NEGATIVE RATES IN A MULTI CURVE FRAMEWORK

CAP PRICING AND VOLATILITY TRANSFORMATION

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SAMMANFATTNING

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This thesis describes how to use an extension of the SABR model, known as the shifted SABR, to solve these issues. Using three different methods, the shifted SABR model is calibrated to EUR cap volatilities based on 6 month EURIBOR. However, market standard is to quote a mixture of 3 month and 6 month cap volatilities, thus the volatilities have to be transformed to a common tenor before calibration. To this end we have developed a technique for volatility transformation in a multi curve framework when rates are negative. The concept is derived by applying Itô's formula on an arbitrage relation between shifted forward rates.

The results show that two of the methods for calibrating the shifted SABR model have a good fit to liquid contracts. However, the methods vary in performance capturing far OTM caps. Our developed volatility transformation technique also works well, and the sensitivity to the potentially unknown correlation between forward rates is low.

The implications are that the extension of the SABR model to the shifted SABR model works fine, both in terms of pricing of caps and volatility transformation in a multi curve framework. Although, it comes to the cost of some additional complexity of extending formulas and arbitrage relations, that used to hold in a positive rate environment.

NYCKELORD

Multi Curve, Volatility Transformation, Negative Rates, shifted SABR, non-standard Tenor, Caps, EUR Market

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Abstract

The SABR model has for a long time been an invaluable tool for capturing the volatility smile and to price financial derivatives not quoted in the market. However, the current negative rate environment in the EUR market has led to numerous challenges for financial institutions. One of the most problematic issues is that the SABR model used for volatility interpolation and extrapolation fails when rates are negative. A related issue is that SABR based techniques for transformation of cap volatilities between different tenors, do not work anymore.

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The results show that two of the methods for calibrating the shifted SABR model have a good fit to liquid contracts. However, the methods vary in performance capturing far OTM caps. Our developed volatility transformation technique also works well, and the sensitivity to the potentially unknown correlation between forward rates is low.

The implications are that the extension of the SABR model to the shifted SABR model works fine, both in terms of pricing of caps and volatility transformation in a multi curve framework. Although, it comes to the cost of some additional complexity of extending formulas and arbitrage relations, that used to hold in a positive rate environment.

Keywords: Multi Curve, Volatility Transformation, Negative Rates, shifted SABR, non-standard Tenor, Caps, EUR Market

Nomenclature

ATM At The Money

ECB European Central Bank

EURIBOR Euro Interbank Offered Rate

FRA Forward Rate Agreement

GBM Geometric Brownian Motion

LIBOR London Interbank Offered Rate

LMM LIBOR Market Model

OIS Overnight Index Swap

OTM Out of The Money

RNVF Risk Neutral Valuation Formula

SABR Stochastic Alpha Beta Rho

SDE Stochastic Differential Equation

SNB Swiss National Bank

VTS Volatility Term Structure

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Chapter 1

Introduction

1.1 Background

In June 2014 the European Central Bank (ECB) did something extremely unusual seen from a historical perspective: they applied negative deposit rates. For many investors this was an entirely new situation, as for a long time, negative rates have been regarded as impossible. The reason for this is intuitively simple: a bond with a negative interest rate is a guaranteed money-loser.

However, the ECB was not the first central bank to apply negative rates. Actually, the Central Bank of Denmark set its deposit rate below zero already in July 2012. Other countries not part of the Eurozone such as Sweden and Switzerland have also followed the ECB by taking similar measures.

The reason for negative interest rates differs depending on the national bank. One can say that the main reason for applying negative rates in the Eurozone and Sweden is to fight the growing threat of deflation. In simple terms this means that policymakers want to make it less attractive to save money. As such lowering the interest rate on savings should potentially make it more attractive to invest the money instead.

Danish low rates in July 2012 were a response to the increasing capital inflow amid the increasing financial distress in the Euro area. In December 2014 the Swiss National Bank (SNB) set their rates negative for the first time since the 60's. The decision was driven by the increased currency appreciation of the Swiss Franc and when the EURCHF cap of 1.20 was abandoned in January 15 2015, rates went even more negative (Arteta et al., 2015). The sharp drop in the 6 month CHF LIBOR can be observed in Figure 1.1.

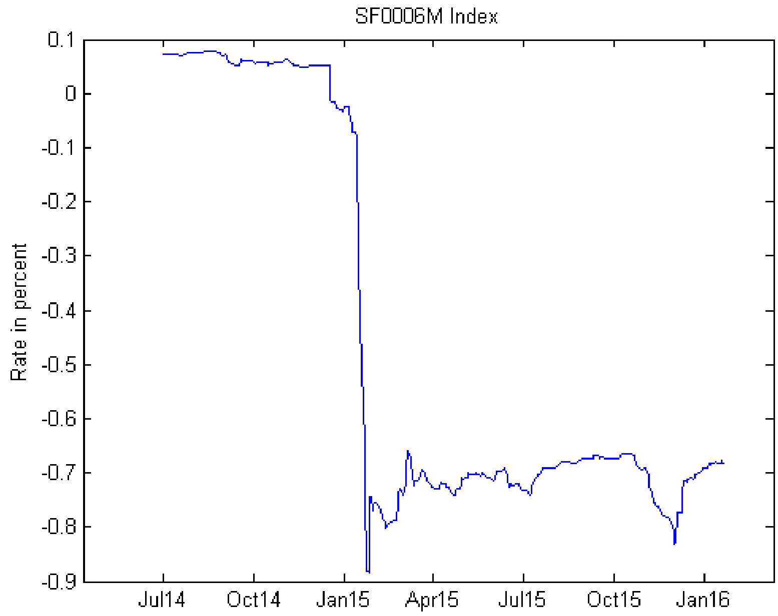


Figure 1.1: *The 6-month CHF LIBOR rate (SF0006M) during the period 01-07-2014 to 22-01-2016. The sharp drop at 15-01-2015 is a reaction to the abandonment of the EURCHF cap of 1.20 CHF/EUR.*

In general, negative interest rates have not yet been used for bank deposits and lending rates to households. However, there is a well published story where a woman in Denmark actually was being paid by the bank for borrowing money[1]. Nonetheless, firms and institutional investors are paying negative interest rates on their deposits. On the other hand, negative interest rates can be circumvented by holding cash which always offers a nominal return of zero. However, this approach renders substantial costs, including costs for secure storage and transport. Furthermore, it is expensive to use cash for transactions involving large amounts or large geographical distances.

In terms of mathematical modelling, negative interest rates have had a considerable impact. Negative rates affect financial institutions in three major areas:

1. Construction of interest rates curves
2. Quotation of volatility surface
3. Interpolation

Problems related to the first area are mainly due to that relations which used to hold when rates are positive are no longer valid. Negative interest rates have also led to a change in the way market quotes volatilities. The Black-76 model that used to be the

[1] For more information see: http://on.ft.com/1FhhLlb

market standard for valuing all vanilla interest rate options does not work anymore. The model only allows for positive rates due to its log-normal nature. The market has proposed two solutions to this. The first is to quote what is known as shifted or displaced Black volatilities. The second is to use normal volatilities.

The third point is related to interpolation of market volatility quotes. Models for volatility interpolation such as SABR fails to cope with negative rates. The prevailing solutions in the market have been to use the shifted SABR or the free boundary SABR.

1.2 Purpose

This thesis intends to investigate different approaches of how negative rates can be coped with when pricing caps. The model under consideration will be the SABR model. We will calibrate the model to implied market cap/floor volatilities using three different methods: Bootstrap, Rebonato and global SABR. An explanation of the methods can be found in chapter 5.

The objective is to calibrate the shifted SABR model to EUR caps in the current negative rate environment. In addition, market quotations of cap/floor volatilities in EUR market assume a multi curve framework[2] and volatilities are based on different underlying tenor. To this end we develop a volatility transformation method to handle negative rates in a multi curve framework.

For the interested reader and to facilitate a more in depth discussion of the applied methods, calibrations will also be performed in the USD cap market (see Appendix B). This calibration study aims to show how to calibrate the SABR model when rates are positive and when working under a single curve framework[3].

The performance of different methods will be evaluated in terms of their capability to reconstruct market prices. The three methods will be evaluated according to three quality measures shown in Figure 1.2. Especially we focus on evaluating the models in their ability to get a good fit to at-the-money (ATM) contracts since these are often the most liquid contracts.

[2]The multi curve framework is the market practice since the credit crises 2007. Essentially, it separates the discounting curve from the forward curve. In addition there are distinct forward curves for each tenor, accounting for the differences in credit and liquidity risk between forward rates with different tenors. The multi curve framework is described in more detail in Chapter 6.

[3]The single curve framework is what used to be the market standard before the credit crises 2007. The framework has one single yield curve which is used for both discounting and forwarding.

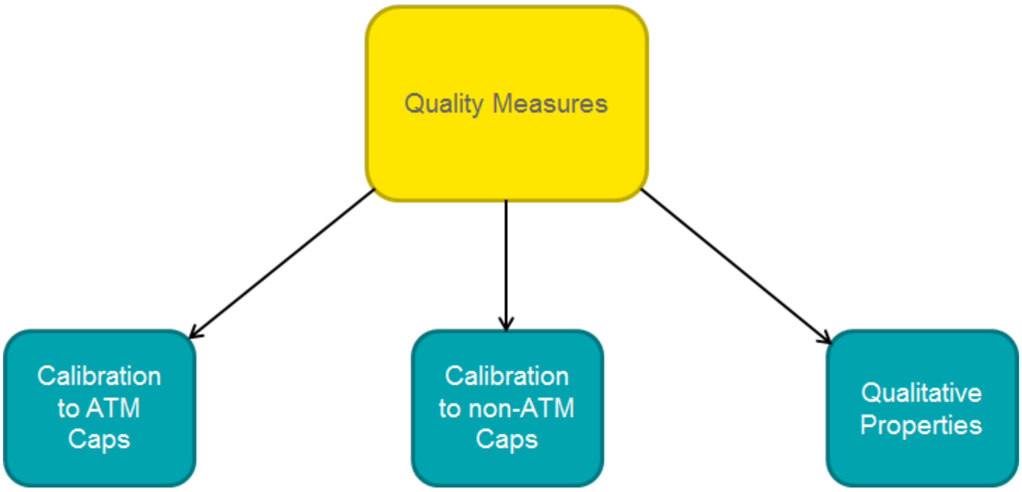


Figure 1.2: *Quality measures for methods of calibrating SABR model. The quality of the methods will be studied in three different ways: ability to calibrate to ATM caps, ability to calibrate to non-ATM caps and qualitative properties. Examples of qualitative properties are for instance ability to include ATM caps or to put different calibration weights on different caps.*

Our contribution consists of delivering three things:

1. A step by step guide of different ways of how one can calibrate a SABR or shifted SABR model to market quotes of cap/floor volatilities.

2. A method for transforming cap volatilities in a multi curve framework when rates are negative.

3. An analysis of the calibrated SABR models to see which method generates the best fit (in a least squares sense) to market data; as well as a comparative analysis between the pros and cons of the methods.

Our first contribution is important seen from a modelling perspective, since the literature on this subject is very scarce and existing literature seldom describe the entire process going from market data to a calibrated model.

The topic of volatility transformation is an important tool for handling current market conventions in the EUR cap market. Previous studies on this have been done by for instance Kienitz (2013) and Andong (2013). However, to our knowledge no one has yet extended the methods so that they become applicable when rates are negative (see sections 7.1.3, 7.2 and 7.3).

We evaluate our methods for SABR calibration to see which method has the best fit to market data. Our motivation for this is that we believe that methods that fit market data poorly is hard to trust when pricing products not in the market such as structured products similar to those in the market.

1.3 Outline

Chapter 2: In this chapter basic stochastic calculus is treated. This involves for instance Itô's lemma, the risk neutral valuation formula and change of numeraire. An advanced reader can skip this chapter without loss of continuity. The chapter also intends to refresh existing knowledge.

Chapter 3: A walkthrough of the the main concepts within interest rate theory are provided in this part. For instance, standard contracts such caplets, caps and swaps are deliberately introduced. Models such as SABR and shifted SABR are also explained.

Chapter 4: The chapter presents the market conventions of how volatilities are quoted in the US and EUR cap market.

Chapter 5: Provides an explanation of the three different methods of how the SABR or shifted SABR model will be calibrated. It includes step by step guides with flow-charts on how to use and implement each calibration method.

Chapter 6: The chapter introduces and motivates the need for a multi curve framework in the EUR market. Further, it includes a description for how curves are constructed and how pricing of derivatives is done.

Chapter 7: This chapter gives a comprehensive description of how volatilities are transformed both in a single and multi curve framework. Especially, it involves one of the main contributions of the thesis by describing how results based on positive rates might be extended so that they become applicable when rates are negative. The chapter finalizes with a transformation example in the EUR cap market.

Chapter 8: In this part, calibrations to the EUR cap market are shown. The calibrations are based on the three methods which were introduced in Chapter 5. The chapter is wrapped up by a comparable analysis between the methods.

Chapter 9: At last, we conclude by discussing the main findings of the thesis and proposing topics for further research.

Chapter 2

Basic Stochastic Calculus

In this chapter, we treat the basic stochastic calculus needed to understand the concepts treated in proceeding chapters. Most of the material is based on the book *Arbitrage Theory in Continuous Time* by Björk (2009). A reader who knows these concepts well can feel free to skip this chapter without loss of continuity. The chapter can also be used to refresh existing knowledge.

2.1 Brownian Motion

A Wiener process is a stochastic Lévy process that is widely used for modelling the dynamics of asset prices. The process is also often called a Brownian motion.

Definition 2.1.1. *A stochastic process W is called a* ***Wiener Process*** *or a* ***Brownian Motion*** *if the following conditions hold:*

1. $W(0) = 0$ *a.s.*

2. *The process W has independent increments, i.e. if $r < s \leq t < u$ then $W(u) - W(t)$ and $W(s) - W(r)$ are independent stochastic variables.*

3. *For $s < t$ the stochastic variable $W(t) - W(s)$ has the Gaussian distribution $\mathcal{N}(0, \sqrt{t-s})$.*

4. *W has continuous trajectories.*

(Björk, 2009, p. 40)

2.2 Itô's Lemma

Itô's lemma is a tool used to find the differential of a function of a time-dependent stochastic process.

Theorem 2.2.1. ***(Itô's formula)*** *Take a vector Wiener process $W = (W_1, \ldots, W_n)$ with correlation matrix ρ as given, and assume that the vector process $X = (X_1, \ldots, X_k)^*$ has a stochastic differential. Then the following hold.*

- *For any $\mathcal{C}^{1,2}$-function f, the stochastic differential of the process $f(t, X(t))$ is given by*

$$df(t, X(t)) = \frac{\partial f}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}dX_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}dX_i dX_j,$$

with the formal multiplication table

$$\begin{cases} (dt)^2 = 0, \\ dt \cdot dW_i = 0, \quad i = 1, \ldots, n, \\ dW_i \cdot dW_j = \rho_{ij}dt. \end{cases}$$

- *If, in particular, $k = n$ and dX has the structure*

$$dX_i = \mu_i dt + \sigma_i dW_i, \quad i = 1, \ldots, n,$$

where $\mu_1, \ldots, \mu_n$ and $\sigma_1, \ldots, \sigma_n$ are scalar processes, then the stochastic differential of the process $f(t, X(t))$ is given by

$$df = \left\{ \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \mu_i \frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \right\} dt + \sum_{i=1}^{n} \sigma_i \frac{\partial f}{\partial x_i} dW_i.$$

(Björk, 2009, pp. 60-61)

2.3 Quadratic Covariation

In this part we define the concept of quadratic covariation. This will be used when deriving volatility transformation formulas in chapter 7.

Definition 2.3.1. ***(Quadratic covariation)*** *Let X and Y be Itô processes, with dynamics*

$$dX_t = \mu_X(t)dt + \sigma_X(t)dW_t^X,$$
$$dY_t = \mu_Y(t)dt + \sigma_Y(t)dW_t^Y.$$

The process $\langle X, Y \rangle$ is defined by

$$\langle X, Y \rangle = \int_0^t \sigma_X(s)\sigma_Y(s)\rho_{X,Y}(s)ds,$$

or equivalently by its ***quadratic covariation***

$$d\langle X, Y \rangle_t = dX_t dY_t = \sigma_X(t)\sigma_Y(t)\rho_{X,Y}(t)dt,$$

where $\rho_{X,Y}(t)$ is the correlation between the Brownian motions, W_t^X and W_t^Y.

(Björk, 2009, pp. 243-244)

2.4 Filtration and Martingales

When dealing with continuous stochastic processes it is notationally convenient to have a single symbol denoting the information generated by the process. We denote this by $\mathcal{F}_t^X$ and by this we mean the filtration of the process X. The formal definition is as follows.

Definition 2.4.1. *The symbol $\mathcal{F}_t^X$ denotes "the information generated by X on the interval $[0, t]$", or alternatively "what has happened to X over the interval $[0, t]$". If, based upon observations of the trajectory $\{X(s); 0 \leq s \leq t\}$, it is possible to decide whether a given event A has occurred or not, then we write this as*

$$A \in \mathcal{F}_t^X,$$

or say that "A is $\mathcal{F}_t^X$-measurable".

If the value of a given stochastic variable Z can be completely determined given observations of the trajectory $\{X(s); 0 \leq s \leq t\}$, then we also write

$$Z \in \mathcal{F}_t^X.$$

If Y is a stochastic process such that we have

$$Y(t) \in \mathcal{F}_t^X,$$

*for all $t \geq 0$, then we say that Y is **adapted** to the **filtration** $\{\mathcal{F}_t^X\}_{t\geq 0}$.*

Another important concept in stochastic calculus is the notion of martingales. A martingale can be viewed as a fair game where the expected value of the processes at some future time point is equal to its current value. The definition is

Definition 2.4.2. *A stochastic process X is called an $(\mathcal{F}_t)$-**martingale** if the following conditions hold.*

- *X is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.*
- *For all t*
$$\mathbb{E}[|X(t)|] < \infty.$$
- *For all s and t with $s \leq t$ the following relation holds*
$$\mathbb{E}[X(t)|\mathcal{F}_s] = X(s).$$

(Björk, 2009, pp. 43, 47)

2.5 Geometric Brownian Motion

One of the fundamental building blocks for modelling assets is the Geometric Brownian Motion.

Definition 2.5.1. *A stochastic process X_s, $t \leq s \leq T$, is said to follow a Geometric Brownian Motion (GBM) if it satisfies the following stochastic differential equation (SDE)*

$$\begin{aligned} dX_s &= \mu(s) X_s ds + \sigma(s) X_s dW_s, \qquad (2.1) \\ X_t &= x_t, \end{aligned}$$

where W_s is a standard Brownian motion, $\mu(s)$ is the drift term and $\sigma(s)$ is the diffusion term.

The solution to the SDE in (2.1) can be derived in the following way:

Set $Z_s = \ln X_s$. Then apply Itô's formula (Theorem 2.2.1) on Z_s to derive the differential. We have

$$\begin{aligned} Z_s &= \ln X_s, \\ dZ_s &= \frac{\partial Z_s}{\partial X_s} dX_s + \frac{1}{2} \frac{\partial^2 Z_s}{\partial X_s^2} (dX_s)^2 \\ &= \frac{1}{X_s} dX_s - \frac{1}{2} \frac{1}{X_s^2} (dX_s)^2 \\ &= \frac{1}{X_s} \left(\mu_s X_s ds + \sigma_s X_s dW_s \right) - \frac{1}{2X_s^2} \left(\mu_s X_s ds + \sigma_s X_s dW_s \right)^2 \\ &= \left(\mu_s - \frac{1}{2} \sigma_s^2 \right) ds + \sigma_s dW_s. \end{aligned}$$

The solution is then obtained by integration on both sides and transforming back from Z_s to X_s,

$$\begin{aligned} Z_T &= z_t + \int_t^T \left(\mu_s - \frac{1}{2} \sigma_s^2 \right) ds + \int_t^T \sigma_s dW_s, \\ X_T &= x_t \exp \left\{ \int_t^T \left(\mu_s - \frac{1}{2} \sigma_s^2 \right) ds + \int_t^T \sigma_s dW_s \right\}. \qquad (2.2) \end{aligned}$$

If the drift and the diffusion are time independent, i.e. $\mu(s) = \mu$ and $\sigma(s) = \sigma$, one can easily make the integrations above and derive the solution on a closed form formula

$$X_T = x_t \exp \left\{ \left(\mu - \frac{1}{2} \sigma^2 \right) (T - t) + \sigma (W_T - W_t) \right\}.$$

From the derived solution to the SDE (2.2), one can derive the distribution of X_T. Let $Y_T = \int_t^T \left(\mu_s - \frac{1}{2}\sigma_s^2\right) ds + \int_t^T \sigma_s dW_s$, be the logarithm of X_T. Then

$$
\begin{aligned}
\mathbb{E}(Y_T|\mathcal{F}_t) &= \mathbb{E}\left(\int_t^T \left(\mu_s - \frac{1}{2}\sigma_s^2\right) ds + \int_t^T \sigma_s dW_s \middle| \mathcal{F}_t\right) \\
&= \int_t^T \left(\mu_s - \frac{1}{2}\sigma_s^2\right) ds = m(t,T), \\
\mathbb{V}(Y_T|\mathcal{F}_t) &= \mathbb{V}\left(\int_t^T \sigma_s dW_s\right) \\
&= \mathbb{E}\left(\left[\int_t^T \sigma_s dW_s\right]^2 \middle| \mathcal{F}_t\right) \\
&= \mathbb{E}\left(\int_t^T \sigma_s^2 ds \middle| \mathcal{F}_t\right) \\
&= \int_t^T \sigma_s^2 ds = \Sigma^2(t,T).
\end{aligned}
$$

So $Y_T \sim \mathcal{N}\left(m(t,T), \Sigma^2(t,T)\right)$, which implies that X_T is log-normally distributed.

(Björk, 2009, pp. 67-69)

2.6 Feynman-Kac's Formula

In order to state the important theorem of Feynman-Kac, the class $\mathcal{L}^2$ has to be defined.

Definition 2.6.1. *A process g belongs to the class $\mathcal{L}^2[a,b]$ if the following conditions are satisfied*

- $\int_a^b \mathbb{E}[g^2(s)]ds < \infty$.
- *The process g is adapted to the $\mathcal{F}_t^W$- filtration.*

Theorem 2.6.1. ***(Feynman-Kac)*** *Assume that F is a solution to the boundary value problem*

$$
\begin{aligned}
\frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) &= 0, \\
F(T,x) &= \Phi(x).
\end{aligned}
$$

Assume furthermore that the process

$$
\sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s),
$$

is in $\mathcal{L}^2$ where X satisfies the SDE

$$\begin{aligned} dX_s &= \mu(s, X_s)ds + \sigma(s, X_s)dW_s, \\ X_t &= x. \end{aligned}$$

Then F has the representation

$$F(t, x) = \mathbb{E}_{t,x}[\Phi(X_T)]. \tag{2.3}$$

Theorem 2.6.2. ***(General Pricing formula)*** *The arbitrage free price process for the T-claim X is given by*

$$\Pi(t; X) = S_0(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{X}{S_0(T)}\middle|\mathcal{F}_t\right], \tag{2.4}$$

where $\mathbb{Q}$ is a (not necessary unique) martingale measure for the priory given market $S_0, ..., S_N$, with S_0 as the numeraire.

Note that different choices of $\mathbb{Q}$ will in the general case give rise to different price processes.

In particular if S_0 is the money account we have

$$S_0(t) = S_0(0)e^{\int_0^t r(s)ds},$$

where r is the short rate. Then the general pricing formula (2.4) reduces to the well known risk neutral valuation formula (RNVF) .

Theorem 2.6.3. ***(Risk Neutral Valuation Formula)*** *Assuming that there exists a short rate, then the pricing formula takes the form*

$$\Pi(t, X) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}X\middle|\mathcal{F}_t\right], \tag{2.5}$$

where $\mathbb{Q}$ is a (not necessary unique) martingale measure with the money account as a numeraire.

(Björk, 2009, pp. 72-74)

2.7 Likelihood Ratios and Change of Measure

In this section we will briefly present some basic results and definitions on measure theory. It can be viewed as a preparation for the change of numeraire section. First we define an equivalent measure (Åberg, 2010, pp. 176-177).

Definition 2.7.1. *An equivalent martingale measure $\mathbb{Q}$ (EM-measure) is a probability measure with the following two conditions.*

1. *The discounted price processes S_i/S_0 are $\mathbb{Q}$-MG for all $i \in \{0, \ldots, n\}$.*

2. The measures $\mathbb{Q}$ and $\mathbb{P}$ are equivalent.

The definition is important since almost all results in the martingale approach of derivative pricing are stated in terms of the EM-measure $\mathbb{Q}$. In that sense the EM-measure is the connection between the economic and mathematical properties of the market. Before we proceed with the theory of Likelihood ratios (LR) and how to change measure recall that two probability measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent ($\mathbb{P} \sim \mathbb{Q}$) if they have the same zero sets, i.e.

$$\mathbb{P}(A) = 0 \Leftrightarrow \mathbb{Q}(A) = 0, \quad \forall A \in \mathcal{F}.$$

How to change from one measure to another is described by the following theorem.

Theorem 2.7.1. *(**Change of measure**) Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an absolutely continuous probability measure $\mathbb{Q}$. Then there is a stochastic variable L with*

$$L \geq 0 \quad \textit{and} \quad \mathbb{E}^{\mathbb{P}}[L] = 1,$$

such that

$$\begin{aligned} &\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}[\mathbf{1}_A L], \quad \forall A \in \mathcal{F}, \\ &\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}^{\mathbb{P}}[X \cdot L], \quad \forall \mathcal{F}\textit{-measurable } X. \end{aligned}$$

L is called a Likelihood Ratio (LR) for the change of measure. Furthermore, if $\mathbb{P}$ and $\mathbb{Q}$ are equivalent the LR above fulfills

$$\begin{aligned} &L > 0 \quad \textit{and} \\ &\mathbb{P}(A) = \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_A \cdot L^{-1}], \quad \forall A \in \mathcal{F}. \end{aligned}$$

2.8 Change of Numeraire

The complexity of calculations made when pricing a derivative, can very often be reduced by changing the numeraire. Any strictly positive tradeable asset can work as a numeraire and the prices of other assets are then expressed in relative prices. This is a technique that is often used when valuing interest rate derivatives on the market.

Theorem 2.8.1. *Assume an arbitrage free market model with asset prices $S_0, S_1, ..., S_n$ where S_0 is assumed to be strictly positive. Then the following holds:*

- *The market model is free of arbitrage if and only if there exists a martingale measure, $\mathbb{Q}^0 \sim \mathbb{P}$ such that the process*

$$\frac{S_0(t)}{S_0(t)}, \frac{S_1(t)}{S_0(t)}, ..., \frac{S_N(t)}{S_0(t)},$$

  are local martingales under $\mathbb{Q}^0$

- *In order to avoid arbitrage, a T-claim X must be priced according to the formula*

$$\Pi(t;X) = S_0(t)\mathbb{E}^0\left[\frac{X}{S_0(T)}\middle|\mathcal{F}_t\right],$$

where $\mathbb{E}^0$ denotes the expectation under $\mathbb{Q}^0$.

Theorem 2.8.2. *Assume that $\mathbb{Q}^0$ is a martingale measure for the numeraire S_0 on $\mathcal{F}_t$ and assume that S_1 is a positive asset price process such that $S_1(t)/S_0(t)$ is a $\mathbb{Q}^0-$martingale. Define $\mathbb{Q}^1$ on $\mathcal{F}_T$ by the likelihood process*

$$L_0^1(t) = \frac{S_0(0)}{S_1(0)} \cdot \frac{S_1(t)}{S_0(t)}, \quad 0 \le t \le T.$$

Then $\mathbb{Q}^1$ is a martingale measure for S_1.

Combining Theorem 2.8.1 and Theorem 2.8.2, the price of a T-claim X is derived as

$$\begin{aligned}
\Pi(0;X) &= \mathbb{E}^1\left[X\frac{S_1(0)}{S_1(T)}\middle|\mathcal{F}_0\right] \\
&= \mathbb{E}^0\left[X\frac{S_1(0)}{S_1(T)} \cdot L_0^1(T)\middle|\mathcal{F}_0\right] \\
&= \mathbb{E}^0\left[X\frac{S_1(0)}{S_1(T)} \cdot \frac{S_0(0)}{S_1(0)} \cdot \frac{S_1(T)}{S_0(T)}\middle|\mathcal{F}_0\right] \\
&= \mathbb{E}^0\left[X\frac{S_0(0)}{S_0(T)}\middle|\mathcal{F}_0\right].
\end{aligned}$$

(Björk, 2009, pp. 396-416)

Chapter 3

Classical Interest Rate Theory

In introductory coursebooks in economics the interest rates are often assumed to be deterministic. In reality this restriction is not always suitable and in many situations interest rates are stochastic. The purpose of having a stochastic interest rate is that many financial instruments have uncertain future cash flows that can not be priced by a simple discounting using a deterministic rate. Instead, one has to price according to a dynamic interest rate model which is able to generate and assign probabilities to future scenarios. An example of a financial instrument that requires to be priced using a stochastic interest rate is an interest rate caplet. The holder of a caplet will only receive a future payoff if the future interest rate exceeds some prespecified level and in addition the payoff will be determined by the magnitude of the exceedance.

Since the development of the first stochastic interest rate models, the interest rate market has grown exponentially and alongside several alternative models for valuation of products have arisen. Broadly speaking, the interest rate models can be divided into two groups: short rate models and market models.

Short rate models attempt to model the continuously compounded instantaneous short rate. This rate can be viewed as a mathematical concept and it is not observable in the market.

Market models aim to model market rates, such as the forward LIBOR, that can be observed in the market. In this thesis we limit ourselves to consider market models. For information about various short rate models we refer the reader to Chapter 3 in Brigo and Mercurio (2007).

3.1 Short Rate and Zero Coupon Bond

Short rate models are based on the continuously compounded short rate. This rate, denoted $r(t)$, is a purely mathematical object and it can not be observed directly in the market. With the short rate the bank account B_t is given by

$$dB_t = r_t B_t dt \quad \Rightarrow \quad B_t = B_0 \exp\left\{ \int_0^t r_s ds \right\}.$$

Another object that relates to the short rate is the zero coupon bond $p(t, T)$. It denotes the value, at time t, of receiving one unit of currency at time T. The zero coupon bond should satisfy that $p(T, T) = 1$ and one also intuitively think that for $t < T$ we have $p(t, T) < 1$. This is known as the time value of money and it basically means that one unit of currency today is worth more than one tomorrow. However, in a negative rate environment this might not be true and we can have situations where $p(t, T) > 1$. The relation between the zero coupon bond and the short rate is given by the expected value of the discount factor $B(t)/B(T)$

$$p(t, T) = \mathbb{E}\left[\frac{B(t)}{B(T)} \middle| \mathcal{F}_t \right] = \mathbb{E}\left[\exp\left\{ -\int_t^T r_s ds \right\} \middle| \mathcal{F}_t \right]. \tag{3.1}$$

3.2 LIBOR

One of the most common market rates is the LIBOR. This rate is simply-compounded in the sense that it is constant during an interval of time. It is defined as follows.

Definition 3.2.1. ***LIBOR*** *is short for* ***London Interbank Offered Rate*** *and it is the rate one bank offers another bank.* $L[t; T_1, T_2]$ *denotes the forward LIBOR, where* $t < T_1$*. The spot LIBOR is defined as* $L[T_1; T_1, T_2] = L[T_1, T_2]$*, and it denotes the rate on an account during the period* $[T_1, T_2]$*. The length of the interval is called the* ***accrual factor*** *and is referred to as* $\tau = T_2 - T_1$*.*

The relation between zero coupon bonds and the forward LIBOR rate is given by

$$1 + \tau L[t; T_1, T_2] = \frac{p(t, T_1)}{p(t, T_2)}, \quad T_1 < T_2.$$

3.3 FRA Rates

A forward rate agreement (FRA) is an over the counter forward contract between two parties, that determines the rate of interest to be paid or received on an obligation beginning at a future start date. The formal definition is as follows.

Definition 3.3.1. *A forward rate agreement for the period* $[T_1, T_2]$ *is a contract to exchange a payment based on a fix rate* K *against a payment based on the time* T_1 *spot LIBOR rate with tenor* $\tau = T_2 - T_1$*. The exchange of cash flows occurs at time* T_2 *and the notional amount on which the interest applies on is denoted* N*.*

Essentially, the contract allows the holder to lock in the interest rate between two future time points T_1 and T_2. Let us consider the value of the FRA.

