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Upper and Lower Bounds on the Feedback Order, Necessary to Stabilize a Given Rational Transfer Function

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Abstract

This MSc-thesis gives algorithms for calculating upper and lower bounds on the "stability order" of a linear SISO-system, with some rational transfer function B/A. The upper bound is calculated via the resultant matrix of the polynomials, A and B. For the lower bound a linear program is defined. The last part of the thesis solves a problem of invariant theory connected to the problem of determining the "stabilizing order" exactly.

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Preface

This work has been carried out as a co-operation between the Dept. of Mathematics and the Dept. of Automatic Control in Lund, mainly during the summer of 1986. Many people at the departments have contributed with questions and remarks during the work and I am greatful to all of them.

First, I would like to thank my two main supervisors, who have been Prof. Karl Johan Åström and Dr. Gunnar Sparr. Prof. Åström has encouraged me with criticism and constructive questions during the whole work and he has also supplied valuable contacts. Dr. Gunnar Sparr has given a detailed criticism on my manuscripts and helped a lot to make the presentation readable.

I am indebted to Prof. Christopher I. Byrnes, Arizona State University, for proposing the invariant theory approach of chapter 5, and to Dr. Gert Almqvist, who gave invaluable help by proposing good litterature in that area.

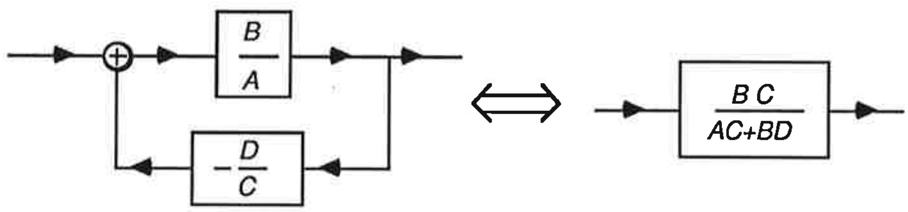
Thanks also to Dr. Bengt Mårtensson and Mats Lilja for many interesting discussions and practical advices, and to Dr. Sven Spanne for supplying the idea of the proof of Lemma 4.1. A special thank to Dr. Bengt Mårtensson for reading and commenting my manuscript.

1. Introduction

Given a system to be controlled, what complexity of the regulator do we need? To answer such a question, we need to define the notions more exactly. In particular, we need a measure of the complexity and criterion on acceptable control properties. This thesis will treat systems that can be represented by a rational transfer function

$$\frac{B(s)}{A(s)} = \frac{b_0 s^m + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}.$$

and regulators that can be represented by an output feedback gain, $-D/C$ like in the following figure:



The order (complexity) of the regulator is defined to be the number $q = \max(\deg C, \deg D)$. (In the applications $\deg D \leq \deg C$.) The rational function, D/C, is called a stabilizer of B/A, if the polynomial $AC + BD$ has all its zeros in the open left half of the complex plane.

Our question can now be restated as a mathematical problem: Given the rational function B/A, what is the lowest order of a stabilizer to B/A?

For example, [Mårtensson] shows that knowledge of this kind is of particular interest in adaptive stabilization. Several attempts have been made to answer the question (e.g. [Anderson-Bose-Jury], [Anderson-Scott]). Still there is a lack of effective methods for computing the order required.

This thesis is an attempt to give easily calculated upper and lower estimates of the number in question. In Chapter 2, we introduce some necessary definitions and notations. Then the methods of estimation are developed in Chapter 3 and 4. Two programs are written to compute the estimates for any particular system. For a given rational function, B/A, the first one tries to calculate a low order stabilizer, $-D/C$, the other computes an order of D/C, for which stabilization is impossible. The codes are supplied in the appendixes A and B. Chapter 5 also includes a some results related to the problem of finding the exact order required.

2. Definitions and Notations

In this chapter the q^*-function is defined and some of its properties are examined. We will use the notation $\mathbb{R}[s]$ for the set of polynomials in one complex variable having real coefficients. A polynomial, $B \in \mathbb{R}[s]$, is called a *Hurwitz polynomial* if all zeros of B have negative real parts. The subset of $\mathbb{R}[s]$ consisting of all Hurwitz polynomials, is denoted $\mathbb{H}[s]$. We will in the sequel refer to rational functions B/A, and implicitly assume that A and $B \in \mathbb{R}[s]$, A monic, and that they are relatively prime. For such a rational function, we define $\deg(B/A) = \max(\deg B, \deg A)$.

DEFINITION 2.1 By a *stabilizer* of B/A, we mean a rational function, D/C, such that $AC+BD \in \mathbb{H}[s]$ and $\deg(AC+BD) = \max(\deg AC, \deg BD)$. The *stabilizing order* of B/A is

$$q^*(B/A) = \min_{D/C \in M} \deg(D/C)$$

where M is the set of stabilizers for B/A. □

Remark 1. It follows that if D/C is a stabilizer of B/A, then B/A is a stabilizer of D/C, B/C is a stabilizer of D/A and C/D is a stabilizer of A/B.

Remark 2. The absence of degree conditions above might seem confusing to a control engineer, since for systems, B/A, with $\deg B \leq \deg A$, he wants compensators, D/C, with $\deg D \leq \deg C$. Theorem 3.3 will show that such a degree condition would make no difference.

It will be convenient to identify each polynomial, $B(s) = b_0 s^m + \cdots + b_m$, with a row matrix $(b_0 \ldots b_m)$. The product $D(s)B(s)$ is then represented by a matrix product

$$(d_0 \;\ldots\; d_k) \begin{pmatrix} b_0 & \ldots & b_m & & 0 \\ & \ddots & & \ddots & \\ 0 & & b_0 & \ldots & b_m \end{pmatrix}_{(k+1)\times(m+k+1)} .$$

In the study of $AC + BD$ we will use the following definition.

Definition 2.2 For two polynomials, $B(s) = b_0 s^m + \cdots + b_m$ and $A(s) = a_0 s^n + \cdots + a_n \in \mathbb{R}[s]$, define $\mathrm{Res}^r_{k,l}(A, B)$ by

$$\mathrm{Res}^r_{k,l}(A,B) = \begin{pmatrix} a_{n+k-r} & \cdots & a_{n+k} \\ \vdots & \ddots & \vdots \\ a_{n-r} & \cdots & a_n \\ b_{m+l-r} & \cdots & b_{m+l} \\ \vdots & \ddots & \vdots \\ b_{m-r} & \cdots & b_m \end{pmatrix}_{(k+l+2)\times(r+1)}$$

where $b_i = a_j = 0$ for $i, j < 0$ as well as for $i > m$ and $j > n$. When $r = k+l+1$ we just write $\mathrm{Res}_{k,l}(A, B)$. □

Example 2.1
If $A(s) = a_0 s^3 + a_1 s^2 + a_2 s + a_3$ and $B(s) = b_0 s^2 + b_1 s + b_2$, then

$$\mathrm{Res}_{1,2}(A,B) = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 \end{pmatrix}.$$

□

The following proposition shows that the q^*-function is always well defined.

Proposition 2.1

$$q^*(B/A) \leq \max(\deg B - 1, \deg A - 1).$$

Proof: In fact, $\mathrm{Res}_{m-1,n-1}(A, B)$ is the ordinary resultant of the polynomials. It is well known [van der Waerden] that $\det \mathrm{Res}_{m-1,n-1}(A, B) \neq 0$ if and only if B and A are relatively prime. Since

$$(\, c_0 \;\; \ldots \;\; c_{m-1} \;\; d_0 \;\; \ldots \;\; d_{n-1}\,)\mathrm{Res}_{m-1,n-1}(A,B)$$

is the coefficient matrix of the polynomial $AC + BD$, where $C(s) = c_0 s^{m-1} + \cdots + c_{m-1}$ and $D(s) = d_0 s^{n-1} + \cdots + d_{n-1}$, this guarantees the existence of polynomials, C_0 and $D_0 \in \mathbb{R}[s]$, with $\deg C_0 \leq \deg B - 1$ and $\deg D_0 \leq \deg A - 1$, such that $AC_0 + BD_0 \in \mathbb{H}[s]$ and $\deg(AC_0 + BD_0) = \deg A + \deg B - 1$. ■

The notion of reciprocal polynomials will be useful in the sequel.

