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ZEROS OF SAMPLED SYSTEMS

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Abstract

The zeros obtained when sampling a time continuous system are explored. Theorems are given which, if the time continuous transfer function is known and of finite order, give estimations of sample intervals for which the time discrete system will have a stable inverse. For the case only the Nyquist curve is known, some results are given.

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Sammanfattning

De nollställen som erhålls vid samplingen av ett tidskontinuerligt system behandlas. Satser ges som, om den tidskontinuerliga överföringsfunktionen är känd och av ändlig ordning, ger uppskattningar på samplingsintervall, för vilka det tidsdiskreta systemet kommer att ha en stabil invers. För fallet att bara Nyquistkurvan är känd ges några resultat.

ZEROS OF SAMPLED SYSTEMS

Bengt Mårtensson

Contents:

1. Introduction

It is well known that zeros outside the unit circle in the transfer function of a time discrete control system severly limits its performance, since it has not a stable inverse. As will be shown i chapter 2, if the time continuous transfer function G(s) is of order n, the sampled transfer function will in general have n-1 zeros, regardless of the number of zeros in G(s). While the sampling maps the poles according to $p \sim e^{ph}$, where h is the sample interval, no such simple transformation exists for the zeros. Further, it is shown in [1] that if the system is sampled fast enough, every time continuous system with a pole excess larger than 2 will give rise to a time discrete transfer function with zeros outside the unit circle.

The purpose of this paper is to give criteria which will guarantee the time discrete transfer function to have a stable inverse.

In chapter 2 we introduce some notations, and give a few basic results. Most of these are well known. The main chapter is chapter 3, where estimates of sample intervals are given, such that the time discrete system will have a stable inverse. Theorems 2-4 are more special, since they deal with different variants of the special case G(0) = 0. Probably the method used is of more importance than the theorems we prove. Finally, in chapter 4, results are given which concerns only with the behavior of the Nyquist curve.

2. Notation and basic facts

We let $N_u(f)$ $(P_u(f))$ denote the number of zeros (poles) of the function f outside the unit circle in the complex plane, while the number of zeros (poles) inside or on the unit circle are denoted by $N_s(f)$ $(P_s(f))$. Finally, the zeros (poles) strictly inside the unit circle are denoted by $N_{as}(f)$ $(P_{as}(f))$.

The argument f will be omitted when no ambiguity can result.

Many results will be based on the following extension of the well-known Rouché's theorem to meromorphic functions:

Theorem Let Ω be a simply connected domain containing a Jordan contour γ. Let f and g be meromorphic functions in Ω, analytic on γ, $f \neq 0$ on γ and assume that $|f(z)| > |g(z)|$ $z \in \gamma$. If we denote the number of zeros (poles) of f inside γ with $N(f)$ $(P(f))$ we have $N(f) - P(f) = N(f+g) - P(f+g)$.

Proof. Consider $I(\lambda) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z) + \lambda g'(z)}{f(z) + \lambda g(z)} dz$

The argument principle gives that $I(0) = N(f) - P(f)$, and $I(1) = N(f+g) - P(f+g)$. Due to the assumptions, $I(\lambda)$ is a continuous function of λ; the argument principle implies that it is integer-valued: hence it is constant. □

Next we state a few wellknown facts about the transformation $\mathbb{D}$: $G(s) \to H(z)$. As definition we take

$$H(z) = \mathbb{D}G = (1 - z^{-1})\ Z\ L^{-1}(G(s)/s) \tag{2.1}$$

Using the inversion formula for the Laplace transform and computing the Z-transform we get

$$H(z) = (1 - z^{-1}) \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{sh}}{z - e^{sh}} \frac{G(s)}{s} ds \tag{2.2}$$

where γ is a real number greater than the real part of all poles of $G(s)/s$. If G is sufficiently small at infinity, the integration contour can be closed either to the left or to the right without changing the value. The integral can the be computed using residue calculus. Closing the integration contour to the right gives

$$H(z) = (1 - z^{-1}) \sum_{k=-\infty}^{\infty} \frac{G[(\log z + 2\pi ik)/h]}{\log z + 2\pi ik} \tag{2.3}$$

Next we consider the special case of G(s) beeing a rational, strictly proper transfer function of order n, with a pole of order n_i in p_i, $i = 1,\ldots,\ell$. If G(s) is analytic at the origin, it can be written in the form

$$G(s) = s\left(\frac{G(0)}{s} + \sum_{i=1}^{\ell} \sum_{j=1}^{n_i} \frac{A_{ij}}{(s - p_i)^j}\right) \tag{2.4}$$

where $\sum_1^{\ell} n_i = n$. (2.1) gives us

$$H(z) = G(0)\, z^{-1} + (1 - z^{-1}) \sum\sum f_{ij}(z) \tag{2.5}$$

where f_{ij} is defined by

$$f_{ij}(z) = A_{ij} \frac{h^{j-1}}{(j-1)!} e^{p_i h} (-z\, d/dz)^{j-1} (z - e^{p_i h})^{-1} \tag{2.6}$$

If G(s) has a pole at the origin, i. e. $p_k = 0$, formulas (2.4) - (2.5) will be valid if we delete the G(0)-term and replace n_k with $n_k + 1$.

Since H(z) has a n_i-tuple pole in $z = e^{p_i h}$, $i = 1,\ldots,\ell$, H(z) must be a rational function of order n (provided that no pole-zero-cancellation occors). It is easily seen from (2.5) and (2.6) that H(z) will in general have n-1 zeros. The only possible exception from this is when

i) a pole-zero-cancellation occurs, or

ii) the coefficient before the leading term in the nominator vanishes. Since this coefficient is an analytic function of h, this can be the case only for isolated $h \in \mathbb{R}^+$, and it can be shown, for example with Rouché's theorem, that in every neighbourhood of such a h, H(z) will possess a zero with arbitrarily large modulus. Therefore, this case lacks interest for us.

We conclude that for all $h \in \mathbb{R}^+$, except possibly at isolated points, the following criterion is valid: H(z) has the same number of unstable poles and zeros, i. e. $N_u = P_u$, if and only if H(z) has one more pole than zero inside or on the unit circle, i. e. $N_s + 1 = P_s$.