The payoff at time T_2 is given by

$$\Pi_{FRA}(T_2) = N\tau \left[L(T_1; T_1, T_2) - K \right].$$

Using the definition of the spot LIBOR

$$L(T_1; T_1, T_2) = \frac{1}{\tau}\left(\frac{1}{p(T_1, T_2)} - 1 \right),$$

we can rewrite the value as follows

$$\Pi_{FRA}(T_2) = N \left[\frac{1}{p(T_1, T_2)} - (K\tau + 1) \right].$$

Now we would like to know what value the cash flow acquires at time t. We begin with the first term within the brackets. The amount $\frac{1}{p(T_1,T_2)}$ at time T_2 is worth $\frac{1}{p(T_1,T_2)} \cdot p(T_1, T_2) = 1$ at time T_1. Discounting this to time t we get that one unit of currency at T_1 is worth $p(t, T_1)$ at time t. The second term gives us $-(K\tau + 1)$ at time T_2. This is worth $-(K\tau + 1)p(t, T_2)$ at time t. Thus, the time t value of the FRA is given by

$$\Pi_{FRA}(t) = N[p(t, T_1) - (K\tau + 1)p(t, T_2)].$$

From the price of a FRA contract we have the following definition.

Definition 3.3.2. *The FRA rate,* $FRA(t; T_1, T_2)$*, is the fixed rate* K *that renders the FRA a fair contract.*

Thus, by setting the value of the FRA contract equal to zero we obtain

$$\Pi_{FRA}(t) = 0 \Leftrightarrow N[p(t, T_1) - (K\tau + 1)p(t, T_2)] = 0,$$

which gives

$$FRA(t, T_1, T_2) = K = \frac{1}{\tau}\left(\frac{p(t, T_1)}{p(t, T_2)} - 1 \right) := F(t; T_1, T_2).$$

We see that the FRA rate equals the current forward rate.

3.4 Caps and Caplets

In this section we discuss LIBOR caps and caplets. Options with simple payoff structures are known as vanillas. One of the most common interest rate vanillas in the market is the cap. A cap can be viewed as a basket of simpler contracts called caplets. The cap contract can be used as a risk management tool to guarantee the holder that otherwise floating rates, such as the LIBOR, will not exceed a specified amount. For instance, a company that takes a loan can use a cap to protect itself against rates higher than the cap rate, also known as the strike of the cap.

To facilitate the definition of a cap we consider a set of increasing maturities $\mathcal{T} = \{T_0, T_1, \ldots, T_N\}$ known as a tenor structure. The distance

$$\tau_k = T_k - T_{k-1}, \quad k = 1, \ldots, N,$$

is known as the tenor. In the market, typical values of τ_i could for instance be 3 or 6 months. With this notation set we can define the LIBOR forward rate in terms of zero coupon bonds.

Definition 3.4.1. *We let $p_k(t)$ denote the zero coupon bond price $p(t, T_k)$ and let $L_k(t)$ denote the LIBOR forward rate, contracted at t, for the period $[T_{k-1}, T_k]$, i.e.*

$$L_k(t) = \frac{1}{\tau_k} \cdot \frac{p_{k-1}(t) - p_k(t)}{p_k(t)}, \quad k = 1, \ldots, N.$$

*The **cap** with strike price K and resettlement dates $T_1, \ldots, T_N$ is a contract which at time T_k, gives the holder of the cap the amount*

$$\Phi_k = \tau_k \cdot \max\left[L_k(T_{k-1}) - K, 0\right], \quad \textit{for each} \quad k = 1, \ldots, N.$$

Thus, a cap can be viewed as a portfolio of individual **caplets** with payoffs $\Phi_1, \ldots, \Phi_N$. We also note that the forward LIBOR rate $L_k(T_{k-1})$ in fact is the spot rate over the period $[T_{k-1}, T_k]$ and it is determined already at time T_{k-1}. This mean that the amount Φ_k is determined at T_{k-1} but not payed until at time T_k. To differ between these times one often calls T_{k-1} the expiry of the option since at this time the underlying LIBOR is known and thus also the payoff. Moreover, T_k is known as the resettlement time or maturity, as this is when the payment is due. The market practice for the pricing of caplets in positive rate environments is to use the Black-76 formula for caplets. This model will be discussed in an upcoming section.

3.5 Swaps and Swaptions

One of the most frequently traded products in the interest rate market is the interest rate swap. The most common swap contract consists of two legs known as the **floating leg** and the **fixed leg**. The term swap comes from that a set of floating rate payments (the floating leg) are exchanged for a set of fixed payments (the fixed leg). The terminology for swaps always refers to the fixed leg. Assuming as before that we have a fixed set of resettlement dates $\mathcal{T} = \{T_0, T_1, \ldots, T_N\}$, this means that a holder of a **receiver swap** with **tenor** $T_N - T_n$ will, at the dates $T_{n+1}, \ldots, T_N$, receive the fixed leg and pay the floating leg. For a **payer swap** the payments go in the other direction. In short we will refer to this swap as the $T_n \times (T_N - T_n)$ swap. To make the above statements more precise we have the following definition.

Definition 3.5.1. *The payments in a $T_n \times (T_N - T_n)$ payer swap are as follows:*

- *Payments will be made and received at $T_{n+1}, T_{n+2}, \ldots, T_N$.*
- *For every elementary period $[T_i, T_{i+1}]$, $i = n, \ldots, N-1$, the LIBOR rate $L_{i+1}(T_i)$ is set at time T_i and the floating leg*

$$\tau_{i+1} \cdot L_{i+1}(T_i),$$

  is received at T_{i+1}.
- *For the same period the fixed leg*

$$\tau_{i+1} \cdot K$$

  is payed at T_{i+1}.

The arbitrage free value, at $t < T_n$, of the floating payment made at T_{i+1} is given by

$$p(t, T_i) - p(t, T_{i+1}),$$

so the total value of the floating side at time t for $t \leq T_n$ equals

$$\sum_{i=n}^{N-1} [p(t, T_i) - p(t, T_{i+1})] = p(t, T_n) - p(t, T_N) = p_n(t) - p_N(t).$$

The total value at time t of the fixed side equals

$$\sum_{i=n}^{N-1} p(t, T_{i+1}) \tau_i K = K \sum_{i=n}^{N-1} \tau_i p(t, T_{i+1}),$$

so the net value $\mathbf{PS_n^N}(t; K)$ of the $T_n \times (T_N - T_n)$ payer swap at time $t < T_n$ is thus given by

$$\mathbf{PS_n^N}(t; K) = p_n(t) - p_N(t) - K \sum_{i=n}^{N-1} \tau_i p(t, T_{i+1}).$$

This expression can be used for defining the **par** or **forward swap rate**.

Definition 3.5.2. *The **par** or **forward swap rate** $R_n^N(t)$ of the $T_n \times (T_N - T_n)$ swap is the value of K for which $\mathbf{PS_n^N}(t; K) = 0$, i.e.*

$$R_n^N(t) = \frac{p_n(t) - p_N(t)}{\sum_{i=n+1}^N \tau_i p(t, T_i)}.$$

In the formula for the par swap rate we can interpret the denominator $\sum_{i=n+1}^N \tau_i p(t, T_i)$ as the value at t of a traded asset. This asset is a buy-and-hold portfolio consisting of n zero coupon bonds where we hold τ_i units of the bond with maturity T_i. This object has in the market become known as the following.

Definition 3.5.3. *For each pair n, k with $n < k$, the process $S_n^k(t)$ is defined by*

$$S_n^k(t) = \sum_{i=n+1}^k \tau_i p(t, T_i),$$

*where S_n^k is referred to as the **accrual factor** or as **the present value of a basis point**.*

With this definition we can express the par swap rate as

$$R_n^N(t) = \frac{p_n(t) - p_N(t)}{S_n^N(t)}, \quad 0 \le t \le T_n.$$

A very important point is that the market does not quote prices for different swaps. Instead there are market quotes for the par swap rates R_n^N. From these one can then easily compute the arbitrage free price for a payer swap with swap rate K by the formula

$$\mathbf{PS_n^N}(t; K) = (R_n^N(t) - K) S_n^N(t).$$

There also exists options on the swaps. These are called **swaptions** which is short for swap options. The definition is as follows.

Definition 3.5.4. *A $T_n \times (T_N - T_n)$ **payer swaption** with **swaption strike** K is a contract which at the exercise date T_n gives the holder the right but not the obligation to enter into a $T_n \times (T_N - T_n)$ swap with the fixed swap rate K. We thus see that the payer swaption is a contingent T_n-claim X_n^N defined by*

$$X_n^N = \max\left[\mathbf{PS_n^N}(T_n; K), 0\right].$$

Using that $\mathbf{PS_n^N}(T_n; K) = (R_n^N(T_n) - K) S_n^N(T_n)$ this can be rewritten as

$$X_n^N = \max[R_n^N(T_n) - K, 0] S_n^N(T_n). \tag{3.2}$$

(Björk, 2009, pp. 428-431)

3.6 Valuation of Caps and Caplets

This section will present two common market practices of pricing caps and caplets. We begin by describing the Black-76 model and then turn our attention to the Bachelier or normal model that has become one of the quoting standards when rates are negative.

3.6.1 Black-76 Model

In the the Black-76 framework the forward LIBOR, $L_k(t)$, is assumed to have the following dynamics under its corresponding forward measure $\mathbb{Q}^{T_k}$

$$dL_k(t) = \sigma_k L_k(t) dW_t^k.$$

The price, $\mathbf{Capl_k}(t)$, of a caplet number k with expiry T_{k-1} and maturity T_k is given by the standard RNVF

$$\mathbf{Capl_k}(t) = \tau_k \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^{T_k} r(s)ds} \cdot \max\left(L_k(T_{k-1}) - K, 0\right) \middle| \mathcal{F}_t \right], \quad k = 1, \dots, N.$$

However, it is much easier to use the T_k-forward measure to calculate the expectation. By changing numeraire we get

$$\begin{aligned} \mathbf{Capl_k}(t) &= \tau_k \mathbb{E}^{T_k} \left[\frac{p(t,T_k)}{p(T_k,T_k)} \cdot \max\left(L_k(T_{k-1}) - K, 0\right) \middle| \mathcal{F}_t \right] \\ &= \tau_i p(t,T_k) \cdot \mathbb{E}^{T_k} \left[\max\left(L_k(T_{k-1}) - K, 0\right) \middle| \mathcal{F}_t \right], \quad k = 1, \dots, N. \end{aligned} \tag{3.3}$$

Computing the expectation above we then obtain the following expression for the price

$$\mathbf{Capl_k^B}(t) = \tau_k p(t,T_k) \mathrm{Bl}(K, L_k(t), \sigma_k \sqrt{T_{k-1} - t}), \quad k = 1, \dots, N. \tag{3.4}$$

where

$$\mathrm{Bl}(K, F, \Sigma) = F \mathcal{N}\left(\frac{\ln(F/K) + \Sigma^2/2}{\Sigma} \right) - K \mathcal{N}\left(\frac{\ln(F/K) - \Sigma^2/2}{\Sigma} \right), \tag{3.5}$$

and $\mathcal{N}$ denotes the standard normal distribution function.

We note that because we assume a constant volatility σ_k we obtain a formula that is very similar to the Black-Scholes formula for a standard European call option on a stock. The Black-76 model implicitly assumes that the LIBOR forward rate is log-normal. This means that it never goes negative. The market practice is to quote prices of caps using Black-76. However, prices are not presented in monetary terms but instead in terms of **implied Black volatilities**. These volatilities can be either **flat volatilities** or **spot volatilities**. Following the definitions in (Björk, 2009, pp. 418-420) they are defined as follows.

Definition 3.6.1. *Given market price data on a tenor structure $\mathcal{T} = \{T_0, T_1, \ldots, T_N\}$ for a fixed date t, the implied Black volatilities are defined as follows.*

- *The implied **flat** or **cap volatilities** $\bar{\sigma}_1^{flat}, \ldots, \bar{\sigma}_N^{flat}$ are defined as the solutions of the equations*

$$\mathbf{Cap_i^{mkt}}(t) = \sum_{k=1}^{i} \mathbf{Capl_k^B}(t; \bar{\sigma}_i^{flat}), \quad i = 1, \ldots, N.$$

- *The implied **forward**, **spot** or **caplet volatilities** $\bar{\sigma}_1^{spot}, \ldots, \bar{\sigma}_N^{spot}$ are defined as solutions of the equations*

$$\mathbf{Capl_i^{mkt}}(t) = \mathbf{Capl_i^B}(t; \bar{\sigma}_i^{spot}), \quad i = 1, \ldots, N.$$

*A sequence of implied volatilities $\bar{\sigma}_1, \ldots, \bar{\sigma}_N$ (cap or caplet) is called a volatility **term structure**. To make the difference between cap and caplet volatilities clear we emphasize that, a cap volatility means that the volatility is constant for all caplets in the cap with maturity T_i. Whereas the caplet volatility is just the implied volatility for caplet number i.*

Here we denote the market price of the cap on the interval $[T_0, T_i]$ as $\mathbf{Cap_i^{mkt}}$. Given the caps it is then easy to get the price of the caplet as

$$\mathbf{Capl_i^{mkt}}(t) = \mathbf{Cap_i^{mkt}}(t) - \mathbf{Cap_{i-1}^{mkt}}(t), \quad i = 1, \ldots, N.$$

Alternatively, if one has observed prices of caplets one can get the market price of caps as

$$\mathbf{Cap_i^{mkt}} = \sum_{k=1}^{i} \mathbf{Capl_k^{mkt}}, \quad i = 1, \ldots, N.$$

The original Black-76 model does not allow for negative interest rates. In order to be able to quote cap prices without the model breaking down one can simply add a shift to the LIBOR rate dynamics. Set $L_k^*(t) = L_k(t) + s$, where s is an arbitrary shift term. Then the shifted LIBOR will follow exactly the same dynamics as the LIBOR rate, hence the formula for the caplet price will stay unchanged, with a slight modification of the strike $K^* = K + s$. The price and volatility from this model will be referred to as **shifted Black**.

3.6.2 Bachelier Model

The Bachelier model is named after the French mathematician Louis Bachelier who was one of the first scientists to mathematically analyze the Brownian motion for option pricing theory. The model dynamics has the following SDE which allows negative forward values

$$dF_t = \sigma dW_t.$$

Here W_t represents the Brownian motion and σ is the volatility. The corresponding solution is

$$F_t = F_0 + \sigma W_t.$$

This means that F_t is normally distributed and as a consequence the model has also become known as the normal model. The price for a caplet on $[T_{k-1}, T_k]$, is given by

$$\mathbf{Capl_k^{Normal}} = p(t, T_k) \left[(L_k(t) - K)\mathcal{N}(d) + \sigma\sqrt{T_{k-1}}\mathcal{N}'(d) \right],$$

where $\mathcal{N}$ is the standard Gaussian cumulative distribution function and N' is the corresponding probability density function. The expression for d is given by

$$d = \frac{L_k(t) - K}{\sigma\sqrt{T_{k-1}}}.$$

Due to its very simple structure it has often difficulties in capturing the dynamics of the forward rate. Hence, the normal distribution model is rarely used for simulation of forward rates. However, in the current low and negative rate environment, it is used for quoting market prices as a complement to Black's model.

3.6.3 ATM Caps and Caplets

A caplet is said to be ATM if its strike equals the current value of the underlying forward rate. In general a cap is a combination of several caplets, with each caplet having a different underlying forward rate. This nature of a cap makes it unclear which of the forward that decides the ATM strike. Market convention is that a cap is said to be ATM if its strike equals the current forward swap rate of the maturity equivalent swap. For example if a cap start at T_n and ends at T_N then the ATM strike K_{ATM} is

$$R_n^N(t) = K_{ATM}.$$

The importance of the ATM cap comes from that it is often the most liquid traded cap in the market.

3.7 LIBOR Market Model

The basic idea behind the LIBOR Market Model (LMM) is to define the LIBOR rates such that each $L_k(t)$ will be log-normal under the corresponding $\mathbb{Q}^{T_k}$ measure, since then all caplet prices in equation (3.3), will follow a Black type formula as in (3.4).

Definition 3.7.1. *If the **LIBOR** forward rates have the dynamics*

$$dL_k(t) = L_k(t)\sigma_k(t)dW^k(t), \qquad k = 1, ..., N, \tag{3.6}$$

*where W^k is a $\mathbb{Q}^{T_k}$-Wiener process, then we say that we have a discrete tenor **LIBOR Market Model** (LMM) with volatilities $\sigma_1, ..., \sigma_N$.*

Given an LMM, we have that the LIBOR rate is just a GBM, hence we obtain from (2.2)

$$L_k(T) = L_k(t) \exp\left\{ \int_t^T \sigma_k(s) dW^k(s) - \frac{1}{2} \int_t^T ||\sigma_k(s)||^2 ds \right\}.$$

In the same way as in the section for geometric Brownian motions, the distribution for $L_k(T)$ conditional on $\mathcal{F}_t$ is derived as log-normal, and we can write

$$L_k(T) = L_k(t) e^{Y_k(t,T)},$$

where $Y_k(t,T)$ is normally distributed with expected value

$$m_k(t,T) = -\frac{1}{2} \int_t^T ||\sigma_k(s)||^2 ds,$$

and variance

$$\Sigma_k^2(t,T) = \int_t^T ||\sigma_k(s)||^2 ds.$$

As the price of a caplet in the LMM is given by the same expression as for the Black-76 model, namely (3.3), and that the LIBOR rate is log-normally distributed in both the Black-76 and the LMM framework, the pricing formula for a caplet is very similar in the two models. Namely, in the LMM framework we get

$$\mathbf{Capl_k^{LMM}}(t) = \tau_k p(t, T_k) \mathrm{Bl}(K, L_k(t), \Sigma_k(t,T)), \quad k = 1, ..., N, \tag{3.7}$$

with Σ_k defined as above and $\mathrm{Bl}(K, F, \Sigma)$ as defined in (3.5). Thus, it becomes clear that the price is given by a Black type formula as in (3.4). (Björk, 2009, pp. 420-422)

3.8 SABR Model

Hagan, Kumar, Lesniewski and Woodward developed the SABR model (Hagan et al., 2002). The name stands for "stochastic, alpha, beta, rho" referring to the model parameters. It is a stochastic volatility model for the forward rate or the forward price of an asset.

Definition 3.8.1. *Assume a forward rate F_t. In the SABR model it is assumed that under the associated measure $\mathbb{Q}^T$, F_t has the following dynamics:*

$$\begin{aligned} dF_t &= V_t F_t^\beta dZ_t, \quad F(0) = f, \\ dV_t &= \nu V_t dW_t, \quad V_0 = \alpha, \end{aligned} \tag{3.8}$$

where Z_t and W_t are $\mathbb{Q}^T$ standard Brownian motions with

$$dZ_t dW_t = \rho dt,$$

and where $\beta \in (0,1]$, ν and α are positive constants and $\rho \in [-1,1]$.

It can be shown that the process in (3.8) is always a martingale when $\beta < 1$. In the limit when $\beta = 1$, i.e. the log-normal case, it can be shown that the rate process is a martingale if and only if $\rho \leq 0$. This of course implies some constraints on the admissible parameter values.

The model is widely used because of its simplicity and that it is possible to derive a closed-form approximation of the implied volatility, which was derived by Hagan, Kumar, Lesniewski and Woodward. The purpose of the SABR model is to fit to the volatility smile that is observed in the market. The term smile comes from that volatilities also vary across different strikes in a pattern that is similar to a smile. For example it can be seen that caplet volatilities for a fixed caplet expiry T are different for different strikes K. A limitation of the SABR model compared to for example LMM is that each forward rate is assumed to exist under its own forward measure $\mathbb{Q}^T$. This means that the model assumes zero correlation between forward rates and as a consequence, it can not price path-dependent derivatives such as an autocap, using the set of calibrated parameters $\phi(T) = \{\alpha(T), \beta(T), \nu(T), \rho(T)\}$.

When dealing with LIBOR forward rates $L_k(t)$ under the $\mathbb{Q}^{T_k}$ dynamics we get that the price, at $t = 0$, of a T_{k-1}-expiry caplet is

$$\mathbf{Capl}(0, T_{k-1}, T_k, K) = \tau_k p(0, T_k) \mathrm{Bl}(K, L_k(0), \sigma_{SABR}(K, L_k(0))\sqrt{T_{k-1}})$$

where $\mathrm{Bl}(K, F, \Sigma)$ is defined in (3.5), $\tau_k = T_k - T_{k-1}$ and K is the strike. The implied Black volatility $\sigma_{SABR}(K, f)$ is given by the following analytical approximation:

$$\sigma_{SABR}(K,f) = \frac{\alpha}{(fK)^{\frac{1-\beta}{2}}\left[1 + \frac{(1-\beta)^2}{24}\left(\ln\left(\frac{f}{K}\right)\right)^2 + \frac{(1-\beta)^4}{1920}\left(\ln\left(\frac{f}{K}\right)\right)^4 + \dots\right]} \cdot \frac{z}{x(z)}\left\{1 + \left[\frac{(1-\beta)^2\alpha^2}{24(fK)^{1-\beta}} + \frac{\rho\beta\nu\alpha}{4(fK)^{\frac{1-\beta}{2}}} + \nu^2\frac{2-3\rho^2}{24}\right]T + \dots\right\}, \tag{3.9}$$

with

$$z = \frac{\nu}{\alpha}(fK)^{\frac{1-\beta}{2}} \ln\left(\frac{f}{K}\right),$$

$$x(z) = \ln\left\{\frac{\sqrt{1 - 2\rho z + z^2} + z - \rho}{1 - \rho}\right\}.$$

In the above derived approximation the dots stand for higher order terms that are usually negligible, especially for contracts with short maturities. The ATM implied volatility is derived by setting $K = F(0) = f$ into equation (3.9):

$$\begin{aligned}\sigma_{\mathrm{ATM}} &= \sigma_{SABR}(f,f) \\ &= \frac{\alpha}{f^{1-\beta}}\left\{1 + \left[\frac{(1-\beta)^2\alpha^2}{24 f^{2-2\beta}} + \frac{\rho\beta\nu\alpha}{4 f^{1-\beta}} + \nu^2\frac{2-3\rho^2}{24}\right]T + \dots\right\}.\end{aligned} \tag{3.10}$$

From (3.10) it is noted that the term that contributes with most impact on the ATM implied volatility is $\alpha/f^{1-\beta}$.

An extension of the SABR model that has become increasingly important in negative rate environments is the **shifted SABR model**. It is defined as follows.

Definition 3.8.2. *Let F_t denote a forward rate, for instance a LIBOR forward rate on some specified interval. Further let s be a positive deterministic shift. Then the shifted SABR model is defined as*

$$\begin{aligned} dX_t &= V_t X_s^{\beta} dW_t, \qquad X_0 = x, \\ dV_t &= \nu V_t dZ_t, \qquad V_0 = \alpha, \end{aligned}$$

where $dW_t dZ_t = \rho dt$, $X_t = F_t + s$ and $X_0 = x = f + s$.

The approximated black volatility corresponding to the shifted SABR process above is

$$\begin{aligned} \sigma_{shift}^{SABR}(x, K_{\mathrm{s}}) = & \frac{\alpha \log(\frac{x}{K_{\mathrm{s}}})}{\int_{K_{\mathrm{s}}}^{x} \frac{dx'}{C(x')}} \cdot \left(\frac{\zeta}{\hat{x}(\zeta)}\right) \cdot \\ & \left\{ 1 + \left[\frac{2\gamma_2 - \gamma_1^2 + 1/x_{\mathrm{av}}^2}{24} \alpha^2 C(x_{\mathrm{av}})^2 + \right.\right. \\ & \left.\left. \frac{1}{4}\rho\alpha\nu\gamma_1 C(x_{\mathrm{av}}) + \frac{2-3\rho^2}{24}\nu^2 \right] T + \ldots \right\}, \end{aligned} \tag{3.11}$$

where K_s is the strike for the shifted process X_t and

$$\begin{aligned} & x = f + s, \qquad x_{\mathrm{av}} = \sqrt{x K_s}, \qquad C(x) = x^{\beta}, \\ & \gamma_1 = \frac{C'(x_{\mathrm{av}})}{C(x_{\mathrm{av}})} = \frac{\beta}{x_{\mathrm{av}}}, \\ & \gamma_2 = \frac{C''(x_{\mathrm{av}})}{C(x_{\mathrm{av}})} = \frac{\beta(\beta-1)}{x_{\mathrm{av}}^2}, \\ & \zeta = \frac{\nu}{\alpha} \frac{x - K_s}{C(x_{\mathrm{av}})}, \qquad \hat{x}(\zeta) = \log\left(\frac{\sqrt{1 - 2\rho\zeta + \zeta^2} - \rho + \zeta}{1-\rho} \right). \end{aligned}$$

The integral found in the first part of the above expression (3.11) for the shifted Black SABR volatility, is derived as

$$\int_{K_s}^{x} \frac{1}{C(x')} dx' = \frac{\left(x^{1-\beta} - K_s^{1-\beta}\right)}{1-\beta}.$$

In order to get the ATM implied volatility for shifted SABR, we have to make an expansion of the right hand side in the above equation. In Hagan et al. (2002) the following expansion is used

$$x^{1-\beta} - K_s^{1-\beta} = (1-\beta)(xK_s)^{\frac{1-\beta}{2}} \log\left(\frac{x}{K_s}\right) \cdot \left\{1 + \frac{(1-\beta)^2}{24}\log\left(\frac{x}{K_s}\right)^2 + \frac{(1-\beta)^4}{1920}\log\left(\frac{x}{K_s}\right) + \dots\right\}.$$

Inserting the expansion into equation (3.11), and setting $K_s = X_0 = x$, we get

$$\begin{aligned}\sigma_{shift}^{\mathrm{ATM}} &= \sigma_{shift}^{SABR}(x,x) \\ &= \frac{\alpha}{x^{1-\beta}}\left\{1 + \left[\frac{(1-\beta)^2\alpha^2}{24x^{2-2\beta}} + \frac{\rho\beta\nu\alpha}{4x^{1-\beta}} + \nu^2\frac{2-3\rho^2}{24}\right]T + \dots\right\}. \end{aligned} \tag{3.12}$$

When fitting the SABR model one has to calibrate four parameters α, β, ν, and ρ for each forward with expiry T. This is often done by minimizing the sum of square differences between the implied volatilities from the SABR model and the market caplet volatilities for caplets with expiry T. The interpretation of the SABR parameters can be found in Table 3.1. Since β and ρ have the same affect on the volatility smile one often runs into problem when trying to optimize all parameters at once. A common approach to circumvent this is to fix β during optimization.

Table 3.1: *The different impact the four parameters in the SABR model have on the volatility curve.*

Parameter	Curve Property	Direction
α	Level	The smile curve shifts upward as α increases
β	Slope	The curve steepens as β decreases
ρ	Slope	The curve steepens as ρ decreases
ν	Curvature	The curvature increases as ν increases

Given the assumption that we have market caplet volatilities, σ_i^{mkt}, for a fix caplet expiry there exists several different ways of estimating the SABR parameters. Common for almost all methods is that they consist of two steps (due to the similar interpretation of β and ρ):

1. Estimate or choose β.
2. Calibrate α, ρ and ν.

The curve created by the ATM volatility, $\sigma_{SABR}(f, f)$, is known as the backbone. The backbone is almost completely determined by β. This can be used to infer β from historical observations of the backbone. Another way is to select β from aesthetic consideration.

Estimating β based on historical observations is done by taking the logarithm of (3.10). Doing so one obtains

$$\ln(\sigma_{\mathrm{ATM}}) = \ln(\alpha) - (1-\beta)\ln(F(0)) + \ln(1 + \dots). \tag{3.13}$$

Consequently, β can be estimated by making a linear regression applied on $(\ln(F(0)), \ln(\sigma_{\text{ATM}}))$, where terms including expiry dates are ignored (Brigo and Mercurio, 2007, p. 508 - 513). As for the aesthetic consideration a common industry practice is to choose $\beta = 0.5$ directly.

Given current market ATM volatility σ_{ATM} we can rearrange (3.10) to obtain the following cubic equation in α

$$\alpha \left\{ 1 + \left[\frac{(1-\beta)^2 \alpha^2}{24 f^{2-2\beta}} + \frac{\rho\beta\nu\alpha}{4 f^{1-\beta}} + \nu^2 \frac{2-3\rho^2}{24} \right] T \right\} - \sigma_{\text{ATM}} f^{1-\beta} = 0. \tag{3.14}$$

Note that when solving this cubic equation we may have three solutions. The value of α is always the smallest real value solution.

This has led to two common approaches for estimating α, ρ, and ν.

1. Let α be a function of ρ and ν according to the relation in (3.14) and choose parameters that minimize the sum of squared differences between model and market volatilities. That is

$$\{\alpha(\rho,\nu), \rho, \nu\} = \arg\min_{\rho,\nu} \sum (\sigma_i^{SABR} - \sigma_i^{mkt})^2.$$

   Note that this approach allows for exact matching of the market ATM volatility.

2. Calibrate α, ρ and ν together

$$\{\alpha, \rho, \nu\} = \arg\min_{\alpha,\rho,\nu} \sum (\sigma_i^{SABR} - \sigma_i^{mkt})^2.$$

   Note that this approach determines α and in turn σ_{ATM}.

Another way to calibrate the SABR model is to introduce a vega weighed cost function (Bianchetti and Carlicchi, 2011).

$$\{\alpha, \rho, \nu\} = \arg\min_{\alpha,\rho,\nu} \sum [\omega_i(\sigma_i^{SABR} - \sigma_i^{mkt})]^2, \tag{3.15}$$

where $\omega_i = \frac{\upsilon_i}{\sum \upsilon_i}$ and υ_i is the Black's vega sensitivity of the caplet with strike K_i. Vega is the measurement of an option's sensitivity to changes in the volatility of the underlying asset. As such it may be expressed as the partial derivative of the asset price with respect to the volatility

$$\upsilon = \frac{\partial \Pi}{\partial \sigma}. \tag{3.16}$$

Calculating the derivative with respect to σ in equation (3.4) we get for Black's formula

$$\upsilon = F_0 \sqrt{T} \mathcal{N}'(d_1),$$

where

$$d_1 = \frac{\ln(\frac{F_0}{K}) + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}.$$

One of the motivations for using vega calibration is that it puts more weight on the near ATM areas of the volatility smile, with high vega sensitivities and market liquidity, and less emphasis to out-of-the-money (OTM) areas with lower vega and liquidity. However, there is a trade-off. Employing vega calibration will often lead to that the left hand side of the smile is inadequately matched. This will result in misspricing of lower strikes for longer maturities.

Chapter 4

Data and Market Conventions

This chapter is meant to contribute to the readers understanding of the market conventions on the cap market. We are for instance explaining how caps are quoted and what the market conventions are on the European and US market.

The interest rate market has for a long time used implied Black volatilities for quotation of interest rate derivatives. Recall that the implied volatility is the volatility that is required in a benchmark pricing model (e.g. Black), for the model price to replicate the market price of the interest rate derivative. Quoting in terms of implied volatility is a convenient way in which one can think about the value of an option. There are many different reasons for using implied volatilities instead of monetary units such as USD, EUR, CHF etc. some of them are

- It works as a communication tool. Traders often prefer implied volatility since it is a more stable measure and does not change as frequently as option prices. Implied volatility removes the effect of parameters not related to the volatility that affects the option price, such as the underlying discount curve, the strike, maturity and tenor. When the price of an underlying asset or interest rate changes, the option price might change while the volatility may not.

- An implied volatility from a simple model, such as Black-76, can help a trader to develop an intuition about the model and use it to structure his or her thinking about option markets.

- It can be used in conjunction with interpolation to work out prices for options that are not observable in the market.

The implied volatilities, discount factors and forward rates, that we are using in this project are accessed through a Bloomberg terminal. Bloomberg has a function called *volatility cube* (VCUB) where cap volatilities are stated from different sources and on different markets. The cap volatilities are changing depending on what source that is used, also available strikes may be different. In this report, we are using data from the source ICAP. They are offering a wide range of strikes that are close to ATM-strikes.

In markets where the interest rates have not yet reached negative values, the implied Black volatilities are fully defined and there are no technical problems. Anyhow, interest rates have gone negative in some markets, such as in Switzerland and in the Eurozone. As concluded before, the implied Black volatilities are not defined for negative values on the underlying LIBOR rate, which bring technical problems into the market quoting system. A solution to this problem brokers have started to quote implied volatilities using a shifted Black or the normal model. The corresponding cap volatilities are then shifted Black volatilities respectively the normal volatilities.

