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Physics Based Wave Generation for the Shallow Water Equations

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Abstract
In this report we study systems governed by the Saint-Venant shallow water equations interacting with boundary conditions including control. We derive closed form formulas for the open-loop behavior. These formulas rely on first, second and third order approximations which are based on a perturbation that we develop. Properties of these solutions are studied and comparisons with existing non-linear solver solutions are carried out. The main advantage of our methodology is to provide computation efficient formulas that can be implemented for real-time calculations.

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Contents

1 Introduction

The goal of the work is to find accurate and explicit formulas for the motion of water waves. The systems under consideration have important boundary conditions, these boundary conditions interact with the water and generate waves trough the Saint-Venant shallow water equations (for background information one may refer to [12]). A prime example of such a system is the water tank as studied by Grundelius [8], Petit and Rouchon [9]. Other example include water channels as studied by Coron, d'Andréa Novel and Bastin [2].

The formulas we derive can be used either for feed-forward control for stabilization purposes (one may use the Prieur and de Halleux approach [10]) or as straight forward simulation techniques. Both benefit from real-time capability. For control applications open-loop trajectories are to be recomputed every time the user updates the desired operating point (tank position). In that context computational speed reduces waiting time for the user. On the other hand fast simulation techniques, as the formulas given in this report, can also be used for interactive realtime computer graphics.

Starting from the first order approximation formulas given in Dubois, Petit and Rouchon [5] we develop a perturbation method that allows us to derive second and third order approximation. A sketch for the method at a general n^{th}-order is also given. Following the classical ideas (see for instance Debnath [3], [4]) we use already computed approximations to derive a new higher order approximation of the partial differential equation, that we show how to solve in a sequence.

Next, properties of these second and third order approximation are studied. We prove that steady-state controllability can not be achieved for second order for a short time. In addition we prove that a set of first order approximated water tanks can be controlled all together provided that their dimensions satisfies rational equations.

For steady-state to steady-state transient we derive bounds for first, second and third order solutions, these serve as sanity checks for numerical simulations.

We compare the different approximations to a Godunov scheme resolution of the non-linear partial differential equation as given in Dubois [5]. We investigate the volume preservation and the shape of evolving wavefronts. A numerical conclusion is that second and third approximation order are more like the solution of the Godunov scheme based numerical method than first order. Also they compare favorably with the Godunov scheme in terms of CPU-time.

The methodology that we develop is extended to the wave-maker problem. We end up with similar formulas that can also be exploited in computer graphics problems. We propose a one-dimensional tank, a two-dimensional tank and a boat on the open sea to illustrate our approach.

1.1 Physics of the water-tank system

We consider a tank of length l, containing a perfect fluid under gravity g, which is submitted to a one-dimensional horizontal motion $D(t)$, as depicted in figure 1. The vertical component of the fluid velocity is considered to be the negligibly small, thus the water velocity is horizontal and only depends on the x-coordinate. The Saint-Venant equations is a suitable model for these shallow water flows.

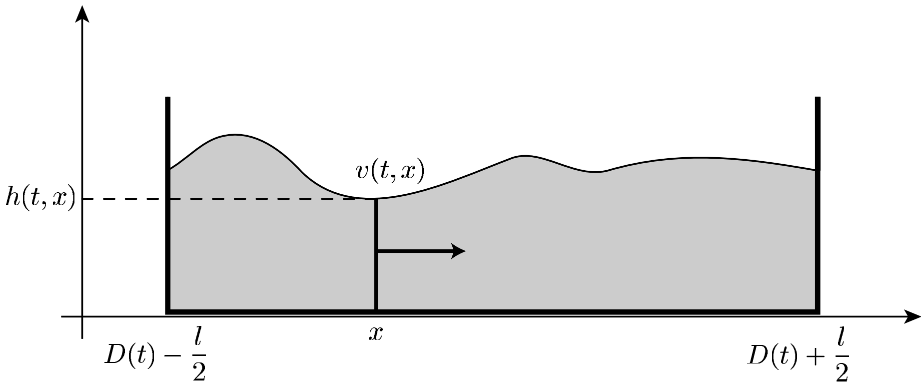

Figure 1: Tank of length l, its midpoint has the x-coordinate $D(t)$

The partial differential equation (PDE) consist of the continuity and momentum equation.

$$PDE \begin{cases} \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hv) = 0 \\ \frac{\partial}{\partial t}(hv) + \frac{\partial}{\partial x}(hv^2 + \frac{g}{2}h^2) = 0 \end{cases}$$

The water velocity is zero at the boundaries of the tank and at time zero the water is at steady state. This implies the following initial and boundary conditions

$$BC \left\{ \begin{array}{l} v(t, D(t) - \frac{l}{2}) = \dot{D}(t) \\ v(t, D(t) + \frac{l}{2}) = \dot{D}(t) \end{array} \right| \forall t \geq 0$$

$$IC \left\{ \begin{array}{r} h(0,x) = h_0 \\ v(0,x) = 0 \end{array} \right| \forall x \in \left[D(t) - \frac{l}{2}, D(t) + \frac{l}{2} \right]$$

where h_0 is the constant water height of steady state.

Notation From now on we denote the partial differential equation as PDE, the boundary condition as BC and initial condition as IC.

1.2 Alternative set of equations

With help of the Riemann invariants (for more details see [9]) the set of equations (PDE, BC, IC) can be rewritten to a more handleable form. Let

$$\begin{cases} z = x - D(t) \\ J_+(t,z) = v(t, z{+}D(t)) - 2\sqrt{g \cdot h(t, z{+}D(t))} \\ J_-(t,z) = v(t, z{+}D(t)) + 2\sqrt{g \cdot h(t, z{+}D(t))} \\ \alpha_+(t,z) = v(t, z{+}D(t)) - \sqrt{g \cdot h(t, z{+}D(t))} - \dot{D}(t) \\ \alpha_-(t,z) = v(t, z{+}D(t)) + \sqrt{g \cdot h(t, z{+}D(t))} - \dot{D}(t) \end{cases}$$

Where (t,z) is the tank local coordinate system. To get back to the original functions these transformations may be used

$$\begin{cases} x = z + D(t) \\ h(t,z) = \dfrac{1}{16g}\left(J_-(t,z) - J_+(t,z)\right)^2 \\ v(t,z) = \dfrac{1}{2}\left(J_+(t,z) + J_-(t,z)\right) \end{cases}$$

The new set of equations writes

$$PDE \begin{cases} \dfrac{\partial J_+}{\partial t} + \alpha_+ \dfrac{\partial J_+}{\partial z} = 0 \\ \dfrac{\partial J_-}{\partial t} + \alpha_- \dfrac{\partial J_-}{\partial z} = 0 \end{cases} \tag{1}$$

$$\begin{gathered} BC \begin{cases} \dfrac{J_+ + J_-}{2}(t, -\dfrac{l}{2}) = \dot{D}(t) \\ \dfrac{J_+ + J_-}{2}(t, \dfrac{l}{2}) = \dot{D}(t) \end{cases} \\ IC \begin{cases} \dfrac{1}{16g}(J_-(0,z) - J_+(0,z))^2 = h_0 \\ \dfrac{J_+(0,z) + J_-(0,z)}{2} = 0 \end{cases} \Bigg| \; z \in [-\dfrac{l}{2}, \dfrac{l}{2}] \end{gathered} \tag{2}$$

From now on the system is represented by (1) and (2) where

$$\begin{cases} \alpha_+ = \dfrac{1}{2}(J_+ + J_-) - \dfrac{1}{4}(J_- - J_+) - \dot{D} \\ \alpha_- = \dfrac{1}{2}(J_+ + J_-) + \dfrac{1}{4}(J_- - J_+) - \dot{D} \end{cases}$$

which equals

$$\begin{cases} \alpha_+ = \dfrac{3}{4}J_+ + \dfrac{1}{4}J_- - \dot{D} \\ \alpha_- = \dfrac{1}{4}J_+ + \dfrac{3}{4}J_- - \dot{D} \end{cases} \tag{3}$$

2 Perturbation Method for the water tank

In [5] a linear approximation to the equations was performed. To obtain higher order approximation with non-linearity we use a series expansion. Let J_+, J_-, α_+ and α_- be expressed by the expansion series

$$
\begin{aligned}
J_+ &= J_+^0 + J_+^1 + J_+^2 + \cdots \\
J_- &= J_-^0 + J_-^1 + J_-^2 + \cdots \\
\alpha_+ &= \alpha_+^0 + \alpha_+^1 + \alpha_+^2 + \cdots \\
\alpha_- &= \alpha_-^0 + \alpha_-^1 + \alpha_-^2 + \cdots
\end{aligned}
$$

In the following we define the k^{th} order approximation as the sum of the first $k+1$ terms in the preceding expansions. Additionally a term with upper index i will be referred to as a i^{th} order term. It should be kept in mind that in this method higher order not necessarily mean smaller magnitude. The PDE (1) writes

$$
\begin{aligned}
\sum_{k=0}^{n} \frac{\partial J_+^k}{\partial t} + \sum_{i=0}^{n} \alpha_+^i \cdot \sum_{k=0}^{n} \frac{\partial J_+^k}{\partial z} = 0 \\
\sum_{k=0}^{n} \frac{\partial J_-^k}{\partial t} + \sum_{i=0}^{n} \alpha_-^i \cdot \sum_{k=0}^{n} \frac{\partial J_-^k}{\partial z} = 0
\end{aligned}
$$

where $n \to \infty$. From now on a more compact notation will be used.

$$
PDE \begin{cases} \sum_{k=0}^{n} \dot{J}_+^k + \sum_{i=0}^{n} \alpha_+^i \cdot \sum_{k=0}^{n} J_{+z}^k = 0 \\ \sum_{k=0}^{n} \dot{J}_-^k + \sum_{i=0}^{n} \alpha_-^i \cdot \sum_{k=0}^{n} J_{-z}^k = 0 \end{cases} \tag{4}
$$

the boundary and initial conditions are

$$
\begin{aligned}
BC &\begin{cases} \frac{1}{2} \sum_{k=0}^{n} \left(J_+^k(t, -\frac{l}{2}) + J_-^k(t, -\frac{l}{2}) \right) = \dot{D}(t) \\ \frac{1}{2} \sum_{k=0}^{n} \left(J_+^k(t, \frac{l}{2}) + J_-^k(t, \frac{l}{2}) \right) = \dot{D}(t) \end{cases} \\
IC &\begin{cases} \frac{1}{16g} \left(\sum_{k=0}^{n} \left(J_-^k(0,z) - J_+^k(0,z) \right) \right)^2 = h_0 \\ \frac{1}{2} \sum_{k=0}^{n} \left(J_+^k(0,z) + J_-^k(0,z) \right) = 0 \end{cases} \Bigg| \; z \in [-\frac{l}{2}, \frac{l}{2}]
\end{aligned} \tag{5}
$$

The kth-order representation of these equations is obtained by only including terms with total order less or equal to k. In a product the total order is obtained by adding the orders, eg. $J_+^1 \alpha_+^2$ is a third order term. To explain how the method work the solution for zero, first, second and third order are now given.

2.1 Zero order

Let the zero order terms be defined as constants, hence the boundary conditions can not be fulfilled at this order but the PDE and the initial conditions can. One of two solutions is

$$\begin{cases} J_+^0 = -2c \\ J_-^0 = 2c \end{cases} \tag{6}$$

where $c = \sqrt{g \cdot h_0}$ is the wave propagation speed. Choosing the other solution would be the same as switching J_+ and J_-, hence giving rise to the same height (h) and velocity (v). The corresponding α-terms can be obtained from Eq. (3) discarding $\dot{D}$ (zero order terms are defined to be constant).

$$\begin{cases} \alpha_+^0 = -c \\ \alpha_-^0 = c \end{cases}$$

The zero order solution corresponds to the steady state, which is also the initial condition.

2.2 First order

2.2.1 Solving the PDE

The first order representation of the PDE, (4), is obtained by taking into account terms with a total order not greater than one. The zero order terms are constants, their derivatives are zero, and can therefore from now on be excluded from the PDE approximations. The first order representation writes

$$\begin{cases} \dot{J}_+^1 + \alpha_+^0 J_{+z}^1 = 0 \\ \dot{J}_-^1 + \alpha_-^0 J_{-z}^1 = 0 \end{cases}$$

which equals

$$\begin{cases} \dot{J}_+^1 - c \cdot J_{+z}^1 = 0 \\ \dot{J}_-^1 + c \cdot J_{-z}^1 = 0 \end{cases} \tag{7}$$

This is a set of two linear Burgers equations with the wave propagation speed c. The characteristics method (see Marsden [1]) gives the general solution

$$\begin{cases} J_+^1(t,z) = \varphi_+(t + \frac{z}{c}) \\ J_-^1(t,z) = \varphi_-(t - \frac{z}{c}) \end{cases}$$

2.2.2 Matching the boundary conditions

Matching the general solution to the boundary conditions according to (5) gives rise to the equations

$$\begin{cases} \frac{J_+^0 + J_+^1 + J_-^0 + J_-^1}{2}(t, -\frac{l}{2}) = \dot{D}(t) \\ \frac{J_+^0 + J_+^1 + J_-^0 + J_-^1}{2}(t, \frac{l}{2}) = \dot{D}(t) \end{cases} \tag{8}$$

so

$$\begin{cases} \varphi_+(t-\dfrac{\Delta}{2})+\varphi_-(t+\dfrac{\Delta}{2})=2\dot{D}(t) \\ \varphi_+(t+\dfrac{\Delta}{2})+\varphi_-(t-\dfrac{\Delta}{2})=2\dot{D}(t) \end{cases}$$

where Δ is the time required for a wave to travel from one end of the tank to the other, $\Delta = \frac{l}{c}$. The same equations in matrix form writes

$$\begin{pmatrix} e^{-s\frac{\Delta}{2}} & e^{s\frac{\Delta}{2}} \\ e^{s\frac{\Delta}{2}} & e^{-s\frac{\Delta}{2}} \end{pmatrix} \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot 2\dot{D} \tag{9}$$

Let

$$M = \begin{pmatrix} e^{-s\frac{\Delta}{2}} & e^{s\frac{\Delta}{2}} \\ e^{s\frac{\Delta}{2}} & e^{-s\frac{\Delta}{2}} \end{pmatrix}$$

Denote the null space of M as

$$\mathcal{N}(M) = \{\pi \,|\, M\cdot\pi = 0\}$$

Its elements are solution to

$$\begin{pmatrix} e^{-s\frac{\Delta}{2}} & e^{s\frac{\Delta}{2}} \\ e^{s\frac{\Delta}{2}} & e^{-s\frac{\Delta}{2}} \end{pmatrix} \cdot \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which implies

$$\begin{cases} \pi_1(t) = -\pi_2(t+\Delta) \\ \pi_1(t+\Delta)-\pi_1(t-\Delta)=0 \\ \pi_2(t+\Delta)-\pi_2(t-\Delta)=0 \end{cases}$$

Hence, the null space elements to M can be written as

$$\begin{pmatrix} -\pi(t+\Delta) \\ \pi(t) \end{pmatrix} \tag{10}$$

where $\pi(t)$ is a 2Δ-periodic function. The general solution to (9) is

$$\begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix} = \begin{pmatrix} -e^{\Delta s} \\ 1 \end{pmatrix} \pi + \text{"}M^{-1}\text{"} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot 2\dot{D}$$

which implies

$$\begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix} = \begin{pmatrix} -e^{\Delta s} \\ 1 \end{pmatrix} \pi + \frac{1}{e^{-s\Delta}-e^{s\Delta}} \begin{pmatrix} e^{-s\frac{\Delta}{2}} & -e^{s\frac{\Delta}{2}} \\ -e^{s\frac{\Delta}{2}} & e^{-s\frac{\Delta}{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot 2\dot{D}$$

$$\begin{cases} \varphi_+ = -e^{\Delta s}\pi + \dfrac{e^{-s\frac{\Delta}{2}}-e^{s\frac{\Delta}{2}}}{e^{-s\Delta}-e^{s\Delta}} \cdot 2\dot{D} \\ \\ \varphi_- = \pi + \dfrac{e^{-s\frac{\Delta}{2}}-e^{s\frac{\Delta}{2}}}{e^{-s\Delta}-e^{s\Delta}} \cdot 2\dot{D} \end{cases} \Rightarrow \begin{cases} \varphi_+ = -e^{\Delta s}\pi + \dfrac{1}{e^{-s\frac{\Delta}{2}}+e^{s\frac{\Delta}{2}}} \cdot 2\dot{D} \\ \\ \varphi_- = \pi + \dfrac{1}{e^{-s\frac{\Delta}{2}}+e^{s\frac{\Delta}{2}}} \cdot 2\dot{D} \end{cases}$$

To make the solution more handleable an extra variable $\tilde{\nu}$ is introduced. Set

$$2\dot{D}(t) = \dot{\tilde{\nu}}(t + \frac{\Delta}{2}) + \dot{\tilde{\nu}}(t - \frac{\Delta}{2})$$

With this definition the general solution to the PDE is

$$\begin{cases} \varphi_+(t) = -\pi(t + \Delta) + \dot{\tilde{\nu}}(t) \\ \varphi_-(t) = \pi(t) + \dot{\tilde{\nu}}(t) \end{cases}$$

