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Underwater Locomotion Using Swimming Cylinders

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Abstract

The object of this thesis is to explore if a planar, almost circular amoeba can move efficiently in an inviscid fluid. We investigate a homogeneous amoeba and a heterogeneous amoeba and find that motion is possible for both using the physical fact that more work is needed when moving a bulky body in a fluid than when moving a small body of the same mass. For our calculations to be valid only small body shape changes may be considered. We find that the greatest amount of motion can usually be achieved by the heterogeneous amoeba and that an in some sense optimal way to control this is to use sinusoidal control inputs. It is concluded that such an amoeba is a practically feasible although not a very efficient form of underwater vehicle.

Key words
Underwater locomotion, Irrotational fluid, Deformation,Lie bracket motion, Optimal control

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Contents

1. Introduction

1.1 Objective

This thesis explores the possibility of locomotion for a planar, deformable body in an inviscid fluid. Motion of a vehicle can be created by propeller, water jet or other means, but swimming is more difficult. Use of fins will create vorticity in a viscous fluid and can thus propel a fish forward, but in a perfect, inviscid fluid no circulation is allowed and consequently no propelling motion can be created that way. This thesis shows that locomotion can be created in several ways and compares their efficiency.

There are many practical applications of such a vehicle [15, 5]. Its major advantages are maneuverability and stealth. Maneuverability is high as the vehicle can change direction almost instantly. Stealth is due to the composition of the vehicle as, contrary to traditional underwater vehicles, no jet or propeller is needed to provide thrust. Areas that would benefit greatly from stealthy vehicles are for example scientific research of the sea bottom and behavioral studies of school of fish or other underwater animals disturbed by noise. Of course military applications can also be found, such as undetectable surveillance of foreign shores. Furthermore, a vehicle able to change its shape has a natural advantage when moving in confined spaces which can be useful for example when investigating shipwrecks.

The thesis project focuses on an amoeba-like, two-dimensional vehicle which can be seen as the cross section of an infinitely long cylinder. Locomotion is created by a step-by-step procedure although the steps must not take place one at a time but can flow freely into one another, see Figure 1.1

1. Move the center of mass (CM) of the amoeba, causing the amoeba to change its position, when the amoeba has a certain shape and thus a certain effective mass while moving through the surrounding fluid.

2. Change the shape of the amoeba

3. Move the CM in the opposite direction. Now the amoeba shape is different and consequently so is its effective mass while moving through the fluid causing it to shift a different amount from Step 1

4. Change the amoeba back to its initial shape, resulting in a translational movement.

We study two separate methods of implementing this procedure. The methods differ in the construction of the amoeba. Firstly an amoeba with a heavy weight inside it and a light deforming shell is considered. This amoeba composition corresponds most clearly to the one depicted in Figure 1.1. Secondly a homogeneous amoeba with all its weight evenly distributed is studied. For this amoeba the center of mass is moved solely by the shape deformation. The first method is shown to give much faster locomotion than

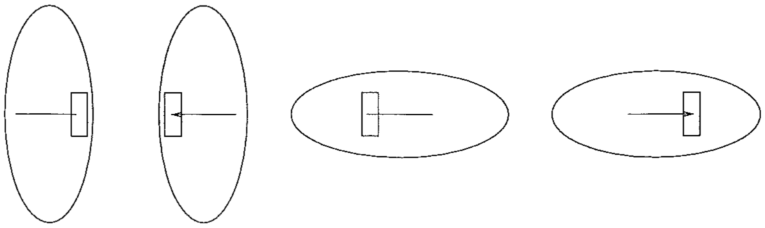

Figure 1.1 General illustration of the step-by-step procedure in Section 1.1. Steps go from left to right.

the second one. This is due to the fact that the surface deformations we can investigate must be very small but the weight inside the first amoeba type can be moved the entire diameter of the amoeba.

1.2 Previous work

Many researchers have been intrigued by this subject and have investigated various variations on this theme. Saffman [11] is generally considered the first to have broached the subject. He showed that a deformable body can move through a perfect fluid without producing vorticity. Blake expanded the case of a homogeneous amoeba [2]. Further research on the subject has been carried out by among others Miloh and Galper [9] and Benjamin and Ellis [1], who studied three-dimensional shapes, and more recently Kelly and Murray [4] and Mason [8]. Section 4 is in fact a generalization of the work of Mason.

1.3 Outline of the thesis

This thesis investigates two different methods of achieving the shape changes and center of mass shifts discussed in Section 1.1. These methods are evaluated with respect to the amount of motion they create compared to the input effort. It is shown that one method of locomotion is more efficient than the other. The optimal control inputs are found; "optimal" in the sense that they move the amoeba a given distance in a given time while minimizing the control effort.

Section 2 describes the general method of finding the velocity of the amoeba. Section 3 discusses the motion of a heterogeneous amoeba as originally suggested in [11] and Section 4 presents the case of a homogeneous amoeba. In Section 5 the amount of motion created by the two different methods of construction for the amoeba are compared.

Appendices A and B are for the benefit of the reader unfamiliar with the mathematics of Lie algebras and the concept of optimal control respectively. They contain only very brief summaries of the aspects of the subjects directly used in the thesis.

1.4 Computational background

A necessary mathematical base for complete understanding of this thesis is knowledge of how vector analysis (see for example [3]) is used to describe flow equations and surface integrals and some insight in the solving of partial differential equations (see for example [14]). Mathematica has been used extensively as a tool for analytical calculations throughout this thesis to integrate, take derivatives, expand in Taylor series, solve equations and simplify expressions. MATLAB was used for some simulations and to generate graphs. The software program RIOTS (Recursive Integration Optimal Trajectory Solver) was used for numerical optimal control computations.

2. Equations of Motion

The aim of this section is to describe the basic physical equations of motion used to find the velocity of the amoeba. Some results from vector analysis and fluid mechanics are needed to perform the calculations.

2.1 Coordinate systems

All vehicles considered have a circular shape when undeformed. A polar coordinate system frame is fixed in the body of the amoeba with its origin in the undeformed amoeba geometric center. The amoeba is then easily described by a radius function in polar coordinates $r(t, \theta)$. Note that only small deformations of the surface are considered in this thesis. Call the body frame origin in world frame rectangular coordinates $(x(t), y(t))$ and its velocity $(\dot{x}, \dot{y})$. The origin of the body frame from which the radius is measured is not, in general, located at the geometric center (centroid) of the amoeba when it is deformed. Call the center of mass in world frame rectangular coordinates C (depending on (x, y) and the setup of the amoeba) and its velocity $\dot{C}$.

2.2 Momentum preservation

The velocity of the amoeba can be found through various physical constraints. We choose to study the momentum of the amoeba, equate it to the momentum of the surrounding fluid and solve for the velocity. The momentum of the amoeba is simply the velocity of the amoeba center of mass times the amoeba mass M: $M\dot{C}(t)$. Exactly how the center of mass is related to the origin of the body frame is determined by the construction of the amoeba. The momentum of a fluid is more difficult to find.

Fluid impulse

The total linear momentum of the fluid is indeterminate but can be approximated by the fluid impulse studied by Kelvin. The fluid impulse varies with the forces acting between the amoeba and the fluid as if it were the momentum of a finite object, equal and opposite in sign to the momentum of the amoeba. For a perfect, irrotational fluid there exists a potential function ϕ such that the velocity field of the fluid is $u = \nabla\phi$ (conservative fields). Kelvin stated that the time integral of the pressures acting on a surface S in a perfect fluid of constant density ρ is given by $K(\phi) = \rho \int_S \phi \; n \; dS$ where the quantity $-K(\phi)$ is the fluid impulse. In Cartesian coordinates $K(\phi)$ is

$$K(\phi) = \rho \int_S \phi \; n \; dS = \rho \int_S \phi \begin{bmatrix} n_x \\ n_y \end{bmatrix} dS. \tag{2.1}$$

So what we have to do is find the potential function ϕ for the amoeba, integrate its normal component around the boundary of the amoeba to find $K(\phi)$, equate this to $M\dot{C}(t)$ and solve for $(\dot{x}, \dot{y})$.

The potential function ϕ

A perfect fluid is incompressible, and therefore (if it is also irrotational) $\nabla \cdot u = 0$ which means that the potential function must fulfill $\nabla \cdot \nabla\phi = \nabla^2\phi = 0$, called Laplace's equation. Application of the separation of variables method (Fourier method) to Laplace's equation gives the standard solution as a series of the form:

$$\phi = (a_0 \log(r) + b_0)(c_0\theta + d_0) + \sum_{n=1}^{\infty}(a_n \frac{1}{r^n} + b_n r^n)(c_n \sin(n\theta) + d_n \cos(n\theta)). \tag{2.2}$$

The constants a_n, b_n, c_n, d_n are determined by the fact that the velocity $u = \nabla\phi$ goes to zero as r approaches infinity (gives $b_n \equiv 0 \ \forall \ n \neq 0$), the requirement that there be no circulation around the amoeba and by the boundary conditions at the amoeba surface.

The boundary conditions at the amoeba surface are that the velocity of the amoeba surface along the surface normal must match the fluid velocity at every point. The fluid velocity is as stated above $\nabla\phi$ and the surface velocity in polar coordinates is

$$u_{\text{surf}} = \begin{bmatrix} \dot{x}\cos\theta + \dot{y}\sin\theta + \dot{r}(t,\theta) \\ -\dot{x}\sin\theta + \dot{y}\cos\theta \end{bmatrix}$$

as vector components can be transformed to polar coordinates from Cartesian coordinates by using

$$\begin{aligned} \mathbf{e}_r &= \cos\theta \ \mathbf{e}_x + \sin\theta \ \mathbf{e}_y. \\ \mathbf{e}_\theta &= -\sin\theta \ \mathbf{e}_x + \cos\theta \ \mathbf{e}_y. \end{aligned} \tag{2.3}$$

where $\mathbf{e}_x$ and $\mathbf{e}_y$ are the unit Cartesian coordinate vectors.

This gives the equation for the boundary conditions in polar coordinates instantaneously centered at the origin of the body frame as

$$(\nabla\phi \cdot n) \mid_{r=R} = (\dot{x}\cos(\theta), -\dot{x}\sin(\theta)) \cdot n + (\dot{y}\sin(\theta), \dot{y}\cos(\theta)) \cdot n + (\dot{r}(t,\theta), 0) \cdot n. \tag{2.4}$$

Writing ϕ as $\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + O(\epsilon^3)$ where each ϕ_i is of the form (2.2), taking $\nabla\phi$ and equating the terms gives all the constants. The unit normal to the amoeba surface is simply given by the gradient of the surface. When computing the gradient in polar coordinates the denominator will contain an r. Since r will be on some form $r = r_0(1 + \epsilon\alpha(\theta))$ to describe the shape changes of the amoeba the gradient often needs to be expanded in a Taylor series requiring us to assume ϵ to be small. Therefore only small shape changes can be considered henceforth.

Equipped with all these functions finding $K(\phi)$ and then solving

$$M\dot{C}(t) = K(\phi) \tag{2.5}$$

for $(\dot{x}, \dot{y})$ is straightforward.

3. Locomotion of a Heterogeneous Amoeba

A conceptually easy way of making an amoeba able to produce locomotion according to the process described in Section 1.1 is to imagine a practically weightless shell with a heavy, movable mass inside it. For such an amoeba moving the heavy mass means moving the center of mass and the boundary shape can be changed separately. So the locomotion is divided into:

1. An shape changing only part of a supposedly very light amoeba shell interacting with the fluid thus producing extra inertial forces and
2. A center of mass (CM) displacement part by moving a heavy weight inside the shell as seen in Figure 3.1.

To prevent the shape change from contributing to the CM-displacement, the shape changes must be symmetric. For now, we also assume the large mass to be moved only in the x-direction.

Our wish is to find the best combination between the two extremes of ΔCM small and $\Delta shape$ big (long and thin shapes) and ΔCM big and $\Delta shape$ small (chubby, almost circular shapes). Unfortunately, the restriction to small deformations mentioned in Section 2.2 limits us to small $\Delta shape$ so our goal is to find the best possible motion for these.