4.1 LIBOR and OIS Rate

In this section we introduce overnight indexed swap (OIS) rate and in line with Hull (2012), we address some important differences between the OIS and LIBOR that became especially important after the credit crises in 2007.

In the US the USD LIBOR is an average of the rates at which banks think they can obtain unsecured funding. The European equivalent is known as the EURIBOR and it follows a similar definition. The EURIBOR is defined as the average of the rates at which banks believe a prime bank[1] can obtain unsecured funding. Both rates are quoted on a range of tenors, where 3 month and 6 month are most important.

Another important rate in the market is the overnight indexed swap. The OIS is a contract between two parties where a fixed rate for a period (e.g. one month, three month, one year, or three years) is exchanged for the geometric average of overnight rates during the period. The relevant overnight rates are the rates in the government-organized interbank market where banks with excess capital reserves lend to banks that need to borrow to meet their reserve requirements.

In the US the overnight rate in the market is known as the fed funds rate. This is what is used in the OIS geometric average calculations. The European equivalent to the fed funds rate is the Eonia (Euro overnight index average). Because no principal is exchanged and since funds are typically exchanged only at maturity there is very little default risk inherent in the OIS market.

This has led to that the OIS swap rate often is regarded as a more appropriate proxy for the risk free rate than the LIBOR. A key indicator of stress in the banking system is the LIBOR-OIS spread[2]. This spread became particularly prevalent during the 2008 finan-

[1] According to Investopedia a bank is said to be a prime bank if it belongs to the top 50 banks (or thereabout) in the world.

[2] The LIBOR-OIS spread is the amount (in bps) by which the 3 month LIBOR exceeds the 3 month overnight indexed swap (OIS) rate.

cial crises, where the LIBOR-OIS spread spiked at 365 basis point on October 10th 2008.

Theoretically banks can borrow at the 3 month OIS rate and the lend the funds to an AA-rated bank at the 3 month LIBOR rate of interest. Thus, the LIBOR-OIS spread can be viewed as a credit spread that compensates the lender for the possibility of the AA-rated bank defaulting during the 3 month period. Usually the the LIBOR-OIS spread is about 10 basis points. However, the greater the LIBOR-OIS spread, the greater the reluctance of other banks to lend to each other because of perceptions about counterparty credit risk.

The eruption of the spreading credit crises that started in the summer of 2007 resulted in the market leaving the single curve framework to adopt a multi curve framework. The key idea of the multi curve setting is to separate the discount curve from the forward curve and to separate forward curves with respect to the underlying tenor.

The purpose of the framework is to be able to account for credit and liquidity premia when pricing financial products. As a result the previous single curve setting is no longer valid. In practice this means that one should estimate one forward curve for each tenor, instead of using one universal forward curve for all tenors.

The new market practice is termed as segmentation and it means that practitioners have taken an empirical approach which is based on the construction of as many forward curves as there are tenors (e.g. 1 month, 3 month, 6 month,...). The forward rates that used to be closely related prior to the credit crunch starting in 2007 are now suddenly different subjects because of each rate having its own liquidity and credit risk. As a consequence future cash flows of a financial contract are generated through the curves associated with the underlying forward rates and then discounted by another curve, which is termed as the discounting curve.

The main focus in this report will be on the topic of a multi curve framework. The theory regarding this framework is presented in Chapter 6, followed by an application of volatility transformation in Chapter 7.

4.2 Cap Contracts in different Markets

When calibrating models using caps it is important to know that contracts do not necessarily have the same definition across different markets. To illustrate this we will compare conventions in the US and EUR market.

In the US market caps are defined with 3 month underlying caplets for all maturities T_i of the underlying cap. Furthermore, there is no caplet on the first 3 month interval. In Figure 4.1 the situation is illustrated for cap maturities up to 10 years.

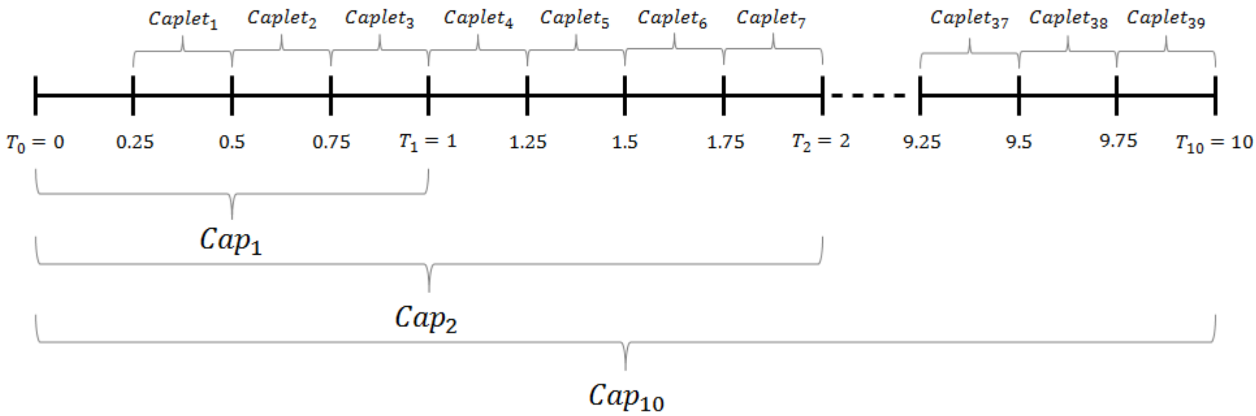


Figure 4.1: *Cap grid for cap maturities up to 10 years in the US market. The caps are composed of 3 month underlying caplets. Note that there is no caplet on the first interval which reflects that the underlying over that interval is deterministic.*

From a modelling perspective this is a very convenient quotation system. As we can see in Figure 4.1 the first cap will be contained within the second cap, and the second cap will be contained in the third and so on. Thus, if one would like to know, say the total price of caplet number 4, 5, 6 and 7 one can take the price difference between the second and first cap.

In the EUR market the convention of how caps are defined are quite different from the US market. In simple terms one can say that there exist market quotes for two different kinds of caps in the market. First, for maturities up to 2 years three caps with maturities 1, 1.5, and 2 years are quoted. These are based on 3 month underlying caplets, with no caplet in the first time slot as illustrated in Figure 4.2.

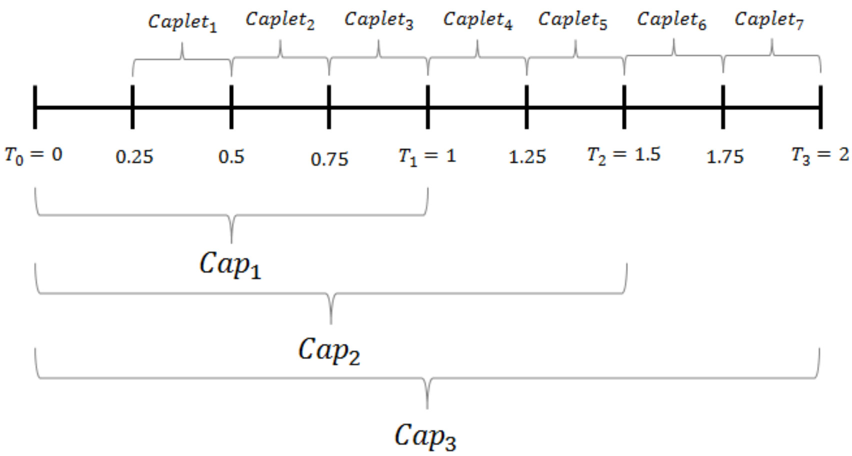


Figure 4.2: *Cap grid for cap maturities up to 2 years in the EUR market. The caps are composed of 3 month underlying caplets. Note that there is no caplet on the first interval.*

The second type of caps exists for maturities from 2 years up to 30 years. These are based on 6 month underlying caplets. The situation is shown in Figure 4.3 where we have chosen to restrict ourselves to 10 years for visibility reasons.

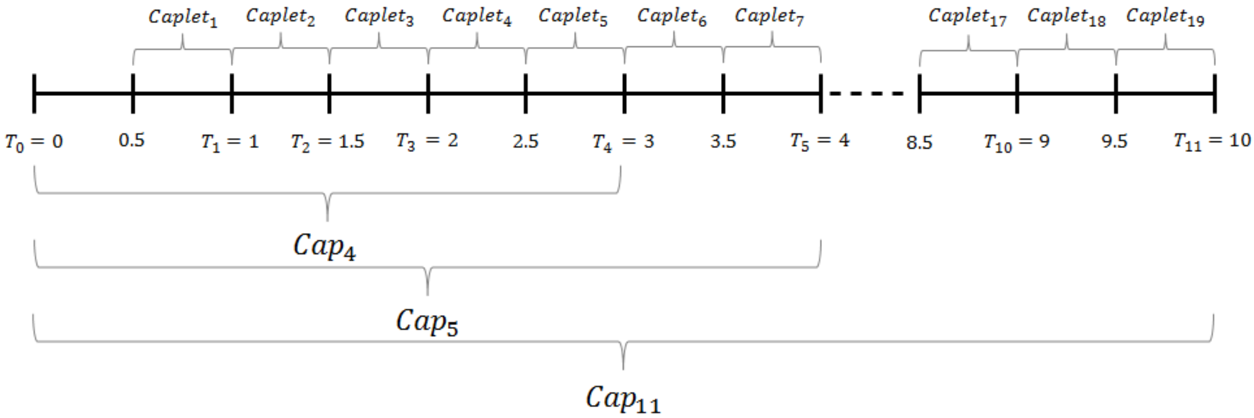


Figure 4.3: *Cap grid for cap maturities up to between 3 and 10 years in the EUR market. The caps are composed of 6 month underlying caplets. Note that there is no caplet on the first interval.*

This quotation pose additional problems when one for instance wants to get caplet prices and calibrate a caplet volatility surface. The reason is that the 2 year cap (cap number 3 in Figure 4.2) is no longer contained in the 3 year cap (cap number 4 in Figure 4.3). To solve this issue we will apply a volatility transformation. This will be discussed further in Chapter 7.

Chapter 5

Methods for calibrating the SABR Model

Three different methods for calibrating a SABR model are presented in this chapter. The first two methods are based on fitting a volatility term structure (VTS) for each strike. This means that for each strike K, we assume that it exists a caplet volatility curve $\sigma_K(T)$, that depends only on the caplet expiry T. Once a VTS is obtained we use the obtained caplet volatilities to calibrate a SABR model for each caplet expiry.

The third method is a global method for obtaining caplet volatilities. The method assumes both an expiry and a strike dependence on the caplet volatility. Here the caplet volatility surface $\sigma(K,T)$, or simply the discrete set of caplet volatilities are described by a set of parameters. There is a direct mapping from the set of parameters to cap prices via pricing of the underlying caplets. This can be used in the calibration by choosing parameters such that the difference between model cap prices and market cap prices are minimized in a least square sense.

In the first two methods we work under the assumption that for each strike we have a set of caplets with increasing expiries $\{T_k^e\}_{k=1}^N$ and maturity dates $\{T_k^m\}_{k=1}^N$. In order to first get a VTS we use the following simple model for the forward rate under the $\mathbb{Q}^{T_k^m}$-measure

$$dL_k(t) = \sigma_k(t)L_k(t)dW_t^k, \quad k = 1, \ldots, N.$$

Here, $\sigma_k(t)$ is the instantaneous volatility of the LIBOR forward rate. Recall from the LMM pricing formula of a caplet in equation (3.7) that the price involves an integrated squared volatility

$$\Sigma^2(t, T_k^e) = \int_t^{T_k^e} \sigma_k(s)^2 ds.$$

Hereafter we will refer to this quantity as the **integrated caplet volatility**. Note that in the Black-76 formula for caplets (3.4), we assume a constant instantaneous forward rate volatility $\sigma_k(s) = \sigma_k$. The integrated caplet volatility then reduces to

$$\Sigma^2(t, T_k^e) = \sigma_k^2(T_k^e - t).$$

Inserting this into the LMM formula (3.7), will give the Black-76 formula for caplets.

In the three methods, different assumptions are made about the caplet volatilities:

- The **first** method assumes a constant instantaneous volatility

  $$\sigma_k(s) = \sigma_k, \quad t < s < T_k^e,$$

  for each LIBOR forward rate $L_k(t; T_k^e, T_k^m)$ defined on the interval $0 < t < T_k^e$. This will correspond to a simple Black-76 model. To infer these volatilities from market cap prices, one can use a method known as caplet stripping.

- The **second** approach is based on assuming a parametric functional form $f(s; \theta)$ for the instantaneous volatility. When using this method we will assume that all LIBOR forward rates have a time homogeneous instantaneous volatility, i.e.,

  $$\sigma_k(s) = f(s; T_k^e, \theta), \quad k = 1, ..., N.$$

  For example, a functional form is to let $f(s; T_k^e, \theta)$ be piecewise constant with constant levels $\theta = (\theta_1, \theta_2, \ldots)$ or to choose more market related forms such as proposed by Rebonato (Rebonato, 2004). Rebonatos function has four parameters; $\theta = (a, b, c, d)$, and are discussed further in a later context.

- Our **third** method is a global SABR method with an interpolated parameter term structure. The SABR model is capable of producing an implied (Black) caplet volatility, see formula (3.9), of the type

  $$\sigma(K) = s(K, T, F, \phi).$$

  This form of the volatility indicates a volatility smile[1] that in general depends on the state variables (the forward rate F and maturity T) and the model parameters ϕ. Using the SABR model we have a set of four parameters, $\phi = \{\alpha, \beta, \rho, \nu\}$. By making the parameters dependent on the expiry T, we may extend the volatility smile to a volatility surface[2]. This means we will have four parameters with term structures $\alpha(T), \beta(T), \rho(T)$, and $\nu(T)$. Hence, our calibration problem boils down to finding the parameter term structures $\phi(T)$ such that the error between the model and market prices are minimized in a least square sense.

[1]The volatility smile illustrates how the spot volatility changes as a function of the strike K

[2]A volatility surface is a two dimensional graph over the spot volatility as a function of both the strike and the maturity

5.1 Method 1: Bootstrap

In the market there does not exist implied forward volatilities for caplets, known as caplet volatilities. Instead, most market data providers quote implied flat volatilities for caps with different tenors. Thus, having market cap volatilities at hand, one needs to infer the caplet volatilities from the quoted market cap prices. This technique is known as caplet stripping.

There are many different ways to obtain caplet volatilities from cap prices. One of the most basic method is known as bootstrapping. This method belongs to the class of caplet stripping schemes where the only assumption made is that caplet volatilities are a function of time to maturity. This means that we assume a volatility term structure, but without a strike dependence. The method is applicable when we have absolute volatility quotes, that is for a fixed strike, we have implied flat volatilities for a set of caps with different maturities.

The bootstrap technique works as follows:

1. For a given strike, put the observed cap prices (obtained from market quoted cap volatilities) in ascending order of maturity.
2. Find the series of price difference for caps

$$\mathbf{Cap_i^{mkt}}(K) - \mathbf{Cap_{i-1}^{mkt}}(K) \quad i = 1, \ldots, M, \tag{5.1}$$

   where by definition $\mathbf{Cap_0^{mkt}} = 0$ and M is the number of observed cap prices and K is a fixed strike.
3. Partition the caplets such that they are assigned to the relevant price difference. For instance, the first set of caplets corresponds to the caplets in the first cap, the second set of caplets corresponds to the second price difference etc.
4. For each partition of caplets, assume that they have a common and constant caplet volatility, and solve for this using the price difference and 1-D root finding.

The bootstrap method above render caplet volatilities that are constant on each interval between two subsequent cap maturities. The situation is shown in Figure 5.1 and we see that for a given interval $[T_{i-1}, T_i]$ all caplets will have the same volatility. For more information about caplet stripping we refer the reader to e.g. White and Iwashita (2014).

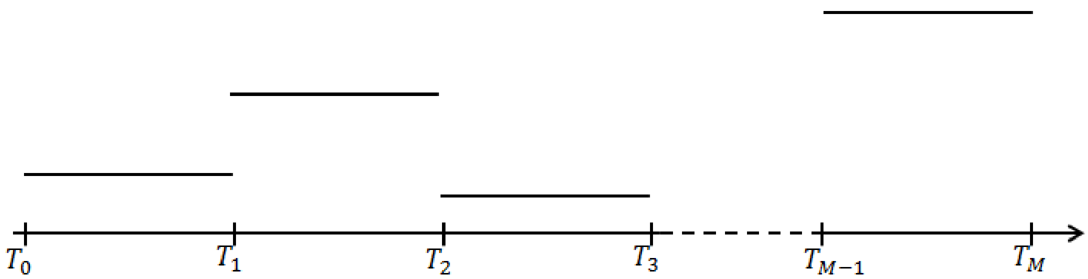


Figure 5.1: *Illustration of piecewise constant caplet volatilities between different cap maturities.*

Due to that ATM strikes are specific for each maturity, the ATM caps are not included in the above mentioned stripping algorithm. In order to incorporate ATM caps, one could build a smile model, such as SABR (or shifted SABR if needed), for each available caplet maturity. Using that model it is possible to obtain prices for caps with strikes that are not quoted on the market. The algorithm that is described by (Zhang and Wu, 2015), is as follows:

1. Strip the fixed-strike caps as described above.
2. Build a smile model (e.g. SABR) for each available caplet maturity based on the grid data obtained in step 1.
3. At each interval and for a given ATM strike K_{ATM}, retrieve a volatility corresponding to that strike. Meaning, construct

$$\mathbf{Cap_i^{mkt}}(K_i^{\text{ATM}}) - \mathbf{Cap_{i-1}^{\text{model}}}(K_i^{\text{ATM}}), \quad i = 1, ..., M,$$

   and back out the caplet volatility $\sigma(K_i^{\text{ATM}})$.

Once the caplet volatilities are obtained one can then use one of the methods described in 3.8 to calibrate SABR parameters for each caplet expiry.

5.2 Method 2: Rebonato

For a given set of caps with different maturities and a common fix strike, there are several different ways of trying to infer the underlying caplet volatility. One way of doing this is to use a functional form for the instantaneous volatility of the forward rate. For a time varying instantaneous volatility $\sigma(s)$ the price of a caplet in the Black or normal model is given by replacing terms involving the constant volatility σ by its time varying counterpart. This gives the following changes

$$\sigma^2(T-t) \to \Sigma^2(t,T) = \int_t^T \sigma(s)^2 ds, \tag{5.2}$$
$$\sigma\sqrt{T-t} \to \Sigma(t,T) = \sqrt{\int_t^T \sigma(s)^2 ds}.$$

An approach proposed by Rebonato (Rebonato, 2004, p. 671- 677) is to model the instantaneous volatility using a parametric continuous function with four parameters. In this case all forward rates in the underlying cap are assumed to have a time varying volatility given by

$$\sigma(s) = f(T-s) = \big[a + b(T-s)\big] \exp\big[-c(T-s)\big] + d. \tag{5.3}$$

Using this function one will get a time homogeneous volatility term structure, i.e. the instantaneous volatilities depend on the time to expiry T only through the function f. One of the advantages of this method is that it is very flexible and has few parameters. It is also very fast and easy to calibrate.

Rebonato (Rebonato, 2004, ch. 21.3) argues that one of the strengths of the method is that it is designed to fit the humped shaped volatility that is often seen in the market. In addition, the function has as few as four parameters and they are easy to interpret. To begin with, as the time to maturity decreases and goes to zeros, we have that

$$\lim_{s \to T} f(T-s) = a + d.$$

Thus, $a + d$ corresponds to the implied volatility of a caplet with a very short expiry vanishingly almost instantaneously. Moreover, as the time to maturity goes to infinity we have that

$$\lim_{T-s \to \infty} f(T-s) = d.$$

This makes it possible to interpret d as the implied volatility of a very long expiry caplet. The parameter b controls the hump in that for $b > 0$ we will have positive hump while $b < 0$ will give a hump in the other direction. Finally, c controls the speed of the fade to the long term instantaneous volatility.

We next illustrate an example of how the parameters (a, b, c, d) can be calibrated. In this thesis we will use the following approach (cf. Figure 5.2).

1. Assume a given set of market cap prices $\{\mathbf{Cap_i^{mkt}}\}_{i=1}^{M}$, with increasing maturity and a common strike.

2. Choose parameters (a, b, c, d) and use these to calculate $\Sigma(t, T)$ for each caplet corresponding to the cap with the largest maturity.

3. Use the calculated integrated caplet volatilities, to obtain a set of cap model prices $\{\mathbf{Cap_i^{model}}\}_{i=1}^{M}$.

4. Compute the cost function given by

$$\sum_{i=1}^{M} \omega_i (\mathbf{Cap_i^{mkt}} - \mathbf{Cap_i^{model}})^2,$$

   where ω_i is the weight associated to the i-th cap price difference. Two natural choices are to weight all price difference equally or to give caps with a shorter maturity T_i a larger weight. This would correspond to $\omega_i = 1$ or $\omega_i = \frac{1}{T_i}$, respectively.

5. Minimize the cost function in step 4 above by feeding step 2-4 into an optimizer. The desired parameters are then those that minimize the cost function.

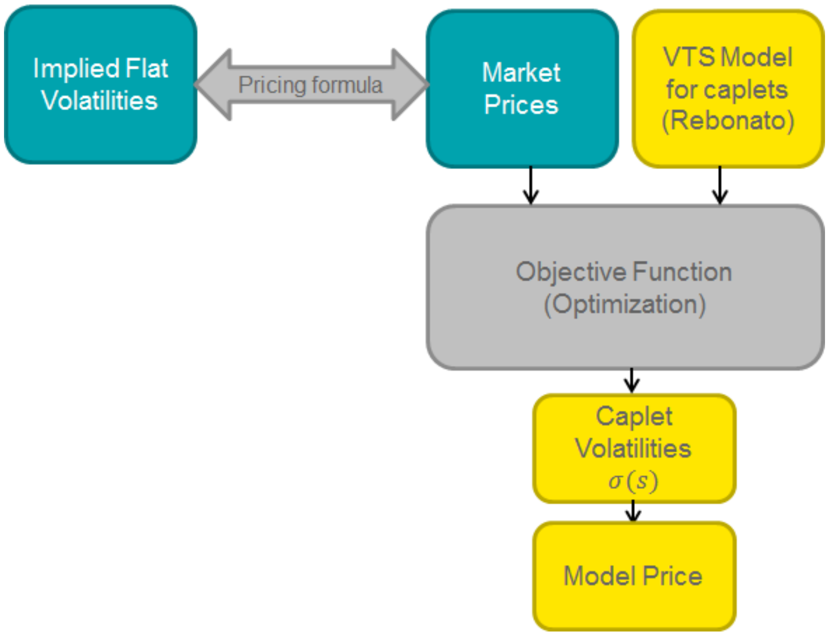


Figure 5.2: *Flow chart describing the process of modelling the caplet instantaneous volatility, by a VTS model, e.g. Rebonato. For a fixed strike, market prices are obtained by corresponding benchmark pricing formula. Then a VTS model is calibrated to the market prices, by minimizing the squared price difference between the prices from the model and the market. Finding the minimum, caplet volatilities $\sigma(s)$ are obtained, as well as model prices.*

When the integrated caplet volatilities have been inferred from a VTS model one can use these to obtain a smile model. A smile model is used to account for the fact that the caplets which have a **common** expiry T but **different** strikes K_j may have different integrated caplet volatilities $\Sigma^2(T, K_j)$. In this report we will use a SABR model to model the volatility smile. In Figure 5.3 the calibration process is presented graphically. In more detail it can be performed as follows.

1. Choose one of the fixed caplet expiries T and the corresponding set of integrated caplet volatilities $\{\Sigma^2(T, K_j)\}_{j=1}^{N}$.
2. Pick SABR-parameters (α, β, ρ, ν).
3. Construct the objective function

$$F(\alpha, \beta, \rho, \nu) = \sum_{j=1}^{N} \left(\frac{1}{\sqrt{T}} \Sigma(T, K_j) - \sigma_{SABR}(T; \alpha, \beta, \rho, \nu) \right)^2 .$$

4. Feed the objective function into an optimizer to obtain the parameters (α, β, ρ, ν) that minimizes F.
5. Repeat step 1-4 above for each available caplet expiry T to receive the term structure for each parameter, i.e, $\alpha = \alpha(T), \rho = \rho(T)$ etc.

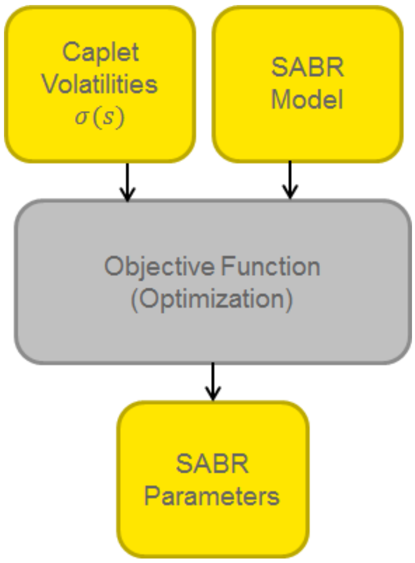


Figure 5.3: *This is a flow chart that illustrates the process of calibrate a smile model, such as SABR. The process is an extension of the one that are described in figure 5.2, meaning that it is dependent on that caplet volatilities are at hand. For a fixed caplet expiry T, the caplet volatilities implied by SABR* (3.9) *is optimized to the integrated caplet volatilities, $\{\Sigma^2(T, K_j)\}_{j=1}^{N}$, that was optimized by a VTS model (Rebonato or Bootstrap). The objective function that are minimized is the squared error between the SABR volatility and the "market" volatility, i.e.* $\min_{\alpha,\beta,\rho,\nu} \sum_{j=1}^{N} \left(\frac{1}{\sqrt{T}} \Sigma(T, K_j) - \sigma_{SABR}(T; \alpha, \beta, \rho, \nu) \right)^2$.

When performing the optimization above, one often run into troubles when trying to optimize both β and ρ at the same time. The reason for this is that both parameters essentially describes the same thing. To circumvent this we will in this thesis fit β as a global parameter. This means β will not depend on T. Instead, we construct a grid of betas $\{\beta_0, \beta_1, \ldots, \beta_B\} \in [0, 1]$ and choose the value of beta that minimizes the global cost function, i.e.,

$$\beta = \underset{\beta^* \in \{\beta_0, \beta_1, \ldots, \beta_B\}}{\arg\min} \sum_T F\big(\alpha(T), \beta^*, \rho(T), \nu(T)\big). \tag{5.4}$$

5.3 Method 3: Global SABR

In this section we discuss how one can fit a SABR model directly to observed cap prices. As the name suggests the method calibrates the volatility term structure and smile simultaneously in one single step. This should be compared to the previously mentioned methods which both can be viewed as a two step approach; first, a VTS is fitted for each strike and then second, these are joined using a smile model.

As already touched upon, the global SABR method is based on trying to find the parameter term structure $\phi(T) = \{\alpha(T), \beta(T), \rho(T), \nu(T)\}$ such that the squared error between the obtained model prices and market prices are as small as possible. The main question is then how to find the optimal functions $\alpha(T), \beta(T), \rho(T)$, and $\nu(T)$. Obviously, one approach would be to define one set of parameters $\{\alpha_i, \beta_i, \rho_i, \nu_i\}$ for each caplet expiry T_i^e on the market and then use linear interpolation. However, this is a very naive approach as it will quickly deteriorate as the number of caplet maturities increase. For instance in Figure 4.1 we have 39 caplet maturities. This will result in trying to optimize $4 \cdot 39 = 156$ variables, which clearly is an almost impossible task.

Our method for determining $\phi(T)$ is instead to use an interpolated parameter term structure. Here we make β a single global parameter and specify a couple of knots for each of the remaining parameters. Then we interpolate between knots using either cubic splines, double quadratic interpolation or pchip. The term pchip stands for piecewise cubic Hermite interpolating polynomial. Information about the double quadratic interpolation used in this thesis can be found in Appendix E.

Using this technique, one only has to determine the optimal knot values for each parameter. The advantage of this method is that we will get a smooth parameter term structure and we get a feasible number of variables to optimize. The drawback is that some manually tuning of the number of knots and knot positions are required.

The calibration procedure is illustrated in Figure 5.4. Below follows a more detailed step by step description.

1. Specify knot positions for each parameter term structure.

2. Specify knot values for each parameter term structure.

3. Interpolate using suitable interpolation method (e.g. cubic splines) between knot values to get a parameter term structure $\phi(T) = \{\alpha(T), \beta(T), \rho(T), \nu(T)\}$. If β is chosen as a global parameter (recommended due to similarities with ρ) put $\beta(T) = \beta$.

4. For each strike K, calculate the set of caplet prices given the current SABR parameters.

5. For each strike K, construct cap prices as sum of caplet prices.

6. Construct the objective/cost function by summarizing the squared difference between model and market cap prices for all cap maturities and strikes. The cost function is given by

$$F\big(\phi(T)\big) = \sum_{T}\sum_{K}\left(\mathbf{Cap}^{\mathbf{mkt}}(T,K) - \mathbf{Cap}^{\mathbf{SABR}}(T,K)\right)^2.$$

7. Find the optimal knot values by minimizing the objective/cost function in step 6 above. This will, via the interpolation, yield the optimal parameter term structure $\phi(T)$.

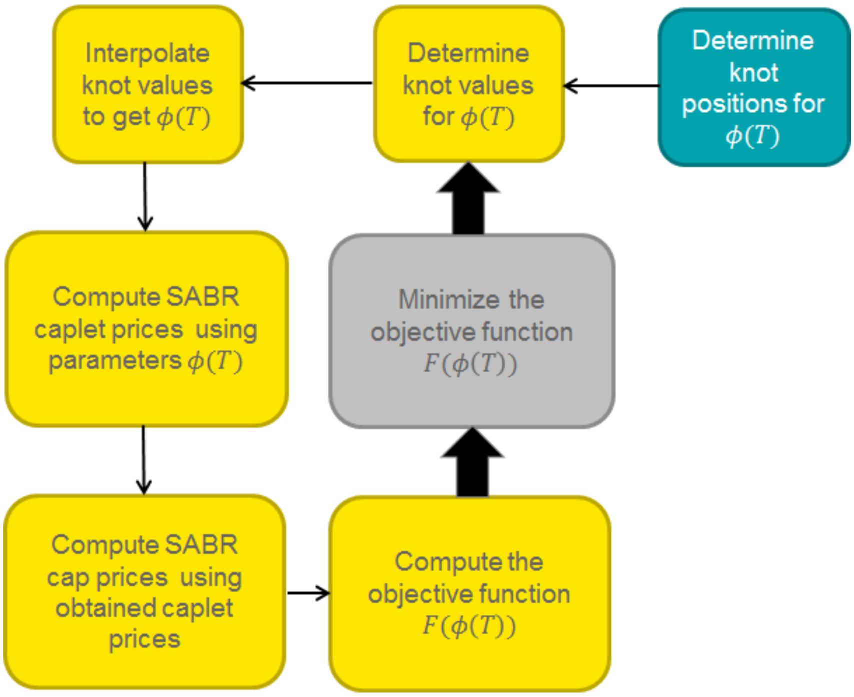


Figure 5.4: *This flow chart illustrates the process of how to perform a global SABR calibration. First knot positions for the parameter term structure* $\phi(T) = \{\alpha(T), \beta(T), \rho(T), \nu(T)\}$ *are chosen. Then the optimization cycle begins as follows. For given knot values and interpolation technique* $\phi(T)$ *is created and used to compute implied caplet volatilities by SABR model. These volatilities are then used to price the underlying caplets. Model cap prices are calculated by summing up the correct number of caplet prices for each cap. Finally, the objective function* $F\big(\phi(T)\big)$ *is formed as the sum of the squared differences between model and market cap prices. Once optimal knot values, minimizing* $F\big(\phi(T)\big)$*, are found the process ceases and our SABR model is calibrated.*

Chapter 6

Multi Curve Setting

In the following chapter we will go through the background and theory of the *multi curve setting*. Since the credit crunch in 2007 this has been a topic for much research. The outline below follows much of the work done by Mercurio (2009) and Bianchetti and Carlicchi (2011), and we recommend the reader to look into these articles for a more in depth discussion of the topic.

6.1 Background to the Multi Curve Framework

Prior to the credit crunch in 2007 most financial institution had a solid understanding of how to deal with interest rate derivatives in terms of valuation. The theory was based on a single yield curve that could be used both for discounting and construction of forward rates with different tenors. However, the credit crises invoked a strong reason for market participants to reconsider their current framework. This led to the development of a multi curve framework.