Definition 2.3 Consider a polynomial, $P \in \mathbb{R}[s]$, of degree n. For $m \geq n$, define its *m:th degree reciprocal polynomial*, P^{*m}, through

$$P^{*m}(s) = s^m P(s^{-1}).$$

We use the notation $P^* = P^{*\deg P}$. □

PROPOSITION 2.2 The reciprocal polynomials have the following properties:

(*i*) $$P_1^{*m_1} P_2^{*m_2} = (P_1 P_2)^{*(m_1+m_2)}$$

in particular $P_1^* P_2^* = (P_1 P_2)^*$

(*ii*) $$(P_1 + P_2)^{*m} = P_1^{*m} + P_2^{*m}$$

(*iii*) $$P \in \mathbb{H}[s] \Leftrightarrow P^* \in \mathbb{H}[s]$$

(*iv*) $$P \in \mathbb{H}[s] \Rightarrow P(0) \neq 0 \Rightarrow \deg P^* = \deg P \Rightarrow (P^*)^* = P$$

for $P_1, P_2 \in \mathbb{R}[s]$ and $m, m_1, m_2 \in \mathbb{Z}$.

Proof: (*i*), (*ii*) and (*iv*) follow immediately from the definition and (*iii*) is true since

$$P(s) = p_0(s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_n) \Rightarrow$$
$$\Rightarrow P^*(s) = s^n P(s^{-1}) = p_0(1-\alpha_1 s)\cdots(1-\alpha_n s).$$

■

3. Upper Bounds

This chapter is devoted to giving upper bounds on the q^*-function for certain cases. The central theorem gives an upper estimate for all kinds of systems. The approach will make use of Routh-Hurwitz' determinant criterion on Hurwitz polynomials [Lancaster-Tismenetsky]. The *Hurwitz matrices* of a polynomial, $A(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_n, a_0 \neq 0$, are defined as

$$\Omega_1(A) = a_1,\ \Omega_2(A) = \begin{pmatrix} a_1 & a_3 \\ a_0 & a_2 \end{pmatrix}, \ldots$$

$$\ldots, \Omega_n(A) = \begin{pmatrix} a_1 & a_3 & a_5 & \ldots & 0 \\ a_0 & a_2 & a_4 & & \vdots \\ 0 & a_1 & a_3 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \ldots & 0 & \ldots & a_n \end{pmatrix},$$

where $a_k = 0$ for $k < 0$ and $k > n$. The *Hurwitz determinants* are $\Delta_k(A) = \det \Omega_k(A)$, $k = 1, \ldots, n$. If $a_0 > 0$, Routh-Hurwitz' theorem says that A is a Hurwitz polynomial if and only if all the Hurwitz determinants are positive.

Remark. By continuity, it follows that $\{(u_1, \ldots, u_n) : u_1 s^{n-1} + \cdots + u_n \in \mathbb{H}[s]\}$ is an open subset of $\mathbb{R}^n$.

Next follows an example that will reveal the most basic idea of this chapter.

EXAMPLE 3.1
Does the transfer function, $B/A = 1/(s^3 + s^2 + 2s - 1)$ have a zero order stabilizer, $D/C = d_0 \in \mathbb{R}$?

The answer is yes as we shall see. The feedback gain $D/C = d_0$ gives $AC + BD = s^3 + s^2 + 2s - 1 + d_0$ and obviously only the last coefficient depends on d_0. The Hurwitz determinants of $AC + BD$ are

$$\Delta_1(AC+BD) = 1, \quad \Delta_2(AC+BD) = \begin{vmatrix} 1 & -1+d_0 \\ 1 & 2 \end{vmatrix} \quad \text{and}$$

$$\Delta_3(AC+BD) = \begin{vmatrix} 1 & -1+d_0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & -1+d_0 \end{vmatrix}.$$

For a sufficiently small positive value on $-1+d_0$, they all become positive and $D/C = d_0$ becomes a stabilizer of B/A. □

In the following lemma, the essence of this example is given a more genaral form.

LEMMA 3.1 Consider a polynomial, $P_\varepsilon \in \mathbb{R}[s]$, whose coefficients are continous functions of $\varepsilon \in \mathbb{R}$. If $P_0 \in \mathbb{H}[s]$ has positive coefficients, then $sP_\varepsilon(s) + \varepsilon \in \mathbb{H}[s]$ for all ε in an interval, $0 < \varepsilon < \delta$.

Proof: Put $n = \deg P_0$. By Routh-Hurwitz' criterion $\Delta_k(P_0) > 0$ for $1 \le k \le n$. If $Q_\varepsilon(s) = sP_\varepsilon(s) + \varepsilon$, we have $\lim_{\varepsilon \to 0} \Delta_k(Q_\varepsilon) = \Delta_k(P_0) > 0$ for $1 \le k \le n$, hence $\Delta(Q_\varepsilon) > 0$ for ε:s in some neighborhood of zero. For $k = n+1$, we have

$$\Delta_{n+1}(Q_\varepsilon) = \begin{vmatrix} \Omega_n(Q_\varepsilon) & 0 \\ * & \varepsilon \end{vmatrix} = \varepsilon \Delta_n(Q_\varepsilon),$$

which is positive for small positive ε:s. The lemma follows by another use of Routh-Hurwitz' criterion. ■

In Example 3.1 the zero order feedback only affected one of the coefficients in the characteristic polynomial, but because of the fortunate values of the other coefficients, it was possible to stabilize the system. This will now be generalized by the use of the matrix notation for polynomials.

EXAMPLE 3.2
Consider $B/A = (s-1)/(s^3 - 3s^2 + s + 5)$ and suppose we are looking for a stabilizer of the form $D/C = d_o s + d_1$. Using the matrix notation of chapter 2, we may write

$$AC + BD = (1 \quad d_0 \quad d_1) \begin{pmatrix} 1 & -3 & 1 & 5 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} = (1 \quad \alpha_1 \quad \delta_0 \quad \delta_1).$$

The lower right submatrix

$$\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

is nonsingular, thus δ_0 and δ_1 can be given arbitrary values. For example, $d_0 = 6$, $d_1 = 5$ gives $\delta_0 = \delta_1 = 0$ and

$$AC + BD = (1 \quad 3 \quad 0 \quad 0) = s^2(s+3).$$

This is not a Hurwitz polynomial, but not very far from. By applying Lemma 3.1 twice and change δ_0 and δ_1 a little bit, we can give a negative real part to the last two zeros as well. This would provide a first order stabilizer, D/C, of B/A. Theorem 3.1 will do the construction in the general case. □

LEMMA 3.2 Consider a rational function, B/A, with $\deg B = m, \deg A = n$, and two integers, k and l, such that $l \le n-1$, $m + l \le n + k$ and $\det \mathrm{Res}_{k-1,l}(A, B) \ne 0$. For any polynomial, $\delta = \delta_0 s^{k+l} + \cdots + \delta_{k+l}$, there are unique polynomials, $C = s^k + c_1 s^{k-1} + \cdots + c_k$, $D = d_1 s^l + \cdots + d_l$ and $R^\delta_{k,l}(B/A) = r_0 s^{n-l-1} + \cdots + r_{n-l-1} \in \mathbb{R}[s]$, such that

$$A(s)C(s) + B(s)D(s) = s^{k+l+1} R^\delta_{k,l}(B/A)(s) + \delta(s).$$

Proof: The determinant condition guarantees that the linear equation

$$(\delta_0 \quad \ldots \quad \delta_{k+l}) = (1 \quad c_1 \quad \ldots \quad c_k \quad d_0 \quad \ldots \quad d_l) \mathrm{Res}_{k,l}(A, B)$$

determines the polynomials C and D uniquely and by that also the third polynomial, $R^\delta_{k,l}(B/A)$, is fixed. ■

If the conditions of this lemma are fulfilled, we say that $R^{\delta}_{k,l}(B/A)$ is well defined. For $\delta = 0$ we write $R_{k,l}(B/A)$ instead of $R^0_{k,l}(B/A)$.

Remark. In the previous example we had $R^{\delta}_{0,1}(B/A) = s + \alpha_1$ and $R^0_{0,1}(B/A) = s + 3$.