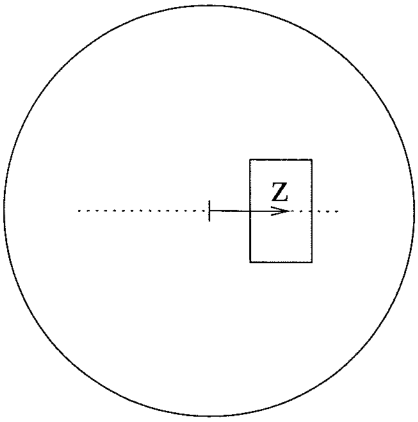

Figure 3.1 The undeformed heterogeneous amoeba. The amoeba origin is described in a fixed world frame. The coordinate l denotes the position of the inner mass in the direction it can move.

3.1 Equations of motion for the heterogeneous amoeba

The heterogeneous amoeba can be described by its radius function $r(t, \theta)$ with small deformations represented by ϵ

$$\begin{aligned} r &= r_0(1+\epsilon k_1\alpha(\theta)) \\ |\alpha(\theta)| &\leq 1 \\ |k_1| &< 1 \end{aligned} \tag{3.1}$$

A general symmetric, normalized function α can be represented as

$$\alpha(\theta) = \frac{\sum\limits_{even\ j} a_j \cos j\theta}{\left(\sum\limits_{even\ j} a_j^2\right)^{1/2}} \tag{3.2}$$

In Appendix C it is shown that, up to order ϵ, the function that gives the largest velocity is given by $a_j = 0$ for all $j > 2$. Of course, when actually constructing a swimming cylinder we would like to make it as simple as possible with few actuators and therefore we choose only the $\cos 2\theta$-oscillation with $a_j = 0$ for all $j > 2$. The normalization constraint on α thus gives $\alpha(\theta) = \cos 2\theta$. This corresponds with intuition: Extra little flutters apart from the original symmetric deformation around the edge of the amoeba should not change the inertial properties that much.

Considered by itself the deformation of (3.1) will yield $K(\phi) = m_{eff}(k_1)\dot{d} = 0$ with $d = (x, y)$ as described by [11]. We have zero resulting fluid impulse because of symmetry and m_{eff} is the effective mass of the amoeba shell. Consider only motion of the inner mass M in one direction with the displacement relative the amoeba origin denoted by l. Pick for example the mass displacement direction x as in Figure 3.1. Extra momentum comes from this displacement and the total momentum for the motion of the amoeba is in the x direction $m_{shell}\dot{x} + M(\dot{x} + \dot{l})$ where m_{shell} is the mass of the shell. As stated earlier, the ratio m_{shell}/M is assumed to be small, and so we can rewrite the amoeba momentum as approximately $M(\dot{x} + \dot{l})$. The momentum equation (2.5) must hold and so this must in turn be equal to the fluid impulse, $K_l = m_{eff}\dot{x}$.

Performing the computations as described in Section 2 with the amoeba surface normal in polar coordinates

$$\mathbf{n} = \begin{bmatrix} (1 - 2k_1^2 \sin{(2\theta)^2})\epsilon^2 + O(\epsilon^3) \\ 2k_1\epsilon \sin 2\theta + 2k_1^2\epsilon^2 \cos 2\theta \sin 2\theta + O(\epsilon^3) \end{bmatrix}$$

and solving (2.4) gives the potential ϕ:

$$\begin{aligned} \phi \;=\; & -r_0(\dot{x}\cos\theta + \dot{y}\sin\theta) \\ & +\epsilon r_0(-\frac{1}{2}r_0\dot{k}_1 \cos 2\theta + k_1(\frac{1}{2}(3\cos\theta - \cos 3\theta)\dot{x} - \frac{1}{2}(3\sin\theta + \sin 3\theta)\dot{y})) \\ & +\epsilon^2(-\frac{1}{8}k_1\dot{k}_1 r_0^2(\cos 4\theta - 4(1+\log r_0)) - \frac{1}{2}k_1^2 r_0((2\cos\theta + \cos 3\theta)\dot{x} + (2\sin\theta - \sin 3\theta)\dot{y})) \end{aligned}$$

The fluid impulse thus becomes

$$K(\phi) = \rho\pi r_0^2 \begin{bmatrix} -1 + 2\epsilon k_1 + O(\epsilon^3) & 0 \\ 0 & -1 - 2\epsilon k_1 + O(\epsilon^3) \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

From this expression it is easy to see what the effective mass matrix is and that it is diagonal.

The control parameters are chosen as k_1 and $k_2 = l/c_2$. c_2 normalizes the stretch M can move. Since the heavy mass cannot be moved out of the amoeba it follows that a careful physical constraint $c_2 < r_0(1 - \epsilon \max(k_1)\max(\alpha)) = r_0(1-\epsilon)$ applies. Insertion of the control parameters and setting the fluid impulse equal to the amoeba momentum yields (in the x-direction)

$$\rho\pi r_0^2(-1 + \epsilon 2k_1 - \epsilon^2 \frac{5k_1^2}{2})\dot{x} \approx M(\dot{x} + c_2\dot{k}_2)$$

Assuming the velocity in the x-direction can be written as a series expansion in ϵ as $\dot{x} = x_0 + \epsilon x_1 + \epsilon^2 x_2 + O(\epsilon^3)$ and solving gives

$$\begin{aligned} \dot{x} &= -\frac{c_2 M \dot{k}_2}{M + \rho\pi r_0^2} - \epsilon\frac{2\rho\pi r_0^2 c_2 M k_1 \dot{k}_2}{(M + \rho\pi r_0^2)^2} \qquad (3.3) \\ &\quad + \epsilon^2 c_2 M k_1^2 \dot{k}_2 \left[\frac{-4(\rho\pi r_0^2)^2}{(M + \rho\pi r_0^2)^3} + \frac{5\rho\pi r_0^2}{2(M + \rho\pi r_0^2)^2}\right] + O(\epsilon^3) \\ &= -c_2 F_0 \dot{k}_2 - \epsilon c_2 F_1 k_1 \dot{k}_2 + \epsilon^2 c_2 F_2 k_1^2 \dot{k}_2 + O(\epsilon^3) \qquad (3.4) \end{aligned}$$

where

$$F_0 = \frac{M}{M + \rho\pi r_0^2} \quad F_1 = \frac{2\rho\pi r_0^2 M}{(M + \rho\pi r_0^2)^2} \quad F_2 = M\left[\frac{-4(\rho\pi r_0^2)^2}{(M + \rho\pi r_0^2)^3} + \frac{5\rho\pi r_0^2}{2(M + \rho\pi r_0^2)^2}\right] \qquad (3.5)$$

Allowing the mass M to move in the y-direction instead and describing this movement analogously as $c_3 k_3(t)$, $k_2 \equiv 0$ gives (by symmetry) the velocity in the y-direction as

$$\dot{y} = -c_3 F_0 \dot{k}_3 + \epsilon c_3 F_1 k_1 \dot{k}_3 + \epsilon^2 c_3 F_2 k_1^2 \dot{k}_3 + O(\epsilon^3) \qquad (3.6)$$

3.2 Lie bracket motion for the heterogeneous amoeba

The Lie bracket motion is the infinitesimal motion that results from flowing around a square defined by two vector fields f and g in three dimensional space. First flow along f for ϵ seconds, then along g for ϵ seconds and then back along $-f$ and $-g$ for another ϵ seconds each. If we take the vector fields to be the control inputs the resulting Lie bracket motion will be the resulting motion of the amoeba after a completed input period.

To find the Lie bracket motion in the x-direction of the world coordinate system we define the system variable $q = (k_1, k_2, x)$. As control inputs we use the rate of change for the changeable parameters: $\dot{k}_1 = u_1$, $\dot{k}_2 = u_2$. The state equation becomes

$$\begin{aligned} \dot{q} &= \begin{bmatrix} \dot{k}_1 \\ \dot{k}_2 \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ -c_2 F_0 - \epsilon c_2 F_1 k_1 + \epsilon^2 c_2 F_2 k_1^2 \end{bmatrix} u_2 \\ &= g_1(q)u_1 + g_2(q)u_2 \end{aligned}$$

To find the motion we take the Lie bracket of the functions working on the input control signals (see Appendix A for a brief note on Lie algebra)

$$[g_1, g_2] = \left[\frac{\partial g_2}{\partial q} g_1(q) - \frac{\partial g_1}{\partial q} g_2(q)\right].$$

Multiplication by [0 0 1] then gives the Lie bracket resulting motion in the x-direction

$$g_x = -\epsilon \frac{2\rho\pi r_0^2 c_2 M}{(M+\rho\pi r_0^2)^2} + \epsilon^2 c_2 M k_1 \left[\frac{-8(\rho\pi r_0^2)^2}{(M+\rho\pi r_0^2)^3} + \frac{5\rho\pi r_0^2}{(M+\rho\pi r_0^2)^2}\right] + O(\epsilon^3)$$

The motion that can be achieved apparently depends on the possible choices of controller parameters. $k_1(t)$ must be periodic because the amoeba must return to its initial shape after a completed cycle and consequently the ϵ^2 term that contains k_1 will on average be a constant. (The special case of $k_1(t) = A \sin \omega t$ even makes the constant zero!) Similarly the Lie bracket motion in the y-direction is given by

$$g_y = +\epsilon \frac{2\rho\pi r_0^2 c_2 M}{(M+\rho\pi r_0^2)^2} + \epsilon^2 c_2 M k_1 \left[\frac{-8(\rho\pi r_0^2)^2}{(M+\rho\pi r_0^2)^3} + \frac{5\rho\pi r_0^2}{(M+\rho\pi r_0^2)^2}\right] + O(\epsilon^3)$$

3.3 Optimal control of the heterogeneous amoeba

Determining the usefulness of a heterogeneous amoeba as discussed above requires more detailed investigation of the motion that can be achieved. Naturally restrictions apply to the controller inputs and other properties of the amoeba. To do this we use the theory of optimal control. For a brief overview of the main results of optimal control, see Appendix B.

The key problems are to define what we mean by best motion and to determine the control inputs that give this best motion. As a performance measurement we choose the control effort that moves the amoeba a given distance in a given time. The optimal control inputs are thus the ones that minimize a minimum control effort performance index

$$J(0) = \frac{1}{2}\int_0^T u^T u dt = \frac{1}{2}\int_0^T (u_1^2 + u_2^2) dt$$

Assume that we can control the rate of change of the variables $u_1 = \dot{k}_1$ and $u_2 = \dot{k}_2$. For a start, to facilitate calculations, we only consider amoeba movement along the x-axis. The equation of motion is

$$\begin{bmatrix} \dot{k}_1 \\ \dot{k}_2 \\ \dot{x} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ (-c_2 F_0 - \epsilon c_2 F_1 k_1 + \epsilon^2 c_2 k_1^2 F_2) u_2 \end{bmatrix} \tag{3.7}$$

with F_0, F_1 and F_2 as in (3.5). Assume also that in the initial state $k_1(0) = k_2(0) = x(0) = y(0) = 0$. At some time T we require a final state $k_1(T) = k_1(0) = 0$, $k_2(T) = k_2(0) = 0$ and $x(T) = d$. The Hamiltonian function is defined by

$$\begin{aligned} H &= \frac{1}{2}(u_1^2 + u_2^2) + \lambda^T \begin{bmatrix} u_1 \\ u_2 \\ (-c_2F_0 - \epsilon c_2F_1k_1 + \epsilon^2 c_2k_1^2F_2)u_2 \end{bmatrix} \\ &= \frac{1}{2}(u_1^2 + u_2^2) + \lambda_1u_1 + \lambda_2u_2 + \lambda_3(-c_2F_0 - \epsilon c_2F_1k_1 + \epsilon^2 c_2k_1^2F_2)u_2 \end{aligned}$$