If the leading term in the nominator vanishes, this can be viewed as an additional zero at infinity. With this interpretation, the criterion will be valid for all $h \in \mathbb{R}^+$.

Two special cases of transfer functions

For the future work we will need the time discrete transfer functions corresponding to two special classes of time continuous transfer functions. The first one concerns the n-tiple integrator:

Lemma 1. $$\mathbb{D}\, s^{-n} = \frac{h^n B_n(z)}{n!\,(z-1)^n} \qquad n = 0,1,\ldots \qquad (2.7)$$

where $B_n(z)$ is defined by

$$B_0(z) = z^{-1}$$

$$B_n(z) = b_1^n z^{n-1} + b_2^n z^{n-2} + \ldots + b_n^n \qquad n = 1,2,\ldots \qquad (2.8)$$

and

$$b_k^n = \sum_{\ell=1}^{k} (-1)^{k-\ell} \ell^n \binom{n+1}{k-\ell} \qquad k = 1,\ldots,n$$

Outline of proof (details are given in [1]): From the preceding and from the mapping of the poles it follows that $H(z) = \mathbb{D}\, s^{-n}$ is on the form

$$H(z) = \frac{b_1 z^{n-1} + \ldots + b_n}{(z-1)^n}$$

Using a step function as input, we get the output $y(t) = t^n/n!$ $t > 0$. Identification gives the desired result. □

Remark. Comparing with (2.4) - (2.6) we get

$$\mathbb{D}\, s^{-n} = (1 - z^{-1})\,(h^n/n!)\,(-z\, d/dz)^n\,(z - 1)^{-1} \qquad (2.9)$$

which can be rewritten as a recursion formula for $B_n(z)$. □

It can be shown, for example with the recursion formula for b_k^n (2.10) in [1], that $b_{n-k+1}^n = b_k^n$. Hence, if $z = z_o$ is a zero of $B_n(z)$, $z = z_o^{-1}$ is also a zero of $B_n(z)$. We claim that the zeros of $B_n(z)$ are simple and lie on the negative real axis, and therefore $B_n(z)$ will have zeros outside the unit circle if $n \geq 3$. Since $B_2(z) = z + 1$ this is obviously true for $n = 2$, and to prove the general case we assume that $B_{n-1}(z)$ has its n-2 zeros at the negative real numbers $x_1 < x_2 < \ldots < x_{n-2} < 0$. Comparison with (2.9) gives that for all n $B_n(z)$ has the same zeros as $C_n(z) = (-z\, d/dz)^n (z - 1)^{-1}$, except for the zero in $z = 0$. Rolles theorem applied to $C_{n-1}(z)$ now gives us that $C'_{n-1}(z)$, and hence $B_n(z)$, has a zero in each of the real intervals $]x_k, x_{k+1}[$, $k = 1,\ldots,n-3$, and in $]x_{n-2}, 0[$. Since $B_0(z)$ and $B_1(z)$ has no zeros, the statement is proved.

<u>Corollary</u>. $$\mathbb{D}\,\frac{s}{(s-p)^j} = (1 - z^{-1})\,\frac{h^{j-1}}{(j-1)!}\,\frac{a^{j-1}\,z\,B_{j-1}\left(\frac{z}{a}\right)}{(z-a)^j} \qquad (2.10)$$

where $j = 1,2,\ldots$ and $a = e^{ph}$.

<u>Proof</u>. $\mathbb{D}\, s/(s-p)^j = \{(1 - z^{-1})\, Z\, L^{-1}(s-p)^{-j}\}\,(z) =$

$= (1 - z^{-1})\,\{Z\, e^{pt}\, L^{-1} s^{-j}\}\,(z) \quad = (1 - z^{-1})\{Z \quad L^{-1} s^{-j}\}\left(\frac{z}{a}\right)$

Since we have

$\{Z\, L^{-1} s^{-j}\}\,(z) = (1 - z^{-1})^{-1}\,\mathbb{D}\, s^{-(j-1)}\,(z)$

lemma 1 gives the desired result. □

<u>Remark 1</u>. We note that the zeros are $z = 1$ and the zeros of $B_{j-1}(z\, e^{-ph})$. If $\mathrm{Re}\, p < 0$, the latter $j-2$ zeros will be inside the unit circle, provided h is sufficiently large. □

<u>Remark 2</u>. Comparing with (2.4) - (2.6) we get

$$f_{ij}(z) = A_{ij}\,(1 - z^{-1})^{-1}\,\mathbb{D}\,\frac{s}{(s-p_i)^j} =$$

$$= A_{ij}\,\frac{h^{j-1}}{(j-1)!}\,\frac{a_i^{j-1}\,z\,B_{j-1}(z/a_i)}{(z-a_i)^j}$$

where $a_i = e^{p_i h}$. Hence $f_{ij}(z)$ has its zeros at $z = 0$ and at the zeros of $B_{j-1}(z\, e^{-p_i h})$, which, if h is large enough, will be inside the unit circle. □

3. Theorems on G(s) known and of finite order

We are now in position to prove several theorems which will provide us with estimates of intervals in which h can be chosen so that the time discrete transfer function will have a stable inverse. From a practical viewpoint, the first one is probably the most interesting.

Theorem 1. Let G(s) be a strictly proper, asymptotically stable transfer function of order n with $G(0) \neq 0$, with a pole of order n_i in p_i, $i = 1,\ldots,\ell$. Assume that G(s) is on the form (2.4) and define

$$A = \max_{i,j} |A_{ij}/G(0)|$$
$$\sigma = -\max_i \operatorname{Re} p_i \qquad (3.1)$$
$$m = \max_i n_i$$

Then H(z) will have a stable invers if

$$\max(1,h^{m-1}) \frac{e^{-\sigma h}}{(1-e^{-\sigma h})^m} < \frac{1}{2An} \qquad (3.2)$$

If m = 1 (i. e. only simple poles) (3.2) can be rewritten as

$$h > \ln(2An+1)/\sigma. \qquad (3.3)$$

The left-hand function in (3.2) is a decreasing function of h if $h > (m-1)/\sigma$; hence there exists an h_o such that (3.2) is fulfilled for $h > h_o$.