THEOREM 3.1 Consider a rational function, B/A, and integers, k, l, m, n, obeying the assumptions of Lemma 3.2. If $R^0_{k,l}(B/A)$ is a Hurwitz polynomial of degree $n - l - 1$, then B/A has a stabilizer D/C with $\deg C = k$, $\deg D = l$. In particular $q^*(B/A) \leq \max(k, l)$.

Proof: For simplicity, we write R^{δ} instead of $R^{\delta}_{k,l}(B/A)$. Suppose R^0 has positive coefficients. $R^{(\delta_0,0,\ldots,0)}$ is a continous function of δ_0, hence by Lemma 3.1, $sR^{(\delta_0,0,\ldots,0)}(s) + \delta_0 \in \mathbb{H}[s]$ for some $\delta_0 > 0$. We can continue choosing $\delta_1, \ldots, \delta_{k+l} > 0$ sufficiently small, such that the polynomial

$$s^{k+l+1}R^{(\delta_0,\ldots,\delta_{k+l})}(s) + \delta_0 s^{k+l} + \ldots + \delta_{k+l} =$$

$$= s\Big(s\Big(\ldots\Big(sR^{(\delta_0,\ldots,\delta_{k+l})}(s) + \delta_0\Big)\ldots\Big) + \delta_{k+l-1}\Big) + \delta_{k+l}$$

becomes a Hurwitz polynomial. Then, by Lemma 3.2, there are polynomials, C and D, C monic, of degree k and l respectively, such that $AC + BD$ is a Hurwitz polynomial of degree $n + k$. Thus D/C is a stabilizer of B/A and we are finished, since the case with negative coefficients in R^0 is completely analogous. ∎

COROLLARY. Suppose B/A is a rational function such that $\deg A = \deg A^* = n$, $\deg B = \deg B^* = m$ and further that B^*/A^* together with the integers k, l, m, n satisfy the assumptions of Lemma 3.2. If $R^0_{k,l}(B^*/A^*)$ is a Hurwitz polynomial of degree $n - l - 1$, then B/A has a stabilizer, D/C, with $\deg D \leq n - m + k$ and $\deg C \leq k$. In particular $q^*(B/A) \leq n - m + k$.

Proof: By Theorem 3.1, B^*/A^* has a stabilizer D_0/C_0, with $\deg C_0 = k$ and $\deg D_0 = l$. Since

$$AC_0^* + Bs^{n-m+k-l}D_0^* = (A^*C_0 + B^*D_0)^* \in \mathbb{H}[s]$$

we may finish the theorem by putting $C = C_0{}^*$ and $D = s^{n-m+k-l}D_0{}^*$. ∎

Remark. Theorem 3.1 has a simple heuristic interpretation. Suppose C_0 and D_0 make $AC_0 + BD_0 \in \mathbb{H}[s]$. Then, when changing C and D from $C = 1, D = 0$ to $C = C_0, D = D_0$, we move the zeros of $AC + BD$ into the left half plane. Unstable zeros have to pass over the imaginary axis or through the "infinity point". In the case of Theorem 3.1 the last $k + l$ unstable poles pass simultaneously through origo. Actually, we move the last zeros to origo, when we choose D/C such that $AC + BD = s^{k+l+1}R_{k+l}(B/A)$. Then we force them all into the left half plane by changing C and D appropriately.

The interpretation of the corollary is essentially the same, but the last $k + l$ zeros pass through the infinity point instead. □

Theorem 3.1 can be given an easier, though less powerful formulation, by means of polynomial division.

For a rational function, A/B, with $\deg B = m \leq \deg A = n$, define the partial remainders, $P_0, P_1, \ldots$, of A/B through the polynomial divisions

$$A = BQ_0 + P_0, \quad \text{for some } Q_0, P_0, \text{ where } \deg P_0 \leq m-1,$$

$$\vdots$$

$$A = s^i BQ_i + P_i, \quad \text{for some } Q_i, P_i, \text{ where } \deg P_1 \leq m+i-1,$$

$$\vdots$$

Obviously $Q_i = 0, P_i = A$ for $i \geq n-m$. All the P_i:s are acquired as partial results when the polynomial division A/B is computed the ordinary way.

THEOREM 3.2 Consider a rational function, B/A, with $\deg B = \deg B^* = m$ and $\deg A = n$. If the r:th partial remainder, P_r, $r \leq n-m$ of the rational A^*/B^*, is a Hurwitz polynomial of full degree, $\deg P_r = m+r-1$, then the inequality $q^*(B/A) \leq n-m-r$ holds.

Proof: The condition $b_m \neq 0$ implies that $\deg B^* = m$ and $\deg Q_r = n-m-r$. We have

$$\det \mathrm{Res}_{-1,n-m-r}(A,B) = \begin{vmatrix} b_m & 0 & \ldots & 0 \\ b_{m-1} & b_m & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ b_{2m-n+r} & b_{2m-n+r+1} & \ldots & b_m \end{vmatrix} \neq 0$$

$$\text{and} \quad A - BQ_r^* = (A^* - s^r B^* Q_r)^{*n} = P_r^{*n} = s^{n-m-r+1} P_r^*,$$

hence $R_{0,n-m-r}(B/A) = P_r^* \in \mathbb{H}[s]$ and the statement follows from Theorem 3.1. ∎

EXAMPLE 3.3
We apply Theorem 3.2 to the rational function $B/A = (s-1)/(s^3-3s^2+s+5)$ from Example 3.2. The division A^*/B^*:

$$\begin{array}{r|l} & -5s^2 - 6s \\ \hline -s+1 & 5s^3 + s^2 - 3s + 1 \\ & \underline{5s^3 - 5s^2} \\ & \quad\; 6s^2 - 3s + 1 \\ & \quad\; \underline{6s^2 - 6s} \\ & \qquad\quad 3s + 1 \end{array}$$

Thus $P_2 = 6s^2 - 3s + 1 \notin \mathbb{H}[s]$ but $P_1 = 3s + 1 \in \mathbb{H}[s]$ and Theorem 3.2 says that $q^*(B/A) \leq 3-1-1 = 1$. □

The stabilizer, D/C, of Example 3.2 is not proper, i.e. $\deg D > \deg C$. This is often not acceptable in practical control engineering. Therefore, it would be interesting to know, whether there is a polynomial, E, with $\deg E > \deg D - \deg C$, such that D/CE is a stabilizer of B/A, or equally, $ACE + BD \in \mathbb{H}[s]$. Theorem 3.3 will answer this question.

THEOREM 3.3 Suppose $P, Q \in \mathbb{R}[s]$, $P + Q \in \mathbb{H}[s]$ and $\deg(P+Q) = \max(\deg P, \deg Q)$. Then, for any positive integer, i, there is an $E \in \mathbb{R}[s]$, of degree i, such that also $PE + Q \in \mathbb{H}[s]$ and $\deg(PE+Q) = \max(\deg P + i, \deg Q)$.

Proof: It is sufficient to prove that the theorem is true for $i = 1$. The corollary then follows by induction over i.

When $\deg P < \deg Q$, the statement is obviously true, since

$$\deg(P(e_0 s + e_1) + Q) \leq \deg(P+Q), \text{ and}$$

$$\lim_{(e_0,e_1)\to(0,1)} (P(e_0 s + e_1) + Q) = P + Q \in \mathbb{H}[s].$$

(See remark in the beginning of the chapter.)

When $\deg Q \leq \deg P$, we set $n = \deg P = \deg(P+Q) = \deg(P+Q)^*$ and put $A = s(P+Q)^*, B = P^{*n}, k = l = 0$ into Theorem 3.1 We get $\det \mathrm{Res}_{-1,0}(s(P+Q)^*, P^{*n})) = p_0 \neq 0$ and $A(s)\cdot 1 + B(s)\cdot 0 = (P+Q)^* s$, i.e. $R^0_{0,1}(A,B) = (P+Q)^* \in \mathbb{H}[s]$ and the theorem shows that there is a number d_0, such that $s(P+Q)^* + P^{*n} d_0 \in \mathbb{H}[s]$. Hence $P(d_0 s + 1) + Q = (s(P+Q)^* + P^{*n} d_0)^{*(n+1)} \in \mathbb{H}[s]$ and we are finished. ∎

Finally, we use Theorem 3.1 and an argument similar to the proof of Lemma 3.1 to prove a well known result about $q^*(B/A)$, when $B \in \mathbb{H}[s]$ (minimum phase systems).

