

ISSN 0280-5316
ISRN LUTFD2/TFRT--5770--SE

Linear Time Periodic Analysis of Dc-Dc converter

Simone Del Favero

Department of Automatic Control
Lund University
June 2006

Department of Automatic Control Lund Institute of Technology Box 118 SE-221 00 Lund Sweden		<i>Document name</i> MASTER THESIS	
		<i>Date of issue</i> June 2006	
		<i>Document Number</i> ISRN LUTFD2/TFRT--5770--SE	
<i>Author(s)</i> Simone Del Favero		<i>Supervisor</i> Andreas Wernrud and Anders Rantzer at Automatic Control in Lund, Sweden Giorgio Picci at Dip.di Ingegneria dell'Informazione Università Padova, Italy	
		<i>Sponsoring organization</i>	
<i>Title and subtitle</i> Linear Time Periodic Analysis of Dc-Dc converters.(Linjär tid-periodisk analys av Dc-Dc omvandlare)			
<i>Abstract</i> <p>Aim of this thesis is to analyze Dc-Dc converters by using the techniques of Linear Periodic Time varying (LTP) systems to estimate the amount of subharmonics injected in the load.</p> <p>Dc-Dc converters are used to transform a Dc input to a Dc output of different voltage. In this thesis we study in particular the so called "switch mode" converters. In this kind of devices the conversion is obtained by using fast commutations of (at least) two switches.</p> <p>Due to the discrete switch-positions these converters are considered a typical example of hybrid systems. Linear models with fixed coefficients (LTI system) give a description of the system inadequate to predict and to analyze harmonic effects, while linear models with coefficients that vary periodically, namely LTP system, can be used effectively to this aim. We use therefore a Linear Time Periodic (LTP) system to describe the converter. This kind of description is much more accurate but the model and the tools used to study it are more complex.</p> <p>In the thesis we first introduce the LTP system theory and its main results. In particular we introduce the concept of Harmonic Transfer Function (HTF).</p> <p>A LTP model for a Dc-Dc converter is then derived and it is shown that this model accurately describes the response of the converter.</p> <p>Furthermore this LTP model is used to analyze the open and closed loop behavior of the system. It is shown that the linear model estimates correctly the amplitude of the subharmonics in the output.</p> <p>The thesis has been developed at the Automatic Control Department, Lund University, Sweden under the supervision of Andreas Wernrud and Anders Rantzer.</p> <p>The Italian supervisor of this thesis is Giorgio Picci, Dipartimento di Ingegneria dell' Informazione, Università degli studi di Padova, Italy.</p>			
<i>Keywords</i> Dc-Dc converter, Pulse Width Modulator, PWM, Linear Time Periodic Systems, LTP, \mbox{Harmonic} Transfer Function, HTF, Harmonic Transfer Matrix, HTM, subharmonics, chaos, subharmonic estimation.			
<i>Classification system and/or index terms (if any)</i>			
<i>Supplementary bibliographical information</i>			
<i>ISSN and key title</i> 0280-5316		<i>ISBN</i>	
<i>Language</i> English	<i>Number of pages</i> 98	<i>Recipient's notes</i>	
<i>Security classification</i>			

Acknowledgment

This year in Lund has been simply great for me.

Both from a professional and a human point of view. To work on my thesis here, in this prestigious university, in this renowned department, in this amazing international environment has been a true privilege.

And there are some persons that specially contributed to make this experience what it has been, something really, really important.

First of all my supervisor Andreas Wernrud, that has been a constant and precious help in my work. He sustained me while I was taking my first insecure steps in a reality completely new to me: the research world. Above all he helped me to find a right approach to a research problem, by helping me to formulate the right questions. I'm really grateful for this.

My second supervisor Anders Rantzer, that has been very cooperative and helpful in my work. I'm specially grateful for the freedom he left me in defining the line I followed in the development of my project.

A huge thanks to my Italian supervisor Giorgio Picci, without whom all this would not have been possible. Thanks for all the technical, human and bureaucratic support he gave me.

Thanks to all the Department of Automatic Control, where I always felt welcome. It has been such a stimulating environment from all points of view and specially from a professional one.

A huge thanks goes to all the friends at the department.

Specially to the ones that became good friends also outside the department: to Mathieu, Rosa, Josep, Jose Luis, Oskar, Francesco and Thomas.

And another huge thanks to Vildanden's crazy corridor (and adopted).

Simply, you made Lund a paradise, my paradise. Thank you so much for all the life we shared, from the parties to the deeper moments.

Finally I can't forget the person that I felt close to me even if they were in Italy... or somewhere else in the world:

First of all my parents, my sisters and my grandparents.

Then all the friends that kept so close contact with me:

the one from the legendary Collegio Mazza, the Copada's one and the group of Marco, Sonia, Marcotto, Davide and Ally.

And all the others that these categories don't include.

Moreover I'm sincerely grateful to my home university, Università degli Studi di Padova, for training me in the field of engineering.

To all these persons:

Thank you very much, grazie infinite, tack så mycket.

Simone

Contents

Abstract	3
Acknowledgment	5
Introduction	9
1 Dc-Dc converter	13
1.1 Pulse Width modulated signal	14
1.2 Pulse Width modulator	16
1.3 Case Study: Step Down Converter	19
2 Linear Time Periodic Systems	21
2.1 Input-Output approach in systems analysis	22
2.1.1 Definition of the Harmonic Transfer Function	24
2.1.2 Exponentially Modulated Periodic signals	27
2.1.3 Harmonic Transfer Matrix	28
2.2 State space approach in systems analysis	29
2.2.1 HTF of LTP Systems in State Space Form	29
2.3 HTF of some basic systems	31
2.3.1 LTI Systems	31
2.3.2 Multiplication Operator	31
2.3.3 LTP systems connection	32
2.4 Frequency analysis techniques for LTP systems	34
3 Properties and Analysis of LTP systems in State Space	38
3.1 Time Domain Techniques for LTP systems	38
3.2 Other Time Domain Techniques for LTP	41
4 Dc-Dc converter model using LTP system	42
4.1 Basic Components Model	42
4.1.1 Linear Filter and Controller	42
4.1.2 Sampler and Holder	42
4.1.3 PWM-modulator Model	44
4.1.4 PWM Model Validation	53
4.1.5 Switching Stage	53
4.2 HTF description of the Controlled Step Down Converter	59
5 Subharmonics and chaos in Dc-Dc converters	62
5.1 Qualitative classification of the response of a dynamical system	63
5.2 Tools used in Chaos analysis	68
5.2.1 Poincarè section	68
5.2.2 Bifurcation Diagram	70
5.2.3 Characteristic multipliers and Lyapunov Exponents	70
5.3 Case Study	74

6 Stability analysis of the periodic system	77
6.1 Subharmonic Analysis	78
6.1.1 $V_{line} = 20$	80
6.1.2 $V_{line} = 23$	83
6.1.3 $V_{line} = 26$	85
6.1.4 $V_{line} = 29$	87
6.1.5 Analysis of Load and Line Tension Variation	89
6.2 Subharmonic Prediction	91
Conclusion	94
Bibliography	97

Introduction

A Dc-Dc converter is a device which transform a Dc input voltage to a Dc output voltage with different amplitude. The conversion can be done using various physical principles but a large and important class of Dc Dc converters is the so called Switch-mode converters. In this kind of devices a switching stage is always present. It comprises at least two semiconductor switches that are turned on and off at a very high frequency.

A low pass filter stage is combined with the switching stage to produce an output voltage with a dominant Dc component and a small ripple. The Dc component of the output can be regulated by choosing the length of the on and off intervals of the semiconductor switches.

Switch-mode Dc-Dc converters represent a class of power electronic circuits that are used in a large variety of applications for their compact size, light weight and due to the high efficiency and their reliability. They constitute the enabling technology in computer power supplies, battery chargers, sensitive and demanding aerospace and medical applications and speed Dc motor drives.

Moreover, due to the discrete switch-positions, these converters are considered a typical example of hybrid systems and to study they become an interesting and important problem that fits perfectly in the hybrid framework.

It is common practice to deal with the switching stage in Dc-Dc converter by making a strong simplification: since the switching stage is always followed by a filter stage, it is usually considered reasonable to study just the Dc-component of the switching stage output, neglecting all the other harmonics. This is equivalent to model the switching stage with a static gain. This approximation simplifies drastically the computation and gives results that are often considered acceptable. However, this simplification hides many phenomena that can be extremely important to predict and to analyze in more advanced application.

For analogous reasons the controller scheme commonly used to drive converters is a simple PI-type controller, tuned according to simple and well developed rules.

In recent years, the power electronics community has become more receptive to sophisticated control loop designs for Dc-Dc converters. This is due to the advances in the technology of digital processors as well as to increased performance demand of modern-days applications.

It is therefore becoming more and more important to develop controllers that, by taking into account the hybrid nature of the switch, improve the performance of this device.

It is also essential to develop efficient and reliable techniques for the analysis of this kind of systems and of their interaction with the global system in which they are inserted. Special attention should be paid in the prediction and the analysis of subharmonics and chaotic behavior, since it has been shown in various work¹ that Dc-Dc converters are systems which often become chaotic.

¹See for instance [10]

This is also the topic and the reason of this thesis.

More specifically in this thesis we will try to use techniques for the frequency analysis of LTP systems to study the properties of the controlled converter presented in [10]. In this paper it is shown that this converter presents subharmonics and chaos when some critical parameters vary.

The basic idea is, once noticed that a LTI model is inadequate to predict subharmonics and chaos, to describe the converter by using a LTP model, namely a linear model with parameters periodically time varying. This kind of models are more complex and require more complex tools to be studied but they allow a more accurate approximation of complex nonlinear systems that present periodic behavior.

The main source of informations about the LTP systems theory has been the PhD thesis of Erik Mölest, [1]. Moreover Erik's thesis has been used as source of inspiration for the presentation of the results in the two sections which concern LTP systems and it has been cited various times.

In the thesis we will report a LTP model for components commonly used in Power Electronics, such as Linear Filters, Samplers and Holders.

Once again these models has been found in Erik's thesis.

Moreover we will derive a new LTP model for PWM-modulators.

Some of these components are present in our example of controlled converter and combining their models we will obtain a LTP model for this global system.

We will then show that the LTP approximated model of the converter describes accurately the response of the system and allow us to estimate the amount of subharmonics injected in the load.

The very important issue of the robustness of the model is also discussed showing that, as experted, the approximate system is not robust enough to remain valid trough the parameter variation that correspond to a brusque change of the system behavior.

The model can hence be used to analyze the harmonics content of the output but not to predict the arise of new subharmonics or chaotic behavior.

Contribution of the thesis :

A first contribution is the development of Linear Time Periodic model for the PWM modulator. This model is still simple but we will show it is quite accurate. Moreover, compared with the other few models of this kind that can be founded in literature, it is more flexible and general, since it can be easily modified to describe various typology of PWM modulators (symmetric, asymmetric, etc.).

A second contribution of the thesis is the application of the LTP techniques as method for analysis of the subharmonics injection in the load.

In particular, we will analyze by mean of these techniques, the effects that a perturbation on the reference signal has on the spectral content of the output. The model will then be modified to take into account the effects of a small amplitude perturbation in the line tension or in the load.

The case of a slow perturbation with big amplitude can also be studied using the method presented in the thesis, by assuming that it is piecewise constant and study the system with the techniques presented for various operating points.

If this assumption is unreasonable than the method we will present is inadequate to the description of the system considered and other methods should be used.

Structure of the report

In section 1 more in detailed information about Dc-Dc converters are presented.

In section 2 we introduce the LTP systems theory, with special attention to the idea of Harmonic Transfer Function (HTF).

In section 3 some results about the generalization of the state space techniques to the LTP case are presented.

These techniques are not directly used in this thesis and therefore this section can be skipped if the reader is mainly interested in how the problem of estimate the subharmonics is solved.

This section, directed to the reader interested in increase his knowledge about LTP system, is mainly a collection of material presented in literature but we hope will help to create a better understanding of this kind of system.

In section 4 the HTF of components commonly used in Power Electronics is derived. Specially important is the LTP model of the PWM modulator, kernel of the LTP model of the converter and one of the main contribution of the thesis.

In section 5 there is a brief and simplified introduction to chaos theory. We present also in this section some results reported in [10].

In section 6 the model developed in section 3 is used to analyze the converter stability, in particular to estimate the amount of harmonics and subharmonics injected in the load.

The issue of the robustness of the model is also discussed in this section.

Other Detail :

Simone Del Favero is a student of Corso di laurea Specialistica in Ingegneria Dell'Automazione (master program in Automation Engineering) of Università degli Studi di Padova, Italy.

The entire work has been done at the department of Automatic Control of Lund University, Sweden within the context of an *ERASMUS* exchange, under the supervision of Andreas Wernrud and Prof. Anders Rantzer.

The Italian supervisor of this thesis is Prof Giorgio Picci, Dipartimento di Ingegneria Elettronica ed Informatica.

1 Dc-Dc converter

A Dc-Dc converter is a device which transform a Dc input voltage to a Dc output voltage with different amplitude. The conversion can be done using various physical principles. A large and important class of Dc Dc converters is the so called Switch-mode converters. In this kind of devices a switching stage is always present. It comprises at least two semiconductor switches that are turned on and off at a very high frequency.

A low pass filter stage is combined with the switching stage to produce an output voltage with a dominant Dc component and a small ripple. The Dc component of the output can be regulated by choosing the length of the on and off intervals of the semiconductor switches.

A further classification of Switch-mode Dc-Dc converters can be given according to the switching frequency:

- In the *fixed frequency dc-dc converters* the switches are periodically turned on at the beginning of the period. The Dc regulation can be done by choosing the instant in which the switches are turned off, i.e choosing the length of the on interval. The periodical switch-on frequency f characterizes the operation of the converter.

A number of circuit topologies with different characteristics have been developed. The most common are step-down converters (or buck converter), step-up converters (or boost converter), buck-boost converters etc.

Two examples of this circuit can be founded in figure 1.

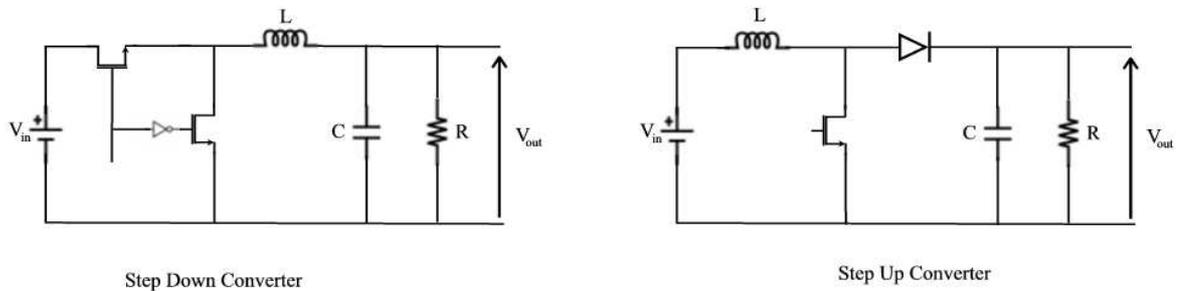


Figure 1: Example of simple Step Up and Step Down Converters

The major advantage of this converters is that the design of the low pass-filter stage is relatively straightforward, since the ripples that are presented in the output can be easily calculated.

On the other hand the basic disadvantage is their limited efficiency, due to the switching losses in the semiconductor that increase linearly with the switching frequency.

- *Resonant dc-dc converters* use a slightly different operation principle. In these circuits a resonant LC stage is intervened between the input voltage and the switching stage. Its rule is to create a resonant current through the semiconductor switch which is turned off when this resonant current drops to zero, leading to a minimal switching losses. Analogously the resonant circuit can be used to create a resonant voltage across the semiconductor switch which is turned off when this resonant tension drops to zero. The Dc component of the output is regulated by changing the length of the off interval. These converters have the main advantage that they have very high efficiency, because of the small dissipation. For that reason they can reach very high frequency. On the other hand to variate the off interval means to variate the operating frequency and this makes the output filter design more difficult.

In this thesis we will restrict our analysis to the case of fixed frequency switch-mode Dc-Dc converters, with special attention to the case of step down converters.

1.1 Pulse Width modulated signal

The signal which drives the switches and the output signal of the switching stage are PWM modulated signals. PWM stay for Pulse Wide Modulation and this kind of signals have well known properties that we report in the following.

A PWM-modulated signal $a(t)$, as it is shown in figure 2, is a a periodic signal of fixed period T which can assume only two values: $A_1 = 1$ and $A_2 = 0$.²

In a period only one switch on-off and one switch off-on are allowed.

The quantity that can variate during the periods is T_{on} , the length of the time interval in which the signal presents amplitude A_1 .³

To be physically meaningful T_{on} must satisfy:

$$0 \leq T_{on} \leq T$$

To normalize this set, it is usually introduced the *duty cycle* d , defined as:

$$d = \frac{T_{on}}{T}$$

The previous constraint becomes, for the duty cycle :

$$0 \leq d \leq 1 \tag{1}$$

²Different physical applications may have different values of A_1 and A_2 . The ranges $[-1, 1]$ and $[0, +V_{cc}]$, for instance, are also very common.

However a generic output range $[A_2, A_1]$ can be simply obtained from the set $[0, 1]$ with multiplication and translation operations:

$$[A_2, A_1] = a[0, 1] + b = (A_1 - A_2)[0, 1] + \frac{A_1 + A_2}{2}$$

Therefore in the following we can suppose that $A_1 \equiv 1$ and $A_2 \equiv 0$.

³We will sometime use the notation $T_{on}(nT)$ to make explicit that T_{on} can variate form the n -th period to $(n + 1)$ -th period.

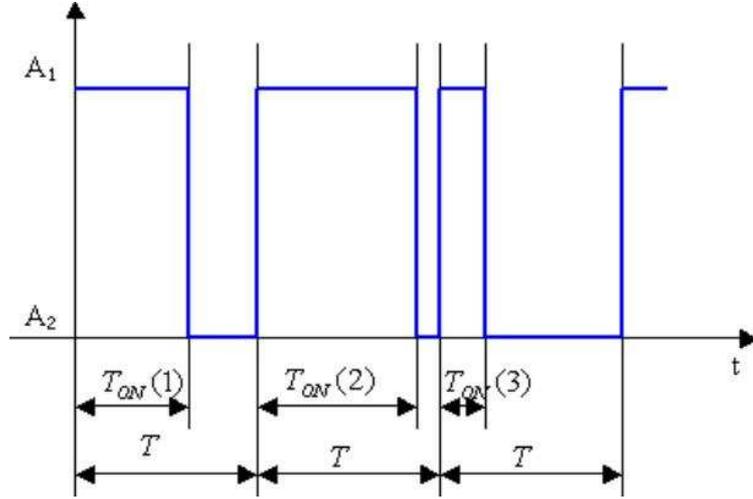


Figure 2: Example of PWM output

As we said, the underlying idea of this kind of modulation is to regulate the mean value of a rectangular signal adjusting the length of the on and off intervals, i.e. adjusting the duty cycle.

For example, if we choose $d(nT) = 1$ then $\bar{a}(t)$, the mean value of $a(t)$, will be $\bar{a}(t) = 1$. Vice versa if $d(nT) = 0$ it will be $\bar{a}(t) = 0$, as shown in figure 3.

Moreover:

$$\bar{a}(t) \triangleq \frac{1}{T} \int_{nT}^{(n+1)T} a(t) dt = \frac{1}{T} \int_{nT}^{nT+T_{on}(nT)} 1 dt = \frac{T_{on}(nT)}{T} = d(nT)$$

This computation shows that in the PWM modulation the information of an analog signal $d(nT)$ is associated to the mean of a digital signal $a(t)$.

If the signal $d(t)$ varies slowly during the PWM periods, to low pass filter $a(t)$ will give a good approximation of the input signal $d(t)$.

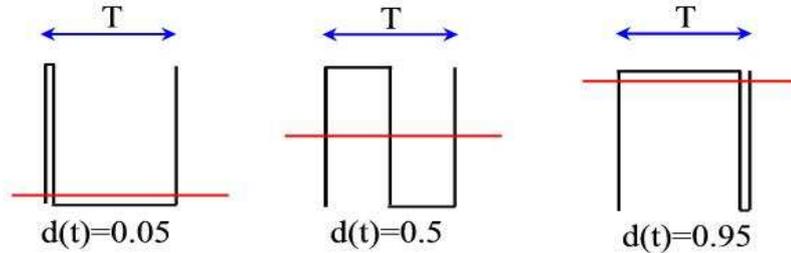


Figure 3: Simple example of the PWM output of various values of $d(nT)$.
In red the mean, $\bar{a}(t)$, of the signal $a(t)$

1.2 Pulse Width modulator

A PWM modulator is often realized comparing the signal $u(t)$ with the periodic signal $m(t)$, as it is showed in figure 4.

There are various possible choices of $m(t)$, which give different kind of modulations, with different properties:

- If $m(t)$ is chosen as a sawtooth wave we have the so called *asymmetric PWM modulation*, in which at the beginning of each period the signal is switched to 1 and during the period, depending on the input signal value, the signal is switched to 0.
- If $m(t)$ is chosen as a triangular wave we have the *symmetric PWM modulation*. In this kind of modulation, to avoid unnecessary switchings, in the even periods the on interval is moved at the end of the period, as it is shown in figure 5. This kind of modulation introduces also less spectral distortion of the input signal.
- sometimes other signals are used, as the current in the inductor of the low pass filter, for reason connected to the specific implementation.

The range of the signal $m(t)$, $[V_L, V_M]$, should be chosen carefully because it defines the saturation range for the modulator.

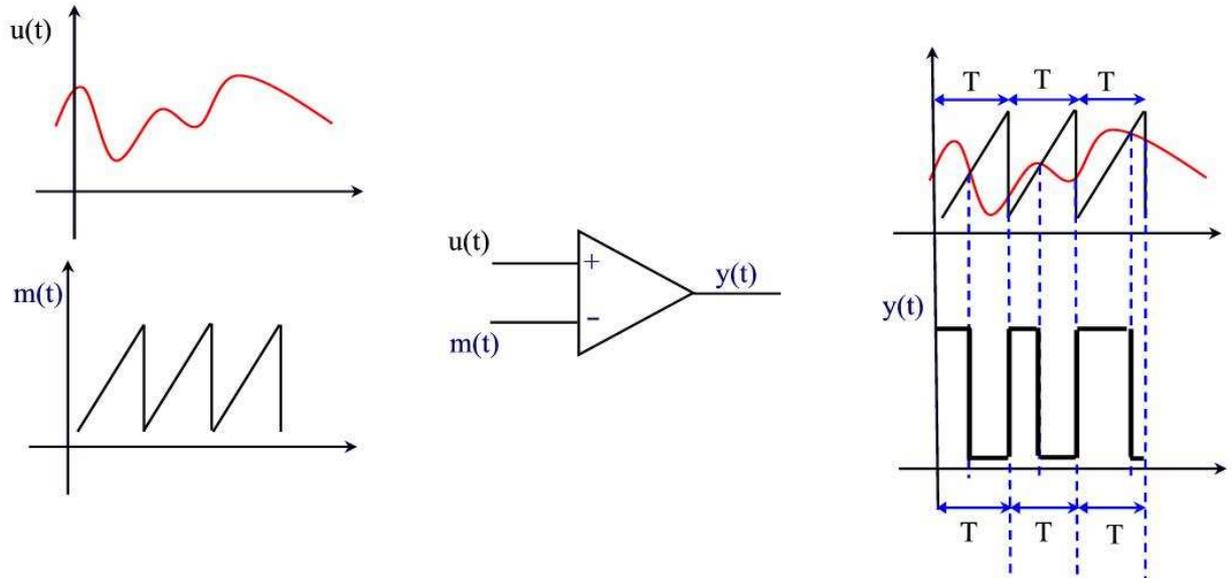


Figure 4: Realization of a PWM Modulator

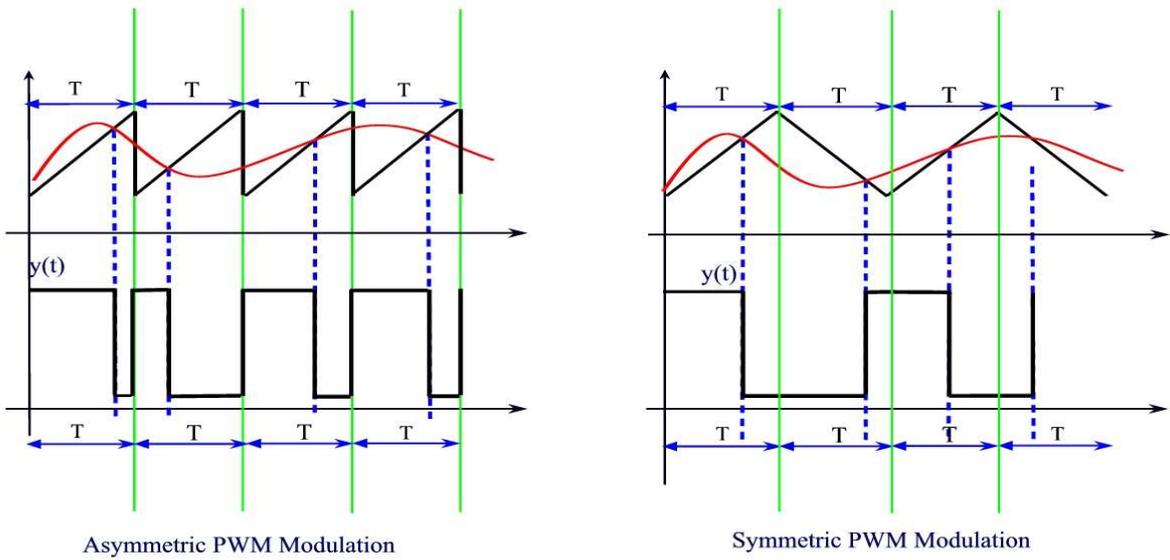


Figure 5: Comparison between Symmetric and Asymmetric Modulation

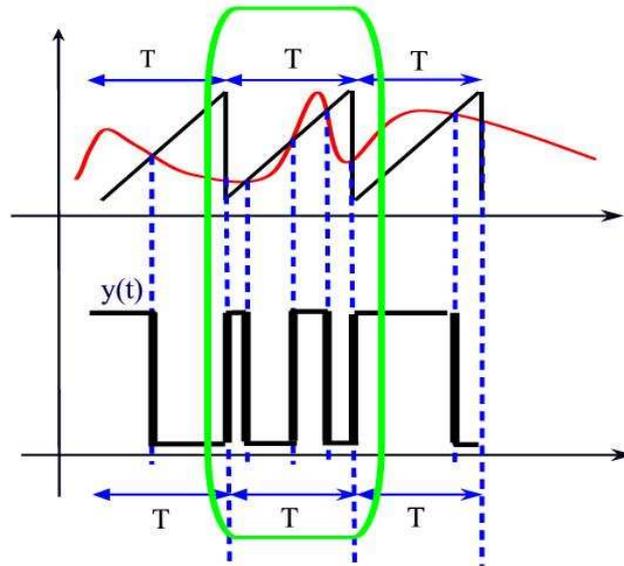


Figure 6: Example of multiple switchings in a period

Note that, with this simple modulation scheme, if the signal $u(t)$ varies too fast the output may present multiple switchings in a period (figure 6). This is an unwanted phenomena, because it introduces more commutations and therefore increases the power dissipation.