The costate equation $-\dot{\lambda} = \frac{\partial H}{\partial q}$ is

$$-\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{bmatrix} = \begin{bmatrix} (-\epsilon c_2F_1 + \epsilon^2 2c_2k_1F_2)u_2 \\ 0 \\ 0 \end{bmatrix}$$

From this it follows that λ_2 and λ_3 are constants and that

$$\lambda_1(t) = \lambda_1(T) + \lambda_3c_2\int_t^T (-\epsilon F_1 + \epsilon^2 2k_1(\tau)F_2)u_2(\tau)d\tau$$

The stationarity condition $\frac{\partial H}{\partial u} = 0$ is

$$\begin{bmatrix} u_1 + \lambda_1 \\ u_2 + \lambda_2 + \lambda_3c_2(-F_0 - \epsilon F_1k_1 + \epsilon^2k_1^2F_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus

$$u_2^{opt} = -\lambda_2 + \lambda_3c_2(F_0 + \epsilon F_1k_1 - \epsilon^2k_1^2F_2) + O(\epsilon^3)$$

$$u_1^{opt} = -\lambda_1(t) = -\lambda_1(T) - \lambda_3c_2\int_t^T (-\epsilon F_1 + \epsilon^2 2k_1(\tau)F_2)u_2^{opt}(\tau)d\tau + O(\epsilon^3)$$

$$\begin{aligned} \dot{k}_1 &= u_1 = -\lambda_1(T) + \epsilon c_2(T-t)(c_2\lambda_3^2F_0 - \lambda_2\lambda_3)F_1 \\ &\quad + \epsilon^2 c_2(c_2\lambda_3^2F_1^2 - (c_2\lambda_3^2F_0 - 2\lambda_2\lambda_3)F_2)\int_t^T k_1(\tau)d\tau \quad + O(\epsilon^3) \end{aligned}$$

The k_1^2 and k_1^3-terms that appear when we insert u_2^{opt} are all included in the $O(\epsilon^3)$ and therefore conveniently need not be considered. Introduce $K_1 = \int_t^T k_1(\tau)d\tau \Rightarrow k_1(t) = -\dot{K}_1(t)$ and this equation becomes

$$\ddot{K}_1 + K_1\epsilon^2c_2(c_2\lambda_3^2F_1^2 - 2(c_2\lambda_3^2F_0 - \lambda_2\lambda_3)F_2) = \lambda_1(T) - \epsilon c_2(T-t)(c_2\lambda_3^2F_0 - \lambda_2\lambda_3)F_1$$

which has the general solution

$$
\begin{aligned}
K_1 &= \frac{A\cos\left(\epsilon\sqrt{c_2(c_2\lambda_3^2F_1^2 - 2(c_2\lambda_3^2F_0 - \lambda_2\lambda_3)F_2)}t + \phi\right)}{\epsilon\sqrt{c_2(c_2\lambda_3^2F_1^2 - 2(c_2\lambda_3^2F_0 - \lambda_2\lambda_3)F_2)}} \\
&+ \frac{\lambda_1(T) - \epsilon c_2F_1(T-t)c_2(\lambda_3^2F_0 - \lambda_2\lambda_3)}{\epsilon^2c_2(c_2\lambda_3^2F_1^2 - 2(c_2\lambda_3^2F_0 - \lambda_2\lambda_3)F_2)}
\end{aligned}
$$

Consequently

$$
\begin{aligned}
k_1 &= A\sin\left(\epsilon\sqrt{c_2(c_2\lambda_3^2F_1^2 - 2(c_2\lambda_3^2F_0 - \lambda_2\lambda_3)F_2)}t + \phi\right) \\
&- c_2F_1\frac{c_2\lambda_3^2F_0 - \lambda_2\lambda_3}{\epsilon c_2(c_2\lambda_3^2F_1^2 - 2(c_2\lambda_3^2F_0 - \lambda_2\lambda_3)F_2)}
\end{aligned}
$$

The initial condition $k_1(0) = 0$ requires that

$$
A\sin(\phi) = -\frac{(c_2\lambda_3^2F_0 - \lambda_2\lambda_3)F_1}{\epsilon\sqrt{c_2(c_2\lambda_3^2F_1^2 - 2(c_2\lambda_3^2F_0 - \lambda_2\lambda_3)F_2)}}
$$

A solution to this equation (not necessarily the optimal solution but a valid one) is $\phi = 0$ and $c_2\lambda_3F_0 = \lambda_2$. The final condition $k_1(T) = 0$ requires that $\sin(c_2\epsilon\lambda_3F_1T) = 0$, i.e. $\lambda_3 = \frac{n\pi}{c_2\epsilon TF_1}$ for some integer n. The control inputs and shape parameters become (up to order ϵ^2)

$$
\begin{aligned}
k_1 &= A\sin\left(\frac{n\pi}{T}t\right) \\
u_1^{opt} &= \dot{k}_1 = A\frac{n\pi}{T}\cos\left(\frac{n\pi}{T}t\right) \\
u_2^{opt} &= c_2\lambda_3\left(\epsilon F_1A\sin\left(\frac{n\pi}{T}t\right) - \epsilon^2F_2A^2\sin\left(\frac{n\pi}{T}t\right)^2\right) \\
&= \frac{n\pi}{TF_1}\left(F_1A\sin\left(\frac{n\pi}{T}t\right) - \epsilon F_2A^2\sin\left(\frac{n\pi}{T}t\right)^2\right) \\
k_2 &= A\left(1 - \cos\left(\frac{n\pi}{T}t\right)\right) - \epsilon\frac{A^2F_2}{4F_1}\left(2\frac{n\pi}{T}t - \sin\left(2\frac{n\pi}{T}t\right)\right)
\end{aligned}
$$

Obviously k_2 and u_2 are dominated by the terms that contain no ϵ, particularly since F_2/F_1 usually is less than zero. On these grounds we ignore the ϵ terms and get

$$
\begin{aligned}
u_2^{opt} &\approx \frac{n\pi}{TF_1}\left(F_1A\sin\left(\frac{n\pi}{T}t\right)\right) \\
k_2 &\approx A\left(1 - \cos\left(\frac{n\pi}{T}t\right)\right)
\end{aligned}
$$

A trivial check shows that k_1 and k_2 fulfill the initial and final conditions for even n. Relaxing these conditions to $k_1(0) = k_1(T)$ and $k_2(0) = k_2(T)$ (which implies that it is not necessary to start from an undeformed shape) makes it possible to remove the constant from k_2 and yet satisfy the constraints for all n. Note that this does not change u_1 or u_2 in any way. The amoeba body origin velocity and position in the world coordinates is (using the above control inputs)

$$\begin{aligned}
\dot{x} &= -\frac{n\pi c_2 F_0}{T} A \sin\left(\frac{n\pi}{T}t\right) - \epsilon \frac{c_2 n\pi A^2}{T} F_1 \sin\left(\frac{n\pi}{T}t\right)^2 \\
x &= c_2 A F_0\left(\cos\left(\frac{n\pi}{T}t\right) - 1\right) - \epsilon \frac{c_2 A^2}{4} F_1\left(\frac{2n\pi}{T}t - \sin\left(\frac{2n\pi}{T}t\right)\right)
\end{aligned}$$

We now have to choose A to fulfill the final condition $x(T) = d$. For even n this means that A must satisfy

$$A^2 n = -\frac{2d}{\epsilon c_2 F_1 \pi} \ \forall \ even \ n \tag{3.8}$$

From this we see that the optimal control input amplitude contains a $\sqrt{\epsilon}$ in the denominator which means that the optimal k_i's can be quite large. But physical restrictions apply to the possible magnitude of the k_i's and so we have to fit d to physically acceptable control inputs. To prevent imaginary A either d or n must be chosen negative.

Replacing the $\dot{x}$ by $\dot{y}$ in (3.7) and performing the optimal control calculations over again with k_1 as before gives

$$k_3 \approx A\left(1 - \cos\left(\frac{n\pi}{T}t\right)\right)$$

with the same A as in (3.8). This means that we can make the amoeba move in any direction in a time T by applying sinusoidal functions where the surface oscillation is 90 degrees out of phase with the inner mass movement as inputs to our model.

$$\begin{aligned}
k_1 &= A \sin\left(\tfrac{n\pi}{T}t\right) & \dot{k}_1 &= A\tfrac{n\pi}{T} \cos\left(\tfrac{n\pi}{T}t\right) \\
k_2 &= A \cos\left(\tfrac{n\pi}{T}t\right) & \dot{k}_2 &= -A\tfrac{n\pi}{T} \sin\left(\tfrac{n\pi}{T}t\right) \\
k_3 &= A \cos\left(\tfrac{n\pi}{T}t\right) & \dot{k}_3 &= -A\tfrac{n\pi}{T} \sin\left(\tfrac{n\pi}{T}t\right)
\end{aligned}$$

3.4 Optimal control using numerical tool RIOTS

The optimal control problem can be numerically solved using RIOTS. It is possible to specify an optimal control problem with initial, final and trajectory constraints and a loss function, guess a control and have the optimal states and control inputs calculated. To use this tool we need to specify reasonable values for the problem constants. A plausible amoeba has

$$M = 1 \ kg \tag{3.9}$$

$$r_0 = 10 \ cm \tag{3.10}$$

$$\rho = 1 \ kg/l \tag{3.11}$$

$$\epsilon = 0.1 \tag{3.12}$$

$$c_2 = r_0(1 - \epsilon) = 9 \ cm \tag{3.13}$$

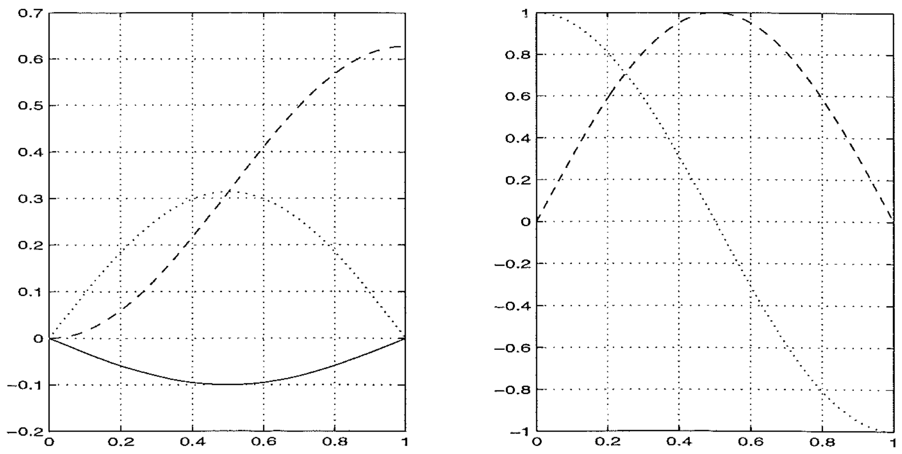

Figure 3.2 The guessed control states (left) and control inputs (right). x (full), k_1 and u_1 (dashed), k_2 and u_2 (dotted).

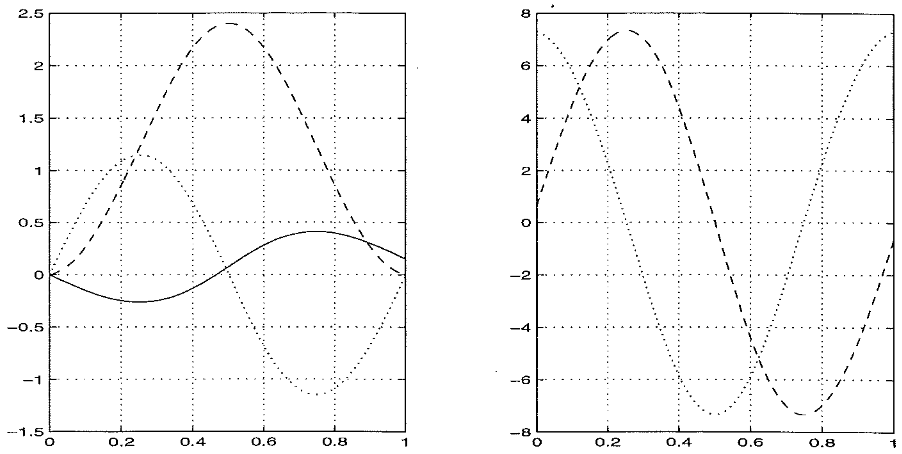

Figure 3.3 The optimal control states (left) and control inputs (right) for the unconstrained problem. x (full), k_1 and u_1 (dashed), k_2 and u_2 (dotted). $d = 0.15r_0$, $T = 1$, $\epsilon = 0.1$.