THEOREM 3.4 Suppose $A(s) = s^n + a_1 s^{n-1} + \cdots + a_n \in \mathbb{R}[s]$, $B(s) = s^m + \cdots + b_m \in \mathbb{H}[s]$ and $r = n - m \geq 1$. Then $q^*(B/A) \leq r - 1$. Further, if $a_1 > b_1$ and $r \geq 2$, the stronger inequality $q^*(B/A) \leq r - 2$ holds.

Proof: Choose any monic $P \in \mathbb{H}[s]$ of degree $r - 1$ and put $B_0/A_0 = A^*/(PB)^{*n}$ and $k = l = 0$ into Theorem 3.1. We get $\det \mathrm{Res}^0_{-1,0}(A_0, B_0) = 1 \neq 0$, $\deg A_0 = n$ and $A_0 \cdot 1 + B_0 \cdot 0 = s(PB)^*$, i.e. $R_{0,0}(B_0/A_0) = (PB)^*$ is well defined and a Hurwitz polynomial of full degree, $n-1$. The theorem gives $A_0 \cdot 1 + B_0 d_0 \in \mathbb{H}$ for some $d_0 \in \mathbb{R}$, thus $Ad_0 + PB = (B_0 d_0 + A_0)^{*n} \in \mathbb{H}[s]$ and we have finished the first part.

If $a_1 > b_1$, choose the polynomial $P(s) = s^{r-2} + p_1 s^{r-3} + \cdots + p_{r-2} \in \mathbb{H}[s]$ in such a way that $Q(s) = s^{n-2} + q_1 s^{n-3} + \cdots + q_{n-2} = B(s)P(s) \in \mathbb{H}[s]$ makes $a_1 > b_1 + p_1 = q_1$. Then we have

$$\lim_{\varepsilon\to 0} \Delta_k(s^2 Q^* + \varepsilon A^*) = \Delta_k(Q^*) > 0, \quad 1 \leq k \leq n-2$$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \Delta_{n-1}(s^2 Q^* + \varepsilon A^*) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \begin{vmatrix} \Omega_{n-2}(s^2 Q^* + \varepsilon A^*) & \mathbf{0} \\ \mathbf{0} & \varepsilon a_1 \end{vmatrix} =$$
$$= a_1 \Delta_{n-2}(Q^*) > 0, \quad \text{and}$$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \Delta_n(s^2 Q^* + \varepsilon A^*) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \begin{vmatrix} \Omega_{n-3}(s^2 Q^* + \varepsilon A^*) & \mathbf{0} \\ \mathbf{0} & \begin{matrix} 1 + \varepsilon a_2 & \varepsilon & 0 \\ q_1 + \varepsilon a_3 & \varepsilon a_1 & 0 \\ q_2 + \varepsilon a_4 & \varepsilon a_2 & \varepsilon \end{matrix} \end{vmatrix} =$$

$$= \lim_{\varepsilon \to 0} \begin{vmatrix} \Omega_{n-3}(s^2Q^* + \varepsilon A^*) & & \mathbf{0} & \\ & 1+\varepsilon a_2 & 1 & 0 \\ \mathbf{0} & q_1 + \varepsilon a_3 & a_1 & 0 \\ & q_2 + \varepsilon a_4 & \varepsilon a_2 & 1 \end{vmatrix} = (a_1 - q_1)\Delta_{n-3}(Q^*) > 0.$$

Consequently, for a sufficiently small c_0, we have $Ac_0 + BP = (s^2Q^* + c_0A^*)^* \in \mathbb{H}[s]$ and $\deg(Ac_0 + BP) = n$, i.e. $q^*(B/A) \leq r - 2$. ∎

The results of this chapter may be used in a computerized search for low order stabilizers of given rational functions. Appendix 1 contains a CTRL-C program for that purpose.

4. Lower Bounds

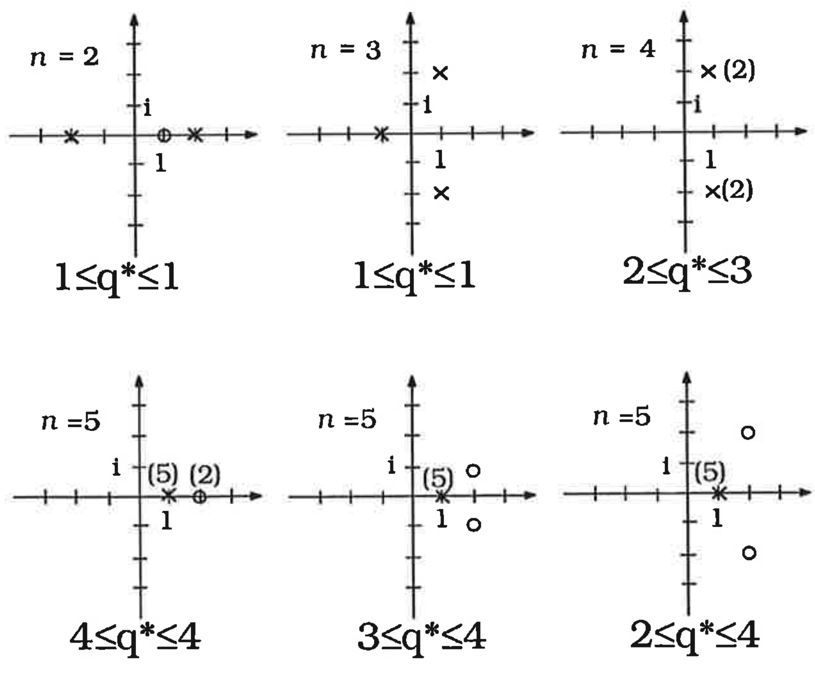
Since all coefficients of a monic Hurwitz polynomial are positive, a necessary condition for $q^*(B/A) \leq k$ is that there are polynomials, C and D, of degree k, which make the coefficients of the polynomial $AC + BD$ positive. Since these coefficients are obtained by matrix multiplication

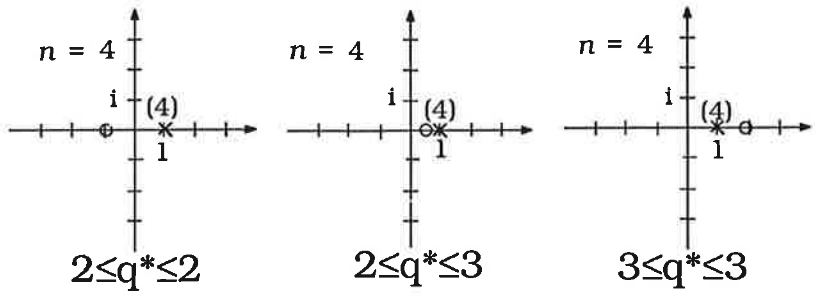
$$AC + BD = (1 \quad c_1 \quad \ldots \quad c_k \quad d_0 \quad \ldots \quad d_k)\mathrm{Res}_{k,k}(A, B)$$

the question is whether there is a linear combination of the rows in the matrix $\mathrm{Res}_{k,k}(A, B)$ having only positive coefficients.

The Pascal-program in Appendix 2 calls the NAG-library for linear programming to check the condition above. The programs of Appendix 1 and 2 together give upper and lower bounds on $q^*(B/A)$ for any rational functions, B/A. Next we give the pole-zero diagrams of some rational functions, together with the respective estimates of q^*.

EXAMPLE 4.1
In the figures, the estimates of $q^*(B/A)$ are shown in some different cases. A zero of A is marked with '×' in the complex plane. Parenthesis are used to indicate multiplicities greater than one. A zero of B is marked with 'o'. For example, the first figure indicates that $q^*\left(\frac{(s-1)}{(s+2)^2(s-2)}\right) = 1$.





The next theorem is just a different formulation of the previous condition.

THEOREM 4.1 If the equation $\mathrm{Res}_{k,l}^{r}(A,B)\delta = 0$, $r = \max(k + \deg A, l + \deg B)$,has a solution, $\delta = (\delta_r \dots \delta_0)^T \neq 0$, with $\delta_0, \dots, \delta_r \geq 0$, then there is no stabilizer, D/C, of B/A, with $\deg C \leq k, \deg D \leq l$, in particular $q^*(B/A) > \min(k,l)$.