It is common to handle this problem sampling and holding $u(t)$ during the period, so that it can't variate. However this patch can not be used in all the physical applications.

1.3 Case Study: Step Down Converter

As we said, in this thesis we will restrict our analysis to the case of switch-mode fixed frequency Dc-Dc converters, with special attention to the case of step down converters. In particular we will study the circuit presented in [10] and reported in figure 7.

It is a Step Down converter whose output is set to a reference value via feedback, using a simple proportional controller. The switches are driven with a PWM modulated signal, which is obtained by comparing the controller output signal with a sawtooth carrier $m(t)$.

All the components are supposed to be ideal. In particular from a control point of view we can imagine the semiconductor switches to be ideal and hence suppose that they instantaneously change from on to off. Moreover they are supposed bidirectional.

The circuit parameters are reported in table 1.

Parameter		Value	Unit of Measurement
Resistance	R	22	Ω
Capacity	C	47	μF
Inductance	L	20	mH
Proportional Gain	K	8.4	/
Reference Tension	V_{ref}	11.3	V
Input Tension	V_{in}	Variable	
Carrier Range	$[V_L, V_U]$	[3.8, 8.2]	V
PWM Period	T	400	μs

Table 1: Parameter of the circuit reported in figure 7

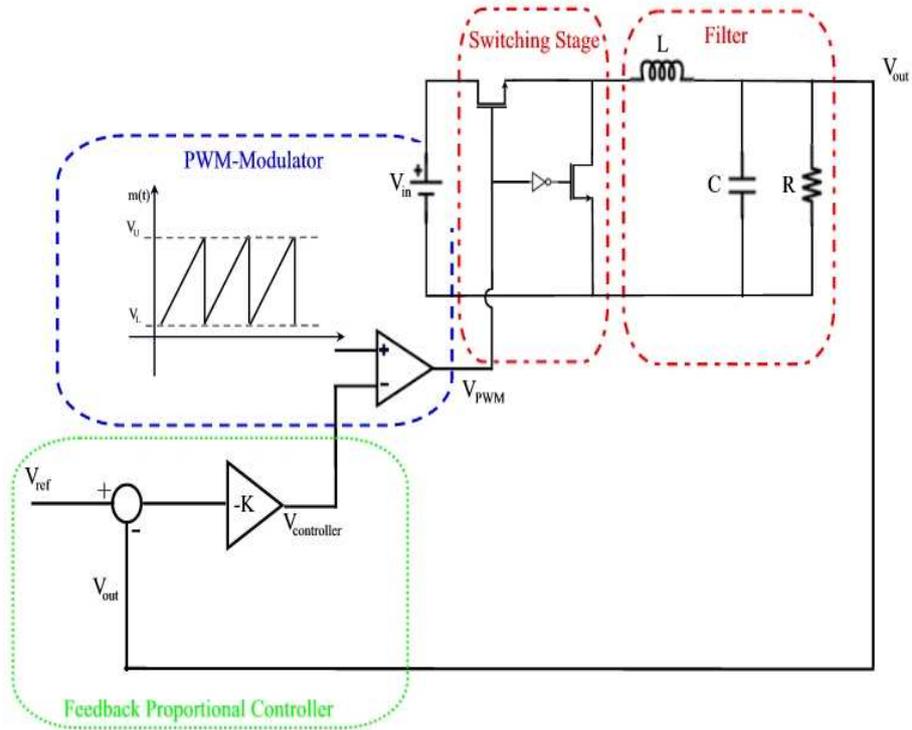


Figure 7: Controlled Step Down Converter

Schematic of the circuit introduced in [10] and which we will study in this thesis. Note a peculiarity: in this case in the PWM modulator the comparison is done according to rule $m(t) < v_{controller}$ and not, as usually happen $m(t) > v_{controller}$. This force us to use a negative gain proportional controller $-K$.

2 Linear Time Periodic Systems

The hybrid nature of a Dc-Dc converter can not be well described by using a LTI model, that hides many important phenomena. In particular a LTI model does not take into account the harmonic injected in the load by the converter, supposed simply neglectable.

To develop a model that allow the description of this phenomena we use therefore a LTP system. This allow us to give a description much more accurate of the converter but require also to use more complex analysis tool.

Linear Periodically Time Varying systems (LTP systems) are linear systems described by coefficients that are not simple constant, as it happen in the case of linear time invariant system, but periodic function of time.

This periodicity in the coefficients is sometimes due to the nature of the system, for instance this is the natural model for physical systems with rotating parts, like electro-mechanic machines.

Other times, as in our case, a LTP approximated model is given to describe a non linear system. In this case the possibility to use variable coefficients allow us to obtain a model much more accurate than the LTI ones.

A LTP model is usually given when the state of the system converges to a periodic equilibrium trajectory. In that case, in fact, the non linear system can be effectively linearized around this equilibrium trajectory.

Another important fact that allows to understand why a LTP model is more close to the original system than the LTI one is the following:

It is well known that the steady state output of a Linear Time Invariant system (LTI) to a sinusoidal input is a sinusoid with the same frequency. This property, often referred as *frequency separation property*, does not hold anymore for a non linear system and the steady state output to a sinusoidal input can contain a number (possibly infinite) of sinusoids of frequency different from the input one. This phenomena, known as *frequency coupling*, is sometimes so strong that it is impossible to find a good linear approximation of the non linear system.

LTP systems can present coupling between frequencies whose difference is a multiple of $1/T$, so they can be used to capture the frequency coupling of a non linear system to have a more accurate model.

The frequency separation property simplifies a lot the analysis of LTI systems and it allows the development of powerful frequency domain techniques for the input-output analysis.

Since the frequency separation property does not hold anymore for LTP system the generalization of this techniques to the LTP case is not trivial, but still possible.

The aim of this section is to introduce this generalized techniques.

2.1 Input-Output approach in systems analysis

A very common approach in linear systems analysis is the so called *input-output approach*. According to this approach a system is described as an operator from the space of the input signals to the space of the output signals.

$$\begin{aligned} \mathcal{S} : \quad \mathcal{I} &\longrightarrow \mathcal{O} \\ x &\longmapsto y(t) = [\mathcal{S}]x(t) \end{aligned}$$

According to the properties of the operator \mathcal{S} we can give the following classification:

A system is called **Linear** if the operator \mathcal{S} is linear, i.e.:

$$[\mathcal{S}][\alpha u_1(t) + \beta u_2(t)] = \alpha[\mathcal{S}]u_1(t) + [\mathcal{S}]u_2(t)$$

In the case of linear systems the operator \mathcal{S} has a special structure:

$$y(t) = \int_{-\infty}^{+\infty} h(t, \tau)x(\tau)d\tau \quad (2)$$

Hence we have that a linear system is modeled with an integral equation.

A system is called **time invariant** if the following property holds:

$$x(t) \xrightarrow{\mathcal{S}} y(t) \quad \Longrightarrow \quad x(t + t_0) \xrightarrow{\mathcal{S}} y(t + t_0) \quad \forall t_0 \in \mathbb{R} \quad (3)$$

This is equivalent to say that the output of the system depends only on the input and not on when the input is applied, hence it means that the system does not change its behavior over the time.

A system is called **periodically time variant** if the following property holds:

$$x(t) \xrightarrow{\mathcal{S}} y(t) \quad \Longrightarrow \quad x(t + t_0) \xrightarrow{\mathcal{S}} y(t + t_0) \quad \forall t_0 = nT \quad (4)$$

This property states that the system periodically behaves in the same way rather than to demand to present the same behavior all over the time as the property of time invariance does. Hence the periodic time invariance is a weaker version of the time invariance property and all the LTI systems are also LTP systems of each period T .

Combining linearity with the above two properties we define two important classes of systems, for which the I-O expression assumes a particular form.

Definition 1

A system is called **Linear Time Invariant (LTI)** if it is both linear and time invariant

Proposition 1

The input-output relation of a LTI system is:

$$y(t) = \int_{-\infty}^{+\infty} \tilde{h}(t - \tau)u(\tau)d\tau = \tilde{h} \circ u \quad (5)$$

where $\tilde{h}(t - \tau) = h(t - \tau, 0)$ and with the symbol \circ is denoted the convolution.

$\tilde{h}(t)$, called *Impulse Response* of the system, is in this case a function of only one variable. Note also that the relation between input and output is described by a convolution which is a well known operator with interesting frequency properties:

Proposition 2

The input-output relation of a LTI system becomes in frequency domain:

$$Y(s) = H(s)U(s)$$

where $Y(s)$ is the Laplace transformation of the output $\mathcal{L}[y(t)]$ and $U(s)$ is the Laplace transformation of the input $\mathcal{L}[u(t)]$.

$H(s)$, called **Transfer Function** of the LTI system, is the Laplace transform of the impulse response:

$$H(s) = \mathcal{L}[\tilde{h}(t)] = \int_0^{+\infty} \tilde{h}(t)e^{-st}dt$$

Analogously to LTI case we give:

Definition 2

A system is called **Linear Periodically Time Variant**, or often simply **Linear Time Periodic (LTP)**, if it is linear and time periodic.

Proposition 3

The input-output relation of a LTP system is:

$$y(t) = \int_{-\infty}^{+\infty} h(t, t - \tau)x(\tau)d\tau \quad (6)$$

where $h(t, t - \tau)$ is, for any fixed τ , T -periodic in t .

In this case $h(t, t - r)$ is a function of two variables, so is not possible to extend straightforwardly the concept of transfer function. This extension is of central interest for this thesis and it will be studied in detail in next section, 2.1.1.

2.1.1 Definition of the Harmonic Transfer Function

The I-O relation for LTP systems (6) is:

$$y(t) = \int_{-\infty}^{+\infty} h(t, t - \tau)x(\tau)d\tau \quad (7)$$

where $h(t, t - \tau)$ is, for any fixed τ , T -periodic in t .

Under appropriate convergence conditions, $h(t, t - \tau)$ can be expressed as a Fourier series:

$$h(t, t - \tau) = \sum_{k=-\infty}^{+\infty} h_k(\tau)e^{jk\omega_0 t} \quad (8)$$

with $\omega_0 = 2\pi f_0 = 2\pi/T$ and $h_k(\tau)$ is the k -th Fourier coefficient of $h(t, t - \tau)$:

$$h_k(\tau) = \frac{1}{T} \int_0^T h(t, t - \tau)e^{-jk\omega_0 t} dt$$

Note that, since $h(t, t - \tau)$ depends also on τ , h_k is function of τ .

Substituting (8) in (7) and supposing that it is allowed to change the operator order, gives:

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} \left(\sum_{k=-\infty}^{+\infty} h_k(\tau)e^{jk\omega_0 t} \right) u(t - \tau)d\tau \\ &= \sum_{k=-\infty}^{+\infty} e^{jk\omega_0 t} \int_{-\infty}^{+\infty} h_k(\tau)u(t - \tau)d\tau \end{aligned} \quad (9)$$

Hence, the output of a LTP system can be consider as the sum of the outputs of infinite LTI systems translated in frequency of a factor $e^{jk\omega_0 t}$, as it is shown in figure 8.

Moreover, (9) can be rewritten as :

$$\begin{aligned} y(t) &= \sum_{k=-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_k(\tau)e^{jk\omega_0 \tau} u(t - \tau)e^{jk\omega_0(t - \tau)} d\tau \\ &= \sum_{k=-\infty}^{+\infty} \left(h_k(\tau)e^{jk\omega_0 \tau} \right) \circ \left(u(\tau)e^{jk\omega_0(\tau)} \right) \end{aligned}$$

Laplace transformation gives:

$$Y(s) = \sum_{k=-\infty}^{+\infty} H_k(s - jk\omega_0)U(s - jk\omega_0) \quad (10)$$

where $H_k(s) = \mathcal{L}[h_k(\tau)]$.

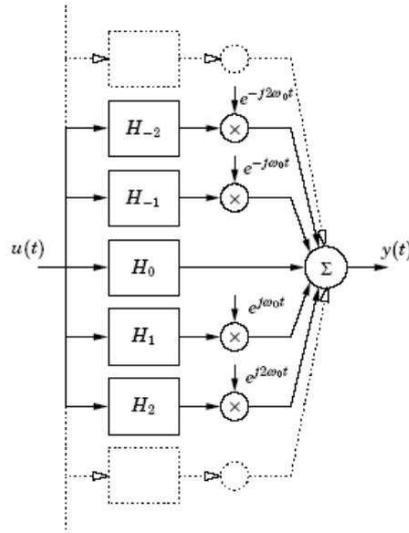


Figure 8: Schematic Representation of (10)

We will now introduce a different notation which will allow us to view a SISO LTP system as MIMO LTI system. In this way the well known MIMO theory can be applied straightforwardly to LTP systems.

Let's constrain s to variate in the complex plane strip given by

$$-\omega_0/2 \leq \text{Im}(s) \leq \omega_0/2$$

and define

$$\begin{aligned} \mathcal{U}(s) &= [\dots, U(s - j\omega_0), U(s), U(s + j\omega_0), \dots]^T \\ &= [\dots, U_{-1}(s), U_0(s), U_1(s), \dots]^T \end{aligned}$$

In this way $\mathcal{U}(s)$ is an equivalent vectorial representation of $U(s)$, where each component $U_k(s)$ of the vector $\mathcal{U}(s)$, includes the information which are contained in $U(s)$ when s variate in the complex plane strip

$$k\omega_0 - \omega_0/2 \leq \text{Im}(s) \leq k\omega_0 + \omega_0/2.$$

Analogously we define $\mathcal{Y}(s)$.

The map from $\mathcal{U}(s)$ to $\mathcal{Y}(s)$ can be written in the matrix form:

$$\mathcal{Y}(s) = \mathcal{H}(s)\mathcal{U}(s) \tag{11}$$

with

$$\mathcal{H}(s) = \begin{bmatrix} \vdots & & & & \vdots \\ \dots & H_{-1,-1} & H_{-1,0} & H_{-1,1} & \dots \\ \dots & H_{0,-1} & H_{0,0} & H_{0,1} & \dots \\ \dots & H_{1,-1} & H_{1,0} & H_{1,1} & \dots \\ \vdots & & & & \vdots \end{bmatrix}$$

$H_{n,m}$ describes the coupling between the spectral content of the the output, around $jn\omega_0$ with the spectral content of the input, around $jm\omega_0$.

$H_{n,m}$ has a special structure, as it can be proved considering:

$$\begin{aligned} Y(s + jn\omega_0) &= \sum_{k=-\infty}^{+\infty} H_k(s + j(n-k)\omega_0)U(s + j(n-k)\omega_0) = \left[m \triangleq n - k \right] \\ &= \sum_{k=-\infty}^{+\infty} H_{n-m}(s + jm\omega_0)U(s + jm\omega_0) \end{aligned} \quad (12)$$

which means that:

$$H_{n,m}(s) = H_{n-m}(s + jm\omega_0)$$

Hence the I-O relation (11) has the form:

$$\begin{bmatrix} \vdots \\ Y(s - j\omega_0) \\ Y(s) \\ Y(s + j\omega_0) \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & & & & \ddots & & \\ \ddots & H_0(s - j\omega_0) & H_{-1}(s - j\omega_0) & H_{-2}(s - j\omega_0) & \ddots & & \\ \ddots & H_1(s) & H_0(s) & H_{-1}(s) & \ddots & & \\ \ddots & H_2(s + j\omega_0) & H_1(s + j\omega_0) & H_0(s + j\omega_0) & \ddots & & \\ \ddots & & & & \ddots & & \end{bmatrix} \begin{bmatrix} \vdots \\ U(s - j\omega_0) \\ U(s) \\ U(s + j\omega_0) \\ \vdots \end{bmatrix}$$

In this way a LTP system is viewed as a MIMO LTI system with infinite inputs and infinite outputs where each component of the input and output vector represents a portion of the spectral content of the signal. It is clear that this kind of model can describe very well the frequency coupling.

Finally we give

Definition 3 (Harmonic Transfer Function)

We define $\mathcal{H}(s)$ **Harmonic Transfer Function** of the system.

Remark 2.1

Note that $\mathcal{H}(s)$ is an infinite dimensional matrix.

Any numerical analysis will be forced to use therefore a truncated approximation of this matrix and to face the question of the approximation error.

A detailed mathematical analysis of the truncation issue can not be developed, for reason of brevity in this thesis. It can be found in [1].

2.1.2 Exponentially Modulated Periodic signals

Another basic property of LTI systems, which gives a deep insight on the meaning of the transfer function, is the following:

Proposition 4 (Frequency separation)

The complex exponential $e^{-s_0 t}$ is eigenfunction of LTI systems, i.e:

$$e^{s_0 t} \xrightarrow[H(s)]{\mathcal{S}} (H(s_0)) \cdot e^{s_0 t}$$

Because of the frequencies coupling, it is no longer true that the complex exponential is an eigenfunction of LTP systems. Anyway there exists an analogous property for LTP systems.

Let's introduce a set of signals that, in LTP systems analysis, will play the role that the complex exponential plays in LTI systems analysis.

Definition 4

We define **Exponentially Modulated Periodic signal (EMP)** a signal that has the form:

$$u(t) = u_p(t)e^{s_0 t}$$

where $u_p(t)$ is a T -periodic signal, that hence can be written as a Fourier series:

$$u(t) = \left(\sum_{k=-\infty}^{+\infty} u_k e^{jk\omega_0 t} \right) e^{s_0 t} = \sum_{k=-\infty}^{+\infty} u_k e^{(s_0 + jk\omega_0)t}$$

So we can give the following extension to LTP systems of the above proposition:

Proposition 5

The Exponentially Modulated Periodic signal $u_p(t)e^{s_0 t}$ is eigenfunction of LTP systems, i.e:

$$u_p(t)e^{s_0 t} = \sum_{k=-\infty}^{+\infty} u_k e^{(s_0 + jk\omega_0)t} \xrightarrow[\mathcal{H}(s)]{\mathcal{S}} y_p(t)e^{s_0 t} = \sum_{k=-\infty}^{+\infty} y_k e^{(s_0 + jk\omega_0)t}$$

Where:

$$\begin{bmatrix} \vdots \\ y_{-1} \\ y_0 \\ y_1 \\ \vdots \end{bmatrix} = \mathcal{H}(s_0) * \begin{bmatrix} \vdots \\ u_{-1} \\ u_0 \\ u_1 \\ \vdots \end{bmatrix}$$

2.1.3 Harmonic Transfer Matrix

In the case of stable LTI systems it is of special interest consider what happen when we use as input a complex exponential with $s = 0$, namely a constant signal $u(t) = U$. As we said before the output will be a constant signal:

$$Y = G(0)U$$

$G(0) = G(s)|_{s=0}$, called *static gain*, is the (steady state) gain of the system when the input is a constant⁴.

Analogously, of special interest is analyze the response of a LTP system to an EMP signal with $s_0 = 0$, namely an ordinary periodic signal with the same period of the LTP system.

As we showed before also the output will be a EMP signal with $s_0 = 0$.

$$u_p(t) = \sum_{k=-\infty}^{+\infty} u_k e^{jk\omega_0 t} \xrightarrow[\mathcal{H}(s)]{\mathcal{S}} y_p(t) = \sum_{k=-\infty}^{+\infty} y_k e^{jk\omega_0 t}$$

And

$$\mathcal{Y} = \mathcal{H}(0)\mathcal{U}$$

In this case \mathcal{U} and \mathcal{Y} are just constant vectors of the Fourier Coefficients of $u(t)$ and $y(t)$ respectively. Therefore $\mathcal{H}(0)$ is a kind of (steady state) matrix gain that give the Fourier coefficients of the output when the input is a T -periodic signal of known Fourier coefficients⁵.

$\mathcal{H}(0)$, usually called *Harmonic Transfer Matrix*, *HTM*⁶ is for LTP systems what the static gain is for LTI .

This steady state response matrix have been developed for several electrical components, for instance, transformer with nonlinear saturated curves, HVDC converters, and static var compensator. A general method for the identification of \mathcal{H} from experimental data have been developed by Möllerstedt, 1998.

The HTM is a powerful description, because it define completely the steady state behavior of the system when a periodic input is applied.

But the information in $\mathcal{H}(0)$ are not sufficient to describe the transient and unfortunately neither to describe stability properties of the system under aperiodic perturbation. This information, as we will be shown later, is contained in $\mathcal{H}(j\omega)$, with $-\omega_0/2 \leq \omega < \omega_0/2$.

⁴To a physical system is not applied a constant but a step. In this case, $G(0)$ should be seen as a *steady state* signals ratio.

⁵To a physical system is not applied a purely periodic signal but a signal that has the form $u_p(t)1(t)$. In this case, to be formal, we should to speak of a *steady state* relation, analogously to the previous note.

⁶In some papers it is called Harmonic Norton Equivalent and is the Jacobian in harmonic balancing of electrical network.

2.2 State space approach in systems analysis

A great number of physical systems can be modeled as a differential equation, as the following:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), t) \\ y(t) = g(x(t), u(t), t) \end{cases} \quad (13)$$

where $x(t) \in \mathbb{R}^n$ represents the state of the system, $u(t) \in \mathbb{R}^m$ the input and $y(t) \in \mathbb{R}^p$ the output of the system.

Based on the properties of f and g we can classify the systems as follow:

- The system (13) is **Linear** if the functions f and g are linear, i.e.:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) + D(t)u(t) \end{cases} \quad (14)$$

- The system (14) is **Linear Time Periodic (LTP)**, if the matrices are periodic functions of t , i.e.:

$$\exists T : \quad A(t+T) = A(t) \quad \forall t$$

and similarly for $B(t)$, $C(t)$ and $D(t)$.

- The system (14) is **Linear Time Invariant (LTI)** if the matrices A, B, C and D are constant, i.e.:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \quad (15)$$

2.2.1 HTF of LTP Systems in State Space Form

As it is known from realization theory a LTI system in state space form (15) has transfer function:

$$G(s) = C(sI - A)^{-1}B + D$$

defined when s is not a singularity of $G(s)$.

In [2] and [6] an analogous expression that give the HTF of a LTP system in state space form (14) is define via direct computation of the response to EMP signals. It is also proved that if the input of the system is EMP also the state vector will be EMP, not only the output.

Consider the a LTP system in the form (14):

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) + D(t)u(t) \end{cases}$$

Where A, B, C, D are T -periodic (matrix) functions and they have (matrix) Fourier coefficients A_k, B_k, C_k, D_k respectively.

Let's define $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the block Toepliz matrices of Fourier coefficients:

$$\mathcal{A} = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & A_0 & A_{-1} & A_{-2} & \ddots \\ \ddots & A_1 & A_0 & A_{-1} & \ddots \\ \ddots & A_2 & A_1 & A_0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

and \mathcal{N} the block diagonal matrix: $\mathcal{N} = \text{blockdiag} \{ jk\omega_0 I \}$.

With this notation the HTF of the LTP system (14) becomes:

$$\mathcal{H}(s) = \mathcal{C}(sI - (\mathcal{A} - \mathcal{N}))^{-1}\mathcal{B} + \mathcal{D}$$

Proof

See[2] and [6] but also in [1].

#

Note the analogy between this expression and the previous one for LTI systems.

This expression becomes sightly more simple in the case that the matrix $A = \hat{A}$ is constant while B, C, D are still periodic. In this case in fact hold the following:

$$H_k(s) = \sum_l C_{k-l}((s + jl\omega_0)I - \hat{A})^{-1}B_l + D_k$$

As can be proved via direct computation from the previous general relation.

2.3 HTF of some basic systems

In this section we will introduce the HTF of some basic systems and the rule to calculate the HTF of the connection of two systems.

2.3.1 LTI Systems

As we said before a LTI system can be considered as a special LTP system periodic of each period T . Because of the frequency separation property of LTI systems, there is no frequency coupling between different portions of the spectral content, hence:

$$H_{m-n}(s) = 0 \quad \forall m \neq n$$

while for $m = n$ the coupling his described by the transfer function $G(s)$:

$$H_0(s) = G(s)$$

hence

$$\mathcal{H}(s) = \begin{bmatrix} \ddots & & \ddots & & \ddots & & \ddots & & \ddots \\ \ddots & G(s - j\omega_0) & 0 & 0 & \ddots & & & & \\ \ddots & 0 & G(s) & 0 & \ddots & & & & \\ \ddots & 0 & 0 & G(s + j\omega_0) & \ddots & & & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & & & & \ddots \end{bmatrix}$$

2.3.2 Multiplication Operator

Let's consider, first of all, the following example:

$$y(t) = \sin(\omega_0 t)$$

If an EMP signal is chosen as input signal $u(t)$, we have that:

$$\begin{aligned} y(t) = \sin(\omega_0 t)u(t) &= \left(\frac{e^{jk\omega_0 t} - e^{-jk\omega_0 t}}{2j} \right) \sum_{k=-\infty}^{+\infty} u_k e^{(s_0 + jk\omega_0)t} = \\ &= \sum_{k=-\infty}^{+\infty} u_k \frac{e^{(s_0 + j(k+1)\omega_0)t} - e^{(s_0 + j(k-1)\omega_0)t}}{2j} = \end{aligned}$$

which can be rewritten as

$$= \sum_{k=-\infty}^{+\infty} \frac{u_{k-1} - u_{k+1}}{2j} e^{(s_0 + jk\omega_0)t}$$

therefore:

$$y_k = \frac{u_{k-1} - u_{k+1}}{2j}$$

Therefore the HTF has the following form:

$$\mathcal{H}(s) = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & 0 & -1 & 0 & \ddots \\ \ddots & 1 & 0 & -1 & \ddots \\ \ddots & 0 & 1 & 0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

More generally it can be proved that the HTF of the system:

$$y = D(t)u$$

where $D(t)$ is a periodic function, is:

$$\mathcal{D} = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & D_0 & D_{-1} & D_{-2} & \ddots \\ \ddots & D_1 & D_0 & D_{-1} & \ddots \\ \ddots & D_2 & D_1 & D_0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

where D_k is k -th Fourier coefficient of $D(t)$.