The rather arbitrarily guessed control inputs and the states corresponding to them are shown in Figure 3.2. During all the calculations the initial and final conditions are $k_1(0) = k_1(T) = k_2(0) = k_2(T) = x(0) = 0$ and $x(T) = d$. The loss function is chosen as in Section 3.3.

With no trajectory constraints and no constraints on the control inputs any displacement d is easy to achieve in one cycle. Figure 3.3 shows the states and control inputs when moving the amoeba $0.15r_0$ (i.e. $d = 0.15$) in one cycle. Closer investigation shows that the control inputs and thus the first two states are very close to sinusoidal, justifying our choice to neglect the ϵ^2 term of u_2^{opt} in Section 3.3. Even with a large ϵ the linear term has very little impact on the states and control inputs as can be seen in Figure 3.4.

The optimal control given by RIOTS obviously corresponds to the analytically found

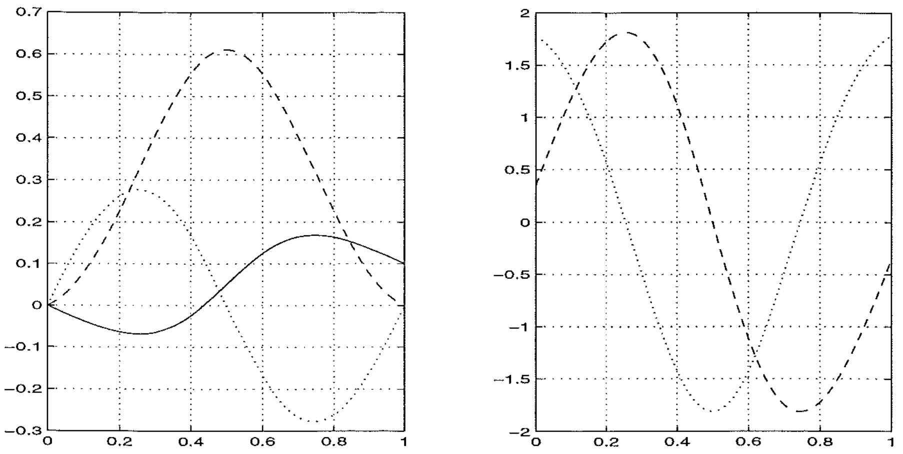

Figure 3.4 The optimal control states (left) and control inputs (right) for the unconstrained problem and a large ϵ. x (full), k_1 and u_1 (dashed), k_2 and u_2 (dotted). $\epsilon = 1$, $d = 0.10r_0$, $T = 1$.

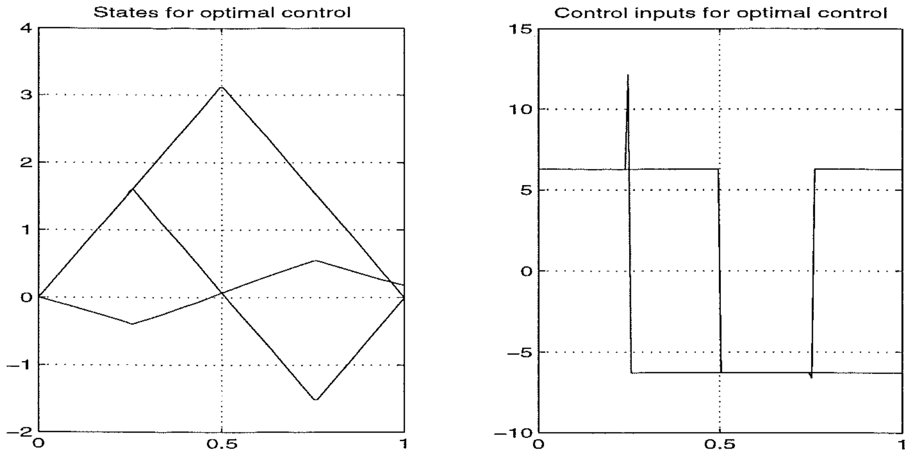

Figure 3.5 The optimal control states (left) and control inputs (right) for the problem with constraints on the control inputs u_i. $d = 0.20r_0$, $T = 1$, $\epsilon = 0.1$.

optimal control in Section 3.3 with $n = -2$. The amplitudes of the numerically found k_1 and k_2 are almost equal and identical to the analytically found amplitudes. k_1 can easily be given a zero mean by subtracting a constant. Since the control inputs are the derivatives of the first two states a constant will not affect the control inputs at all.

In real life constraints must be applied. As stated earlier any distance can be transversed in a single cycle if we allow infinite control inputs. The control inputs are in reality bounded by the constraints that the absolute value of the amplitude of k_1 and k_2 must be less than 1 and that we cannot change the their value infinitely fast. Applying the constraints on k_1 and k_2 immediately gives that the problem is impossible to solve for $d > 0.12r_0$. For larger d either the constraints must be violated or the condition $x(T) = d$ is not satisfied. For smaller d the optimal control is unchanged from the unconstrained problem.

Constraining only the control inputs u_1 and u_2 pushes k_1 and k_2 toward a triangular wave when d increases as seen in Figure 3.5. The control inputs change sign instantaneously which is of course practically impossible since it does not allow the amoeba mass and boundary to slow down before changing their direction.

4. Locomotion of a Homogeneous Amoeba

From Section 3 we conclude that locomotion can be achieved by moving the center of mass and changing the shape of the amoeba separately. But what if we could automatically move the center of mass while changing the shape only? Naturally this requires an amoeba with a different mass distribution than before and we now assume the amoeba to be homogeneous. As the amoeba shape changes the center of mass moves too. Intuitively, we feel it might be possible to produce more efficient motion this way.

4.1 Description of the homogeneous amoeba

To make a practically feasible amoeba we prefer it to have as few actuators as possible. Therefore we choose a restricted set of possible deformations, determined by the shape parameters $k_1(t)$ and $k_2(t)$ as follows: Fix again a frame in the body of the amoeba with the origin in the undeformed amoeba center. Let the radius of the amoeba in polar coordinates be described by

$$r(t,\theta) = R(t,\theta) = r_0[1 + \epsilon(k_1(t)\cos(n\theta) + k_2(t)\sin(m\theta))] \tag{4.1}$$

where m and n are integers determining how many sine waves will fit around the amoeba surface.

It follows from the Scallop Theorem [13] that a single mode of deformation (i.e. $k_1 \equiv 0$ or $k_2 \equiv 0$) cannot create a resulting translational motion. The reason is that the displacements induced by a forward motion are cancelled by the return motion. This is confirmed by computations showing that all physically possible shape parameters determining the single mode of deformation always bring the amoeba back to where it started.

A combination of two deformation modes as used above in (4.1) can give motion only in one direction, along a straight line. Obviously, the cross terms (involving both k_1 and k_2) create the motion. But in which direction and why? Understanding this means understanding the symmetry of the motion. Take for example $n = 2$ and $m = 3$, as in Figure 4.1. The oscillation determined by the shape parameter k_1 is symmetric around both axes and the oscillation determined by the shape parameter k_2 is symmetric around the y-axis. Intuitively, we feel that motion is impossible in the x-direction because the fluid is equally displaced to both sides of the y-axis at all times and so no resulting force should be generated.

More rigorously, conjecture that motion is possible along the positive x-axis when the combination of shapes of the amoeba is symmetric around the y-axis. Then reflexion of the shape in the y-axis (e.g. $x \to -x$) should give motion in the opposite direction, e.g. the negative x-axis. Clearly, this is not possible, because the shape is exactly the same in both cases and so cannot cause motion in opposite directions just because we

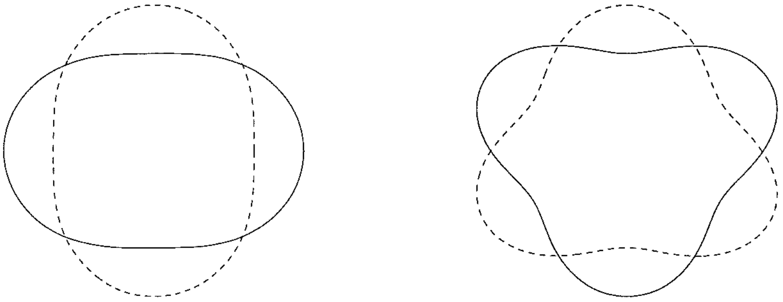

Figure 4.1 Exaggerated deformation shapes for the homogeneous amoeba with deformations $\epsilon \cos 2\theta$ (left) and $\epsilon \sin 3\theta$ (right).

fiddle with the mathematics. Therefore, the only possible motion is $\dot{x} = 0$. Appendix E contains the calculations in Section 4 redone with x replaced by $-x$ confirming this.

Addition of a $k_3(t)\cos(l\theta)$ term makes movement in any direction possible, but does not change the properties of the equations of motion in any significant way. It simply adds motion in a direction perpendicular to the first, for which the reasoning and equations of motion are analogous. Since the k_3-term gives no new information about the possible amount of motion, we ignore it to facilitate computation.

4.2 The equations of motion for the homogeneous amoeba

The origin of the body frame from which the radius is measured is not, in general, located at the geometric center (centroid C) of the amoeba as it wiggles forth. However, the center of mass always coincides with the centroid since the amoeba is assumed to be homogeneous. The geometric center has world frame coordinates

$$C(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \frac{1}{A(t)} \int_0^{2\pi} \int_0^{r(t,\theta)} \begin{bmatrix} r' \cos\theta \\ r' \sin\theta \end{bmatrix} r' \, dr' d\theta \tag{4.2}$$

if the body frame **origin** has world frame rectangular coordinates (x, y) and the area of the amoeba is $A(t)$. The area under the time-varying deformation is

$$A(t) = \pi r_0^2 [1 + \frac{1}{2}\epsilon^2 (k_1(t)^2 + k_2(t)^2)] \tag{4.3}$$

Consequently the body frame has velocity $(\dot{x}, \dot{y})$ in rectangular coordinates which gives the centroid the velocity:

$$\dot{C}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} + \frac{d}{dt} \left(\frac{1}{A(t)} \int_0^{2\pi} \int_0^{r(t,\theta)} \begin{bmatrix} r' \cos\theta \\ r' \sin\theta \end{bmatrix} r' \, dr' d\theta \right) \tag{4.4}$$

The unit normal to the surface, given by the gradient of the surface, is at any point (in these polar coordinates)

$$\mathbf{n} = \begin{bmatrix} 1 - \frac{1}{2}\epsilon^2(k_2 m \cos(m\theta) - k_1 n \sin(n\theta))^2 \\ \epsilon(k_1 n \sin(n\theta) - k_2 m \cos(m\theta)) + \epsilon^2(k_1 \cos(n\theta) + k_2 \sin(m\theta))(k_2 m \cos m\theta - k_1 n \sin n\theta) \end{bmatrix}$$

This gives the equation for the boundary conditions in polar coordinates instantaneously centered at the origin of the body frame as

$$(\nabla\phi \cdot n)\mid_R = (\dot{x}\cos\theta, -\dot{x}\sin\theta)\cdot n + (\dot{y}\sin\theta, \dot{y}\cos\theta)\cdot n + \epsilon r_0(\dot{k}_1\cos(n\theta) + \dot{k}_2\sin(n\theta), 0)\cdot n$$

Writing ϕ as $\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + O(\epsilon^3)$ where each ϕ_i is of the form (2.2), taking $\nabla\phi$ and equating the terms gives all the constants. A typical expression while doing these laborious calculations is

$$-\frac{m-n+1}{r^{m-n+2}}C_{m-n+1} = F(k_1, k_2, \dot{k}_1, \dot{k}_2, r_0, m, n)$$

From this we can see that for certain combinations of (m, n), in this case $m = n - 1$, the coefficients simply disappear. Let us for the time being consider only such (m, n) that make all expansion coefficients non-zero.

