On Steady Ideal Flows with Nonvanishing Vorticity in Cylindrical Domains

Douglas Svensson Seth

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Lund University

Faculty of Science
Centre for Mathematical Sciences
Mathematics

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Om Stationära Ideala Flöden med Nollskild Vorticitet i Cylindriska Domäner

Douglas Svensson Seth

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Erik Wahlén

Abstract

A paper by Alber shows the existence of steady, inviscid incompressible flows with nonvanishing vorticity for general three-dimensional domains with smooth boundary. In this thesis we show the existence of such flows in cylindrical domains under the conditions that there is no flow through the side of the cylinder, nonzero flow into the cylinder at all points in the bottom, and nonzero flow out of the cylinder at all points in the top. The flow is constructed by adding a perturbation with nonvanishing vorticity to an already existing flow with no vorticity. To show that this indeed gives us another flow we use a fixed point argument. This can be done if we put certain restrictions on the boundary conditions that define the vorticity of the perturbation.

Populärvetenskaplig Sammanfattning

Ekvationerna som beskriver flöden i fluider kallas Navier-Stokes ekvationer och trots att de formulerades på 1800-talet är kunskapen om lösningar fortfarande långt ifrån fullständing. Forskningen som rör detta går ofta framåt genom att specialfall som förenklar ekvationerna undersöks. Så är även fallet i detta arbete. Antagenden som görs är att fluidens hastighet och trycket är konstant i tiden, att fluiden saknar viskositet (rör sig utan friktion) och att den är inkompressibel. Under dessa antaganden tillsammans med antagandet att vorticiteten (ett mått på rotationen i vätskan) är noll är mycket redan utrett. Om man däremot ställer kravet att vorticiteten inte ska vara noll finns desto mindre kunskap. Det har dock visats att i tre dimensioner existerar sådana flöden i områden vars rand är glatt, vilket i princip betyder att det inte finns några hörn eller skarpa kanter. Det som görs i detta arbetet är att visa att sådana flöden även existerar i cylindriska områden under antagandet att det inte sker något flöde genom manteln och att allt flöde in i cylindern sker genom botten och allt flöde ut ur cylindern sker genom toppen.

Contents

1 Introduction

We study the steady flow of an inviscid, incompressible medium through a cylinder $\Omega = U \times (0, L) \subset \mathbb{R}^2 \times \mathbb{R}$, where U is an open, bounded, simply connected subset of $\mathbb{R}^2$ with C^∞ boundary and $0 < L < \infty$ (see Figure 1). For functions and variables in $\mathbb{R}^n$ we use subscripts to denote the components, i.e. if $x \in \mathbb{R}^3$ then $x = (x_1, x_2, x_3)$. The cylinder is oriented so that its cross section with a plane given by $x_3 = l$, $0 \leq l \leq L$, is $U \times \{l\}$. To denote a cross section of this type for a particular x_3 we use $\Omega_{x_3} = U \times \{x_3\}$.

Mathematically, the problem of our interest is given by the incompressible Euler equations

$$(v \cdot \nabla)v + \nabla p = 0 \qquad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div} v = 0 \qquad \text{in } \Omega, \tag{1.2}$$

with boundary condition

$$n \cdot v = \phi \qquad \text{on } \partial\Omega, \tag{1.3}$$

where v is the velocity field of the flow, p is the pressure and n is the exterior unit normal. We will also assume that $\phi < 0$ on the bottom of the cylinder Ω_0, $\phi > 0$ on the top Ω_L and $\phi = 0$ on $\partial U \times (0, L)$. This means that the flow into the cylinder is limited to the bottom and that the flow out is limited to the top. The goal of this thesis is to construct flows with nonzero vorticity, that is, with $\operatorname{curl} v \not\equiv 0$.

We begin with a brief review of previous results in order to put the present contribution into context. Throughout this discussion, we replace the cylinder Ω by a smooth, simply connected, bounded domain $\Gamma \subset \mathbb{R}^3$. Surprisingly little is known about steady three-dimensional ideal flows with nonzero vorticity. In contrast, irrotational flows, characterized by $\operatorname{curl} v = 0$, are very well-understood. An irrotational flow in Γ can be described by a velocity potential Φ, with $\Delta\Phi = 0$ and $v = \nabla\Phi$. The harmonic function Φ is uniquely determined up to a constant by its normal derivative $\partial\Phi/\partial n$ on $\partial\Gamma$. Hence, v is completely specified by its normal component $\phi = v \cdot n$ on the boundary. Here, ϕ is an arbitrary, sufficiently smooth function satisfying the compatibility condition $\int_{\partial\Gamma} \phi \, dS = 0$. By using the relation

$$(v \cdot \nabla)v = \nabla \left(\frac{1}{2}|v|^2\right) - v \times \operatorname{curl} v$$

