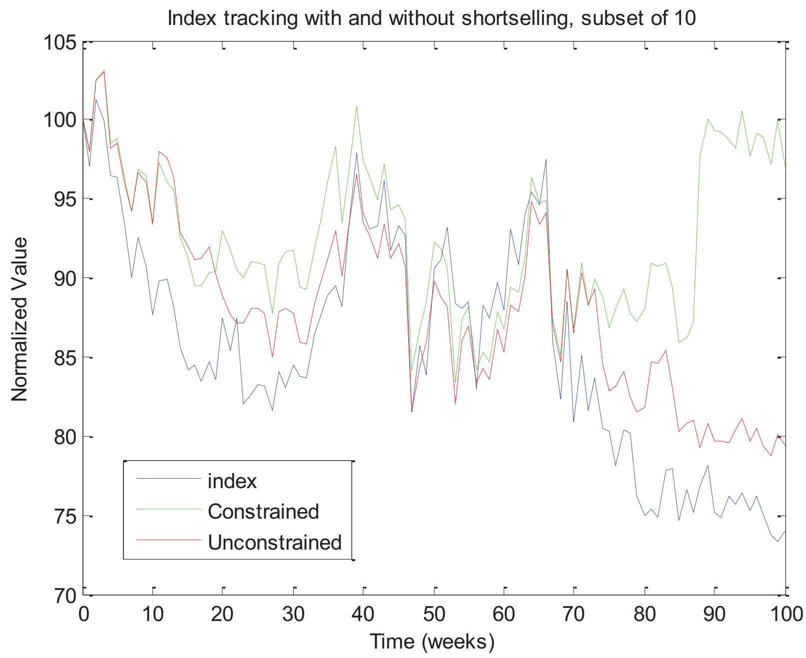

Figure 4.9. The portfolio value of the unconstrained and constrained Index tracking problem, with real data and a subset of 10 stocks, with not allowing shortselling.

As mentioned before, some problems with too much data were encountered when using SeDuMi, which is the reason why no further investigation with this method was performed on the additional significantly larger index. When going from the Risk adjusted wealth maximization problem to the Index tracking problem the state vector, of size *n*, will increase with the number of stocks in the index, *q*. This will then extend this multivariable index problem setup a lot. It is then obvious that with every added stock in the index, and thereby an increased state vector, the problem will drastically enlarge. One big contribution to the enlargement is the increasing size of the matrices containing the state vector. All matrices containing the states will be affected, but the absolute most critical factor will be the matrix of the covariance constraint which increases with the size of the index and the subset *l*, seen below:

$$\begin{bmatrix} \Sigma(t+1) & * & * & * \\ (A\Sigma(t)+BU(t))^T & \Sigma(t) & * & * \\ (C\Sigma(t)+DU(t))^T & 0 & \Sigma(t) & * \\ (C\bar{x}(t)+D\bar{u}(t))^T & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} (n\times n) & * & * & * \\ (n\times n) & (n\times n) & * & * \\ (n\times q)\times n & 0 & (n\times n)\times q & * \\ (l\times q)\times n & 0 & 0 & 1 \end{bmatrix}$$

The matrix of the covariance constraint will approximately have a size of $(n^2 \times n^2)$, since *q=n-1*. This means that an index of 29 stocks, with 30 states, will have a number of matrix inequalities where the matrices contain more than 900^2 elements. To exemplify how big this matrix will be with a larger index, an index of 999 stocks, with 1000 states, this gives a matrix size with more than 10^{12} elements.

To handle large portfolios the setup of the optimization problem must be simplified to avoid previously mentioned problems with too large matrices. The way this could be done is by representing the index as one state. The problem will then have two states, the wealth and the index instead of the wealth and all the stock prices in the index. The index is modeled as a stock and this is possible because the behavior of the index is very similar to a stocks behavior in the financial market.

$$I(t+1) = \big(1 + \mu_I + w_I(t)\big) I(t)$$

This will make the problem setup of large portfolios significantly simpler and manageable to compute and this can be seen in the problem setup below. The optimization problem in its final form is stated as before:

$$\max \sum_{t=0}^{N-1} f^T \begin{bmatrix} \bar{x}(t) \\ \bar{u}(t) \end{bmatrix} - Tr\big(MP(t)\big) - Tr\left(\widehat{M}S(t)\right) + \phi^T \bar{x}(N) - Tr\left(\Phi' \mathrm{P(N)}\right) - \mathrm{Tr}\left(\widehat{\Phi}' \mathrm{S(N)}\right)$$

subject to:

$$\bar{x}(t+1) = A\bar{x}(t) + B\bar{u}(t), \qquad for\ 0 \le t \le N-1 \text{ and } \bar{x}(0) = x_0$$

$$\begin{bmatrix} \Sigma(t+1) & * & * & * \\ (A\Sigma(t) + BU(t))^T & \Sigma(t) & * & * \\ (C\Sigma(t) + DU(t))^T & 0 & \Sigma(t) & * \\ (C\bar{x}(t) + D\bar{u}(t))^T & 0 & 0 & 1 \end{bmatrix} \succcurlyeq 0, \qquad \text{for t} = 1, .., \text{N} - 1$$

$$\begin{bmatrix} \Sigma(1) & * \\ (Cx(0) + D\bar{u}(0))^T & 1 \end{bmatrix} \succcurlyeq 0$$

$$\begin{bmatrix} P(t) & * & * \\ [\Sigma(t) \quad U^T(t)] & \Sigma(t) & * \\ [\bar{x}^T(t) \quad \bar{u}^T(t)] & 0 & 1 \end{bmatrix} \succcurlyeq 0, \qquad \text{for t} = 0, .., \text{N} - 1$$

$$\begin{bmatrix} S(t) & * \\ [\Sigma(t) \quad U(t)^T] & \Sigma(t) \end{bmatrix} \succcurlyeq 0, \qquad \text{for t} = 0, .., \text{N} - 1$$

The new constants of the cost function (3) for this new adjusted setup of the normalized Index tracking objective (8) are presented below:

$$M = \begin{bmatrix} \frac{100}{W(0)} \\ -\frac{100}{I(0)} \\ 0^{l\times 1} \end{bmatrix} \begin{bmatrix} \frac{100}{W(0)} & -\frac{100}{I(0)} & 0^{1\times l} \end{bmatrix}$$

$$\Phi = \begin{bmatrix} \frac{100}{W(0)} \\ -\frac{100}{I(0)} \end{bmatrix} \begin{bmatrix} \frac{100}{W(0)} & -\frac{100}{I(0)} \end{bmatrix}$$

$$g = \widehat{M} = \phi = \widehat{\Phi} = 0$$

This can be compared to the old constants of the cost function (3) for the normalized original Index tracking setup (8) which for comparison are presented below:

$$M = \begin{bmatrix} \frac{100}{W(0)} \\ -\alpha \frac{100}{I(0)} \\ 0^{3\times 1} \end{bmatrix} \begin{bmatrix} \frac{100}{W(0)} & -\alpha^T \frac{100}{I(0)} & 0^{1\times 3} \end{bmatrix} \quad \text{where } \alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_q \end{bmatrix}$$

$$\Phi = \begin{bmatrix} \frac{100}{W(0)} \\ -\alpha \frac{100}{I(0)} \end{bmatrix} \begin{bmatrix} \frac{100}{W(0)} & -\alpha^T \frac{100}{I(0)} \end{bmatrix}$$

$$g = \widehat{M} = \phi = \widehat{\Phi} = 0$$

This new simpler setup with the fact that the entire problem decreases with the decreasing state vector gives a significantly simpler problem. The new expected return is calculated as a mean of the weighted returns of the stock in the index over time. The covariance is now calculated between the selected stocks in the subset and the mean of the weighted returns of the stocks in the index at each time.

The tracking of MSCI World index with the new setup is first performed with a subset of 10 stocks, selected over the whole index, and the tracking error is very small, see Figure 4.10. This shows very promising results for tracking larger portfolios with a relative small subset of stocks.