In order to prove this theorem we first need a lemma:

Lemma 2. Let $f_{ij}(z)$ be defined by (2.6). If $|z| = 1$ we have

$$|f_{ij}(z)| \le \frac{|A_{ij}|\, h^{j-1}\, e^{\operatorname{Re} p_i h}}{(1-e^{\operatorname{Re} p_i h})^j}$$

Proof. Let g_{ij} be defined by

$$g_{ij}(z) = (-z\, d/dz)^{j-1} (z - e^{p_i h})^{-1}$$

We have to prove that

$$|g_{ij}(z)| \le (j-1)!/(1-e^{\operatorname{Re} p_i h})^j$$

If $j > 1$ and $|z| = 1$ we have

$$|g_{ij}| = |-z\, d/dz\, g_{ij-1}| = |z||g'_{ij-1}| = |g'_{ij-1}|$$

Hence

$$|g_{ij}| = |g_{i1}^{(j-1)}| = \left|\frac{(-1)^{j-1}(j-1)!}{(z-e^{p_i h})^j}\right| \le \frac{(j-1)!}{(1-e^{\operatorname{Re} p_i h})^j}$$

which proves the lemma. □

Proof of the theorem. We can without restriction assume that $G(0) = 1$. If so, (2.5) gives us

$$H(z) = z^{-1} + (1 - z^{-1})\sum_{ij}\sum f_{ij}(z) \tag{2.5'}$$

where f_{ij} is defined in (2.6). If the first term dominates absolutely on the unit circle, Rouché's theorem will give us that $N_{as}\{H(z)\} - P_{as}\{H(z)\} = N_{as}(z^{-1}) - P_{as}(z^{-1}) = 0 - 1 = -1$. The criterion in chapter 2 will then give us that $N_u\{H(z)\} = 0$ since $H(z)$ due to assumptions is asymptotically stable. For $|z| = 1$ lemma 2 and elementary inequalities give the estimation

$$|(1 - z^{-1})\Sigma\Sigma f_{ij}| \leq 2\,\Sigma\Sigma|A_{ij}|\; h^{j-1} \frac{e^{\mathrm{Re}\, p_i h}}{(1 - e^{\mathrm{Re}\, p_i h})^j} \leq$$

$$\leq 2nA \max(1,h^{m-1})\; e^{-\sigma h}/(1 - e^{-\sigma h})^m \tag{3.4}$$

Since the first term in (2.5') dominating absolutely on the unit circle is the same as the above estimated quantity beeing < 1, (3.2) follows.

The derivation of (3.3) from (3.2) is elementary. Finally, if we observe that $h^{m-1}\, e^{-\sigma h}$ has a negative derivative for $h > (m-1)/\sigma$, the last statement follows immediately. □

Probably the qualitative part of the theorem is the most important: All time continuous transfer functions satisfying the conditions will give rise to a time-discrete transfer function which has a stable inverse provided it is sampled slowly enough. This qualitative part, in the case of simple poles, was proved in [1].

Example 1. Consider $G(s) = \frac{1 - s}{(s + 2)(s + 3)}$, which has an unstable inverse in the time continuous case. Calculation of $H(z)$ shows that $H(z)$ has a stable inverse if $h > 1.2485$. (3.3) gives us the estimation $h > 1.8055$. □

The following example shows that when two poles approaches each other to form a double pole in the limit, theorem 1 can give very weak estimations.

Example 2. Consider $G_\varepsilon(s) = \frac{1 + \varepsilon}{(s + 1)(s + 1 + \varepsilon)}$ $\varepsilon > 0$

Clearly

$G_\varepsilon(s) \to \frac{1}{(s + 1)^2} = G_o(s)$

Theorem 1 gives that $\mathbb{D}G_\varepsilon$ has a stable inverse if $h > \ln[4(1 + \varepsilon^{-1}) + 1]$, which grows towards infinity when $\varepsilon \to 0$. However, $\mathbb{D}G_o$ has a stable inverse if $\max(1,h) \frac{e^{-h}}{(1 - e^{-h})^2} < \frac{1}{4}$, i. e. $h > 2.4665$. □

In cases like this, it might be possible to use the technique of the proof directly.

Remark 1. With poles on the imaginary axis, i. e. $\sigma = 0$, it is in general not true that the sampled system will have a stable inverse just if the sample interval is large enough. A simple counter example is $G(s) = s^{-3}$, where $H(z)$ has a zero $-2-\sqrt{3} \approx -3.732$. See also examples 4 and 5 in [1]. □

Remark 2. If we have one or more unstable poles, different things can happen. Say for example $p_1 = 1$ is a simple, unstable pole, while all the other poles are asymptotically stable. From (2.4) - (2.6) it follows that $f_{11} = A_{11} e^{p_1 h}/(z - e^{p_1 h}) \to A_{11}$ when $h \to \infty$. For large h, $H(z)$ will then come arbitrarily close to $G(0) z^{-1} + (1 - z^{-1})A_{11}$. If $|A_{11}| < |G(0)|/2$ the first term will dominate absolutely on the unit circle, and hence $H(z)$ must have an unstable zero. If $|A_{11}| \leq |G(0)|/2$ it is not possible to use this technique. (Direct calculation shows us that $H(z)$ will asymptotically have an unstable zero if and only if $|A_{11}| \leq |G(0)|/2$.) □

If $G(0) = 0$ the situation is slightly more complicated, which will be shown by the complexity of the following theorems. It follows from (2.5) that $H(z)$ will always have a zero at $z = 1$. In general, it is not possible to get the other zeros inside the unit circle just by choosing h "large enough". However, if $G(s)$ possesses a real pole, closer to the imaginary axis than all the rest, the conclusion will still hold. More precisely, we have the following theorem:

Theorem 2. Let $G(s)$ be a strictly proper, asymptotically stable transfer function of finite order with $G(0) = 0$. We assume that

$p_1 = -\sigma$ is a p-tiple pole, and

$\mathrm{Re}\, p_i \lneq -\sigma \quad i = 2,\dots,\ell$

If $\mathrm{Re}\, p_k = -\sigma$ for some $k > 1$ we assume that these poles has multiplicity $< p$. Under these circumstances, $H(z)$ will have a stable inverse provided h is large enough.