2.2.3 Fitting the initial conditions

According to (5) the initial conditions with $z \in [-\frac{l}{2}, \frac{l}{2}]$ lead to

$$\begin{cases} J_-^0(0,z) + J_-^1(0,z) - (J_+^0(0,z) + J_+^1(0,z)) = 4 \cdot \sqrt{g \cdot h_0} \\ J_-^0(0,z) + J_-^1(0,z) + J_+^0(0,z) + J_+^1(0,z) = 0 \end{cases}$$

$$\begin{cases} J_-^1(0,z) - J_+^1(0,z) = 0 \\ J_-^1(0,z) + J_+^1(0,z) = 0 \end{cases} \Rightarrow \begin{cases} J_+^1(0,z) = 0 \\ J_-^1(0,z) = 0 \end{cases}$$

The initial conditions will be the same for higher orders.

$$\begin{cases} J_+^n(0,z) = 0 \\ J_-^n(0,z) = 0 \end{cases}, z \in \left[-\frac{l}{2}, \frac{l}{2}\right], n = 1, 2, 3, ... \tag{11}$$

The initial condition rewrite in terms of φ_+ and φ_- as

$$\begin{cases} \varphi_+(s) = -\pi(s + \Delta) + \dot{\tilde{\nu}}(s) = 0 \\ \varphi_-(s) = \pi(s) + \dot{\tilde{\nu}}(s) = 0 \end{cases}, s \in \left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$$

Thus, π is not only 2Δ-periodic but Δ-antiperiodic. Set

$$\dot{\nu}(s) = \pi(s) + \dot{\tilde{\nu}}(s)$$

one easily get

$$\dot{\nu}(s) = 0\,, s \in \left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right] \tag{12}$$

and

$$2\dot{D}(t) = \dot{\nu}(t + \frac{\Delta}{2}) - \pi(t + \frac{\Delta}{2}) + \dot{\nu}(t - \frac{\Delta}{2}) - \pi(t - \frac{\Delta}{2})$$

Yet π is Δ-antiperiodic so

$$2\dot{D}(t) = \dot{\nu}(t + \frac{\Delta}{2}) + \dot{\nu}(t - \frac{\Delta}{2})$$

In conclusion the unique solution is

$$\begin{cases} J_+^1(t,z) = \dot{\nu}(t + \frac{z}{c}) \\ J_-^1(t,z) = \dot{\nu}(t - \frac{z}{c}) \end{cases} \tag{13}$$

The corresponding α-terms are determined from equation (3), including $\dot{D}$ this time.

$$\begin{cases} \alpha_+^0 + \alpha_+^1 = \frac{3}{4}(J_+^0 + J_+^1) + \frac{1}{4}(J_-^0 + J_-^1) - \dot{D} \\ \alpha_-^0 + \alpha_-^1 = \frac{1}{4}(J_+^0 + J_+^1) + \frac{3}{4}(J_-^0 + J_-^1) - \dot{D} \end{cases}$$

Using equations (6,13) and the fact that $2\dot{D}(t) = \dot{\nu}(t + \frac{\Delta}{2}) + \dot{\nu}(t - \frac{\Delta}{2})$, the expression can be rewritten to

$$\begin{cases} \alpha_+^1 = \frac{3}{4}\dot{\nu}(t + \frac{z}{c}) + \frac{1}{4}\dot{\nu}(t - \frac{z}{c}) - \frac{1}{2}\Big(\dot{\nu}(t + \frac{\Delta}{2}) + \dot{\nu}(t - \frac{\Delta}{2})\Big) \\ \alpha_-^1 = \frac{1}{4}\dot{\nu}(t + \frac{z}{c}) + \frac{3}{4}\dot{\nu}(t - \frac{z}{c}) - \frac{1}{2}\Big(\dot{\nu}(t + \frac{\Delta}{2}) + \dot{\nu}(t - \frac{\Delta}{2})\Big) \end{cases} \tag{14}$$

The first order solution corresponds to the solution to the linearized model used in [5].

2.2.4 Summary first order

Set of equations:

$$PDE \begin{cases} \dot{J}_+^1 - c \cdot J_{+z}^1 = 0 \\ \dot{J}_-^1 + c \cdot J_{-z}^1 = 0 \end{cases}$$

$$BC \begin{cases} (J_+^1 + J_-^1)(t, -\frac{l}{2}) = 2\dot{D}(t) \\ (J_+^1 + J_-^1)(t, \frac{l}{2}) = 2\dot{D}(t) \end{cases}$$

$$IC \begin{cases} J_+^1(0, z) = 0 \\ J_-^1(0, z) = 0 \end{cases}, z \in \left[-\frac{l}{2}, \frac{l}{2}\right]$$

Solution:

$$\begin{cases} J_+^1(t, z) = \dot{\nu}(t + \frac{z}{c}) \\ J_-^1(t, z) = \dot{\nu}(t - \frac{z}{c}) \end{cases}$$

where

$$2\dot{D}(t) = \dot{\nu}(t + \frac{\Delta}{2}) + \dot{\nu}(t - \frac{\Delta}{2})$$

and

$$\dot{\nu}(s) = 0\,, s \in \left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$$

2.3 Second order

2.3.1 Solving the PDE

The second order representation of (4) is

$$\begin{cases} \dot{J}_+^1 + \dot{J}_+^2 + \alpha_+^0 J_{+z}^1 + \alpha_+^0 J_{+z}^2 + \alpha_+^1 J_{+z}^1 = 0 \\ \dot{J}_-^1 + \dot{J}_-^2 + \alpha_-^0 J_{-z}^1 + \alpha_-^0 J_{-z}^2 + \alpha_-^1 J_{-z}^1 = 0 \end{cases}$$

Subtracting first order (7) the equations can be written in the same form as (7) but with two forcing terms f_+ and f_-.

$$\begin{cases} \dot{J}_+^2 - c \cdot J_{+z}^2 = -\alpha_+^1 J_{+z}^1 = f_+ \\ \dot{J}_-^2 + c \cdot J_{-z}^2 = -\alpha_-^1 J_{-z}^1 = f_- \end{cases}$$

According to (14) the forcing terms are

$$\begin{aligned} f_+(t,z) &= -\left(\frac{3}{4}\dot{\nu}(t+\frac{z}{c}) + \frac{1}{4}\dot{\nu}(t-\frac{z}{c}) - \frac{1}{2}\left(\dot{\nu}(t+\frac{\Delta}{2}) + \dot{\nu}(t-\frac{\Delta}{2})\right)\right) \cdot \\ &\qquad\qquad \ddot{\nu}(t+\frac{z}{c}) \cdot \frac{1}{c} \\ f_-(t,z) &= \left(\frac{1}{4}\dot{\nu}(t+\frac{z}{c}) + \frac{3}{4}\dot{\nu}(t-\frac{z}{c}) - \frac{1}{2}\left(\dot{\nu}(t+\frac{\Delta}{2}) + \dot{\nu}(t-\frac{\Delta}{2})\right)\right) \cdot \\ &\qquad\qquad \ddot{\nu}(t-\frac{z}{c}) \cdot \frac{1}{c} \end{aligned} \tag{15}$$

To handle this problem another change of coordinates $(t,z) \mapsto (\xi_+, \xi_-)$ is applied. Let

$$\begin{cases} \xi_+ = t - \dfrac{z}{c} \\ \\ \xi_- = t + \dfrac{z}{c} \end{cases}$$

and inversely

$$\begin{cases} t = \dfrac{\xi_+ + \xi_-}{2} \\ \\ z = (\xi_- - \xi_+) \cdot \dfrac{c}{2} \end{cases} \tag{16}$$

In the following a function defined in the coordinate system (t,z) is noted with a hat in the coordinate system (ξ_+, ξ_-). Eg.

$$\begin{aligned} J_+ &\longleftrightarrow J_+(t,z) \\ \hat{J}_+ &\longleftrightarrow \hat{J}_+(\xi_+,\xi_-) \end{aligned}$$

with

$$\begin{cases} \hat{J}_+(\xi_+,\xi_-) = J_+(\dfrac{\xi_+ + \xi_-}{2}, (\xi_- - \xi_+) \cdot \dfrac{c}{2}) \\ J_+(t,z) = \hat{J}_+(t - \dfrac{z}{c}, t + \dfrac{z}{c}) \end{cases}$$

Classically

$$\begin{cases} \left(\dfrac{\partial \hat{J}_+^2}{\partial \xi_+}\right)_{\xi_-} = \left(\dfrac{\partial J_+^2}{\partial t}\right)_z \cdot \left(\dfrac{\partial t}{\partial \xi_+}\right)_{\xi_-} + \left(\dfrac{\partial J_+^2}{\partial z}\right)_t \cdot \left(\dfrac{\partial z}{\partial \xi_+}\right)_{\xi_-} \\ \left(\dfrac{\partial \hat{J}_-^2}{\partial \xi_-}\right)_{\xi_+} = \left(\dfrac{\partial J_-^2}{\partial t}\right)_z \cdot \left(\dfrac{\partial t}{\partial \xi_-}\right)_{\xi_+} + \left(\dfrac{\partial J_-^2}{\partial z}\right)_t \cdot \left(\dfrac{\partial z}{\partial \xi_-}\right)_{\xi_+} \end{cases} \tag{17}$$

Further (16) implies

$$\begin{cases} \left(\dfrac{\partial t}{\partial \xi_+}\right)_{\xi_-} = \dfrac{1}{2} \\ \left(\dfrac{\partial t}{\partial \xi_-}\right)_{\xi_+} = \dfrac{1}{2} \end{cases} \qquad \begin{cases} \left(\dfrac{\partial z}{\partial \xi_+}\right)_{\xi_-} = -\dfrac{c}{2} \\ \left(\dfrac{\partial z}{\partial \xi_-}\right)_{\xi_+} = \dfrac{c}{2} \end{cases}$$

Hence (17) can be rewritten as

$$\begin{cases} \left(\dfrac{\partial \hat{J}_+^2}{\partial \xi_+}\right)_{\xi_-} = \dfrac{1}{2}\left(\left(\dfrac{\partial J_+^2}{\partial t}\right)_z - c\left(\dfrac{\partial J_+^2}{\partial z}\right)_t\right) \\ \left(\dfrac{\partial \hat{J}_-^2}{\partial \xi_-}\right)_{\xi_+} = \dfrac{1}{2}\left(\left(\dfrac{\partial J_-^2}{\partial t}\right)_z + c\left(\dfrac{\partial J_-^2}{\partial z}\right)_t\right) \end{cases}$$

In coordinate system (t, z) the problem to solve is

$$\begin{cases} \dfrac{\partial J_+^2}{\partial t} - c\dfrac{\partial J_+^2}{\partial z} = f_+^2 \\ \\ \dfrac{\partial J_-^2}{\partial t} + c\dfrac{\partial J_-^2}{\partial z} = f_-^2 \end{cases}$$

In coordinate system (ξ_+, ξ_-) it becomes

$$\begin{cases} 2 \cdot \left(\dfrac{\partial \hat{J}_+^2}{\partial \xi_+}\right)_{\xi_-} = \hat{f}_+^2 \\ 2 \cdot \left(\dfrac{\partial \hat{J}_-^2}{\partial \xi_-}\right)_{\xi_+} = \hat{f}_-^2 \end{cases}$$

Its general solution is given by:

$$\begin{cases} \hat{J}_+^2(\xi_+, \xi_-) = \dfrac{1}{2}\displaystyle\int_{\xi_-}^{\xi_+} \hat{f}_+^2(s, \xi_-)\ ds + \hat{\varphi}_+^2(\xi_-) \\ \hat{J}_-^2(\xi_+, \xi_-) = \dfrac{1}{2}\displaystyle\int_{\xi_+}^{\xi_-} \hat{f}_-^2(\xi_+, s)\ ds + \hat{\varphi}_-^2(\xi_+) \end{cases}$$

Where the starting points of the integration may be arbitrary chosen, as long as they are not dependant on the integration variable. For practical reasons the starting points have been chosen to ξ_- and ξ_+.

The solution can be separated in two parts, the homogenous and the particular. The particular parts are the integrals which when differentiated according to the PDE gives rise to the forcing terms. The homogenous parts are the general solutions to the homogenous equation known from first order. The forcing terms (15) gives rise to the following solutions.

$$J_+^2(t,z) = -\frac{\ddot{\nu}(t+\frac{z}{c})}{8\cdot c}\Bigg[\nu(t-\frac{z}{c}) - \nu(t+\frac{z}{c}) - 3\dot{\nu}(t+\frac{z}{c})\cdot\frac{2z}{c} - 4\left(\nu(t+\frac{\Delta}{2}) + \nu(t-\frac{\Delta}{2}) - \nu(t+\frac{z}{c}+\frac{\Delta}{2}) - \nu(t+\frac{z}{c}-\frac{\Delta}{2})\right)\Bigg] + \varphi_+^2(t+\frac{z}{c}) \tag{18}$$

$$J_-^2(t,z) = \frac{\ddot{\nu}(t-\frac{z}{c})}{8\cdot c}\Bigg[\nu(t+\frac{z}{c}) - \nu(t-\frac{z}{c}) + 3\dot{\nu}(t-\frac{z}{c})\cdot\frac{2z}{c} - 4\left(\nu(t+\frac{\Delta}{2}) + \nu(t-\frac{\Delta}{2}) - \nu(t-\frac{z}{c}+\frac{\Delta}{2}) - \nu(t-\frac{z}{c}-\frac{\Delta}{2})\right)\Bigg] + \varphi_-^2(t-\frac{z}{c}) \tag{19}$$

To simplify the expression without loss of generality, and thus making it more practical for computations, the general homogenous parts can be redefined as follows.

$$\Phi_+^2(t+\frac{z}{c}) = -\frac{\ddot{\nu}(t+\frac{z}{c})}{8\cdot c}\Bigg[-\nu(t+\frac{z}{c}) + 4\cdot\nu(t+\frac{z}{c}+\frac{\Delta}{2}) + 4\cdot\nu(t+\frac{z}{c}-\frac{\Delta}{2})\Bigg] + \varphi_+^2(t+\frac{z}{c})$$

$$\Phi_-^2(t-\frac{z}{c}) = \frac{\ddot{\nu}(t-\frac{z}{c})}{8\cdot c}\Bigg[-\nu(t-\frac{z}{c}) + 4\cdot\nu(t-\frac{z}{c}+\frac{\Delta}{2}) + 4\cdot\nu(t-\frac{z}{c}-\frac{\Delta}{2})\Bigg] + \varphi_-^2(t-\frac{z}{c})$$

Hence a more compact expression for the solution is obtained

$$J_+^2(t,z) = \underbrace{-\frac{\ddot{\nu}(t+\frac{z}{c})}{8\cdot c}\Bigg[\nu(t-\frac{z}{c}) - 6\dot{\nu}(t+\frac{z}{c})\cdot\frac{z}{c} - 4\left(\nu(t+\frac{\Delta}{2}) + \nu(t-\frac{\Delta}{2})\right)\Bigg]}_{J_{p+}^2(t,z)} \underbrace{+\,\Phi_+^2(t+\frac{z}{c})}_{J_{h+}^2(t,z)}$$

$$J_-^2(t,z) = \underbrace{\frac{\ddot{\nu}(t-\frac{z}{c})}{8\cdot c}\Bigg[\nu(t+\frac{z}{c}) + 6\dot{\nu}(t-\frac{z}{c})\cdot\frac{z}{c} - 4\left(\nu(t+\frac{\Delta}{2}) + \nu(t-\frac{\Delta}{2})\right)\Bigg]}_{J_{p-}^2(t,z)} \underbrace{+\,\Phi_-^2(t-\frac{z}{c})}_{J_{h-}^2(t,z)}$$

2.3.2 Matching the boundary conditions

The boundary conditions for second order, subtracting the terms of zero and first order (8), yield

$$\left.\begin{aligned}\left(\underset{h}{J}^2_+ + \underset{p}{J}^2_+ + \underset{h}{J}^2_- + \underset{p}{J}^2_-\right)(t, -\frac{l}{2}) = 0\\ \left(\underset{h}{J}^2_+ + \underset{p}{J}^2_+ + \underset{h}{J}^2_- + \underset{p}{J}^2_-\right)(t, \frac{l}{2}) = 0\end{aligned}\right\} \tag{20}$$

The particular solutions have the following symmetry properties

$$\begin{cases}\underset{p}{J}^2_+(t, -\frac{l}{2}) = -\underset{p}{J}^2_-(t, \frac{l}{2})\\ \underset{p}{J}^2_+(t, \frac{l}{2}) = -\underset{p}{J}^2_-(t, -\frac{l}{2})\end{cases}$$

Set $A(t) = \underset{p}{J}^2_+(t, \frac{l}{2}) - \underset{p}{J}^2_+(t, -\frac{l}{2})$. Now (20) can be written as

$$\begin{cases}\Phi^2_+(t - \frac{\Delta}{2}) + \Phi^2_-(t + \frac{\Delta}{2}) = A(t)\\ \Phi^2_+(t + \frac{\Delta}{2}) + \Phi^2_-(t - \frac{\Delta}{2}) = -A(t)\end{cases}$$

with

$$\begin{aligned}A(t) = \frac{\ddot{\nu}(t + \frac{\Delta}{2})}{8 \cdot c}&\left[3\dot{\nu}(t + \frac{\Delta}{2}) \cdot \Delta + 3\nu(t - \frac{\Delta}{2}) + 4\nu(t + \frac{\Delta}{2})\right]\\ &+ \frac{\ddot{\nu}(t - \frac{\Delta}{2})}{8 \cdot c}\left[3\dot{\nu}(t - \frac{\Delta}{2}) \cdot \Delta - 3\nu(t + \frac{\Delta}{2}) - 4\nu(t - \frac{\Delta}{2})\right]\end{aligned}$$