Proof: The condition implies that

$$(1 \quad c_1 \quad \dots \quad c_k \quad d_0 \quad \dots \quad d_l)\mathrm{Res}_{k,l}^{r}(A,B)(\,\delta_r \quad \dots \quad \delta_0\,)^T = 0,$$

so D/C with $\deg C = k$ and $\deg D \leq l$ can not be a stabilizer since the polynomial $AC + BD$ must contain at least one negative coefficient. By Theorem 3.3 no stabilizer of lower degree can exist either and we are finished. ∎

EXAMPLE 4.2
Consider

$$\frac{B(s)}{A(s)} = \frac{s-2}{(s+1)(s-2+i)(s-2-i)} = \frac{s-2}{s^3 - 3s^2 + s + 5}.$$

Gaussian elimination gives

$$\mathrm{Res}_{1,1}^{4}(A,B)\delta = 0 \Leftrightarrow \begin{pmatrix} 1 & -3 & 1 & 5 & 0 \\ 0 & 1 & -3 & 1 & 5 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} \delta_4 \\ \delta_3 \\ \delta_2 \\ \delta_1 \\ \delta_0 \end{pmatrix} = 0$$

$$\Leftrightarrow \delta^T = (\,1 \quad 5 \quad 4 \quad 2 \quad 1\,)\delta_0,$$

and by Theorem 4.1 we have $q^*(B/A) \geq 2$. Since the system is of third order, equality must hold. □

We will now use Theorem 4.1 to find a class of systems that need a full order regulator for stabilization.

THEOREM 4.2 Suppose

$$A(s) = s^n + a_1 s^{n-1} + \dots + a_n = (s - \beta_1)\cdots(s - \beta_n), \quad \beta_1, \dots, \beta_n \in \mathbb{R}^+.$$

Then $1/A$ has no stabilizer D/C with $\deg D < n - 1$. In particular, $q^*(1/A) = n - 1$.

To prove this theorem we will need a lemma.

LEMMA 4.1 The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = c_1{\beta_1}^x + \cdots + c_n{\beta_n}^x$$

where $\beta_1, \ldots, \beta_n \in \mathbb{R}^+$ and $c_1, \ldots, c_n \in \mathbb{R}$, has at most $n-1$ zeros.

Proof: The statement is true for $n = 1$. Assume that it is true for $n = m$. We have

$$0 = c_1{\beta_1}^x + \cdots + c_{m+1}{\beta_{m+1}}^x \Leftrightarrow -c_{m+1} = c_1\left(\frac{\beta_1}{\beta_{m+1}}\right)^x + \cdots + \left(\frac{\beta_m}{\beta_{m+1}}\right)^x.$$

Thus, putting $g(x) = c_1(\beta_1/\beta_{m+1})^x + \cdots + c_m(\beta_m/\beta_{m+1})^x$, we want to prove that the equation $g(x) = -c_{m+1}$ has at most m solutions. This follows however from the facts that

$$g'(x) = c_1 \ln(\beta_1/\beta_{m+1})\left(\frac{\beta_1}{\beta_{m+1}}\right)^x + \cdots + c_1 \ln(\beta_m/\beta_{m+1})\left(\frac{\beta_m}{\beta_{m+1}}\right)^x$$

by the induction assumption, has at most $m-1$ zeros, and that between every two zeros of g, there must be a zero of g' (Rolle's theorem). The proof is completed by induction over n. ∎

Now we are ready to prove the theorem.

Proof of Theorem 4.2: Define $(\delta_k)_{k=0}^{\infty}$ through the difference equation

$$\delta_{k+n} + a_1\delta_{k+n-1} + \cdots + a_n\delta_k = 0$$

and the initial conditions $\delta_0 = \cdots = \delta_{n-2} = 0$, $\delta_{n-1} = 1$. If the numbers $\beta_1, \ldots, \beta_n$ are all different, it follows from the theory of difference equations that there are $c_1, \ldots, c_n$, such that $\delta_k = c_1{\beta_1}^k + \cdots + c_n{\beta_n}^k$, $k \geq 1$. Lemma 4.1 together with the initial conditions show that $\delta_k > 0$ for $k \geq n$. By continuity the inequalities $\delta_k \geq 0$ must hold even if A has multiple zeros. It is obvious from the definition of $\delta = (\delta_1 \cdots \delta_{n+k})^T$, that

$$\mathrm{Res}_{k,n-2}^{n+k}(A, 1)\delta =$$

$$= \begin{pmatrix} 1 & a_1 & & \cdots & a_n & & 0 \\ & \ddots & \ddots & & & \ddots & \\ & & 1 & a_1 & a_2 & \cdots & a_n \\ & & & & 1 & & 0 \\ & & 0 & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} \delta_{n+k} \\ \vdots \\ \delta_0 \end{pmatrix} = 0.$$

The statements to be proved now follow from Theorem 4.1 and the fact that $q^*(B/A) \leq \max(\deg B - 1, \deg A - 1)$ (chapter 2.). ∎

EXAMPLE 4.3

$$q^*\left(\frac{1}{(s-i)(s+i)(s-1)^3}\right) \geq 2$$

since if D/C is a stabilizer of $\frac{1}{(s-i)(s+i)(s-1)^3}$, then $\frac{D}{C(s+i)(s-i)}$ must be a stabilizer of $\frac{1}{(s-1)^3}$, thus Theorem 4.2 implies that $\deg D \geq 2$. □

Finally follows another proof that $q^*(1/(s-1)^n) = n-1$. This may seem unnecessary, but being a bit more explicit, it could serve as an illustration of the previous proof. The convention $\binom{n}{i} = 0$ for $i < 0$ and $i > n$, will be convenient.

LEMMA 4.2

$$\binom{j-1}{i}\binom{n+i}{j} + \binom{j-1}{i-1}\binom{n+i-1}{j} = \binom{n+i-1}{i}\binom{n}{j-i},$$

for $i \geq 0$, $n, j \geq 1$.

Proof: For $1 \leq i \leq j-1 \leq n+i-2$ we have

$$\binom{j-1}{i}\binom{n+i}{j} + \binom{j-1}{i-1}\binom{n+i-1}{j} =$$

$$\frac{(j-1)!(n+i)!}{i!(j-1-i)!j!(n+i-j)!} + \frac{(j-1)!(n+i-1)!}{(i-1)!(j-i)!j!(n+i-j-1)!} =$$

$$= \frac{(j-1)!(n+i-1)! \cdot nj}{i!(j-i)!j!(n+i-j)!} = \binom{n+i-1}{i}\binom{n}{j-i}.$$

The generalization to $i \geq 0, n, j \geq 1$ includes nothing more than checking some special cases were some of the terms are zero. This is left to the reader. ■

THEOREM 4.3 The rational function $1/(s-1)^n$ has no stabilizer D/C with $\deg D < n-1$. In particular, $q^*(1/(s-1)^n) = n-1$.

Proof: Put $\delta = (\,\delta_{n+k} \;\; \ldots \;\; \delta_0\,)^T$, with $\delta_i = \binom{i}{i-n+1}$. Then

$$\mathrm{Res}^{n+k}_{k,n-2}((s-1)^n, 1)\delta =$$

$$= \begin{pmatrix} 1 & -n & & \cdots & & 1 & & 0 \\ & \ddots & \ddots & & & & \ddots & \\ & & 1 & -n & \binom{n}{2} & \cdots & & 1 \\ & & & & 1 & & & 0 \\ & & 0 & & & \ddots & & \\ & & & & & & & 1 \end{pmatrix} \begin{pmatrix} \binom{n+k}{1+k} \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Only the first $k+1$ elements of this product need to be calculated:

$$\sum_{i=0}^{j}(-1)^{j-i}\binom{n}{j-i}\binom{n+i-1}{i} =$$

$$= \sum_{i=0}^{j}(-1)^{j-i}\left[\binom{j-1}{i}\binom{n+i}{j} + \binom{j-1}{i-1}\binom{n-i-1}{j}\right] =$$