For $G(s)$ having only single poles we give this theorem in a slightly stronger version:

Theorem 2'. Let $G(s)$ be as in theorem 2, of order n, and with single poles. Define

$$\alpha = -\max_{i=2,\dots,n} \mathrm{Re}\, p_i$$

$$A = \max_{i=2,\dots,n} \left|\mathrm{res}_{p_i}\, G(s)/s\right| \,/\, \left|\mathrm{res}_{p_1}\, G(s)/s\right|$$

Then $H(z)$ will have a stable inverse if

$$e^{(\alpha-\sigma)h}\,\frac{1-e^{-\alpha h}}{1+e^{-\sigma h}} > (n-1)A \qquad (3.5)$$

The left-hand function is an increasing function of h.

Proof theorem 2. If $G(s)$ is on the form (2.4) $H(z)$ is given by (2.5):

$$H(z) = (1 - z^{-1})\Sigma\Sigma f_{ij}(z)$$

where f_{ij} is defined in (2.6). It follows from (2.6) and our assumptions that f_{1p} will dominate absolutely on $|z| = 1$ for sufficiently large h. It was shown in remark 2 after the corollary to lemma 1 that for h sufficiently large, f_{ij} has all its zeros inside the unit circle. Rouché's theorem now gives that $N_{as}\{H(z)\} - P_{as}\{H(z)\} = N_{as}\{f_{1p}\} - P_{as}\{f_{1p}\} = -1$. Since the factor $(1 - z^{-1})$ does not affect $N_u\{H(z)\} - P_u\{H(z)\}$ the theorem is proved. □

The proof of theorem 2' is similar:

Proof theorem 2'. With obvious notation we have

$$H(z) = (1 - z^{-1})\Sigma f_i(z) = (1 - z^{-1})\Sigma A_i \frac{e^{p_i h}}{z - e^{p_i h}}$$

If

$$|f_1| \geq \left|\sum_2^n f_i\right| \qquad (3.6)$$

when $|z| = 1$ $N_u\{H(z)\}$ will be $= 0$.

Now, if $|z| = 1$

$$|f_1| \geq |A_1| \frac{e^{-\sigma h}}{1 + e^{-\sigma h}}$$

and

$$|f_i| \leq \max_{k=2,\ldots,n} |A_k| \frac{e^{-\alpha h}}{1 - e^{-\alpha h}}$$

(3.6) will clearly be fulfilled if

$$|A_1| \frac{e^{-\sigma h}}{1 + e^{-\sigma h}} > (n - 1) \max_{k=2,\ldots,n} |A_k| \frac{e^{-\alpha h}}{1 - e^{-\alpha h}}$$

which is the same as (3.5). □

If $G(0) = 0$ and the conditions in theorem 2 are not fulfilled, $H(z)$ may have unstable zeros for arbitrarily large h. An example of this is given in example 3. We will give two theorems concerning this phenomena. If $G(s)$ has a complex-conjugated pair of poles which are closer to the imaginary axis than the other poles, theorem 3 will give intervals in which the sample interval can be chosen so that $H(z)$ will have a stable inverse. Theorem 4, which is given more sketchier, is more general, and deals with the case of $G(s)$ having several poles with the same real part, one complex-conjugated pole pair of these having larger multiplicity than the rest.

<u>Theorem 3</u>. Let $G(s)$ be an asymptotically stable, strictly proper transfer function on the form (2.4) with $G(0) = 0$, and of order $n \geq 3$. Assume that

$p_1 = -\sigma + i\omega$

$p_2 = -\sigma - i\omega$

are simple poles, where

$-\sigma > \text{Re } p_i \quad i = 3,\ldots,\ell$ and

$\omega > 0$

Define

$A = \max |A_{ij}|/|A_{11}|$

$\phi = \arg A_{11}$

$\alpha = - \max \text{Re } p_i$

$m = \max n_i$

where all max is to be taken over $i = 3,\ldots,\ell$.

Under these circumstances H(z) will have a stable inverse if

$$|\cos(\omega h + \phi)| > |\cos\phi|\, e^{-\sigma h} + \frac{n-2}{2} A \max(1, h^{m-1})\, e^{-(\alpha-\sigma)h} \frac{(1+e^{-\sigma h})^2}{(1-e^{-\alpha h})^m} \qquad (3.7)$$

For large h, the right hand function is monotonically decreasing, and approaches 0 asymptotically.

Proof. Define

$$G_1(s) = s\left(\frac{A_{11}}{s-p_1} + \frac{A_{12}}{s-p_2}\right)$$

(If $A_{11} \neq -A_{12}$ this is not a strictly proper transfer function. It can, however, be an additive part of a strictly proper transfer function.) Since $G_1(s)$ is real, we can write $A_{11} = \rho e^{i\phi}$, $A_{12} = \rho e^{-i\phi}$. Simple computations give us

$$\mathbb{D}\, G_1 = H_1(z) = (1 - z^{-1})\, 2\rho e^{-\sigma h} \frac{\cos(\omega h + \phi)\, z - e^{-\sigma h} \cos\phi}{(z - e^{p_1 h})(z - e^{p_2 h})} \qquad (3.8)$$

With the notation above, in (2.4) and (2.6), we have

$$G(s) = G_1(s) + \sum_{i=3}^{\ell} \sum_{j=1}^{n_i} \frac{A_{ij}}{(s-p_i)^j}$$

(2.5) gives us

$$H(z) = H_1(z) + (1 - z^{-1}) \Sigma\Sigma f_{ij}(z)$$

Now we use an estimation very similar to (3.4):

$$|\Sigma\Sigma f_{ij}| \le (n-2) \max_{i=3,\ldots,\ell} |A_{ij}| \max(1, h^{m-1}) \frac{e^{-\alpha h}}{(1 - e^{-\sigma h})^m}$$

(3.8) now gives us that if (3.7) is fulfilled the term $(1 - z^{-1})^{-1} H_1(z)$ will dominate absolutely in $(1 - z^{-1})^{-1} H(z)$ on the unit circle. Since (3.7) also implies that $(1 - z^{-1})^{-1} H_1(z)$ has its only zero inside the unit circle, the theorem is proved. □

Consider the following example from [1]!