The general solution to this system can be obtained in the same manner as for first order.

$$\begin{pmatrix}\Phi^2_+\\ \Phi^2_-\end{pmatrix} = \begin{pmatrix}-e^{\Delta s}\\ 1\end{pmatrix}\pi + \frac{1}{e^{-s\Delta} - e^{s\Delta}}\begin{pmatrix}e^{-s\frac{\Delta}{2}} & -e^{s\frac{\Delta}{2}}\\ -e^{s\frac{\Delta}{2}} & e^{-s\frac{\Delta}{2}}\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix} \cdot A$$

which implies

$$\begin{cases}\Phi^2_+ = -e^{\Delta s}\pi + \dfrac{e^{-s\frac{\Delta}{2}} + e^{s\frac{\Delta}{2}}}{e^{-s\Delta} - e^{s\Delta}} \cdot A\\ \\ \Phi^2_- = \pi - \dfrac{e^{-s\frac{\Delta}{2}} + e^{s\frac{\Delta}{2}}}{e^{-s\Delta} - e^{s\Delta}} \cdot A\end{cases} \Rightarrow \begin{cases}\Phi^2_+ = -e^{\Delta s}\pi + \dfrac{1}{e^{-s\frac{\Delta}{2}} - e^{s\frac{\Delta}{2}}} \cdot A\\ \\ \Phi^2_- = \pi - \dfrac{1}{e^{-s\frac{\Delta}{2}} - e^{s\frac{\Delta}{2}}} \cdot A\end{cases}$$

Set

$$A(t) = \tilde{\eta}(t - \frac{\Delta}{2}) - \tilde{\eta}(t + \frac{\Delta}{2}) \tag{21}$$

This yields the homogenous solution where $\pi(t)$ is a 2Δ-periodic function (see (10)).

$$\begin{cases}\Phi^2_+(t) = -\pi(t + \Delta) + \tilde{\eta}(t)\\ \\ \Phi^2_-(t) = \pi(t) - \tilde{\eta}(t)\end{cases}$$

2.3.3 Fitting the initial conditions

Initial conditions are (see (11))

$$\begin{cases} J_+^2(0,z) = \underset{p}{J}_+^2(0,z) + \underset{h}{J}_+^2(0,z) = 0 \\ J_-^2(0,z) = \underset{p}{J}_-^2(0,z) + \underset{h}{J}_-^2(0,z) = 0 \end{cases}, \forall z \in [-\frac{l}{2}, \frac{l}{2}]$$

and it is known from first order (12) that

$$\dot{\nu}(s) = 0, \ \ \forall s \in [-\frac{\Delta}{2}, \frac{\Delta}{2}]$$

which implies

$$\begin{cases} \underset{p}{J}_+^2(0,z) = 0 \\ \underset{p}{J}_-^2(0,z) = 0 \end{cases}, \forall z \in [-\frac{l}{2}, \frac{l}{2}]$$

hence

$$\begin{cases} -\pi(s+\Delta) + \tilde{\eta}(s) = 0 \\ \\ \pi(s) - \tilde{\eta}(s) = 0 \end{cases}, \forall s \in \left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$$

This implies that $\pi(s)$ is not only 2Δ-periodic but also Δ-periodic and that

$$\pi(s) = \tilde{\eta}(s), \ \ s \in \left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$$

Set $\eta(s) = \tilde{\eta}(s) - \pi(s)$, then

$$\eta(s) = 0\,, \ \ s \in \left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$$

and (21) writes

$$A(t) = \eta(t - \frac{\Delta}{2}) + \pi(t - \frac{\Delta}{2}) - \eta(t + \frac{\Delta}{2}) - \pi(t + \frac{\Delta}{2})$$

In addition $\pi(s)$ is Δ-periodic so

$$A(t) = \eta(t - \frac{\Delta}{2}) - \eta(t + \frac{\Delta}{2})$$

The unique homogenous solution is

$$\begin{cases} \underset{h}{J}_+^2(t,z) = \eta(t + \frac{z}{c}) \\ \underset{h}{J}_-^2(t,z) = -\eta(t - \frac{z}{c}) \end{cases}$$

and the total solution for second order (sum of homogenous and particular solution) is

$$\begin{cases} J_+^2 = \underset{p}{J}_+^2 + \underset{h}{J}_+^2 \\ J_-^2 = \underset{p}{J}_-^2 + \underset{h}{J}_-^2 \end{cases}$$

It can be noted that[1]

$$\begin{cases} \underset{p}{J}{}^2_+(t,z) = -\underset{p}{J}{}^2_-(t,-z) \\ \underset{h}{J}{}^2_+(t,z) = -\underset{h}{J}{}^2_-(t,-z) \\ J^2_+(t,z) = -J^2_-(t,-z) \end{cases} \tag{22}$$

The second order α-terms can be derived from equation (3) but are not written here because of their high number of terms. $\dot{D}$ have already been taken care of in first order (see (14)) so for this and higher orders the α-definition writes

$$\begin{cases} \alpha^i_+ = \frac{3}{4}J^i_+ + \frac{1}{4}J^i_- \\ \alpha^i_- = \frac{1}{4}J^i_+ + \frac{3}{4}J^i_- \end{cases}, i \geq 2$$

2.3.4 Summary second order

PDE:

$$\begin{cases} \dot{J}^2_+ - c \cdot J^2_{+z} = -\left(\frac{3}{4}\dot{\nu}(t+\frac{z}{c}) + \frac{1}{4}\dot{\nu}(t-\frac{z}{c}) - \frac{1}{2}\left(\dot{\nu}(t+\frac{\Delta}{2}) + \dot{\nu}(t-\frac{\Delta}{2})\right)\right) \cdot \\ \qquad\qquad \ddot{\nu}(t+\frac{z}{c}) \cdot \frac{1}{c} \\ \dot{J}^2_- + c \cdot J^2_{-z} = \left(\frac{1}{4}\dot{\nu}(t+\frac{z}{c}) + \frac{3}{4}\dot{\nu}(t-\frac{z}{c}) - \frac{1}{2}\left(\dot{\nu}(t+\frac{\Delta}{2}) + \dot{\nu}(t-\frac{\Delta}{2})\right)\right) \cdot \\ \qquad\qquad \ddot{\nu}(t-\frac{z}{c}) \cdot \frac{1}{c} \end{cases}$$

BC:

$$\begin{cases} (J^2_+ + J^2_-)(t, -\frac{l}{2}) = 0 \\ (J^2_+ + J^2_-)(t, \frac{l}{2}) = 0 \end{cases}$$

IC:

$$\begin{cases} J^2_+(0,z) = 0 \\ J^2_-(0,z) = 0 \end{cases}, \forall z \in \left[-\frac{l}{2}, \frac{l}{2}\right]$$

Solution:

$$\begin{cases} J^2_+(t,z) = \eta(t+\frac{z}{c}) + \underset{p}{J}{}^2_+(t,z) \\ J^2_-(t,z) = -\eta(t-\frac{z}{c}) + \underset{p}{J}{}^2_-(t,z) \end{cases}$$

where

$$A(t) = \eta(t-\frac{\Delta}{2}) - \eta(t+\frac{\Delta}{2})$$

$$\eta(s) = 0 \, , s \in \left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$$

[1] Hence to save computational effort, J^2_- could be computed from J^2_+.

with

$$A(t) = \frac{\ddot{\nu}(t+\frac{\Delta}{2})}{8\cdot c}\left[3\dot{\nu}(t+\frac{\Delta}{2})\cdot\Delta + 3\nu(t-\frac{\Delta}{2}) + 4\nu(t+\frac{\Delta}{2})\right] + \frac{\ddot{\nu}(t-\frac{\Delta}{2})}{8\cdot c}\left[3\dot{\nu}(t-\frac{\Delta}{2})\cdot\Delta - 3\nu(t+\frac{\Delta}{2}) - 4\nu(t-\frac{\Delta}{2})\right]$$

and

$$\underset{p}{J}{}^2_+(t,z) = -\frac{\ddot{\nu}(t+\frac{z}{c})}{8\cdot c}\left[\nu(t-\frac{z}{c}) - 6\dot{\nu}(t+\frac{z}{c})\cdot\frac{z}{c} - 4\left(\nu(t+\frac{\Delta}{2}) + \nu(t-\frac{\Delta}{2})\right)\right]$$

$$\underset{p}{J}{}^2_-(t,z) = \frac{\ddot{\nu}(t-\frac{z}{c})}{8\cdot c}\left[\nu(t+\frac{z}{c}) + 6\dot{\nu}(t-\frac{z}{c})\cdot\frac{z}{c} - 4\left(\nu(t+\frac{\Delta}{2}) + \nu(t-\frac{\Delta}{2})\right)\right]$$

2.4 Third order

The expressions are getting bigger and of practical reasons some are not shown explicitly for third order.

2.4.1 Solving the PDE

The third order representation of (4), including terms with total order less or equal to three with equalities from known lower orders subtracted, writes

$$\begin{cases} \dot{J}^3_+ - c\cdot J^3_{+z} = -\alpha^1_+ J^2_{+z} - \alpha^2_+ J^1_{+z} = f^3_+ \\ \dot{J}^3_- + c\cdot J^3_{-z} = -\alpha^1_- J^2_{-z} - \alpha^2_- J^1_{-z} = f^3_- \end{cases}$$

Where the righthand sides are known functions (they can be found in appendix A.1). Actually, as the fourth order terms $-\alpha^2_+ J^2_{+z}$ and $-\alpha^2_+ J^2_{+z}$ are known they could be included, but they are not due to the high number of terms they give rise to. These equations have the same form as for second order and can be solved the same way.

$$\begin{cases} \hat{J}^3_+(\xi_+,\xi_-) = \dfrac{1}{2}\displaystyle\int_{\xi_-}^{\xi_+} \hat{f}^3_+(s,\xi_-)\;ds + \hat{\varphi}^3_+(\xi_-) \\ \hat{J}^3_-(\xi_+,\xi_-) = \dfrac{1}{2}\displaystyle\int_{\xi_+}^{\xi_-} \hat{f}^3_-(\xi_+,s)\;ds + \hat{\varphi}^3_-(\xi_+) \end{cases}$$

As in second order the solution can be expressed as

$$\begin{cases} J^3_+ = \underset{p}{J}{}^3_+ + \underset{h}{J}{}^3_+ \\ J^3_- = \underset{p}{J}{}^3_- + \underset{h}{J}{}^3_- \end{cases} \Leftrightarrow \begin{cases} J^3_+(t,z) = \underset{p}{J}{}^3_+(t,z) + \varphi^3_+(t+\dfrac{z}{c}) \\ J^3_-(t,z) = \underset{p}{J}{}^3_-(t,z) + \varphi^3_-(t-\dfrac{z}{c}) \end{cases}$$

The particular solutions have a high number of terms and are presented in appendix A.1. The solutions have the following symmetric properties

$$\underset{p}{J}{}^3_+(t,z) = \underset{p}{J}{}^3_-(t,-z) \tag{23}$$

2.4.2 Matching the boundary conditions

Similarly to second order the boundary condition equations for third order writes

$$\begin{cases} \left(\underset{h}{J}{}^3_+ + \underset{p}{J}{}^3_+ + \underset{h}{J}{}^3_- + \underset{p}{J}{}^3_-\right)(t, -\frac{l}{2}) = 0 \\ \left(\underset{h}{J}{}^3_+ + \underset{p}{J}{}^3_+ + \underset{h}{J}{}^3_- + \underset{p}{J}{}^3_-\right)(t, \frac{l}{2}) = 0 \end{cases}$$

Due to the symmetries in the particular solutions (23) the boundary condition equations can be simplified as usual

$$\begin{cases} \underset{p}{J}{}^3_+(t, -\frac{l}{2}) = \underset{p}{J}{}^3_-(t, \frac{l}{2}) \\ \underset{p}{J}{}^3_+(t, \frac{l}{2}) = \underset{p}{J}{}^3_-(t, -\frac{l}{2}) \end{cases}$$

Set

$$A^3(t) = -\underset{p}{J}{}^3_+(t, \frac{l}{2}) - \underset{p}{J}{}^3_+(t, -\frac{l}{2})$$

and the equation can be written as

$$\begin{cases} \varphi^3_+(t - \frac{\Delta}{2}) + \varphi^3_-(t + \frac{\Delta}{2}) = A^3(t) \\ \varphi^3_+(t + \frac{\Delta}{2}) + \varphi^3_-(t - \frac{\Delta}{2}) = A^3(t) \end{cases}$$

As for first order the general solution is

$$\begin{pmatrix} \varphi^3_+ \\ \varphi^3_- \end{pmatrix} = \begin{pmatrix} -e^{\Delta s} \\ 1 \end{pmatrix} \pi + \frac{1}{e^{-s\Delta} - e^{s\Delta}} \begin{pmatrix} e^{-s\frac{\Delta}{2}} & -e^{s\frac{\Delta}{2}} \\ -\, e^{s\frac{\Delta}{2}} & e^{-s\frac{\Delta}{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot A^3$$

where $\pi(s)$ is 2Δ-periodic function. The equations can be reduced to

$$\begin{cases} \varphi^3_+ = -e^{\Delta}\pi + \dfrac{1}{e^{-s\frac{\Delta}{2}} + e^{s\frac{\Delta}{2}}} \cdot A^3 \\ \\ \varphi^3_- = \pi + \dfrac{1}{e^{-s\frac{\Delta}{2}} + e^{s\frac{\Delta}{2}}} \cdot A^3 \end{cases}$$

If A^3 is set

$$A^3(t) = \tilde{\eta}^3(t + \frac{\Delta}{2}) + \tilde{\eta}^3(t - \frac{\Delta}{2})$$

the boundary fitted solution is

$$\begin{cases} \varphi^3_+(t) = -\pi(t + \Delta) + \tilde{\eta}^3(t) \\ \\ \varphi^3_-(t) = \pi(t) + \tilde{\eta}^3(t) \end{cases}$$

2.4.3 Fitting the initial conditions

According to (11) the initial conditions for third order are

$$\begin{cases} J_+^3(0,z) = 0 \\ J_-^3(0,z) = 0 \end{cases}, \forall z \in [-\frac{l}{2}, \frac{l}{2}]$$

$$\dot{\nu}(s) = 0, \forall s \in [-\frac{\Delta}{2}, \frac{\Delta}{2}] \Rightarrow \begin{cases} \underset{p}{J}{}_+^3(0,z) = 0 \\ \underset{p}{J}{}_-^3(0,z) = 0 \end{cases}, \forall z \in [-\frac{l}{2}, \frac{l}{2}]$$

hence

$$\begin{cases} \varphi_+^3(s) = -\pi(s+\Delta) + \tilde{\eta}^3(s) = 0 \\ \\ \varphi_-^3(s) = \pi(s) + \tilde{\eta}^3(s) = 0 \end{cases}, \forall s \in [-\frac{\Delta}{2}, -\frac{\Delta}{2}]$$

thus, $\pi(s)$ is Δ-antiperiodic. Set

$$\eta^3(s) = \pi(s) + \tilde{\eta}^3(s)$$

hence

$$\eta^3(s) = 0, s \in [-\frac{\Delta}{2}, -\frac{\Delta}{2}]$$

$$A^3(t) = \eta^3(t + \frac{\Delta}{2}) - \pi(t + \frac{\Delta}{2}) + \eta^3(t - \frac{\Delta}{2}) - \pi(t - \frac{\Delta}{2})$$

Yet π is Δ-antiperiodic, hence

$$A^3(t) = \eta^3(t + \frac{\Delta}{2}) + \eta^3(t - \frac{\Delta}{2})$$

So given the initial conditions the unique third order homogenous solution is

$$\begin{cases} \underset{h}{J}{}_+^3(t,z) = \varphi_+^3(t + \frac{z}{c}) = \eta^3(t + \frac{z}{c}) \\ \underset{h}{J}{}_-^3(t,z) = \varphi_-^3(t - \frac{z}{c}) = \eta^3(t - \frac{z}{c}) \end{cases}$$

Hence,

$$J_+^3(t,z) = J_-^3(t,-z)$$

and once again J_-^3 can be computed from J_+^3 to save computational effort.

2.4.4 Summary third order

PDE:

$$\begin{cases} \dot{J}_+^3 - c \cdot J_{+z}^3 = f_+^3 \\ \dot{J}_-^3 + c \cdot J_{-z}^3 = f_-^3 \end{cases}$$

BC:

$$\begin{cases} (J_+^3 + J_-^3)(t, -\frac{l}{2}) = 0 \\ (J_+^3 + J_-^3)(t, \frac{l}{2}) = 0 \end{cases}$$

IC:

$$\begin{cases} J_+^3(0,z) = 0 \\ J_-^3(0,z) = 0 \end{cases}, \forall z \in \left[-\frac{l}{2}, \frac{l}{2}\right]$$

Solution:

$$\begin{cases} J_+^2(t,z) = \eta^3(t+\frac{z}{c}) + \underset{p}{J}{}_+^3(t,z) \\ J_-^2(t,z) = \eta^3(t-\frac{z}{c}) + \underset{p}{J}{}_-^3(t,z) \end{cases}$$

where

$$A^3(t) = \eta^3(t+\frac{\Delta}{2}) + \eta^3(t-\frac{\Delta}{2})$$

$$\eta^3(s) = 0\,, s \in \left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$$

The explicit formulas for f_+^3, f_-^3, $\underset{p}{J}{}_+^3$, $\underset{p}{J}{}_-^3$ and A^3 can be found in appendix A.1.