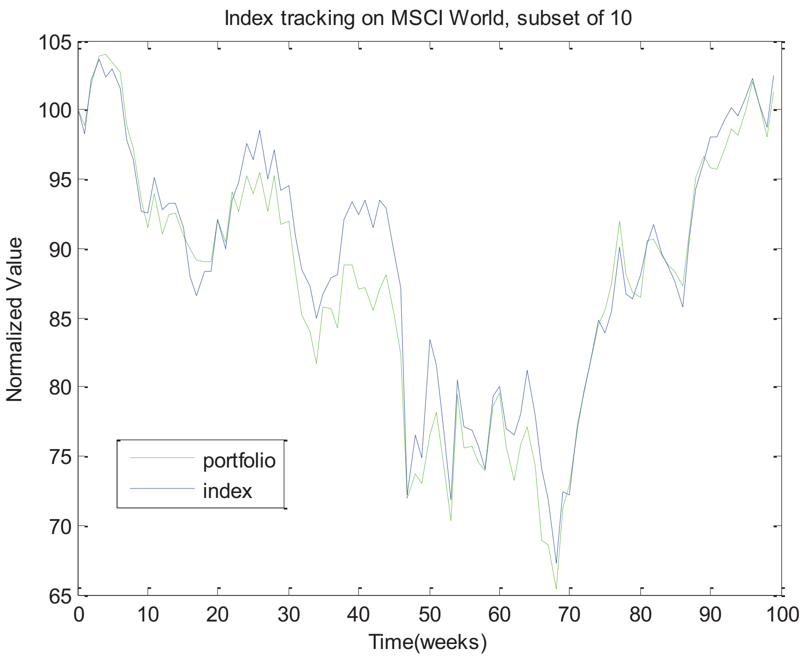

Figure 4.10. MSCI World index and the optimized portfolio value of the Index tracking problem, with real data and a subset of 10 stocks.

This result motivates an even larger subset, and index tracking with a subset of 25 stocks is performed, Figure 4.11. The result gives very close tracking.

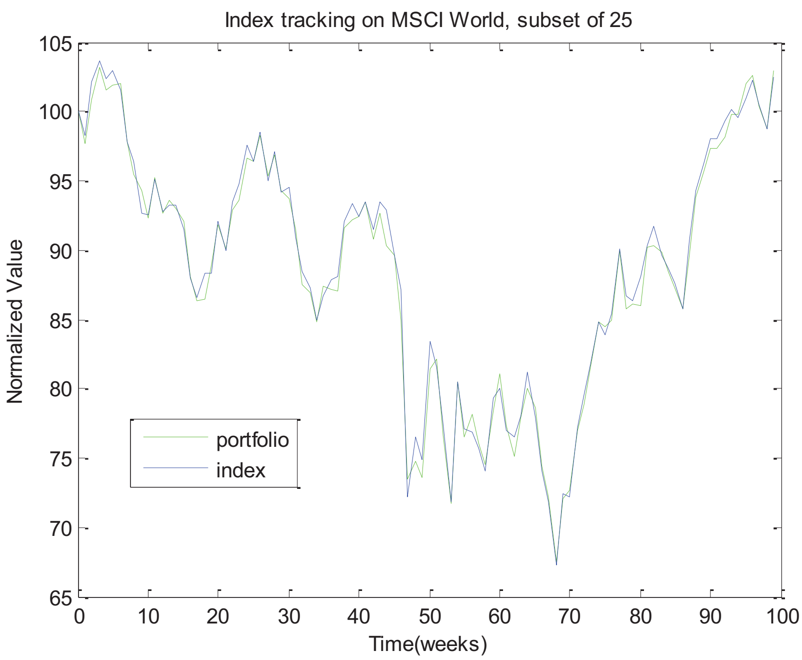

Figure 4.11. MSCI World index and the optimized portfolio value of the Index tracking problem, with real data and a subset of 25 stocks.

This tracking approach is also performed on the smaller Amex Biotechnology index to ensure its functionality. It shows close tracking result and this removes this methods uncertainty. The next to investigate would be to perform this tracking setup where covariance and mean data are calculated on one data set and tested on another to ensure that the knowledge of the future is not affecting the result to the better.

With knowledge of the future performance of the market it could be interesting to look at how the Risk adjusted wealth maximization would have performed during this time period. The result shows that with the knowledge of the declining values of the market, in this case represented by the Amex Biotechnology index, shortselling could be very profitable. This can clearly be seen when the weights of the unrestricted Risk adjusted wealth problem under this time period was studied, where almost all stocks was given negative values. If a constraint of no shortselling is added to the problem a less aggressive approach will show more realistic results, see Figure 4.12.