4.3 (m, n) chosen to make the ϕ coefficients non-zero

In the general case, being careful to choose (m, n) to avoid division by zero, the coefficients of the series expansion of ϕ to second order in ϵ become

$$\begin{aligned} \phi_0 &= -r_0^2\dot{x}\cos\theta/r - r_0^2\dot{y}\sin\theta/r \\ \phi_1 &= (a_0\log(r) + b_0)(c_0\theta + d_0) + \frac{1}{r^{n-1}}(c_{n-1}\sin((n-1)\theta) + d_{n-1}\cos((n-1)\theta)) \\ &\quad + \frac{1}{r^n}(c_n\sin(n\theta) + d_n\cos(n\theta)) + \frac{1}{r^{n+1}}(c_{n+1}\sin((n+1)\theta) + d_{n+1}\cos((n+1)\theta)) \\ &\quad + \frac{1}{r^{m-1}}(c_{m-1}\sin((m-1)\theta) + d_{m-1}\cos((m-1)\theta)) + \frac{1}{r^m}(c_m\sin(m\theta) \\ &\quad + d_m\cos(m\theta)) + \frac{1}{r^{m+1}}(c_{m+1}\sin((m+1)\theta) + d_{m+1}\cos((m+1)\theta)) \end{aligned}$$

$$
\begin{aligned}
\phi_2 \;=\;& (e_0 \log(r) + f_0)(g_0\theta + h_0) + \frac{1}{r}(g_1 \sin\theta + h_1 \cos\theta) \\
&+ \frac{1}{r^{2n-1}}(h_{2n-1}\sin((2n-1)\theta) + g_{2n-1}\cos((2n-1)\theta)) \\
&+ \frac{1}{r^{2n}}(h_{2n}\sin(2n\theta) + g_{2n}\cos(2n\theta)) \\
&+ \frac{1}{r^{2n+1}}(h_{2n+1}\sin((2n+1)\theta) + g_{2n+1}\cos((2n+1)\theta)) \\
&+ \frac{1}{r^{2m-1}}(h_{2m-1}\sin((2m-1)\theta) + g_{2m-1}\cos((2m-1)\theta)) \\
&+ \frac{1}{r^{2m}}(h_{2m}\sin(2m\theta) + g_{2m}\cos(2m\theta)) \\
&+ \frac{1}{r^{2m+1}}(h_{2m+1}\sin((2m+1)\theta) + g_{2m+1}\cos((2m+1)\theta)) \\
&+ \frac{1}{r^{m+n+1}}(h_{m+n+1}\sin((m+n+1)\theta) + g_{m+n+1}\cos((m+n+1)\theta)) \\
&+ \frac{1}{r^{m-n+1}}(h_{m-n+1}\sin((m-n+1)\theta) + g_{m-n+1}\cos((m-n+1)\theta)) \\
&+ \frac{1}{r^{m+n-1}}(h_{m+n-1}\sin((m+n-1)\theta) + g_{m+n-1}\cos((m+n-1)\theta)) \\
&+ \frac{1}{r^{m-n-1}}(h_{m-n-1}\sin((m-n-1)\theta) + g_{m-n-1}\cos((m-n-1)\theta)) \\
&+ \frac{1}{r^{m+n}}(h_{m+n}\sin((m+n)\theta) + g_{m+n}\cos((m+n)\theta)) \\
&+ \frac{1}{r^{m-n}}(h_{m-n}\sin((m-n)\theta) + g_{m-n}\cos((m-n)\theta))
\end{aligned}
$$

The expansion coefficients of the ϕ's can be found in Appendix D. In the case of singularities, as mentioned above, the coefficients containing the singularities are simply zero, i.e. $n = 1$ gives $c_{n-1} = 0$ and $d_{n-1} = 0$.

Insertion of $r(t, \theta)$ in (4.2) and (4.4) for the case of **general** (m, n) shows that the centroid C has the world frame coordinates

$$C(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \tag{4.5}$$

and that the centroid velocity consequently is (in world frame coordinates)

$$\dot{C}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix}$$

Thus, the centroid C coincides with the body frame origin (x, y) in the general case. Intuitively, we feel that motion of the amoeba in the fluid is difficult to create if the centroid never moves from the body frame origin. The following verifies this reasoning.

PROPOSITION 4.1

For a homogeneous amoeba described by (4.1), call the center of mass of the amoeba C and the coordinates of the body frame origin of the amoeba in fixed world coordinates

(x, y). If

$$C(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad \forall\, t > 0$$

i.e. the center of mass always coincides with the body frame origin then the velocity of the center of mass $\dot{C}(t)$ will be

$$\dot{C}(t) \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \forall\, t > 0$$

up to order ϵ^2 and no locomotion of the amoeba will result. □

Proof: To find the velocity of the centroid (described by (4.4)) the velocity of the body frame origin $(\dot{x}, \dot{y})$ must be found. This can be done by equating the momentum of the fluid to the momentum of the amoeba as described in Section 2.2 with $K(\phi)$ from (2.1) where the normal vector in Cartesian coordinates is

$$\begin{bmatrix} n_x \\ n_y \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \epsilon \begin{bmatrix} \sin\theta\ (k_2 m \cos(m\theta) - k_1 n \sin(n\theta)) \\ \cos\theta\ (-k_2 m \cos(m\theta) + k_1 n \sin(n\theta)) \end{bmatrix}$$

$$-\tfrac{1}{2}\epsilon^2 [k_2 m \cos(m\theta) - k_1 n \sin(n\theta)] *$$

$$* \begin{bmatrix} 2\sin\theta(k_1 \cos(n\theta) + k_2 \sin(m\theta)) + \cos\theta\ (k_2 m \cos(m\theta) - k_1 n \sin(n\theta)) \\ -2\cos\theta(k_1 \cos(n\theta) + k_2 \sin(m\theta)) + \sin\theta\ (k_2 m \cos(m\theta) - k_1 n \sin(n\theta)) \end{bmatrix} + O(\epsilon^3)$$

The momentum equation (2.5) then becomes (with M the mass of the amoeba)

$$M \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \rho\pi r_0^2 \begin{bmatrix} -\dot{x} + \epsilon(\dot{x} f_1 + \dot{y} f_2) + \epsilon^2(\dot{x} g_1 + \dot{y} g_2) \\ -\dot{y} + \epsilon(\dot{x} f_3 + \dot{y} f_4) + \epsilon^2(\dot{x} g_3 + \dot{y} g_4) \end{bmatrix} \tag{4.6}$$

where f_i and g_i are functions of $(k_1, k_2, \dot{k}_1, \dot{k}_2)$.

Assuming that $\dot{x}$ can be written as a series expansion in ϵ as $\dot{x} = x_0 + \epsilon x_1 + \epsilon^2 x_2 + O(\epsilon^3)$ and $\dot{y}$ equivalently inspection of equation (4.6) yields that all the terms of the solution vanish and

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Subsequently, the centroid velocity becomes

$$\dot{C}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + O(\epsilon^3)$$

□

4.4 Lie bracket motion for the homogeneous amoeba in the y-direction

The discussion in the beginning of this chapter told us that motion in the x-direction would be impossible with the considered surface deformations, so we need only investigate $\dot{C}_y$ further. To find the Lie bracket motion in the y-direction we define the system variable $q = (k_1, k_2, C_y)$. The state equation becomes

$$\begin{aligned} \dot{q} &= \begin{bmatrix} \dot{k}_1 \\ \dot{k}_2 \\ \dot{C}_y \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_2 & (4.7) \\ \dot{q} &= g_1(q)u_1 + g_2(q)u_2 & (4.8) \end{aligned}$$

Just as in Section 3.2 we take the Lie-bracket of the functions working on the input control signals

$$[g_1, g_2] = \left[\frac{\partial g_2}{\partial q} g_1(q) - \frac{\partial g_1}{\partial q} g_2(q)\right]$$

Multiplication by [0 0 1] gives the motion in the y-direction. The (m, n) that maximize this motion are the best ones to use. Obviously, when (m, n) are chosen to make the coefficients non-zero, no motion can ever be generated.

4.5 Amoeba Velocity When ϕ-Coefficients Are Zero

The computations from Section 4.1 are repeated for all possible singular cases (which in turn contain singular cases) with $m, n > 0$. They are messy but uncomplicated. Only a few combinations of (m, n) yield any motion at all and are presented in Table 4.1 with

$$F = \frac{2\rho\pi r_0^2}{\rho\pi r_0^2 + M}$$

As stated earlier the symmetry discussion in Section 4.1 tells us that motion in the x-direction is absolutely zero up to any order epsilon.

The combinations of (m, n) that yield motion at all have in common that the center of mass computed from (4.2) does not coincide with the body frame origin (a necessary condition, see Proposition 1). As a typical example

$$r(t, \theta) = r_0(1 + \epsilon(k_1(t)\cos 2\theta + k_2(t)\sin 3\theta + k_3(t)\cos 3\theta)) \tag{4.9}$$

has the center of mass

$$C(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \epsilon^2 r_0 \begin{bmatrix} k_1(t)k_3(t) \\ k_1(t)k_2(t) \end{bmatrix} + O(\epsilon^3)$$

	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
$n=1$	0	0	0	0	0	0
$n=2$	$\epsilon^2 r_0 F$	0	$-\epsilon^2 r_0 F$	0	0	$\ddots$
$n=3$	0	0	0	$-\epsilon^2 r_0 F$	0	$\ddots$
$n=4$	0	0	0	0	$-\epsilon^2 r_0 F$	$\ddots$
$n=5$	0	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$

Table 4.1 Lie bracket movement in the y-direction for different combinations of (m,n). All expressions are up to order ϵ^3 only. All Lie bracket movement in the x-direction is absolutely zero.

and center of mass velocity

$$\dot{C}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} + \epsilon^2 r_0 \begin{bmatrix} k_1(t)\dot{k}_3(t) + \dot{k}_1(t)k_3(t) \\ k_1(t)\dot{k}_2(t) + \dot{k}_1(t)k_2(t) \end{bmatrix} + O(\epsilon^3)$$

Working through the calculations to find $\dot{x}$ and inserting gives a centroid velocity of

$$\dot{C}(t) = \epsilon^2 r_0 F \begin{bmatrix} \dot{k}_1(t)k_3(t) \\ \dot{k}_1(t)k_2(t) \end{bmatrix}$$

Obviously, all combinations producing locomotion give motion of order ϵ^2. Any practical homogeneous device we could build would have to oscillate enormously fast to be able to move at even a moderate speed. Clearly we need to consider other amoeba shapes that might give a higher motion to rate of oscillation ratio. Unfortunately the computations become incredibly complicated very fast when we try shapes that are not a simple combination of sine waves. Although any shape can be expanded in a Fourier Series consisting of sine waves it is normally very long and might even be infinite. This makes the potential (and thus the boundary condition Equation (2.4)) very complex and intractable to compute.

Constant area oscillations do not improve facilitate the computations, the major difficulties come in when the gradient is taken (for computation of the surface normal and the boundary condition equation) as this contains division by r. Switching to Cartesian coordinates and considering square amoeba shapes does no better. The division by r is avoided but Laplace's Equation with Neumann boundary conditions (i.e. $\frac{\partial u}{\partial n} = g(x)$) has no simple solution.