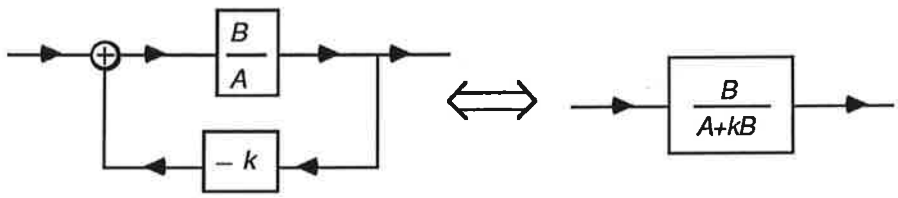
$$(-1)^j\binom{j-1}{-1}\binom{n-1}{j} + (-1)^0\binom{j-1}{j}\binom{n+j}{j} = 0$$

for $1 \leq j \leq k+1$. The rest of the elements are trivially zero, i.e. $\mathrm{Res}^{n+k}_{k,n-2}((s-1)^n, 1)\delta = 0$, and the theorem follows from Theorem 4.1. ■

5. An Invariant Theory Approach.

This chapter, I guess, deserves its own introduction. The one who first proposed the stabilizing order, q^*, as a subject for my master thesis, was Prof. C.I. Byrnes when visiting Lund in April 1986. He asked me to try to determine the ring of all output feedback invariant polynomials in the coefficients of a transfer function. This ring is interesting, since also the q^*-function is an output feedback invariant function of those coefficients [Byrnes-Crouch]. Theorem 5.1 gives a set of generators of the ring while Theorem 5.2, totally independent of the rest of the chapter, will prove that $q^*(B/A)$ is a function of some such output feedback invariant polynomials in the coefficients of A and B.

The action of feedback with constant gain, k, on a finite dimensional linear system is illustrated below:



This motivates the following notations and definitions. Let Γ be the the group of all matrices of the form

$$\sigma_k = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}, \quad k \in \mathbb{R},$$

under multiplication. Γ is obviously a subgroup of $SL(2)$.

DEFINITION 5.1 The group Γ operates on polynomials, $I : \mathbb{R}^{2n} \to \mathbb{R}$, in the following way:

$$(\sigma_k \cdot I)(x_1, y_1, \ldots, x_n, y_n) = I(x_1, y_1 + kx_1, \ldots, x_n, y_n + kx_n).$$

□

Observe that

$$\sigma_k \cdot (\sigma_l \cdot I) = (\sigma_k \sigma_l) \cdot I = \sigma_{k+l} \cdot I$$
$$\sigma_k \cdot (I_1 I_2 + I_3) = (\sigma_k \cdot I_1)(\sigma_k \cdot I_2) + \sigma_k \cdot I_3$$

for polynomials, $I, I_1, I_2, I_3 : \mathbb{R}^{2n} \to \mathbb{R}$, and $k, l \in \mathbb{R}$.

DEFINITION 5.2 A polynomial,$I(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$, in $2n$ variables, is said to be a *polynomial invariant of the group* Γ if $I = \sigma_k \cdot I$ for all $\sigma_k \in \Gamma$. The set of polynomial invariants of Γ in $2n$ variables forms a subring, $R(n)$, of $\mathbb{R}[x_1, \ldots, y_n]$. □

By [Byrnes-Crouch], $(x_i)_{i=1}^n$ and $(x_i y_j - x_j y_i)_{1 \leq i,j \leq n}$ are elements in $R(n)$. These correspond to the fact that the zeros of a system, as well as the branch-points of the rootlocus, are output feedback invariants. Below (Theorem 4.1) it will be proved that they generate all of $R(n)$. First we need some definitions and lemmata. Lemmata 5.1 and 5.2 are found in [Schur].

DEFINITION 5.3 A polynomial $I(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$, is said to be *homogeneous of degree r in the pair (x_i, y_i)*, if every term in I contains exactly r factors from the pair (x_i, y_i). □

LEMMA 5.1 Every element in $R(n)$ can be written as a sum of terms in $R(n)$, all of them being homogeneous in each pair $(x_j, y_j), 1 \leq j \leq n$.

Proof: For a fixed j, any $I \in R(n)$ can be written as a sum, $I = H_m + H_{m-1} + \cdots + H_0$, where $H_i, 0 \leq i \leq m$, are homogeneous of degree i in (x_j, y_j). We would like to prove that the terms, H_i, are also invariants of Γ.

For $\sigma_k \in \Gamma$ it is clear that $\sigma_k \cdot H_i$ is also homogeneous of degree i in (x_j, y_j) and from the polynomial identity

$$I = \sigma_k \cdot I = \sigma_k \cdot H_m + \sigma_k \cdot H_{m-1} + \cdots + \sigma_k \cdot H_0,$$

follows that $H_i \in R(n)$ for all i. Repeated use of the argument for all pairs (x_j, y_j) completes the proof. ■

LEMMA 5.2 (POLARIZATION.) Given any polynomial $I \in R(n)$, homogeneous in each pair (x_j, y_j), there is another polynomial $\hat{I} \in R(m), m \geq n$, homogeneous of degree 0 or 1 in each pair (x_j, y_j), such that

$$I(x_1, \ldots, y_n) = \hat{I}(x_1, \ldots, y_n, x_{i_{n+1}}, y_{i_{n+1}}, \ldots, x_{i_m}, y_{i_m})$$

for some values of the indexes i_l, $1 \leq i_l \leq n$, $n+1 \leq l \leq m$.

Proof: Suppose $I(x_1, \ldots, y_n) \in R(n)$. Then for a fixed i, $1 \leq i \leq n$,

$$\hat{I}_i(x_1, \ldots, y_n, x_{n+1}, y_{n+1}) = x_{n+1}\frac{\partial I}{\partial x_i} + y_{n+1}\frac{\partial I}{\partial y_i}$$

belongs to $R(n+1)$.This is evident since

$$\begin{aligned}
\sigma_k \cdot \hat{I}_i &= x_{n+1}\Big(\sigma_k \cdot \frac{\partial I}{\partial x_i}\Big) + (y_{n+1} + kx_{n+1})\Big(\sigma_k \cdot \frac{\partial I}{\partial y_i}\Big) = \\
&= x_{n+1}\left[\sigma_k \cdot \frac{\partial I}{\partial x_i} + k\Big(\sigma_k \cdot \frac{\partial I}{\partial y_i}\Big)\right] + y_{n+1}\Big(\sigma_k \cdot \frac{\partial I}{\partial y_i}\Big) = \\
&= x_{n+1}\frac{\partial(\sigma_k \cdot I)}{\partial x_i} + y_{n+1}\frac{\partial(\sigma_k \cdot I)}{\partial y_i} = \hat{I}_i.
\end{aligned}$$

Also, if I is of degree r in (x_i, y_i), we have $\hat{I}_i(x_1, \ldots, y_n, x_i, y_i) = rI(x_1, \ldots, y_n)$ so I can be recovered from $\hat{I}_i$. Obviously, the new polynomial invariant is of a lower degree in (x_i, y_i), and by repeated use of the construction we get a polynomial invariant with the desired property. ■

By Lemma 5.1 and 5.2, in searching for a set of generators of $R(n)$ it suffices to consider Γ-invariant polynomials, which are linear in each of a number of variable pairs, (x_i, y_i). Let $S(n)$ denote the set of all such polynomials $\mathbb{R}^{2n} \to \mathbb{R}$.

LEMMA 5.3 No polynomial in $S(n), n \geq 1$ has a term with more y-factors than x-factors.