Example 3. Let $G(s) = \dfrac{s}{[(s+1)^2 + 1](s+2)}$

The zeros of H(z) are z = 1 and

$$z_2 = \frac{e^{-2h}(\sin h + \cos h) - e^{-h}}{e^{-h} + \sin h - \cos h} \approx \frac{-1}{1 + e^{h}(\sin h - \cos h)}$$

Fig 1 shows z_2 as a function of h.

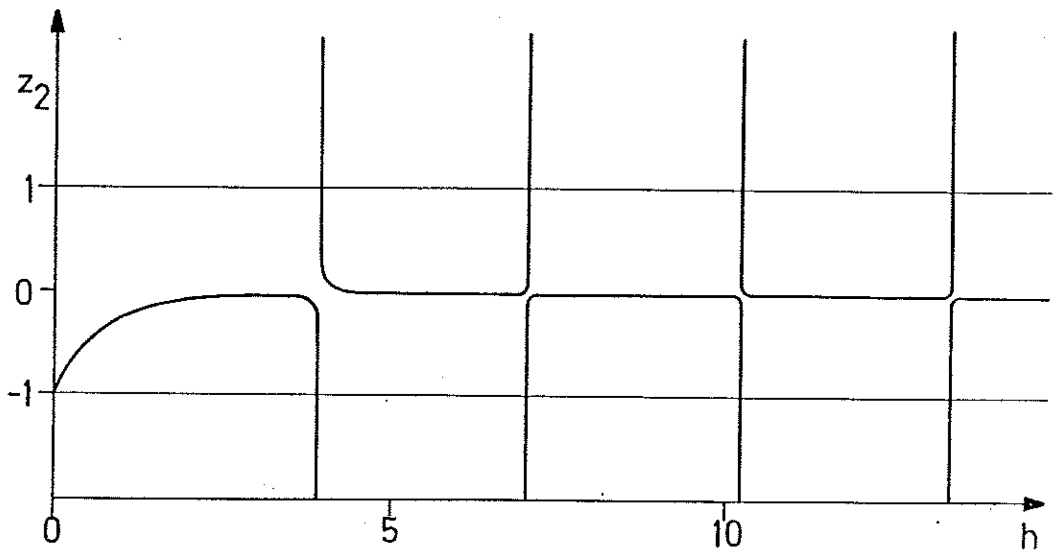

Fig 1. The zero z_2.

We list some of the first intervals in which h can be chosen so H(z) will have a stable inverse, together with the estimations given by (3.7):

Exact intervals	Intervals given by (3.7)
0 - 3.926623	1.323840 - 3.897698
3.954452 - 7.067378	3.954629 - 7.067375
7.068583 - 10.210176	7.069789 - 10.210123
10.210207 -	10.210231 -

□

Next we prove the following, purely qualitative, extension of theorem 3:

Theorem 4. Let G(s) be a strictly proper, asymptotically stable transfer function on the form (2.4) with G(0) = 0, and of order $n \geq 3$. Assume that

$\text{Re } p_i \leq -\sigma \quad i = 1,\ldots,\ell$

where equality holds for the first 2k+1 poles, i. e.

$p_{2m-1,2m} = -\sigma \pm i\omega_m, \ \omega_m > 0, \quad m = 1,\ldots,k$

$p_{2k+1} = -\sigma$

Further, we assume that $n_1 > 1$ and

$n_1 > n_j \quad j = 3,\ldots,2k+1$

Under these circumstances, if h is large enough, H(z) will have a stable inverse, except for h in neighbourhoods of the points $\{h \in \mathbb{R}^+: \cos(\omega_1 h + \phi) = 0\}$, where $\phi = \arg A_{1n_1}$. When h is increasing these neighbourhoods become arbitrarily small.

Remark. It is not a restriction to assume that G(s) has a real pole $p_{2k+1} = -\sigma$, because if it has not, it can be represented in (2.4) as a single pole with $A_{2k+1,1} = 0$. □

Proof. For typographical reasons, we define $n' = n_1 - 1 = n_2 - 1 > 0$. Let the part of H(z) which corresponds to A_{1n_1} and A_{2n_1} be denoted by $H_1(z)$. We have to show that for suitable h

i) $(1 - z^{-1})^{-1} H_1(z)$ dominates absolutely over $(1 - z^{-1})(H(z) - H_1(z))$ on the unit circle.

ii) $H_1(z)$ has only stable zeros.

Since G(s) is real, we can write

$A_{1n_1} = \rho e^{i\phi}$, $A_{2n_1} = \rho e^{-i\phi}$

(2.10) and some simple calculations give us

$$(1 - z^{-1})^{-1} H_1(z) = \frac{h^{n'} z}{n'!} \rho e^{-n'\sigma h} \cdot$$

$$\frac{e^{i(n'\omega_1 h+\phi)}(z-e^{p_2 h})^{n_1} B_{n'}[ze^{(\sigma-i\omega_1)h}] + e^{-i(n'\omega_1 h+\phi)}(z-e^{p_1 h})^{n_1} B_{n'}[ze^{(\sigma+i\omega_1)h}]}{(z^2 - 2e^{-\sigma h}\cos\omega_1 h \cdot z + e^{-2\sigma h})^{n_1}}$$

Since $b_1^{n'} = 1$, the nominator on the last line will for large h come arbitrarily close to

$$2\, z^{2n'}\, e^{(n'-1)\sigma h} \cos(\omega_1 h + \phi)$$

If h is sufficiently large, and selected so that the cosine-term is not too small, it will follow that $H_1(z)$ has only stable zeros. Finally, the statement i) is proved using the same type of estimations as in the preceding proofs. □