4.6 Optimal control of the homogeneous amoeba

Evaluation of this locomotion means repeating the optimal control calculations from Section 3.2 with the same definition of "optimal" control, i.e. minimizing the minimum control effort performance index

$$J(0) = \frac{1}{2}\int_0^T u^T u dt = \frac{1}{2}\int_0^T (u_1^2 + u_2^2)dt$$

Pick the typical amoeba described in (4.9) above. These calculations are performed in [8] and repeated here for the convenience of the reader. Assume that we can control the rate of change of the shape variables $u_1 = \dot{k}_1$ and $u_2 = \dot{k}_2$. For a start, to facilitate calculations, we only consider amoeba movement along the y-axis. Thus the equation of motion is

$$\begin{bmatrix} \dot{k}_1 \\ \dot{k}_2 \\ \dot{C}_y \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \epsilon^2 r_0 F k_2 u_1 \end{bmatrix}$$

Assume also that in the initial state $k_1(0) = k_2(0) = C_x(0) = 0$. At some time T we require a final state $k_1(T) = k_2(T) = 0$ and $C_x(T) = d$. The Hamiltonian function is defined by

$$\begin{aligned} H &= \frac{1}{2}(u_1^2 + u_2^2) + \lambda^T \begin{bmatrix} u_1 \\ u_2 \\ \epsilon^2 r_0 F u_1 \end{bmatrix} \\ &= \frac{1}{2}(u_1^2 + u_2^2) + \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3(\epsilon^2 r_0 F u_1) \end{aligned}$$

The costate equation $-\dot{\lambda} = \frac{\partial H}{\partial q}$ is

$$-\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda_3 \epsilon^2 r_0 F u_1 \\ 0 \end{bmatrix}$$

From this we see that λ_1 and λ_3 are constants, while

$$\lambda_2(t) = \lambda_2(T) + \lambda_3 F \epsilon^2 r_0 \int_t^T u_1(\tau)d\tau$$

The stationarity condition $\frac{\partial H}{\partial u} = 0$ is

$$\begin{bmatrix} u_1 + \lambda_1 + \lambda_3 \epsilon^2 r_0 F k_2 \\ u_2 + \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus

$$\begin{aligned} u_1^{opt} &= -\lambda_1 - \lambda_3 \epsilon^2 r_0 F k_2 + O(\epsilon^3) \\ u_2^{opt} &= -\lambda_2(t) = -\lambda_2(T) + \lambda_3 \epsilon^2 r_0 F \int_t^T \lambda_1 + \lambda_3 \epsilon^2 r_0 F k_2(\tau)d\tau + O(\epsilon^3) \\ \dot{k}_2 &= u_2 = -\lambda_2(T) + \epsilon^2 r_0 (T-t)\lambda_1 \lambda_3 F + \epsilon^4 \lambda_3^2 r_0^2 F^2 \int_t^T k_2(\tau)d\tau \end{aligned}$$

Introduce the variable $K_2 = \int_t^T k_2(\tau)d\tau \Rightarrow k_2(t) = -\dot{K}_2(t)$ and this equation becomes

$$\ddot{K}_2 + K_2\epsilon^4\lambda_3^2 r_0^2 F^2 = \lambda_2(T) - \epsilon r_0(T-t)\lambda_1\lambda_3 F$$

which gives

$$k_2 = A\sin(\epsilon^2\lambda_3 r_0 F t + \phi) - \frac{\lambda_1}{\lambda_3\epsilon^2 r_0 F}$$

The initial condition $k_2(0) = 0$ requires that

$$A\sin(\phi) = \frac{\lambda_1}{\lambda_3\epsilon^2 r_0 F}$$

A solution to this equation is $\phi = 0$ and $\lambda_1 = 0$. The requirement that $k_2(T) = 0$ gives that $\sin(\lambda_3\epsilon^2 r_0 F T) = 0$, or$\lambda_3 = \frac{n\pi}{\epsilon^2 r_0 F T}$ for some integer n. The control inputs and shape parameters become (up to order ϵ^2)

$$\begin{aligned}
k_2 &= A\sin(\frac{n\pi}{T}t) \\
u_2^{opt} &= \dot{k}_2 = A\frac{n\pi}{T}\sin(\frac{n\pi}{T}t) \\
u_1^{opt} &= -A\frac{n\pi}{T}\sin(\frac{n\pi}{T}t) \\
k_1 &= A(-1+\cos(\frac{n\pi}{T}t))
\end{aligned}$$

The amoeba center of mass velocity and position (in world coordinates) are

$$\begin{aligned}
\dot{C}_y &= -\epsilon^2 r_0 F A^2\frac{n\pi}{T}\sin(\frac{n\pi}{T}t)^2 \\
C_y &= -\epsilon^2 r_0 F A^2\frac{n\pi}{T}\left[\frac{t}{2} - \frac{\sin(2n\pi/T)t}{4n\pi/T}\right]
\end{aligned}$$

To make sure that C_y fulfills the condition $C_y(T) = d$ it follows that

$$A^2 n = -\frac{2d}{\epsilon^2 r_0 F\pi}\ \forall\ even\ n$$

From this expression we find that the optimal control input amplitude contains a ϵ in the denominator which means that the optimal k_i's will be quite large, even bigger than for the heterogeneous amoeba. Physical restrictions still apply to the possible magnitude of the k_i's and so we have to fit d to physically acceptable control inputs. To avoid imaginary amplitudes either d or n should be chosen negative.

Allowing a third oscillation term k_3 causes motion in the x-direction by symmetry with k_1 as before. By relaxing the constraints on k_1 to $k_1(0) = k_1(t) \neq 0$ the constant is removed and k_1 is a pure sinusoid. Note that u_1 is not affected by this. So we can make the amoeba move in any direction by applying sinusoidal functions where the surface oscillations are 90 degrees out of phase with each other.

Since the analytical expressions are so straightforward no numerical optimal control calculations in RIOTS were performed for the homogeneous amoeba.

5. Comparison Between Two Types of Amoeba

5.1 Which amoeba gives the best Lie bracket motion?

Comparison of the Lie bracket motion for the homogeneous amoeba and the heterogeneous amoeba is usually favorable for the heterogeneous one. The best possible Lie bracket motion for the homogeneous amoeba is (see Section 4.3)

$$g_{hom} = -\epsilon^2 r_0 \frac{2\rho\pi r_0^2}{M + \rho\pi r_0^2} + O(\epsilon^3)$$

For the heterogeneous amoeba the corresponding expression is (see Section 3.2)

$$g_{het} = -\epsilon \frac{2\rho\pi r_0^2 c_2 M}{(M + \rho\pi r_0^2)^2} + \epsilon^2 c_2 M k_1 \frac{5M\rho\pi r_0^2 - 3\rho^2\pi^2 r_0^4}{(M + \rho\pi r_0^2)^3} + O(\epsilon^3)$$

The reason why g_{hom} only contains terms of second order in ϵ and higher and g_{het} contains an ϵ-term is that the second control input does not have to be small in order for the calculations to be valid. Neglecting the ϵ^2-terms after insertion of $c_2 = r_0 - \epsilon k_1^{max}$ in g_{het} we find that the heterogeneous amoeba motion exceeds the homogeneous amoeba motion when

$$\epsilon < \frac{M}{M + \rho\pi r_0^2} < 1$$

Clearly, if the amoeba density is very small compared to the fluid density ($M \ll \rho\pi r_0^2$) $\frac{M}{M+\rho\pi r_0^2}$ is very small, even smaller than ϵ and the homogeneous motion exceeds the heterogeneous motion. So when the amoeba is very light compared to the fluid the homogeneous construction works. Otherwise the heterogeneous amoeba should give the best results. The ϵ-component of g_{het} has its largest value for $M = \rho\pi r_0^2$. This is verified by numerical calculation in MATLAB investigating the value of g_{het} for different combinations of M and r_0.

5.2 Which amoeba gives the highest average velocity?

Assume the inputs to the homogeneous amoeba to be

$$\begin{aligned} k_1(t) &= \cos(\omega t) \\ k_2(t) &= -\sin(\omega t) \\ k_3(t) &= -\sin(\omega t) \end{aligned}$$

The centroid velocity becomes

$$\dot{C}(t) = \epsilon^2 r_0 \frac{2\rho\pi r_0^2}{\rho\pi r_0^2 + M} \begin{bmatrix} \omega \sin(\omega t)^2 \\ \omega \sin(\omega t)^2 \end{bmatrix} + O(\epsilon^3) \tag{5.1}$$

Thus the average velocity is on the order of $\epsilon^2 \omega r_0$. In other words, each shape change cycle moves the amoeba a distance on the order of $\epsilon^2 r_0$.

Assume the inputs to the heterogeneous amoeba to be

$$\begin{aligned} k_1(t) &= \sin(\omega t) \\ k_2(t) &= -\cos(\omega t) \\ k_3(t) &= -\cos(\omega t) \end{aligned}$$

This gives the velocity

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} -c_2 F_0 \omega \sin(\omega t) - \epsilon c_2 F_1 \omega \sin(\omega t)^2 + \epsilon^2 c_2 F_2 \omega \sin(\omega t)^3 \\ -c_3 F_0 \omega \sin(\omega t) + \epsilon c_3 F_1 \omega \sin(\omega t)^2 + \epsilon^2 c_3 F_2 \omega \sin(\omega t)^3 \end{bmatrix} \tag{5.2}$$

and thus the average velocity of the heterogeneous amoeba is on the order of $\epsilon c_2 \omega \approx \epsilon r_0 \omega$. This is of the order $1/\epsilon$ more motion than for the homogeneous amoeba. This means that for $\epsilon = 0.1$, 100 oscillations are needed to move the homogeneous amoeba one body radius while only 10 oscillations are required to move the heterogeneous amoeba the same distance.

6. Conclusions

6.1 Results

The objective of this thesis is to investigate the possibility of motion for an infinitely long cylinder in an inviscid fluid. Only two dimensional motion perpendicular to the cylinder axis is considered. This means that it is sufficient to consider a cross section of the cylinder. Locomotion is created by changing the shape and center of mass of the amoeba. By exploiting the fluid inertial properties this creates a propelling motion. Two methods of locomotion are investigated: a heterogeneous amoeba consisting of a light shell and a heavy mass movable independently of each other and a homogeneous amoeba where deformation of the amoeba surface changes both the shape and the center of mass. Only small surface deformations are allowed for our calculations to be valid.

The heterogeneous amoeba can normally move faster because the enclosed mass can be shifted much further than the surface can be deformed. The Lie bracket motion of the heterogeneous amoeba exceeds the Lie bracket motion of the homogeneous amoeba when $\epsilon < M/(M + \rho\pi r_0^2)$. Clearly a homogeneous amoeba is better if the amoeba density is very small compared to the fluid density. Maximum Lie bracket motion of the heterogeneous amoeba occurs when the density of the amoeba equals the density of the fluid and is on the order of ϵr_0 to be compared to the maximum Lie bracket motion of the homogeneous amoeba which is on the order of $\epsilon^2 r_0$.

Considering the average velocities of the two amoeba types gives similar results. The average motion of the heterogeneous amoeba is on the order of $\epsilon^2 r_0$ and the average motion of the homogeneous amoeba is on the order of ϵr_0, i.e. a factor $1/\epsilon$ larger. This means that for $\epsilon = 0.1$, 100 oscillations are needed to move the homogeneous amoeba one body radius while only 10 oscillations are required to move the heterogeneous amoeba the same distance.

6.2 Future Work

Before a vehicle operating like a swimming cylinder can be built there is much more work to be done. This thesis merely scratches the surface of the subject. The most immediate thing to investigate seems to be ways of turning and steering. For the case of the heterogeneous amoeba two weights would be necessary for moving in the plane but this would violate the condition of the weightless amoeba shell. The possible coupling between two weights moving simultaneously has not been investigated and may give rise to complications. Perhaps a way of turning could be devised so one weight would be enough. An option worth looking into might be to place one weight on a circular rail inside the amoeba. Introducing vorticity into the system makes the problem more complicated by removing the irrotationality of the fluid. Still it may be possible to take advantage of this for example by attaching a fin to the back of the amoeba for steering purposes.

In all the previous discussion of the heterogeneous amoeba it has been assumed that the mass can move only as far as the amoeba allows when at its smallest. It would be interesting to apply a more generous constraint on the weight trajectory so it could move all the way to the boundary of the amoeba when it is expanded. The optimal control analysis with this constraint unfortunately seems to be complicated. It would also be interesting to perform an optimal control analysis where “optimal” denotes maximum traversed distance in one motion cycle.