Note that the same result can be obtained straightforwardly using what we said in section 2.2.1.

2.3.3 LTP systems connection

The connection between two LTP systems of period T_1 and T_2 is a LTP system of period T_c if and only if $T_1/T_2 \in \mathbb{Q}$.

In that case the connection can be considered periodic of period T_c equal to the Least Common Multiple between T_1 and T_2 :

$$\begin{aligned} T_c = LCM(T_1, T_2) & \quad T_c = N * T_1 & \quad N, M \in \mathbb{N} \\ & \quad T_c = M * T_2 \end{aligned}$$

Since T_c is a multiple of T_1 and T_2 , the two systems can be both modeled as T_c periodic and the usual algebraic MIMO systems operation are straightforwardly extended to HTF. Hence we have that:

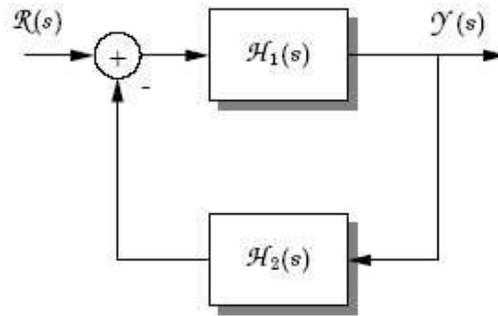


Figure 9: Feedback connection of LTP systems

- The HTF for two LTP systems in parallel is:

$$\mathcal{H}_{\parallel} = \mathcal{H}_1 + \mathcal{H}_2$$

- The HTF for two LTP systems in series is:

$$\mathcal{H}_{12} = \mathcal{H}_1 \mathcal{H}_2$$

- The HTF for two LTP systems in feedback, namely connected as it is shown in figure (9), becomes:

$$\mathcal{H}_{cl} = (I + \mathcal{H}_1 \mathcal{H}_2)^{-1} \mathcal{H}_1$$

Note that the above formulas are valid only if both the two systems are periodic of the same period.

Therefore the HTF of a complex system, made by the connection of a many basic systems, can then be derived with the usual systems algebra.

2.4 Frequency analysis techniques for LTP systems

Bode Plot

In LTI systems analysis it is very common to illustrate the frequency response of a system by means of the so-called *Bode Plot*. It is a couple of two-dimensional subplots with the frequency on the x-axis and on the y-axis, respectively, gain and phase. We can imagine to construct it by sending in input to the system a complex exponential of frequency $j\omega$, $e^{j\omega t}$. The output will be, thanks to the frequency separation property, a complex exponential with same frequency $j\omega$ and of amplitude $A = G(j\omega) = |G(j\omega)|e^{j\Phi(j\omega)}$. The Bode plots are obtained plotting $|G(j\omega)|$ and $\Phi(j\omega)$ against ω . This plot are also often in logarithmic scale to increase the readability.

An example of Bode plot for LTI systems is reported in figure 10.

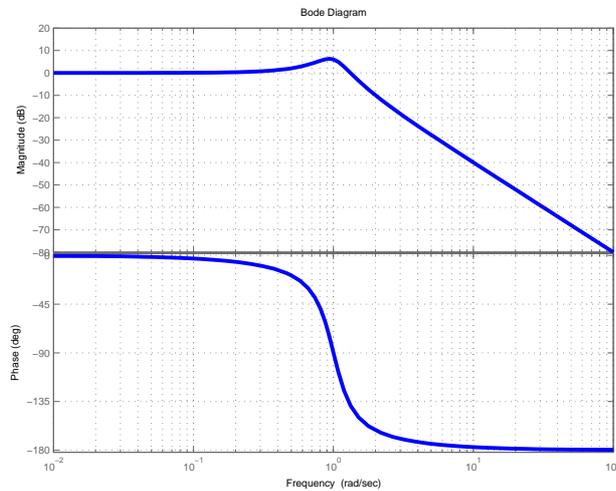


Figure 10: An example of a Bode Plot of a LTI system

Analogously we can construct a Bode plot for LTP system, but in this case it will be a three-dimensional plot with the input frequency on the x-axis, output frequency on y-axis and gain or phase in z-axis. Suppose that the input to the system is a complex exponential of frequency $j\omega$, $e^{j\omega t}$. The output will be, because of the frequencies coupling, a EMP signal that can be view as the sum of a number of complex exponential at frequency $j(\omega + k\omega_0)$. So for each couple $(f_{in}, f_{out}) = (\omega, \omega + k\omega_0)$ it can be defined a complex gain $|\tilde{G}|e^{j\Phi}$. The bode plot is then the 3D-plot of $|\tilde{G}((f_{in}, f_{out}))|$ or $\Phi(f_{in}, f_{out})$ against $(f_{in}, f_{out}) \quad \forall f_{in}, f_{out}$.

Note that this plot can be easily obtained computing $\mathcal{H}(j\omega)$ for $-\omega_0/2 \leq \omega < \omega_0/2$.

An example of Bode plot for LTP systems is reported in figure 11. It is common practice, in the LTP case, to draw only the Gain diagram in linear scale.

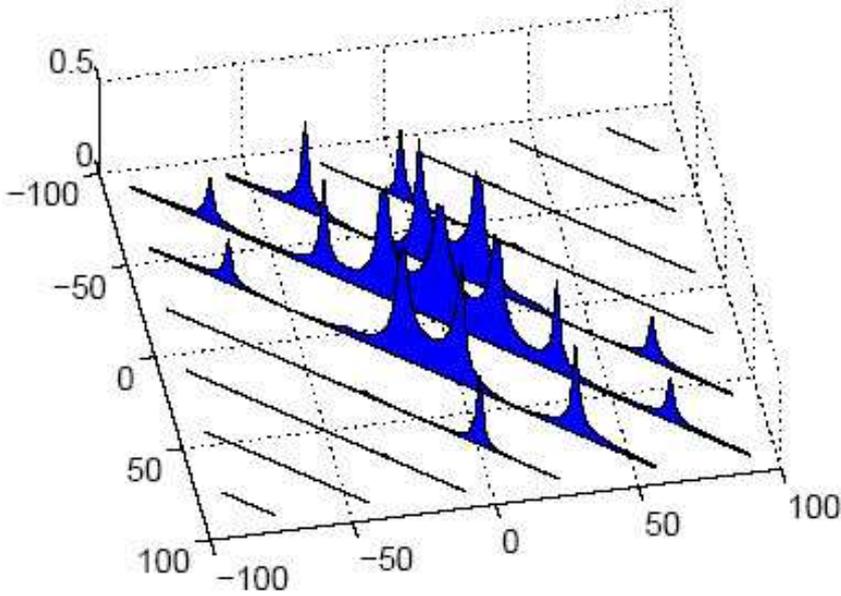


Figure 11: An example of a Bode Plot of a LTP system.

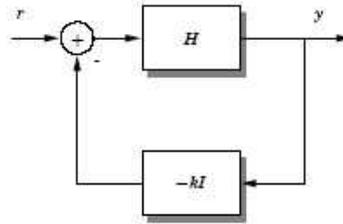


Figure 12: Feedback scheme

Nyquist Criterion

Dealing with LTI systems it is common to use the well known *Nyquist Criterion* to analyze the stability of the closed loop system illustrated in figure 12 using only informations about the open loop system H .

In the LTI case the stability can be investigated by plotting the curve defined by $G(s)_{s=j\omega}$ for $-\infty < \omega < +\infty$ and counting the number of encirclement of the point $-1/k$ that this curve does.

The Nyquist criterion can be extended to LTP case by using the generalized Nyquist criterion for MIMO systems and the HTF of the open loop system. This has been done by Wereley, 1991 ([2]), as reported in [1]:

Theorem 6

Assume a linear, periodic, causal input-output relation between y and u described by the HTF $\mathcal{H}(s)$. Denote by $\{\lambda_i(s)\}_{i=-\infty}^{+\infty}$ the eigenvalues of the double infinity matrix $\mathcal{H}(s)$ for s varying in the contour of figure 13. The eigenvalues produce a number of closed curves in the complex plane called eigen-loci of the HTF. The closed loop system in figure 12 is L_2 -stable from r to y if and only if the total number of counterclockwise encirclements of the point $-1/k$ of this curves equals the number of open-loop right half plane poles of $\mathcal{H}(s)$ (hence zero if \mathcal{H} is stable)

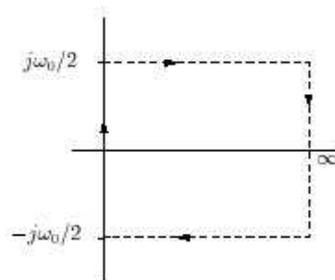


Figure 13: The integration contour in the Nyquist Criterion for LTP systems

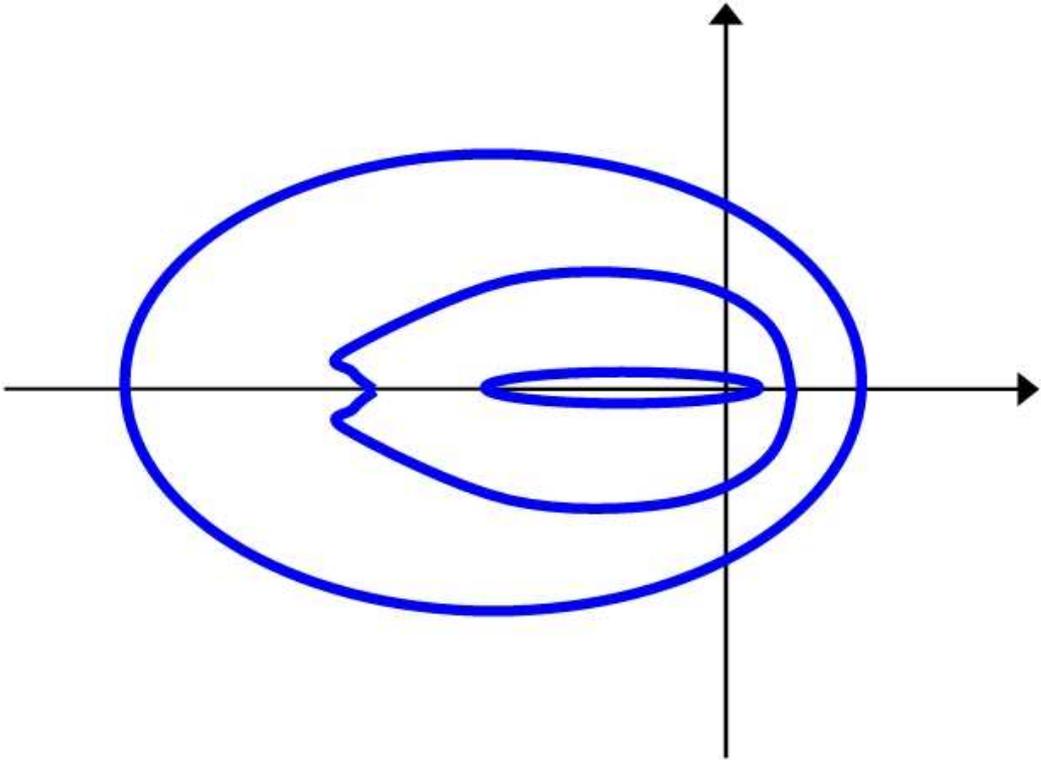


Figure 14: Example of Nyquist plot for a LTP system

3 Properties and Analysis of LTP systems in State Space

Thanks to an important (and quite old) results, known as Floquet Decomposition theorem, the consolidate theory for the analysis of LTI systems in state space can be generalized quite straightforwardly to LTP systems. In section we discuss this generalization.

The techniques here reported are not directly used in this thesis, therefore, if the reader is mainly interested in how the problem of the subharmonics quantification has been solved, this section should be skipped.

3.1 Time Domain Techniques for Analysis of Periodic Systems

As we said a linear system is described in state space by the equation:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) + D(t)u(t) \end{cases} \quad (16)$$

If the matrices A, B, C and D are T -periodic function of the time the system is linear time periodic. If the matrices A, B, C and D are constant the system is linear time invariant.

To analyze the properties of a linear system in the form (16) can be difficult. For example, even if the eigenvalues of the dynamic matrix $A(t)$ always remain in the left half plane, stability can not be guaranteed.

Anyway the consolidate theory for the analysis of LTI systems can be extended quite straightforwardly to LTP systems thanks to Floquet Decomposition theorem. It says that there always exists that a T -periodic state transformation, called *Floquet Decomposition*, that transforms the system into a similar state space form where the new dynamic matrix is constant $\hat{A}(t) = \hat{A}$. Since a periodic state transformation does not change the stability properties of the system, all the stability theory developed for LTI systems can be reused for LTP systems, by analyzing \hat{A} instead that $A(t)$.

Consider the system:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad x(t_0) = x_0 \quad (17)$$

It can be proved that the solution $x(t)$ has the form:

$$x(t) = \Phi(t, t_0)x_0 + \int_0^t \Phi(t, \tau)B(\tau)u(\tau)d\tau \quad (18)$$

Where $\Phi(t, \tau)$, called *state transition matrix*, satisfies:

$$\frac{\partial}{\partial t}\Phi(t, \tau) = A(t)\Phi(t, \tau), \quad \Phi(\tau, \tau) = I \quad (19)$$

In the case of LTI systems the analytical expression of $\Phi(t, \tau)$ is given by:

$$\Phi(t, \tau) = e^{A(t-\tau)}$$

It is also well known that a LTI system is asymptotically stable if and only if the eigenvalues of A are in the left half complex plane.

Let's consider now the case of a LTP system.

Notice, first of all, that in this case, since $A(t)$ is T -periodic, also $\Phi(t, \tau)$ is periodic, i.e.:

$$\Phi(t + T, \tau + T) = \Phi(t, \tau)$$

Further, we give the following two definitions:

Definition 5

The matrix

$$\Phi_T(t) \triangleq \Phi(t + T, t)$$

is called **Monodromy Matrix** of the system at the time t .

The eigenvalues of $\Phi_T(t)$ are called the **Characteristic Multipliers**.

$\Phi_T(t)$ is a T -periodic matrix function of t which describe how the state evolves after one period when no input signal is applied, as equation (18) suggest:

$$x(t_0 + T) = \Phi_T(t_0)x(t_0)$$

Consider, for instance, $t_0 = 0, t_0 = T \dots t_0 = nT$. Equation above gives:

$$\begin{aligned} x(T) &= \Phi_T(0)x(0) \\ x(T + 1) &= \Phi_T(T)x(T) = \Phi_T(0)x(T) \\ &\vdots \\ x((n + 1)T) &= \Phi_T(nT)x(nT) = \Phi_T(0)x(nT) \end{aligned}$$

Hence, the monodromy matrix $\Phi_T(t_0)$ can be seen as dynamic matrix of the discrete system that describes the LTP system sampled in $t_0, t_0 + T \dots t_0 + nT$.

We have hence that, as it can be expected, a periodic system sampled exactly with $T_{\text{semp}} = T$, can be described with a time invariant system. The sampling acts in this case as a stroboscope.

Moreover it can also be proved that the eigenvalues of $\Phi_T(t_0)$ are independent on t_0 . It is clear therefore that $\Phi_T(t_0)$ and the characteristic multipliers play a central role in the stability analysis of the system.

Let's introduce now Floquet results:

Theorem 7 (Floquet, 1883)

For any LTP system in state space form (14) there exists a T -periodic state transformation $x = P(t)v$, that transforms the system to a similar one with constant (generally complex-valued) dynamic matrix:

$$\begin{cases} \dot{v}(t) = \hat{A} v(t) + \hat{B}(t)u(t) \\ y(t) = \hat{C}(t)v(t) + \hat{D}(t)u(t) \end{cases}$$

The transformation $x = P(t)v$ is called **Floquet decomposition**.

Note that, even if $A(t)$ is real, \hat{A} might become a complex matrix.

Corollary 8

In LTP systems the state transition matrix has the form:

$$\Phi(t, \tau) = P(t)e^{\hat{A}(t-\tau)}P^{-1}(\tau)$$

Inserting this expression into equation (19), we obtain:

Corollary 9

$$\frac{d}{dt}P(t) = A(t)P(t) + P(t)\hat{A} \quad P(0) = P_0 \quad (20)$$

This result can be used to compute $P(t)$.

Further we have that the monodromy matrix of the system have the form:

$$\Phi_T(t) = \Phi(t + T, t) = P(t + T)e^{\hat{A}(t+T-t)}P(t) = P(t)e^{\hat{A}T}P(t)$$

where the last equality is due to the fact that $P(t)$ is a periodic transformation. By choosing $P(0) = I$ as initial condition of the differential equation (20) we have that:

$$\Phi_T(0) = \Phi(T, 0) = e^{\hat{A}T}$$

This shows the relation between \hat{A} and $\Phi(T, 0)$: $\hat{A}T$ is the logarithm of $\Phi(T, 0)$. Since a periodic coordinate transformation can not change the stability property of a system, the following results can be extend from LTI theory :

Definition 6

The **Lyapunov exponents of the system** are the eigenvalues of \hat{A} .

Proposition 10

A T -periodic linear system is asymptotically stable if and only if all the Lyapunov exponents are in the open left half plane.

As we said before the eigenvalues of the monodromy matrix $\Phi_T(t)$ do not depend on time. In particular $\text{eig}\{\Phi_T(t)\} = \text{eig}\{\Phi_T(0)\} = \text{eig}\{e^{AT}\}$. Hence we have:

Proposition 11

A LTP is asymptotically stable if and only if all its characteristic multipliers, i.e. the eigenvalues of the monodromy matrix, are inside the open unit disc.

Also Lyapunov theorem has a quite straightforward extension to the periodic case:

Proposition 12

A LTP is asymptotically stable if the Lyapunov equation

$$\frac{d}{dt}P(t) = P(t)A^T(t) + A(t)P(t) + Q(t)$$

has a unique positive definite solution $P(t)$, for $Q(t)$ periodic positive definite. The solution has the form:

$$P(t) = \Phi^T(0, t)P_0\Phi(0, t) - \int_0^t \Phi^T(s, t)Q(s)\Phi(s, t)ds$$

3.2 Other Time Domain Techniques for Periodic Systems

Also other time domain methods for LTI systems are often straightforward to generalize to periodic systems. In the following we will enumerate, for sake of completeness, some results presented in [1] :

The periodic pair $(A(t), B(t))$ is said to be *Stabilizable* if there exist a T -periodic $L(t)$ so that $A(t) - B(t)L(t)$ is stable.

Likewise, the periodic pair $(A(t), C(t))$ is said to be *Detectable* if there exist a T -periodic $K(t)$ so that $A(t) - K(t)C(t)$ is stable.

Pole placement can not be used in a straightforward way, since the eigenvalues of LTP systems are time varying. One possible strategy would be to find a feedback gain $L(t)$ that gives the desired characteristics multipliers.

Also linear quadratic optimal control can be extended. Consider the problem to find the feedback law $u(t) = -L(t)x(t)$ that minimize the quadratic cost function:

$$J = \int_0^{+\infty} (x^T Q x + u^T R u) dt$$

If some technical condition are fulfilled, the optimal feedback gain is given by:

$$L(t) = R^{-1}B^T(t)X(t)$$

where $X(t)$ is the unique T -periodic positive definite solution of periodic Riccati equation

$$-\frac{d}{dt}X(t) = A^T(t)X(t) + X(t)A(t) - X(t)B(t)R^{-1}B^T(t)X(t) + Q$$

4 Dc-Dc converter model using LTP system

In this section we will introduce a LTP approximated model for some basic components that are commonly used in Power Electronic Converters. We will analyze with particular attention PWM modulators, components with hard non linearity non trivial to be modeled. For this components a LTP model is developed and validated via time simulation.

Finally a LTP model for the step-down converter presented in section 1.3, is derived by using these basic component models.

4.1 Basic Components Model

Some basic components can be recognized in all the power electronic converters. The HTF of for these components are derived in the following:

4.1.1 Linear Filter and Controller

In the most of practical implementations a power electronic stage is followed by a low pass filter stage. Sometime it is not necessary to add an extra filtering stage because the low pass filtering action is performed directly by the load.

The most natural way to model the low pass filter is to describe it as a LTI system.

If a LTP model is required it can be obtained from the previous applying the method that we described in section 2.3.

4.1.2 Sampler and Holder

Sometimes in power electronic schemes it is introduced a sampler and holder circuit, for instance to avoid some unwanted variations of the input signal during the PWM period and to prevent in this way multiple switchings. To give a model of this component it is useful to consider separately the effect of the Sampler and of the Holder.

Sampler

We can imagine to describe the output $y(t)$ of a sampler as a train of pulses with of area $u(t_k)$:

$$y(t) = \sum_k \delta(t - kT)u(t_k)$$

since $\delta(t) = 0 \forall t \neq 0$, the previous formula can be rewritten as

$$y(t) = \sum_k \delta(t - kT)u(t) \quad (21)$$

Equation (21) gives an IO description of the sampler as a multiplication operation: $y(t) = D_s(t)u(t)$, where:

$$D_s(t) = \sum_k \delta(t - kT)$$

which is a periodic function of period T . This proves that the sampler is a LTP system that, as known, introduces frequency coupling between frequencies separate by a multiple of the sampling frequency.

As reported in section 2.3 the HTF of a system described with the multiplication with a periodic function is:

$$\mathcal{D} = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & D_0 & D_{-1} & D_{-2} & \ddots \\ \ddots & D_1 & D_0 & D_{-1} & \ddots \\ \ddots & D_2 & D_1 & D_0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

where D_k is the k -th Fourier coefficient of $D(t)$, i.e. the k -th sample of the discrete signal $\hat{D}(k\omega_0)$, the Fourier transform of the periodic signal $D(t)$. $\hat{D}(k\omega_0)$ is defined as:

$$\begin{aligned} \hat{D}(k\omega_0) &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} D(t) \cdot e^{-jk\omega_0 t} dt = \\ &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{n=-\infty}^{+\infty} \delta(t - kT) \cdot e^{-jk\omega_0 t} dt = \\ &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) \cdot e^{-jk\omega_0 t} dt = \\ &= \frac{1}{T} \quad \forall k \end{aligned}$$

Hence the HTF becomes:

$$\mathcal{D} = \frac{1}{T} \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & 1 & 1 & 1 & \ddots \\ \ddots & 1 & 1 & 1 & \ddots \\ \ddots & 1 & 1 & 1 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

Holder

Holders are LTI systems, as it can be easily verified. The easiest way to obtain the transfer function of this component is to consider the impulse response, which is a square pulse with duration equal to the sampling period T :

$$h(t) = 1(t) - 1(t - T) \quad \xrightarrow{\mathcal{L}} \quad H(s) = \frac{1 - e^{-sT}}{s}$$

4.1.3 PWM-modulator Model

As we said in section 1.2 a PWM modulator is often realized comparing the signal $u(t)$ with the periodic signal $m(t)$, as it is showed in figure 15.

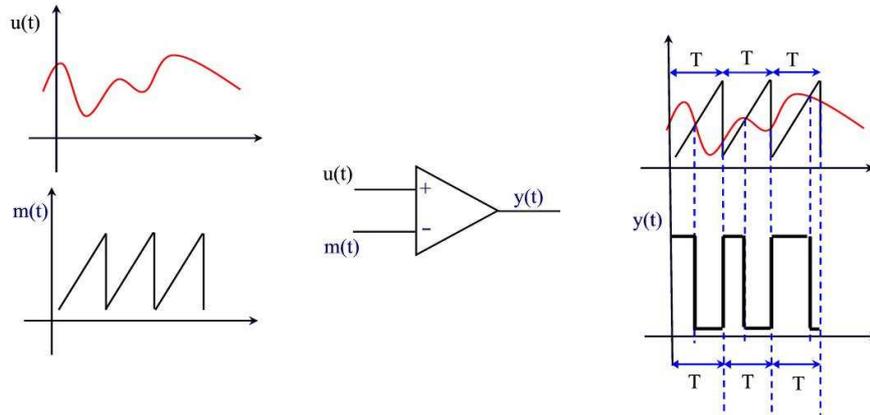


Figure 15: Realization of a PWM Modulator

To create a model for the PWM modulator we must, first of all, give a mathematical description of the comparison operation

$$y(t) = (u(t) > m(t)) = \begin{cases} 1 & \text{if } u(t) > m(t) \\ 1/2 & \text{if } u(t) = m(t) \\ 0 & \text{if } u(t) < m(t) \end{cases}$$

or, equivalently, of the operation

$$y(t) = (u(t) - m(t) > 0) = \begin{cases} 1 & \text{if } u(t) - m(t) > 0 \\ 1/2 & \text{if } u(t) - m(t) = 0 \\ 0 & \text{if } u(t) - m(t) < 0 \end{cases} \quad (22)$$

A simple way to model the logical operation $\cdot > 0$ is to use the nonlinear function step, $1(\cdot)$. Hence the previous relation (22) becomes:

$$y(t) = 1(u(t) - m(t))$$

where $m(t)$ is a known periodic function.

First of all note this system is strongly nonlinear, so there is no way to give a exact linear model and least of all a LTI or LTP one.

Example 4.1 (Non linearity of PW modulator)

Suppose $m(t) \in [0, 1]$.

If $u_1(t) = 0.5$ we will have a perfect square wave of amplitude 1 in output, $y|_{u_1}$.

If $u_2(t) = 1$ we will have output will be the constant $y|_{u_2} \cong 1$.

But $u_1(t) = 2 * u_2$, therefore, if the system would be linear that the output to u_2 will be

$$y|_{u_2} = y|_{2*u_1} = 2 \cdot y|_{u_1}$$

that is a perfect square wave of amplitude 2, and not

$$y|_{u_2} \cong 1$$

as it actually happend.

However, we can linearize the system around a periodic trajectory and obtain a approximate linear model.