Time delays

One of the pleasant properties of time discrete systems is that given a time continuous transfer function with a time delay, i. e. a system of infinite order, it will, when sampled, become a system of finite order. It is possible to use the technique developed in this chapter to show that if $G(0) \neq 0$ it is also possible to get a system with a stable inverse, which can never be the case in the time continuous case. For G(s) beeing a rational function times an exponential, with only single poles, we have the following theorem:

Theorem 5. Let $G(s) = e^{-s\tau} G_1(s)$, where $G_1(s)$ is a rational, strictly proper, asymptotically stable transfer function with $G(0) \neq 0$. We write $G(s)$ on the form

$$G(s) = e^{-s\tau} s \left(\frac{G(0)}{s} + \sum_{i=1}^{n} \frac{A_i}{s - p_i} \right)$$

and define

$$A = \max_i |A_i e^{-p_i \tau}| / |G(0)|$$

Then $H(z)$ will have a stable inverse if

$$h > \max\{\tau, \ln(2An + 1)/\sigma\}$$

Remark. Note that the condition is asymptotically linear in τ. □

Proof. We use the representation (2.2) of the time discrete transfer function. If $h > \tau$ the integration contour can be closed to the left, and the integral evaluated using residue calculus. Doing so, we get

$$H(z) = G(0) z^{-1} + (1 - z^{-1}) \sum A_i \frac{e^{p_i (h - \tau)}}{z - e^{p_i h}}$$

If we apply theorem 1 with A_{i1} replaced by $A_i e^{-p_i \tau}$ the theorem follows. □

Remark. It is obviously possible to prove similar theorems for the multiple pole case. □

4. Nyquist curve oriented methods

Next we consider the situation of only $G(i\omega), \omega \in \mathbb{R}$, beeing known, and when no assumption of for example finite order of the system can be made. Since $G(s)$ is meromorphic, the knowledge of $G(i\omega)$ implies that $G(s), s \in \mathbb{C}$, is completely determined (theoretically!). We use the representation (2.2) of $H(z)$ and assume that $G(s)$ is so small at infinity in the half plane $\operatorname{Im} s < \gamma$ that the integration contour can be closed to the left. If $G(s)$ has only finitely many poles, evaluation of the integral using residue calculus will give us a rational time discrete transfer function. We can then use the methods of chapter 3, and the problem will be solved. Now the Bolzano-Weierstrass' theorem implies that a function which is bounded at infinity will have only finitely many poles, and is therefore rational. This means that every strictly proper transfer function will satisfy the condition above. Further, if $G(s)$ is a meromorphic function such that the complex plane can be divided into two parts, $G(s)$ beeing bounded at infinity in one and analytic in the other, the conclusion of finitely many poles will still hold. Therefore the situation with $G(s)$ beeing a strictly proper transfer function times an exponential (i. e. a time delay) is also solved, provided $e^{sh}G(s)$ is small at infinity in the sense made precise above.

For the case of the Nyquist curve beeing strictly below the real axis, we have the following theorem:

Theorem 6. Let $G(s)$ be a strictly proper transfer function with $\operatorname{Im} G(i\omega) < 0$, $\omega > 0$. Then

i) If $G(0) > 0$ then N_u $(= N_u\{H(z)\})= P_u$

ii) If $G(0) < 0$ then $N_u = P_u - 1$

iii) If $G(0) = 0$ and $G'(0) \neq 0$ then $N_u = P_u - 1$

Further, if $G(s)$ is an asymptotically stable transfer function, then $N_u = 0$.

Proof. For the proof we are going to use the argument principle to compute $N_{as}\{zH(z)\} - P_{as}\{zH(z)\}$. (2.3) gives us $zH(z) = (z-1)F(z)$, $z \neq 1$, where

$$F(z) = \sum_{k=-\infty}^{\infty} \frac{G[(\log z + 2\pi ik)/h]}{\log z + 2\pi ik} \qquad z \neq 1$$

First we assume $G(0) \neq 0$. We evaluate $zH(z)$ at the unit circle, $z = e^{i\phi}$, where $-\pi < \phi \leq \pi$

$$zH(z) = (e^{i\phi} - 1) \sum \frac{G[i(\phi + 2\pi k)/h]}{i(\phi + 2\pi k)} \qquad \phi \neq 0$$

Since $G(-i\omega) = \overline{G(i\omega)}$ we have

$$\frac{\pi}{2} < \arg F(e^{i\phi}) < \frac{3\pi}{2} \qquad \phi \neq 0$$

and, because

$$\frac{\pi}{2} < \arg (e^{i\phi} - 1) < \frac{3\pi}{2} \qquad \phi \neq 0$$

we deduce that the image of the unit circle under the mapping given by $zH(z)$ avoids the negative real axis, except possibly at the point corresponding to $\phi = 0$, i. e. $H(1) = G(0)$. If $G(0) > 0$ the image curve cannot encircle the origin, and if $G(0) < 0$ it will do so exactly once. To prove that it will encircle the origin in a positive direction, we have to prove that it is "going down" when it passes the negative real axis. If $d/dz\ \{zH(z)\}(1) \neq 0$, this follows immediately from conformallity. Otherwise, we can always make a suitable perturbation of the contour so that the conclusion will still be valid.

If $G(0) = 0$ and $G'(0) \neq 0$, then $F(z)$ is analytic and $\neq 0$ on all of the unit circle. The condition $\text{Im}\, G(i\omega) < 0$, $\omega > 0$, implies that the image of the unit circle under the mapping given by $F(z)$ lies strictly to the left of the origin.