The multi curve framework has since then become the accepted paradigm and is used in modern pricing of interest rate derivatives. The main reason for the new framework is that the credit crises led to an increased spread between rates that used to be mutually consistent. As an example we can take a look at some empirical evidence in Figure 6.1. Prior to 2007 the basis spread between 3 month EURIBOR and 3 month Eonia was negligible. However, as we can observe in Figure 6.1, things really started to change in the third quarter of 2007. Suddenly, the spread between the rates were becoming significant and it reached a maximum of 206.4 basis point in the beginning of the fourth quarter of 2008.

The reason for the spread is that the market started to recognize the increased credit risk prevalent in the EURIBOR. This resulted in a significant positive spread over the Eonia. As previously mentioned in Chapter 4, overnight transactions are regarded as nearly risk free. This has resulted in that market are using the OIS rate as the best proxy for the risk free rate.

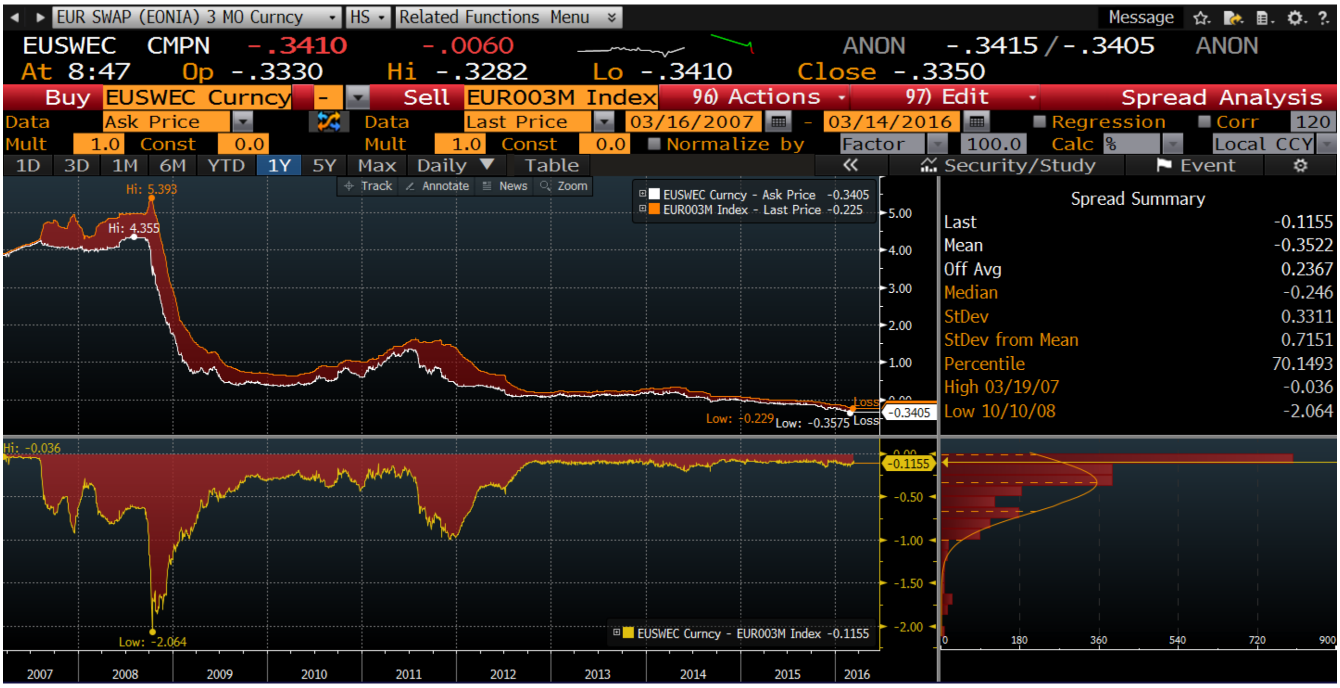


Figure 6.1: *Historical comparison of EURIBOR3M (yellow) and OIS3M (white). Prior to the third quarter in 2007 the rates were closely connected with a negligible spread. The credit crises starting in August 2007 brought and end to this and since 2007 there has been a significant spread recognizing the difference in liquidity and credit risk between the rates. The figure is obtained from a Bloomberg platform.*

In addition, it became apparent that lending to another bank involved various amount of risk depending on the underlying tenor of the rate. For instance, lenders were no longer indifferent of receiving a 6 month EURIBOR semiannually or the 3 month EURIBOR rolled over quarterly. Consequently, rates with different tenors that used to be closely related to each other became different subjects, due to differences in credit and liquidity risk.

The differences in liquidity and credit risk between tenors have given rise to a market segmentation of forward rates. There no longer exists a unique yield curve that is indifferent to all tenors, as in the single curve setting. Instead, the multi curve framework has a distinct yield curve for each market tenor and this curve is used to construct the forwarding curve for that specific tenor.

To summarize, one can say that the multi curve framework consists of two important parts

1. A unique curve for discounting of future cash flows. This curve is often chosen to be the OIS curve due to its risk free nature.
2. Distinct yield curves corresponding to different tenors in order to construct forward rates on different tenors.

Thus, to account for the multi curve structure above it is crucial to build one curve for the discounting of future cash flows and build distinct curves corresponding to the different tenors used for computing the forward rates. Hereafter, the discounting curve and the forwarding (also called fixing or funding) curves will be denoted C_d and C_m respectively, where $m = \{m_1, \ldots, m_k\}$ denotes the different relevant tenors.

6.2 Revisiting the FRA Rate

Recall from section 3.3 that the FRA rate is the rate which renders the FRA a fair contract. Let T_1 and T_2 define a future period $[T_1, T_2]$ and let t be the present date. The FRA rate, $FRA(t; T_1, T_2)$, could then be viewed as the fixed rate to be exchanged at T_2 for the LIBOR rate, $L(T_1, T_2)$, so that the present value of the corresponding FRA is zero.

Let $\Pi_{FRA}(t)$ be today's price of the FRA. Under the $\mathbb{Q}_d^{T_2}$ measure (whose numeraire is $p_d(t, T_2)$ the discount factor from t to T_2 related to the discount curve) the process

$$\frac{\Pi_{FRA}(t)}{p_d(t, T_2)},$$

is a martingale. The martingale property can be expressed by

$$\frac{\Pi_{FRA}(t)}{p_d(t, T_2)} = \mathbb{E}_t^{\mathbb{Q}_d^{T_2}}\left[\frac{\Phi_{FRA}(T_2)}{p_d(T_2, T_2)}\right],$$

where $\Phi_{FRA}(T_2)$ is the payoff of the FRA at T_2, i.e

$$\Phi_{FRA}(T_2) = N\omega\big[L(T_1, T_2) - K\big],$$

$p_d(T_2, T_2) = 1$ and $\omega = \pm 1$ depending on the counterparty side. Putting the fixed rate, $K = FRA(t; T_1, T_2)$ yields $\Pi_{FRA}(t) = 0$, that is

$$N\omega\mathbb{E}_t^{\mathbb{Q}_d^{T_2}}\big[L(T_1, T_2) - FRA(t; T_1, T_2)\big] = 0,$$

or equivalently

$$\mathbb{E}_t^{\mathbb{Q}_d^{T_2}}\big[L(T_1, T_2)\big] - \mathbb{E}_t^{\mathbb{Q}_d^{T_2}}\big[FRA(t; T_1, T_2)\big] = 0.$$

$FRA(t; T_1, T_2)$ is known at time t and thus $\mathcal{F}_t$-measurable so,

$$FRA(t; T_1, T_2) = \mathbb{E}_t^{\mathbb{Q}_d^{T_2}}\big[L(T_1, T_2)\big].$$

The LIBOR rate $L(T_1, T_2)$ is defined by

$$L(T_1, T_2) = \frac{1}{T_2 - T_1}\left[\frac{1}{p_d(T_1, T_2)} - 1\right]. \tag{6.1}$$

In the single curve frame work where the same curve is used for both discounting and funding, the forward rate $F_d(t;T_1,T_2) = \frac{1}{T_2 - T_1}\left[\frac{p_d(t,T_1)}{p_d(t,T_2)} - 1\right]$ corresponds to the FRA rate $FRA(t;T_1,T_2)$ on the time period $[T_1,T_2]$, because:

$$FRA(t;T_1,T_2) = \mathbb{E}_t^{\mathbb{Q}_d^{T_2}}\big[L(T_1,T_2)\big] = \mathbb{E}_t^{\mathbb{Q}_d^{T_2}}\big[F_d(T_1;T_1,T_2)\big] = F_d(t;T_1,T_2),$$

since, the forward rate $F_d(t;T_1,T_2)$ is a martingale under $\mathbb{Q}_d^{T_2}$.

In the multi curve framework $L(T_1,T_2) \neq F_d(T_1;T_1,T_2)$, since the discount factors in (6.1) are not the ones corresponding to the discount curve. Instead $L(T_1,T_2)$ is defined as

$$L(T_1,T_2) = \frac{1}{T_2 - T_1}\left[\frac{1}{p_m(T_1,T_2)} - 1\right], \tag{6.2}$$

where $p_m(T_1,T_2)$ is the discount factor on the LIBOR forward curve with tenor m. As such the discount factors to be used will now depend on the underlying tenor and consequently, they will be calculated from the corresponding forward curve and not from the discount curve.

The new multi curve framework will invalidate the equality

$$FRA(t;T_1,T_2) = F_d(t;T_1,T_2),$$

which used to hold in the single curve framework. Instead a spread appears between the two quantities. Moreover, by the law of iterated expectations we have

$$\mathbb{E}_t^{\mathbb{Q}_d^{T_2}}\big[FRA(T_1;T_1,T_2)\big] = \mathbb{E}_t^{\mathbb{Q}_d^{T_2}}\left[\mathbb{E}^{\mathbb{Q}_d^{T_1}}\big\{L(T_1,T_2)|\mathcal{F}_{T_1}\big\}\right],$$

and since $L(T_1,T_2)$ is known at time T_1

$$\mathbb{E}_t^{\mathbb{Q}_d^{T_2}}\big[FRA(T_1;T_1,T_2)\big] = \mathbb{E}_t^{\mathbb{Q}_d^{T_2}}\big[L(T_1,T_2)\big] = FRA(t;T_1,T_2). \tag{6.3}$$

This means that under $\mathbb{Q}_d^{T_2}$, $FRA(t;T_1,T_2)$ is a martingale. This will be an important property when pricing interest rate derivatives in the multi curve framework.

6.3 Pricing in the Multi Curve Framework

In the previous section we argued that the FRA rate is a martingale under the forward measure. Consider a derivative with a simple payoff $\Phi(T)$ at time T. Under the forward measure $\mathbb{Q}_d^T$ the arbitrage-free price today at time t is given by

$$\Pi(t) = p_d(t,T)\mathbb{E}_t^{\mathbb{Q}_d^T}\big[\Phi(T)\big], \tag{6.4}$$

where $p_d(t,T)$ is the discount factor obtained from the discounting curve C_d. Assuming a stochastic short rate $r(t)$ we have

$$p_d(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}\right].$$

In general the payoff $\Phi(T)$ is a function of the LIBOR. The expectation of the LIBOR in (6.4) will then lead to the appearance of the FRA rate in the pricing formula. Hence, the FRA rate is one of the fundamental quantities of the multi curve framework.

Remark 6.1. *We consider the risk free rate to be the OIS rate. As such the discounting curve C_d will be based on the OIS rate and*

$$p_d(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_{OIS}(s)ds}\right].$$

6.3.1 Pricing Algorithm

Below we summarize how to price interest derivatives in the multi curve framework. The algorithm can be summarized into six steps.

1. Construct the discounting curve C_d by choosing the most liquid vanilla interest rate instruments with different tenors. Then use a bootstrap procedure to compute the discount factors $p_d(t, T)$.

2. Construct distinct forward curves $C_{m_1}, \ldots, C_{m_k}$ using the proper selections of distinct set of vanilla interest rate market instruments, each belonging to a specific tenor category and bootstrapping procedure.

3. For each interest rate coupon $i = \{1, \ldots, n\}$, compute the relevant FRA rates with tenor m applying the following formula

$$FRA_m(t; T_{i-1}, T_i) = \frac{1}{\tau_m(T_{i-1}, T_i)} \left[\frac{p_m(t, T_{i-1})}{p_m(t, T_i)} - 1 \right],$$

   where p_m's are the discount factors from the forward curve of tenor m and $\tau_m(T_{i-1}, T_i)$ is the day count year fraction between T_{i-1} and T_i.

4. Compute the cash flows c_i as the expectation of the corresponding coupon payoff Φ_i with respect to the discounting T_i-forward measure $\mathbb{Q}_d^{T_i}$ corresponding to the numeraire $p_d(t, T_i)$, taken from the discount curve C_d:

$$c_i(t, T_i) = \mathbb{E}_t^{\mathbb{Q}^{T_i}} \left[\Phi_i \right].$$

5. Compute relevant discount factors $p_d(t, T_i)$ from the discounting yield curve C_d.

6. Compute the price, $\Pi(t)$, of the derivative at time t as the sum of the discounted cash flows:

$$\Pi(t) = \sum_{i=1}^{n} p_d(t, T_i) c_i(t, T_i).$$

6.3.2 Pricing of Caplets

Recall the pricing formula for a caplet in the classical Black-76 formula (3.3). In that expression one is using that the forward rate L_k is a martingale under the forward measure $\mathbb{Q}^{T_k}$. However, in the multi curve setting, the martingale property is not fully obtained, and so the pricing formula requires some extra attention. In fact, since the the pricing measure is now the forward measure $\mathbb{Q}_d^{T_k}$ for the curve C_d, the caplet price at time t becomes

$$\mathbf{Capl}_k(t) = \tau_k p_d(t, T_k)\mathbb{E}^{\mathbb{Q}_d^{T_k}}\big[\max(L_k(T_{k-1}) - K, 0)|\mathcal{F}_t\big]. \tag{6.5}$$

Moreover, the relation between the FRA and LIBOR is explained in equation (6.3). And at the reset time T_{k-1}, the two rates are equal $L_k(T_{k-1}) = FRA(T_{k-1})$. Replacing L_k by FRA in (6.5) will render a new pricing formula

$$\mathbf{Capl}_k(t) = \tau_k p_d(t, T_k)\mathbb{E}^{\mathbb{Q}_d^{T_k}}\big[\max(FRA(T_{k-1}) - K, 0)|\mathcal{F}_t\big]. \tag{6.6}$$

The FRA rate is by definition a martingale under the measure $\mathbb{Q}_d^{T_k}$. And so the pricing formula derived in (3.3) will be simular to the one derived for the multi curve setting. The only difference is that the underlying is the FRA, and the discount factor is derived from curve C_d. (Mercurio, 2009)

6.4 The Modern SABR Model

In order to get a modern version of the SABR model applicable to the multi curve framework one has to adjust the classical version of Hagan et al. in (3.8) in two ways. First, the classical forward rate has to be replaced by the modern FRA-rate. Second, the $\mathbb{Q}^{T_k}$-forward LIBOR measure with the single curve numeraire $p(t, T_k)$ is replaced by the modern $\mathbb{Q}_d^{T_k}$-forward LIBOR measure associated with the discounting numeraire $p_d(t, T_k)$. Conceptually, this means that the SABR volatility formula in (3.9) now take the FRA-rate as input, but remains otherwise unaltered.

Let $F_{x,k}(t)$ be the FRA-rate time t on $[T_{k-1}, T_k]$, corresponding to tenor m_x with $x = 1, \ldots, k$. Then, the SABR dynamics are

$$\begin{aligned} dF_{x,k}(t) &= V_t(F_{x,k}(t))^{\beta} dZ_t, \quad F_{x,k}(0) = f, \\ dV_t &= \nu V_t dW_t, \quad V_0 = \alpha, \end{aligned} \tag{6.7}$$

where Z_t and W_t are $\mathbb{Q}_d^{T_k}$ standard Brownian motions with

$$dZ_t dW_t = \rho dt,$$

and $\beta \in (0, 1]$, ν and α are positive constants and $\rho \in [-1, 1]$.

As the SABR volatility formula in 3.9 remains unchanged the procedure for fitting the SABR model is still applicable. Hence, the SABR calibration is applied to each volatility smile based on the stripped caplet volatilities of caplets with the same expiry date T_{k-1}, underlying FRA rate $F_{x,k}(t)$ and different strikes. As such the main idea is still to obtain SABR parameters α, β, ρ, ν minimizing the sum of the squared distance between market and implied SABR volatilities. The result is a set of SABR parameters describing the smile structure for each distinct caplet expiry T_{k-1}.

Chapter 7

Volatility Transformation

In this chapter we consider the problem of deriving cap volatilities of non-standard tenors given quotes for standard tenors. As previously touched upon, we have a tenor issue in the EUR market. Broker's (e.g. ICAP) apply a standard where caps are quoted against 3 month forward rates up to 2 years, and against 6 month forward rates on longer cap maturities. To solve this issue, we would like to find a way of obtaining volatilities for caps based on a long tenor, for instance 6 months, given market quotes for caps with a short tenor, for instance 3 month. For completeness we also consider the problem of transforming long tenor volatilities to short ones.

The proposed methodology is inspired by the work of Kienitz (2013). However, we extend the results so that they are applicable when rates are negative. We first treat volatility transformation in a single curve setting and then use these results to derive an algorithm for transformation in a multi curve setting. In addition, we look into the problem of transforming the market observed smile for non-standard tenors by applying the shifted SABR model. To summarize, our contribution is as follows.

1. Detailed derivation of the algorithm for transforming cap volatilities for m month forward rates to volatilities on n month forward rates where $n = km$ and $k \in \mathbb{N}$. The results are based on a single curve setting and can be regarded as a review of the work by Kienitz (2013).

2. Extension of the above algorithm, making it applicable for negative rates as well. To this end we apply a deterministic shift and Itô's formula.

3. Extending the algorithm from 2, to a multi curve setting taking the money market basis spread into account. To achieve this we use displaced diffusions.

4. Algorithm for transforming cap volatilities from long tenor to short tenor in both a single curve and multi curve setting.

5. Transformation of the smile/skew for cap volatilities to rates calculated on non-standard forward rates using algorithms 1-4, and the shifted SABR model.

7.1 Transforming Cap Volatilities in a Single Curve Setting

In this section we consider the transformation of cap volatilities to non-standard tenors in a single curve setting. To achieve this, one needs to have access to caplet volatilities. These can be obtained by bootstrapping as described in section 5.1. Once caplet volatilities are transformed, the caps can be priced. As such, one can easily obtain cap volatilities by inverting Black's formula. The single curve framework means that we assume that there is one single curve for discounting and one curve for forwarding. The presentation below is based on the theory developed by Kienitz (2013). However, we extend the methods so that they are applicable in a negative rate environment. This extension is important when we discuss the multi curve setting since we apply this method to the OIS curve.

7.1.1 From short to long periods

We consider the case where we have caplet volatilities for m month tenor and want to find volatilities for n month tenor, where $n > m$. As an example we can take the EUR market. For cap maturities up to 2 year we have 3 month caplet volatilities. In order to get a joint tenor grid, we want to find corresponding 6 month caplet volatilities.

The transformation technique that will be used is based on an arbitrage relation between simple forward rates and Itô's lemma. Denote the time today by t and let $F(t;T_i,T_j) = F_{i,j}$ be a forward rate, e.g. the LIBOR forward rate, on the interval $[T_i,T_j]$. Further, let $\tau_{i,j} = T_j - T_i$. In Figure 7.1 we show the situation where we have three different rates $F_{1,2} = F(t;T_1,T_2)$, $F_{2,3} = F(t;T_2,T_3)$ and $F_{1,3} = F(t;T_1,T_3)$.

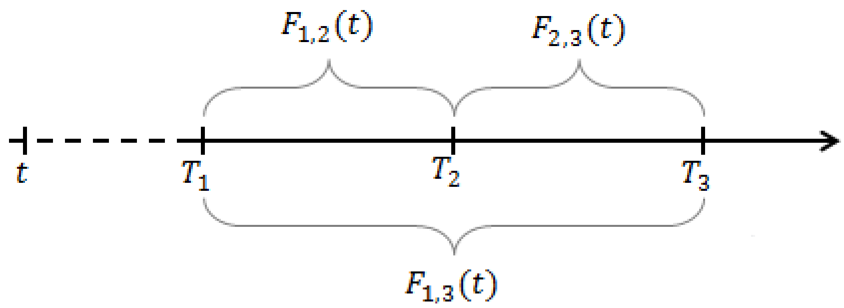


Figure 7.1: *The simple forward rates on their respective intervals.*

Assuming no arbitrage the following relation must hold

$$(1 + \tau_{1,2}F_{1,2})(1 + \tau_{2,3}F_{2,3}) = 1 + \tau_{1,3}F_{1,3}. \tag{7.1}$$

This can be rewritten in the following way

$$F_{1,3} = \frac{\tau_{1,2}}{\tau_{1,3}} F_{1,2}(1 + \tau_{1,3} F_{2,3}) + \frac{\tau_{2,3}}{\tau_{1,3}} F_{2,3}. \tag{7.2}$$

If we let $\sigma_{i,j}$ denote the caplet volatility connected to $F_{i,j}$ we have that under some arbitrary but unspecified measure the rates will have the following dynamics

$$\begin{aligned} dF_{1,2} &= \mu_1 dt + \sigma_{1,2} F_{1,2} dW_1, \\ dF_{2,3} &= \mu_2 dt + \sigma_{2,3} F_{2,3} dW_2, \\ dF_{1,3} &= \mu_3 dt + \sigma_{1,3} F_{1,3} dW_3. \end{aligned} \tag{7.3}$$

Where the quadratic covariation of the dW_1 and dW_2 is given by

$$d \langle W_1, W_2 \rangle_t = \rho dt.$$

We know that since $F_{1,3}$ will be a martingale under $\mathbb{Q}^{T_3}$, it will not have any drift term, i.e. the dynamics are

$$dF_{1,3} = \sigma_{1,3} F_{1,3} dW_3.$$

However, it is also possible to obtain the dynamics by applying Itô's formula on equation (7.2). Hence, we will derive a second expression for the dynamics, and combine the two in order to solve for the unknown volatility, $\sigma_{1,3}$. The calculations of deriving the dynamics are based on Itô's formula (2.2.1). The steps are straightforward but tedious, so we choose not to present them here. The result is

$$dF_{1,3} = \frac{\sigma_{1,2}}{\tau_{1,3}} \left(\tau_{1,2} F_{1,2} + \tau_{1,2} \tau_{2,3} F_{1,2} F_{2,3} \right) dW_1 + \frac{\sigma_{2,3}}{\tau_{1,3}} \left(\tau_{2,3} F_{2,3} + \tau_{1,2} \tau_{2,3} F_{1,2} F_{2,3} \right) dW_2.$$

Now, as we have two different expressions for $dF_{1,3}$ under $\mathbb{Q}^{T_3}$ we can take the quadratic variation of both sides in the following expression

$$\sigma_{1,3} F_{2,3} dW_3 = \frac{\sigma_{1,2}}{\tau_{1,3}} \left(\tau_{1,2} F_{1,2} + \tau_{1,2} \tau_{2,3} F_{1,2} F_{2,3} \right) dW_1 + \frac{\sigma_{2,3}}{\tau_{1,3}} \left(\tau_{2,3} F_{2,3} + \tau_{1,2} \tau_{2,3} F_{1,2} F_{2,3} \right) dW_2.$$

Doing this we obtain

$$
\begin{aligned}
d\left\langle F_{1,3}, F_{1,3}\right\rangle_t &= \sigma_{1,3}^2 F_{1,3}^2 dt \\
&= \Bigg[\left(\frac{\sigma_{1,2}}{\tau_{1,3}}\right)^2 (\tau_{1,2}F_{1,2} + \tau_{1,2}\tau_{2,3}F_{1,2}F_{2,3})^2 \\
&+ \left(\frac{\sigma_{2,3}}{\tau_{1,3}}\right)^2 (\tau_{2,3}F_{2,3} + \tau_{1,2}\tau_{2,3}F_{1,2}F_{2,3})^2 \\
&+ 2\rho\left(\frac{\sigma_{1,2}\sigma_{2,3}}{\tau_{1,3}^2}\right)(\tau_{1,2}F_{1,2} + \tau_{1,2}\tau_{2,3}F_{1,2}F_{2,3}) \\
&\cdot (\tau_{2,3}F_{2,3} + \tau_{1,2}\tau_{2,3}F_{1,2}F_{2,3}) \Bigg] dt.
\end{aligned}
$$

Solving for $\sigma_{1,3}$ we obtain

$$\sigma_{1,3}^2 = V_1(t)^2\sigma_{1,2}^2 + V_2(t)^2\sigma_{2,3}^2 + 2\rho V_1(t)V_2(t)\sigma_{1,2}\sigma_{2,3},$$

where

$$V_j(t) = \frac{\tau_{j,j+1}F_{j,j+1}(t) + \tau_{1,2}\tau_{2,3}F_{1,2}(t)F_{2,3}(t)}{\tau_{1,3}F_{1,3}(t)}, \quad j = 1, 2.$$

Making an approximation by freezing the time argument, proposed by for instance Brigo and Mercurio (2007), we obtain the following

$$\sigma_{1,3}^2 \approx V_1(0)^2\sigma_{1,2}^2 + V_2(0)^2\sigma_{2,3}^2 + 2\rho V_1(0)V_2(0)\sigma_{1,2}\sigma_{2,3}. \tag{7.4}$$

For the general setting, where the n month period is a decomposition of k rates $F_{j,j+1}$, $j = 1, \ldots, k$ of some period with volatilities $\sigma_{j,j+1}$ and correlation matrix $(\rho_{i,j})$, we find using Itô's formula that

$$\sigma_{1,k+1}^2 = \sum_{j=1}^{k} \sigma_{j,j+1}^2 V_j^2 + \sum_{i \neq j} \rho_{i,j}\sigma_{j,j+1}\sigma_{i,i+1}V_jV_i, \tag{7.5}$$

where

$$V_j = \frac{\tau_{j,j+1}F_{j,j+1} + \sum \tau_{j,j+1}F_{j,j+1} * \{\tau_{i,i+1}F_{i,i+1}\}_{i \neq j}}{\tau_{1,k+1}F_{1,k+1}}.$$

Here $\sum \tau_{j,j+1}F_{j,j+1} * \{\tau_{i,i+1}F_{i,i+1}\}_{i \neq j}$ should be interpreted as the sum of all possible products between the factor $\tau_{j,j+1}F_{j,j+1}$ and the factors in the set $\{\tau_{i,i+1}F_{i,i+1}\}_{i \neq j}$ Thus, for instance in the case when we have three periods ($k = 3$) and $j = 1$ we obtain

$$V_1 = \frac{\tau_{1,2}F_{1,2} + \tau_{1,2}F_{1,2}(\tau_{2,3}F_{2,3} + \tau_{3,4}F_{3,4} + \tau_{2,3}F_{2,3}\tau_{3,4}F_{3,4})}{\tau_{1,4}F_{1,4}}.$$

Remark 7.1.

One can also assume a time varying volatility as for example in LMM. The general approximation formula then becomes

$$\nu_{1,3}^2 \approx V_1(0)^2\nu_{1,2}^2 + V_2(0)^2\nu_{2,3}^2 + 2\rho V_1(0)V_2(0)\nu_{1,2}\nu_{2,3},$$

where

$$\nu_{j,j+1}^2 = \frac{1}{\tau_{j,j+1}}\int_0^{T_j} \sigma_{j,j+1}^2(t)dt.$$

In practice, we only work with constant volatilities since this is the kind of volatilities that brokers quote.

7.1.2 From long to short periods

Market practitioners also face the problem of going from long to short tenors. For instance in the EUR market we might want to obtain a cap volatility surface on a 3 month tenor. Since caps with maturity larger than 2 years are based on 6 month forward, we need to transform 6 month volatilities to 3 month volatilities. This gives rise to an ill determined system as two or more volatilities have to be inferred from one single volatility. To stay in the 6 month and 3 month example we have to find two 3 month cap volatilities based on each available 6 month volatility. The situation gets even worse if we want to obtain 1 month volatilities based on 6 month volatilities. Then we face an ill determined system in 6 dimensions instead of 2, as in the 6 month to 3 month setting.

Once more we used the expression derived in equation (7.4). Since, the terms V_j do not depend on the volatility, the 6 to 3 month problem essentially consists of finding solutions (x, y) to the equation

$$c \approx ax^2 + bx^2 + 2\rho abxy, \quad a, b, c \text{ given.}$$

The solutions to this equation lies on a curve, hence they are infinitely many. The simplest approach of solving would be assuming equal 3 month caplet volatilities, i.e. assuming $x = y$. Below we present a more sophisticated method based on the suggestions of Kienitz (2013).

We take the last available volatility as an anchor point and calculate the remaining $k-1$ volatilities in equation 7.5 using a parametric function. For example in the EUR 6 month to 3 month case we can take the last available 3 month volatility as an anchor point. An example of a parametric function is the parsimonious Rebonato like function given by

$$\sigma(t) = \gamma_1\Big\{\big[(T_i^e - t)\gamma_2 + \gamma_3\big]\exp\big[-(T_i^e - t)\gamma_4\big] + \gamma_5\Big\}.$$

For a given set of real parameters $\{\gamma_i\}_{i=1}^{5}$ this will together with (7.5) give us a set of n month volatilities $\bar{\sigma}^i_{nM}$. The optimal parameters are then found by minimizing the sum of squared differences between model and given volatilities σ^i_{nM}. That is, we minimize

$$F(\gamma_1, \dots, \gamma_5) = \sum_i (\bar{\sigma}^i_{nM} - \sigma^i_{nM})^2.$$

Remark 7.2. *So far we have disregarded how the correlations $\rho_{i,j}$ could be obtained. On one hand, we can estimate correlations from historical data. On the other hand, we can also choose the correlations as free parameters. The latter alternative gives us freedom of controlling the results. As indicated by Brigo and Mercurio (Brigo and Mercurio, 2007, p. 304) one would expect the correlation between consecutive froward rates to be very close to one.*

7.1.3 Extension to handle negative rates

When the forward rates, $F_{j,j+1}$, are negative the log-normality assumption in equation (7.3) is no longer valid. To allow for negative rates we consider shifting the rates. To this end we introduce a deterministic shift s, so that the shifted rates $X_{j,j+1} = F_{j,j+1} + s$, follow a log-normal process. Considering the example of going from short to long periods when $k = 2$, we get the following dynamics,

$$\begin{aligned} dX_{1,2} &= \mu_1 dt + \sigma_{1,2} X_{1,2} dW_1, \\ dX_{2,3} &= \mu_2 dt + \sigma_{2,3} X_{2,3} dW_2, \\ dX_{1,3} &= \mu_3 dt + \sigma_{1,3} X_{1,3} dW_3. \end{aligned} \tag{7.6}$$

The advantage of this method is that the volatilities can be viewed as shifted Black volatilities. The structure of equation (7.6) is exactly the same as in equation (7.3). Thus, one might be deluded to believe that the transformation in (7.4) is applicable on $X_{j,j+1}$. However, this is not the case, since the introduction of a shift invalidates the arbitrage relation between forward rates in (7.1).