Proof: Suppose l is the greatest number of y-factors present in any of the terms of the polynomial $I(x_1, \dots, y_n) \in S(n)$. If $l > m = n - l$, then for some choise of the indexes, $i_1, \dots, i_n \in \{1,2\}$, the polynomial $J(x_1, y_1, x_2, y_2) = I(x_{i_1}, y_{i_1}, \dots, x_{i_n}, y_{i_n}) \in R(2)$ would be homogeneous of degrees l and m in the pairs (x_1, y_1) and (x_2, y_2) respectively and also contain some term with exactly l y-factors. We want to show that such a polynomial does not exist. In other words, assuming that

$$J(x_1, y_1, x_2, y_2) =$$

$$= \begin{pmatrix} x_1^l & x_1^{l-1}y_1 & \dots & y_1^l \end{pmatrix} \begin{pmatrix} c_{1,1} & \cdots & c_{1,m} \\ & & \vdots \\ \vdots & & c_{l-m+1,m} \\ & \cdot^{\cdot^{\cdot}} & \\ c_{l,1} & & 0 \end{pmatrix} \begin{pmatrix} x_2^m \\ x_2^{m-1}y_2 \\ \vdots \\ y_2^m. \end{pmatrix} \in R(2),$$

$m < l$, we want to prove that $c_{l-i+1,i} = 0,\ 1 \leq i \leq m$. Note that

$$\begin{aligned} \sigma_k \cdot \begin{pmatrix} x_1^l & x_1^{l-1}y_1 & \dots & y_1^l \end{pmatrix} &= \begin{pmatrix} x_1^l & x_1^{l-1}(y_1 + kx_1) & \dots & (y_1 + kx_1)^l \end{pmatrix} \\ &= \begin{pmatrix} x_1^l & x_1^{l-1}y_1 & \dots & y_1^l \end{pmatrix} S_l(k) \end{aligned}$$

$$\text{where} \quad S_l(k) = \begin{pmatrix} 1 & \binom{1}{0}k & \dots & \binom{l}{0}k^l \\ & 1 & \ddots & \vdots \\ & & \ddots & \binom{l}{l-1}k \\ 0 & & & 1 \end{pmatrix}_{(l+1)\times(l+1)}.$$

The fact that $\sigma_k \cdot J = J$ may be written in terms of $C = (c_{i,j})$.

$$C = S_l(k) C S_m(k)^T \Rightarrow S_l(-k)C = S_l(k)^{-1}C = C S_m(k)^T$$

$$\begin{pmatrix} * & & \\ & & c_{l-m,m} - (l-m+1)kc_{l-m+1,m} \\ & \cdot^{\cdot^{\cdot}} & c_{l-m+1,m} \\ c_{l-1,1} - lkc_{l,1} & \cdot^{\cdot^{\cdot}} & \\ c_{l,1} & & 0 \end{pmatrix} = S_l(-k)C =$$

$$= C S_m(k)^T = \begin{pmatrix} & * & & \\ & & & c_{l-m,m} \\ & & \cdot^{\cdot^{\cdot}} & c_{l-m+1,m} \\ & c_{l-2,2} + 2kc_{l-2,3} & \cdot^{\cdot^{\cdot}} & \\ c_{l-1,1} + kc_{l-1,2} & c_{l-1,2} & & \\ c_{l,1} & & & 0 \end{pmatrix} \Rightarrow$$

$$\Rightarrow 0 = c_{l-m+1,m} = c_{l-m+2,m-1} = \dots = c_{l,1}.$$

By that we are finished. ∎

Now we are ready to prove the main theorem of this chapter.

THEOREM 5.1 For $n \geq 1$, let $P(n)$ be the subring of $\mathbb{R}[x_1, \ldots, y_n]$ generated by the polynomials $(x_i)_{i=1}^n$ and $(x_i y_j - x_j y_i)_{1 \leq i,j \leq n}$. Then $R(n) = P(n)$.

Proof: It is easy to check that these polynomials belongs to $R(n)$, so that $P(n) \subset R(n)$. It remains to prove that $R(n) \subset P(n)$. Lemmata 5.1 and 5.2 show that it is sufficient to show that $S(n) \subset P(n)$. To do this, we use induction over n. Obviously the only elements in $S(1)$ are the multiples of x_1, so the theorem is true for $n = 1$.

Now suppose that $S(m) \subset P(m)$ for $1 \leq m \leq n$. Let $I \in S(n+1)$. Then $I(x_1, \ldots, y_{n+1}) = x_{n+1} J_a(x_1, \ldots, y_n) + y_{n+1} J_b(x_1, \ldots, y_n)$ where J_a and J_b are linear in every pair $(x_i, y_i), 1 \leq i \leq n$. We have

$$\begin{aligned} I = \sigma_k \cdot I &= x_{n+1}(\sigma_k \cdot J_a) + (y_{n+1} + k x_{n+1})(\sigma_k \cdot J_b) = \\ &= x_{n+1}(\sigma_k \cdot J_a + k(\sigma_k \cdot J_b)) + y_{n+1}(\sigma_k \cdot J_b), \end{aligned}$$

so $I \in S(n+1)$ implies $\sigma_k \cdot J_a = J_a - k(\sigma_k \cdot J_b)$ and $\sigma_k \cdot J_b = J_b$. Hence $J_b \in S(n) \subset P(n)$ (the induction assumption). Since $I = x_{n+1} J_a + y_{n+1} J_b$ and $I \in S(n+1)$ Lemma 5.3 shows that every term in J_b must have more x-factors than y-factors. By these facts, it is possible to write $J_b = x_1 I_1 + \ldots + x_n I_n$, where $I_1, \ldots, I_n \in P(n-1)$. Put $J_c = y_1 I_1 + \ldots + y_n I_n$ to get

$$\begin{aligned} \sigma_k \cdot (J_a + J_c) &= \sigma_k \cdot J_a + (y_1 + k x_1) I_1 + \ldots + (y_n + k x_n) I_n = \\ &= J_a - k(\sigma_k \cdot J_b) + k J_b + J_c = J_a + J_c, \end{aligned}$$

i.e. $J_a + J_c \in S(n) \subset P(n)$. Thus, since

$$\begin{aligned} I &= x_{n+1}(J_a + J_c) + y_{n+1} J_b - x_{n+1} J_c = \\ &= x_{n+1}(J_a + J_c) + (x_1 y_{n+1} - x_{n+1} y_1) I_1 + \ldots + (x_n y_{n+1} - x_{n+1} y_n) I_n, \end{aligned}$$

it follows that $I \in P(n+1)$ and we are finished. ∎

The next theorem is completely independent of the theory above, and shows how polynomials in $P(n)$ arise in the stydy of the q^*-function.

THEOREM 5.2 The quantities $(b_i)_{i=1}^n$ and $(b_i a_j - b_j a_i)_{1 \leq i,j \leq n}$ determine the number

$$q^* \left(\frac{b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n} \right)$$

uniquely.

Proof: Suppose that B/A_1 and B/A_2 are rational functions and $B' A_1 - B A_1' = B' A_2 - B A_2'$. Then $(A_1/B)' = (A_2/B)'$, i.e. $A_1 = A_2 + kB$ and $q^*(B/A_1) = q^*(B/A_2)$. This shows that $q^*(B/A)$ is a well defined function of B and $B'A - BA'$, and since the coefficients of $B'A - BA'$ are all linear combinations of elements in $(b_i a_j - b_j a_i)_{1 \leq i,j \leq n}$, the proof is complete. ∎

References.

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Appendix A.