Finally, since $N_u \geq 0$, case ii and iii of the theorem implies that $G(s)$ is unstable, which proves the final statement. □

Remark 1. If the leading term in the nominator of $H(z)$ vanishes for some h, it should be interpreted as an additional zero at infinity in order for the theorem to be fully correct. Compare chapter 2. □

Remark 2. If $G(i\omega) > 0$, $\omega > 0$, it is possible to prove a theorem similar to theorem 6 just by exchanging certain inequalities. It is however a better idea to consider $-G(s)$. □

The conditions on G in theorem 6 are very strong, and will not be satisfied by many systems in the real world. It is a natural question to ask if the conditions can be weakened slightly, for example if $\mathrm{Im}\, G(i\omega) < 0$ when $0 < \omega < \Omega$, is there an h_o such that $h > h_o$ will imply that the sampled system has a stable inverse? The following example will show that without further conditions, there is no such h_o.

Before giving the example we prove the following simple lemma:

Lemma 3. Let $G(s) = \dfrac{1}{s^2 + 2\zeta\omega_o s + \omega_o^2}$ $\qquad 0 < \zeta < 1$

and

$f_{\zeta,\omega_o}(\omega) = \mathrm{Re}\,[G(i\omega)/i\omega] \qquad \omega \in \mathbb{R}\setminus\{0\}$

Then

i) $f_{\zeta,\omega_o}(\pm\omega_o) = -\dfrac{1}{2\zeta\omega_o^3} \to -\infty,\ \zeta \to 0+$

ii) $f_{\zeta,\omega_o}(\omega) < 0 \qquad \omega \in \mathbb{R}\setminus\{0\}$

iii) $f_{\zeta,\omega_o}(\omega) \to 0,\ \zeta \to 0+,\quad \omega \neq \pm\omega_o$

Proof. We have

$$f_{\zeta,\omega_o}(\omega) = -\frac{2\zeta\omega_o}{(\omega_o^2 - \omega^2)^2 + 4\zeta^2\omega_o^2\omega^2}$$

From this the lemma follows immediately. □

Example 4. Let

$$G(s) = \frac{1}{s^2 + 2\zeta_1 s + 1} - \frac{1}{s^2 + 2\zeta_2\omega_o s + \omega_o^2} =$$

$$= \frac{2(\zeta_2\omega_o - \zeta_1)s + \omega_o^2 - 1}{(s^2 + 2\zeta_1 s + 1)(s^2 + 2\zeta_2\omega_o s + \omega_o^2)}$$

We assume that $\omega_o \gg 1$ and $\zeta_1 > \zeta_2 > 0$ are "small". It is clear from theorem 1 that for ζ_1 and ζ_2 fixed, it is possible to find a h_o such that $h > h_o$ implies that $\mathbb{D}G = H(z)$ has a stable inverse. For the parameter values $\omega_o = 10$, $\zeta_1 = 0.1$, $\zeta_2 = 0.05$, the argument curve and the Nyquist curve are shown in fig 2.

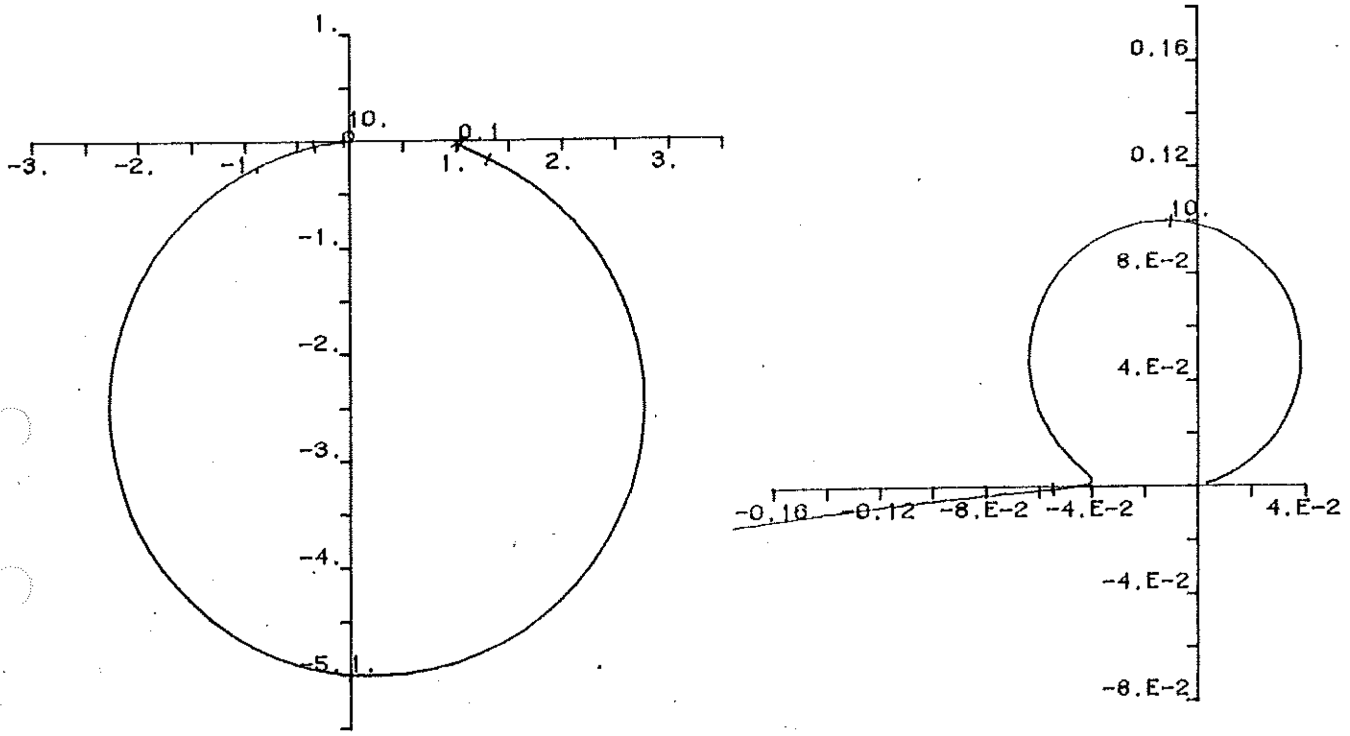

Fig 2a. The Nyquist curve with detail magnification.