To do this in a generic case consider a nonlinear static function

$$y = f(u).$$

Let' define $u_{nom}(t)$ the nominal input and $y_{nom}(t)$ the nominal output; moreover let's call $\Delta u(t)$ and $\Delta y(t)$ the perturbation of the nominal input and output respectively. Hence it holds that:

$$u(t) = u_{nom}(t) + \Delta u(t) \quad \text{and} \quad y(t) = y_{nom}(t) + \Delta y(t).$$

If f is continuous, with first order derivate continuous, it can be written as a Taylor series around the point (u_{nom}, y_{nom}) :

$$y(t) = y_{nom} + \left(\frac{d}{du} f(u) \Big|_{u=u_{nom}} \right) \Delta u + \mathcal{O}(\Delta u^2) \quad \text{for} \quad u \rightarrow u_{nom}$$

Or, using other words:

$$\Delta y = y - y_{nom} \cong \left(\frac{d}{du} f(u) \Big|_{u=u_{nom}} \right) \Delta u \quad (23)$$

for $u(t)$ in a neighborhood of $u_{nom}(t)$.

Model using the distribution δ

In our case, $f(x) = 1(x - m(t))$ is not continuous. However, using the distribution theory, the previous formula (23) still holds and in our case gives:

$$\begin{aligned}\Delta y(t) = y(t) - y_{nom}(t) &\cong \left(\frac{d}{du} \left(1(u(t) - m(t)) \right) \Big|_{u=u_{nom}} \right) \Delta u(t) \\ &\cong \delta(u_{nom}(t) - m(t)) \Delta u(t)\end{aligned}$$

Hence the system can be approximated by the product with the time varying function:

$$D(t) = \delta(u_{nom}(t) - m(t))$$

That is a train of pulses in the nominal commutation instants t_n^* :

$$t_n^* \in \{t \mid u_{nom}(t) - m(t) = 0\}.$$

$$D(t) = \sum_{n=-\infty}^{+\infty} \delta(t - t_n^*)$$

If u_{nom} is a periodic function of period T_1 , if $m(t)$ is a periodic function of period T_2 and if $T_1/T_2 \in \mathbb{Q}$ than also $D(t)$ will be a periodic function.

The period T_D of $D(t)$ is the Least Common Multiple of T_1 and T_2 ,

$$T_D = LCM(T_1, T_2).$$

In the special case of u_{nom} constant, $D(t)$ will be periodic of period T_2 .

The HTF of a system described by a multiplication with a periodic function, as we said in section 2.3, is:

$$\mathcal{D} = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & D_0 & D_{-1} & D_{-2} & \ddots \\ \ddots & D_1 & D_0 & D_{-1} & \ddots \\ \ddots & D_2 & D_1 & D_0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

where D_k is the k -th Fourier coefficient of $D(t)$, that is exactly the k -th sample of the discrete signal $\hat{D}(k\omega_0)$, the Fourier transform of the periodic signal $D(t)$. $\hat{D}(k\omega_0)$ is defined as:

$$\begin{aligned}\hat{D}(k\omega_0) &= \frac{1}{T} \int_{-T/2}^{T/2} D(t) \cdot e^{-jk\omega_0 t} dt = \\ &= \frac{1}{T} \int_0^T \sum_{n=-\infty}^{+\infty} \delta(t - t_n^*) \cdot e^{-jk\omega_0 t} dt =\end{aligned}$$

Because of in this case we are integrating over a finite interval, the infinite sum becomes the finite sum of the pulses in $[0, T]$

$$\begin{aligned} &= \frac{1}{T} \sum_{t_n^* \in [0, T]} \int_0^T \delta(t - t_n^*) \cdot e^{-jk\omega_0 t} dt = \\ &= \frac{1}{T} \sum_{t_n^* \in [0, T]} e^{-jk\omega_0 t_n^*} \end{aligned}$$

If the nominal input $u_{nom}(t)$ gives only one switching per period, as in the most of real cases, than the above sum reduces to only one element.

Note, first of all, that $\hat{D}(k\omega_0)$ can be calculated only when u_{nom} and hence t_n^* are known.

Note also that, as usual, this matrix is infinite dimensional.

But in this case we are not allowed to make a truncated approximation of the harmonic transfer function, because:

$$\left| e^{j\omega_0 t_n^* k} \right| = 1 \quad \forall k$$

So

$$(e^{j\omega_0 t_n^*})^k \not\rightarrow 0 \quad \text{for} \quad k \rightarrow +\infty$$

Hence we can not use this model to describe the behavior a PWM stand alone. However if the PWM modulator is followed by a linear low pass filter, as it usually happen in the practical application, the coefficients of global HTF $HTF_{\text{PWM+LP Filer}}$ will go to zero for (m, n) growing:

$$HTF_{\text{PWM+LP Filer}}(m, n) \xrightarrow{(m, n) \rightarrow +\infty} 0$$

This implies, because of the diagonal structure of the HTF of a LTI system, that we can use a truncated approximation of the harmonic transfer function of the PWM we obtained before.

Model using a smooth approximation of $1(\cdot)$

To derive the HTF of a PWM modulator in a way such that it can be truncated we can use a smooth approximation, $g(x)$, of the function $1(x)$. $g(x)$ can be chosen continuous with first derivate continuous (function of class \mathcal{C}^1) so that the Taylor expansion can be applied directly without use the distribution theory. Some possible approximating functions which have been studied in this work are:

- arctangent:

$$1(x) \cong f(x) = \frac{1}{\pi} \arctan(\gamma \cdot x) + 0.5$$

With $\gamma > 0$ is a positive constant.

The quality of the approximation increases for γ increasing:

$$f(x) \xrightarrow{\gamma \rightarrow +\infty} 1(x)$$

- hyperbolic tangent:

$$1(x) \cong f(x) = \frac{1}{2} \tanh(\gamma \cdot x) + 0.5$$

With $\gamma > 0$ is a positive constant.

Again, the quality of the approximation increases for γ increasing:

$$f(x) \xrightarrow{\gamma \rightarrow +\infty} 1(x)$$

- smooth approximation of the function sign:

It is known that $\text{sgn}(x) = x/|x|$, and it known as well that $1(x) = \frac{1}{2} \text{sgn}(x) + 0.5$.

This suggest to approximate $1(x)$ as follow:

$$1(x) \cong f(x) = \frac{1}{2} \frac{x}{|x| + \epsilon} + 0.5$$

With $\epsilon > 0$ is a positive constant.

In this case the quality of the approximation increases for ϵ decreasing:

$$f(x) \xrightarrow{\epsilon \rightarrow 0} 1(x)$$

- using a spline

The degree of the polynomials which make the spline should be chosen ≥ 2 so that the first order derivate can be made continuous in the conjunction point. This ensures that the spline is a \mathcal{C}^1 function.

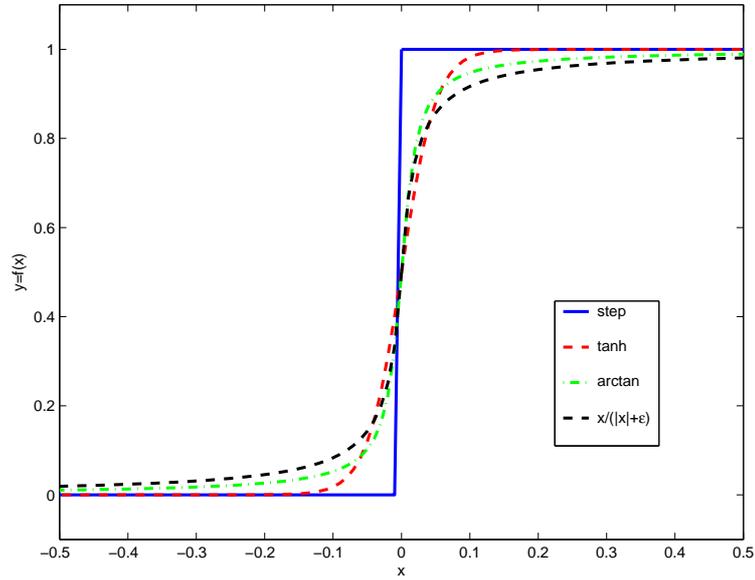


Figure 16: Comparison between the various possible smooth functions $g(x)$ that can be used to approximate $1(x)$

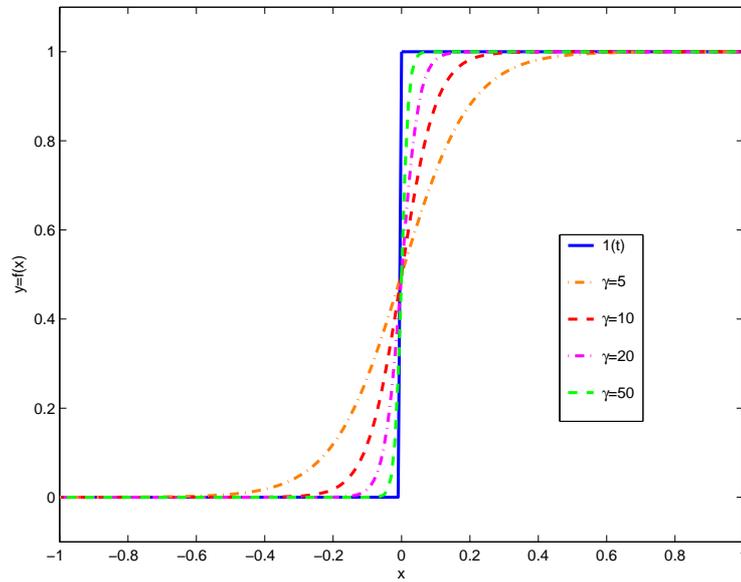


Figure 17: hyperbolic tangent approximation of $1(x)$ for various values of γ

Once the approximating function $f(x)$ has been chosen we can obtain a linear model, using Taylor expansion.

As we said previously, an approximation of Δy can be obtained from Δu according to equation (23):

$$\Delta y = y - y_{nom} \cong \left(\frac{d}{du} f(u - m) \Big|_{u=u_{nom}} \right) \Delta u$$

Hence, we can view the system as a product with a known function $D(t)$

$$D(t) = \frac{d}{du} f(u(t) - m(t)) \Big|_{u=u_{nom}}$$

Again, if both $u(t)$ and $m(t)$ are periodic and the ratio between their period is rational $D(t)$ will be a periodic function. Depending on which function we choose as approximation for $1(\cdot)$ the expression of $D(t)$ changes. For instance if we choose the hyperbolic tangent approximation $D(t)$ becomes:

$$D(t) = \frac{\gamma}{2 \cdot \cosh^2 \left(\gamma \cdot (u_{nom}(t) - m(t)) \right)}$$

Some examples of the function $D(t)$ for various values of γ are shown in figures 18 - 20. It is clear that $D(t)$ is an approximation of the train of pulses described in the previous section. Note also that for $\gamma \rightarrow +\infty$ the function $D(t)$ goes to the train of pulses:

$$D(t) \rightarrow \sum_{n=-\infty}^{+\infty} \delta(t - t_n^*)$$

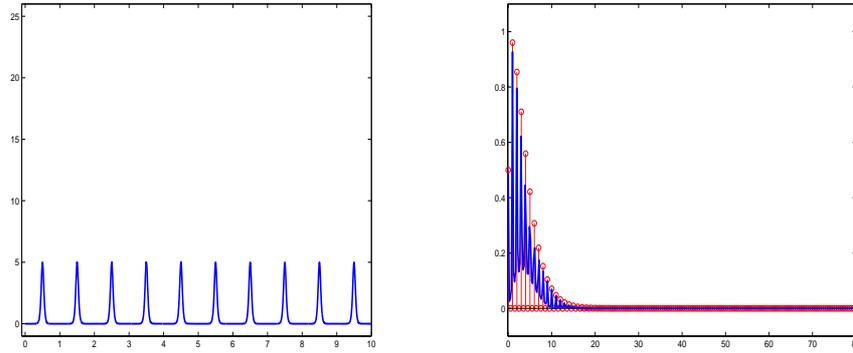
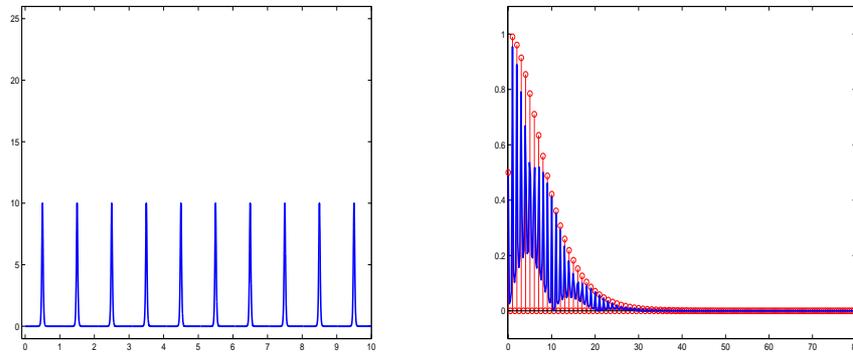
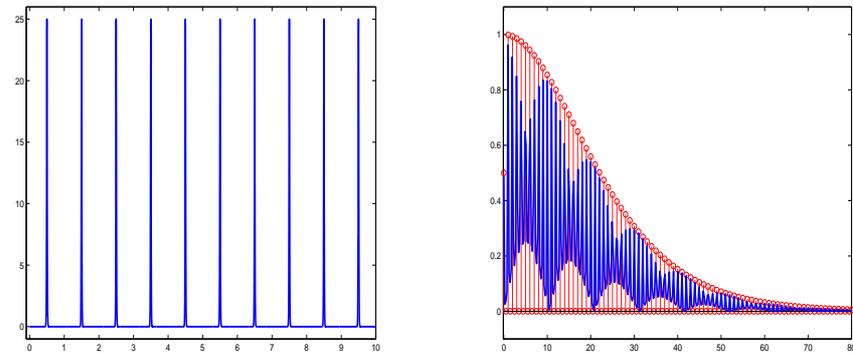
This implies also the band of the signal $D(t)$ increases obliging us to keep more frequencies when we are truncating and hence to have a bigger HTF. This effect is illustrated in figures 18 - 20.

Moreover, the quality of the Taylor approximation decreases when we increase the quality of the approximation of the step function $1(\cdot)$. Hence we are obliged to tune the parameter δ manually, to find a good trade off between a good approximation of the step function and a good Taylor approximation of the smooth function. This tuning is done performing various simulations and comparing the actual output with the approximated one, as shown in figures 21-23. It is important to consider various nominal trajectories, because the quality of the approximation depends strongly on the nominal trajectory.

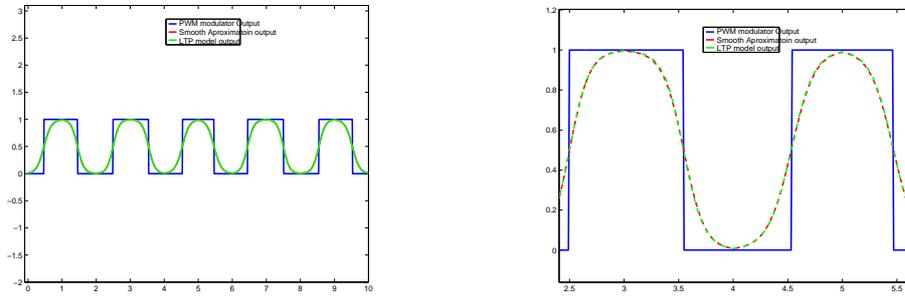
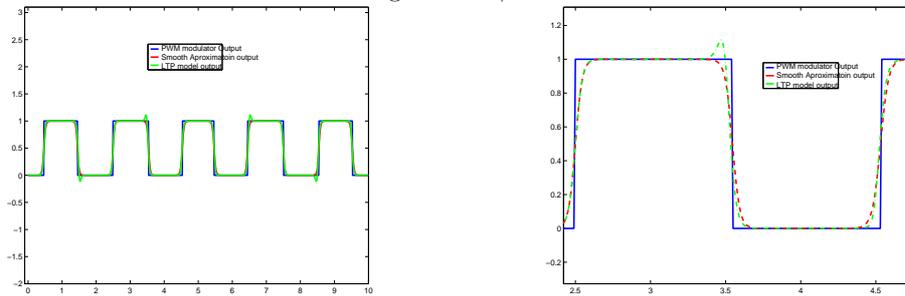
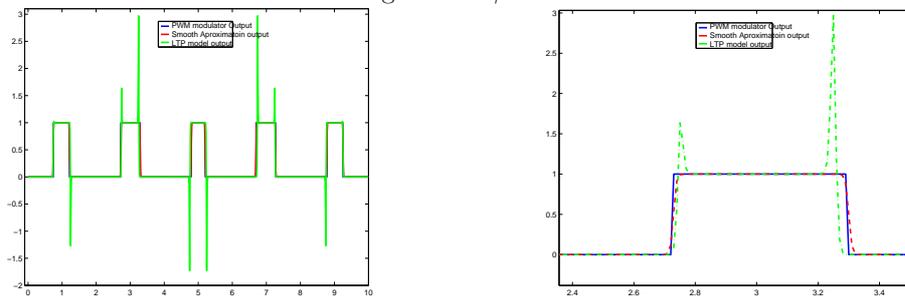
Analogous consideration can also be done for the other approximations of $1(\cdot)$.

Various tests have been performed and they showed that the best function to approximate the step for our purpose is the arctangent.

γ has been chosen $\gamma = 20$.

Figure 18: $\gamma = 10$ Figure 19: $\gamma = 20$ Figure 20: $\gamma = 50$

The left figures represent $D(t)$ and the right ones $\hat{D}(f)$ for various choices of γ and in the case of the hyperbolic tangent approximation of the step. Note that for γ rising, $D(t)$ becomes more close to the pulses train defined in the previous section and increases its band.

Figure 21: $\gamma = 5$ Figure 22: $\gamma = 20$ Figure 23: $\gamma = 100$

These figures show the effect in the approximation of a variation of the parameter γ . If γ is too small, for example $\gamma = 5$ (figures 21), the approximation of the step function is not good enough. On the other hand if γ is too big, for example $\gamma = 100$ (figures 23), we have a very good approximation of the step but a very poor Taylor approximation of the smooth function, as it is clearly shown in the zoomed right picture of 23. Therefore the best solution is to tune manually the parameter γ to have a good global approximation that is a trade off between a good approximation of the step and a good Taylor approximation. We choose $\gamma = 20$.

4.1.4 PWM Model Validation

A model validation in time domain is done comparing the actual output of the PWM with the output of the approximated model. Figures 24-28 show that the LTP model describes the behavior of the system very well.

Remark 4.1

For our purpose it is of special interest to use a constant PWM input as nominal trajectory. The results of the model validation procedure in the case of constant input of are presented in figures 24 - 27 and they show that the approximation is still very good.

These results may be surprising because it is usually very difficult to have a good approximation of a function $g(x)$ similar to a step, linearizing around a fixed point. To linearize around a fixed point means in fact to approximate the non linearity $g(x)$ with one line of fixed slope, but in a non-linearity like $g(x)$ the slope variates so much in a so short interval that the approximation is unavoidably bad.

But in our case, even if the input is a constant $u(t) = U$, we are not linearizing $g(x)$ around a static point. The non-linearity is linearized, in fact, around the periodic trajectory $U - m(t)$. This mean that $g(x)$ is described with a time varying parameter that is its slope along $U - m(t)$ and not just in a point. So instead to describe the non linearity with a slope we describe it with a set of slopes, and this enable a much more accurate approximation.

4.1.5 Switching Stage

Power electronic switches are used to increase the power of a digital signal by increasing the current that the signal can induce on the load. If we assume that the switches are ideal and that the PWM signal assumes values $[0, 1]$ we can see the switching stage, since it does not change the shape of a signal, as a gain $K = V_{in}$.

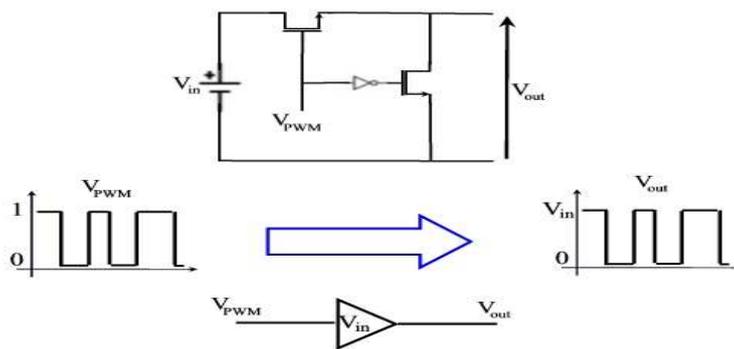
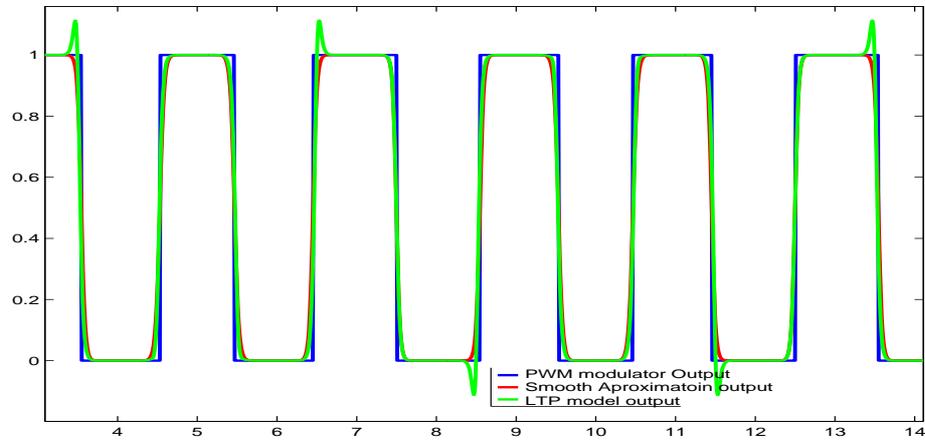


Figure 33: Model of the Switching Stage

Figure 24: Model validation, $u_{nom}(t) = 0.5$

The output of the approximated model (in green) of the PWM-modulator linearized around the periodic trajectory $u_{nom}(t) = 0.5$ is compared with the output of the actual PWM-modulator (in blue) and with the output of the model obtained using $\tanh(\cdot)$ as smooth nonlinear approximation of $1(\cdot)$ (in red).

γ has been chosen $\gamma = 20$.

$m(t)$ is in this case a sawtooth wave in the range $[0, 1]$

A perturbation $u_p(t) = 0.05 \cos(0.3\omega_0 t)$ has been added to nominal input.

These pictures show that even if subject to a perturbation of the 10% of the nominal trajectory the PWM-linear model gives a good approximation of the actual output.

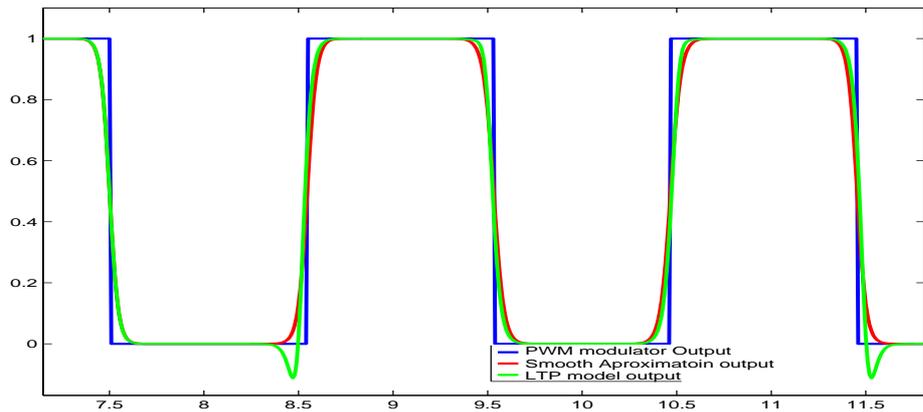
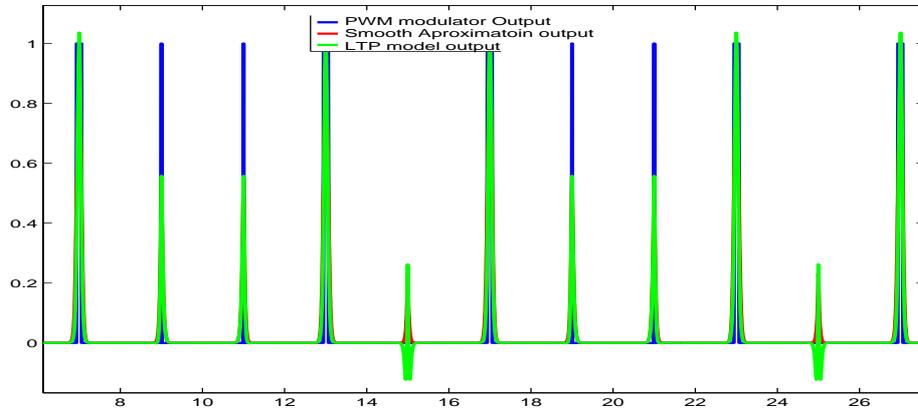


Figure 25: Zoom of the previous picture

Figure 26: Model Validation, $u_{nom}(t) = 0.1$

Analogously to the previous pictures the output of the linearized model is compared with the actual output and with a smooth approximation of the actual output. Once again γ has been chosen $\gamma = 20$.

In this case the nominal input is $u_{nom}(t) = 0.1$.

As in the previous pictures the input perturbation has been chosen $u_p(t) = 0.05 \cos(0.3\omega_0 t)$.

These pictures show that the quality of the approximation around this "difficult" trajectory decrease. It is due to the fact in this case we have very fast commutations, with short T_{on} ($T_{on_{min}} = 0.05$). So fast commutations are very difficult to be captured with a linear model.

Anyhow this very fast commutations are significantly power consuming and they decrease strongly the efficiency of the converter. So in the practical application this operating zone is always avoid as much as possible.