2.5 nth order

To solve for a higher arbitrary order the solving method follows from the one exposed before for second or third order.

2.5.1 The set of equations

As the PDE (4) is the same for J_+ and J_- it is here represented by one equation without subindex. The nth order approximation to the PDE is obtained by including terms with total order less or equal to the approximation order. The PDE for order $m-1$ is

$$\sum_{k=0}^{m-1} \dot{J}^k + \sum_{k+i\leq m-1} \alpha^i \cdot J_z^k = 0$$

and for order m:

$$\sum_{k=0}^{m} \dot{J}^k + \sum_{k+i\leq m} \alpha^i \cdot J_z^k = 0$$

The expression for order $m-1$ equals zero so we can subtract its terms from the expression for order m without changing it's value. This gives the following expression for a general order n.

$$\dot{J}^n + \sum_{k+i=n} \alpha^i \cdot J_z^k = 0$$

which equals

$$\dot{J}^n + \sum_{k=0}^{n} \alpha^k \cdot J_z^{n-k} = 0$$

Since the zero order terms are constants $J_z^0 = 0$, so

$$\dot{J}^n + \alpha^0 \cdot J_z^n = -\sum_{k=1}^{n-1} \alpha^k \cdot J_z^{n-k}$$

In the full form the set of equations is

$$\begin{cases} \dot{J}_{+}^{n} + \alpha_{+}^{0} \cdot J_{+z}^{n} = f_{+}^{n} \\ \dot{J}_{-}^{n} + \alpha_{-}^{0} \cdot J_{-z}^{n} = f_{-}^{n} \end{cases}$$

which implies

$$\begin{cases} \dot{J}_{+}^{n} - c \cdot J_{+z}^{n} = f_{+}^{n} \\ \dot{J}_{-}^{n} + c \cdot J_{-z}^{n} = f_{-}^{n} \end{cases}$$

The left hand side is the linear Burgers equation and the right hand side forcing terms are known functions that can be calculated from lower orders.

$$\begin{cases} f_{+}^{n} = -\sum_{k=1}^{n-1} \alpha_{+}^{k} \cdot J_{+z}^{n-k} \\ f_{-}^{n} = -\sum_{k=1}^{n-1} \alpha_{-}^{k} \cdot J_{-z}^{n-k} \end{cases}$$

2.5.2 Repeating structure

Proposition 1. *The equation*

$$J_{+}^{k}(t,z) = (-1)^{k+1} J_{-}^{k}(t,-z) \tag{24}$$

holds for any $k \geq 0$

We are going to prove it by induction. Equation (24) holds for the explicit expressions we derived from zero to third order. Let $k = 3$

$$J_{+}^{k}(t,z) = (-1)^{k+1} J_{-}^{k}(t,-z)$$

Differentiation with respect to z gives

$$J_{+z}^{k}(t,z) = (-1)^{k} J_{-z}^{k}(t,-z) \tag{25}$$

The forcing terms for next order write

$$f_{+}^{k+1} = -\sum_{q=1}^{k} \alpha_{+}^{q} \cdot J_{+z}^{k+1-q}$$

$$f_{-}^{k+1} = -\sum_{q=1}^{k} \alpha_{-}^{q} \cdot J_{-z}^{k+1-q}$$

or

$$f_{+}^{k+1} = -\sum_{q=1}^{k} \left(\left(\frac{3}{4} J_{+}^{q} + \frac{1}{4} J_{-}^{q} \right) J_{+z}^{k+1-q} \right) + \dot{D} \cdot J_{+z}^{k}$$

$$f_{-}^{k+1} = -\sum_{q=1}^{k} \left(\left(\frac{3}{4} J_{-}^{q} + \frac{1}{4} J_{+}^{q} \right) J_{-z}^{k+1-q} \right) + \dot{D} \cdot J_{-z}^{k}$$

Using equations (24) and (25) we get

$$f_+^{k+1}(t,z) = -\sum_{q=1}^{k} \left(\left(\frac{3}{4}(-1)^{q+1} J_-^q(t,-z) + \frac{1}{4}(-1)^{q+1} J_+^q(t,-z) \right) \cdot \right.$$
$$\left. (-1)^{k+1-q} \cdot J_{-z}^{k+1-q}(t,-z) \right) + (-1)^k \dot{D} \cdot J_{-z}^k(t,-z)$$

Hence the symmetry property of (24) implies a similar symmetry in the forcing terms of the following order

$$f_+^{k+1}(t,z) = (-1)^k \cdot f_-^{k+1}(t,-z)$$

The same equality in the (ξ_+, ξ_-) coordinate system writes

$$\hat{f}_+^{k+1}(\xi_+, \xi_-) = (-1)^k \cdot \hat{f}_-^{k+1}(\xi_-, \xi_+)$$

The particular solutions

$$\begin{cases} \hat{J}_{\substack{+\\p}}^{k+1}(\xi_+, \xi_-) = \dfrac{1}{2} \displaystyle\int_{\xi_-}^{\xi_+} \hat{f}_+^{k+1}(s, \xi_-)\ ds \\ \hat{J}_{\substack{-\\p}}^{k+1}(\xi_+, \xi_-) = \dfrac{1}{2} \displaystyle\int_{\xi_+}^{\xi_-} \hat{f}_-^{k+1}(\xi_+, s)\ ds \end{cases}$$

also have symmetry properties according to

$$\hat{J}_{\substack{+\\p}}^{k+1}(\xi_+, \xi_-) = (-1)^k \cdot \frac{1}{2} \int_{\xi_-}^{\xi_+} \hat{f}_-^{k+1}(\xi_-, s)\ ds$$

so

$$\hat{J}_{\substack{+\\p}}^{k+1}(\xi_+, \xi_-) = (-1)^k \cdot \hat{J}_{\substack{-\\p}}^{k+1}(\xi_-, \xi_+)$$

In the (t, z) coordinate system one gets

$$J_{\substack{+\\p}}^{k+1}(t,z) = (-1)^k \cdot J_{\substack{-\\p}}^{k+1}(t,-z) \tag{26}$$

The homogenous parts of the total solution are

$$\begin{cases} J_{\substack{+\\h}}^{k+1}(t,z) = \varphi_+^{k+1}(t + \dfrac{z}{c}) \\ J_{\substack{-\\h}}^{k+1}(t,z) = \varphi_+^{k+1}(t - \dfrac{z}{c}) \end{cases}$$

and the boundary condition equations writes

$$\begin{cases} \left(J_{\substack{+\\h}}^{k+1} + J_{\substack{+\\p}}^{k+1} + J_{\substack{-\\h}}^{k+1} + J_{\substack{-\\p}}^{k+1} \right)(t, -\dfrac{l}{2}) = 0 \\ \left(J_{\substack{+\\h}}^{k+1} + J_{\substack{+\\p}}^{k+1} + J_{\substack{-\\h}}^{k+1} + J_{\substack{-\\p}}^{k+1} \right)(t, \dfrac{l}{2}) = 0 \end{cases}$$

Equation (26) give rise to the following symmetries

$$\begin{cases} J_{\substack{+\\p}}^{k+1}(t, \dfrac{l}{2}) = (-1)^k \cdot J_{\substack{-\\p}}^{k+1}(t, -\dfrac{l}{2}) \\ J_{\substack{+\\p}}^{k+1}(t, -\dfrac{l}{2}) = (-1)^k \cdot J_{\substack{-\\p}}^{k+1}(t, \dfrac{l}{2}) \end{cases}$$

Set

$$A^{k+1}(t) = (-1)^{k+1} \cdot \underset{p}{J}_{+}^{k+1}(t, \frac{l}{2}) - \underset{p}{J}_{+}^{k+1}(t, -\frac{l}{2})$$

then the boundary condition equations writes

$$\begin{cases} \varphi_{+}^{k+1}(t - \frac{\Delta}{2}) + \varphi_{-}^{k+1}(t + \frac{\Delta}{2}) = A^{k+1}(t) \\ \varphi_{+}^{k+1}(t + \frac{\Delta}{2}) + \varphi_{-}^{k+1}(t - \frac{\Delta}{2}) = (-1)^{k} \cdot A^{k+1}(t) \end{cases}$$

and the solution fitted to the initial conditions is

$$\begin{cases} \underset{h}{J}_{+}^{k+1}(t, z) = \eta^{k+1}(t + \frac{z}{c}) \\ \underset{h}{J}_{-}^{k+1}(t, z) = (-1)^{k} \cdot \eta^{k+1}(t - \frac{z}{c}) \end{cases}$$

where

$$A^{k+1}(t) = \eta^{k+1}(t - \frac{\Delta}{2}) + (-1)^{k} \cdot \eta^{k+1}(t + \frac{\Delta}{2})$$

and

$$\eta^{k+1}(s) = 0, \ \ \forall s \in \left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$$

The homogenous solution have the same symmetry property as the particular

$$\underset{h}{J}_{+}^{k+1}(t, z) = (-1)^{k} \cdot \underset{h}{J}_{-}^{k+1}(t, -z)$$

The total solution for order $k + 1$ is

$$\begin{cases} J_{+}^{k+1} = \underset{h}{J}_{+}^{k+1} + \underset{p}{J}_{+}^{k+1} \\ J_{-}^{k+1} = \underset{h}{J}_{-}^{k+1} + \underset{p}{J}_{-}^{k+1} \end{cases}$$

and have the same symmetry properties as for order k, (24).

$$J_{+}^{k+1}(t, z) = (-1)^{k} J_{-}^{k+1}(t, -z)$$

Therefore the solving method can be repeated and the same structure is obtained for all orders.

3 Open loop control

The first order approximation is identical to the solution of a linear model derived in [9]. There the system was proven to be steady-state controllable, in other words there exists a way to move the tank from steady state to a new position and arrive there in steady state. The controllability can be explained by the fact that all system variables of the first order approximation write in terms of ν. ν is a flat output for the system (see [6] and [7]). To control the system one may simply control its flat output.

$$\begin{cases} D(t) = \dfrac{\nu(t+\frac{\Delta}{2}) + \nu(t-\frac{\Delta}{2})}{2} \\ h(t,z) = \dfrac{1}{16g}\left(\dot{\nu}(t-\dfrac{z}{c}) - \dot{\nu}(t+\dfrac{z}{c})\right)^2 \\ v(t,z) = \dfrac{1}{2}\left(\dot{\nu}(t+\dfrac{z}{c}) + \dot{\nu}(t-\dfrac{z}{c})\right) \end{cases} \Bigg| \; t \geq 0, z \in [-\frac{l}{2}, \frac{l}{2}]$$

The tank position D converges in finite time to ν provided that ν gets to an equilibrium and they are identical when ν has been constant for a time period of Δ seconds. In addition, when ν has been constant for a period of Δ seconds the surface settles, $h = h_0$ and $v = 0$.

3.1 Steady-state controllability of the second order system

Steady-state controllability of second order is not proven. In this report we show that there exists no motion, such that $\dot{D}$ has finite support smaller than 2Δ, that cancels the second order system. This implies that one may not move the second order approximation of the tank system from steady state to steady state in less than 2Δ transient time.

Definition The term cancellation will be used in the sense that cancellation is obtained if the following holds for all times after the tank stopped moving.

$$\begin{aligned} h(t,z) &= h_0 \\ v(t,z) &= 0 \end{aligned}$$

3.1.1 Lower bound on motion cancellation time for the second order system

Consider a motion D applied on a tank which is at $t = 0$ in steady-state. Let $\dot{D}$ be a non-zero differentiable function with a finite support included in an interval less than 2Δ. Further let $\dot{D}(0) = 0$ and

$$\dot{D}(t) = 0, \;\; \forall t > 2\Delta$$

To obtain cancellation (as defined before) of the second order system the following must hold for all $t > 2\Delta$ and for all $z \in [-\frac{l}{2}, \frac{l}{2}]$.

$$\begin{cases} h(t,z) = \dfrac{1}{16g}\left(\displaystyle\sum_{k=0}^{2}\left(J_-^k(t,z) - J_+^k(t,z)\right)\right)^2 = h_0 \\ v(t,z) = \dfrac{1}{2}\displaystyle\sum_{k=0}^{2}\left(J_+^k(t,z) + J_-^k(t,z)\right) = 0 \end{cases}$$

The zero order terms are constants, $J_+^0 = -2c$ and $J_-^0 = 2c$, so the equations can be simplified to

$$\begin{cases} \sum_{k=1}^{2} J_-^k(t,z) - \sum_{k=1}^{2} J_+^k(t,z) = 0 \\ \sum_{k=1}^{2} J_-^k(t,z) + \sum_{k=1}^{2} J_+^k(t,z) = 0 \end{cases}$$

which implies

$$\begin{cases} J_+^1(t,z) + J_+^2(t,z) = 0 \\ J_-^1(t,z) + J_-^2(t,z) = 0 \end{cases} \Leftrightarrow \begin{cases} J_+^1(t,z) + J_+^2(t,z) = 0 \\ J_-^1(t,-z) + J_-^2(t,-z) = 0 \end{cases}$$

the terms have the following symmetry properties (see section 2.2.4 and equation (22))

$$\begin{cases} J_+^1(t,z) = J_+^2(t,-z) \\ J_-^1(t,z) = -J_-^2(t,-z) \end{cases}$$

hence

$$J_+^1(t,z) = 0, \quad J_-^1(t,z) = 0, \quad J_+^2(t,z) = 0, \quad J_-^2(t,z) = 0$$

J_+^1 can be expressed in terms of ν (see section 2.2.4)

$$J_+^1(t,z) = \dot{\nu}(t + \frac{z}{2}) = 0$$

So to obtain cancellation the following must hold

$$\dot{\nu}(t + \frac{z}{c}) = 0, \ \ \forall t > 2\Delta, \ \ \forall z \in [-\frac{l}{2}, \frac{l}{2}]$$

which implies

$$\dot{\nu}(t) = 0, \ \ \forall t > \frac{3\Delta}{2}$$

an additional constraint is obtained from the first order system (see section 2.2.4)

$$\dot{\nu}(t) = 0, \ \ \forall t \in [-\frac{\Delta}{2}, \frac{\Delta}{2}]$$

This implies that the second order system can only be cancelled if $\dot{\nu}$ has a finite support included in $\left(\frac{\Delta}{2}, \frac{3\Delta}{2}\right]$. In lemma 1 it is shown that there exist no non-zero function that cancels the second order system.

Lemma 1. *For a* $\dot{D}(t)$

$$\dot{D}(t) = \frac{\dot{\nu}(t - \frac{\Delta}{2}) + \dot{\nu}(t + \frac{\Delta}{2})}{2}$$

where $\dot{\nu}$ *is chosen as a non-zero differentiable function with finite support included in* $\left(\frac{\Delta}{2}, \frac{3\Delta}{2}\right]$ *(first order cancellation is achieved) then cancellation of the second order system (as defined in section 2.3.4) can not be achieved.*

Proof. To obtain second order cancellation the following must hold

$$h(t^*, z) = h_0$$
$$v(t^*, z) = 0$$

where t^* is used to denote every time after the tank stopped moving, $(t^* \geq 2\Delta)$.

$$\begin{cases} h(t^*, z) = \dfrac{1}{16g}\left(\displaystyle\sum_{k=0}^{2}\left(J_-^k(t^*, z) - J_+^k(t^*, z)\right)\right)^2 = h_0 \\ v(t^*, z) = \dfrac{1}{2}\displaystyle\sum_{k=0}^{2}\left(J_+^k(t^*, z) + J_-^k(t^*, z)\right) = 0 \end{cases} \tag{27}$$

By hypothesis $\dot{\nu}(t^*) = 0$ and $z \in [-\frac{l}{2}, \frac{l}{2}]$ then from section 2.2.4 and 2.3.4

$$\begin{cases} J_+^1(t^*, z) = 0 \\ J_-^1(t^*, z) = 0 \\ \underset{p}{J}{}_+^2(t^*, z) = 0 \\ \underset{p}{J}{}_-^2(t^*, z) = 0 \end{cases}$$

Further $J_+^0 = -2c$ and $J_-^0 = 2c$ hence the equations (27) reduce to

$$\begin{cases} \underset{h}{J}{}_-^2(t^*, z) - \underset{h}{J}{}_+^2(t^*, z) = 0 \\ \underset{h}{J}{}_+^2(t^*, z) + \underset{h}{J}{}_-^2(t^*, z) = 0 \end{cases}$$

which implies

$$\begin{cases} \underset{h}{J}{}_-^2(t^*, z) = 0 \\ \underset{h}{J}{}_+^2(t^*, z) = 0 \end{cases}$$

Yet from section 2.3.4

$$\begin{cases} \underset{h}{J}{}_+^2(t, z) = \eta(t + \dfrac{z}{c}) \\ \underset{h}{J}{}_-^2(t, z) = -\eta(t - \dfrac{z}{c}) \end{cases}$$

So

$$\eta(t^* + \frac{z}{c}) = 0$$

To obtain cancellation of second order (28) must hold. In summary

$$\eta(t) = 0, \ \ \forall t > \frac{3\Delta}{2} \tag{28}$$

Besides

$$\begin{cases} \eta(t + \dfrac{\Delta}{2}) = \eta(t - \dfrac{\Delta}{2}) - A(t) \\ \eta(t) = 0\,, \ \ \forall t \in \left[-\dfrac{\Delta}{2}, \dfrac{\Delta}{2}\right] \end{cases}$$