The aim of this section is to derive a corresponding arbitrage relation as in (7.1). Once an expression is obtained we use it together with Itô's formula to derive a transformation formula for shifted Black volatilities. Returning to the "6 month to 3 month" example, we start by considering a two period case ($k = 2$) with $\tau_{1,2}$ and $\tau_{2,3}$ representing two year fraction intervals between two subsequent caplets, and $\tau_{1,3} = \tau_{1,2} + \tau_{2,3}$. Recall the arbitrage relation (7.1),

$$1 + 2\tau_{1,3} F_{1,3} = (1 + \tau_{1,2} F_{1,2})(1 + \tau_{2,3} F_{2,3}).$$

Now, replacing $F_{j,j+1}$ by $X_{j,j+1} = F_{j,j+1} + s$, $j = 1, 2$, we get

$$1 + \tau_{1,3}(X_{1,3} - s) = \big[1 + \tau_{1,2}(X_{1,2} - s)\big]\big[1 + \tau_{2,3}(X_{2,3} - s)\big],$$

$$\begin{aligned}
\Rightarrow X_{1,3} &= \frac{\big[1 + \tau_{1,2}(X_{1,2} - s)\big]\big[1 + \tau_{2,3}(X_{2,3} - s)\big] + s\tau_{1,3} - 1}{\tau_{1,3}} \\
&= \frac{\tau_{1,2}X_{1,2} + \tau_{2,3}X_{2,3} + \tau_{1,2}\tau_{2,3}X_{1,2}X_{2,3}}{\tau_{1,3}} \\
&\quad + s \cdot \frac{-\tau_{2,3} - \tau_{1,2} - \tau_{1,2}\tau_{2,3}(X_{1,2} + X_{2,3} - s)}{\tau_{1,3}} + s.
\end{aligned}$$

Applying Itô's formula in the same way as before and freezing the time argument we end up with the following transformation formula

$$\sigma_{1,3}^2 \approx U_1^2(0)\sigma_{1,2}^2 + U_2^2(0)\sigma_{2,3}^2 + 2U_1(0)U_2(0)\sigma_{1,2}\sigma_{2,3}\rho, \tag{7.7}$$

where

$$\begin{aligned}
U_1 &= \frac{\tau_{1,2}X_{1,2} + \tau_{1,2}\tau_{2,3}X_{1,2}(X_{2,3} - s)}{X_{1,3}\tau_{1,3}}, \\
U_2 &= \frac{\tau_{2,3}X_{2,3} + \tau_{1,2}\tau_{2,3}X_{2,3}(X_{1,2} - s)}{X_{1,3}\tau_{1,3}}.
\end{aligned}$$

In the general setting, where the n month period is a decomposition of k rates $F_{j,j+1}$, volatilities $\sigma_{j,j+1}$ and a correlation matrix $(\rho_{i,j})$, $j = 1, ..., k$, we find using Itô's formula that

$$\sigma_{1,k+1}^2 = \sum_{j=1}^{k} \sigma_{j,j+1}^2 U_j^2 + \sum_{i \neq j} \rho_{i,j}\sigma_{j,j+1}\sigma_{i,i+1}U_jU_i,$$

where

$$U_j = \frac{\tau_{j,j+1}X_{j,j+1} + \sum \tau_{j,j+1}X_{j,j+1} * \{\tau_{i,i+1}(X_{i,i+1} - s)\}_{i \neq j}}{\tau_{1,k+1}X_{1,k+1}}.$$

Here $\sum \tau_{j,j+1}X_{j,j+1} * \{\tau_{i,i+1}(X_{i,i+1} - s)\}_{i \neq j}$ is interpreted as the sum of all possible products between the factor $\tau_{j,j+1}X_{j,j+1}$ and the terms in the set $\{\tau_{i,i+1}(X_{i,i+1} - s)\}_{i \neq j}$.

The above formula is one of the main contribution of this thesis. With this formula we are able to go from 3 month to 6 month shifted Black volatilities in a similar way as we would transform regular Black volatilities when the rates are positive.

7.1.4 Transforming the smile

When ATM volatilities for non-standard tenors have been found, using for instance one of the techniques above, we need to find a way of reconstructing the smile. This means finding non-ATM volatilities for caplets of the non-standard tenor. To this end we will use a method based on the SABR model. As indicated by Kienitz (2013) it works a follows.

1. Calculate SABR parameters $\{\alpha(T), \beta, \rho(T), \nu(T)\}$ for caplet expiries T of the standard tenor.
2. Transform, ATM volatilities from standard tenor to non-standard tenor.

   Since we have little knowledge about the smile for a non-standard tenor we could assume that volatilities have the same smile shape and preserve this shape with respect to moneyness.
3. In order to preserve the same shape of the smile we fix the parameters β, ρ, and ν previously calibrated for the standard tenor. This means we only have to find α for the non-standard tenor. This, is done by using either equation (3.10) or (3.12) (depending on if using SABR or shifted SABR) to solve for α.
4. Use the obtained SABR model to infer non-ATM caplet volatilities for the non-standard tenor.

Employing this algorithm we have found a SABR model with the same smile shape for moneyness but with a transformed ATM volatility to a non-standard tenor. Doing so we are able to obtain non-ATM volatilties for a non-standard tenor. Recalling the interpretation of α in Table 3.1 it has the affect of changing the level of the smile. Hence, the proposed method will shift the whole smile in a consistent way.

7.2 Transforming Cap Volatilities in a Multi Curve Setting

This far we have considered the extrapolation problem for both Black and shifted Black volatilities in a single curve setting. However, after the credit crises in 2007 the market has begun to adopt a multi curve setting. As mentioned in Chapter 6, this means that the discounting curve and the forwarding curves are separated. Furthermore, forwards are segmented based on tenor due to differences in liquidity and credit risk. In order to take these changes into account, we develop a method based on displaced diffusions. The main idea is to transform the volatility on the forward curve to the OIS curve and then apply the same techniques for volatility transformation as in the single curve setting. When the volatility has been transformed it is mapped back to the forward curve.

In order to take the money market spread into account we denote

$F_{1,2}^{OIS}(t) := F^{OIS}(t; T_1, T_2) :=$ forward rate calculated on the OIS curve,

$F_{1,2}^{nM}(t) := F^{nM}(t; T_1, T_2) :=$ FRA rate retrieved from a forward curve where

$n = 1, 3, 6, 12$ corresponding to EURIBOR1M, EURIBOR3M, EURIBOR6M, EURIBOR12M.

We notice that for some given dates T_1 and T_2,

$$F_{1,2}^{nM}(t) = F_{1,2}^{OIS}(t) + b_{1,2}^{nM}. \tag{7.8}$$

Where $b_{1,2}^{nM}$ denotes the money market basis spread which has to be added to the OIS forward rate, to get the FRA rate on the n month forward curve.

Remark 7.3. *At this point we want to emphasize that although $b_{1,2}^{nM}$ is a constant, the model nowhere assumes a constant nor a deterministic basis spread. As mentioned by Andong (2013) one should instead think of $b_{1,2}^{nM}$ as a way of quoting. The idea is similar to when we obtain an implied Black volatility from Black's formula using today's forward rate. We use the current forward rate but this does not mean that we assume it is constant. The basis spread can for instance be time dependent or even stochastic. In this way, the transformation method only assumes that it is the current spot value that is used for quoting. Furthermore, it is natural to suggest a non-negative restriction on the basis spread. A situation where $b_{1,2}^{nM}$ is negative would imply that the OIS-curve no longer is considered as a risk free curve. This would of course not make sense in a multi curve setting.*

Next we consider a model for $F_{1,2}^{nM}$. On one hand, we can assume that it follows some log-normal process given by

$$dF_{1,2}^{nM}(t) = \ldots dt + \sigma_{1,2} F_{1,2}^{nM}(t) dW(t), \tag{7.9}$$

where $\sigma_{1,2}$ is obtained from market quotations. On the other hand, we also have a representation of the process as a displaced diffusion model

$$dF_{1,2}^{OIS}(t) = dF_{1,2}^{nM}(t) = \ldots dt + \sigma_{1,2}(F_{1,2}^{OIS}(t) + b_{1,2}^{nM}) dW(t).$$

Consequently, we may interpret $\sigma_{1,2}$ as a displaced diffusion volatility σ_{DD} and $b_{1,2}^{nM}$ as the displacement b.

The additional problem in the multi curve setting is that after 2007, equation (7.1) does not hold true since we now have different yield curves for generating FRA rates of each rate tenor. However, we have the following relation

$$(1 + \tau_{1,2} F_{1,2}^{OIS}(t))(1 + \tau_{2,3} F_{2,3}^{OIS}(t)) = 1 + \tau_{1,3} F_{1,3}^{OIS}(t). \tag{7.10}$$

This equation holds true due to the risk free nature of the OIS curve. Thus, our goal is to convert the displaced diffusion volatility on the forward curve to a Black volatility on the OIS curve. The reason why we want this is, once we have obtained black volatilities, then we can apply the previously developed single curve transformation technique.

In line with Kienitz (2013) we let σ denote the ATM implied Black volatility on the

OIS curve. Then as demonstrated by Kienitz the displaced diffusion volatility σ_{DD} can be written as

$$\sigma_{DD} = \frac{F^{OIS}}{F^{OIS} + b}\xi = \beta\xi,$$

with the following expansion formulas

$$\sigma = \xi + \frac{1-\beta^2}{24}\beta^3 T + \frac{7 - 10\beta^2 + 3\beta^4}{1920}\beta^5 T^2 + \mathcal{O}(\xi^7), \tag{7.11}$$

$$\xi = \sigma + \frac{\beta^2 - 1}{24}\sigma^3 T + \frac{3 - 10\beta^2 + 7\beta^4}{1920}\sigma^5 T^2 + \mathcal{O}(\sigma^7). \tag{7.12}$$

Going from a short period of m months to long period of n months ($n = km$), we can apply the formulas as follows:

1. We first use equation (7.11) to transform short period displaced diffusion volatilities, with displacements $b^{mM}_{j,j+1}$, to Black volatilities. This corresponds to moving from the the forward curve to the OIS curve.
2. Then, since (7.10) is valid we can employ the previous developed transformation technique (cf. equation (7.5)), to get a single Black volatility for the long period.
3. We then use equation (7.12) together with a new spread $b^{nM}_{1,k+1}$ (and thus a new β) to get the corresponding displaced diffusion volatility for the long period.

So far, we have implicitly assumed (by our log-normal assumption in (7.9)) that both $F^{nM}_{1,2}$ and $F^{OIS}_{1,2}$ are positive. But what to we do if both rates are negative? To this end we proceed as before and shift both rates using a deterministic shift s. Hence, we define

$X^{OIS}_{1,2}(t) := F^{OIS}(t; T_1, T_2) + s :=$ shifted forward rate calculated on the OIS curve.
$X^{nM}_{1,2}(t) := F^{nM}(t; T_1, T_2) + s :=$ shifted FRA rate retrieved from a forward curve where
$n = 1, 3, 6, 12$ corresponding to EURIBOR1M, EURIBOR3M, EURIBOR6M, EURIBOR12M.

Doing this, (7.8) is still valid for the shifted rates and we are still able to assume log-normality, i.e. the dynamics are

$$dX^{OIS}_{1,2}(t) = dX^{nM}_{1,2}(t) = \ldots dt + \sigma_{1,2}(X^{OIS}_{1,2}(t) + b^{nM}_{1,2})dW(t).$$

As such we are able to view $\sigma_{1,2}$ as a Black volatility that has been displaced twice. First by the shift s and then by the spread $b^{nM}_{1,2}$. However, compared to the ATM implied shifted Black volatility, σ, on the OIS curve we still have a net displacement of $b_{1,2}^{nM}$. Consequently, we may conceptually still view $\sigma_{1,2}$ as a displaced diffusion volatility σ_{DD} and use the exact same algorithm as above with the slight modifications that σ_{DD} is given by

$$\sigma_{DD} = \frac{X^{OIS}}{X^{OIS} + b}\xi = \beta\xi.$$

and that we use formula (7.7) instead of (7.5).

Remark 7.4. *We want to stress that the market practice is to quote shifted Black cap volatilities together with the corresponding shift s. For example in the EUR market we currently have $s = 3\,\%$. The volatility $\sigma_{1,2}$ on the forward curve can then be obtained by bootstrapping the cap volatilities. As a consequence it is important to use the given market value of s when using the above tranformation method to map the volatility from the forward curve to the OIS curve.*

7.3 Transformation example in the EUR Market

The convention in e.g. EUR and CHF market, of using 3 month underlying caplets in caps with a maturity up to 2 years and 6 month caplets in caps with a maturity longer than 2 years, adds some complexity to the caplet stripping. The problem arises when we try to take the difference between the 3 and 2 year cap to get the forward cap price. The complication comes from that the 2 year cap consists of 7 successive 3 month caplets whereas the 3 year cap consists of 5 successive 6 month caplets. Thus, it is not correct to infer the forward cap price, by taking the price difference between the 3 and 2 year cap.

Instead, in order to get the correct price difference, we would like to have a 2 year cap that is based on 6 month caplets. Unfortunately, these caps are not directly observable in the market. To solve this we suggest the following approach

- First we strip the caps with maturities 1, 1.5 and 2 years using the bootstrap technique described in section 5.1. This render caplet volatilities for the underlying 3 month caplets.
- Once we have the 3 month caplet volatilities we use these to obtain 6 month caplet volatilities. This is done by discarding the first 3 month caplet volatility[1], which leaves us with 6 consecutive 3 month caplet volatilities which are pairwised transformed to become 3 consecutive 6 month caplet volatilities.
- As we now have our 6 month caplet volatilities, we can use these to price a 2 year cap based on 6 month caplets. This, cap is somewhat fictitious since its price is not quoted in the market. However, it is used for obtaining the correct price of a forward cap between year 2 and 3.

[1]Recall that for our 3 year cap, the first caplet is due first after 6 months, hence the first 3 month caplet volatility is neglected

- We then continue to strip the remaining caps in the usual way by taking the price difference between the 4 and 3 year cap etc.

Below we provide a step by step guide, together with obtained results, of how to transform 3 month caplet volatilities to 6 month caplet volatilities in the European market. We emphasize that the transformation is applied only for caps with maturity up to 2 years. Cap volatilities corresponding to caps with maturity more than 2 years are by the convention already based on 6 month caplets and as such they do not need to be transformed.

Throughout this example we are only considering caps with maturities up to two years, i.e. caps that are constructed by 3 month caplets.

1. Begin to construct your discount curve (from OIS) and forward curve (FRA rate) at correct dates.

2. The transformation algorithm is based on the assumption that

$$(1 + \tau_{1,2} F_{1,2}^{\mathrm{OIS}})(1 + \tau_{2,3} F_{2,3}^{\mathrm{OIS}}) = 1 + \tau_{1,3} F_{1,3}^{\mathrm{OIS}}.$$

   The assumption is checked, and we have an error in the range of 10^{-5} which we believe is acceptable since similar results have been obtained in related studies (Andong, 2013).

3. Start with pricing your caps according to the standard pricing formulas that were derived in (6.6).

4. Bootstrap out (cf. section 5.1) the caplet volatilities corresponding to the fixed strike grid. Due to that we want as good fit as possible for ATM caps, we are only considering strikes from -0.75 to 0.25. Fortunately, in this example the ATM-strikes for caps up to two years are the same (cf. Table A.1). So the corresponding caplet volatilities are backed out in the same way as those on the fixed strike grid.

5. Calibrate a SABR model to the obtained set of caplet volatilities. Use vega weights (3.15) to maximize the precision on ATM strikes .

6. Construct implied SABR volatilities for ATM caplets using equation (3.12). Note that the ATM caplet strike is not the same as ATM strikes for caps. The ATM-strike for caplets is the current value of the underlying FRA-rate.

7. Make the transformations as explained in section 7.2. In this step we are taking σ_{DD} to be the implied ATM volatility from the shifted SABR model, that was calculated in the step before. The transformed caplet volatilities are presented in Table 7.1.

8. Keep SABR parameters based on 3 month caplet, taking $\beta_{\mathrm{6m}} = \beta_{\mathrm{3m}}$, $\rho_{\mathrm{6m}} = \rho_{\mathrm{3m}}(T_{\mathrm{6m}})$ and $\nu_{\mathrm{6m}} = \nu_{\mathrm{3m}}(T_{\mathrm{6m}})$, where T_{6m} is the expiry dates for the 6 month caplets. The last parameter α_{6m}, is then derived by solving for α in equation (3.14), choosing the smallest real root.

9. Use the obtained SABR parameters to construct cap prices based on 6 month caplets and back out corresponding shifted Black cap volatilities. The cap volatilities obtained are found in Table A.3.

Remark 7.5. *We recognize that it is important to get a good SABR calibration for 3 month caplets, in order to reconstruct 3 month based cap prices reasonably well. This is important since the calibrated SABR model is used to obtain ATM caplet volatilities, which are inputs to the volatility transformation that eventually result in 6 month based ATM caplet volatilities.*[2].

Table 7.1: *Presentation of the obtained transformed caplet volatilities. The displaced 3 month caplet volatilities σ_{DD}^{3m}, are obtained from the implied ATM SABR volatilities* (3.12)*. Then, using the relation in* (7.11)*, we are obtaining Black volatilities σ^{3m}. The rebasing formula* (7.7)*, is applied when transforming 3 month Black caplet volatilities to 6 month Black caplet volatilities, while the correlation between the subsequent 3 month Brownian motions is set to $\rho = 0.9$. At last, displaced 6 month caplet volatilities are received by applying the formula that was derived in equation* (7.12)*. In the table, expiry has the unit years counted from start date and the volatilities are in percent.*

Caplet Volatilities

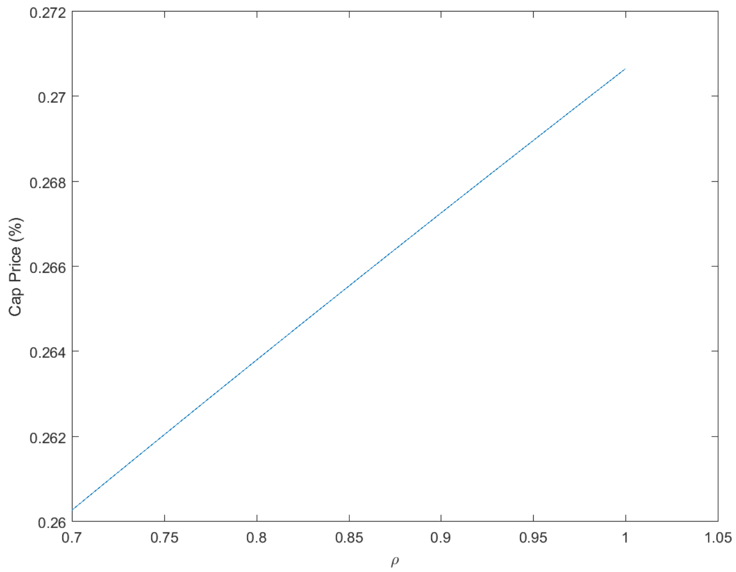
Expiry	σ_{DD}^{3m}	σ^{3m}	Expiry	σ^{6m}	σ_{DD}^{6m}
0.25	0.0772	0.0807			
0.5	0.0769	0.0806	0.5	0.0782	0.0713
0.75	0.0769	0.0810			
1	0.0934	0.0982	1	0.0949	0.0863
1.25	0.0934	0.0979			
1.5	0.1135	0.1191	1.5	0.1153	0.1050
1.75	0.1135	0.1192			

7.4 Sensitivity of Correlation

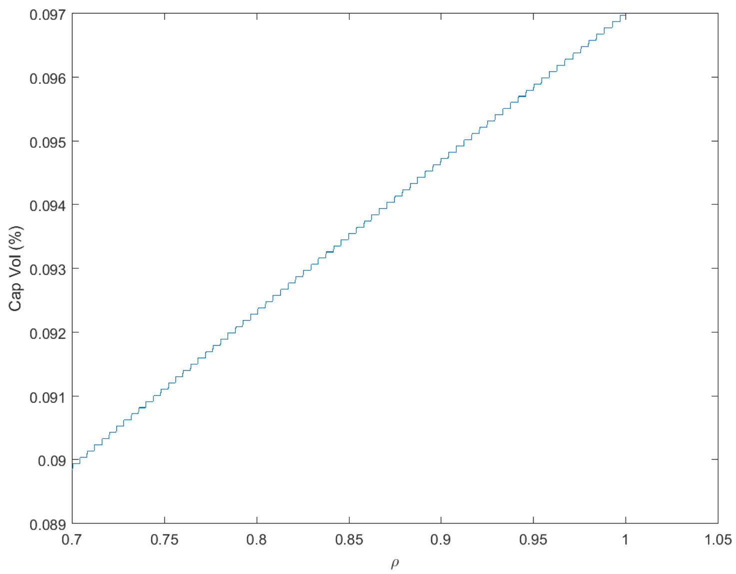
A small analysis regarding the choice of correlation between forward rates is presented in this section. In this study we explore in what way cap prices and volatilities are changing, depending on the choice of correlation, ρ, when transforming from 3 month caplet volatilities to 6 month caplet volatilities (cf. section 7.3).

In Figure 7.2, the cap maturity is set to 2 years, and the strike is fixed to ATM. For that fixed set of state parameters, cap prices and volatilities are calculated for different values of correlation coefficient $\rho \in [0.7, 1]$. We notice that the price and volatility are both increasing with ρ. Figure 7.2a, shows a function looking very linear, but further inspection reveals that it actually has a small concave property. Similarly, the function in Figure 7.2b is pretty linear, but zooming in, one can notice that it has a stepwise constant property.

[2]In introductory courses in computer science one often hear the phrase: "Crap in crap out". This adage is meant to emphasize that regardless of the correctness of the logic built into the program, no answer can be valid if the input is erroneous. We believe this is certainly applicable in this setting as well.



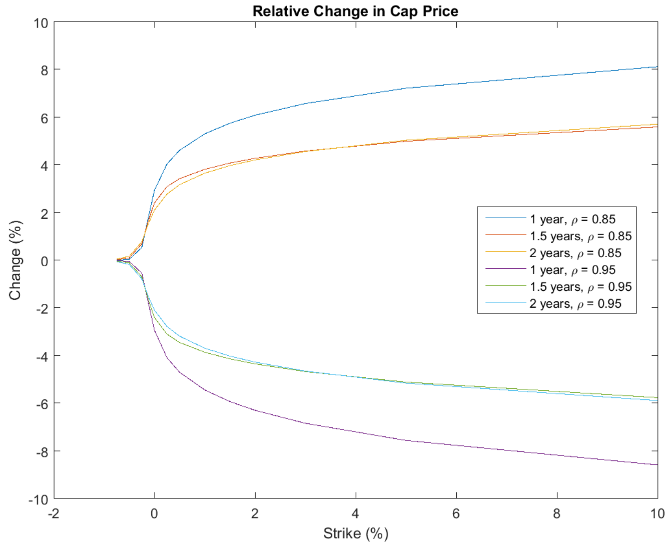
(a) *Cap price as a function of ρ. The function is increasing and almost linear.*



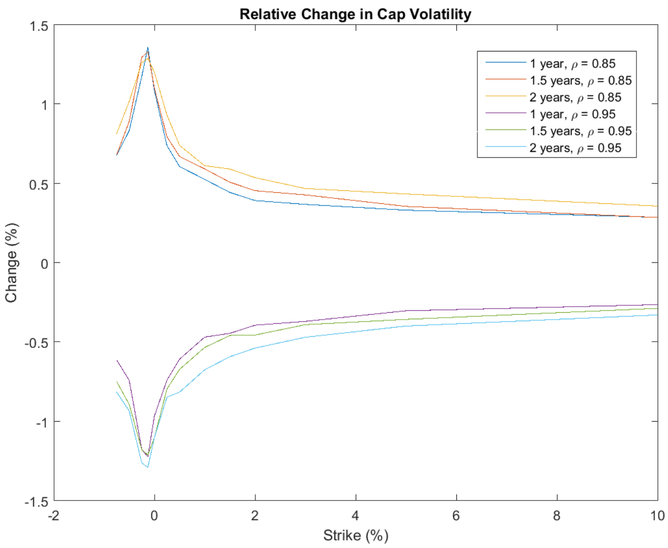
(b) *Cap volatility as an increasing function of ρ.*

Figure 7.2: *The sensitivity of ρ is here expressed as change of cap price, respectively cap volatility. The cap maturity and strike are fixed at 2 years respectively ATM strike.*

In the previous section, the caplet volatilities on the European market were transformed using $\rho = 0.9$. It is therefore interesting to investigate the relative change in cap prices respectively cap volatilities, that occur when changing ρ by ± 0.05. The relative changes are presented in Figure 7.3. In the price domain, the relative changes are increasing as the cap gets further OTM. If we instead take a look at the cap volatility domain, the absolute relative changes are largest for strikes near ATM, and they are converging against $\approx \pm 0.5\%$ for the other strikes. This indicates that for low strikes, the choice of ρ is not of major importance. However, if one is interesting in pricing caps with higher strikes, one should use a sufficient technique when estimating ρ. An overview of cap volatilities corresponding to different ρ, can be found in Appendix A in Table A.5.



(a) *Cap prises resulting from putting $\rho = 0.9$ are compared to cap prises that corresponds to $\rho = 0.85$ respectively $\rho = 0.95$. Prices decrease when changing to $\rho = 0.85$, hence a positive relative change. There is a symmetry around the strike axes, therefore relative changes will get negative when increasing the correlation to $\rho = 0.95$.*



(b) *Cap volatilities that are backed out from cap prices when setting $\rho = 0.9$, are compared to cap volatilities resulting from setting $\rho = 0.85$ respectively $\rho = 0.95$. There is a symmetry around the strike axis. The cap volatilities are increasing with ρ, hence the changes are positive for $\rho = 0.85$, and negative for $\rho = 0.95$. The maximum value of the relative change, takes place for strikes near ATM.*

Figure 7.3: *Illustrates the sensitivity of cap prices and volatilities when changing ρ. In the price domain, the changes are increasing when strikes get further OTM. The absolute relative errors in the volatility domain, are taking their largest values for strikes near ATM. All relative changes are measured in percent.*

Chapter 8

Calibration of shifted SABR to EUR Caps

In the following chapter, calibration results on the European market are presented. In the EUR market there are two market quotes to choose from, normal and shifted Black cap volatilities. We will focus on shifted Black volatilities, which are currently quoted with a shift of 3%. The calibration of the shifted SABR model is based on 6 month caplets, using ICAP data on implied cap volatilities (cf. Table A.1 and A.3). To achieve this we must first obtain a cap volatility surface based solely on 6 month caplets. As such, we use the results obtained in section 7.3 which led to the cap volatilities within the yellow area in Table A.3. Since we do not know the exact value of the correlation between FRA rates we have to make a choice regarding this parameter. In line with Andong (2013) we choose to put $\rho = 0.9$. This implies that there is, as one would expect, a strong positive linear dependence between FRA rates on subsequent intervals.

Furthermore, we decide to only use data for caps with maturities up to 10 years, and for strikes up to 3%. The reason is that we do not want caps that are very far OTM and illiquid to inflict on the calibration results.

8.1 Method 1: Bootstrap

Once the tranformation step above is completed we are stripping out the caplet volatilities using the data set of cap volatilities in Table A.3. The resulting constant caplet volatilities are depicted in Figure 8.1.

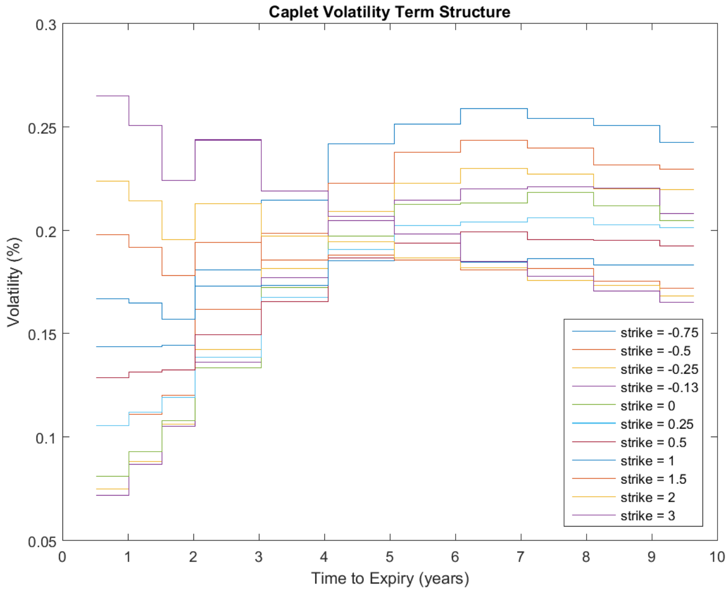


Figure 8.1: *Caplet volatilities that are backed out in a bootstrapping approach. There is one volatility term structure for each strike.*

The next step is to fit shifted Black SABR volatilities to the backed out caplet volatilities on a fixed strike grid. To this end we use vega weighted SABR calibration and let β be a global parameter. The resulting shifted SABR parameters are presented in Figure 8.2. The corresponding caplet volatility surface, constructed by the shifted SABR model, is shown in Figure 8.3.[1]

[1]We want to emphasize that we were trying to insert ATM cap volatilities into the calibration as described in section 5.1. Since this extra step just resulted in even worse pricing errors, we decided to exclude ATM from the calibration step.

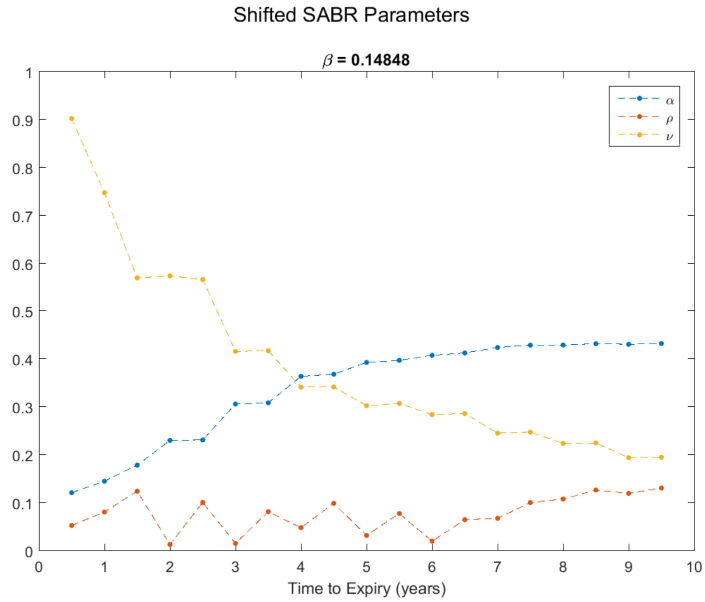


Figure 8.2: *Shifted SABR parameters resulting from bootstrapped caplet volatilities in the EUR market.*

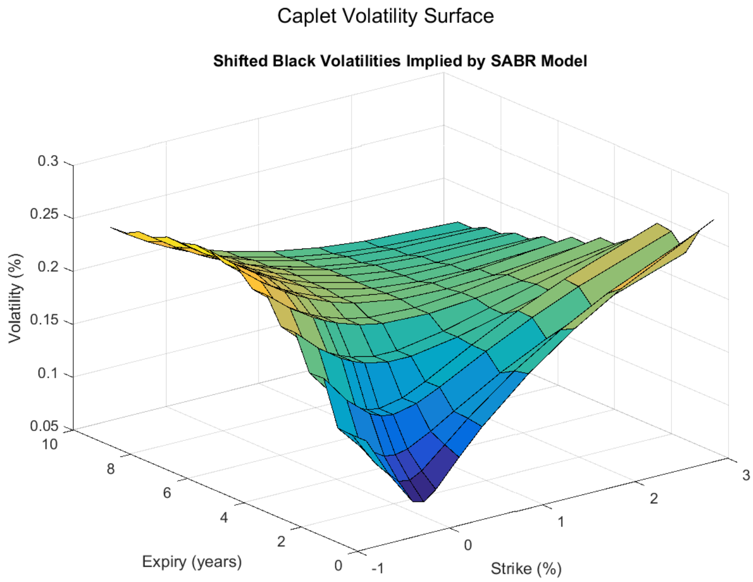
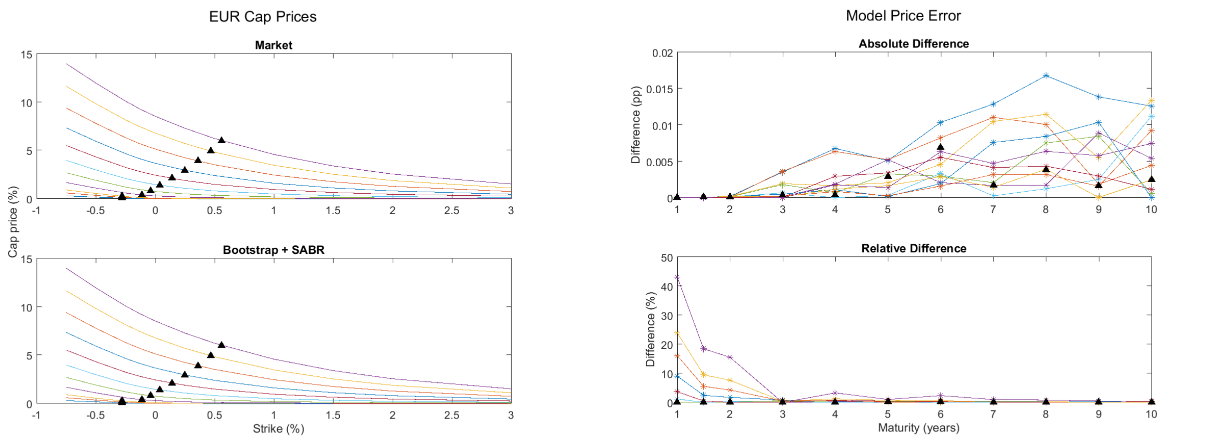


Figure 8.3: *Caplet volatility surface, where the whole surface is based on 6 month caplets on the European market.*

Finally, cap prices are reconstructed using the obtained volatilities coming from the shifted SABR model. These are then compared to corresponding market price, except for the caps with maturities up to two years. These are compared to the cap prices that resulted from the transformation step, which were the prices used for calibration. The prices and errors are presented in Figure 8.4 together with a compensating Table D.1.