This is a CTRL/C-program computing low order stabilizers according to chapter 3.

```
disp(['Given two polynomials, A and B, with degA>=2 and degA>=degB>=0,'
      'the procedure tries to calculate polynomials, C and D, of a    '
      'relatively low degree, such that AC+BD has all its zeros in the'
      'left half of the complex plane.                                ']);
m=size(B);
m=m(2)-1;
n=size(A);
n=n(2)-1;
if min([n-2,n-m])<0, ...
  disp(['You have to define A and B as two row matrices according to the';...
        'conditions given above, before calling the procedure.          ']),...
  return,...
end;
//
//Define the resultant matrix.
//
Psia=A;
for i=1:n-1, Psia=[Psia,0*ones(i,1);0*ones(1,i),A];
if n=m,P=B;else,P=[0*ones(1,n-m),B];
Psib=P;
for i=1:n-1, Psib=[Psib,0*ones(i,1);0*ones(1,i),P];
Psi=[Psia;Psib];
q0=n-1;
t0=3;
//
//Check the minimum phase case.
//
if min([m-1,max(real(roots(B)))])<0, ...
  t0=2; ...
  C=1; ...
  if n=m,D=1;P=1;q0=0;...
  else,...
    if n=m+1,D=1;P=[0 1];q0=0; ...
    else, ...
      D=poly(-ones(1,n-m-1))';P=[0 1];q0=n-m-1; ...
      if m>=1,u=A(2)/A(1)-B(2)/B(1);else,u=A(2)/A(1);end, ...
      if u>0, ...
        if n=m+2,D=1;P=[0 0 1];q0=0; ...
```

```
          else,...
            D=poly(-.5*u*ones(1,n-m-2)/(n-m-2))';P=[0 0 1];q0=n-m-2; ...
          end, ...
        end, ...
      end, ...
    end, ...
    while max(real(roots(A+conv(conv(D,P),B))))>=0,D=2*D; ...
  end,
  //
  //t=0: Put poles in origo.
  //t=1: Put poles in infinity.
  //
  for t=0:1, ...
    for k=0:min([q0-t*(n-m),n-1])-1, ...
      for l=0:min([q0-1+t*n,n-2,n-m+k]), ...
        Alpha=Psi([n-k+1:n,2*n-l:2*n] ,[2*n-k-l:2*n]); ...
        if cond(Alpha)<1D+03, ...
          Beta=Psi([n-k+1:n,2*n-l:2*n] ,[n-k:2*n-k-l-1]); ...
          CD=-[A(n-l+1:n+1),0*ones(1,k)]/Alpha; ...
          R0=A(1:n-l)+CD*Beta; ...
          si=max(real(roots(R0))); ...
          if si<0, ...
            if R0(1)<> 0, ...
              q=t*(n-m+k)+(1-t)*max([l,k]); ...
              if q<q0, ...
                q0=q;k0=k;l0=l;t0=t;Alp0=Alpha;r1=R0(1); ...
                if k>0,C=[1,CD(1:k)];else,C=1;end, ...
                D=CD(k+1:k+l+1); ...
                qklts=[q,k,l,t,si]; ...
              end, ...
            end, ...
          end, ...
        end, ...
      end, ...
    end, ...
    if t0<>1, ...
      A=A(n+1:-1:1);B=B(m+1:-1:1); ...
      Psia=Psia([n:-1:1],[2*n:-1:1]); ...
      Psib=Psib([n:-1:1],[1:n-m,2*n:-1:n-m+1]); ...
      Psi=[Psia;Psib];...
    end, ...
  end;
  if t0=3,...
    disp('No such polynomials of degree less than degA-1 are found.'),...
    return,...
  end;
  //
  //Move the last poles into the left half plane.
  //
```

```
if t0<2, ...
  k=k0; ...
  l=l0; ...
  Delta=0*ones(1,k+l+1);...
  for j=1:k+l+1, ...
    sig=0; ...
    Beta=Psi([n-k+1:n,2*n-l:2*n] ,[n-k:2*n-k-l+j-1]); ...
    Delta(j)=r1; ...
    while sig>=0, ...
      sig0=sig; ...
      CD=(Delta-[A(n-l+1:n+1),0*ones(1,k)])/Alp0 ; ...
      R0=Psi(1,1:n-l+j)+CD*Beta; ...
      sig=abs(R0(2))*max(real(roots(R0))); ...
      Delta(j)=Delta(j)/2; ...
    end, ...
    r1=Delta(j); ...
  end;...
  if k>0,C=[1,CD(1:k)];else,C=1;end;...
  D=CD(k+1:k+l+1);...
  if n+k-l-m>0,P=[0*ones(1,n+k-l-m),1];else,P=1;end, ...
  if t0=1, ...
    A=A(n+1:-1:1);B=B(m+1:-1:1); ...
    C=C(k+1:-1:1);D=conv(D(l+1:-1:1),P(n+k-l-m+1:-1:1)); ...
    P=1;R0(1)=R0(n+k+1); ...
  end; ...
end,
//
//Make the regulator proper.
//
while size(D')>size(C'),...
  ACBD=conv(A,C)+conv(B,conv(D,P)); ...
  C=[ACBD(1)*A(1)/abs(ACBD(1)),C];P=[0,P];...
  loop='C(1)=C(1)/2;C=2*C;D=2*D;';...
  while max(real(roots(conv(A,C)+conv(B,conv(D,P)))))>=0,]loop[;...
end,...
end,
A,B,C,D,
zero=roots(conv(A,C)+conv(B,conv(D,P))),
```

An execution in CTRL-C would look like this:

```
[>  a=[1 -2 0 8 -12 8]

 A      =

     1.   -2.    0.    8.  -12.    8.
```

```
[>  roots(a)

ANS    =

  -2.0000 + 0.0000i
   1.0000 + 1.0000i
   1.0000 + 1.0000i
   1.0000 - 1.0000i
   1.0000 - 1.0000i


[>  b=[1 0 3 6 10]

B      =

    1.     0.     3.     6.    10.


[>  roots(b)

ANS    =

   1.0000 + 2.0000i
   1.0000 - 2.0000i
  -1.0000 + 1.0000i
  -1.0000 - 1.0000i


[>  do qlow

GIVEN TWO POLYNOMIALS, A AND B, WITH DEGA>=2 AND DEGA>=DEGB>=0,
THE PROCEDURE TRIES TO CALCULATE POLYNOMIALS, C AND D, OF A
RELATIVELY LOW DEGREE, SUCH THAT AC+BD HAS ALL ITS ZEROS IN THE
LEFT HALF OF THE COMPLEX PLANE.

A      =

    1.    -2.     0.     8.   -12.     8.



B      =

    1.     0.     3.     6.    10.



C      =
```

```
   1.0D+04 *

   0.0001    3.4359    6.5536


D      =

   1.0D+05 *

  -0.3432    0.0381    2.4408


ZERO   =

 -24.2256 + 0.0000i
  -0.9724 +13.7610i
  -0.9724 -13.7610i
  -4.8951 + 7.4825i
  -4.8951 - 7.4825i
  -0.6446 + 2.7620i
  -0.6446 - 2.7620i
```

Appendix B.

This is a Pascal-program that calls a NAG-program to compute a lower bound on q^* according to chapter 4.

```
program underq(input,output);
const max=20;
      bigmax=400;
      maxint=1E+20;
type  vector=array[1..max] of double;
      bigvector=array[1..bigmax] of double;
      intvect=array[1..max] of integer;
      matrix=array[1..max] of vector;
var   m,n,i,j,k,liwork,lwork,ifail: integer;
      istate,iwork:intvect;
      objlp:double;
      b,a,bl,bu,x,clamda:vector;
      work:bigvector;
      r: matrix;
procedure E04MBF(itmax,msglvl,n,nclin,nctotl,nrowa:integer;
                 a:matrix;
                 bl,bu,cvec:vector;
                 linobj:boolean;
                 var x:vector;
                 var istate:intvect;
                 var objlp:double;
                 var clamda:vector;
                 var iwork:intvect;
                 var liwork:integer;
                 var work:bigvector;
                 var lwork,ifail:integer                  );extern;
begin
   lwork:=bigmax;
   liwork:=max;
   bu[1]:=1;
   bl[1]:=bu[1];
   for i:=2 to max do
   begin
      bu[i]:=maxint;
      bl[i]:=0;
   end;
   writeln('Give the coefficients of the numerator, B.');
```

```
    i:=1;
    while not eoln do
    begin
       read(b[i]);
       i:=i+1;
    end;
    m:=i-2;
    readln;
    writeln('Give the coefficients of the denominator, A.');
    i:=1;
    while not eoln do
    begin
       read(a[i]);
       i:=i+1;
    end;
    n:=i-2;
    k:=-1;
    ifail:=1;
    while (k<n-1) and (ifail<>0) do
    begin
    (****************************
    Compute the resultant matrix.
    ****************************)
       k:=k+1;
       for i:=2 to 2*k+2 do
          bl[i]:=-maxint;
       for i:=1 to 2*k+2 do
          for j:=1 to 2*n do
             r[i,j]:=0;
       for i:=1 to k+1 do
          for j:=1 to n+1 do
             r[i,i-1+j]:=a[j];
       for i:=1 to k+1 do
          for j:=1 to m+1 do
             r[k+1+i,n-m+i-1+j]:=b[j];
       (****************************
        Search for a feasable point.
        ****************************)
       E04MBF(-1,-1,2*k+2,n+k+1,n+3*k+3,max,r,bl,bu,bl,false,x,istate,objlp,
              clamda,iwork,liwork,work,lwork,ifail);
    end;
    writeln('qstar>=',k);
    write('D=');
    for i:=k+2 to 2*k+2 do
       write(x[i]:6:3);
    writeln;write('C=');
    for i:=1 to k+1 do
       write(x[i]:6:3);
end.
```

An execution of the program looks like this:

```
Give the coefficients of the numerator, B.
1 0 3 6 10
Give the coefficients of the denominator, A.
1 -1 -4 16 -20 12
qstar>=         1
D=-0.917 5.917
C= 1.000 1.917
```