This implies

$$\eta(t) = \eta(t - \Delta - N \cdot \Delta) - \sum_{k=0}^{N} A(t - \frac{\Delta}{2} - k \cdot \Delta)$$

For any $t > \frac{3\Delta}{2}$ there exists a unique natural number N_0 such that $(t - \Delta - N_0 \cdot \Delta) \in \left(-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$. Hence

$$\eta(t) = -\sum_{k=0}^{N_0} A(t - \frac{\Delta}{2} - k \cdot \Delta)$$

As we know it, A can be expressed in terms of ν. This gives

$$\eta(t) = -\sum_{k=0}^{N_0} \Bigg(\frac{\ddot{\nu}(t - k \cdot \Delta)}{8 \cdot c} \left[3\dot{\nu}(t - k \cdot \Delta) \cdot \Delta + 3\nu(t - \Delta - k \cdot \Delta) + 4\nu(t - k \cdot \Delta)\right]$$
$$+ \frac{\ddot{\nu}(t - \Delta - k \cdot \Delta)}{8 \cdot c} \left[3\dot{\nu}(t - \Delta - k \cdot \Delta) \cdot \Delta - 3\nu(t - k \cdot \Delta) - 4\nu(t - \Delta - k \cdot \Delta)\right] \Bigg)$$

where by hypothesis $\ddot{\nu}(t) = 0$ and $\dot{\nu}(t) = 0$ for $t \notin \left(\frac{\Delta}{2}, \frac{3\Delta}{2}\right]$. There exists a unique natural number N_1 such that

$$t - N_1 \cdot \Delta \in \left(\frac{\Delta}{2}, \frac{3\Delta}{2}\right]$$

The only terms in the sum that can have a non zero value are the one corresponding to $k = N_1$ or $k = N_1 - 1$. So $\eta(t)$ can be written

$$\begin{aligned} \eta(t) = -&\frac{\ddot{\nu}(t - N_1 \cdot \Delta)}{8c} \left(3\dot{\nu}(t - N_1 \cdot \Delta)\Delta - 4\nu(t - N_1 \cdot \Delta)\right) \\ &- \frac{\ddot{\nu}(t - N_1 \cdot \Delta)}{8c} \left(3\dot{\nu}(t - N_1 \cdot \Delta)\Delta + 4\nu(t - N_1 \cdot \Delta)\right) \end{aligned} \tag{29}$$

which implies

$$\eta(t) = -\frac{3\Delta}{4c} \ddot{\nu}(t - N_1 \cdot \Delta)\dot{\nu}(t - N_1 \cdot \Delta)$$

According to (28) the following must hold to obtain second order cancellation.

$$-\frac{3\Delta}{4c} \ddot{\nu}(t - N_1 \cdot \Delta)\dot{\nu}(t - N_1 \cdot \Delta) = 0$$

hence

$$\frac{d}{dt}\left(\dot{\nu}(t - N_1 \cdot \Delta)^2\right) = 0, \ \forall t - N_1 \cdot \Delta \in \left(\frac{\Delta}{2}, \frac{3\Delta}{2}\right]$$

which is equivalent to

$$\frac{d}{dt}\left(\dot{\nu}(t)^2\right) = 0, \ \forall t \in \left(\frac{\Delta}{2}, \frac{3\Delta}{2}\right]$$

hence $\dot{\nu}(t)$ must be a constant over $\left(\frac{\Delta}{2}, \frac{3\Delta}{2}\right]$. This constant has to be zero due to that $\dot{\nu}$ is differentiable and $\dot{\nu}(\frac{\Delta}{2}) = 0$. Due to that $\dot{\nu}$ has to be a nonzero function by hypothesis cancellation of the second order system can not be achieved. □

3.2 Multiple tank regulation

As stated in section 3 the first order model is steady-state controllable, thus there exist ways to move a tank in steady state to a new position and arrive in steady state. Here a method to control two tanks with different dimensions is presented.

Consider a tank with length l_1 and mean height h_1, the corresponding Δ is

$$\Delta_1 = \frac{l_1}{\sqrt{gh_1}}$$

The tank is steady-state controlled if the flat output is fed with a function whose derivative has finite support. Let the tank flat output ν_1 be fed with such a function σ_1, the tank motion is given by ν_1

$$\begin{aligned} \nu_1(t) &= \sigma_1(t) \\ D_1(t) &= \frac{1}{2}\left(\nu_1(t+\frac{\Delta_1}{2}) + \nu_1(t-\frac{\Delta_1}{2})\right) \end{aligned}$$

hence

$$D_1 = \frac{1}{2}\left(e^{s\frac{\Delta_1}{2}} + e^{-s\frac{\Delta_1}{2}}\right)\sigma_1$$

Now consider a second tank with different dimensions

$$\Delta_2 = \frac{l_2}{\sqrt{gh_2}}$$

Similarly as for the first tank the second is steady-state controlled if ν_2 is fed with a function σ_2 where $\dot{\sigma}_2$ has finite support.

$$\begin{aligned} \nu_2(t) &= \sigma_2(t) \\ D_2(t) &= \frac{1}{2}\left(\nu_2(t+\frac{\Delta_2}{2}) + \nu_2(t-\frac{\Delta_2}{2})\right) \end{aligned}$$

hence

$$D_2 = \frac{1}{2}\left(e^{s\frac{\Delta_2}{2}} + e^{-s\frac{\Delta_2}{2}}\right)\sigma_2$$

Let the two tanks have the same motion $D = D_1 = D_2$.

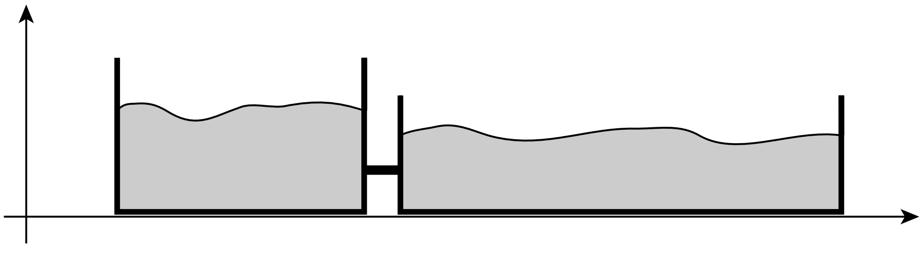

Figure 2: Two tanks with different dimensions but the same motion

Thus the following equation is valid.

$$\left(e^{s\frac{\Delta_1}{2}} + e^{-s\frac{\Delta_1}{2}}\right)\sigma_1 = \left(e^{s\frac{\Delta_2}{2}} + e^{-s\frac{\Delta_2}{2}}\right)\sigma_2 \tag{30}$$

The two tanks are regulated(steady-state controlled) by the same motion if one of the σ's can be written as an explicit function of the other so that both $\dot{\sigma_1}$ and $\dot{\sigma}_2$ have finite support. This is possible for a certain case, when

$$\Delta_2 = (2k+1)\Delta_1\,,\ k \in \mathbb{N}.$$

Under that hypothesis the right hand side in (30) writes

$$\left(e^{s\frac{(2k+1)\Delta_1}{2}} + e^{-s\frac{(2k+1)\Delta_1}{2}}\right)\sigma_2$$

which can always be factorized as (see Lemma 2)

$$\left(e^{s\frac{\Delta_1}{2}} + e^{-s\frac{\Delta_1}{2}}\right)P(e^{s\frac{\Delta_1}{2}}, e^{-s\frac{\Delta_1}{2}})\sigma_2$$

Where P is a polynomial of finite degree if k is finite. So equation (30) writes

$$\sigma_1 = P(e^{s\frac{\Delta_1}{2}}, e^{-s\frac{\Delta_1}{2}})\sigma_2$$

Hence the two tanks are regulated by the same motion D. The second tank is regulated as usual

$$\begin{aligned}\nu_2(t) &= \sigma_2(t)\\ D(t) &= \frac{1}{2}\left(\nu_2(t-\frac{\Delta_2}{2}) + \nu_1(t-\frac{\Delta_2}{2})\right)\end{aligned}$$

and the flat output of the first tank is fed with a function adapted by the second tank

$$\begin{aligned}\nu_1(t) &= \sigma_1 = P(e^{s\frac{\Delta_1}{2}}, e^{-s\frac{\Delta_1}{2}})\sigma_2\\ D(t) &= \frac{1}{2}\left(\nu_1(t-\frac{\Delta_1}{2}) + \nu_1(t-\frac{\Delta_1}{2})\right)\end{aligned}$$

From these results it can be seen that two tanks, with Δ_a respectively Δ_b, can be regulated by the same motion D if it exist a Δ_c such that

$$\begin{cases}\Delta_c = (2k_a+1)\Delta_a, k_a \in \mathbb{N}\\ \Delta_c = (2k_b+1)\Delta_b, k_b \in \mathbb{N}\end{cases}$$

then D would be constructed to regulate a tank with Δ_c and the tanks with Δ_a and Δ_b would be regulated by the same D.

Lemma 2. *The expression*

$$\left(e^{s\frac{(2k+1)\Delta}{2}} + e^{-s\frac{(2k+1)\Delta}{2}}\right) \tag{31}$$

where k is a finite natural number can always be factorized as

$$\left(e^{s\frac{\Delta}{2}} + e^{-s\frac{\Delta}{2}}\right)P(e^{s\frac{\Delta}{2}}, e^{-s\frac{\Delta}{2}})$$

where P is a polynomial of finite degree $2k$ in the variables $e^{s\frac{\Delta}{2}}$ and $e^{-s\frac{\Delta}{2}}$.

Proof. Define

$$a = e^{s\frac{\Delta}{2}}$$
$$b = e^{-s\frac{\Delta}{2}}$$

then (31) writes

$$a^{2k+1} + b^{2k+1} \tag{32}$$

As $2k+1$ is an odd number, $a = -b$ is a root to the equation

$$a^{2k+1} + b^{2k+1} = 0$$

(32) can therefore always be factorized as

$$(a+b)P(a,b)$$

where P is a polynomial. Additionally

$$deg(a^{2k+1} + b^{2k+1}) = deg(a+b) + deg(P)$$

hence

$$deg(P) = 2k$$

and as k is finite so is the degree of P. $\square$

4 Results and analysis

4.1 Bounds for the Perturbation Method

We do not prove convergence for the perturbation method, however we give bounds for a specific type of motion for the derived orders. These are summarized by equations (33,34,38,39).

We consider function $[0,1] \ni t \mapsto \sigma(t)$ be a $\mathcal{C}^\infty$ step function that moves from 0 to 1 when t goes from 0 to 1, that has the following bounds

$$\begin{aligned}
&\sigma(t) \in [0,1] \\
&|\sigma(t)| \leq C_0 = 1 \\
&|\dot{\sigma}(t)| \leq C_1 \\
&|\ddot{\sigma}(t)| \leq C_2 \\
&|\sigma^{(3)}(t)| \leq C_3 \\
&\vdots \\
&|\sigma^{(i)}(t)| \leq C_i, \ i \geq 0
\end{aligned}$$

where C_i are constants. For an example of such a function see appendix A.2. Let

$$\nu(t) = \begin{cases} 0, \ \ for \ t \leq 0 \\ \sigma(\frac{t}{T}), \ \ for \ 0 < t < T \\ 1, \ \ for \ t \geq T \end{cases}$$

which give rise to the following bounds for ν

$$\begin{aligned}
&|\nu(t)| \leq 1\\
&|\dot{\nu}(t)| \leq \frac{C_1}{T}\\
&|\ddot{\nu}(t)| \leq \frac{C_2}{T^2}\\
&|\nu^{(3)}(t)| \leq \frac{C_3}{T^3}\\
&\vdots\\
&|\nu^{(i)}(t)| \leq \frac{C_i}{T^i},\ i \geq 0
\end{aligned}$$

4.1.1 Zero order

The zero order terms are constants so

$$\begin{aligned}
J^0_+ &= -2c\\
J^0_- &= 2c
\end{aligned} \tag{33}$$

4.1.2 First order

The bounds are the same as for $\dot{\nu}$ according to equation (13).

$$\begin{aligned}
|J^1_+| &\leq \frac{C_1}{T}\\
|J^1_-| &\leq \frac{C_1}{T}
\end{aligned} \tag{34}$$

4.1.3 Second order

According to (15) the bounds for the second order forcing terms are

$$\begin{aligned}
|f^2_+| &\leq \frac{2C_1C_2}{cT^3}\\
|f^2_-| &\leq \frac{2C_1C_2}{cT^3}
\end{aligned}$$

The second order solution decomposed in to their homogenous and particular parts writes

$$\begin{aligned}
J^2_+ &= \underset{p}{J}{}^2_+ + \underset{p}{J}{}^2_+\\
J^2_- &= \underset{p}{J}{}^2_- + \underset{p}{J}{}^2_-
\end{aligned}$$

and the particular solutions in their non-simplified version ((18) and (19)) are

$$\begin{aligned}
\underset{p}{J}{}^2_+(t,z) = -\frac{\ddot{\nu}(t+\frac{z}{c})}{8\cdot c}\Bigg[&\nu(t-\frac{z}{c}) - \nu(t+\frac{z}{c}) - 3\dot{\nu}(t+\frac{z}{c})\cdot\frac{2z}{c}\\
&- 4\left(\nu(t+\frac{\Delta}{2}) + \nu(t-\frac{\Delta}{2}) - \nu(t+\frac{z}{c}+\frac{\Delta}{2}) - \nu(t+\frac{z}{c}-\frac{\Delta}{2})\right)\Bigg]
\end{aligned} \tag{35}$$

$$
\begin{aligned}
\underset{p}{J}{}^2_{-}(t,z) = \frac{\ddot{\nu}(t-\frac{z}{c})}{8\cdot c}\Bigg[&\nu(t+\frac{z}{c}) - \nu(t-\frac{z}{c}) + 3\dot{\nu}(t-\frac{z}{c})\cdot\frac{2z}{c} \\
&- 4\left(\nu(t+\frac{\Delta}{2}) + \nu(t-\frac{\Delta}{2}) - \nu(t-\frac{z}{c}+\frac{\Delta}{2}) - \nu(t-\frac{z}{c}-\frac{\Delta}{2})\right)\Bigg]
\end{aligned} \tag{36}
$$

By Rolle's theorem there exists a $\xi_a \in \left[t-\frac{z}{c}, t+\frac{z}{c}\right]$ such that

$$\nu(t+\frac{z}{c}) = \nu(t-\frac{z}{c}) + \frac{2z}{c}\dot{\nu}(\xi_a)$$

and a $\xi_b \in \left[t-\frac{z}{c}+\frac{\Delta}{2}, t+\frac{\Delta}{2}\right]$ such that

$$\nu(t-\frac{z}{c}+\frac{\Delta}{2}) = \nu(t+\frac{\Delta}{2}) - \frac{z}{c}\dot{\nu}(\xi_b)$$

Using Rolle's theorem one more time for the ν terms in (35) and (36) we get

$$\underset{p}{J}{}^2_{+}(t,z) = -\frac{\ddot{\nu}(t+\frac{z}{c})}{8\cdot c}\left[-\frac{2z}{c}\dot{\nu}(\xi_1) - 3\dot{\nu}(t+\frac{z}{c})\cdot\frac{2z}{c} - \frac{4z}{c}\dot{\nu}(\xi_2) - \frac{4z}{c}\dot{\nu}(\xi_3)\right]$$

$$\underset{p}{J}{}^2_{-}(t,z) = \frac{\ddot{\nu}(t+\frac{z}{c})}{8\cdot c}\left[\frac{2z}{c}\dot{\nu}(\xi_4) + 3\dot{\nu}(t+\frac{z}{c})\cdot\frac{2z}{c} - \frac{4z}{c}\dot{\nu}(\xi_5) - \frac{4z}{c}\dot{\nu}(\xi_6)\right]$$

Hence, we can bound

$$
\begin{aligned}
|\underset{p}{J}{}^2_{+}| &\leq \frac{2\Delta C_1 C_2}{T^3 c} \\
|\underset{p}{J}{}^2_{-}| &\leq \frac{2\Delta C_1 C_2}{T^3 c}
\end{aligned}
$$

On the other hand $A(t)$ is defined by

$$A(t) = \underset{p}{J}{}^2_{+}(t,\frac{l}{2}) - \underset{p}{J}{}^2_{+}(t,-\frac{l}{2})$$

So

$$|A| \leq \frac{4\Delta C_1 C_2}{T^3 c}$$

At last $\eta(t)$ is connected to $A(t)$ by

$$A(t) = \eta(t-\frac{\Delta}{2}) - \eta(t+\frac{\Delta}{2}) \tag{37}$$

where

$$\eta(t) = 0, \ \ t \in \left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$$

$$A(t) = 0, \ \begin{cases} t < 0 \\ t > T \end{cases}$$

So η can only accumulate values from A through (37) during the transition time, that is (at most) $T/\Delta + 1$ times, thus

$$|\eta| \leq \frac{T+\Delta}{\Delta} \cdot \frac{4\Delta C_1 C_2}{T^3 c}$$

Finally the bound for second order solution is

$$\begin{aligned} |J_+^2| &\leq \frac{2C_1C_2}{T^2 c}\left(\frac{3\Delta}{T}+2\right) \\ |J_-^2| &\leq \frac{2C_1C_2}{T^2 c}\left(\frac{3\Delta}{T}+2\right) \end{aligned} \tag{38}$$

4.1.4 Third order

Due to the high number of terms a precise bound has not been derived. However, using similar methods as for second order (excluding Rolle's Theorem) on the third order expressions (see appendix A.1) it can be seen that J_+^3 and J_-^3 is bounded by

$$\begin{aligned} |J_+^3| &\leq \sum_{k=3}^{N} \frac{\widetilde{C}_k}{T^k} \\ |J_-^3| &\leq \sum_{k=3}^{N} \frac{\widetilde{C}_k}{T^k} \end{aligned}$$

Where N is finite and $\widetilde{C}_k$ is a constant (consisting of an algebraic expression of the constants c, Δ, C_1, C_2, C_3 etc.). For not small T the dominant term is of magnitude $1/T^3$ or smaller. So for not small T third order terms are bounded by

$$\begin{aligned} |J_+^3| &\leq \frac{\widetilde{C}}{T^3} \\ |J_-^3| &\leq \frac{\widetilde{C}}{T^3} \end{aligned} \tag{39}$$

where $\widetilde{C}$ is a constant.