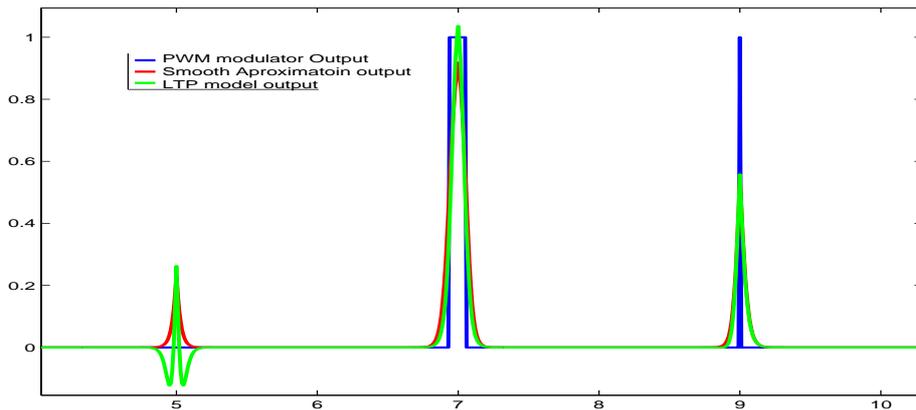


Figure 27: Zoom of the previous picture

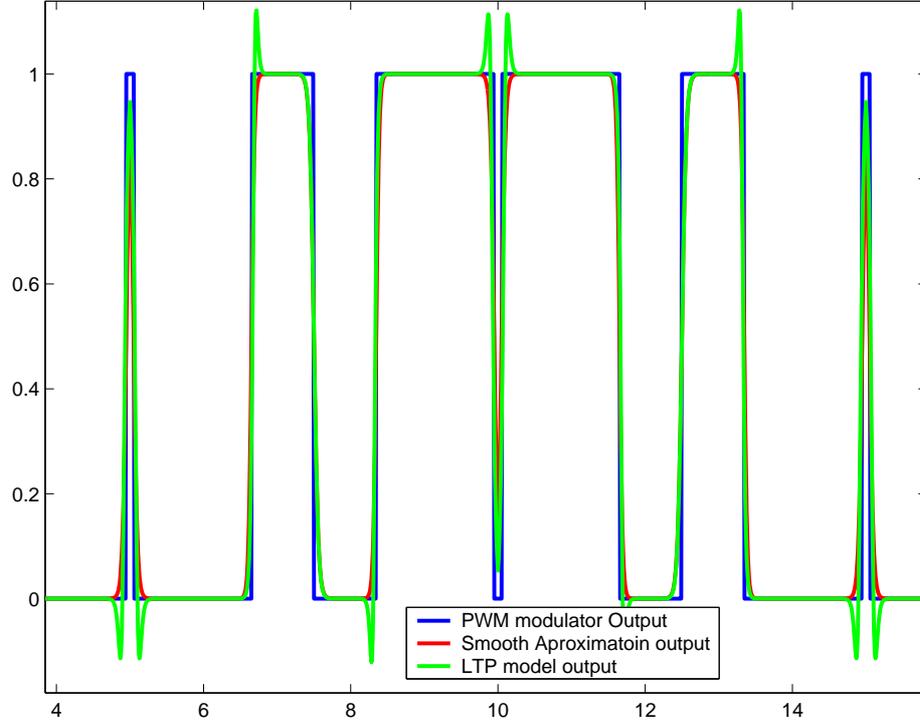


Figure 28: Model Validation, $u_{nom}(t) = 0.5 + 0.4 \cos(\omega_0 t/10)$

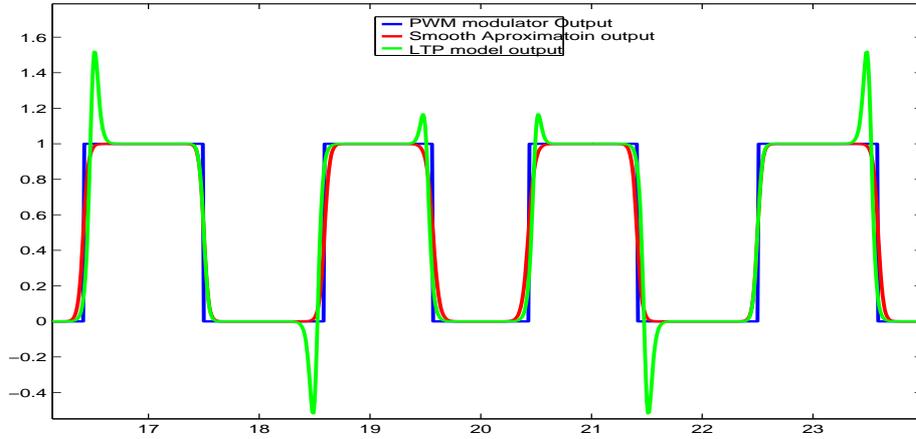
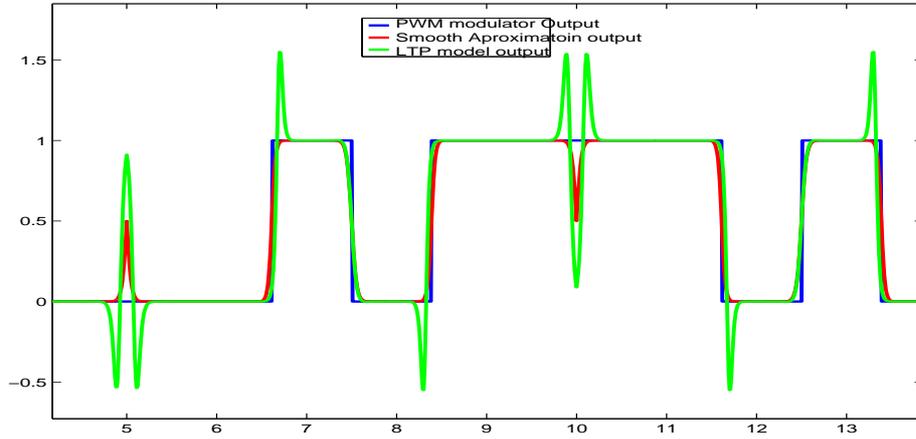
Analogously to the previous pictures the output of the linearized model is compared with the actual output and with a smooth approximation of the actual output. Once again γ has been chosen $\gamma = 20$.

In this case the nominal input is $u_{nom}(t) = 0.5 + 0.4 \cos(\omega_0 t/10)$.

As in the previous pictures the input perturbation has been chosen

$$u_p(t) = 0.05 \cos(0.3\omega_0 t).$$

This nominal trajectory has the propriety to cover almost all the input range $[0, 1]$ giving very short, normal and very long T_{on} . As we notice previously the model has some problem in capture the very fast commutations, but the approximation can be considered acceptable for most of the applications.

Figure 29: Nominal input: $u_{nom}(t) = 0.5$ Figure 30: Nominal input: $u_{nom}(t) = 0.5 + 0.4 \cos(\omega_0 t/10)u(t)$

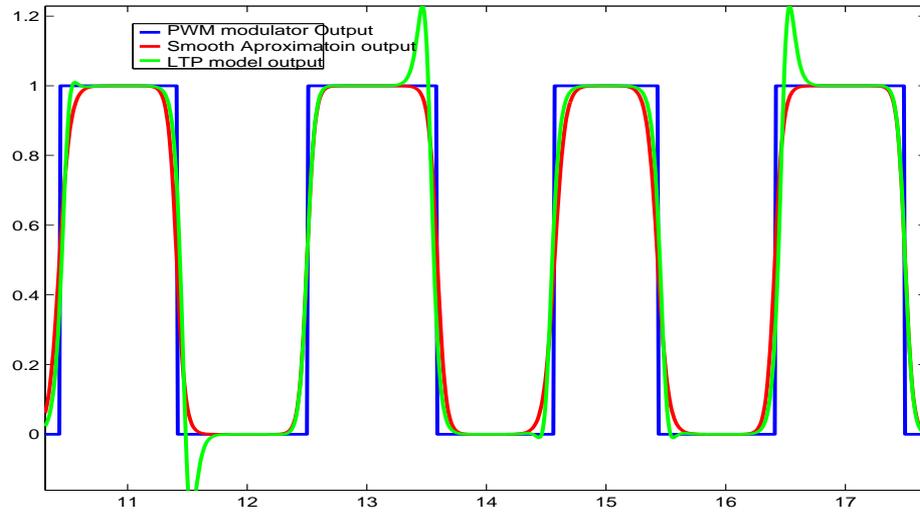
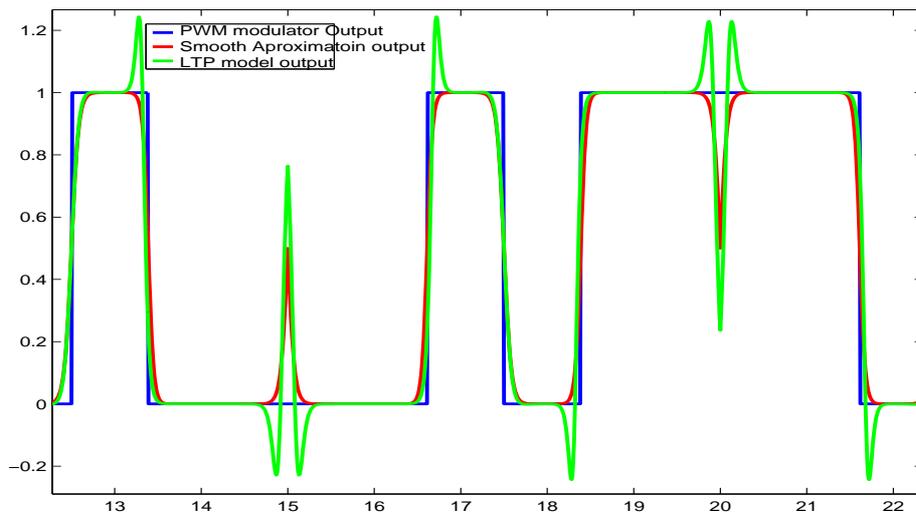
Analogously to the previous pictures the output of the linearized model is compared with the actual output and with a smooth approximation of the actual output. Once again γ has been chosen $\gamma = 20$.

In this case the amplitude of the input perturbation is increased from the 10% of the nominal amplitude to the 20% of the nominal amplitude, by choosing

$$u_p(t) = 0.1 \cos(0.3\omega_0 t).$$

The pictures show that quality of the approximation decrease, as we expect. In fact the small picks in the commutation instants increase their amplitude from the 15% of the previous case to the 45%.

Note also that, it can happen that $T_{on} \rightarrow 0$ or $T_{on} \rightarrow T_{PWM}$. In that case the approximation error becomes unavoidably rather big.

Figure 31: Nominal input: $u_{nom}(t) = 0.5$ Figure 32: Nominal input: $u_{nom}(t) = 0.5 + 0.4 \cos(\omega_0 t / 10)$

Pictures (31) and (32) have been obtained with the same conditions of the previous two but retuning γ , which has been chosen in this case $\gamma = 13$. In this way we can increase slightly the quality of the approximation.

Even if the perturbation has still an amplitude of 0.1 the amplitude of the pick is decreased from the 45% of the previous simulation to 22%.

4.2 HTF description of the Controlled Step Down Converter

The Controlled Step Down Converter (figure 34) can be represented by means of the block diagram reported in figure 35.

So far we have introduced the HTF of all the blocks constituting the diagram hence it is quite straightforward to use the techniques described in section 2 to analyze the global properties of the closed loop system. For example, since V_{in} is unknown and varying, it is natural to analyze the stability properties of the Controlled Converter with the Nyquist criteria, as we will do in section 6. This will allow us to predict the emerge of subharmonics.

However there are some things to pay attention to in modeling and connecting the sub-blocks:

1. It is important to choose accurately the nominal trajectory u_{nom} around which to linearize the PWM modulator. The best way to do this is to determinate u_{nom} via simulation, choosing a realistic V_{in}
2. To obtain a LTP model, u_{nom} must be chosen periodic. If the signal measured in the simulation is not periodic it should be decomposed in a periodic part and in a aperiodic disturbance:

$$u_{simulation} = u_{periodic} + u_{aperiodic} \quad \Longrightarrow \quad u_{nom} \triangleq u_{periodic}$$

3. As we discuss previously the approximate model of the PWM will be time periodic of period T_{PWM} if and only if the ratio between the period T_{nom} of the nominal input u_{nom} and the period T_m of the modulation function $m(t)$ is rational: $T_m/T_{nom} \in \mathbb{Q}$.
In this case it holds that:

$$T_{PWM} = \text{LCM}(T_{nom}, T_m)$$

This is very important in the presence of subharmonics. In this case in fact $T_{nom} = T/n$ and hence $T_{PWM} = T_{nom}$ while usually it happens that $T_{nom} = nT$ and hence $T_{PWM} = T_m$.

This point can introduce errors difficult to be recognized and corrected.

4. As we said previously, the usual block algebra holds only if all the components are modeled as LTP systems of the same period T .

In our scheme all the blocks, except for the PWM modulator, are LTI systems, and hence LTP of any period, in particular of period T_{PWM} . In this case it is therefore sufficient to choose

$$T = T_{PWM}.$$

Conversely, if other LTP blocks are present (for instance a sampler, as it happened in some case), it is important to remember to fix:

$$T = \text{LCM}(T_{PWM}, T_{sys1}, T_{sys2}, T_{sys3}, \dots)$$

Also this detail can cause insidious errors.

Another two specifications about some peculiarities of this circuit might be useful:

1. in the PWM modulator the comparison is done according to rule $m(t) < v_{controller}$ and not, as it usually happen $m(t) > v_{controller}$. This force us to modify a little bit the function $D(t)$. Arguing as in section 4.1.3 we obtain:

$$D(t) = \frac{-\gamma}{2 \cdot \cosh^2 \left(\gamma \cdot (u_{nom}(t) - m(t)) \right)}$$

This explains also the negative proportional gain in the controller, used to compensate the coefficient $-\gamma$ on the numerator.

2. In section 4.1.3 the parameter γ was tuned to $\gamma = 20$. In that case we had a modulating function $m(t)$ which variates in the range $[0, 1]$. In this scheme the modulating function range is $[3.8 \ 11.3]$ and the parameter γ should be rescaled according to:

$$\gamma' = 20 * \frac{1 - 0}{11.3 - 3.8} = 2.66 \cong 3$$

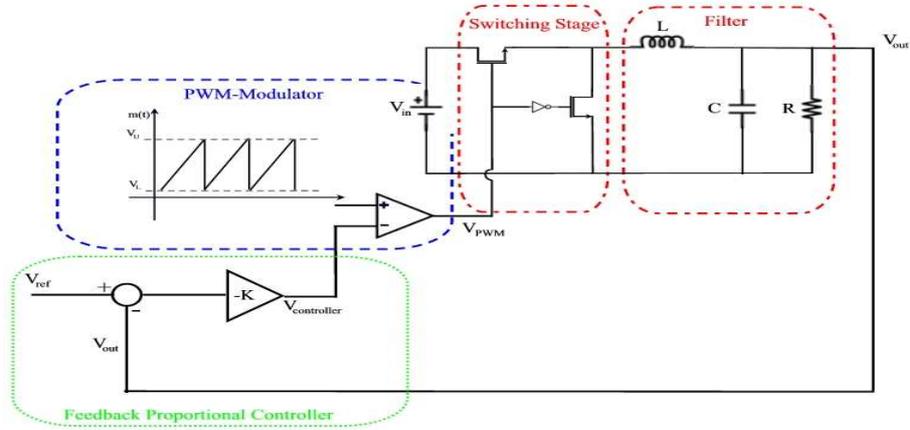


Figure 34: Controlled Step Down Converter

Parameter		Value	Unit of Measurement
Resistance	R	22	Ω
Capacity	C	47	μF
Inductance	L	20	mH
Proportional Gain	K	8.4	/
Reference Tension	V_{ref}	11.3	V
Input Tension	V_{in}	Variable	
Carrier Range	$[V_L, V_U]$	[3.8, 8.2]	V
PWM Period	T	400	μs

Table 2: Parameter of the circuit reported in figure 34

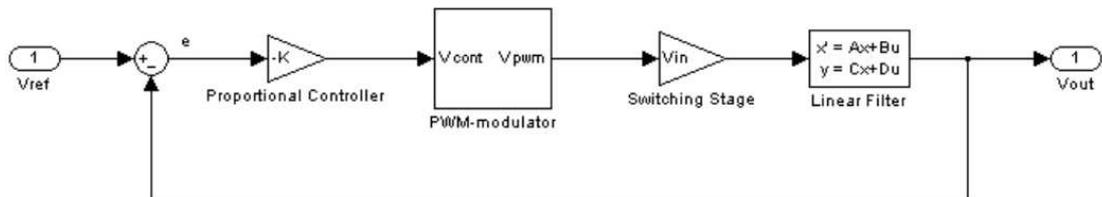


Figure 35: Block Diagram of the Controlled Step Down Converter

5 Subharmonics and chaos in Dc-Dc converters

Dealing with nonlinear systems, even with simple ones, very strange behaviors can arise. This fact has always fascinated scientists and in the last 30 years a new discipline, called chaos theory, has given a deeper understanding of this kind of phenomena.

A field extremely rich of non-linear systems that often exhibit chaotic behaviors is the power electronic field. In particular Dc-Dc converters have often been taken as interesting case studies from Chaos scientists. Nowadays, important explanations have been found to what switching pioneers defined “unusually high noisiness” of this circuits.

Naturally, to develop mathematical tools which allow us to predict the emergence of chaos or subharmonics is extremely important also from a practical point of view. In this way it will be possible to predict and to avoid that the circuit operates in chaotic mode.

In this thesis we will use for this purpose the techniques introduced in section 2. In particular, the case of the controlled step down converter presented in section 1.3 will be analyzed. In fact it has been shown in [10] that this circuit presents chaotic behavior.

Since chaos theory is very complex and it has a very difficult mathematical formalization, we will introduce just some of the main results, the ones we will use in the following for the analysis of our case study.

Moreover to introduce these results in a general way is rather complex and requires the use of evolute mathematical tools. Hence we prefer to introduce them in a less general fashion, considering only non autonomous systems.

In fact, the Controlled Dc-Dc Converter can be easily modeled as an non autonomous periodic system with period T_{PWM} .

A more general and formal dissertation about chaos theory can be found in [9].

5.1 Qualitative classification of the response of a dynamical system

In this section we will give a qualitative classification of non-linear systems based on the asymptotic behavior of solutions $x(t)$ and we will introduce some of the basic concepts of chaos theory, for instance we will try to clarify the meaning of the word “chaos” itself.

Consider the non autonomous⁷ system:

$$\dot{x}(t) = f(x(t), t) \quad x(t_0) = x_0 \quad (24)$$

where $f(x, t)$ is a T -periodic function in t .

Let's call the solution of the above differential equation:

$$x(t) = \phi_t(x_0, t_0)$$

We will refer to this solution sometimes with the notation $x(t)$ and other times with the notation $\phi_t(x_0, t_0)$ to remark the dependence of the solution from the initial conditions (x_0, t_0) .

Moreover let's introduce:

Definition 7

An orbit $x_a(t) = \phi_t(x_{0a}, t_0)$ is called **attractive** if there exists an open set \mathcal{I} , $x_{0a} \in \mathcal{I}$, so that:

$$\phi_t(x_0, t_0) \rightarrow x_a(t) = \phi_t(x_{0a}, t_0) \quad \forall x_0 \in \mathcal{I}$$

\mathcal{I} , the neighborhood of x_{0a} , is called **Domain of Attraction**.

In the following we will classify systems according to the properties of their attractive orbits.

A first classification is:

- Unbounded or (Divergent) behavior of x_a .
- Bounded behavior of x_a .

It is known that, in the special case of LTI systems, there is only one possible bounded attractive trajectory: the constant trajectory $x(t) \equiv x_{eq}$.

Moreover in a LTI system there can exist only one point x_{eq} and $\mathcal{I} = \mathbb{R}^n$.⁸

On the contrary in the case of nonlinear systems there is a wide variety of possible bounded behaviors.

First of all there can exist more than one equilibrium point x_{eq} .

Moreover the solutions can tend to a time varying periodic trajectory instead of a fixed equilibrium point. But these are also more complex possibilities.

In the following we enumerate the possible attracting orbits.

⁷non autonomous is equivalent, in this formulation, to time varying

⁸except for the pathological case of $A = 0$, where all the point of the state space are equilibrium points.

Equilibrium Point :

$$x_a(t) \equiv x_{eq}, \forall t.$$

The plot drawn by the trajectory in the state space is in this case just a point.

Periodic Solution :

$x_a(t)$ is a T -periodic function, called *Limit Cycle*.

Note that the convergence to an equilibrium point can be viewed as a special case of convergence to the limit cycle $x_{cl} \equiv x_{eq}, \forall t$.

Naturally the periodic function x_a can contain (and usually do) also harmonics at frequencies multiple of $1/T$.

The plot drawn by the trajectory is in this case a closed curve in state space. $x_a(t)$ makes a revolution of the curve (at least) once per period.

Subharmonics :

$x_a(t)$ is a kT -periodic function.

It is quite surprising that a T periodic system gives a kT -periodic solution.

On an engineering point of view this phenomena can be very problematic, specially when the system is connected to other systems, because subharmonics interaction often leads to instabilities.

Subharmonics introduce also distortion in the signal band and, furthermore, emerging of subharmonics is index of non linearity and often of possible-chaotic behavior.

The plot of a trajectory with subharmonics is again a closed curve in the state space, but this time it is required k periods to complete one tour.

Quasi-periodic solution :

x_a is a quasi-periodic signal.

Quasi-periodic waveforms are very special signals which are not strictly periodic but they are the result of composition of periodic functions whose periods ratio are not rational.

Consider, for instance, an amplitude modulated signal:

$$x(t) = \cos(t) \cos(\sqrt{2}t)$$

where the ratio between the two frequencies is not rational:

$$\frac{f_2}{f_1} = \frac{\sqrt{2}}{1} = \sqrt{2} \notin Q$$

$x(t)$ is not periodic, as it is shown in figure 36(b). However the spectra of this signal (reported also in figure 36(b)) is much simpler than the one of an "ordinary" aperiodic signal because is constitute just of 2 frequencies.

A more complex 2-quasi periodic signal is reported in figure 36(c). Once again the spectra is constitute just by 2 frequencies and their multiples.

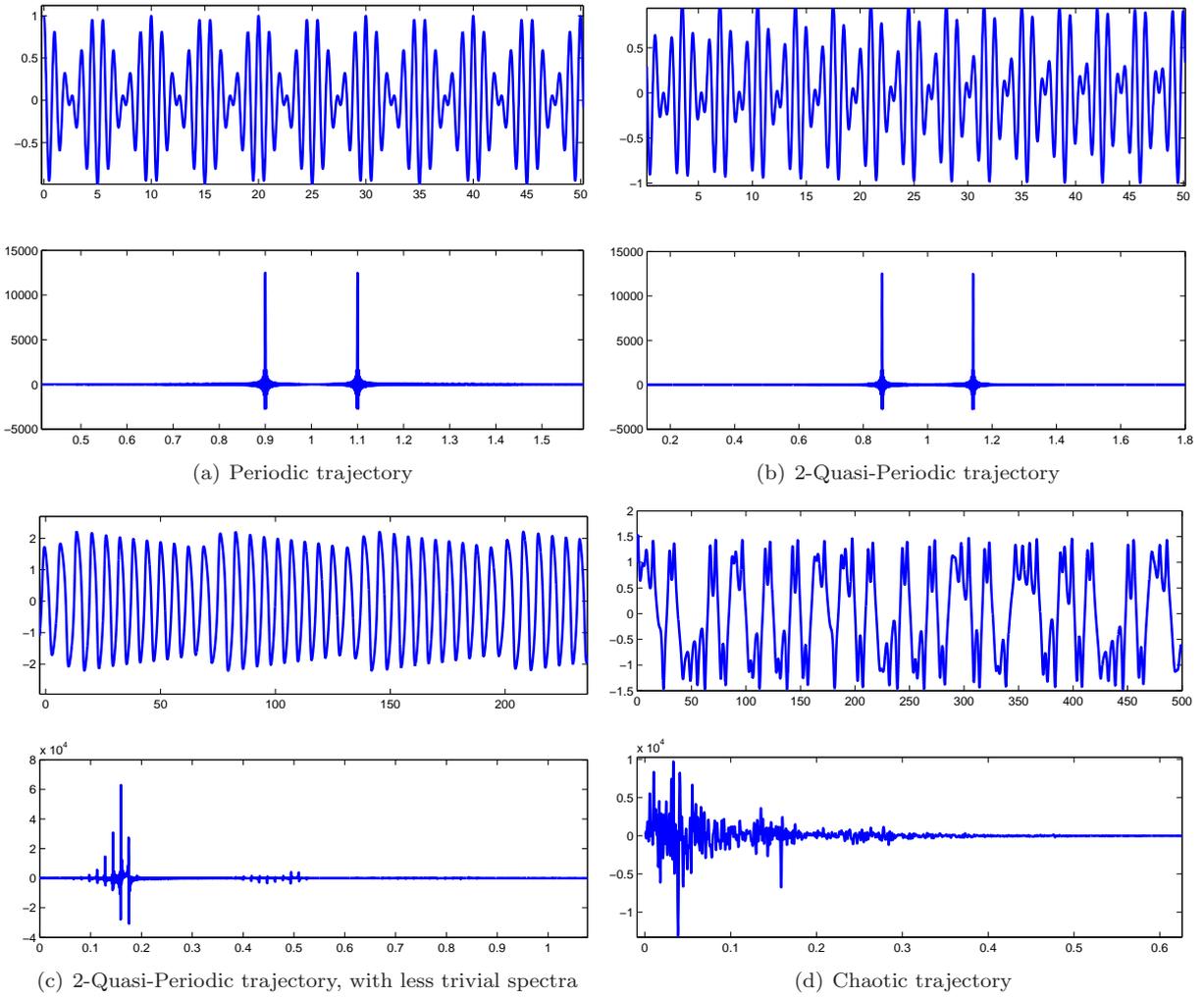


Figure 36: Time and Frequency representation of a Periodic, a Quasi-periodic and a Chaotic Trajectory

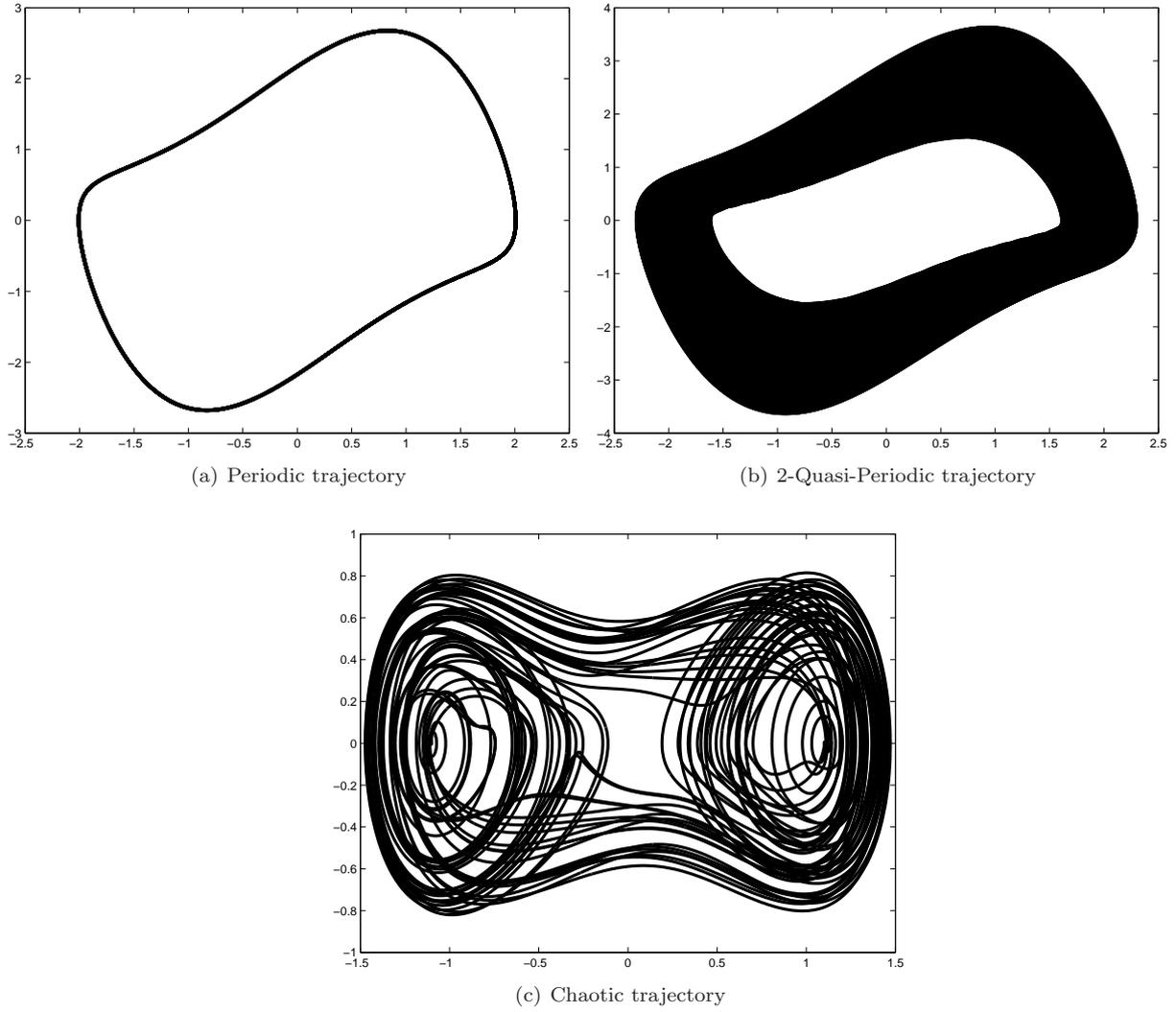


Figure 37: State Space representation of a Periodic, a Quasi-periodic and a Chaotic Trajectory

A formal definition of a quasi-periodic signal is:

Definition 8

A Quasi-periodic signal is a signal that can be written as:

$$x(t) = \sum_i h_i(t)$$

where h_i is a T_i periodic solution.