(a) *Market cap prices (top) are compared to the ones that are reconstructed by the SABR model (bottom). The model price structure is looking similar to the market price structure.*

(b) *The absolute errors between the market and model prices are on the top graph, and corresponding relative errors are presented in the bottom graph.*

Figure 8.4: *Cap prices from the market is compared to the reconstructed model prices are depicted to the left. The resulting price errors are presented in the right figure. Black triangles stands for caps with ATM strikes.*

8.2 Method 2: Rebonato

Rebonato's function for instantaneous forward rate volatility (5.3), is fitted to market prices of caps with a fix strike. The optimization process of the Rebonato parameters follows the flow chart that is illustrated in Figure 5.2. For each fixed strike, the Rebonato parameters are estimated and the integrated caplet volatility $\Sigma^2(T, K)$, is calculated (c.g. (5.2)). The estimated parameters are found in Appendix C in Table C.1. The corresponding integrated caplet volatilities are illustrated in Figure 8.5. We note that the volatility increases as the strikes decreases.

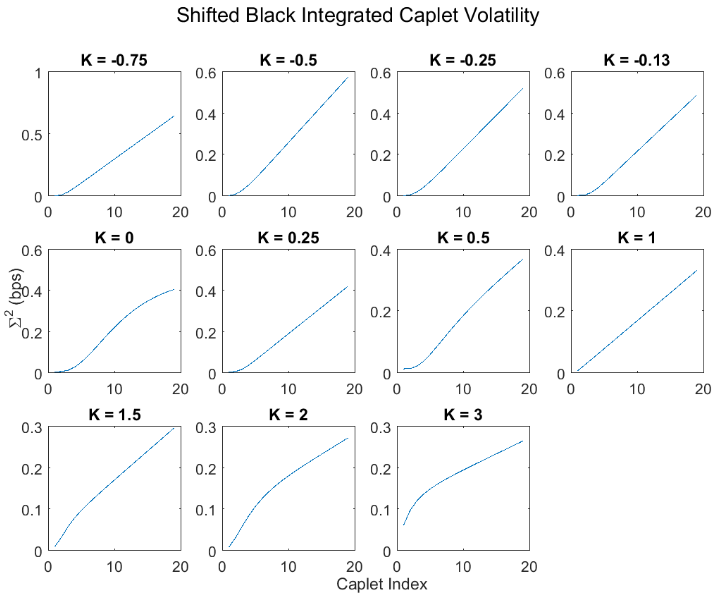


Figure 8.5: *Estimated Rebonato integrated caplet volatility (in bps) for each strike (K). This is the up integrated caplet volatility that is inserted to the Black type pricing formula* (3.7)*, in order to receive the model cap price from constituent caplet prices.*

Furthermore, the obtained integrated caplet volatilities are used for calibrating the shifted SABR model. The SABR parameters are estimated in the way that are described in Figure 5.3. So for each fixed maturity, the smile model parameters are estimated, with one exception for β which is estimated globally. The parameter estimates are presented in Figure 8.6.

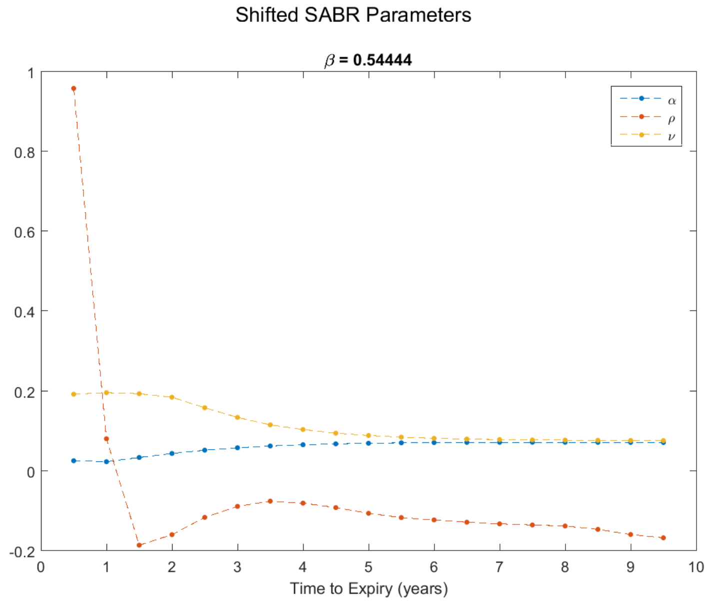


Figure 8.6: *Estimated shifted SABR parameters for EUR market with caplet expiry up to 10 years.*

The caplet volatility surface that is derived from the SABR parameters is illustrated in Figure 8.7.

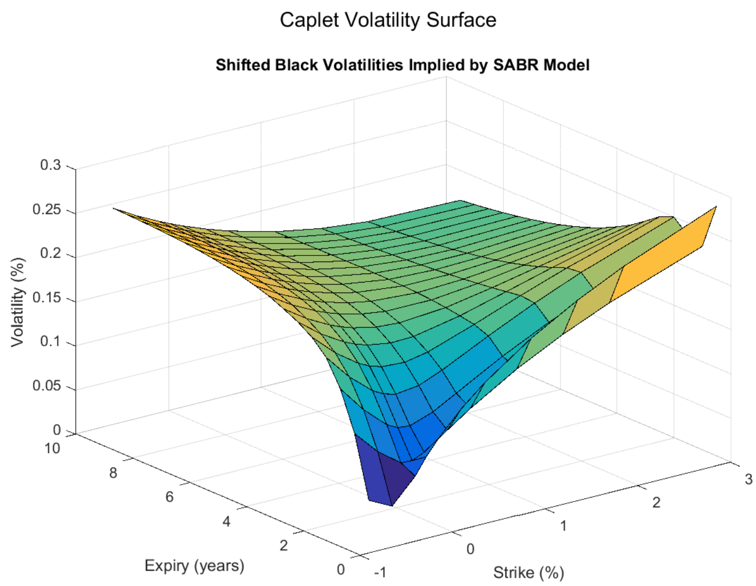
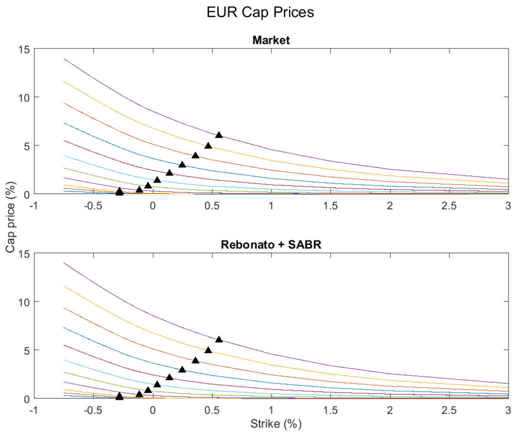
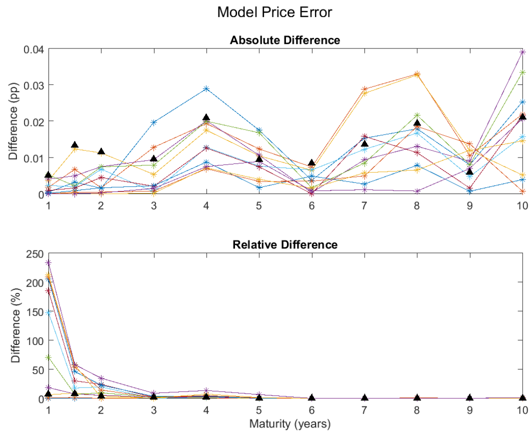


Figure 8.7: *Caplet volatility surface for shifted Black volatilities implied by SABR parameters. The volatility smiles are found by looking at the Volatility-Strike plane and the volatility term structure is illustrated in the Volatility-Expiry plane.*

In order to get some sense of how well the method works, we are reconstructing the prices from the SABR parameters that we received by calibration with Rebonato's function. In Figure 8.8a, the reconstructed prices are plotted together with the original market prices. We are also calculating the absolute and relative errors between the model and market prices, which are presented in Figure 8.8b. The exact values can be found in Table D.2 in Appendix D.



(a) *Cap prices from EUR market (top) and reconstructed prices from shifted SABR model (bottom). Each line corresponds to a cap maturity. The lowest price corresponds then to the cap with maturity 1 year, and the highest price corresponds to the cap with maturity 10 years.*



(b) *Absolute difference between model and market prices (top), where the errors are growing for larger maturities. Relative errors between model and market prices (bottom). We can observe a clear peak for caps with maturity 1 year.*

Figure 8.8: *Illustration of how well the shifted SABR model reconstruct the prices. To the left are cap prices from model and market plotted, and to the right are the price errors shown. At-the-money caps are represented by black triangles.*

8.3 Method 3: Global SABR

The results presented in this section are based on the Global SABR method which is explained in Figure 5.4. This approach does not need an initial step of first fitting a caplet volatility term structure, however it needs some manually tuning on the choice of knots in the term structure of the SABR parameters.

We are constructing cap prices based on the volatilities from Table A.3. Further on, knot points are tuned by comparing resulting price errors. The number of knots and where they are placed are presented in Table 8.1.

Table 8.1: *Presentation of the choices for knot positions and quantity. The placement column has unit years from today. The interpolation methods "spline" and "pchip" are interpolations commands used in Matlab.*

Knots for Global SABR method

Parameter	Number of Knots	Placement	Interpolation Method
α	6	$0, 1, 3, 5, 7, 10$	"spline"
ρ	3	$0, 5, 10$	"pchip"
ν	6	$0, 1, 3, 5, 7, 10$	"spline"

Calibrated SABR parameters, resulting from the above mentioned knot specifications, are depicted in Figure 8.9.

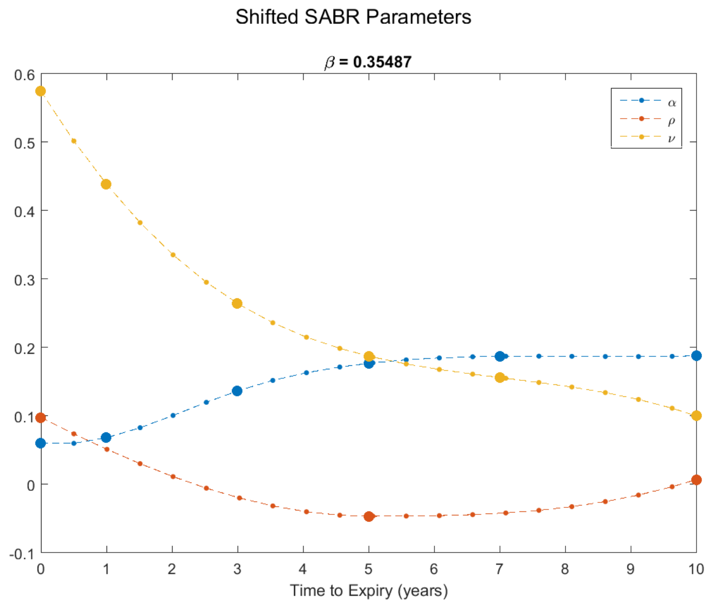


Figure 8.9: *Shifted Black parameters for shifted SABR model on EUR market. The large dots indicate where the knots are placed.*

From the set of above presented parameters, a caplet volatility surface is constructed and illustrated in Figure 8.10. The surface is really smooth which reflects that the SABR parameters are constructed by interpolation methods.

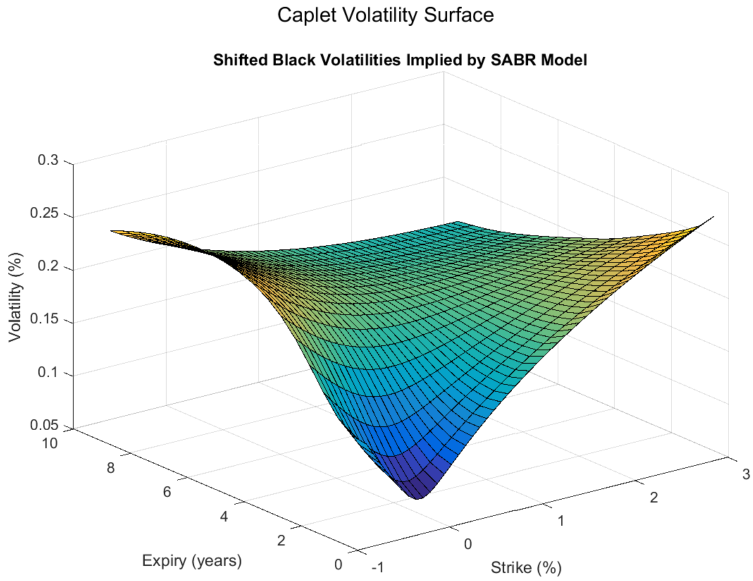
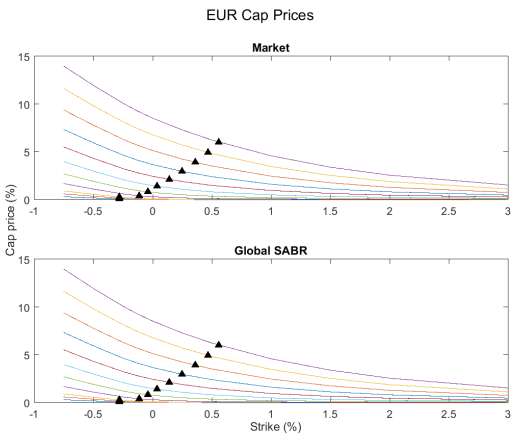
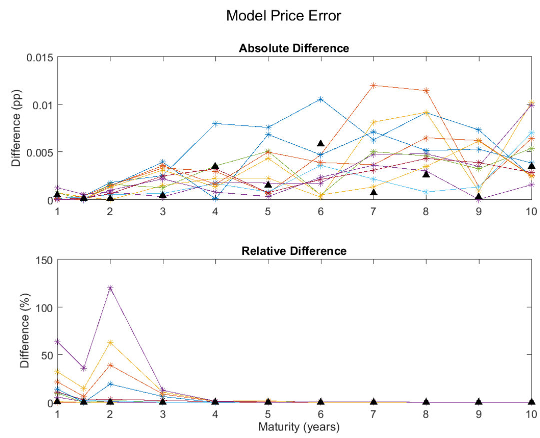


Figure 8.10: *Shifted Black caplet volatilities corresponding to the shifted SABR parameters that are illustrated in Figure 8.9.*

Using the SABR caplet volatilities that we have received, we are reconstructing the cap prices. These are then compared to actual market prices by calculating the absolute and relative price errors. The result is illustrated in Figure 8.11 and complementing tables are found in Appendix D in Table D.3.



(a) *EUR market cap prices (top) compared to reconstructed shifted Black SABR cap prices (bottom).*



(b) *Absolute (top) and relative (bottom) differences between the cap prices from the market and those that are reconstructed by the shifted Black volatilities coming from the calibrated shifted SABR model. Relative errors are peaking for caps with short maturities and large strikes.*

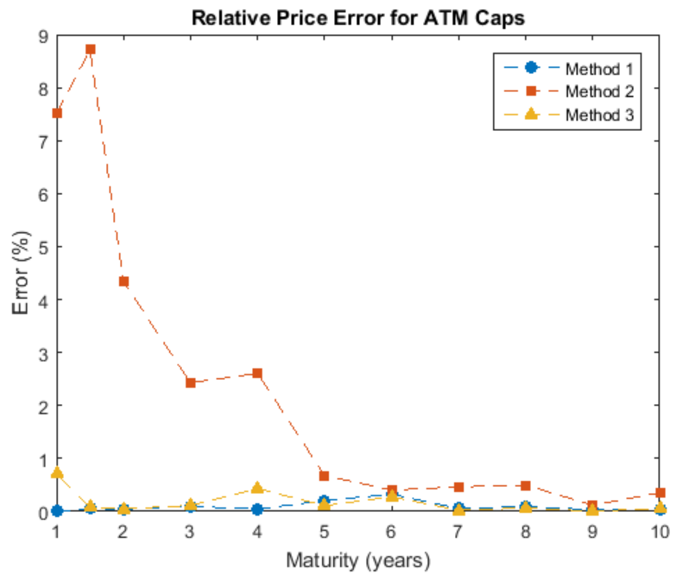
Figure 8.11: *To the left are cap prices, coming from the EUR market, compared to prices that are reconstructed by the calibrated shifted SABR parameters. Each line represents a maturity (1 - 10 years) and black triangles are denoting the ATM cap prices. To the right are absolute and relative price differences constructed. Each line represents a fixed strike except for black triangles that are denoting the error for ATM caps.*

8.4 Analysis of shifted SABR Calibration

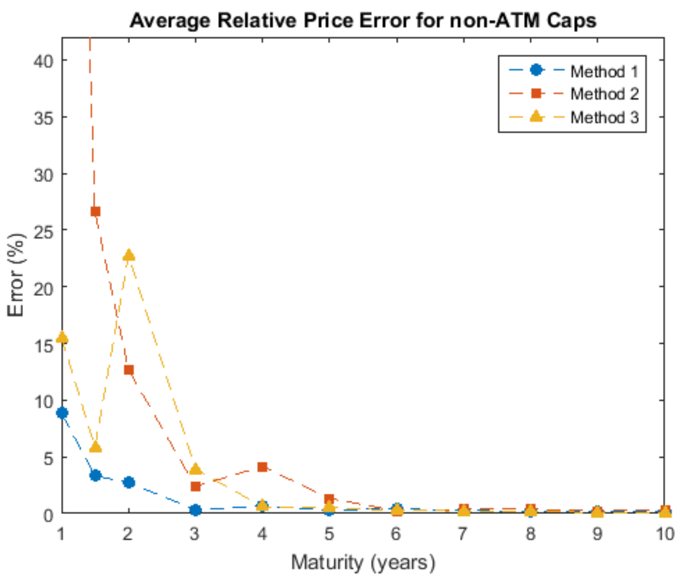
In this section we will examine the calibration results of the shifted SABR model in the EUR market. The analysis will be based on a comparison of our three proposed methods. The calibration takes place in a multi curve setting. Looking at Figures 8.4a, 8.8a, 8.11a the overall impression of the methods is good. All methods successfully generate cap prices that have the correct shape in that they are decreasing with strike and are similar to the market prices. To make a more in depth analysis of the methods they will be evaluated against three quality measures as described in Figure 1.2.

First, their ability to fit ATM cap prices will be studied. The relative errors to ATM market cap prices are shown in Figure 8.12a. Here it is seen that the bootstrap calibration (method 1) and the global SABR calibration (method 3), both outperforms Rebonato (method 2). The bootstrap method is slightly better then the global SABR method, this is probably due to that we use vega weights in bootstrap calibration. One should also consider that ATM caps are not included in the bootstrap method, hence the method do not use all information that is provided.

In Figure 8.12b we have tried to boil down the overall relative price error of the methods into one single measure. Because a fix cap maturity offers non-ATM caps with different strikes, we have chosen to calculate the average relative error for each cap maturity. From the figure we find Rebonato to be worst while Bootstrap and global SABR perform similar with Bootstrap being better to calibrate to short maturities caps and global SABR for long maturity caps.



(a) *ATM relative errors between market prices and model prices generated by each method. Rebonato is performing poorly in comparison to bootstrap and global SABR.*



(b) *Average relative price errors, calculated for each cap maturity. Bootstrap outperforms the other two methods, for maturities up to four years. From 5 years and forward, the methods produce a similar goodness of fit.*

Figure 8.12: *Presentation of how well the three methods (Bootstrap, Rebonato and Global SABR) are performing considering the two quality meassures, ATM errors (top) and average errors (bottom).*

A more detailed view of the relative errors can be found in Figures 8.4b, 8.8b, and 8.11b and in Appendix D. The overall trend in relative error is that all methods have some problems to match prices of caps with short maturities and large strikes. Although, this may seem as a very undesirable property we believe this has to do with the fact that most of these caps are out-of-the-money and illiquid. The reason for this is that their strikes are well above the ATM strikes. Due to their short time to maturity, it is unlikely that the EURIBOR will increase that much. Hence, few people believe it is worth investing in these caps. As a consequence, these caps cost almost nothing and the relative errors become very sensitive.

In addition, there is a large difference between the relative errors. Especially Rebonato produces extremely high errors for caps with short maturity and high strikes. This can also be seen in Appendix D in Table D.2. We think this is partly due to the common issue mentioned above. Furthermore, one should be aware of that the fitted Rebonato parameters (see Appendix C), are not able to perfectly reconstruct cap prices. This implies that the calibration of SABR to Rebonato caplet volatilities can never perform better than the bootstrap approach, hence Rebonato will be the worse performing method.

As observed in Figures 8.2, 8.6 and 8.9, the SABR parameters are quite different. For example, β is varying a lot depending on what method is used. In the literature, various techniques for estimating β are stated. However, the general view seems to be that β is not that important. For instance Rouah proposes to put $\beta = 0.5$. Despite this, to illustrate the character of the different methods, we have chosen to optimize β in a global way.

In order to get a better fit for caps with strikes close to ATM, one could try some different things. First, one could cut the data set, and discard cap volatilities that are far out-of-the-money. In this setting one would only use liquid contracts for calibration. However, using this approach one might end up with a small set of data, which could render a poor fit. Our next suggestion, that might prevent the lack of data, is to use interpolation in the strike domain. We believe this approach should be used with cautiousness. As if there are too large steps between the strikes, there is a risk that one is calibrating to interpolated prices which do not catch the right structures in the original market data. At last, one could use interpolation methods in the cap maturity domain. This could be preferable in the bootstrap approach, where a interpolation would lead to a more realistic looking term structure as we do not have to force the caplet volatilities to be constant on each interval between cap maturities (especially on the USD market where we have 3 month tenor). As always, one should be careful when using interpolation methods as those may lead to inaccurate results.

Finally, we make a *qualitative* comparison of the three methods based on our findings from the calibration. The pros and cons of the methods are listed below.

Table 8.2: *Pros and cons of qualitative properties corresponding to the Bootstrap method when calibrating SABR.*

Bootstrap

Pros	Cons
Simple and fast approach for obtaining constant caplet volatilities.	Coarse caplet volatility term structure for fix strike since it is assuming piecewise constant caplet volatility between cap maturities.
Obtained caplet volatilities, can reconstruct market cap prices perfectly.	Rather difficult to incorporate ATM caps in bootstrapping procedure due to different strikes of ATM Caps. This can be resolved by building a smile model as described in section 5.1. However, this comes at the price of building a model upon a model which might lead to inaccurate results.
Ability to employ vega weighted calibration and as such give more importance to near ATM-strikes.	

Table 8.3: *Pros and cons regarding the qualitative properties of Rebonato method when calibrating SABR.*

Rebonato

Pros	Cons
Generates a smooth Volatility Term Structure, which render in a nice looking volatility surface.	Somewhat theoretical inconsistent as it in the step towards obtaining caplet volatilities assumes the instantaneous volatility of forward LIBOR to be strike dependent. This is because one Rebonato functional form is fitted for each available market strike.
Only 4 parameters have to be estimated which are able to capture the main volatility term structure.	Obtained Rebonato caplet volatilities are unable to reconstruct market cap prices.
	Can not include ATM into calibration.

Table 8.4: *Qualitative pros and cons of Global SABR method.*

Global SABR	
Pros	**Cons**
Everything is estimated in one single step.	Is in need for serious tuning. A badly chosen set of node points render in hideous parameters and large price errors.
The ATM volatilities are in a natural way included into the calibration.	The calibration results depends on the chosen interpolation technique between knots. That is we get different results depending if we choose spline, pchip or double quadratic interpolation.
Does not need to transform cap volatilities into caplet volatilities before calibration.	

Chapter 9

Conclusions

The current negative rate environment has led to difficulties in using the traditionally SABR model for volatility interpolation and extrapolation. In this thesis we have shown how to use the shifted SABR model to resolve this issue. The model has been applied in the EUR market for both calibration to caps and volatility transformation in a multi curve setting.

Three different methods for calibrating the SABR model have been analysed. We believe that all methods give reasonable results, but our recommendation is to use bootstrap or global SABR method based on their performance validated by the quality measures: calibration to ATM caps, calibration to non-ATM caps and qualitative properties (cf. Figure 1.2).

We have also developed a multi curve setting based method for transforming cap volatilities between different tenors when rates are negative. The method has successfully been tested in the EUR market going from 3 month to 6 month cap volatilities. By its nature, the method requires the correlation ρ, between forward rates as input, however the analysis in section 7.4 indicates that the sensitivity of the choice of ρ does not seem to be crucial for the method.

Considering the quality of the obtained transformation and calibration results, there does not seem to be any apparent problem with the shifted SABR model compared to the standard SABR model. The main take away is thus, that using the framework developed in this thesis would alleviate many of the problematic issues related to cap pricing under negative rates.

Although, the obtained results seem satisfying there are several areas that could be examined further. One can for instance try to estimate the correlation between forward rates using historical data and a parsimonious functional form as proposed by for instance Rebonato (Rebonato, 2004, ch. 22). This would then eliminate some of the arbitrariness involved by fixing the correlation at 0.9.

Another area that one can develop is the automation of the global SABR method. As previously mentioned, it is currently in need of a manually tuning step, where the number of knots, their placement and the interpolation method is predetermined. An algorithm that finds the best set of knots and finds the optimal interpolation methods would probably both improve the efficiency and quality of the method. Another suggestion is that one could analyze if smoothing splines can improve the fit.

Further, there is room for improvement of the bootstrap approach; at the moment, ATM caps are not included to the calibration in a sufficient way. A suggestion is to try a sequential technique, where ATM caps are included to the calibration simultaneously with non-ATM caps by using previously calibrated SABR parameters.

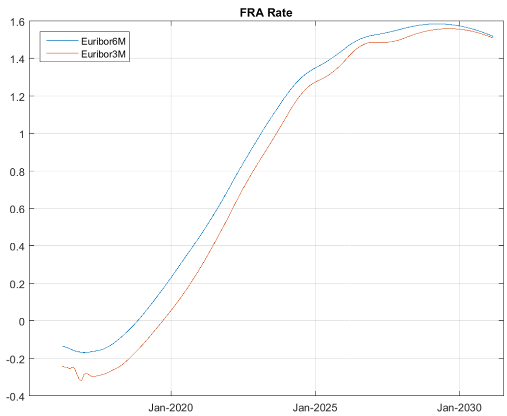
At last, we would like to point out that we have restricted ourselves to studying shifted Black volatilities in this thesis. The SABR model also provides an approximation similar to (3.9) but for implied volatilities based on the normal (Bachelier) model. As a consequence, a suggested topic for further research would be to develop volatility transformation techniques based on normal volatilities and then use normal volatilities to calibrate the shifted SABR model.

Appendices

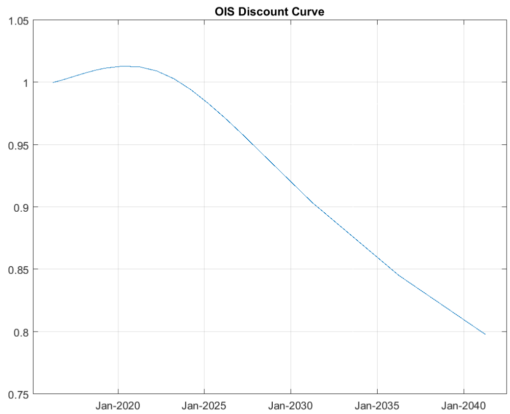
Appendix A

Data

A.1 EUR



(a) *The underlying FRA rates are negative in the beginning of the period. The 6 month EURIBOR is always above the 3 month EURIBOR partly due to the increased credit risk in a 6 month EURIBOR contract.*



(b) *The Discount curve constructed from the OIS rate. The discount factor is larger than one for short maturities, this is a clear sign of a negative rate environment.*

Figure A.1: *To the left are the underlying FRA rates, both 3 month and 6 month EURIBOR, illustrated. To the right are the discount factor, constructed from the OIS curve. The set of curves are obtained from a Bloomberg platform as of March 30th 2016.*

Table A.1: *Shifted Black cap volatilities on the EUR market, for maturities up to 2 years. The volatilities are based on 3 month caplets. The data is gathered from ICAP on March 30th 2016. Both strikes and volatilities have the units percentage. Note that ATM strikes are the same for the three cap maturities. Further, the table only presents volatilities with three decimal points but in reality they are stated with a much higher accuracy. Hence, the volatilities corresponding to a strike of* 0.5%, *tend to be alike, but in reality there are some small differences hidden in the fourth decimals.*

SHIFTED BLACK CAP VOLATILITIES

Term	ATM strike	ATM	-0.75	-0.5	-0.25	-0.13	0	0.25
1 year	-0.28	0.0781	0.120	0.097	0.076	0.085	0.100	0.129
1.5 years	-0.28	0.0866	0.125	0.104	0.084	0.093	0.106	0.133
2 years	-0.28	0.0969	0.130	0.113	0.095	0.102	0.113	0.136

Term	ATM strike	0.5	1	1.5	2	3	5	10
1 year	-0.28	0.156	0.201	0.238	0.270	0.322	0.399	0.518
1.5 years	-0.28	0.156	0.197	0.230	0.259	0.305	0.373	0.476
2 years	-0.28	0.156	0.192	0.221	0.246	0.287	0.346	0.433

Table A.2: *Expiry and maturity dates for 3 month caplets in the 2 year maturity cap.*

Expiry Date	Maturity Date
2016-06-29	2016-09-28
2016-09-29	2016-12-28
2016-12-29	2017-03-29
2017-03-30	2017-06-28
2017-06-29	2017-09-27
2017-09-28	2017-12-27
2017-12-28	2018-03-27

Table A.3: *Shifted Black cap volatilities on the EUR market, for maturities up to 30 years. The volatilities are based on a 6 month tenor. The caps with maturities up to two years (marked with) are obtained from the volatility transformation method which is explained in section 7.2. The data set is downloaded from ICAP, March 30th 2016. Strikes and volatilities are presented in percentage.*

Shifted Black Cap Volatilities

Term	ATM Strike	ATM	-0.75	-0.5	-0.25	-0.13	0	0.25
1 year	-0.28	0.077	0.144	0.105	0.075	0.072	0.081	0.106
1.5 years	-0.28	0.085	0.144	0.109	0.083	0.081	0.089	0.111
2 years	-0.28	0.095	0.144	0.115	0.093	0.091	0.097	0.115
3 years	-0.11	0.114	0.168	0.142	0.118	0.114	0.117	0.13
4 years	-0.04	0.138	0.189	0.165	0.142	0.138	0.139	0.147
5 years	0.04	0.158	0.207	0.183	0.162	0.158	0.157	0.162
6 years	0.14	0.172	0.219	0.197	0.177	0.172	0.171	0.173
7 years	0.25	0.180	0.228	0.207	0.188	0.182	0.18	0.18
8 years	0.36	0.185	0.233	0.213	0.195	0.189	0.187	0.185
9 years	0.47	0.186	0.236	0.216	0.199	0.194	0.191	0.188
10 years	0.56	0.186	0.237	0.218	0.202	0.196	0.193	0.19
12 years	0.73	0.184	0.235	0.218	0.203	0.198	0.195	0.19
15 years	0.89	0.179	0.23	0.215	0.202	0.198	0.194	0.189
20 years	1.01	0.175	0.223	0.211	0.2	0.196	0.192	0.187
25 years	1.04	0.173	0.219	0.208	0.198	0.194	0.191	0.186
30 years	1.03	0.171	0.216	0.205	0.196	0.193	0.189	0.184