4.2 Simulation setup

In the following sections simulations were made for a tank with length 1m and depth 0.1m subject to a 1m translation during a transition time of T seconds. The motion is adapted to cancel the first order system.

4.3 Formulas implementation

The method have been implemented in Matlab up to third approximation order. The nature of the solutions makes it quite easy to implement and requires a relatively small computational effort, small enough to do realtime interactive simulations.

The height and velocity in the (t,z) local coordinate system of the tank is obtained from the formulas

$$\begin{cases} h(t,z) = \dfrac{1}{16g}\left(\displaystyle\sum_{k=0}^{3}\left(J_-^k(t,z) - J_+^k(t,z)\right)\right)^2 \\ v(t,z) = \dfrac{1}{2}\displaystyle\sum_{k=0}^{3}\left(J_+^k(t,z) + J_-^k(t,z)\right) \end{cases}$$

Further, J_+ and J_- are functions of ν, η, η^3 and some of their derivatives. To calculate height and speed within the tank at a specific time t, values of the above mentioned functions only have to be known in the interval $\left[t - \frac{\Delta}{2}, t + \frac{\Delta}{2}\right]$. If the motion is defined by the function $D(t)$, then ν and its derivatives can easily be obtained from D. It is known that

$$\nu(t + \frac{\Delta}{2}) = 2D(t) - \nu(t - \frac{\Delta}{2})$$

$$\nu(t) = 0\,, \forall t \in \left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$$

and derivation of the expressions gives

$$\dot{\nu}(t + \frac{\Delta}{2}) = 2\dot{D}(t) - \dot{\nu}(t - \frac{\Delta}{2})$$

$$\dot{\nu}(t) = 0\,, \forall t \in \left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$$

For the third approximation order the fourth derivative of ν is used, so to get smooth solutions we let $D(t) \in \mathcal{C}^4$. Then let $D(t)$ be discretized with a time step dt. Hence, ν and its derivatives are known at $\frac{\Delta}{dt}$ points in the interval $\left[t - \frac{\Delta}{2}, t + \frac{\Delta}{2}\right]$. This induces a space discretization $dz = c \cdot dt$.

All the functions describing the state of the tank, h, v, D, can be calculated from ν and its derivatives and as mentioned above they have to be stored for a Δ long time interval.

4.4 Numerical comparison with Godunov scheme based algorithm

Simulations with tank parameters set as described above were made and compared with a Godunov scheme based numerical method (for more details see [5]). The tank is subject to a move D that cancels first order and $\dot{D}$ has a finite support included in $[0, T]$. The first, second and third order approximation of the height ($h_1(t,z)$, $h_2(t,z)$, $h_3(t,z)$) as a function of t and z were calculated, for discrete values of t and z. The Godunov numerical method used as reference is dissipative and its height ($h_G(t,z)$) will settle with time. As a measurement of the deviation from $h_G(t,z)$ for the different orders the 2-norm over space is applied followed by the mean value over a time interval. The time interval is chosen to be $[0, \frac{5T}{4}]$, that is, slightly longer than the transient time. N_t is the number of discrete time points in the interval.

$$E_1 = \frac{1}{N_t}\sum_{k=1}^{N_t}\left|h_G(t,z) - h_1(t,z)\right|_2$$

$$E_2 = \frac{1}{N_t} \sum_{k=1}^{N_t} \left| h_G(t,z) - h_2(t,z) \right|_2$$

$$E_3 = \frac{1}{N_t} \sum_{k=1}^{N_t} \left| h_G(t,z) - h_3(t,z) \right|_2$$

The deviations have been calculated for different transition times and smoothness levels of the motion D. In table 1 $D \in \mathcal{C}^\infty$, the function is a Gevrey function based upon an unpublished work of François Malrait, see appendix A.2. In table 2 D is a piecewise polynomial such that $D \in \mathcal{C}^5$ and in table 3 $D \in \mathcal{C}^4$.

The tables indicates that longer transition times imply greater likeliness to the numerical scheme. It can also be observed that the second and third order corresponds better than first for long transition times, and this property is consistent for different degrees of smoothness.

For each of these simulations the mean computational speed for the perturbation method was 5 to 9 times faster than for the Godunov method, measured in (t,z) mesh points per second.

	T= $4s$	T= $5s$	T= $6s$	T= $7s$	T= $8s$	T= $9s$
E_1	0.0205	0.0114	0.00391	0.00204	0.000931	0.000629
E_2	0.0195	0.00661	0.00307	0.00181	0.000707	0.000557
E_3	0.0341	0.00622	0.00306	0.00188	0.000715	0.000562

Table 1: Deviation for a $\mathcal{C}^\infty$ function

	T= $4s$	T= $5s$	T= $6s$	T= $7s$	T= $8s$	T= $9s$
E_1	0.0413	0.00651	0.00188	0.000899	0.000494	0.000301
E_2	0.0451	0.00439	0.000995	0.000545	0.000313	0.000202
E_3	0.0648	0.00479	0.0011	0.000562	0.000322	0.000202

Table 2: Deviation for a $\mathcal{C}^5$ function

	T= $4s$	T= $5s$	T= $6s$	T= $7s$	T= $8s$	T= $9s$
E_1	0.0236	0.00412	0.00133	0.000717	0.000371	0.000253
E_2	0.0205	0.00211	0.000796	0.000475	0.000258	0.000178
E_3	0.024	0.00235	0.000819	0.000491	0.000259	0.000178

Table 3: Deviation for a $\mathcal{C}^4$ function

4.5 Volume preservation

The perturbation method do not preserve volume exactly at least not for a finite order. An analytical expression for the volume deviation has not been derived but series of numerical simulations have been made. The volume $V(t)$ has been numerically estimated by integrating the surface height using the trapezoidal rule. Table 4 shows volume deviation for different orders and transition times, the motion is a $\mathcal{C}^5$ piecewise polynomial and the tank dimensions are as described before. As a measurement of the volume deviation, the maximum deviation over the time period $[0, \frac{5T}{4}]$ is used, represented in percentage of the true volume (V_0).

$$\max_{t \in [0, \frac{5T}{4}]} \frac{|V(t) - V_0|}{V_0} \cdot 100$$

	T= $4s$	T= $5s$	T= $6s$	T= $7s$	T= $8s$	T= $9s$
1^{st}	1.66	0.61	0.269	0.135	0.0744	0.0443
2^{nd}	0.521	0.017	0.00321	0.00127	0.000661	0.000369
3^{rd}	1.18	0.0493	0.00452	0.000605	0.000173	6.43e-005

Table 4: Maximum volume deviation in percentage for different orders and transition times

The tables indicate that volume deviation decreases with transition time and for large transition times higher order implies smaller volume deviation.

4.6 Evolving wavefronts

In this simulation the tank has been moved a short distance to the right in a short time. This movement gives rise to two travelling waves, one going to the right and one to the left. The figures (3,4,5) show a zoom-in on the right going wave profiles for different approximation orders. The figures on the left are at a time shortly after the tank stopped moving and the figures on the right about 0.6 seconds later.

It can be observed that the linear first order model gives rise to a travelling wave with constant wave profile while for higher order approximations nonlinear terms are present and the wave profile evolves as the wave propagates. All three orders are quite similar directly after the tank motion, although second order shows a steeper right side and third even steeper. Another property is that they share the same group velocity c.

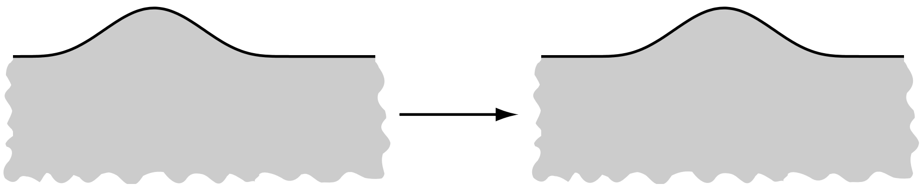

Figure 3: Zoom-in on a first order wavefront

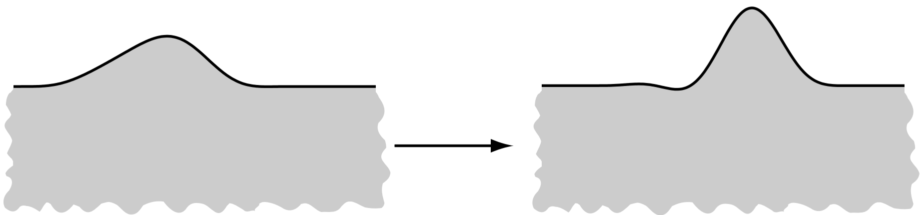

Figure 4: Zoom-in on a second order wavefront

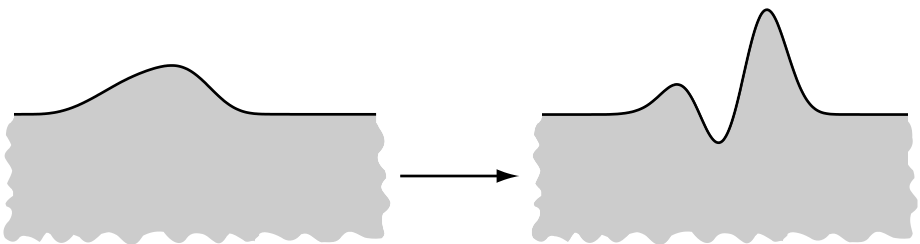

Figure 5: Zoom-in on a third order wavefront

5 Perturbation method applied to the wave-maker problem

The perturbation method can be applied to other situations than the tank problem. Here it is used for a wave-maker problem where a non-viscid liquid with a non zero height over the interval $[D(t), \infty]$ is governed by the Saint-Venant equations. In addition the liquid is under the constraint that the horizontal velocity at the boundary $(D(t))$ is equal to the velocity of the border $(\dot{D}(t))$. In the initial condition of the system the liquid is at steady-state with constant height h_0 and the border has the x-coordinate zero for all times equal to or before 0, $D(t) = 0, \forall t \leq 0$.

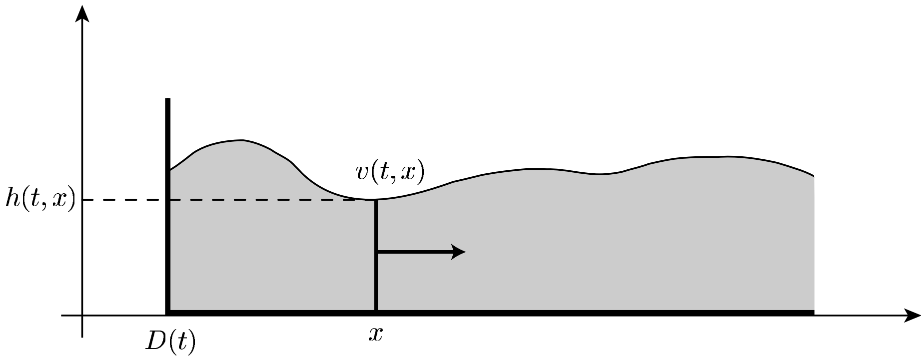

Figure 6: Liquid with one boundary with the x-coordinate $D(t)$

The PDE is the same as for the tank problem

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}(hv) = 0 \\ \dfrac{\partial}{\partial t}(hv) + \dfrac{\partial}{\partial x}(hv^2 + \dfrac{g}{2}h^2) = 0 \end{cases}$$

the boundary condition writes

$$v(t, D(t)) = \dot{D}(t)$$

and the steady-state initial condition implies

$$\begin{cases} h(0, x) = h_0 \\ v(0, x) = 0 \end{cases}$$

where $x \in [D(t), \infty]$.

5.1 Change of variables

Applying the same change of variables as for the one-dimensional tank (see section 1.2) the following set of equations is obtained.

$$
PDE \begin{cases} \dfrac{\partial J_+}{\partial t} + \alpha_+ \dfrac{\partial J_+}{\partial z} = 0 \\ \dfrac{\partial J_-}{\partial t} + \alpha_- \dfrac{\partial J_-}{\partial z} = 0 \end{cases}
$$

$$
BC \quad \frac{J_+ + J_-}{2}(t, 0) = \dot{D}
$$

$$
IC \begin{cases} \dfrac{1}{16g}(J_-(0, z) - J_+(0, z))^2 = h_0 \\ \dfrac{J_+(0, z) + J_-(0, z)}{2} = 0 \end{cases}
$$

The location of the border in the coordinate system (t, z) is $z = 0$ for all time. Using the same decomposition of J_+, J_-, α_+ and α_- as in the tank problem, the system can be solved in a similar way.

5.2 Zero order

Zero order terms are defined as constants and therefore can not fulfill the BC, but only the PDE and the IC. The chosen steady-state solution is

$$
\begin{cases} J_+^0 = -2c \\ J_-^0 = 2c \end{cases}
$$

with the corresponding α-terms

$$
\begin{cases} \alpha_+^0 = -c \\ \alpha_-^0 = c \end{cases}
$$

It can easily be shown that the initial condition for any higher order k writes

$$
\left. \begin{cases} J_+^k(0, z) = 0 \\ J_-^k(0, z) = 0 \end{cases} \right| , \forall z \geq 0
$$

5.3 First order

The set of equations are

$$
PDE \begin{cases} \dfrac{\partial J_+^1}{\partial t} - c \dfrac{\partial J_+^1}{\partial z} = 0 \\ \dfrac{\partial J_-^1}{\partial t} + c \dfrac{\partial J_-^1}{\partial z} = 0 \end{cases}
$$

$$
BC \quad \frac{J_+^1 + J_-^1}{2}(t, 0) = \dot{D}
$$

$$
IC \begin{cases} J_+^1(0, z) = 0 \\ J_-^1(0, z) = 0 \end{cases}
$$

The characteristics method gives the general solution to the PDE

$$\begin{cases} J_+^1(t,z) = \varphi_+(t + \frac{z}{c}) \\ J_-^1(t,z) = \varphi_-(t - \frac{z}{c}) \end{cases}$$

Fitting the solution to the initial conditions, taking consideration positive z and t, we get

$$\begin{cases} J_+^1(t,z) = 0 \\ J_-^1(t,z) = 0, \ \ t - \frac{z}{c} \leq 0 \end{cases}$$

Adding the boundary condition constraint give the unique solution

$$\varphi_+(t) + \varphi_-(t) = 2\dot{D}(t)$$

which implies

$$\varphi_-(t - \frac{z}{c}) = 2\dot{D}(t - \frac{z}{c})$$

so the total solution is

$$\begin{cases} J_+^1(t,z) = 0 \\ J_-^1(t,z) = 2\dot{D}(t - \frac{z}{c}) \end{cases}$$

The corresponding α-terms are

$$\begin{cases} \alpha_+^1 = \frac{3}{4}J_+^1 + \frac{1}{4}J_-^1 - \dot{D} \\ \alpha_-^1 = \frac{1}{4}J_+^1 + \frac{3}{4}J_-^1 - \dot{D} \end{cases}$$

which implies

$$\begin{cases} \alpha_+^1 = \frac{1}{2}\dot{D}(t - \frac{z}{c}) - \dot{D}(t) \\ \alpha_-^1 = \frac{3}{2}\dot{D}(t - \frac{z}{c}) - \dot{D}(t) \end{cases}$$

5.4 Second order

The set of equations for second order, according to the perturbation method, only including terms with order less or equal to 2 derive to

$$PDE \begin{cases} \frac{\partial J_+^2}{\partial t} - c\frac{\partial J_+^2}{\partial z} = -\alpha_+^1 J_{+z}^1 \\ \frac{\partial J_-^2}{\partial t} + c\frac{\partial J_-^2}{\partial z} = -\alpha_-^1 J_{-z}^1 \end{cases}$$

$$BC \quad \frac{J_+^2 + J_-^2}{2}(t,0) = 0$$

$$IC \begin{cases} J_+^2(0,z) = 0 \\ J_-^2(0,z) = 0 \end{cases}$$

The forcing terms in the PDE expressed in terms of D

$$\begin{cases} \dfrac{\partial J_+^2}{\partial t} - c\dfrac{\partial J_+^2}{\partial z} = 0 \\ \dfrac{\partial J_-^2}{\partial t} + c\dfrac{\partial J_-^2}{\partial z} = \left(\dfrac{3}{2}\dot{D}(t-\dfrac{z}{c}) - \dot{D}(t)\right) \cdot \dfrac{2}{c}\ddot{D}(t-\dfrac{z}{c}) = f_-^2(t,z) \end{cases}$$

Here the same method as for the tank problem can be applied, invoking the coordinate change $(t,z) \mapsto (\xi_+, \xi_-)$.