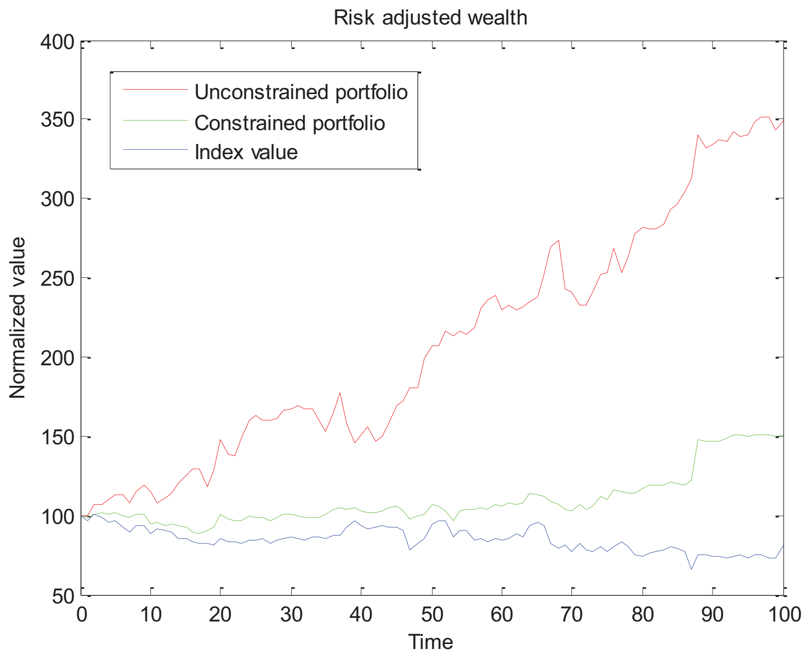

Figure 4.12. The index value and the portfolio value of the Risk adjusted wealth, with real data, with and without allowing shortselling.

Chapter 5

5. Conclusions

The theoretical examples that were investigated first gave an understanding of the method and its limitations. In the Risk adjusted wealth maximization problem different risk levels of λ was evaluated to see how the method considers the variance. It shows that a smaller λ gave a riskier profile on the portfolio and larger fluctuation over time. The λ should be chosen according to the preferences of the portfolio manager. The first example is calculating the receding horizon allocations successfully and can handle added constraints. Since it is more common use active management on this type of portfolios in reality, the index tracking problem was further investigated. The next example, involving Index tracking, showed that it is possible to track an index with a relatively small tracking error. This problem can also handle constraints successfully, but thereby not saying that these constraints produces better index tracking then the unconstrained problem. This tracking is being countered by controlling the value of the portfolio. The movements of the wealth are in that case being prevented so therefore the focus should lie on controlling the different assets for the closest tracking result.

It is clearly seen in the historical portfolios from Danske Bank that a larger subset of stocks gives a closer tracking of the index because of the increased amount of stocks to choose from. This can be seen in the Index tracking with real data of the smaller index which gives satisfying results for small subsets but even better when full replication is used.

To summarize the method investigated in this thesis the first drawback that can be concluded is that transaction costs are not included. Small changes can therefore lead to profit losses for the portfolio if the predicted profit of the changes is too small to compensate the transaction costs. A second drawback is the problems of handling large portfolios in the Index tracking example. Here the state vector will increase with the number of stocks in the index. This will then extend this multivariable index problem setup a lot. It is then obvious that with every added stock in the index, and thereby an increased state vector, the problem will drastically enlarge. The most critical factor in the setup in the enlargement of the problem is the increasing size of the matrices containing the inequality of the covariance which increases a lot with the size of the index and the subset of the index.

To handle the large indexes the setup of the optimization problem was simplified with a new approach by representing the index as one state. The problem will then have two states, the wealth and the index instead of the wealth and all the stock prices in the index. The index was then modeled as a stock and that is possible because the behavior of the index is very similar to a stocks behavior in the financial market. The simplified problem setup could then manage to optimize index tracking portfolios to the larger indexes. The result gave a very close tracking and can therefore be seen as a good alternative for the large indexes.

Another example that was performed was the Risk adjusted wealth maximization problem with real data. With knowledge of the future performance of the market it was interesting to look at how the Risk adjusted wealth maximization would have performed during this time period. The result shows that with the knowledge of the declining values of the market, shortselling could be very profitable. It is off course not possible to know the future behavior but it is interesting to see that the method takes advantage of this knowledge. Due to limited data over time the expected return vector and the covariance matrix had to be calculated over the same time period as the later performed simulation. This is, as mentioned, not a correct way of using the data because the simulation is done with already calculated values, which is equal to already having some knowledge of the future. But since this is a mean over the whole time period and since the interest lies in trying the method out this is being tolerated. All results calculated with this real data should therefore be look at with a critical eye.

5.1 Extensions to the thesis

If more time were given one way of proceeding with this thesis would be to include transaction costs in the program. If this were to be done the optimal portfolio would consider the transaction cost in contrast to profit when selecting the different portfolio weights. For example a small change in a weight might be put to zero instead because it would not pay off to have this small weight in the portfolio because of the transaction costs it will bring to it. Another way of proceeding would be to have data for a longer time period so that the mean of the return vector and the covariance matrix could be calculated on a different time period than the simulation to get a more realistic simulation and reality-based situation. This to be sure of avoiding misleading possibly false good results.

The model describing the states could have been more studied, where one idea could be to evaluate the random variable. It might be better to distribute the random variable in another way, to better describe the stock market. One way of approximating the stock market could be to use a simulated variance and thereby change the variance over time in the distribution. For example, this could be done with a volatility model, such as the ARCH-model.

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