Term	ATM Strike	0.5	1	1.5	2	3	5	10
1 year	-0.28	0.129	0.167	0.198	0.224	0.265	0.323	0.408
1.5 years	-0.28	0.131	0.165	0.192	0.215	0.251	0.302	0.375
2 years	-0.28	0.132	0.159	0.182	0.200	0.229	0.270	0.327
3 years	-0.11	0.144	0.17	0.192	0.211	0.242	0.286	0.348
4 years	-0.04	0.155	0.172	0.188	0.202	0.226	0.261	0.31
5 years	0.04	0.167	0.178	0.188	0.198	0.215	0.242	0.281
6 years	0.14	0.175	0.181	0.187	0.193	0.207	0.229	0.262
7 years	0.25	0.181	0.182	0.185	0.189	0.198	0.216	0.245
8 years	0.36	0.184	0.183	0.184	0.185	0.191	0.205	0.23
9 years	0.47	0.186	0.183	0.182	0.182	0.185	0.196	0.218
10 years	0.56	0.187	0.183	0.18	0.179	0.18	0.188	0.207
12 years	0.73	0.187	0.181	0.176	0.173	0.172	0.174	0.187
15 years	0.89	0.185	0.178	0.172	0.168	0.164	0.161	0.166
20 years	1.01	0.182	0.175	0.169	0.164	0.158	0.151	0.149
25 years	1.04	0.181	0.173	0.167	0.162	0.155	0.147	0.141
30 years	1.03	0.18	0.172	0.166	0.16	0.153	0.145	0.136

Table A.4: *Expiry and maturity dates for 6 month caplets in the cap with a maturity of 10 years. Note that by the construction of caps (where for instance the 4 year cap is included in the 5 year cap) this will automatically give information about expiry and maturity for caplets in caps with maturity less than 10 years.*

Expiry Date	Maturity Date
2016-09-29	2017-03-29
2017-03-30	2017-09-27
2017-09-28	2018-03-27
2018-03-28	2018-09-26
2018-09-27	2019-03-27
2019-03-28	2019-09-26
2019-09-27	2020-03-29
2020-03-30	2020-09-28
2020-09-29	2021-03-29
2021-03-30	2021-09-28
2021-09-29	2022-03-29
2022-03-30	2022-09-28
2022-09-29	2023-03-29
2023-03-30	2023-09-27
2023-09-28	2024-03-26
2024-03-27	2024-09-26
2024-09-27	2025-03-27
2025-03-28	2025-09-28
2025-09-29	2026-03-29

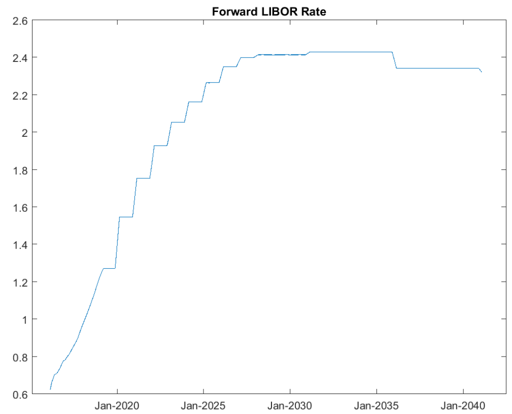
Table A.5: *This table is a part of the analysis of the correlation coefficent ρ, made in section 7.4. Transformations from 3 month caplets to 6 month caplets are made for three different values of $\rho \in [0.85, 0.9, 0.95]$. Rows that are colored in light grey , corresponds to caps with smaller strikes, and the ones colored in a darker tone , corresponds to caps with larger strikes. Both strikes and volatilities have the unit percentage.*

ρ	Term	ATM Strike	ATM	-0.75	-0.5	-0.25	-0.13	0	0.25
	1 year	-0.28	0.0704	0.1428	0.1046	0.0738	0.0709	0.0800	0.1049
0.85	1.5 years	-0.28	0.0852	0.1426	0.1085	0.0818	0.0796	0.0876	0.1097
	2 years	-0.28	0.1036	0.1428	0.1140	0.0917	0.0897	0.0963	0.1143
ρ	Term	ATM Strike	0.5	1	1.5	2	3	5	10
	1 year	-0.28	0.1278	0.1661	0.1970	0.2228	0.2639	0.3222	0.4066
0.85	1.5 year	-0.28	0.1301	0.1639	0.1910	0.2136	0.2496	0.3005	0.3738
	2 years	-0.28	0.1309	0.1584	0.1805	0.1988	0.2280	0.2686	0.3260

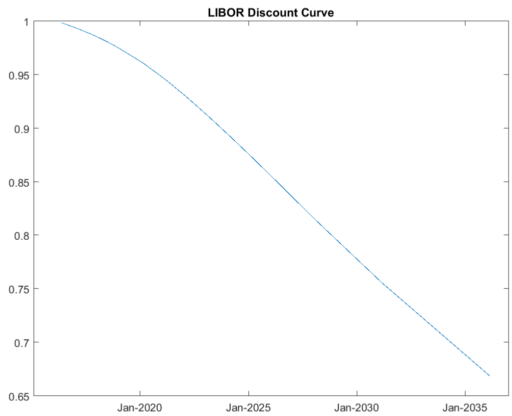
ρ	Term	ATM Strike	ATM	-0.75	-0.5	-0.25	-0.13	0	0.25
	1 year	-0.28	0.0713	0.1438	0.1055	0.0747	0.0719	0.0809	0.1057
0.90	1.5 years	-0.28	0.0863	0.1436	0.1095	0.0829	0.0807	0.0886	0.1105
	2 years	-0.28	0.1050	0.1439	0.1151	0.0929	0.0909	0.0975	0.1153
ρ	Term	ATM Strike	0.5	1	1.5	2	3	5	10
	1 year	-0.28	0.1286	0.1670	0.1979	0.2236	0.2648	0.3232	0.4078
0.90	1.5 years	-0.28	0.1310	0.1648	0.1920	0.2146	0.2507	0.3016	0.3749
	2 years	-0.28	0.1318	0.1594	0.1815	0.1999	0.2291	0.2697	0.3271

ρ	Term	ATM Strike	ATM	-0.75	-0.5	-0.25	-0.13	0	0.25
	1 year	-0.28	0.0722	0.1446	0.1063	0.0756	0.0728	0.0816	0.1064
0.95	1.5 years	-0.28	0.0875	0.1446	0.1104	0.0839	0.0816	0.0896	0.1114
	2 years	-0.28	0.1064	0.1451	0.1162	0.0940	0.0921	0.0985	0.1163
ρ	Term	ATM Strike	0.5	1	1.5	2	3	5	10
	1 year	-0.28	0.1294	0.1678	0.1987	0.2245	0.2658	0.3242	0.4089
0.95	1.5 years	-0.28	0.1318	0.1657	0.1929	0.2155	0.2517	0.3026	0.3760
	2 years	-0.28	0.1329	0.1604	0.1826	0.2010	0.2302	0.2708	0.3282

A.2 USD



(a) *3 month forward LIBOR on the USD market.*



(b) *Discont curve constructed from the 3 month forward LIBOR.*

Figure A.2: *The forward LIBOR curve (left) with corresponding discount curve (right). The rate is not negative, and therefore is the discount rate is less than one. The curves are collected February 22th 2016.*

Table A.6: *Black cap volatilities on the USD market, for maturities up to 20 years. The volatilities are based on a 3 month tenor. Both the strikes and the volatilities have unit percentage. The data set is downloaded from ICAP, February 22th 2016.*

BLACK CAP VOLATILITIES

Term	ATM Strike	ATM	0.25	0.5	0.75	1	1.5	2
1 year	0.72	0.603	1.044	0.726	0.596	0.587	0.541	0.506
2 years	0.79	0.761	1.197	0.906	0.773	0.711	0.631	0.585
3 years	0.88	0.840	1.488	1.081	0.898	0.797	0.672	0.595
4 years	0.99	0.843	1.674	1.169	0.957	0.838	0.690	0.601
5 years	1.12	0.786	1.640	1.154	0.952	0.830	0.678	0.583
6 years	1.23	0.737	1.662	1.142	0.940	0.817	0.664	0.568
7 years	1.33	0.692	1.642	1.125	0.922	0.800	0.648	0.554
8 years	1.42	0.654	1.641	1.108	0.906	0.785	0.634	0.541
9 years	1.49	0.624	1.677	1.097	0.895	0.772	0.622	0.530
10 years	1.57	0.595	1.746	1.088	0.882	0.759	0.610	0.518
12 years	1.69	0.539	1.737	1.026	0.830	0.715	0.576	0.491
15 years	1.82	0.490	2.098	0.998	0.793	0.680	0.546	0.465
20 years	1.96	0.441	2.779	0.913	0.714	0.611	0.491	0.418

Term	ATM Strike	ATM	2.5	3	4	5	6	7
1 year	0.72	0.603	0.509	0.510	0.506	0.509	0.512	0.513
2 years	0.79	0.761	0.554	0.530	0.499	0.485	0.486	0.473
3 years	0.88	0.840	0.546	0.515	0.485	0.477	0.477	0.481
4 years	0.99	0.843	0.542	0.504	0.465	0.450	0.445	0.446
5 years	1.12	0.786	0.519	0.477	0.430	0.410	0.402	0.401
6 years	1.23	0.737	0.504	0.460	0.410	0.388	0.379	0.377
7 years	1.33	0.692	0.489	0.444	0.392	0.369	0.360	0.358
8 years	1.42	0.654	0.477	0.432	0.379	0.355	0.345	0.343
9 years	1.49	0.624	0.466	0.422	0.369	0.344	0.333	0.331
10 years	1.57	0.595	0.455	0.411	0.358	0.332	0.321	0.318
12 years	1.69	0.539	0.433	0.391	0.339	0.311	0.298	0.293
15 years	1.82	0.490	0.410	0.369	0.319	0.293	0.280	0.275
20 years	1.96	0.441	0.367	0.331	0.284	0.259	0.248	0.243

Appendix B

Calibration of SABR Model to USD Caps

The calibration results for USD caps are presented in this chapter. In order to calibrate a SABR model to a cap market, one has to make some preparatory transformations of the implied flat volatilities (cap volatilities) in order to get the caplet volatilities. As explained in Chapter 5, there are different methods for obtaining caplet volatilities. The results that are presented below are based on the three different approaches which we explained in Chapter 5. We end the chapter by analysing the methods based on the quality measures in Figure 1.2.

The data that calibrations will be fitted to, is presented in Table A.6. However, we will only include caps with maturities up to 10 years respectively caps with strikes up to 5%. The reason for this is that we do not want to include caps that are "too much" OTM. The lowest strike in the data set, 0.25%, is also neglected because a preparatory analysis showed that these caps were not trustworthy.

B.1 Method 1: Bootstrap

The caplet stripping method is the first approach that is discussed in this chapter. Firstly, forward cap prices are constructed in the same spirit as explained in section 5.1. On each interval, the caplet volatilities are backed out and the smile model SABR is fitted to the obtained caplet volatilities.

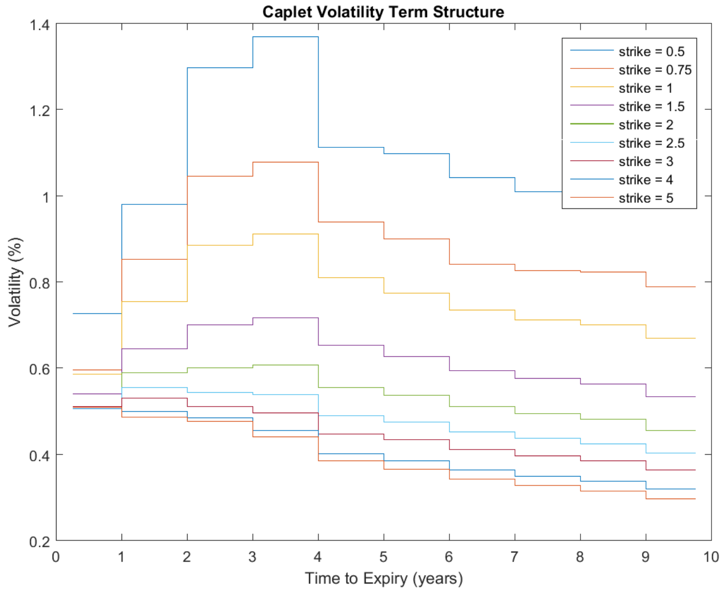


Figure B.1: *Caplet volatilities resulting from backing out volatilities from forward cap prices.*

The ATM strikes are not yet included. The proposed method of including ATM strikes (cf. section 5.1) is used when ATM caps are included into the calibration. However, this approach increases the price errors on the ATM strikes. Hence, we decide not to include the ATM volatilities into the calibration. The SABR parameters resulting from the bootstrap approach, without ATM strikes, are presented in Figure B.2. The caplet volatility surface that corresponds to the callibrated set of SABR parameters are found in Figure B.3.

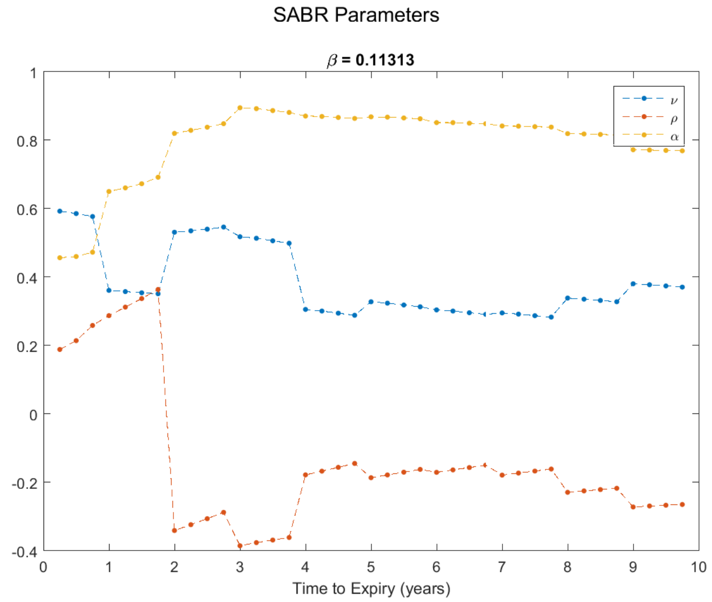


Figure B.2: *Estimated SABR parameters for USD market with caplet maturity up to 10 years.*

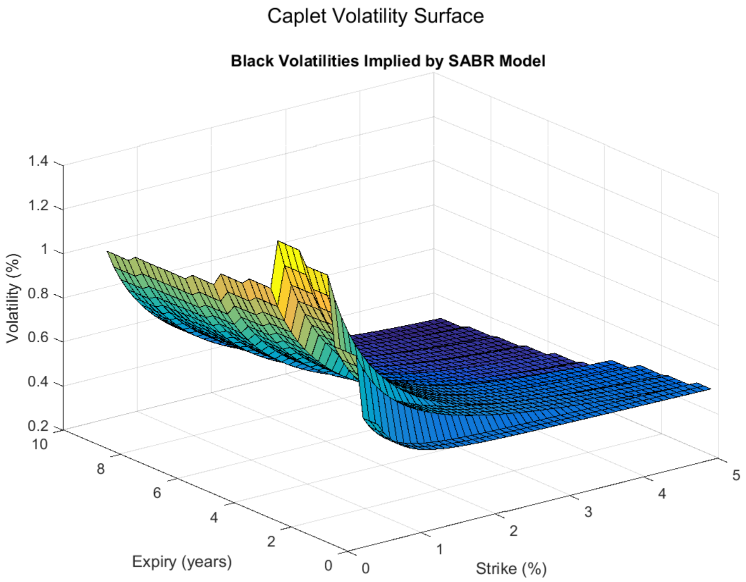
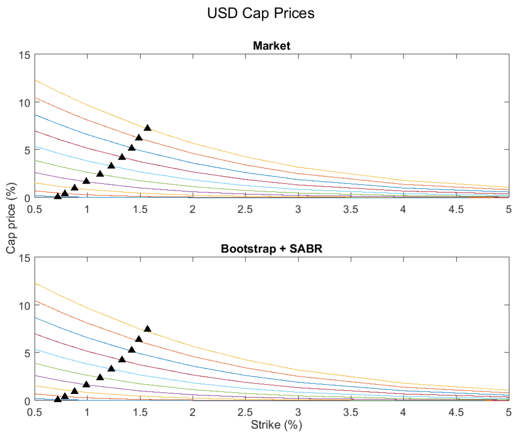
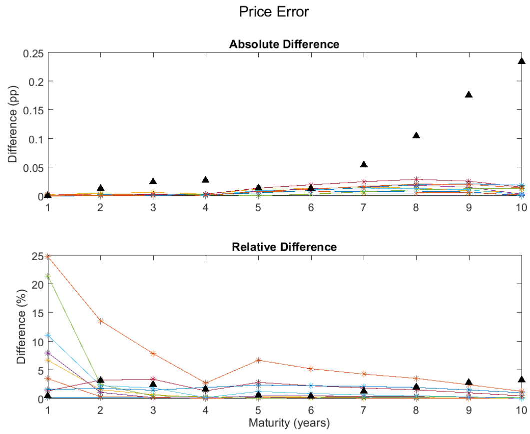


Figure B.3: *Caplet volatility surface that is based on the bootstrap approach. The assumption that the caplet volatilities are piecewise constant is reflected in the VTS.*

The cap prices are reconstructed using the Black caplet volatilities that resulted from the SABR calibration. We are comparing the reconstructed prices to the ones on the market, and also computing the absolute and relative price errors. This is illustrated in Figure B.4 and in Table D.4.



(a) *Cap prices on USD market (top) and corresponding cap prices that are reconstructed by the calibrated SABR model (bottom). Each line corresponds to a maturity (1 - 10 years).*



(b) *Absolute errors (top) between market and model prices. Each line corresponds to a fixed strike. One should notice that the absolute errors are increasing for larger maturities. Relative errors (bottom) between market and model are taking their largest values for short cap maturities.*

Figure B.4: *In the left figure, market and model prices are calculated for caps on the USD market. To the right, the corresponding relative and absolute errors are calculated and illustrated. At-the-money caps are represented by black triangles.*

B.2 Method 2: Rebonato

Rebonato's function for instantaneous forward rate volatility (5.3), is fitted to market prices. The optimization process of the Rebonato parameters follows the flow chart that is illustrated in Figure 5.2. For each fixed strike, the Rebonato parameters are estimated and the integrated caplet volatility $\Sigma^2(T, K)$, is calculated (cf. (5.2)). The estimated parameters are found in Appendix C in Table C.2. The corresponding integrated caplet volatilities are illustrated in Figure B.5.

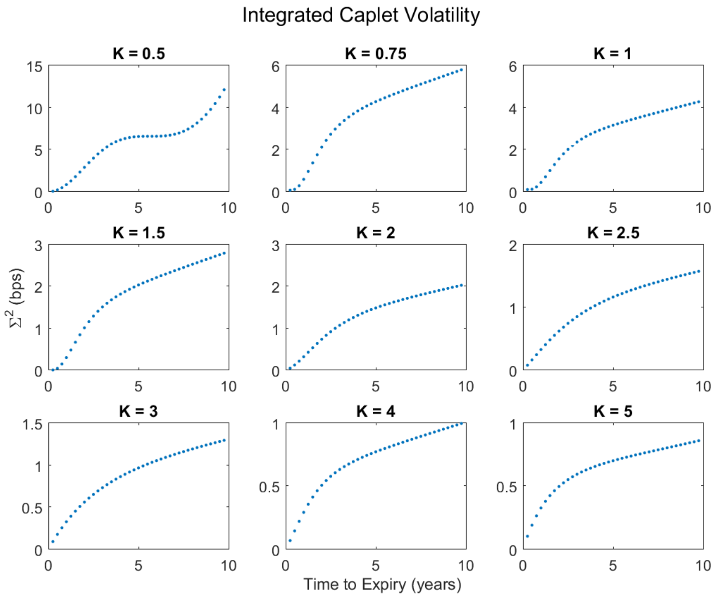


Figure B.5: *Estimated Rebonato integrated caplet volatilities for each strike (K). This is the up integrated caplet volatility that is inserted to the Black type pricing formula 3.7, in order to receive the model cap price from constituent caplet prices.*

Furthermore, the obtained integrated caplet volatilities are used for fitting the SABR model (3.8). The SABR parameters are estimated in the way that are described in Figure 5.3. So for each fixed maturity, the smile model parameters are estimated, with one exception for β which is estimated globally. The parameter estimates are presented in Figure B.6.

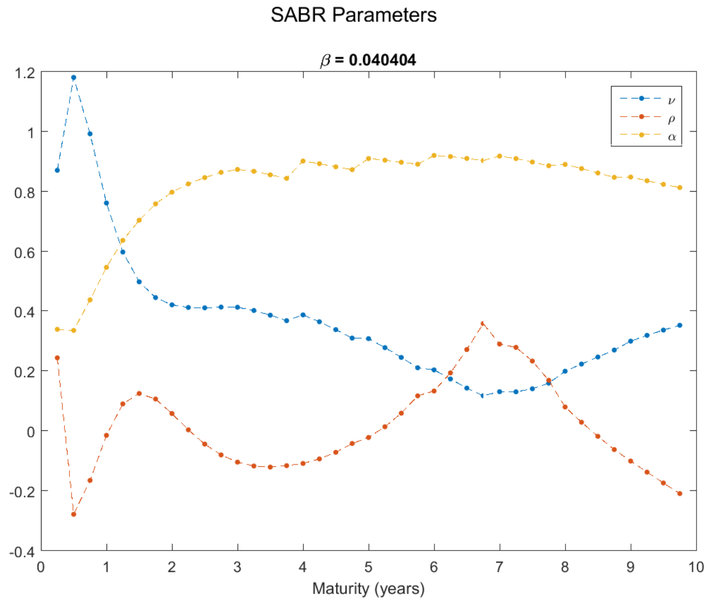


Figure B.6: *Estimated SABR parameters for USD market.*

The caplet volatility surface that is derived from the SABR parameters is illustrated in Figure B.7. The hump in the VTS for low strikes is distinct at about expiry 2 years.

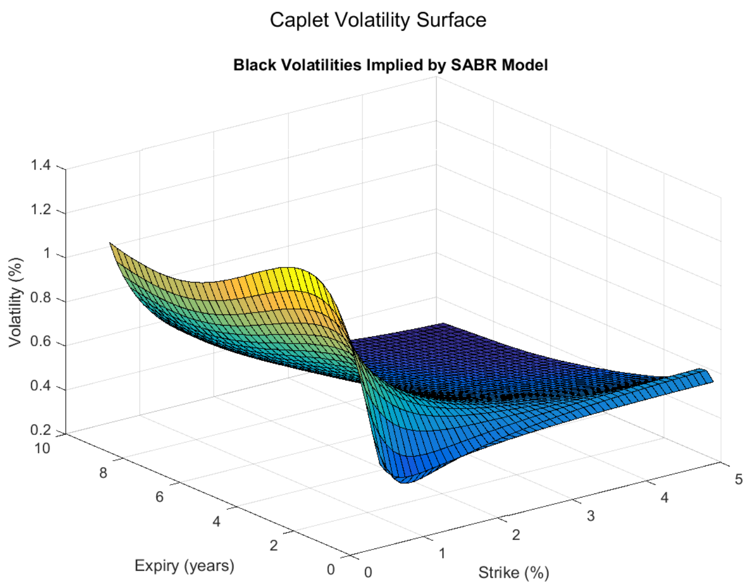
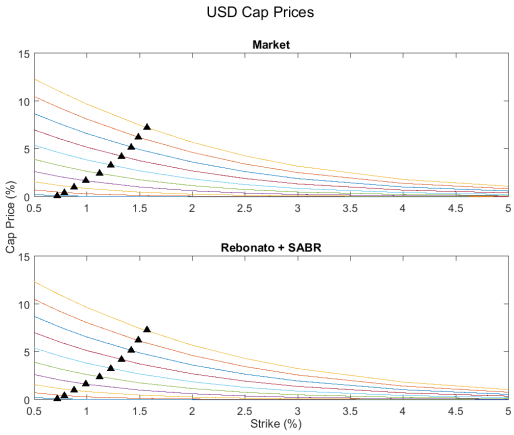
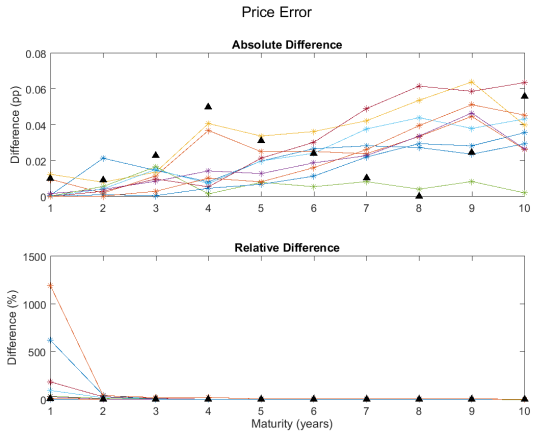


Figure B.7: *Caplet volatility surface for Black volatilities implied by SABR parameters. The volatility smiles are found by looking at the Volatility-Strike plane and the volatility term structure is illustrated in the Volatility-Maturity plane.*

In order to get some sense of how well the methods work, we are reconstructing the prices from the SABR parameters that we received by calibration with Rebonato's function. In Figure B.8a, the reconstructed prices are plotted together with the original market prices. We are also calculating the absolute and relative errors between the model and market prices, which are presented in Figure B.8b. The exact values can be found in Table D.5 in Appendix D.



(a) *Cap prices from USD market (top) and reconstructed prices from SABR model (bottom). Each line corresponds to a cap maturity. The lowest price corresponds then to the cap with maturity 1 year, and the highest price corresponds to the cap with maturity 10 years.*



(b) *Absolute difference between model and market prices (top), where the errors are growing for larger maturities. Relative errors between model and market prices (bottom). We can observe a clear peak for caps with maturity 1 year.*

Figure B.8: *Illustration of how well the model reconstruct the prices. To the left are cap prices from model and market plotted, and to the right are the price errors shown. At the money caps are represented by black triangles.*

B.3 Method 3: Global SABR

The results presented in this section are based on the Global SABR method which is explained in Figure 5.4. This approach does not need an initial step of first fitting a caplet volatility term structure, however it needs some manually tuning on the choice of knots in the term structure of the SABR parameters. Furthermore, knot points are tuned by comparing resulting price errors. The number of knots and where they are placed for this study are presented in Table B.1.

Table B.1: *Presentation of the chosen knots and corresponding interpolation methods used in the Global SABR method. The placement column hace unit years from today.*

KNOTS FOR GLOBAL SABR METHOD

Parameter	Number of Knots	Placement	Interpolation Method
ν	6	$1, 2, 3, 5, 7, 10$	"pchip"
α	6	$1, 2, 3, 5, 7, 10$	"pchip"
ρ	3	$0, 5, 10$	"pchip"

With the above knot specifications we go on to calibrate the global SABR model as described in Figure 5.4. The obtained optimal knot values and the parameter term structures are shown in Figure B.9.

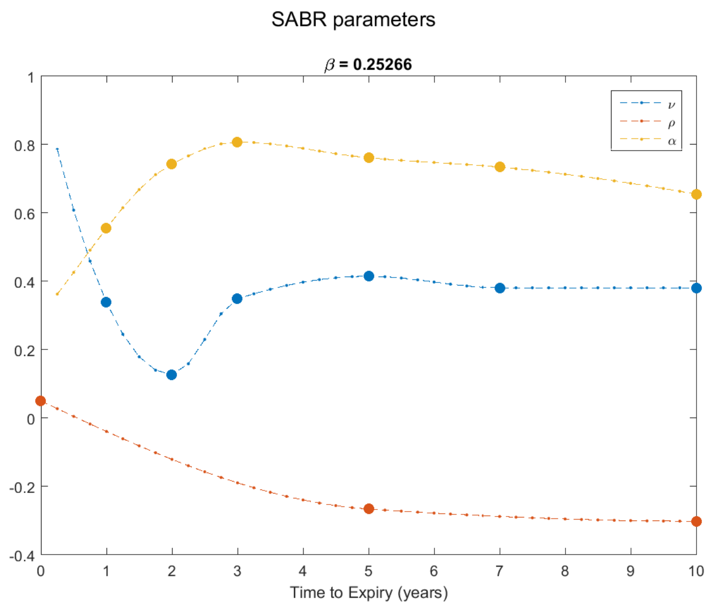


Figure B.9: *Calibrated SABR parameters using global SABR calibration. The knot placements are illustrated by large dots. Interpolation between knot values is done using piecewise cubic Hermite polynomial interpolation.*

From the obtained set of SABR parameters the resulting caplet volatility surface is constructed. This is shown in Figure B.10.

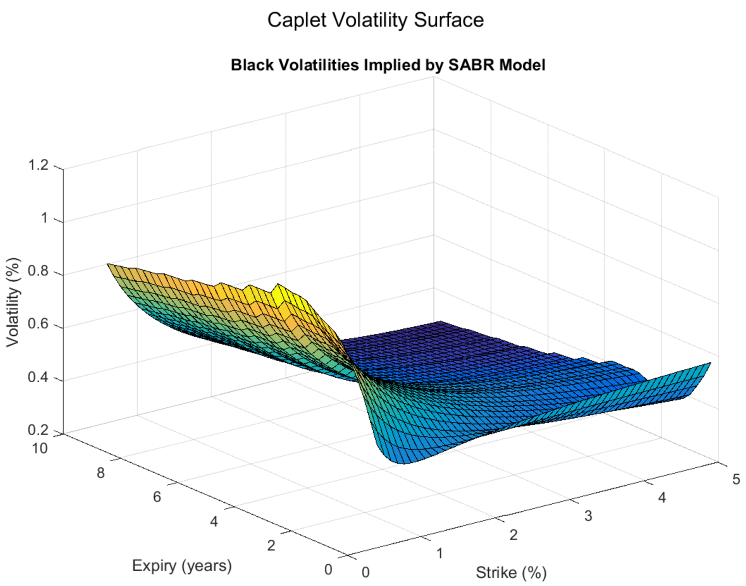
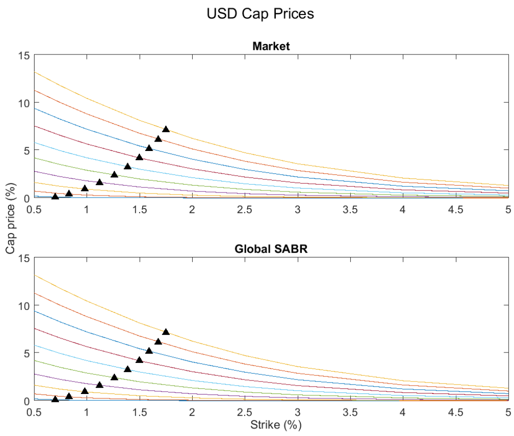
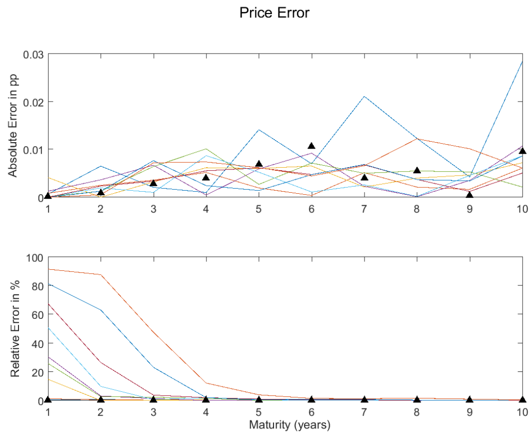


Figure B.10: *Caplet volatility surface for Black volatilities implied by the calibrated SABR model.*

We then use the caplet volatility surface from the SABR model to reconstruct cap prices. The cap prices from the model are compared to market cap prices in Figure B.11. In this figure we also show the absolute and relative errors between model and market prices. Complementing results can be found in Appendix D in Table D.6.