$$\begin{cases} \hat{J}_+^2 = \hat{\varphi}_+^2(\xi_-) \\ \hat{J}_-^2 = \dfrac{1}{2}\displaystyle\int_{\xi_+}^{\xi_-} \hat{f}_-^2(\xi_+, s)ds + \hat{\varphi}_-^2(\xi_+) \end{cases}$$

We can solve $\hat{J}_-^2$ in terms of D

$$\hat{J}_-^2 = \frac{1}{2}\int_{\xi_+}^{\xi_-} \left(\frac{3}{2}\dot{D}(\xi_+) - \dot{D}\left(\frac{\xi_+ + s}{2}\right)\right) \cdot \frac{2}{c}\ddot{D}(\xi_+)ds + \hat{\varphi}_-^2(\xi_+)$$

or

$$\hat{J}_-^2 = \frac{3}{2c}\dot{D}(\xi_+)\ddot{D}(\xi_+)(\xi_- - \xi_+) - \frac{2\ddot{D}(\xi_+)}{c}\left(D\left(\frac{\xi_+ + \xi_-}{2}\right) - D(\xi_+)\right) + \hat{\varphi}_-^2(\xi_+)$$

Hence, the general solution to the PDE in coordinate system (t,z) writes

$$\begin{cases} J_+^2(t,z) = \varphi_+^2(t+\dfrac{z}{c}) \\ J_-^2(t,z) = \ddot{D}(t-\dfrac{z}{c})\left(\dfrac{3z}{c^2}\dot{D}(t-\dfrac{z}{c}) - \dfrac{2}{c}\left(D(t) - D(t-\dfrac{z}{c})\right)\right) + \varphi_-^2(t-\dfrac{z}{c}) \end{cases}$$

The initial conditions

$$\begin{cases} J_+^2(0,z) = 0 \\ J_-^2(0,z) = 0 \end{cases}$$

together with that $D(t) = 0, \forall t \leq 0$ implies

$$\begin{cases} \varphi_+^2(t+\dfrac{z}{c}) = 0 \\ \varphi_-^2(t-\dfrac{z}{c}) = 0, \;\; for\; t-\dfrac{z}{c} \leq 0 \end{cases}$$

Hence $J_+^2(t,z) = 0$ and the boundary condition reduces to

$$J_-^2(t,0) = 0$$

which implies

$$\varphi_-^2(t) = 0, \forall t$$

As a conclusion the unique second order solution is

$$\begin{cases} J_+^2(t,z) = 0 \\ J_-^2(t,z) = \ddot{D}(t-\frac{z}{c})\left(\frac{3z}{c^2}\dot{D}(t-\frac{z}{c}) - \frac{2}{c}\left(D(t) - D(t-\frac{z}{c})\right)\right) \end{cases}$$

The z term in the solution will give rise to high wave amplitude at long distances from the boundary, which has been numerically verified. So the second order solution to the wave maker problem do not produce credible results for large z, however for intervals $z \in [D(t), L]$ were L is chosen not too big the solution has numerically shown to have realistic properties.

The corresponding α-terms are

$$\begin{cases} \alpha_+^2 = \frac{3}{4}J_+^2 + \frac{1}{4}J_-^2 = \frac{1}{4}J_-^2 \\ \alpha_-^2 = \frac{1}{4}J_+^2 + \frac{3}{4}J_-^2 = \frac{3}{4}J_-^2 \end{cases}$$

5.5 Third order

The set of equations for third order are

$$PDE \begin{cases} \frac{\partial J_+^3}{\partial t} - c\frac{\partial J_+^3}{\partial z} = -\alpha_+^1 J_{+z}^2 - \alpha_+^2 J_{+z}^1 = f_+ \\ \frac{\partial J_-^3}{\partial t} + c\frac{\partial J_-^3}{\partial z} = -\alpha_-^1 J_{-z}^2 - \alpha_-^2 J_{-z}^1 = f_- \end{cases}$$

$$BC \quad \frac{J_+^2 + J_-^2}{2}(t,0) = 0$$

$$IC \begin{cases} J_+^2(0,z) = 0 \\ J_-^2(0,z) = 0 \end{cases}$$

The forcing terms in the PDE can be written in terms of D and $J_+^1(t,z) = J_+^2(t,z) = 0$ so $f_+ = 0$. In analogy with second order the PDE solution writes

$$\begin{cases} \hat{J}_+^3 = \hat{\varphi}_+^3(\xi_-) \\ \hat{J}_-^3 = \frac{1}{2}\int_{\xi_+}^{\xi_-} \hat{f}_-^3(\xi_+, s)ds + \hat{\varphi}_-^3(\xi_+) \end{cases}$$

The initial and boundary conditions force the homogenous solution to be zero. After some calculations we get the unique third order solution which is

$$J_+^3(t,z) = 0$$

and

$$
\begin{aligned}
J_-^3(t,z) = -3\,\dot{D}(t-\frac{z}{c})D^{(3)}(t-\frac{z}{c})\mathrm{D}(t-\frac{z}{c})zc^{-2}& \\
+\,3\,\dot{D}(t-\frac{z}{c})D^{(3)}(t-\frac{z}{c})\mathrm{D}(t)zc^{-2}& \\
-\,9/2\ \dot{D}(t-\frac{z}{c})\left(\ddot{D}(t-\frac{z}{c})\right)^2 z^2c^{-3} - 3\,\ddot{D}(t-\frac{z}{c})\left(\dot{D}(t-\frac{z}{c})\right)^2 zc^{-2}& \\
+\,3\,\left(\ddot{D}(t-\frac{z}{c})\right)^2\mathrm{D}(t)zc^{-2} - 3\,\left(\ddot{D}(t-\frac{z}{c})\right)^2\mathrm{D}(t-\frac{z}{c})zc^{-2}& \\
+\,5\,\dot{D}(t-\frac{z}{c})\ddot{D}(t-\frac{z}{c})\mathrm{D}(t)c^{-1} + 2\,D^{(3)}(t-\frac{z}{c})\mathrm{D}(t-\frac{z}{c})\mathrm{D}(t)c^{-1}& \\
-\,5\,\dot{D}(t-\frac{z}{c})\ddot{D}(t-\frac{z}{c})\mathrm{D}(t-\frac{z}{c})c^{-1} - 9/4\,D^{(3)}(t-\frac{z}{c})\left(\dot{D}(t-\frac{z}{c})\right)^2 z^2c^{-3}& \\
-\,D^{(3)}(t-\frac{z}{c})\left(\mathrm{D}(t)\right)^2 c^{-1} - D^{(3)}(t-\frac{z}{c})\left(\mathrm{D}(t-\frac{z}{c})\right)^2 c^{-1}& \\
-\,\ddot{D}(t-\frac{z}{c})c^{-1}\int_{t-\frac{z}{c}}^{t}\left(\dot{D}(s)\right)^2 ds&
\end{aligned}
$$

6 Implementations of the perturbation method

In the previous sections we studied the wave propagations in a one-dimensional domain due to the interaction of the water and its boundary conditions. As a first approximation many real world physical systems can be considered as a collection of such simple systems as the one-sided (wave-maker problem) and the two-sided (water-tank problem) system.

More precisely a two-dimensional rectangular tank gives rise to waves that are at first order approximation the superposition of waves travelling in orthogonal directions. Its structure is similar to two one-dimensional orthogonal water tanks for which our methodology applies. Though at second and third order dynamics are not completely decoupled we give results that have some realistic features (shape of the wavefronts and absence of perfect symmetries) but are of course debatable.

At last we found it interesting to try to recreate the waves generated by a boat travelling in a straight line. A large set of one-sided systems (wave-maker problem) was used, fed by the hull profile and the boat position and velocity.

6.1 The one-dimensional tank

An algorithm to simulate the one-dimensional tank problem was constructed following the ideas described in section 4.3.

Figure 7 shows a snapshot of a real-time interactive simulation of a 2m long water tank. The vertical scale of the upper figure is enlarged to clarify the wave profile. The lower part shows the water velocity as a function of z, and it can be seen that the water velocity is the same at both boundaries. The boundary velocity is negative so at the snapshot moment the tank was moving leftwards.

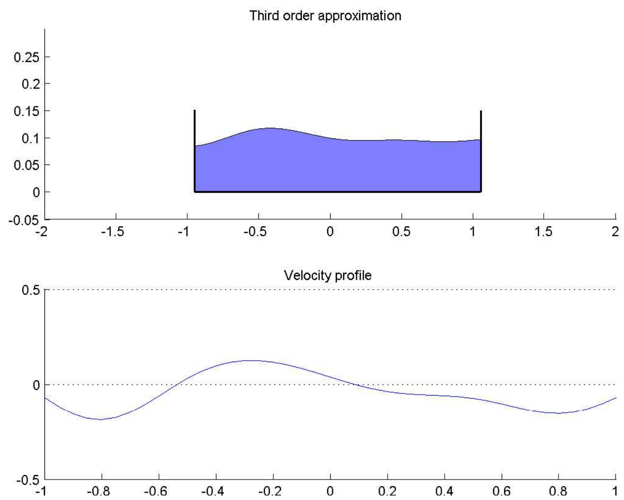

Figure 7: One-dimensional tank simulation

6.2 The two-dimensional tank

The algorithm used for the one-dimensional tank is here used as a primer for a two-dimensional rectangular tank. Under the approximation that perpendicular waves are decoupled the surface height is estimated by superposition of two perpendicular one-dimensional tanks. The one-dimensional tanks are subject to the vector component of the two-dimensional motion in the their direction.

$$
\begin{aligned}
H(t,x,y) &= h_x(t,x) + h_y(t,y) \\
\vec{V}(t,x,y) &= v_x(t,x)\vec{e_x} + v_y(t,y)\vec{e_y} \\
\vec{D}(t) &= D_x\vec{e_x} + D_y\vec{e_y}
\end{aligned}
$$

Under this approximation rotational flows are impossible. Figure 8 show a snapshot from the interactive tank simulation. As for the one-dimensional tank the vertical scale is enlarged to clarify height differences.

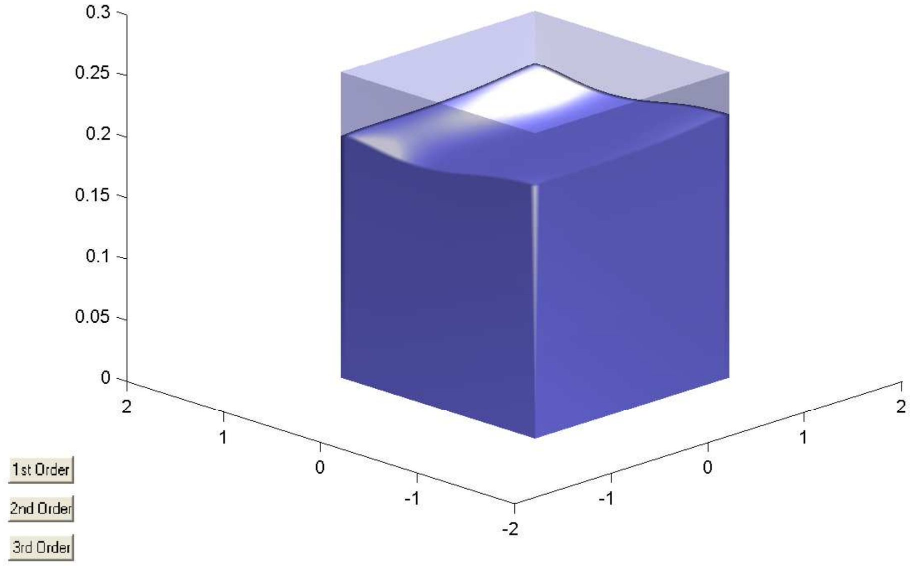

Figure 8: Two-dimensional tank simulation

The water motion is calculated in realtime according to the user defined motion. Additionally, the approximation order can be changed during simulation with the buttons to the left.

6.3 Boat induced water waves

A scenario with a boat in the open ocean is simulated with the formulas from the wave-maker problem. The boat in consideration will only move in a straight line and the water is approximated by two arrays of parallel one-sided tanks perpendicular to the boat direction. Each tank array point out from the boat route with the boundaries placed at the symmetry axis. When the boat passes between a pair of tanks (as seen in figure 9) the hull separates the tank boundaries and thus inducing two waves travelling perpendicular to the boat route.

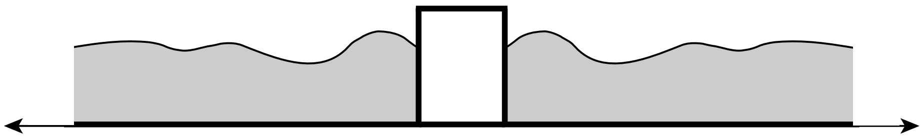

Figure 9: Boat hull between two one-sided tanks

The horizontal profile of the hull is described by a smooth $\mathcal{C}^4$ function and the tank boundary displacement D is calculated from the hull profile and the boat position. This equation provide all necessary information (D, $\dot{D}$, $\ddot{D}$, $D^{(3)}$) for the third order solution to the wave-maker problem. However, the wave maker produces waves that are not realistic for long distances from the boundary. Therefore it has been necessary to introduce damping for some geometrical settings and boat velocities.

To add some ocean characteristics, not induced by the boat, an approximation of the ocean surface is superimposed, once again using one-sided systems. This time the tanks are oriented in 2 arrays in orthogonal directions and are fed with periodic input signals for wave generation. For texturing, filtered white noise is added following [11]. Finally, the water between the tanks is linearly interpolated by the renderer.

Figure 10 show a photograph of boat induced water waves to the left and a simulation imitating the photograph to the right.

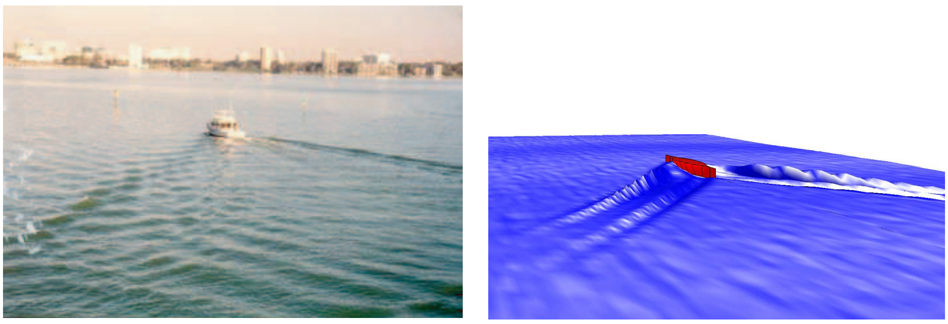

Figure 10: Boat induced water waves

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A Appendix

A.1 Third order functions

The explicit expressions for $\underset{p}{J}{}^3_+$, f^3_- and A^3 are as follows. Additionally the symmetric properties

$$f^3_-(t,z) = -f^3_+(t,-z)$$
$$\underset{p}{J}{}^3_+(t,z) = \underset{p}{J}{}^3_+(t,-z)$$

hold.