Furthermore, it must exist a finite set of base frequencies $\{\hat{f}_1, \dots, \hat{f}_p\}$ with the following properties:

- i It is linear independent, i.e. it does not exist a nonzero set of integers $\{k_1, \dots, k_p\}$, $k_i \in \mathbb{Z}$ such that:

$$k_1 \hat{f}_1 + \dots + k_p \hat{f}_p = 0$$

- ii It forms a finite integral base for f_i , that is:

$$f_i = k_1 \hat{f}_1 + \dots + k_p \hat{f}_p$$

for each f_i , for some integers $\{k_1, \dots, k_p\}$, $k_i \in \mathbb{Z}$

Note that the base frequencies are not uniquely defined, but p is.

A quasi-periodic waveform with p base frequencies is called **p -quasi-periodic**.

Plotting a quasi-periodic trajectory in state space we will not obtain a closed curve but an annular-shape region, uniformly filled-in during the various iterations. An example can be found in figure 37(b).

Chaos :

There is no generally accepted definition of chaos. From a practical point of view chaos can be defined as none of the above but still a bounded behavior. Examples of chaotic behavior are reported in figures (36(d)) and (37(c)). The spectrum plot is noise-like and quite different from the quasi-periodic case, that has instead an evident prevalence of two frequencies and their multiples. Also in the state space plot we can see a quite strong difference with the previous case. In the chaos example in fact the area of the state space is not uniformly filled-in, and have a more complex geometry with respect to the annular-like shape of the quasi periodic trajectory.

This intuitive argumentation can be mathematically formalized by saying that the trajectory in the p-quasi-periodic case is diffeomorphic to a p-hyper-torus⁹ while the trajectory in the chaotic case is diffeomorphic to a fractal¹⁰.

⁹it means that we can obtain the curve as image of a torus under a diffeomorphism.

A function f is a diffeomorphism if f^{-1} exists and both $D[f]$ and $D[f^{-1}]$ exist and are continuous. A diffeomorphism a kind of elastic coordinate transformation that may modify the shape of the objects but not their geometrical properties.

¹⁰for a formal definition of fractal see [9]

5.2 Tools used in Chaos analysis

There are some tools which are typically used to analyze nonlinear systems and to predict the local stability of a solution. Those are Poincarè map, Bifurcation Diagram, Characteristic multipliers and Lyapunov Exponents.

5.2.1 Poincarè section

This tools has a very general and rather elaborate definition. It can be introduced for both autonomous nonlinear systems and for non autonomous nonlinear periodic systems, but the definitions are quite different and the one developed for the autonomous case is rather complex. Since in this thesis we are interested in non-autonomous systems, we will give the definition only in this case.

Definition 9 (Poincarè map for Non Autonomous Systems)

The Poincarè map P for a T -periodic system is a function that describes where the state x^* is carried from the solution $\phi_t(x^*, t_0)$ after T seconds:

$$x^* \longmapsto P(x^*) = \phi_T(x^*, t_0)$$

Therefore the Poincarè map P can be seen as a discrete dynamic system that describes the behavior of the original system in the instants $t_0, t_0 + T, t_0 + 2T, \dots$:

$$x(t_0 + (n + 1)T) = P(x(t_0 + nT)) \quad (25)$$

Note that (25) is a system (discrete, nonlinear and) **time invariant**.

The Poincarè map works in fact as a stroboscope for the system $\dot{x} = f(x, t)$.

Note the analogy with what we have done in section 3 .

Note also that¹¹

$$\tilde{x}(kT) = P^k(x^*)$$

is the discrete version of the trajectory $\phi_t(x^*, t_0)$ sampled with periodicity T :

$$\tilde{x}(kT) = P^k(x^*) = \phi_{kT}(x^*, t_0) = x(kT)$$

Example 5.1

For example, consider the particular case of a LTP system, i.e. a system where $f(x, t)$ is linear.

In this case the system can be written as:

$$\dot{x} = A(t)x \quad x(t_0) = x_0$$

Therefore the Poincarè map is a linear time invariant map defined by the product of x with the monodromy matrix $\Phi(t_0)$ ¹²:

$$P(x) = \Phi_T(t_0)x$$

¹¹With the symbol P^k we mean composition of P k times.

¹²introduced in 3.1

As the intuition suggests, there exists a correspondence between the asymptotic behavior of the system and the asymptotic behavior of the Poincarè map. It is interesting to analyze this link:

Attracting solution :

A solution of the nonlinear system is attracting if and only if its discrete version is an attracting solution for the discrete system described by the Poincarè map.

Bounding behavior :

First of all (and particularly intuitive) a trajectory is bounded if and only if the Poincarè map gives a bounded evolution.

Equilibrium Point :

as we said an equilibrium point defines a trajectory that degenerate into a point in state space. Hence the Poincarè trajectory will be also a point.

Periodic Solution :

a T -periodic solution defines a closed curve in state space. An integer number of tours of this curve is done in exactly T seconds, therefore sampling the curve each T seconds, we will obtain always the same point $x(t_0)$. Hence the plot of the Poincarè trajectory will also be just a point.

This means also that a periodic solution will give an equilibrium point in the Poincarè map.

k -Subharmonics :

Also in this case the state space trajectory is a closed curve but this time it is required kT seconds to complete a tour of the curve. Hence, sampling each T second, we will obtain k points in the state space, visited periodically.

This is equivalent to say that the discrete system converges to a k -periodic (discrete) orbit.

Quasi-periodic solution :

To understand what happen in this case consider first of all a 2-quasi periodic solution, with one of the base frequencies $f_1 = 1/T$. In all the sampling instants the present sampled point is close to the previous one but not exactly the same, because the solution is not periodic. Plotting the points obtained iterating the Poincarè map, in the case of a 2-quasi periodic waveform we will obtain a close curve in state space, instead of some separate points. The curve in the 2-quasi periodic case is circular-like, or in more strict words, diffeomorphic to a circumference.

If the solution is p -quasi periodic we will obtain a more complex figure, that will be diffeomorphic to a $(p - 1)$ -torus.

Chaos :

The Poincarè orbits of a chaotic system are distinctive and often quite beautiful. Some typical orbits are shown in figures 38(d), 39(b) and 43 . Looking at these orbits, it is immediately clear that they do not lie on simple geometrical objects as is the case of periodic and quasi-periodic behavior. This intuition can be formalized by saying that the Poincarè orbit is diffeomorphic to a fractal.

5.2.2 Bifurcation Diagram

Often in a dynamic system there are uncertain parameters.

It is interesting to study how the qualitative behavior of a trajectory varies when the parameters vary.

To do this consider one of the varying parameters, α .

Consider then the evolution of the state obtained by iterating the Poincarè map:

$$x(kT) = P^k(x_0).$$

The behavior of $x(kT)$ characterizes the qualitative behavior of the solution, as we described in the previous section. If we restrict our analysis to the i -th component of the steady state solution and we plot it against the varying parameter α we obtain a bi-dimensional plot like the one reported in figure 40. If in correspondence of the value α_1 of the parameter we have just one point the solution is 1-periodic, if we have two points the solution is two periodic, if there is a huge number of spread point the solution is quasi periodic or chaotic.

5.2.3 Characteristic multipliers and Lyapunov Exponents

In non linear systems theory, the *Characteristic Multipliers* have a definition a little bit more general than the one that we gave in section 3.1. Characteristic Multipliers are used to determine if a periodic solution of the non linear system (24) is stable. The underlying idea of this technique is that the periodic solution is locally asymptotically stable if and only if the correspondent fixed point of the Poincarè map is locally asymptotically stable.

The properties of the fixed point of the Poincarè map, that is a (non linear, discrete and) **time invariant** map can be easily analyzed with the usual techniques, for instance by checking the eigenvalues of the Jacobian of $P(x)$, evaluated in x^* , $D[P(x^*)]$. If the eigenvalues are inside the open unit disc the nonlinear system is locally stable, if they are outside the unit closed disc the nonlinear system is locally unstable.

We give:

Definition 10

The eigenvalues of the $D[P(x^)]$ are called **Characteristic Multipliers**.*

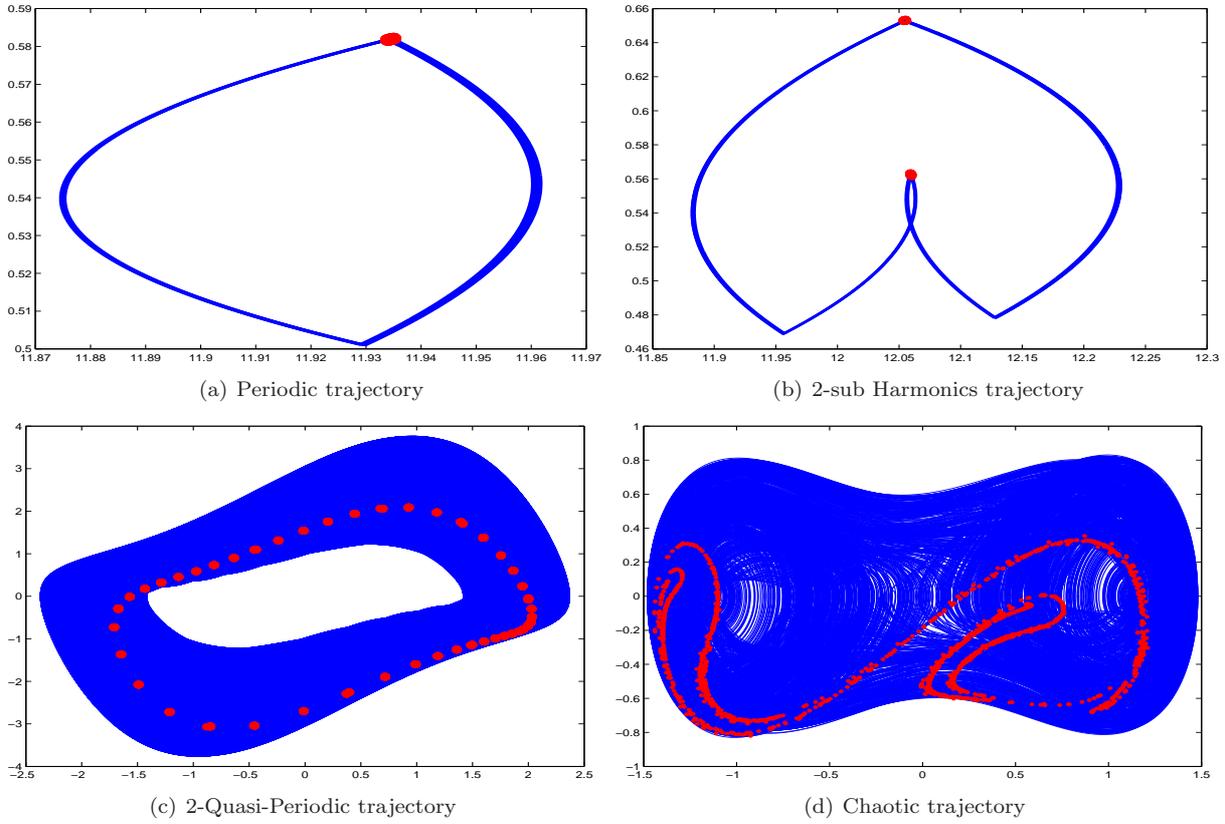


Figure 38: State Space representation of trajectories and of the sequence of point of the Poincarè Map

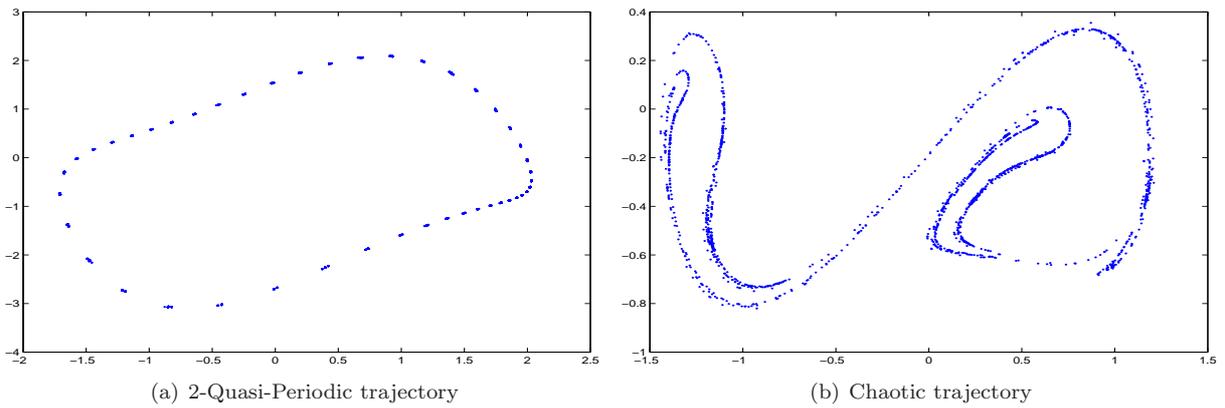


Figure 39: State Space representation of the sequence of point of the Poincarè Map

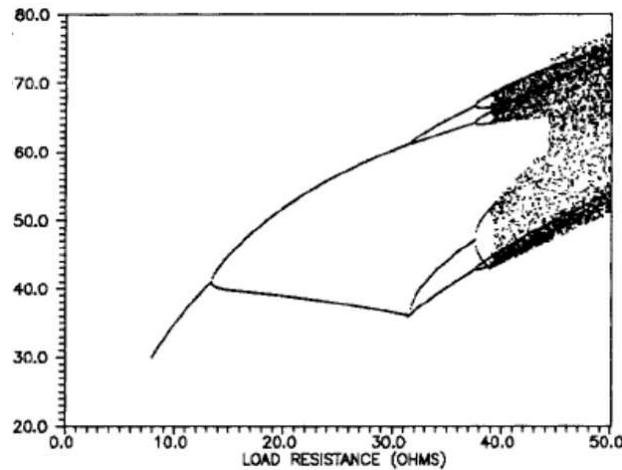


Figure 40: Example of a Bifurcation Diagram

From what we said above follows that:

Proposition 13

A periodic trajectory is locally asymptotically stable if all its characteristic multipliers are inside the open unit disc, unstable if all its characteristic multipliers are outside the closed unit disc.

Remark 5.1

In the special case of LTP systems, since $P(x) = \Phi_T(t_0)x$, we have that:

$$D[P(x^*)] = \Phi_T(t_0)$$

Therefore we have that $D[P(x^*)]$ is (for all x^*) the Monodromy Matrix and this is coherent with the definition of Characteristic Multiplier we give in section 3.1: *“The eigenvalues of the Monodromy matrix are called Characteristic multipliers”*.

However the definition we gave in this section is more general because it can be applied also to non linear systems that have periodic solution.

Remark 5.2

As we said, the approach of this thesis is to attempt to give an approximated LTP model of a system and then to analyze the properties of this with the LTP techniques. The approach used in definition 10 in proposition 13 is instead to create an exact Poincarè map for the nonlinear system and then linearize this exact map.

The analogies and differences between the two approaches are therefore evident.

In non linear systems theory there is a further generalization of the concept of the characteristic multipliers, the so called *Lyapunov Exponents*.

In the case of periodic orbits Lyapunov Exponents $\{\lambda_i\}$ and Characteristic multipliers $\{m_i\}$ are related by the simple relation:

$$\lambda_i = \frac{1}{T} \ln |m_i|$$

But, contrary to what happen with the Characteristic Multipliers, Lyapunov Exponents can be defined also for quasi-periodic and chaotic orbits.

A formal definition of Lyapunov Exponents can be found in [9].

As the intuition suggest, proposition 13 becomes in the case of Lyapunov Exponents

Proposition 14

A trajectory is locally asymptotically stable if all its Lyapunov Exponents are in the open left half plane, unstable if all its Lyapunov Exponents are in the open right half plane.

Other important tools used in chaos theory are the *Dimensions* of the attractor. These powerful tools have a rather complex mathematical formalization which can be found in [9].

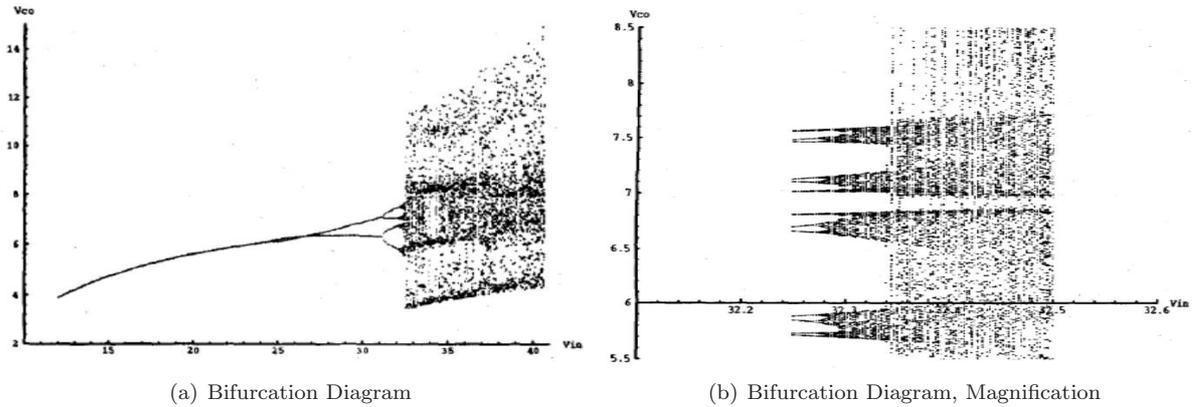


Figure 41: Bifurcation Diagram of $v_{controller}$ taking V_{in} as bifurcation parameter

5.3 Case Study

In [10] there is a detailed study of the circuit presented in section 1.3 with the techniques presented in this section.

We will report in the following some of the results presented in that paper, in particular the ones which show how the behavior of the system changes varying the parameter V_{in} .

In section 6 we will then analyze the same problem using the techniques introduced in section 2.

First of all, we report in figures 41(a) and 41(b) the bifurcation diagram of $v_{controller}$ taking V_{in} as bifurcation parameter.

It shows that increasing V_{in} the attracting trajectory, which is initially 1-periodic, becomes 2-periodic, then 4-periodic and so on until it becomes chaotic ($32, 29 < V_{in}$).

In [10] the trajectory stability analysis is performed using the Characteristic Multipliers Loci. These are the plots of the Characteristic Multipliers $\lambda_i(V_{in})$ in the complex plane when V_{in} varies.

As we said a trajectory is stable if its characteristic multipliers are inside the unit open disc, while is unstable when the multipliers are outside the unit closed disc. Hence when the Characteristic Multipliers Locus crosses the unit disc the trajectory changes its behavior.

In figure 42(a) it is reported the Characteristic Multipliers Locus for the 1-periodic trajectory. For $V_{in} < 24$ the characteristic multipliers are complex conjugate and they move on a circle of radius $\cong 0.82$ and therefore the orbit is asymptotically stable. Near to $V_{in} = 24$ the characteristic multipliers become both real and when $V_{in} = 25$ one of the characteristic multipliers has norm greater than one and the orbit becomes unstable.

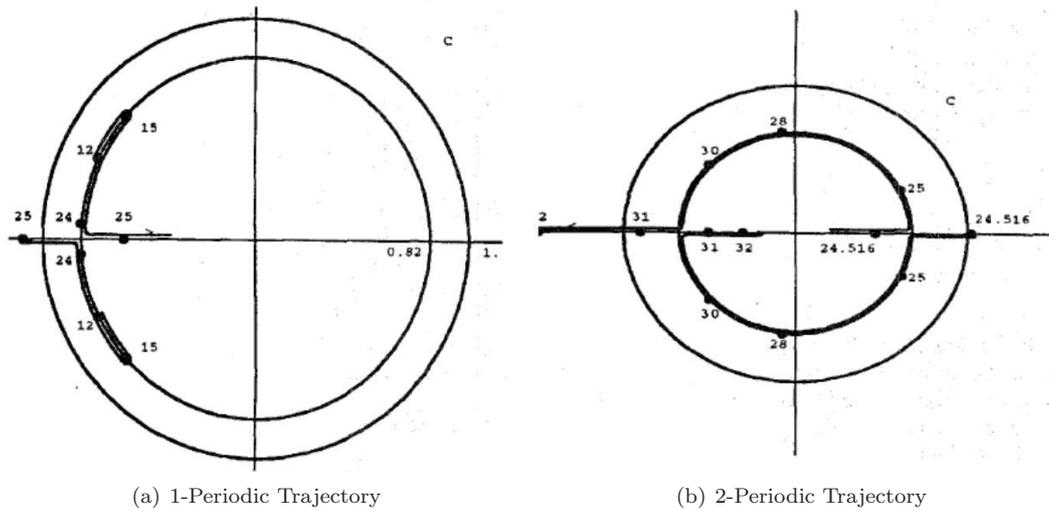


Figure 42: Characteristic Multiplier Loci for the 1-Periodic and 1-Periodic Trajectories

This is in perfect accord with the bifurcation diagram that shows that for $V_{in} \cong 24.5$ the orbit switch from 1-periodic to 2-periodic.

In figure 42(b) is reported the Characteristic Multiplier Locus of the 2-periodic trajectory. For $V_{in} < 24.5$ one of the characteristic multipliers are real, positive and with norm greater than one. This means that the orbit is unstable, in accord with the fact that the attracting orbit is 1-periodic. Near to $V_{in} = 24.5$ the unstable characteristic multiplier enters in the unit circle and the 2-periodic orbit becomes stable. The characteristic multipliers remain inside the unit circle until $V_{in} < 31$ then for a value of V_{in} in $[31 \ 32]$ one characteristic multiplier exits from the unit circle and the trajectory becomes unstable. This corresponds to the arise of a 4-periodic orbit.

Note that the stability analysis procedure predicts very well the behavior of the system and it is very accurate in the determination of the bifurcation point.

Note also that with this kind of analysis we can just predict the loss of asymptotic stability for an orbit, that suggests that another orbit will become attractive. But no prediction can be done about the shape of this orbit. For instance there is no guarantee that when a 1-periodic orbit becomes unstable a 2-periodic one becomes stable. It can happen, for instance, that after the 1-periodic trajectory a 4-periodic one, or even a chaotic one, becomes stable.

Finally we report in figure 43 the plot of the points obtained iterating the Poincaré map in the case of Chaotic Orbit. This set of points is called sometimes *strange attractor* of the Chaotic Trajectory.