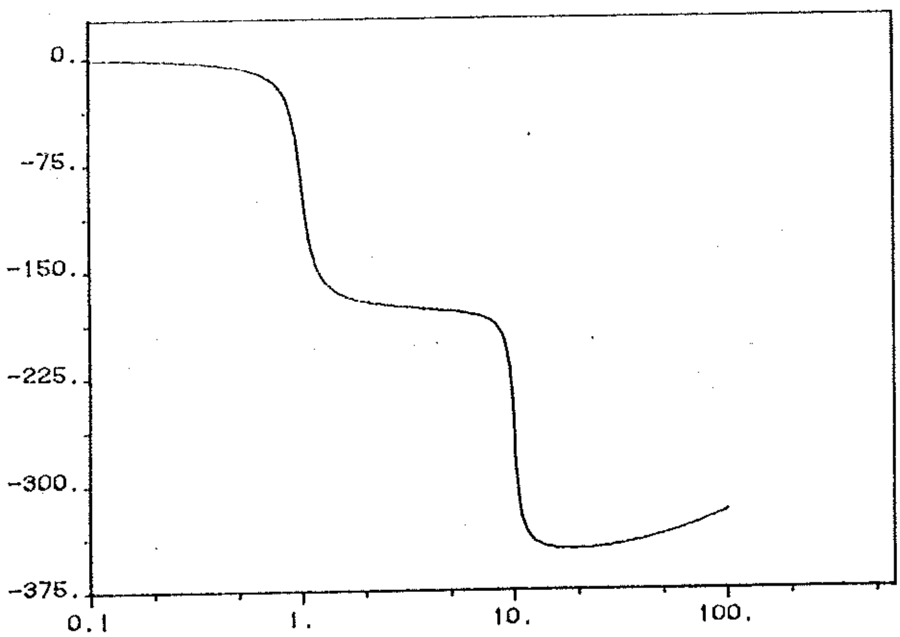

Fig 2b. The argument curve.

It is obvious that there exists a ω_1 ($\simeq\sqrt{\omega_o}$) such that $-\pi < \arg G(i\omega) < 0$ when $0 < \omega < \omega_1$.

We are going to show that for every h_o there is an $h \geq h_o$ and a ζ_2 so small that $H(z)$ has an unstable zero. (In theorem 1, this corresponds to letting $\sigma \to 0+$.)

Define $F(z) = (1 - z^{-1})^{-1} H(z) = \sum_{k=-\infty}^{\infty} \frac{G[(\log z + 2\pi ik)/h]}{\log z + 2\pi ik}$

If the leading term in the nominator of $H(z)$ does not vanish, the following is valid: $H(z)$ has a stable inverse if and only if $F(z)$ has the same number of poles and zeros inside the contour shown in fig 3, where $r > 0$ is selected so small that the contour avoids all poles and zeros inside the unit circle.

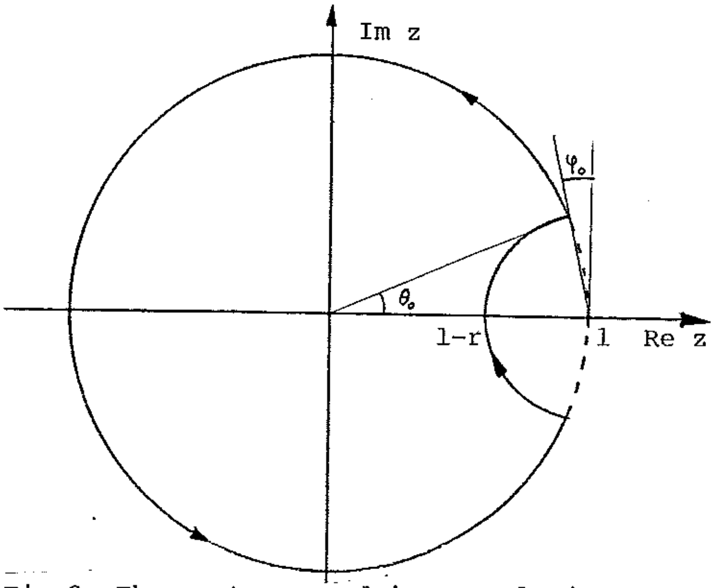

Fig 3. The contour used in example 4.

If the leading term in the nominator vanishes for some h, we showed in chapter 2 that in a neighbourhood of this h, H(z) has a zero of arbitrarily large modulus, so this case will not rupture our results. Now we select a $h \geq h_o$ such that $(2k_o + 1)\pi/h = \omega_o$ for some integer k_o. If so, $(\phi + 2\pi k)/h \neq \omega_o$ for all integers k if ϕ is not an odd integer multiple of π.

We let the small "semicircle" with radius r be parametrized by $z = 1 - re^{i\phi}$, where $-\frac{\pi}{2} + \phi_o \leq \phi \leq \frac{\pi}{2} - \phi_o$. In a neighbourhood of $z = 1$ F(z) can be written

$$F(z) = \frac{G(0) + rK_1(z)}{-r + r^2 K_2(z)} e^{-i\phi} + K_3(z)$$

where $K_k(z)$ $k = 1,2,3$ are bounded. Letting $r \to 0$, the image of the small "semicircle" will come close to a semicircle which lie in the left half plane, except possibly at its ends.

Next we examine the curve $F(e^{i\theta})$, $\theta_o < \theta < 2\pi - \theta_o$. Let $f_{\zeta_1,1}$ and f_{ζ_2,ω_o} be as in lemma 3. Then we have

$$\text{Re } F(e^{i\theta}) = \frac{1}{h}\left(\sum_{k=-\infty}^{\infty} f_{\zeta_1,1}\left(\frac{\theta + 2\pi k}{h}\right) - \sum_{k=-\infty}^{\infty} f_{\zeta_1,\omega_o}\left(\frac{\theta + 2\pi k}{h}\right)\right)$$

Lemma 3 gives us that for ζ_1 fixed, we can select a ζ_2 so small that Re $F(e^{i\theta}) < 0$, except in a neighbourhood of $\theta = \pi$, where it will be positive. Since the image curve is symmetric with respect to the real axis it will encircle the origin exactly once. Hence $H(z)$ has a zero outside the unit circle. □

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