$$\begin{aligned}
\underset{p}{J}{}^3_+(t,z) = &-\frac{1}{64}\,\ddot{\nu}(t+\frac{z}{c})\nu(t+\frac{z}{c})\dot{\nu}(t-\frac{z}{c})c^{-2} - 1/16\,\nu^{(3)}(t+\frac{z}{c})\nu(t-\frac{\Delta}{2})\nu(t-\frac{z}{c})c^{-2}\\
&+ 1/4\,\nu^{(3)}(t+\frac{z}{c})\nu(t-\frac{\Delta}{2})\nu(t+\frac{\Delta}{2})c^{-2} + 3/8\,\ddot{\nu}(t+\frac{z}{c})\dot{\nu}(t+\frac{z}{c})\nu(t+\frac{\Delta}{2})c^{-2}\\
&- 1/16\,\nu^{(3)}(t+\frac{z}{c})\nu(t-\frac{z}{c})\nu(t+\frac{\Delta}{2})c^{-2} + 3/8\,\ddot{\nu}(t+\frac{z}{c})\dot{\nu}(t+\frac{z}{c})\nu(t-\frac{\Delta}{2})c^{-2}\\
&+ 1/16\,\ddot{\nu}(t+\frac{z}{c})\dot{\nu}(t-\frac{z}{c})\nu(t+\frac{\Delta}{2})c^{-2} - \frac{9}{64}\,\ddot{\nu}(t+\frac{z}{c})\dot{\nu}(t+\frac{z}{c})\nu(t-\frac{z}{c})c^{-2}\\
&- \frac{3}{64}\,z\ddot{\nu}(t+\frac{z}{c})\left(\dot{\nu}(t-\frac{z}{c})\right)^2 c^{-3} + 3/8\,z\left(\ddot{\nu}(t+\frac{z}{c})\right)^2\nu(t-\frac{\Delta}{2})c^{-3}\\
&+ \frac{9}{16}\,z^2\dot{\nu}(t+\frac{z}{c})\left(\ddot{\nu}(t+\frac{z}{c})\right)^2 c^{-4} + \frac{9}{32}\,z^2\nu^{(3)}(t+\frac{z}{c})\left(\dot{\nu}(t+\frac{z}{c})\right)^2 c^{-4}\\
&+ \frac{1}{128}\,\nu^{(3)}(t+\frac{z}{c})\left(\nu(t-\frac{z}{c})\right)^2 c^{-2} - \frac{5}{128}\,\ddot{\nu}(t+\frac{z}{c})\int_{t+\frac{z}{c}}^{t-\frac{z}{c}}(\dot{\nu}(s))^2\,dsc^{-2}\\
&+ 3/4\,z\ddot{\nu}(t+\frac{z}{c})\eta(t+\frac{z}{c})c^{-2} - \frac{3}{32}\,z\left(\ddot{\nu}(t+\frac{z}{c})\right)^2\nu(t-\frac{z}{c})c^{-3}\\
&+ 3/8\,z\left(\ddot{\nu}(t+\frac{z}{c})\right)^2\nu(t+\frac{\Delta}{2})c^{-3} + \frac{9}{16}\,z\ddot{\nu}(t+\frac{z}{c})\left(\dot{\nu}(t+\frac{z}{c})\right)^2 c^{-3}\\
&+ 3/4\,z\dot{\nu}(t+\frac{z}{c})\dot{\eta}(t+\frac{z}{c})c^{-2} + 1/2\,\dot{\eta}(t+\frac{z}{c})\nu(t+\frac{\Delta}{2})c^{-1} + 1/2\,\dot{\eta}(t+\frac{z}{c})\nu(t-\frac{\Delta}{2})c^{-1}\\
&+ 1/8\,\nu^{(3)}(t+\frac{z}{c})\left(\nu(t-\frac{\Delta}{2})\right)^2 c^{-2} - 1/8\,\dot{\eta}(t+\frac{z}{c})\nu(t-\frac{z}{c})c^{-1}\\
&+ 1/8\,\ddot{\nu}(t+\frac{z}{c})\int_{t+\frac{z}{c}}^{t-\frac{z}{c}}\eta(s)dsc^{-1} + 1/16\,\ddot{\nu}(t+\frac{z}{c})\dot{\nu}(t-\frac{z}{c})\nu(t-1/2\,\Delta)c^{-2}\\
&+ 1/8\,\nu^{(3)}(t+\frac{z}{c})\left(\nu(t+\frac{\Delta}{2})\right)^2 c^{-2} - \frac{3}{32}\,z\dot{\nu}(t+\frac{z}{c})\nu^{(3)}(t+\frac{z}{c})\nu(t-\frac{z}{c})c^{-3}\\
&+ 3/8\,z\dot{\nu}(t+\frac{z}{c})\nu^{(3)}(t+\frac{z}{c})\nu(t-\frac{\Delta}{2})c^{-3} + 3/8\,z\dot{\nu}(t+\frac{z}{c})\nu^{(3)}(t+\frac{z}{c})\nu(t+\frac{\Delta}{2})c^{-3}
\end{aligned}$$

$$
\begin{aligned}
f_-^3(t,z) = -\frac{1}{32c^3} \Bigg(& -4\,\ddot{\nu}(t+\frac{z}{c})\ddot{\nu}(t-\frac{z}{c})\nu(t+\frac{\Delta}{2})c - 4\,\ddot{\nu}(t+\frac{z}{c})\ddot{\nu}(t-\frac{z}{c})\nu(t-\frac{\Delta}{2})c \\
& - 6\,\dot{\nu}(t+\frac{z}{c})\left(\ddot{\nu}(t-\frac{z}{c})\right)^2 z - 16\,\dot{\nu}(t+\frac{\Delta}{2})\dot{\eta}(t-\frac{z}{c})c^2 + \ddot{\nu}(t-\frac{z}{c})\left(\dot{\nu}(t+\frac{z}{c})\right)^2 c \\
& - 18\,\nu^{(3)}(t-\frac{z}{c})\left(\dot{\nu}(t-\frac{z}{c})\right)^2 z + 18\,\ddot{\nu}(t-\frac{z}{c})\left(\dot{\nu}(t-\frac{z}{c})\right)^2 c + 8\,\dot{\nu}(t+\frac{z}{c})\dot{\eta}(t-\frac{z}{c})c^2 \\
& + 24\,\dot{\nu}(t-\frac{z}{c})\dot{\eta}(t-\frac{z}{c})c^2 - 36\,\dot{\nu}(t-\frac{z}{c})\left(\ddot{\nu}(t-\frac{z}{c})\right)^2 z + 12\,\dot{\nu}(t+\frac{\Delta}{2})\left(\ddot{\nu}(t-\frac{z}{c})\right)^2 z \\
& - 16\,\dot{\nu}(t-\frac{\Delta}{2})\dot{\eta}(t-\frac{z}{c})c^2 + 12\,\dot{\nu}(t-\frac{\Delta}{2})\left(\ddot{\nu}(t-\frac{z}{c})\right)^2 z - 3\left(\ddot{\nu}(t-\frac{z}{c})\right)^2 \nu(t+\frac{z}{c})c \\
& + 12\left(\ddot{\nu}(t-\frac{z}{c})\right)^2 \nu(t+\frac{\Delta}{2})c + 12\left(\ddot{\nu}(t-\frac{z}{c})\right)^2 \nu(t-\frac{\Delta}{2})c - 8\,\ddot{\nu}(t-\frac{z}{c})\eta(t+\frac{z}{c})c^2 \\
& + 24\,\ddot{\nu}(t-\frac{z}{c})\eta(t-\frac{z}{c})c^2 - 6\,\dot{\nu}(t+\frac{z}{c})\nu^{(3)}(t-\frac{z}{c})\dot{\nu}(t-\frac{z}{c})z \\
& - \dot{\nu}(t+\frac{z}{c})\nu^{(3)}(t-\frac{z}{c})\nu(t+\frac{z}{c})c + 4\,\dot{\nu}(t+\frac{z}{c})\nu^{(3)}(t-\frac{z}{c})\nu(t+\frac{\Delta}{2})c \\
& + 4\,\dot{\nu}(t+\frac{z}{c})\nu^{(3)}(t-\frac{z}{c})\nu(t-\frac{\Delta}{2})c + 9\,\dot{\nu}(t+\frac{z}{c})\ddot{\nu}(t-\frac{z}{c})\dot{\nu}(t-\frac{z}{c})c \\
& - 3\,\dot{\nu}(t-\frac{z}{c})\nu^{(3)}(t-\frac{z}{c})\nu(t+\frac{z}{c})c + 12\,\dot{\nu}(t-\frac{z}{c})\nu^{(3)}(t-\frac{z}{c})\nu(t+\frac{\Delta}{2})c \\
& + 12\,\dot{\nu}(t-\frac{z}{c})\nu^{(3)}(t-\frac{z}{c})\nu(t-\frac{\Delta}{2})c + 12\,\dot{\nu}(t+\frac{\Delta}{2})\nu^{(3)}(t-\frac{z}{c})\dot{\nu}(t-\frac{z}{c})z \\
& + 2\,\dot{\nu}(t+\frac{\Delta}{2})\nu^{(3)}(t-\frac{z}{c})\nu(t+\frac{z}{c})c - 8\,\dot{\nu}(t+\frac{\Delta}{2})\nu^{(3)}(t-\frac{z}{c})\nu(t+\frac{\Delta}{2})c \\
& - 8\,\dot{\nu}(t+\frac{\Delta}{2})\nu^{(3)}(t-\frac{z}{c})\nu(t-\frac{\Delta}{2})c - 12\,\dot{\nu}(t+\frac{\Delta}{2})\ddot{\nu}(t-\frac{z}{c})\dot{\nu}(t-\frac{z}{c})c \\
& - 2\,\dot{\nu}(t+\frac{\Delta}{2})\ddot{\nu}(t-\frac{z}{c})\dot{\nu}(t+\frac{z}{c})c + 12\,\dot{\nu}(t-\frac{\Delta}{2})\nu^{(3)}(t-\frac{z}{c})\dot{\nu}(t-\frac{z}{c})z \\
& + 2\,\dot{\nu}(t-\frac{\Delta}{2})\nu^{(3)}(t-\frac{z}{c})\nu(t+\frac{z}{c})c - 8\,\dot{\nu}(t-\frac{\Delta}{2})\nu^{(3)}(t-\frac{z}{c})\nu(t+\frac{\Delta}{2})c \\
& - 8\,\dot{\nu}(t-\frac{\Delta}{2})\nu^{(3)}(t-\frac{z}{c})\nu(t-\frac{\Delta}{2})c - 12\,\dot{\nu}(t-\frac{\Delta}{2})\ddot{\nu}(t-\frac{z}{c})\dot{\nu}(t-\frac{z}{c})c \\
& - 2\,\dot{\nu}(t-\frac{\Delta}{2})\ddot{\nu}(t-\frac{z}{c})\dot{\nu}(t+\frac{z}{c})c - 6\,\ddot{\nu}(t-\frac{z}{c})\ddot{\nu}(t+\frac{z}{c})\dot{\nu}(t+\frac{z}{c})z \\
& + \ddot{\nu}(t-\frac{z}{c})\ddot{\nu}(t+\frac{z}{c})\nu(t-\frac{z}{c})c \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
A^3(t) = {} & \frac{5}{128}\,\frac{\ddot\nu(t-\frac{\Delta}{2})\int_{t-\frac{\Delta}{2}}^{t+\frac{\Delta}{2}}\left(\dot\nu(s)\right)^2 ds}{c^2} - 3/8\,\frac{\dot\eta(t+\frac{\Delta}{2})\nu(t-\frac{\Delta}{2})}{c} \\
& + \frac{5}{128}\,\frac{\ddot\nu(t+\frac{\Delta}{2})\int_{t+\frac{\Delta}{2}}^{t-\frac{\Delta}{2}}\left(\dot\nu(s)\right)^2 ds}{c^2} - 1/8\,\frac{\nu^{(3)}(t-\frac{\Delta}{2})\left(\nu(t-\frac{\Delta}{2})\right)^2}{c^2} \\
& - 1/8\,\frac{\ddot\nu(t-\frac{\Delta}{2})\int_{t-\frac{\Delta}{2}}^{t+\frac{\Delta}{2}}\eta(s)ds}{c} - \frac{3}{64}\,\frac{\ddot\nu(t-\frac{\Delta}{2})\dot\nu(t+\frac{\Delta}{2})\nu(t-\frac{\Delta}{2})}{c^2} \\
& - \frac{15}{64}\,\frac{\dot\nu(t-\frac{\Delta}{2})\ddot\nu(t-\frac{\Delta}{2})\nu(t+\frac{\Delta}{2})}{c^2} - 3/16\,\frac{\nu^{(3)}(t-\frac{\Delta}{2})\nu(t-\frac{\Delta}{2})\nu(t+\frac{\Delta}{2})}{c^2} \\
& - 3/8\,\frac{\dot\eta(t-\frac{\Delta}{2})\nu(t+\frac{\Delta}{2})}{c} + 3/16\,\frac{\nu^{(3)}(t-\frac{\Delta}{2})\dot\nu(t-\frac{\Delta}{2})\nu(t-\frac{\Delta}{2})\Delta}{c^2} \\
& + \frac{9}{64}\,\frac{\nu^{(3)}(t-\frac{\Delta}{2})\dot\nu(t-\frac{\Delta}{2})\nu(t+\frac{\Delta}{2})\Delta}{c^2} - \frac{9}{64}\,\frac{\nu^{(3)}(t+\frac{\Delta}{2})\dot\nu(t+\frac{\Delta}{2})\nu(t-\frac{\Delta}{2})\Delta}{c^2} \\
& - 3/16\,\frac{\nu^{(3)}(t+\frac{\Delta}{2})\dot\nu(t+\frac{\Delta}{2})\nu(t+\frac{\Delta}{2})\Delta}{c^2} - \frac{3}{64}\,\frac{\ddot\nu(t+\frac{\Delta}{2})\nu(t+\frac{\Delta}{2})\dot\nu(t-\frac{\Delta}{2})}{c^2} \\
& - 3/8\,\frac{\ddot\nu(t+\frac{\Delta}{2})\dot\nu(t+\frac{\Delta}{2})\nu(t+\frac{\Delta}{2})}{c^2} - 3/16\,\frac{\nu^{(3)}(t+\frac{\Delta}{2})\nu(t-\frac{\Delta}{2})\nu(t+\frac{\Delta}{2})}{c^2} \\
& - 1/16\,\frac{\ddot\nu(t+\frac{\Delta}{2})\dot\nu(t-\frac{\Delta}{2})\nu(t-\frac{\Delta}{2})}{c^2} + \frac{9}{64}\,\frac{\left(\ddot\nu(t-\frac{\Delta}{2})\right)^2\nu(t+\frac{\Delta}{2})\Delta}{c^2} \\
& + 3/8\,\frac{\ddot\nu(t-\frac{\Delta}{2})\eta(t-\frac{\Delta}{2})\Delta}{c} - \frac{15}{64}\,\frac{\ddot\nu(t+\frac{\Delta}{2})\dot\nu(t+\frac{\Delta}{2})\nu(t-\frac{\Delta}{2})}{c^2} \\
& - \frac{9}{128}\,\frac{\nu^{(3)}(t-\frac{\Delta}{2})\left(\dot\nu(t-\frac{\Delta}{2})\right)^2\Delta^2}{c^2} + \frac{9}{32}\,\frac{\ddot\nu(t-\frac{\Delta}{2})\left(\dot\nu(t-\frac{\Delta}{2})\right)^2\Delta}{c^2} \\
& - \frac{9}{64}\,\frac{\dot\nu(t-\frac{\Delta}{2})\left(\ddot\nu(t-\frac{\Delta}{2})\right)^2\Delta^2}{c^2} - \frac{9}{64}\,\frac{\left(\ddot\nu(t+\frac{\Delta}{2})\right)^2\nu(t-\frac{\Delta}{2})\Delta}{c^2} \\
& - \frac{9}{128}\,\frac{\nu^{(3)}(t+\frac{\Delta}{2})\left(\dot\nu(t+\frac{\Delta}{2})\right)^2\Delta^2}{c^2} - 3/16\,\frac{\left(\ddot\nu(t+1/2\,\Delta)\right)^2\nu(t+\frac{\Delta}{2})\Delta}{c^2} \\
& + \frac{3}{128}\,\frac{\ddot\nu(t+\frac{\Delta}{2})\left(\dot\nu(t-\frac{\Delta}{2})\right)^2\Delta}{c^2} - 3/8\,\frac{\dot\nu(t+\frac{\Delta}{2})\dot\eta(t+\frac{\Delta}{2})\Delta}{c} \\
& - \frac{9}{64}\,\frac{\dot\nu(t+\frac{\Delta}{2})\left(\ddot\nu(t+\frac{\Delta}{2})\right)^2\Delta^2}{c^2} - 3/8\,\frac{\ddot\nu(t+\frac{\Delta}{2})\eta(t+\frac{\Delta}{2})\Delta}{c} \\
& - \frac{9}{32}\,\frac{\ddot\nu(t+\frac{\Delta}{2})\left(\dot\nu(t+\frac{\Delta}{2})\right)^2\Delta}{c^2} + 3/16\,\frac{\left(\ddot\nu(t-\frac{\Delta}{2})\right)^2\nu(t-\frac{\Delta}{2})\Delta}{c^2} \\
& + 3/8\,\frac{\dot\nu(t-\frac{\Delta}{2})\dot\eta(t-\frac{\Delta}{2})\Delta}{c} - \frac{3}{128}\,\frac{\ddot\nu(t-\frac{\Delta}{2})\left(\dot\nu(t+\frac{\Delta}{2})\right)^2\Delta}{c^2} \\
& - 1/16\,\frac{\ddot\nu(t-\frac{\Delta}{2})\dot\nu(t+\frac{\Delta}{2})\nu(t+\frac{\Delta}{2})}{c^2} - 3/8\,\frac{\dot\nu(t-\frac{\Delta}{2})\ddot\nu(t-\frac{\Delta}{2})\nu(t-\frac{\Delta}{2})}{c^2} \ldots
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{9}{128}\,\frac{\nu^{(3)}(t+\frac{\Delta}{2})\left(\nu(t-\frac{\Delta}{2})\right)^2}{c^2}-1/8\,\frac{\nu^{(3)}(t+\frac{\Delta}{2})\left(\nu(t+1/2\,\Delta)\right)^2}{c^2}\\
&\qquad-\frac{9}{128}\,\frac{\nu^{(3)}(t-\frac{\Delta}{2})\left(\nu(t+\frac{\Delta}{2})\right)^2}{c^2}-1/8\,\frac{\ddot{\nu}(t+\frac{\Delta}{2})\int_{t+\frac{\Delta}{2}}^{t-\frac{\Delta}{2}}\eta(s)ds}{c}\\
&\qquad\qquad-1/2\,\frac{\dot{\eta}(t+\frac{\Delta}{2})\nu(t+\frac{\Delta}{2})}{c}-1/2\,\frac{\dot{\eta}(t-\frac{\Delta}{2})\nu(t-\frac{\Delta}{2})}{c}
\end{aligned}
$$

A.2 Gevrey function

The function ϕ used for simulations is based upon an unpublished work of François Malrait and is defined by

$$
\phi(\tau)=\begin{cases} L+\Delta L & if \quad \tau\geq 1,\\ L+\Delta L g(\tau) & if \quad 1>\tau>0,\\ L & if \quad \tau\leq 0\end{cases}
$$

where

$$
g(\tau)=\frac{f(\tau)}{f(\tau)+f(1-\tau)},\qquad \tau\in[0,1]
$$

and

$$
f(\tau)=\begin{cases} e^{-\frac{1}{\tau}} & if \quad \tau>0,\\ 0 & if \quad \tau\leq 0,\end{cases}
$$

The derivatives of ϕ are bounded by

$$
\left|\phi^{(k+1)}(\tau)\right|\leq\frac{\Delta L e^2 4^k k!^2}{\sqrt{2\pi}}
$$