Yet another topic for further studies is investigation of amoeba configurations that make it easy to control the trajectory the amoeba follows. All we know so far is how to make the amoeba move from one point to another but the trajectory is roundabout. At times this might be inconvenient and so a more direct way of controlling the amoeba path is desirable.

A. Some Results of Lie Algebra

A very brief note on the specialized mathematics used is provided for greater understanding of the calculations.

The *Lie bracket* is the infinitesimal motion that results from flowing around a square defined by two vector fields f and g in three-dimensional space. First the flow along f for ϵ seconds, then along g for ϵ seconds and then back along $-f$ and $-g$ for another ϵ seconds each, see Figure A.1. It is defined as

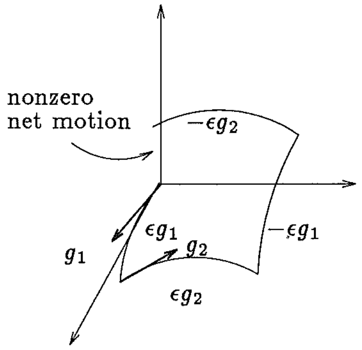

Figure A.1 Lie bracket motion. (Figure courtesy of [10])

$$[f,g](q) = \frac{\partial g}{\partial q} f(q) - \frac{\partial f}{\partial q} g(q)$$

The Lie bracket is used here to determine how far the amoeba will move with the specified control inputs, where f is the flow of one control input and g is the flow of the other. The variable equation is

$$\dot{\mathbf{q}} = \mathbf{f}(\mathbf{q})u_1 + \mathbf{g}(\mathbf{q})u_2$$

Typically the amoeba movement in the investigated direction is included in $\mathbf{q}$ and the shape variables k make up the rest of it. The control inputs are taken to be the rate of change of the shape variable.

B. Some Results of Optimal Control

For the reader unfamiliar with the concept of optimal control this section is recommended for comprehension of the thesis. It contains a brief summary of the most important theorems in optimal control used here. There is no shortage of textbooks on this subject, and for example [6] can be recommended for further study.

Preliminaries

A few remarks on the calculus of variations are necessary for attaining the Optimal Control theory results. Define the increment $\delta x(t)$ as the change in $x(t)$ when t is fixed. Then (with T the final time)

$$dx(T) = \delta x(T) + \dot{x}(T)dT \tag{B.1}$$

Leibniz's Rule for functionals says that if

$$J(x) = \int_{t_0}^{T} h(x(t), t)dt$$

where J and h are real scalar functions and $h_x = \frac{\delta h}{\delta x}$ it holds that

$$dJ = h(x(T), T)dT - h(x(t_0), t_0)dt_0 + \int_{t_0}^{T} [h_x^T(x(t), t)\delta x]dt$$

We will use these results in the next section.

Problem description

The system to be controlled is described by the nonlinear and time variant equation

$$\dot{x}(t) = f(x, u, t) \quad t \geq t_0 \tag{B.2}$$

where t_0 is a fixed starting time, T a fixed final time and $x(t_0)$ and $x(T)$ are the initial states and final states respectively. A physically relevant control performance index on the form

$$J(t_0) = \phi(x(T), T) + \int_{t_0}^{T} L(x(t), u(t), t)dt$$

is applied with ϕ a final weighting function depending on the final condition and the final time and L a weighting function for the states and control inputs during the time the optimization is performed. The physical relevance can mean that for example time, fuel or energy is minimized.

The optimal control problem is to find system inputs $u^*(t)$ during $[t_0, T]$ that drive the system along a trajectory $x^*(t)$ such that $J(t_0)$ is minimized and that a final state constraint

$$\psi(x(T), T) = 0 \tag{B.3}$$

for a given function ψ is satisfied.

Finding the optimal control

Solving this problem requires Lagrange multiplier theory. Introduce the associated multiplier $\lambda(t)$ to (B.2) and ν to (B.3). The augmented performance index becomes

$$J'(t) = \phi(x(T), T) + \nu^T \psi(x(T), T) + \int_{t_0}^{T} [L(x, u, t) + \lambda^T(t)(f(x, u, t) - \dot{x})]dt$$

Insertion of the Hamiltonian function

$$H(x, u, t) = L(x, u, t) + \lambda^T f(x, u, t)$$

associated with this gives

$$J'(t) = \phi(x(T), T) + \nu^T \psi(x(T), T) + \int_{t_0}^{T} [H(x, u, t) - \lambda^T \dot{x}]dt$$

Lagrange theory now says that the constrained minimum of J is attained at the unconstrained minimum of J', where $dJ' = 0$ for all independent increments δ in its arguments. Applying (B.1) to J' we get

$$\begin{aligned} dJ' &= (\phi_x + \psi_x^T \nu - \lambda)^T dx \,|_T + (\phi_t + \psi_t^T \nu + H - \lambda^T \dot{x} + \lambda^T \dot{x}) d \,|_T + \phi^T \,|_T \, d\nu \\ &\quad - (H - \lambda^T \dot{x} + \lambda^T \dot{x}) dt \,|_{t_0} + \lambda^T dx \,|_{t_0} + \int_{t_0}^{T} [(H_x + \dot{\lambda})^T \delta x + H_u^T \delta u + (H_\lambda - \dot{x})^T \delta\lambda] dt \end{aligned}$$

t_0 is fixed and so is $x(t_0)$ meaning that $dt_0 = 0$ and $dx(t_0) = 0$. Equating all other coefficients of increments to zero one at a time yields the
State equation

$$\dot{x} = \frac{\partial H}{\partial \lambda} = f \quad t \geq t_0$$

Costate equation

$$-\dot{\lambda} = \frac{\partial H}{\partial x} = \frac{\partial f^T}{\partial x}\lambda + \frac{\partial L}{\partial x} \quad t \leq T$$

Stationarity condition

$$0 = \frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \frac{\partial f^T}{\partial u}\lambda$$

Boundary condition

$$(\phi_x + \psi_x^T \nu - \lambda)^T \mid_T dx(T) + (\phi_t + \psi_t^T \nu + H) \mid_T dT = 0$$

and of course $\psi(T) = 0$ according to the problem description. Straightforward equation solving now gives us the optimal control. Notice that the costate equation is a dynamic equation evolving backwards in time. We also see that finding λ is a necessary step on the way to finding the optimal input $u^*(t)$.

C. Heterogeneous Amoeba: Higher Order Symmetric Boundary Oscillations

Proposition C.1

For a heterogeneous amoeba described by (3.1) and with the general symmetric

$$\alpha(\theta) = \frac{\sum_{even\ i} a_i \cos i\theta}{\left(\sum_{even\ i} a_i^2\right)^{1/2}}.$$

the higher modes ($n > 2$) of the boundary oscillation do not contribute to the Lie bracket motion up to order ϵ. □

Consequently we need only use the boundary oscillation with $n = 2$ when constructing the amoeba.

Proof: Dominant impact means that it affects terms of low order in ϵ. Consequently we have to show that the ϵ- component of $K(\phi)$ has a maximum for $a_i = 0$ for all $i \neq 2$ and that the ϵ-component of the Lie bracket has this as well follows.

Doing all the computations from Section 2.1 up to order ϵ to calculate $K(\phi)$ in the x-direction from (2.1) gives

$$\begin{aligned}
\phi\,|_{r=R} &= -r_0(\dot{x}\cos\theta + \dot{y}\sin\theta) \\
&+ \epsilon\left(\frac{k_1 r_0}{\left(\sum_{even\ i} a_i^2\right)^{\frac{1}{2}}}(\dot{x}\cos\theta + \dot{y}\sin\theta)\sum_{even\ i} a_i\cos i\theta + \sum_{i=1}^{n}\frac{i d_i}{r_0^i}\cos i\theta + \sum_{i=1}^{n}\frac{i c_i}{r_0^i}\sin i\theta\right) \\
n_x\,|_{r=R} &= \cos\theta - \epsilon\sin\theta\frac{k_1\sum_{even\ i} i a_i \sin i\theta}{\left(\sum_{even\ i} a_i^2\right)^{1/2}} \\
dS &= r_0\left(1 + \epsilon\frac{k_1\sum_{i=1}^{n} a_i\cos i\theta}{\left(\sum_{even\ i} a_i^2\right)^{1/2}}\right) \qquad \text{(C.1)}
\end{aligned}$$

The only terms containing ϵ^1 that come from multiplying these together and to be integrated from 0 to 2π are

$$\begin{aligned} dK &= -r_0(\dot{x}\cos\theta + \dot{y}\sin\theta)\cos\theta \frac{k_1 \sum_{i=1}^{n} a_i \cos i\theta}{\left(\sum_{even\ i} a_i^2\right)^{1/2}} \\ &+ r_0(\dot{x}\cos\theta + \dot{y}\sin\theta) r_0 \sin\theta \frac{k_1 \sum_{even\ i} i a_i \sin i\theta}{\left(\sum_{even\ i} a_i^2\right)^{1/2}} \\ &+ r_0 \cos\theta \left(\frac{k_1 r_0}{\left(\sum_{even\ i} a_i^2\right)^{1/2}} \dot{x}\cos\theta + \dot{y}\sin\theta) \sum_{even\ i} a_i \cos i\theta + \sum_{i=1}^{n} \frac{i d_i}{r_0^i} \cos i\theta + \sum_{i=1}^{n} \frac{i c_i}{r_0^i} \sin i\theta \right) \end{aligned} \tag{C.2}$$

Basic calculus tells us that many of these terms disappear when integrated. In particular of all the constants in ϕ only d_1 remains. Also all terms containing $\sin n\theta$ or $\cos n\theta$ with $n > 2$ disappear and this leaves us with

$$K_x(\phi) = \rho\pi r_0^2(-1 + \epsilon \frac{2\rho\pi r_0^2 k_1 a_2}{\left(\sum_{even\ i} a_i^2\right)^{1/2}})\dot{x} + O(\epsilon^2);$$

Equating the fluid impulse and the amoeba momentum as described in Section 3.1 gives

$$M(\dot{x} + c_2\dot{k}_2) = \rho\pi r_0^2(-1 + \epsilon \frac{2\rho\pi r_0^2 k_1 a_2}{\left(\sum_{even\ i} a_i^2\right)^{1/2}})\dot{x}$$

Write $\dot{x}$ as a series expansion in ϵ: $\dot{x} = x_0 + \epsilon x_1 + \epsilon^2 x_2 + O(\epsilon^3)$ and solve for $\dot{x}$. Only components containing k_1 will affect the Lie bracket i.e not x_0 but x_1 and x_2. For smallϵ x_2 will have little influence on $\dot{x}$, so the only noteworthy influence on the Lie bracket is

$$\rho\pi r_0^2 \epsilon \frac{2\rho\pi r_0^2 k_1 a_2}{\left(\sum_{even\ i} a_i^2\right)^{1/2}} \dot{x}$$

which clearly has a maximum for $a_i = 0$ when $i \neq 2$. This means that of all possible symmetric amoeba deformations the basic symmetric boundary oscillation $\cos 2\theta$ has dominant impact on the amoeba Lie bracket motion.