(a) *Cap prices from USD market (top) and reconstructed prices from SABR model (bottom). Each line corresponds to a cap maturity. The lowest price corresponds then to the cap with maturity 1 year, and the highest price corresponds to the cap with maturity 10 years.*



(b) *Absolute difference between the model and the market prices (top), where the errors are growing for larger maturities. Relative errors between model and market prices (bottom). Each line coresond to a fixed market strike and maturities for caps is between 1 and 10 year.*

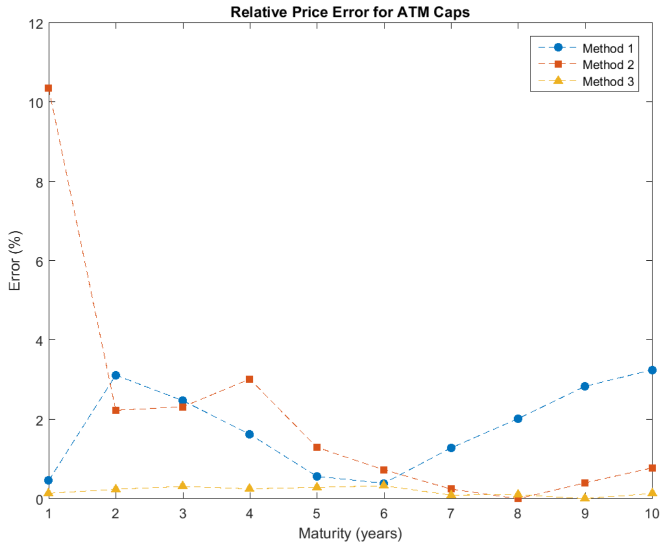
Figure B.11: *Illustration of how well the model reconstruct the prices. To the left are cap prices from model and market plotted, and to the right are the price errors presented. At-the-money caps are represented by black triangles.*

B.4 Summary of SABR Calibration to USD Caps

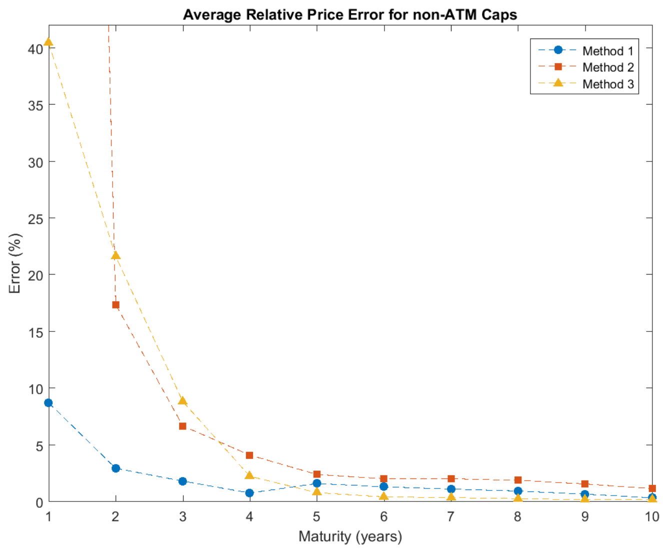
In this section we will summarize the calibration results on USD caps. A more in deep analysis of the methods are presented on the European caps, in section 8.4.

First, their ability to fit ATM cap prices will be studied. The relative errors to ATM market cap prices are shown in Figure B.12a. Here it is seen that the global SABR calibration (method 3) outperforms both bootstrap (method 1) and Rebonato (method 2). This may be explained by the fact that ATM cap prices are easily incorporated into the calibration of this model whereas bootstrap and Rebonato do not use ATM caps. Further more, global SABR uses an interpolation method, which probably is preferable when dealing with a short tenor (3 months) and a course term grid (1 year = 4 caplets), as for bootstrap caplet volatilities will be constant on each term interval.

In Figure B.12b we have tried to boil down the overall relative price error of the methods into one single measure. Because a fix cap maturity offers non-ATM caps with different strikes, we have chosen to calculate the average relative error for each cap maturity. From the figure we find Rebonato to be worst while Bootstrap and global SABR perform similar with Bootstrap being better to calibrate to short maturities caps and global SABR for long maturity caps.



(a) *ATM relative errors between market prices and model prices generated by each method. Method 1 and method 2 are performing poorly in comparison to method 3.*



(b) *Average relative price errors, calculated for each cap maturity. Method 1 outperforms the other two methods, for maturities up to four years. From 5 years and forward, method 3 is the best method considering this criteria.*

Figure B.12: *Presentation of how well the three methods (Bootstrap, Rebonato and Global SABR) are performing considering the two quality meassures, ATM errors (top) and average errors (bottom).*

A more detailed view of the relative errors can be found in Figures B.4b, B.8b, and B.11b and in Appendix D. The overall trend in relative error is that all methods have some problems to match prices of caps with short maturities and large strikes. Although, this may seem as a very undesirable property we believe this has to do with the fact that most of these caps are out of the money and illiquid. The reason for this is that their strikes are well above the ATM strikes. Due to their short time to maturity, it is unlikely that the LIBOR will increase that much. Hence, few people believe it is worth investing in these caps. As a consequence, these caps cost almost nothing and the relative errors become very sensitive.

In addition, there is a large difference between the relative errors. Especially Rebonato produces extremely high errors for caps with short maturity and high strikes. This can also be seen in Appendix D in Table D.5. We think this is partly due to the common issue mentioned above, but also due to the different shape of the volatility surface in Figure B.7. If we compare this to the volatility surfaces of method 1 and 3 (Figure B.3 and B.10), we observe that for caplets with low expiries and high strikes, Rebonato has a different shape with a positive hump instead of a flat or negative hump.

Appendix C

Parameter Estimates (Rebonato)

Table C.1: *Rebonato parameters calibrated to the European market. There is one set of parameters (a,b,c,d) for each fixed strike.*

Parameter	-0.75	-0.5	-0.25	-0.13	0	0.25
a	-0.1012	-0.1012	-0.3682	-0.0679	-0.2279	-0.1479
b	-3.0193	-3.0193	-0.4515	-2.3005	0.2573	-2.4293
c	3.4317	3.4317	1.8852	2.7435	0.3273	2.9527
d	0.2755	0.2755	0.2531	0.2443	0.0484	0.2254

Parameter	0.5	1	1.5	2	3
a	-0.4582	-0.3528	-0.4956	-0.1023	0.2823
b	0.2411	-0.8807	0.9322	0.3441	-0.0168
c	0.6655	7.9189	1.9050	0.9918	0.8658
d	0.1888	0.1890	0.1660	0.1392	0.1236

Table C.2: *Rebonato parameters calibrated to USD cap prices, on set of parameters for each fixed strike.*

Parameter	0.5	0.75	1	1.5	2	2.5	3	4	5
a	3.9560	-1.5617	-1.5751	-0.7545	0.0791	0.3135	0.3996	0.2971	0.4714
b	2.6075	3.4081	3.1250	1.9967	0.6359	0.2331	-0.0356	0.5269	-0.0989
c	0.2900	1.0649	1.0796	1.0398	0.7283	0.5448	0.2065	0.9807	0.3084
d	-3.6354	0.5462	0.4697	0.3877	0.3070	0.2503	0.2275	0.2118	0.2051

Appendix D

Calibration Errors

In this section tables with calibration errors are presented. In the tables, relative errors between 1% and 5% are marked with , errors $> 5\%$ are marked with and relative errors $< 1\%$ are unmarked.

D.1 EUR

Table D.1: *Errors between reconstructed cap prices from shifted Black volatilities implied by the shifted SABR model and the EUR market prices. The shift is set to 3% and the method used for calibration is method 1, bootstrap. The absolute errors are quoted in percentage points, and the relative errors are quoted in percentage.*

Bootstrap + SABR

Absolute errors

Term	ATM Strike	ATM	-0.75	-0.5	-0.25	-0.13	0	0.25	0.5	1	1.5	2	3
1 year	-0.28	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.5 years	-0.28	0.0001	0.0001	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001	0.0001	0.0000
2 years	-0.28	0.0001	0.0002	0.0001	0.0002	0.0001	0.0001	0.0001	0.0000	0.0001	0.0002	0.0002	0.0001
3 years	-0.11	0.0004	0.0035	0.0036	0.0019	0.0001	0.0018	0.0007	0.0001	0.0005	0.0002	0.0001	0.0000
4 years	-0.04	0.0004	0.0068	0.0064	0.0017	0.0019	0.0007	0.0000	0.0030	0.0011	0.0009	0.0012	0.0018
5 years	0.04	0.0029	0.0051	0.0052	0.0021	0.0052	0.0033	0.0003	0.0034	0.0002	0.0003	0.0017	0.0014
6 years	0.14	0.0069	0.0104	0.0082	0.0046	0.0021	0.0030	0.0033	0.0056	0.0020	0.0016	0.0029	0.0063
7 years	0.25	0.0017	0.0129	0.0111	0.0105	0.0017	0.0021	0.0003	0.0042	0.0076	0.0032	0.0014	0.0047
8 years	0.36	0.0038	0.0168	0.0101	0.0115	0.0018	0.0075	0.0012	0.0043	0.0084	0.0032	0.0039	0.0064
9 years	0.47	0.0016	0.0139	0.0019	0.0055	0.0089	0.0085	0.0025	0.0030	0.0104	0.0016	0.0000	0.0058
10 years	0.56	0.0025	0.0126	0.0044	0.0134	0.0054	0.0006	0.0112	0.0012	0.0000	0.0092	0.0025	0.0075

Relative errors

Term	ATM Strike	ATM	-0.75	-0.5	-0.25	-0.13	0	0.25	0.5	1	1.5	2	3
1 year	-0.28	0.0029	0.0050	0.0081	0.0327	0.0556	0.0019	1.2000	3.7141	9.0696	16.1243	23.9236	43.1093
1.5 years	-0.28	0.0577	0.0130	0.0074	0.0615	0.0097	0.1111	0.0440	0.5117	2.4612	5.5484	9.5171	18.4506
2 years	-0.28	0.0458	0.0211	0.0099	0.0785	0.0586	0.1473	0.3438	0.0532	1.7582	4.2833	7.6554	15.5476
3 years	-0.11	0.0950	0.2102	0.3296	0.3269	0.0300	0.6179	0.4054	0.0657	0.8296	0.5300	0.4279	0.0000
4 years	-0.04	0.0499	0.2513	0.3321	0.1386	0.1970	0.0937	0.0027	0.8338	0.5495	0.6494	1.2767	3.3015
5 years	0.04	0.2072	0.1285	0.1756	0.0986	0.2983	0.2235	0.0311	0.4213	0.0366	0.0847	0.7162	1.0520
6 years	0.14	0.3253	0.1879	0.1902	0.1416	0.0738	0.1217	0.1769	0.3747	0.2038	0.2478	0.6361	2.3713
7 years	0.25	0.0578	0.1753	0.1858	0.2248	0.0410	0.0568	0.0089	0.1729	0.4732	0.2869	0.1772	1.0098
8 years	0.36	0.0978	0.1787	0.1293	0.1815	0.0308	0.1464	0.0293	0.1235	0.3436	0.1810	0.3063	0.8513
9 years	0.47	0.0330	0.1191	0.0189	0.0672	0.1194	0.1246	0.0442	0.0618	0.2990	0.0629	0.0026	0.5199
10 years	0.56	0.0416	0.0901	0.0371	0.1326	0.0584	0.0072	0.1535	0.0187	0.0010	0.2723	0.0965	0.4886

Table D.2: *Errors between reconstructed cap prices from shifted Black volatilities implied by the shifted SABR model and the EUR market prices. The shift is set to 3% and the method used for calibration is method 2, Rebonato. The absolute errors are quoted in percentage points, and the relative errors are quoted in percentage.*

REBONATO + SABR

Absolute errors

Term	ATM Strike	ATM	-0.75	-0.5	-0.25	-0.13	0	0.25	0.5	1	1.5	2	3
1 year	-0.28	0.0052	0.0005	0.0019	0.0037	0.0041	0.0053	0.0024	0.0010	0.0002	0.0001	0.0000	0.0000
1.5 years	-0.28	0.0135	0.0033	0.0068	0.0124	0.0050	0.0023	0.0020	0.0017	0.0009	0.0005	0.0003	0.0001
2 years	-0.28	0.0116	0.0016	0.0017	0.0113	0.0075	0.0075	0.0069	0.0046	0.0017	0.0005	0.0000	0.0003
3 years	-0.11	0.0096	0.0198	0.0129	0.0054	0.0095	0.0080	0.0006	0.0022	0.0023	0.0008	0.0002	0.0016
4 years	-0.04	0.0210	0.0290	0.0194	0.0176	0.0202	0.0200	0.0128	0.0126	0.0088	0.0070	0.0072	0.0075
5 years	0.04	0.0095	0.0176	0.0124	0.0105	0.0107	0.0169	0.0077	0.0075	0.0018	0.0035	0.0041	0.0090
6 years	0.14	0.0086	0.0038	0.0074	0.0065	0.0004	0.0017	0.0066	0.0001	0.0049	0.0035	0.0016	0.0009
7 years	0.25	0.0137	0.0154	0.0289	0.0277	0.0094	0.0085	0.0123	0.0159	0.0028	0.0050	0.0058	0.0012
8 years	0.36	0.0195	0.0180	0.0331	0.0328	0.0131	0.0217	0.0168	0.0114	0.0079	0.0186	0.0066	0.0008
9 years	0.47	0.0059	0.0080	0.0106	0.0120	0.0090	0.0081	0.0048	0.0016	0.0008	0.0138	0.0121	0.0070
10 years	0.56	0.0211	0.0254	0.0220	0.0146	0.0391	0.0335	0.0159	0.0216	0.0040	0.0008	0.0052	0.0208

Relative errors

Term	ATM Strike	ATM	-0.75	-0.5	-0.25	-0.13	0	0.25	0.5	1	1.5	2	3
1 year	-0.28	7.5090	0.1717	1.0845	6.5017	18.8588	70.8738	147.8751	185.3628	205.8165	209.4251	212.3295	233.7123
1.5 years	-0.28	8.7143	0.5471	1.9349	9.4192	7.8311	7.3636	18.1604	30.4215	46.5124	54.2672	57.6726	58.4261
2 years	-0.28	4.3367	0.1752	0.3090	4.8162	5.6000	9.6643	19.9254	24.5101	23.0850	14.6448	1.6914	34.9997
3 years	-0.11	2.4305	1.1767	1.1654	0.9087	2.2718	2.7205	0.3550	1.9118	3.7163	2.0263	0.8451	9.1544
4 years	-0.04	2.6068	1.0749	1.0132	1.4520	2.1165	2.6713	2.5473	3.5672	4.3608	5.2807	7.7596	13.9108
5 years	0.04	0.6800	0.4417	0.4156	0.5023	0.6074	1.1520	0.7144	0.9229	0.3565	1.0668	1.7531	6.8509
6 years	0.14	0.4043	0.0686	0.1718	0.2014	0.0142	0.0710	0.3507	0.0037	0.5143	0.5446	0.3423	0.3254
7 years	0.25	0.4649	0.2092	0.4854	0.5932	0.2281	0.2327	0.4162	0.6604	0.1715	0.4463	0.7218	0.2528
8 years	0.36	0.4984	0.1912	0.4240	0.5186	0.2306	0.4215	0.3956	0.3238	0.3221	1.0571	0.5164	0.1090
9 years	0.47	0.1203	0.0688	0.1081	0.1469	0.1209	0.1191	0.0840	0.0339	0.0217	0.5443	0.6410	0.6323
10 years	0.56	0.3517	0.1816	0.1838	0.1448	0.4203	0.3929	0.2173	0.3464	0.0867	0.0237	0.2044	1.3603

Table D.3: *Errors between reconstructed cap prices from shifted Black volatilities implied by the shifted SABR model and the EUR market cap prices. The absolute errors are quoted in percentage points, and the relative errors are quoted in percentage. The shift is set to 3%. The SABR parameters are calibrated using method 3, Global SABR.*

GLOBAL SABR

Absolute errors

Term	ATM Strike	ATM	-0.75	-0.5	-0.25	-0.13	0	0.25	0.5	1	1.5	2	3
1 year	-0.28	0.0005	0.0001	0.0001	0.0007	0.0013	0.0007	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000
1.5 years	-0.28	0.0001	0.0004	0.0004	0.0003	0.0006	0.0000	0.0003	0.0002	0.0000	0.0001	0.0001	0.0001
2 years	-0.28	0.0001	0.0018	0.0015	0.0000	0.0008	0.0016	0.0005	0.0006	0.0014	0.0015	0.0013	0.0010
3 years	-0.11	0.0005	0.0025	0.0033	0.0015	0.0003	0.0013	0.0007	0.0025	0.0040	0.0036	0.0031	0.0022
4 years	-0.04	0.0035	0.0080	0.0030	0.0022	0.0018	0.0036	0.0017	0.0033	0.0001	0.0016	0.0013	0.0008
5 years	0.04	0.0015	0.0076	0.0007	0.0023	0.0018	0.0051	0.0008	0.0007	0.0069	0.0050	0.0044	0.0004
6 years	0.14	0.0058	0.0106	0.0047	0.0003	0.0018	0.0005	0.0036	0.0021	0.0047	0.0039	0.0005	0.0024
7 years	0.25	0.0007	0.0063	0.0120	0.0081	0.0048	0.0051	0.0022	0.0031	0.0071	0.0037	0.0014	0.0036
8 years	0.36	0.0026	0.0091	0.0115	0.0092	0.0048	0.0046	0.0008	0.0043	0.0052	0.0065	0.0035	0.0030
9 years	0.47	0.0003	0.0073	0.0014	0.0009	0.0035	0.0033	0.0014	0.0039	0.0053	0.0062	0.0062	0.0000
10 years	0.56	0.0035	0.0025	0.0064	0.0102	0.0099	0.0054	0.0070	0.0029	0.0039	0.0025	0.0025	0.0016

Relative errors

Term	ATM Strike	ATM	-0.75	-0.5	-0.25	-0.13	0	0.25	0.5	1	1.5	2	3
1 year	-0.28	0.7098	0.0369	0.0714	1.2777	5.7987	9.9630	10.7268	11.1455	14.2004	21.7388	32.3152	63.6690
1.5 years	-0.28	0.0884	0.0674	0.1181	0.2258	0.9106	0.0146	2.5464	2.9384	0.1335	6.3973	14.8869	36.2797
2 years	-0.28	0.0519	0.1987	0.2738	0.0041	0.6155	2.1139	1.5905	3.4826	19.3613	39.3582	63.0286	120.4448
3 years	-0.11	0.1144	0.1512	0.3028	0.2529	0.0767	0.4314	0.3991	2.2258	6.3990	8.9538	10.9145	12.8898
4 years	-0.04	0.4331	0.2968	0.1562	0.1859	0.1902	0.4744	0.3394	0.9198	0.0536	1.2462	1.4448	1.4872
5 years	0.04	0.1104	0.1909	0.0231	0.1088	0.1000	0.3478	0.0759	0.0920	1.3780	1.5290	1.8814	0.2687
6 years	0.14	0.2763	0.1915	0.1094	0.0086	0.0623	0.0207	0.1910	0.1441	0.4925	0.6048	0.1133	0.8834
7 years	0.25	0.0250	0.0857	0.2017	0.1743	0.1151	0.1376	0.0739	0.1292	0.4437	0.3292	0.1678	0.7777
8 years	0.36	0.0666	0.0973	0.1470	0.1452	0.0846	0.0895	0.0193	0.1233	0.2109	0.3698	0.2717	0.4045
9 years	0.47	0.0064	0.0630	0.0141	0.0112	0.0470	0.0480	0.0237	0.0808	0.1531	0.2455	0.3281	0.0044
10 years	0.56	0.0582	0.0176	0.0536	0.1009	0.1069	0.0631	0.0962	0.0460	0.0842	0.0749	0.0969	0.1037

D.2 USD

Table D.4: *Errors between reconstructed cap prices from SABR model, that was calibrated by the caplet stripping approach, and the USD market prices. The absolute errors are qouted in percentage points, and the relative errors are quoted in percentage.*

BOOTSTRAP + SABR

Absolute errors

Term	ATM Strike	ATM	0.5	0.75	1	1.5	2	2.5	3	4	5
1 year	0.72	0.0004	0.0004	0.0030	0.0024	0.0004	0.0002	0.0000	0.0000	0.0000	0.0000
2 year	0.79	0.0131	0.0018	0.0015	0.0045	0.0013	0.0012	0.0005	0.0003	0.0000	0.0001
3 year	0.88	0.0244	0.0030	0.0029	0.0059	0.0008	0.0012	0.0025	0.0026	0.0004	0.0011
4 year	0.99	0.0270	0.0012	0.0004	0.0030	0.0014	0.0021	0.0007	0.0031	0.0019	0.0013
5 year	1.12	0.0136	0.0073	0.0074	0.0118	0.0084	0.0002	0.0093	0.0136	0.0050	0.0075
6 year	1.23	0.0128	0.0084	0.0086	0.0127	0.0122	0.0029	0.0111	0.0193	0.0093	0.0119
7 year	1.33	0.0539	0.0076	0.0045	0.0144	0.0161	0.0071	0.0124	0.0248	0.0145	0.0165
8 year	1.42	0.1045	0.0082	0.0053	0.0141	0.0185	0.0105	0.0129	0.0288	0.0195	0.0207
9 year	1.49	0.1759	0.0069	0.0056	0.0086	0.0151	0.0127	0.0106	0.0256	0.0210	0.0200
10 year	1.57	0.2346	0.0003	0.0017	0.0064	0.0029	0.0125	0.0022	0.0153	0.0185	0.0140

Relative errors

Term	ATM Strike	ATM	0.5	0.75	1	1.5	2	2.5	3	4	5
1 year	0.72	0.4608	0.2177	3.4804	6.6891	8.0004	21.3922	11.0543	1.3552	1.5977	24.7723
2 year	0.79	3.1192	0.2593	0.3320	1.5505	1.0888	2.4456	2.2652	3.2281	1.7807	13.6016
3 year	0.88	2.4750	0.1953	0.2547	0.6968	0.1657	0.4747	1.8635	3.3809	1.4674	7.8675
4 year	0.99	1.6279	0.0443	0.0189	0.1853	0.1384	0.3468	0.1864	1.3578	1.9657	2.7142
5 year	1.12	0.5670	0.1864	0.2309	0.4458	0.4745	0.0157	1.2547	2.8360	2.3360	6.7046
6 year	1.23	0.3911	0.1565	0.1887	0.3319	0.4534	0.1545	0.8811	2.2557	2.2504	5.2177
7 year	1.33	1.2824	0.1081	0.0748	0.2786	0.4281	0.2654	0.6539	1.8628	2.1275	4.2754
8 year	1.42	2.0199	0.0940	0.0700	0.2141	0.3749	0.2909	0.4920	1.5297	1.9409	3.5315
9 year	1.49	2.8366	0.0658	0.0604	0.1065	0.2442	0.2737	0.3107	1.0201	1.5178	2.4215
10 year	1.57	3.2531	0.0023	0.0153	0.0659	0.0392	0.2192	0.0513	0.4804	1.0309	1.2857

Table D.5: *Errors between reconstructed cap prices from Rebonato + SABR model and the USD market prices. The absolute errors are qouted in percentage points. and the relative errors are quoted in percentage.*

REBONATO + SABR

Absolute errors

Term	ATM Strike	ATM	0.5	0.75	1	1.5	2	2.5	3	4	5
1 year	0.72	0.0100	0.0008	0.0097	0.0124	0.0017	0.0003	0.0002	0.0001	0.0000	0.0000
2 year	0.79	0.0094	0.0214	0.0020	0.0080	0.0041	0.0057	0.0043	0.0030	0.0011	0.0003
3 year	0.88	0.0228	0.0146	0.0116	0.0137	0.0086	0.0167	0.0151	0.0098	0.0005	0.0030
4 year	0.99	0.0501	0.0082	0.0369	0.0408	0.0143	0.0015	0.0073	0.0055	0.0046	0.0103
5 year	1.12	0.0312	0.0198	0.0251	0.0338	0.0128	0.0080	0.0199	0.0215	0.0069	0.0082
6 year	1.23	0.0240	0.0267	0.0252	0.0363	0.0188	0.0055	0.0242	0.0303	0.0114	0.0162
7 year	1.33	0.0104	0.0284	0.0239	0.0423	0.0229	0.0084	0.0378	0.0491	0.0221	0.0262
8 year	1.42	0.0001	0.0274	0.0336	0.0538	0.0338	0.0041	0.0440	0.0617	0.0295	0.0396
9 year	1.49	0.0246	0.0237	0.0448	0.0640	0.0467	0.0084	0.0381	0.0588	0.0284	0.0513
10 year	1.57	0.0558	0.0295	0.0263	0.0400	0.0267	0.0021	0.0433	0.0636	0.0357	0.0455

Relative errors

Term	ATM Strike	ATM	0.5	0.75	1	1.5	2	2.5	3	4	5
1 year	0.72	10.3527	0.3735	11.2398	34.3669	30.2334	34.7932	94.9246	184.9160	620.0964	1191.5526
2 year	0.79	2.2333	3.0374	0.4338	2.7872	3.4401	11.0982	18.8384	28.9887	46.4570	41.4558
3 year	0.88	2.3200	0.9395	1.0047	1.6034	1.8617	6.7153	11.2460	12.9857	1.8722	21.9788
4 year	0.99	3.0191	0.3144	1.7764	2.4853	1.4220	0.2475	1.9792	2.4242	4.8900	21.4844
5 year	1.12	1.2974	0.5081	0.7781	1.2759	0.7290	0.6939	2.6937	4.4869	3.1749	7.3317
6 year	1.23	0.7322	0.4973	0.5521	0.9461	0.6994	0.2988	1.9234	3.5433	2.7434	7.1127
7 year	1.33	0.2466	0.4061	0.3967	0.8187	0.6083	0.3110	1.9961	3.6961	3.2432	6.8068
8 year	1.42	0.0013	0.3155	0.4420	0.8143	0.6858	0.1147	1.6833	3.2717	2.9391	6.7714
9 year	1.49	0.3969	0.2260	0.4838	0.7883	0.7559	0.1822	1.1143	2.3379	2.0459	6.2168
10 year	1.57	0.7741	0.2394	0.2404	0.4125	0.3565	0.0365	1.0153	1.9994	1.9853	4.1844

Table D.6: *Errors between reconstructed cap prices from global SABR model and the USD market prices. The absolute errors are qouted in percentage points, and the relative errors are quoted in percentage.*

GLOBAL SABR

Absolute errors

Term	ATM Strike	ATM	0.5	0.75	1	1.5	2	2.5	3	4	5
1 year	0.72	0.0001	0.0007	0.0009	0.0041	0.0013	0.0001	0.0001	0.0000	0.0000	0.0000
2 year	0.79	0.0009	0.0065	0.0025	0.0000	0.0037	0.0013	0.0020	0.0024	0.0013	0.0005
3 year	0.88	0.0029	0.0021	0.0036	0.0032	0.0067	0.0064	0.0010	0.0034	0.0077	0.0072
4 year	0.99	0.0040	0.0010	0.0052	0.0062	0.0005	0.0101	0.0087	0.0055	0.0024	0.0074
5 year	1.12	0.0069	0.0141	0.0020	0.0060	0.0059	0.0027	0.0052	0.0062	0.0014	0.0062
6 year	1.23	0.0106	0.0070	0.0004	0.0066	0.0092	0.0072	0.0011	0.0048	0.0047	0.0045
7 year	1.33	0.0040	0.0211	0.0052	0.0021	0.0023	0.0050	0.0026	0.0068	0.0069	0.0066
8 year	1.42	0.0055	0.0123	0.0021	0.0040	0.0001	0.0055	0.0002	0.0037	0.0037	0.0122
9 year	1.49	0.0003	0.0043	0.0017	0.0047	0.0035	0.0053	0.0046	0.0012	0.0034	0.0101
10 year	1.57	0.0096	0.0284	0.0061	0.0072	0.0107	0.0021	0.0087	0.0051	0.0086	0.0061

Relative errors

Term	ATM Strike	ATM	0.5	0.75	1	1.5	2	2.5	3	4	5
1 year	0.72	0.1379	0.3857	1.2354	14.7731	30.4421	25.7159	50.6810	67.3691	81.4153	91.4997
2 year	0.79	0.2417	0.9247	0.5623	0.0081	3.2075	2.7932	9.8604	26.4975	63.1289	87.7450
3 year	0.88	0.3116	0.1285	0.2975	0.3565	1.3330	2.3033	0.6563	3.7903	23.1366	47.5850
4 year	0.99	0.2516	0.0363	0.2337	0.3480	0.0420	1.4339	1.9676	1.9884	2.0296	12.1735
5 year	1.12	0.2881	0.3369	0.0560	0.2080	0.2992	0.2034	0.5903	1.0353	0.4961	4.0102
6 year	1.23	0.3283	0.1208	0.0078	0.1559	0.3070	0.3379	0.0752	0.4630	0.9011	1.4827
7 year	1.33	0.0947	0.2797	0.0787	0.0379	0.0551	0.1654	0.1189	0.4333	0.8173	1.3194
8 year	1.42	0.1070	0.1312	0.0259	0.0559	0.0023	0.1359	0.0064	0.1708	0.3057	1.6420
9 year	1.49	0.0053	0.0377	0.0167	0.0531	0.0521	0.1036	0.1192	0.0429	0.2089	0.9974
10 year	1.57	0.1342	0.2149	0.0520	0.0692	0.1317	0.0344	0.1830	0.1421	0.4166	0.4713

Appendix E

Double Quadratic Interpolation

Let us briefly review the double quadratic interpolation that will be used in the global SABR fit approach for interpolating between knots in the parameter term structure. Other applications of the method are illustrated by for instance Iwashita (2014), who uses it for smile interpolation.

Having a set of points (x_i, y_i) we fit a quadratic polynomial $f_i(x) = a_i + b_i x + c_i x^2$ for each consecutive triplet of data points such that $f_i(x_i) = y_i$, $f_i(x_{i+1}) = y_{i+1}$, and $f_i(x_{i+2}) = y_{i+2}$ are satisfied. Then for a given x such that $x_i < x < x_{i+1}$ the interpolant F_i is given by a linearly weighted sum between f_{i-1} and f_i

$$F_i(x) = \frac{x - x_i}{x_{i+1} - x_i} f_{i-1}(x) + \frac{x_{i+1} - x}{x_{i+1} - x_i} f_i(x).$$

Note that for the leftmost and rightmost intervals then the interpolants are a single quadratic polynomial. The interpolation method is not shape-preserving and in general the C^2 smoothness may be smeared out due to the weighting sum. However, the method is at least C^1 continuous everywhere and widely used due to its semi-local property and analytical expression of interpolant.

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Popular Science Summary

Negative Rates when dealing with Caps

Negative rates have become a more common view on the leading markets. This creates problems for financial institutions, as some of their existing models break down. We have analyzed three methods, used for calibration to caps, and developed a new technique that deals with some of the problems related to negative rates.

In June 2014 the European Central Bank (ECB) did something extremely unusual seen from a historical perspective: they applied negative deposit rates. For financial institutions and investors this was an entirely new situation. Historically, negative rates have been regarded as impossible. The economic intuition behind this is that negative rates actual means you have to pay for lending money.

The motivation for negative interest rates differs depending on the national bank. The main reason in the Eurozone and Sweden is to fight the growing threat of deflation. Policymakers want to make it less attractive to save money. As such lowering the interest rate on savings should potentially make it more attractive to invest the money instead.

Financial institutions use statistical models in different areas such as pricing and risk validation. An example is the SABR model which for a long time has been an invaluable tool for capturing the volatility smile and to price cap contracts not quoted in the market. However, the current negative rate environment in the European market has led to numerous challenges for financial institutions. One of the most problematic issues is that the SABR model used for volatility interpolation and extrapolation fails when rates are negative. A related issue is that SABR based techniques for transformation of cap volatilities, do not work anymore.

Our work[1] focuses on the pricing of cap contracts available on the European market. Caps are contracts based on a set of simpler contracts known as caplets. Each caplet provides the holder an insurance guaranteeing that the rate connected to the caplet

[1]The full version is called: *Negative Rates in a Multi Curve Framework - Cap Pricing and Volatility Transformation.*

won't exceed a predetermined level. Caps are quoted in terms of volatilities, which can be seen as a measure for the variability of the underlying rates. Moreover, EUR cap contracts are offered on different underlying rate index. Depending on the maturity of the cap, it may be based on 3 or 6 months rates. In addition, the discount curve used when pricing the caps, is not constructed from the underlying indexes. This is known as the multi curve setting and brings additional complexity to the calibration when trying to unify caps under the same index rate, i.e. *volatility transformation* to the same tenor.

A suggested solution to the problem is to use an extension of the SABR model, known as the shifted SABR. The extended model adds a positive shift to the negative rate, so that the shifted process is modeled by the ordinary SABR. In our study we use three different methods to show how to calibrate the shifted SABR model to cap volatilities based on 6 months EURIBOR. Nonetheless, market standard is to quote a mixture of cap volatilities, thus the volatilities have to be transformed to a common tenor before calibration. To this end we have developed a technique for volatility transformation in a multi curve framework when rates are negative. The concept is derived by applying Itô's formula on a arbitrage relation between shifted forward rates[2].

The results show that two of the methods for calibrating the shifted SABR model have a good fit to liquid contracts. Though, the methods vary in performance capturing cheap caps which are out of the money. Our developed volatility transformation technique also works well, and the sensitivity to the potentially unknown correlation between forward rates is low.

The implications of the study are that the extension of the SABR model to the shifted SABR model works fine, both in terms of pricing of caps and volatility transformation in a multi curve framework. Although, it comes to the cost of some additionally complexity of extending formulas and arbitrage relations, that used to hold in a positive rate environment.

[2]The derivation is explained in the full version report, in Ch. 7.

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