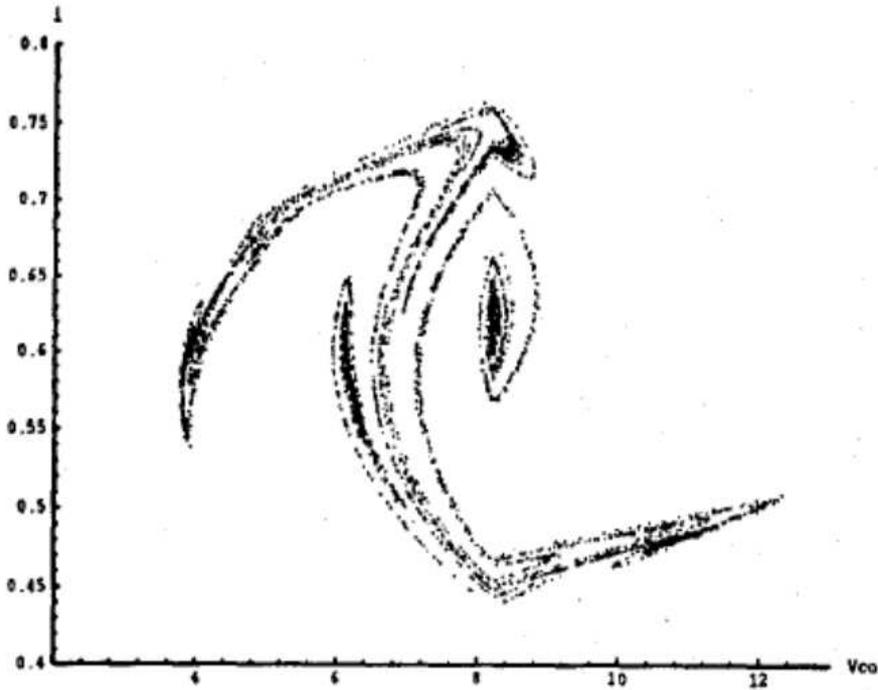


Figure 43: Strange Attractor for $V_{in} = 35$

6 Stability analysis of the periodic system

In this section we will use the LTP model introduced in section 4 to analyze the behavior of our case study. Two different kinds of problems will be analyzed:

First the problem to estimate the spectral content of the output of a controlled Dc-Dc converter is considered.

This information is extremely useful if someone is interested to analyze the interaction of the converter with the remaining part of the complex system in which the converter is inserted. In particular the method presented will allow us to quantify the “amount of subharmonics” the system injects in the load.

In this first part V_{line} is considered constant and known and the nominal input around which we linearize the PWM modulator is determined by simulation.

If V_{line} changes we recalculate the LTP model for the new value, to increase the accuracy of the prediction.

Still from the point of view of someone interested to study the interaction between the converter and the remaining part of the complex system in which the converter is inserted, there is another important application: This model can be, in fact, effectively used to perform global stability analysis using the techniques presented in section 2.

A second and problem that we tried to solve using LTP models and techniques is the prediction of subharmonics arise.

We know in fact that subharmonics and chaotic trajectory can arise when some critical parameters, for instance V_{line} , vary.

The idea in this case is to create a LTP model for a certain value of V_{line} and to use this model, that we have seen experimentally to be somehow roust to small variation of V_{line} , to predict the behavior of the system also for other values of the parameter V_{line} . In particular we wondered if the model is robust enough to remain reliable even when V_{line} variations produce a change of qualitative behavior, namely when subharmonics arise.

This would mean that the LTP model would be able to predict the arise of subharmonics.

It comes out that the brusque change in the properties of the system makes the linear approximated model unreliable.

This outcome was expected and suggested by the experience with the LTI approximated model, but this result was not sure a priori and it was therefore import to explore this possible application of LTP model.

Note that, nevertheless, *if we relinearize the model* for the new value of V_{line} we can still estimate properly the amount of subharmonics injected in the load.

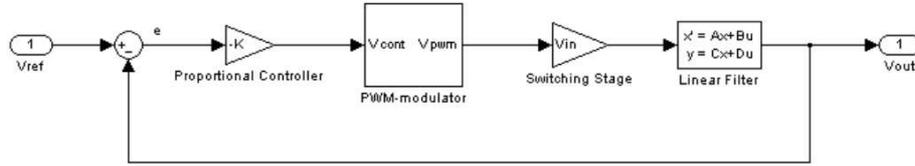


Figure 44: Block scheme of our case study

6.1 Subharmonic Analysis

Consider now the first problem we introduced, namely the problem to understand how the converter interacts with the system in which it is inserted. In particular we try to quantify the amount of subharmonics the converter injects in his load.

To do this let's consider first of all the block scheme representation of our case study, presented in figure 44. We want to obtain the HTF of the closed loop system by combining the HTFs of the single blocks. The majority of the blocks are LTI and it is therefore quite simple to obtain the LTP description from the LTI one. The critical block is the PWM modulator. The first information we need to determinate the HTF of this component is the nominal input trajectory. Since the system has a feedback it can become complex to obtain an analytical expression for the nominal input and the simplest choice is to compute it by simulation.

As we said it is important to obtain a LTP model that the input is periodic. Let's call the input period T_{in} . As we explained in section 4 the model we obtain is T-periodic LTP system with:

$$T = \text{LCM}(T_{in}, T_m)$$

If there are 2-subharmonics in the output they are reported via feedback to the PWM input, therefore:

$$T_{in} = 2 \cdot T_m$$

and hence $T = T_{in} = 2 \cdot T_m$.

In the case of 1-harmonics in the output

$$T_{in} = T_m$$

and hence $T = T_{in} = T_m$.

Anyway, in order to use everywhere the same convention, we can consider also in this case the system $2 \cdot T_m$ - periodic¹³.

Once the nominal input is known and period T is chosen, we can compute the HTF that describes the PWM modulator, as explained in section 4.

¹³Remember that every signal T periodic is also NT -periodic, for all $N \in \mathbb{N}$

The closed loop HTF is then computed according to the usual block algebra.

To estimate now the injection of subharmonics in the output is enough to notice that the HTF contains informations of the coupling between input and output frequency and allows us, once the input spectra is known to estimate the spectral content of the output.

In the following pages we report and comment, for various values of the parameter V_{line} , some plots obtained using the informations contained in the HTF of the closed loop system. Note that in this case, to obtain the maximum accuracy in the prediction, we recompute for each value of V_{line} the nominal PWM input trajectory and the HTF of the PWM by linearizing around this trajectory.

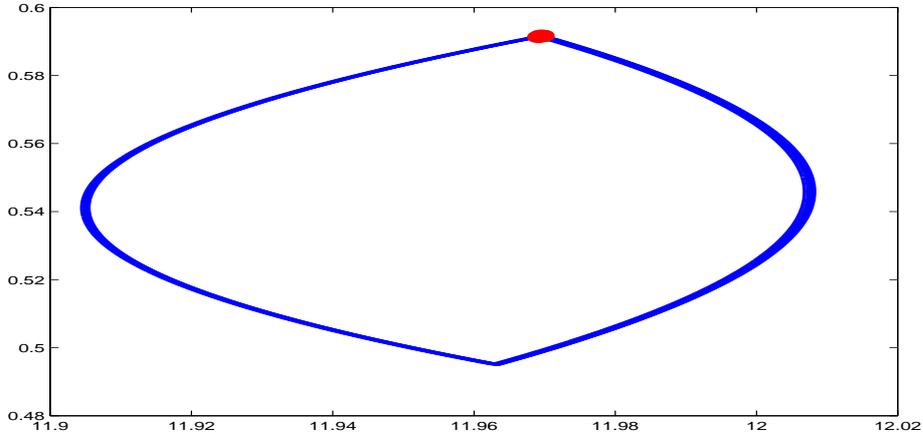


Figure 45: Curve drawn by the attracting trajectory in state space

6.1.1 $V_{line} = 20$

Figure 45 reports in blue the plot drawn by the steady state trajectory in state space and in red the steady state of the Poincarè map. According to what we said in the previous section this figure shows that the system is 1-periodic and that, therefore, there is not subharmonics injection in the output.

For this reason all the coefficients that describe coupling between harmonic and subharmonics are null as we can see in figures 46 and 47.

Figure 46 is a plot of the spectrum of the output when as input we have a sinusoidal perturbation of amplitude 1 and with frequency $f = \frac{1}{2T}$, namely the frequency of the first subharmonic. In this case the output will have only components at the frequencies $\frac{1}{2T} + N \cdot \frac{1}{T}$.

This means that there is coupling only between frequencies spaced by multiple of $\frac{1}{T}$ and no harmonics-subharmonics coupling.

Analogously figure 47 is a plot of the spectrum of the output when as input we have a sinusoidal perturbation of amplitude 1 and with frequency $f = \frac{1}{T}$, namely the frequency of the first harmonic. In this case the output will have only components at the frequencies $(N + 1) \cdot \frac{1}{T}$.

Once again, this means coupling only between frequencies spaced by multiple of $1/T$ and no harmonics-subharmonics coupling.

The absence of harmonics-subharmonics coupling is also the reason for the special structure of the Bode Plot presented in figure 48. We can in fact clearly notice that there is always one diagonal null between two non null diagonal, because all the coefficients that describes subharmonics-harmonics coupling are null.

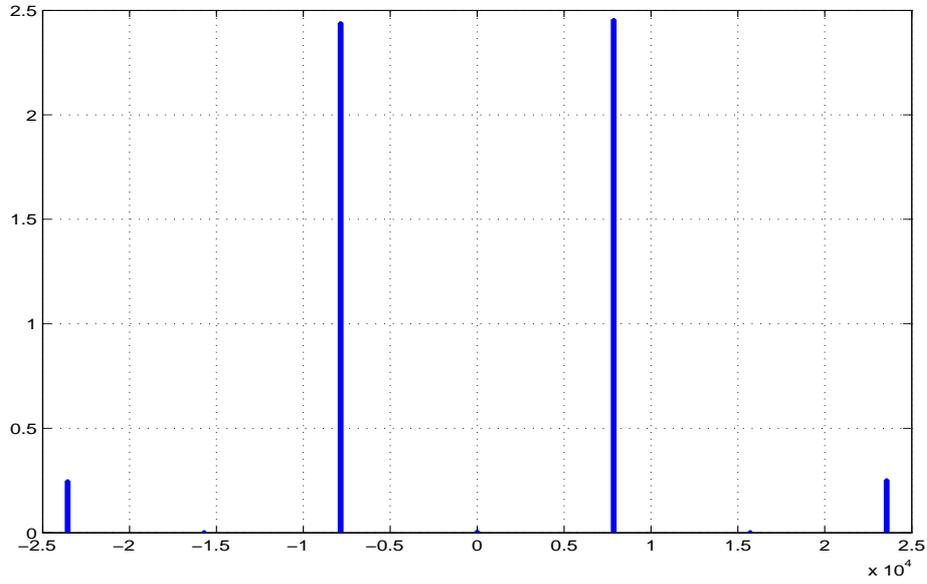


Figure 46: Spectrum of the output when the input is a sinusoidal perturbation of frequency $T = 2 \cdot T_{PWM}$, namely a 2-subharmonic

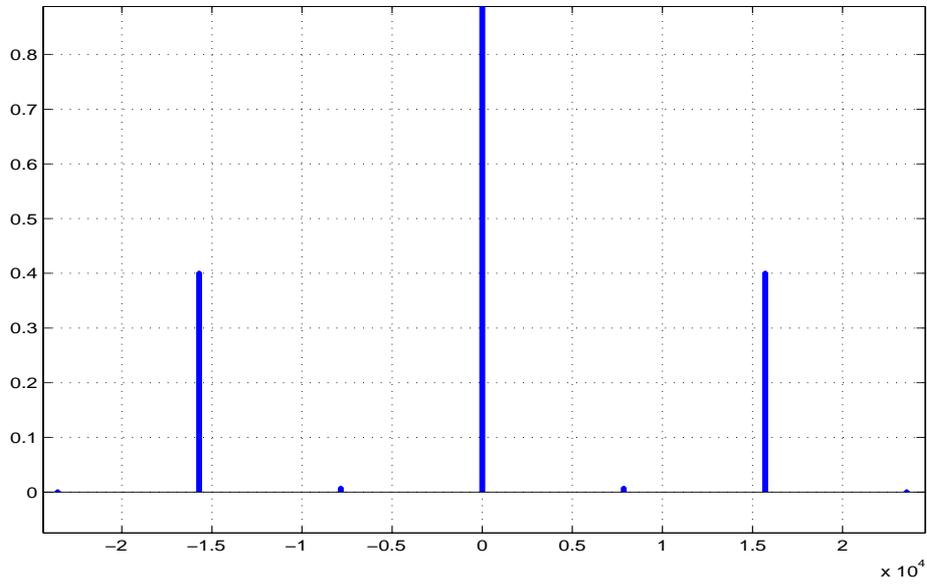


Figure 47: Spectrum of the output when the input is a sinusoidal perturbation of frequency $T = T_{PWM}$, namely a 1-harmonic

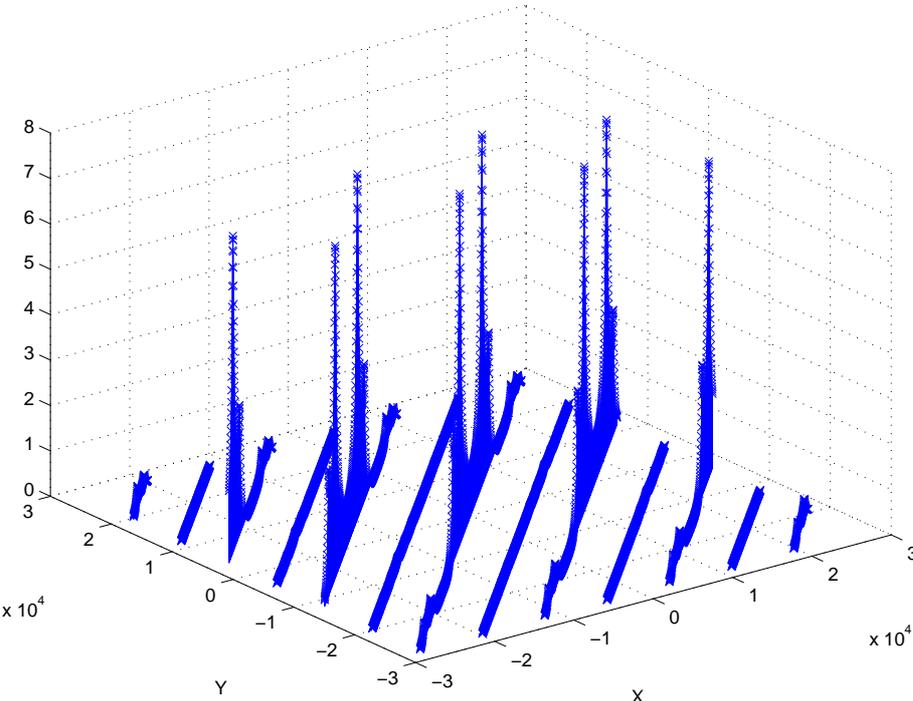


Figure 48: Bode Plot of the closed loop

6.1.2 $V_{line} = 23$

Also in this case the system is 1-periodic. Considerations and the interpretation of the figures are therefore analogous to the case $V_{line} = 20$.

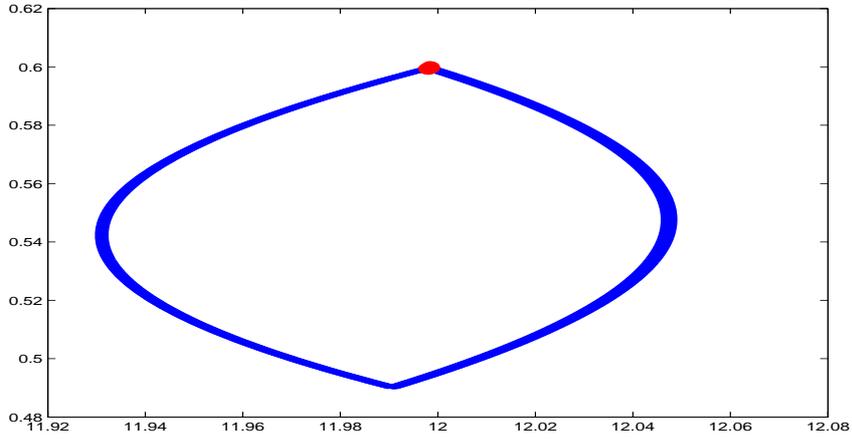


Figure 49: Curve drawn by the attracting trajectory in state space

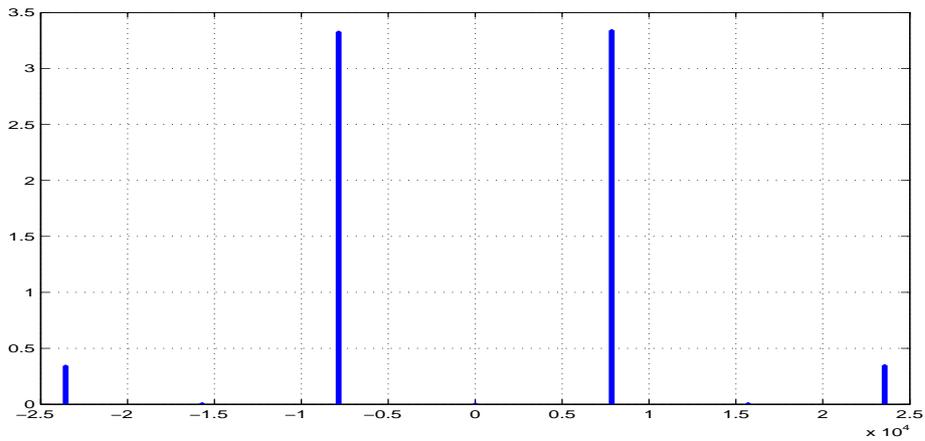


Figure 50: Spectrum of the output when the input is a sinusoidal perturbation of frequency $T = 2 \cdot T_{PWM}$, namely a 2-subharmonic

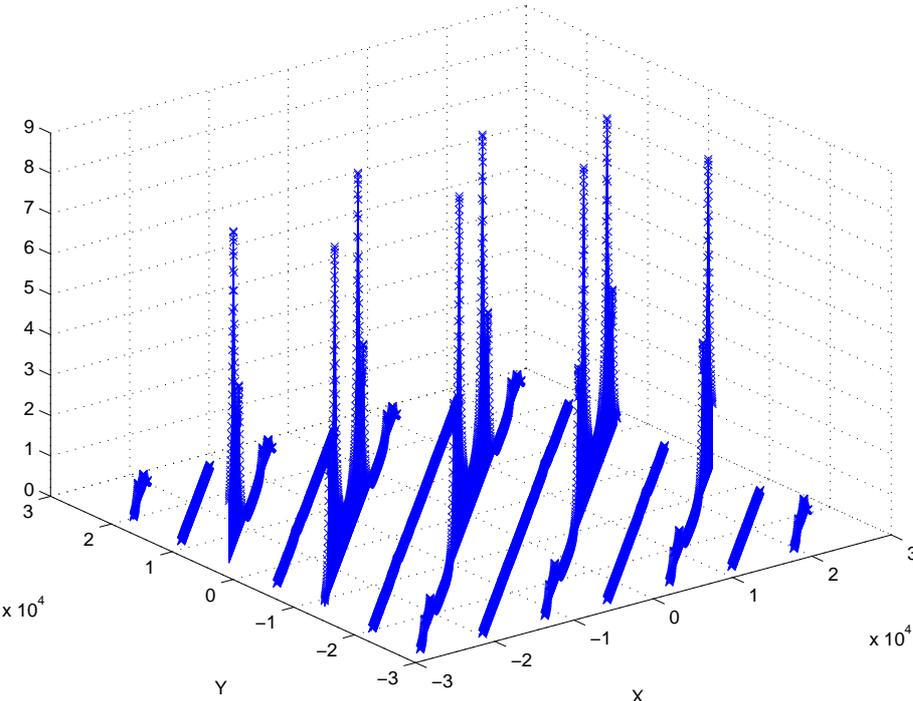


Figure 51: Bode Plot of the closed loop

6.1.3 $V_{line} = 26$

In this case the system becomes 2-periodic as it is shown in figure 52 and according to the bifurcation diagram presented in the previous section (figure 41). Therefore, in this case, the subharmonics-harmonics coupling coefficients becomes non-null as it can be observed in figure 53.

The subharmonics-harmonics coupling modify also the structure of the Bode plot. Comparing, for instance, figure 54 with figure 48, it is obvious the null diagonals have now disappeared.

Note however that figures 53 and 54 not only give us a qualitative information about the “presence of subharmonics” but give us quantitative information about the amount of subharmonics injected in the load.

This information has an obvious extreme practical interest.

Note moreover that the information contained in the HTF can be used to estimate the amount of harmonics injected in the load and not only the amount of subharmonics.

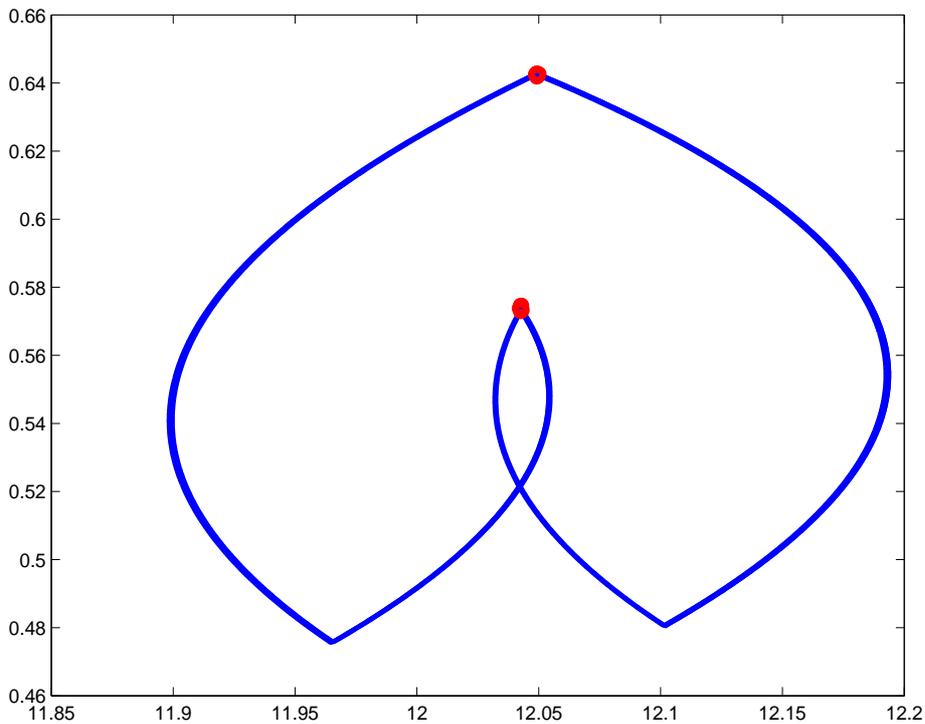


Figure 52: Curve drawn by the attracting trajectory in state space

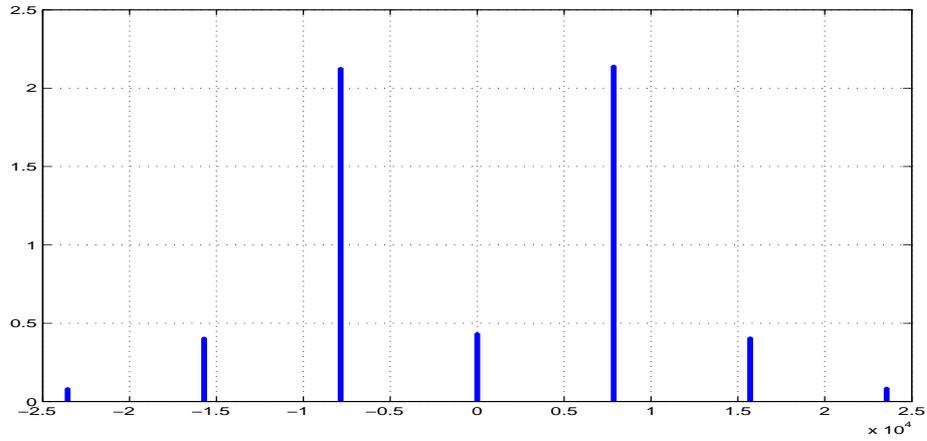


Figure 53: Spectrum of the output when the input is a sinusoidal perturbation of frequency $T = 2 \cdot T_{PWM}$, namely a 2-subharmonic

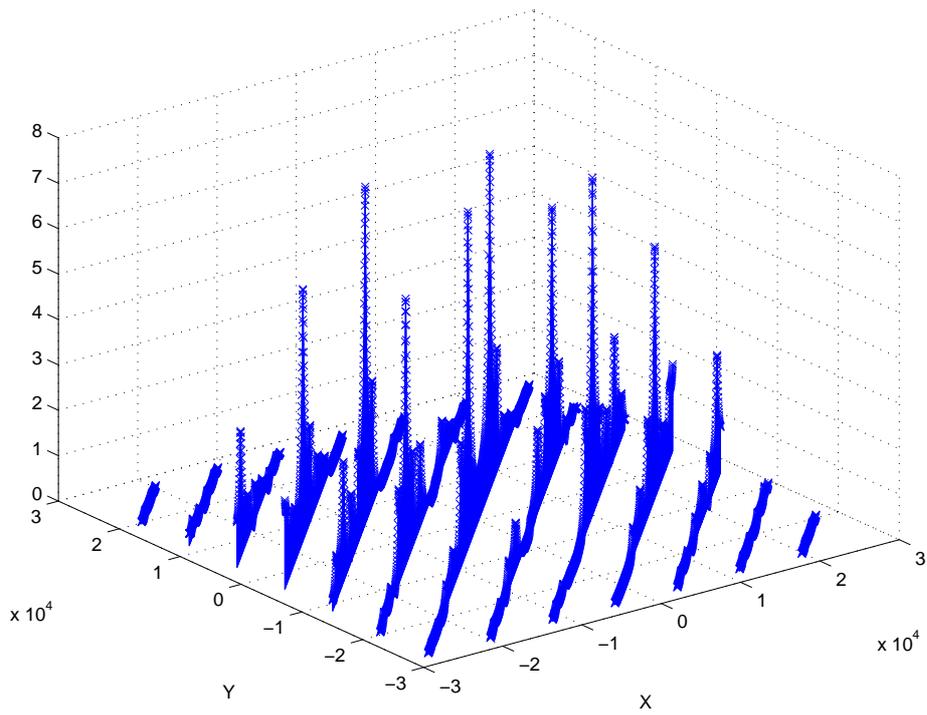


Figure 54: Bode Plot of the closed loop

6.1.4 $V_{line} = 29$

Also in this case the system is 2-periodic. Considerations and the interpretation of the figures are therefore analogous to the case $V_{line} = 26$.