□

An an illustration, choose

$$\alpha(\theta) = \frac{1}{\sqrt{a_2^2 + a_4^2}}(a_2\cos(2\theta) + a_4\cos(4\theta))$$

The fluid impulse is

$$K(\phi) = \rho\pi r_0^2 \begin{bmatrix} \dot{x}(-1 + \epsilon\frac{2a_2k_1}{\sqrt{a_2^2+a_4^2}} - \epsilon^2\frac{k_1^2}{2(a_2^2+a_4^2)}(5a_2^2 - 10a_2a_4 + 13a_4^2)) + O(\epsilon^3) \\ \dot{y}(-1 + \epsilon\frac{-2a_2k_1}{\sqrt{a_2^2+a_4^2}} - \epsilon^2\frac{k_1^2}{2(a_2^2+a_4^2)}(5a_2^2 + 10a_2a_4 + 13a_4^2)) + O(\epsilon^3) \end{bmatrix}$$

Solving the momentum equation for $\dot{x}$ as above gives

$$\begin{aligned}
x_0 &= -\frac{Mc_2\dot{k}_2}{M+\rho\pi r_0^2} \\
x_1 &= -\frac{2\rho\pi r_0^2 c_2 M a_2 k_1 \dot{k}_2}{(M+\rho\pi r_0^2)^2\sqrt{a_2^2+a_4^2}} \\
x_2 &= c_2 M k_1^2 \dot{k}_2 \left[\frac{-4(\rho\pi r_0^2)^2 a_2^2}{(M+\rho\pi r_0^2)^3(a_2^2+a_4^2)} + \frac{\rho\pi r_0^2(5a_2^2-10a_2a_4+13a_4^2)}{2(M+\rho\pi r_0^2)^2(a_2^2+a_4^2)}\right]
\end{aligned}$$

Doing a Lie bracket calculation gives resulting motion in the x-direction as

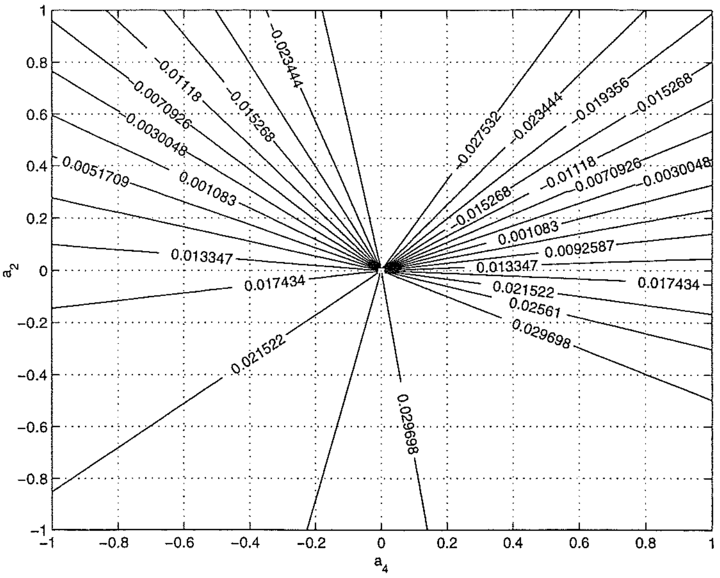

Figure C.1 Level curves for g_3 for different combinations of a_2 and a_4 showing the effect of the ϵ^2 term.

$$\begin{aligned}
g_3 &= -\epsilon\frac{2\rho\pi r_0^2 c_2 M a_2}{(M+\rho\pi r_0^2)^2\sqrt{a_2^2+a_4^2}} \\
&+ \epsilon^2 c_2 M k_1 \dot{k}_2 \left[\frac{-8(\rho\pi r_0^2)^2 a_2^2}{(M+\rho\pi r_0^2)^3(a_2^2+a_4^2)} + \frac{\rho\pi r_0^2(5a_2^2-10a_2a_4+13a_4^2)}{(M+\rho\pi r_0^2)^2(a_2^2+a_4^2)}\right] + O(\epsilon^3)
\end{aligned}$$

Obviously, for sufficiently small ϵ the first term is dominant and we need only consider this. Maximizing the ϵ-term with respect to a_2 and a_4 gives the highest absolute values for $a_4 = 0$ and (because of the normalization) any a_2. This was verified using MATLAB with fictional constant values, see Figure C.1. Adding a small, positive a_4 for increasing values of a_2 gives a tiny little bit more motion from the ϵ^2-term, but really very little, as expected from the reasoning above.

D. The Potential ϕ

The coefficients in the series expansion of the potential ϕ (4.5) are

$$c_0 = 0 \quad d_0 = 0$$
$$d_{n-1} = r_0^n k_1 \dot{x}$$
$$d_n = -\frac{r_0^{n+2}}{n} \dot{k}_1$$
$$d_{n+1} = -r_0^{n+2} k_1 \dot{x}$$
$$d_{m-1} = r_0^m k_2 \dot{y}$$
$$d_m = 0$$
$$d_{m+1} = r_0^{m+2} k_2 \dot{y}$$
$$c_{n-1} = -r_0^n k_1 \dot{y}$$
$$c_n = 0$$
$$c_{n+1} = -r_0^{n+2} k_1 \dot{y}$$
$$c_{m-1} = r_0^m k_2 \dot{x}$$
$$c_m = -\frac{r_0^{m+2}}{m} \dot{k}_2$$
$$c_{m+1} = -r_0^{m+2} k_2 \dot{x}$$

$$f_0 = 0 \quad g_0 = 0 \quad h_0 = 1$$
$$e_0 = \tfrac{1}{2} r_0^2 (k_1 \dot{k}_1 + k_2 \dot{k}_2)$$
$$h_1 = -\tfrac{1}{2} r_0^2 (k_1^2 (2n-1) + k_2^2 (2m-1)) \dot{x}$$
$$h_{2m-1} = -\frac{r_0^{2m}}{4(2m-1)} k_2^2 (3 - 8m + 4m^2) \dot{x}$$
$$h_{2m} = -\frac{r_0^{2m+2}}{4m} k_2 \dot{k}_2 (1 + 2m)$$
$$h_{2m+1} = \tfrac{1}{4} r_0^{2m+2} k_2^2 (2m+1) \dot{x}$$
$$h_{2n-1} = -\frac{r_0^{2n}}{4(2n-1)} k_1^2 (3 - 8n + 4n^2) \dot{x}$$
$$h_{2n} = -\frac{r_0^{2n+2}}{4n} k_1 \dot{k}_1 (1 + 2n)$$
$$h_{2n+1} = -\tfrac{1}{4} r_0^{2n+2} k_1^2 (2n+1) \dot{x}$$
$$h_{m+n+1} = \tfrac{1}{2} r_0^{2+m+n} k_1 k_2 (1 + m + n) \dot{y}$$
$$h_{m+n-1} = \frac{r_0^{m+n}}{2(m+n-1)} k_1 k_2 (3 - 4m + m^2 - 4n + 2mn + n^2) \dot{y}$$
$$h_{m-n+1} = -\frac{r_0^{2+m-n}}{2(m-n+1)} k_1 k_2 ((n-1)^2 - m^2) \dot{y}$$
$$h_{m-n-1} = -\frac{r_0^{m-n}}{2(m-n-1)} k_1 k_2 (n^2 - (m-1)^2) \dot{y}$$
$$h_{m+n} = 0 \quad h_{m-n} = 0$$

$g_1 = -\frac{1}{2} r_0^2 (k_1^2 (2n-1) + k_2^2 (2m-1)) \dot{y}$

$g_{2m-1} = r_0^{2m} / (4(2m-1)) \; k_2^2 (3 - 8m + 4m^2) \dot{y}$

$g_{2m} = 0$

$g_{2m-1} = -\frac{1}{4} r_0^{2m+2} k_2^2 (1 + 2m) \dot{y}$

$g_{2n-1} = -r_0^{2n} / (4(2n-1)) \; k_1^2 (3 - 8n + 4n^2) \dot{y}$

$g_{2n} = 0$

$g_{2n+1} = -\frac{1}{4} r_0^{2n+2} k_1^2 (1 + 2n) \dot{y}$

$g_{m+n+1} = -\frac{1}{2} r_0^{m+n+2} k_1 k_2 (m + n + 1) \dot{x}$

$g_{m+n-1} = r_0^{m+n} / (2(m+n-1)) \; k_1 k_2 (3 - 4m + m^2 - 4n + 2mn + n^2) \dot{x}$

$g_{m-n-1} = r_0^{m-n} / (2(m-n-1)) \; k_1 k_2 ((1-m)^2 - n^2) \dot{x}$

$g_{m-n+1} = -r_0^{m-n+2} / (2(m-n+1)) \; k_1 k_2 (m^2 - (n-1)^2) \dot{x}$

$g_{m+n} = -r_0^{m+n+2} / (2(m+n)) \; (\dot{k}_1 k_2 + k_1 \dot{k}_2)(m + n + 1) \dot{x}$

$g_{m-n} = -r_0^{m-n+2} / (2(m-n)) \; (\dot{k}_1 k_2 (1 - m + n) + k_1 \dot{k}_2 (1 + m - n))(m + n + 1) \dot{x}$

E. Homogeneous Amoeba: Motion Perpendicular to the Symmetry Axis

That motion perpendicular to the symmetry axis of a homogeneous amoeba is impossible is shown in this section. For any combination of amoeba oscillation shapes symmetric around the y-axis, reflexion in the y-axis gives, of course, the same shape at all times. The reflexion corresponds to a change of coordinates $x \rightarrow -x$. The calculations in Section 4.2 are on the whole the same with this new x, but some signs change. The fluid velocity now becomes

$$\begin{aligned} \mathbf{e}_r &= -\cos\theta\ \mathbf{e}_x + \sin\theta\ \mathbf{e}_y \\ \mathbf{e}_\theta &= \sin\theta\ \mathbf{e}_x + \cos\theta\ \mathbf{e}_y \end{aligned}$$

and the boundary equation becomes

$$(\nabla\phi \cdot n)\ |_R = (-\dot{x}\cos\theta, \dot{x}\sin\theta) \cdot n + (\dot{y}\sin\theta, \dot{y}\cos\theta) \cdot n + \epsilon r_0(\dot{k}_1 \cos(n\theta) + \dot{k}_2 \sin(n\theta), 0) \cdot n$$

The normal vector in Cartesian coordinates is

$$\begin{bmatrix} n_x \\ n_y \end{bmatrix} = \begin{bmatrix} -\cos\theta \\ \sin\theta \end{bmatrix} + \epsilon \begin{bmatrix} \sin\theta\ (k_2 m \cos(m\theta) + k_1 n \sin(n\theta)) \\ \cos\theta\ (-k_2 m \cos(m\theta) + k_1 n \sin(n\theta)) \end{bmatrix}$$

$$-\tfrac{1}{2}\epsilon^2[k_2 m \cos(m\theta) - k_1 n \sin(n\theta)]*$$

$$* \begin{bmatrix} -2\sin\theta(k_1\cos(n\theta) + k_2 \sin(m\theta)) - \cos\theta\ (k_2 m \cos(m\theta) - k_1 n \sin(n\theta)) \\ -2\cos\theta(k_1\cos(n\theta) + k_2 \sin(m\theta)) + \sin\theta\ (k_2 m \cos(m\theta) - k_1 n \sin(n\theta)) \end{bmatrix} + O(\epsilon^3)$$

Performing the calculations again give a new potential function ϕ and consequently a new fluid impulse K. The expression for the centroid position (4.5) remains unchanged. We are interested in how the x-component of the solution to $M\dot{C} = K(\phi)$ changes. Obviously, we need to study the fluid impulse closer.

What actually happens is that all terms on the right hand side of the boundary equation (2.4) containing $\dot{x}$ switch signs, causing only coefficients in ϕ containing $\dot{x}$ to change.

Consider the fluid impulse K as a series expansion in ϵ,

$$K(\phi) = K_0 + \epsilon K_1 + \epsilon^2 K_2 + O(\epsilon^3)$$

K_0 will remain unchanged as the integrated $\dot{x}$-terms not containing any ϵ change signs in two places, ϕ and n. Consequently, the centroid displacement term not containing any ϵ will be the same. K_1 can change, but as K_1 never contains any cross terms (i.e.

$k_1 k_2$) it will never contribute to the creation of any resulting amoeba motion. K_2 on the other hand does contain cross terms, but the only change is (as stated above) in terms that also contain $\dot{x}$ and these never contribute to motion as seen in (4.6).

From this follows that no change in motion is possible because of a change in coordinate system, just as expected. Following thorough the reasoning from Section 4.1, the only possible motion in the x direction that does not change when the shape is reflected in the y-axis (as motion should start in the opposite direction after reflexion) is $\dot{x} = 0$, proving that no motion perpendicular to the symmetry axis of the amoeba can exist.

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