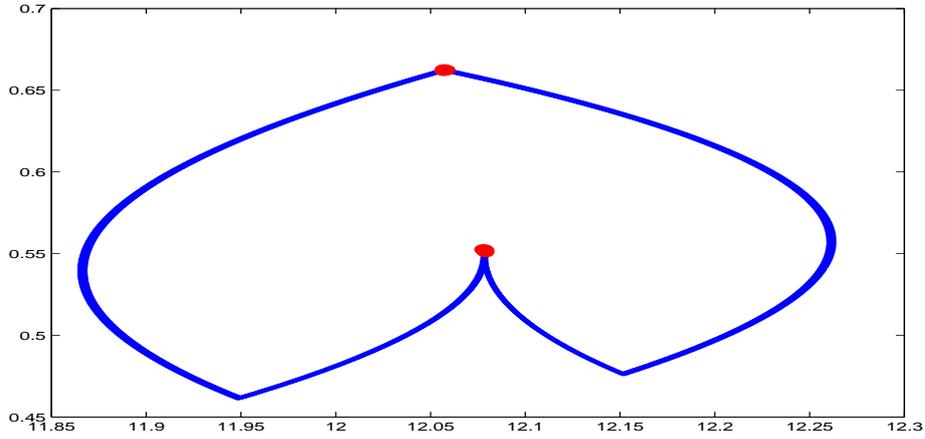


Figure 55: Curve drawn by the attracting trajectory in state space

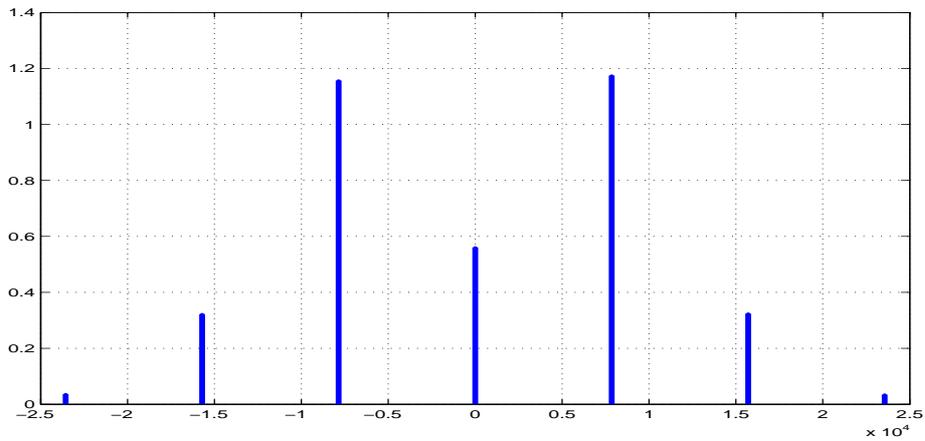


Figure 56: Spectrum of the output when the input is a sinusoidal perturbation of frequency $T = 2 \cdot T_{PWM}$, namely a 2-subharmonic

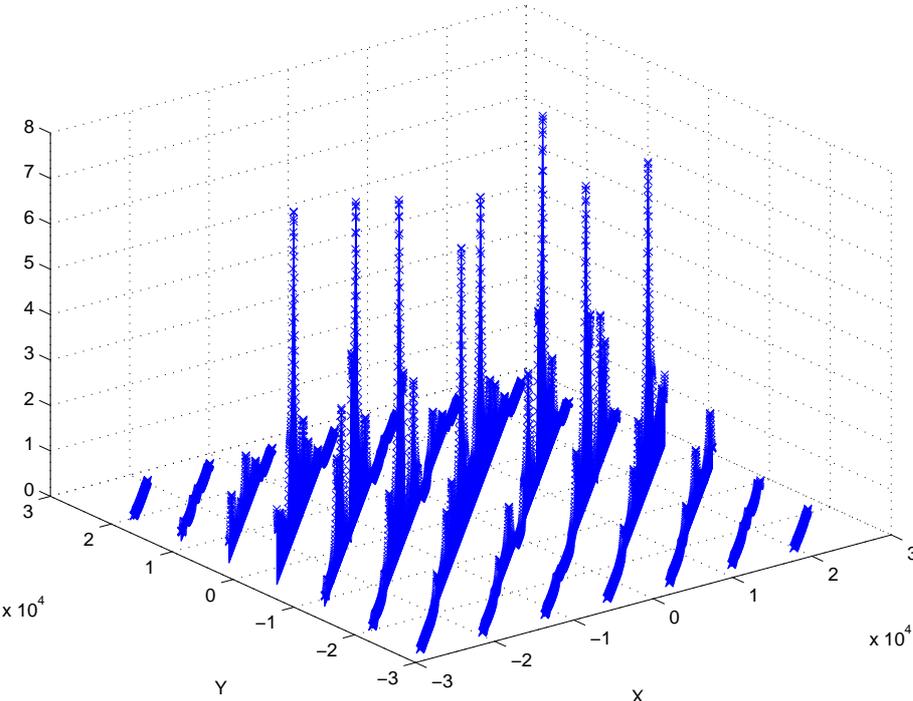


Figure 57: Bode Plot of the closed loop

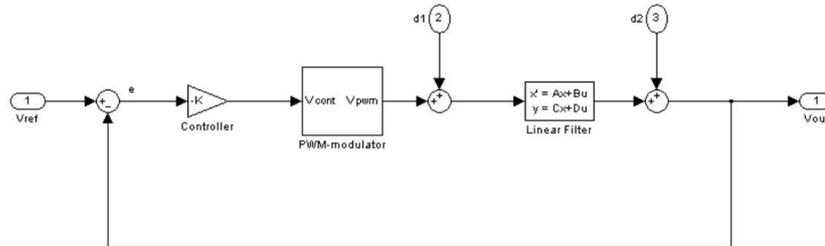


Figure 58: Block scheme of our case study with 2 new inputs used to model load and line tension variation.

6.1.5 Analysis of Load and Line Tension Variation

The important issue of the analysis of **load** and **line tension** variations can be handle in an analogous way, at least for variation with small amplitude.

Let's add to the scheme presented of figure 44 two new disturbances, as reported in figure 58. The signal d_1 can be used to model line tension variations while d_2 can be used to model load variations.

Once the HTF of the PWM modulator is available it is easy to obtain the HTF of the closed loop system that has as input one of the disturbances, namely

$$\text{HTF}_{d_1 \rightarrow v_{out}}(s) \quad \text{and} \quad \text{HTF}_{d_2 \rightarrow v_{out}}(s)$$

for instance using the usual block algebra.

These matrices can be used to quantify the amount of subharmonics injected in the load when the spectral content of the perturbations is known.

In figures 59 and 60 we report the the bode plots of the two HTF matrices for

$$V_{line_{nom}} = 20.$$

Remark 6.1

Note that this method can be used only if the amplitude of the perturbation is small enough not to modify too much the linearization trajectory, so that we can suppose the linear model still valid.

If instead the amplitude of the perturbation is too big and this moves too much the operating point we can not rely anymore on the linear approximation.

Anyway, if the variation is slow compared to the frequency of the PWM, we can assume this variation to be piecewise constant. The system can then be studied

around various operating points, under the assumption that the dynamic effects due to the transition from an operating point to another can be neglected.

If this assumption is unreasonable the method presented so far becomes inadequate to the description of the system and other techniques should be developed.

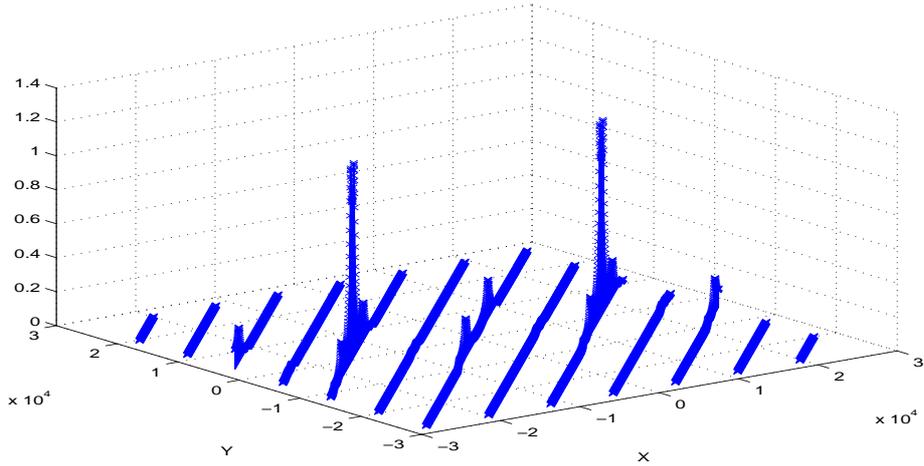


Figure 59: $\text{HTF}_{d_1 \rightarrow v_{out}}(s)$ for $V_{line} = 20$

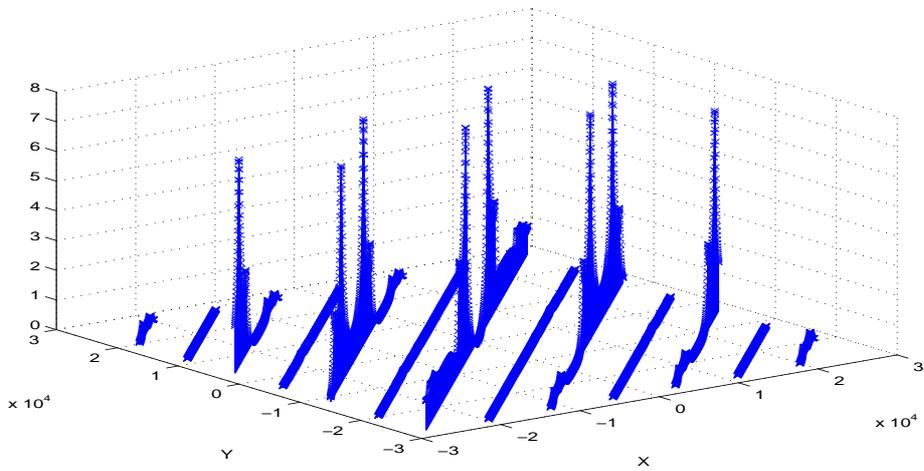


Figure 60: $\text{HTF}_{d_2 \rightarrow v_{out}}(s)$ for $V_{line} = 20$

6.2 Subharmonic Prediction

Finally, we try to predict the arise of subharmonics in our case study using LTP model and techniques.

The idea in this case is to create a LTP model for a certain value of V_{line} and to use this model, that we have seen experimentally to be somehow roust to small variation of V_{line} , to predict the behavior of the system also for other values of the parameter V_{line} .

In particular we are interested in what happens when we create our model for V_{line} very close to a bifurcation value V_{line}^* , namely a boundary value between two zones where the system exhibits a different qualitative behavior.

Consider for instance

$$V_{line}^* = 24.5$$

The system exhibits a 1-periodic behavior for $V_{line} < 24.5$ and it becomes 2-periodic for $V_{line} > 24.5$.

Suppose to compute a LTP model for $V_{line} \lesssim 24.5$, let's say 24, and we tried to use this model to predict the behavior of the system for $24.5 \lesssim V_{line}$, let's say 25. Two scenarios are now possible:

- Thanks to its robustness the model is able to predict the arise of subharmonics. This means that the coefficients that measure the coupling between harmonics and subharmonics in the HTF become non-zero.
- The brusque change in the properties of the system that occurs around V_{line}^* makes the linear approximated model unreliable, as usually happen when we deal with linearized model. In this case the coefficients that describe the harmonics-subharmonics coupling remain null.

Figures 61-64 show that the possibility that come out to be true is the second.

As we said this outcome was the expected one but, at the same time, it was not possible to exclude a priori the first possibility.

It was therefore import to explore this possible application of LTP model and to understand the validity limits of this description.

Note that, nevertheless, *if we relinearize the model* for the new value of V_{line} we can still estimate properly the amount of subharmonics injected in the load.

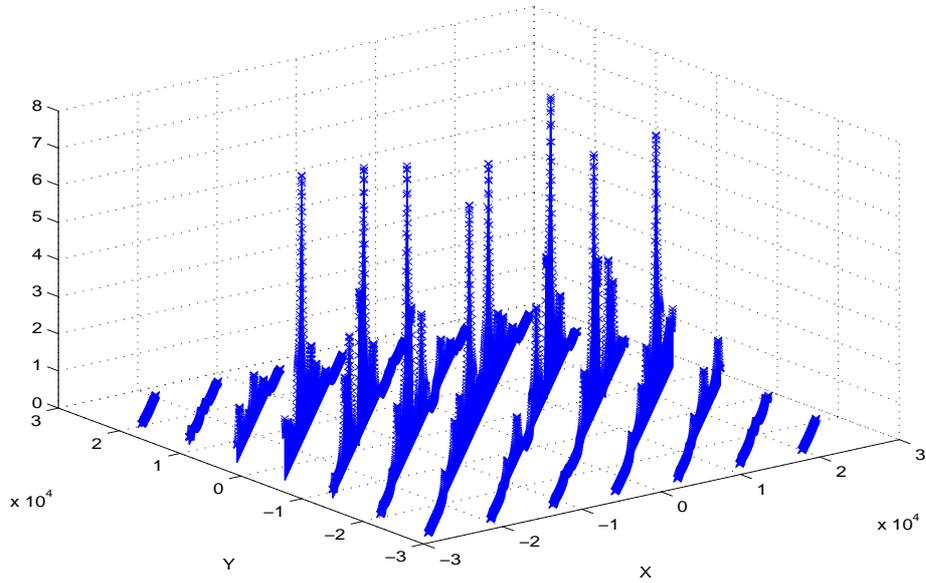


Figure 61: Bode Plot of the closed loop. The LTP model of the PWM is computed for the right value of the line tension, $V_{line} = 25$.

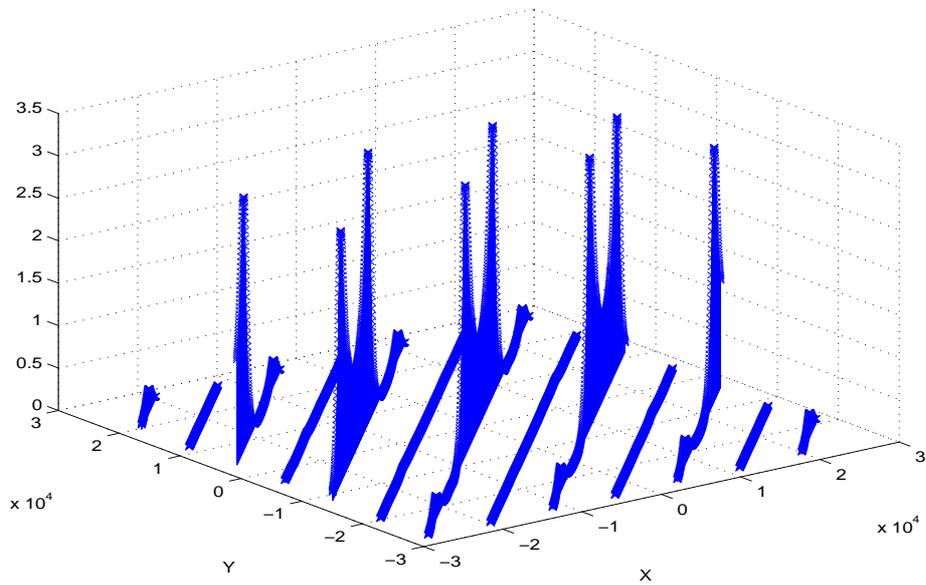


Figure 62: Bode Plot of the closed loop. The LTP model of the PWM is computed for $V_{line} = 24$ and it is supposed still reliable for $V_{line} = 25$.

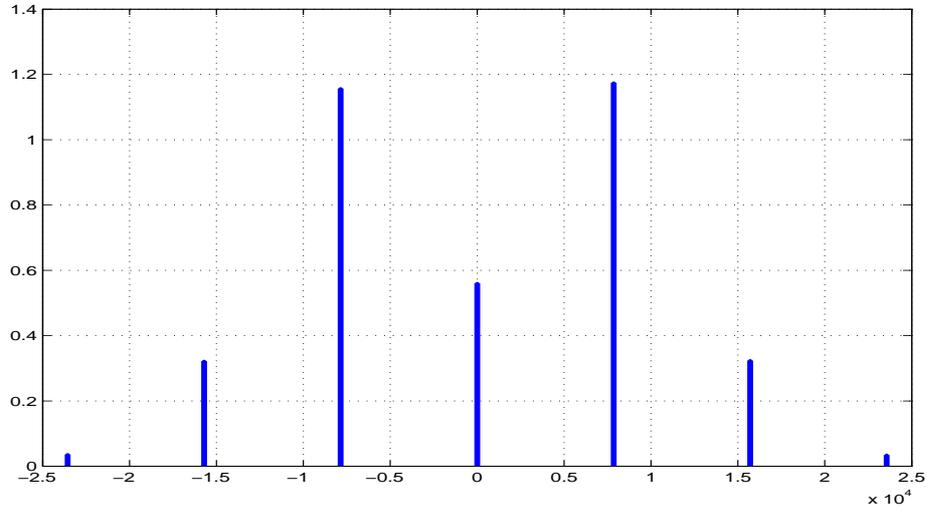


Figure 63: Spectrum of the output when the input is a sinusoidal perturbation of frequency $T = 2 \cdot T_{PWM}$, namely a 2-subharmonic. The LTP model of the PWM is computed for the right value of the line tension, $V_{line} = 25$.

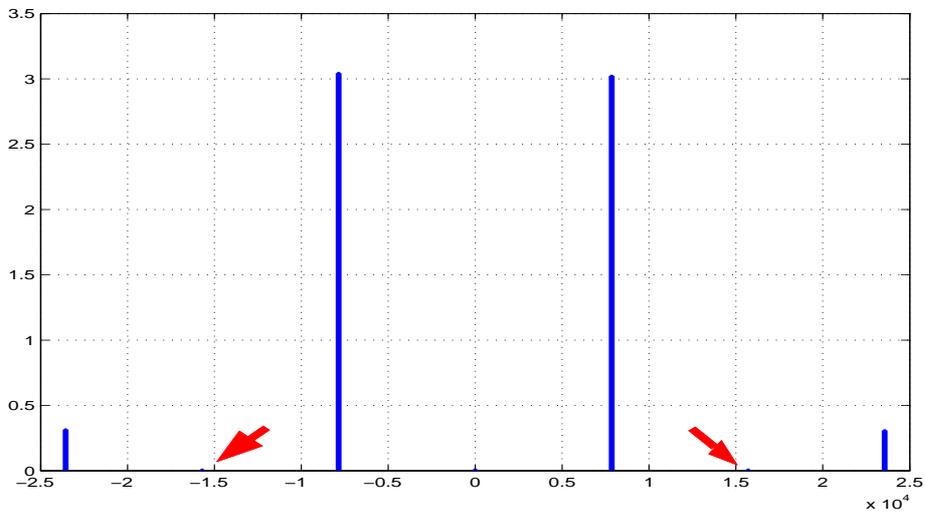


Figure 64: Spectrum of the output when the input is a sinusoidal perturbation of frequency $T = 2 \cdot T_{PWM}$, namely a 2-subharmonic. The LTP model of the PWM is computed for $V_{line} = 24$ and it is supposed still reliable for $V_{line} = 25$.

Conclusions

Switch-mode Dc-Dc converters represent a class of power electronic circuits that are used in a large variety of applications for their compact size, light weight and due to the high efficiency and their reliability. They constitute the enabling technology in computer power supplies, battery chargers, sensitive and demanding aerospace and medical applications and speed Dc motor drives.

Their importance is still increasing and in the recent years, the power electronics community has become more receptive to sophisticated control loop designs. This is due to the advances in the technology of digital processors as well as to increased performance demand of modern-days applications.

As remarked in Hycon work package 4a, a problem of extreme interest in the industrial environment is the analysis of the subharmonics injected in the load by the converter.

This is also the problem we analyzed in this thesis.

Since a linear model with constant coefficients does not describe harmonics coupling, it is not adequate to solve this problem and we have been forced, therefore, to use more complex models.

In particular we chose, instead to abandon completely the class of linear model, to model converters with a linear system with time varying coefficients.

By linearizing the converter around its periodic input we obtained a linear periodically time variant model, that we showed to describe well the behavior of the converter, in particular the harmonics interaction.

For Linear Time Periodic systems there exists a nice theory that generalizes the well known theory for frequency analysis of LTI systems.

The main results of this theory have been reported in the thesis.

By using a LTP model we have obtained therefore a reliable model that is analyzable with simple techniques. Moreover these techniques are similar to the techniques used by practitioner engineers.

Particular attention has been paid to introducing the concept of Harmonic Transfer Function, a generalization of the Transfer Function to LTP systems. This is a natural choice of tool when analyzing harmonics interaction and therefore to estimate the amount of harmonics injected in the load.

An example of Dc-Dc converter has then been analyzed and the subharmonics injection has been estimated.

Summary of main results :

A first contribution is the development of Linear Time Periodic model for the PWM modulator. This model is still simple but quite accurate.

Moreover, compared with the other few models of this kind that can be founded in literature, it is more flexible and general, since it can be easily modified to describe various topology of PWM modulators (symmetric, asymmetric, etc.).

A second contribution of the thesis is the application of the LTP techniques as method for analysis of the subharmonic injection in the load.

In particular, we analyzed what effects that a perturbation on the reference signal has on the spectral content of the output.

To model the effects of a small amplitude perturbation in the line tension or in the load we have introduced two disturbances in appropriate points of the loop. The effects that this perturbation has on the output could then be analyzed by using a procedure completely analogous to the case of a perturbation in the reference.

If the perturbation has an amplitude too big to consider the linearization valid we can not use the LTP approximation anymore.

To deal with this case, if the perturbation can be considered slow compared to dynamic of the system, we can assume that it is piecewise constant and study the system with the techniques presented for various operating points.

If this assumption is unreasonable then the method we presented is inadequate to the description of the system considered and other methods should be developed and used.

Further work :

Naturally the work presented here can be improved in various aspects. We report, as conclusion of the thesis, some possible improvements and some ideas on how to achieve them.

First of all, further tests about the accuracy of the estimation of the subharmonics injected should be performed.

It would be particularly interesting to improve the way we modeled the effects of a load and line tension variation. In this thesis they have been modeled as simple disturbance added in appropriate points of the loop. An interesting possibility to make the model more resemblant to the physical reality is to “filter” the disturbance before it enter in the loop, possibly using a LTP system.

An experimental validation of the model and its predictions should be then performed.

Moreover, different and more complex topologies of converters should be studied following the outline defined in thesis.

Finally it would be extremely interesting to improve the model so that it becomes robust against variations of load or line tension. Unfortunately to achieve this result it is probably necessary to change approach and to abandon the class of linear model.

References

- [1] Erik Möllerstedt, *Dynamic Analysis of Harmonics in Electrical Systems*, PhD Thesis, Department of Automatic Control, Lund University, 2000.
- [2] Norman M. Wereley, *Analysis and Control of Linear Periodically Varying Systems Dynamic*, PhD Thesis, Department of Aeronautic and Astronautic, MIT.
- [3] Henrik Sandberg, *Nonlinear Modelling of Locomotive Propulsion System and Control*, Master Thesis, Department of Automatic Control, Lund University.
- [4] Erik Möllerstedt, Bo Bernhardsson *A Harmonic Transfer Function Model for Diode Converter Train*, In IEEE Power Engineering Society Winter Meeting, Singapore, January 2000.
- [5] Erik Möllerstedt, Bo Bernhardsson *Out of Control Because of Harmonics-An Analysis of the Harmonic Response of an Inverter Locomotive*, IEEE Control Systems Magazine, 20:4, pp. 70–81, August 2000.
- [6] Norman M. Wereley and Steven R. Hall, *Frequency Response of Linear Periodic Systems Dynamic*, Proceeding of the 29th IEEE Conference on Decision and Control; Honolulu, Hawaii, December 5-7 1999.
- [7] Piet Vanassche, Georges Gielen, Willy Sansen, *Symbolic Modeling of Periodically Time-Varying Systems Using Harmonic Transfer Matrices*, IEEE transactions on Computer-Aided Design of Integrated Circuits and Systems, Vol. 21, No. 9, pp. 1011-1024, September 2002.
- [8] Adam Semlyen, *s-Domain methodology for assessing the small signal stability of complex systems in non-sinusoidal steady state.*, IEEE transactions on Power Electronics Vol. 14, No. 1, pp. 132-137, February 1999.
- [9] Thomas S. Parker, Leon O. Chua *Chaos: a tutorial for Engineers*, Proceeding of the IEEE, Vol. 75, No. 8, pp. 982-1008, January 1996.
- [10] Enric Fossas, Gerard Olivar *Study of Chaos in the Buck Converter*, IEEE transactions on Circuits and Systems-I: Fundamental theory and application, Vol. 43, No. 1, pagg. 13-25, January 1996.
- [11] Soumitro Banerjee and Krishnendu Chakrabarty, *Nonlinear Modeling and Bifurcations in the Boost Converter*, IEEE transactions on Power Electronics Vol. 13, No. 2, pp. 252-260, March 1998.
- [12] Krishnendu Chakrabarty, Goutam Poddar and Soumitro Banerjee *Bifurcation Behavior of the Buck Converter*, IEEE transactions on Power Electronics Vol. 11, No. 3, pp. 439-447, May 1996.

- [13] Jhonathan H. B. Deane and Dacid C. Hamill *Instability, Subharmonics and Chaos in Power Electronic Systems* , IEEE transactions on Power Electronics Vol. 5, No. 3, pp. 260-268, July 1990.
- [14] Fan dong Tan and Raymond S. Ramshaw *Instabilities of a Boost Converter System Under Large Parameter Variation*, IEEE transactions on Power Electronics Vol. 5, No. 3, pp. 260-268, July 1990.
- [15] C. K. Tse *Study of Chaos in the Buck Converter*, IEEE transactions on Circuits and Systems-I: Fundamental theory and application, Vol. 41, No. 1, pagg. 16-23, January 1994.
- [16] Guohui Yuan, Soumitro Banerjee, Edward Ott and James A. Yorke *Border-Collision Bifurcation in the Buck Converter*, IEEE transactions on Circuits and Systems-I: Fundamental theory and application, Vol. 41, No. 1, pagg. 16-23, January 1994.