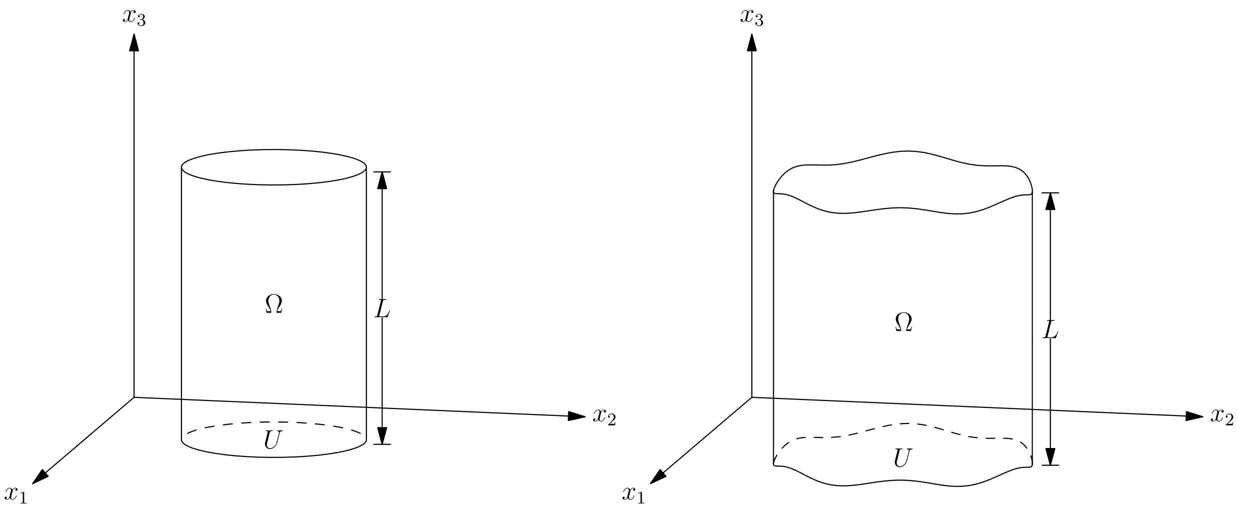
we can rewrite equation (1.1) as

$$\nabla \left(\frac{1}{2}|v|^2 + p\right) = v \times \operatorname{curl} v, \tag{1.4}$$

showing that the Euler equations are automatically satisfied for an irrotational flow if v is given by a harmonic potential and the pressure is defined by

$$p = -\frac{1}{2}|v|^2$$

(again, p is only unique up to an additive constant). This discussion implies that the normal component of the velocity field is not enough to uniquely determine the flow if we



(a) Standard circular cylinder. (b) Cylinder with an non-circular base.

Figure 1: Examples of a possible cylinder Ω.

allow for nonvanishing vorticity. The expression $H = \frac{1}{2}|v|^2 + p$ appearing above is called the Bernoulli function. Equation (1.4) says that the Bernoulli function is constant along integral curves of the flow and that it is identically constant throughout Γ if the flow is irrotational.

Steady flows with vorticity have mostly been studied in the two-dimensional setting, say with $v = (v_1(x_1, x_3), 0, v_3(x_1, x_3))$ (Γ now being unbounded and uniform in the x_2-direction). In this case, there is a stream function ψ, such that $(v_1, v_3) = (-\frac{\partial \psi}{\partial x_3}, \frac{\partial \psi}{\partial x_1})$ and ψ is constant on the integral curves of v. Moreover, the vorticity only has one nontrivial component, $\operatorname{curl} v = (0, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, 0)$, which is also constant on the integral curves of v. In the absence of stagnation points and closed integral curves this implies that there is a functional relationship between the vorticity and the stream function. This allows one to replace Euler's equations by a semilinear elliptic equation $\Delta \psi = F(\psi)$ for some function F (see e.g. [5]). The function ϕ is the tangential derivative of ψ along $\partial\Gamma$, so that ϕ determines ψ up to a constant for a given F (under some technical hypotheses).

However, one would also like to know if the flow can be determined uniquely only in terms of boundary conditions. It's not difficult to see that F will be determined if the nontrivial vorticity component is given on the inflow part of the boundary $\partial\Gamma_- = \{x \in \partial\Gamma \colon \phi(x) < 0\}$ (under some nondegeneracy conditions). Another possible choice is to specify the Bernoulli function H on $\partial\Gamma_-$, since equation (1.4) implies that $\nabla H \times n$ equals $\phi \operatorname{curl} v$ in view of the fact that $n \cdot \operatorname{curl} v = 0$. Both of these boundary conditions therefore give rise to well-posed boundary value problems. For a discussion of the nonuniqueness issues which arise when stagnation points and closed integral curves are allowed we refer to [24].

In the three-dimensional setting, the most well-studied solutions with vorticity are axisymmetric flows and Beltrami fields (or force-free fields)[1]. The axisymmetric case can however be reduced to a two-dimensional problem set in a cross-section of the fluid and again there is a simplified formulation in terms of the Stokes stream function. The Beltrami fields are on the other hand genuinely three-dimensional and are characterized by the condition $\operatorname{curl} v \parallel v$, so that H is constant throughout Γ. They are often associated with chaotic behavior, a famous example being the so-called ABC flows [5, Example II.1.9]. The boundary value problem for axisymmetric flows and Beltrami fields has for example been studied in [6, 7, 16]. We will return to this in Section 7.

The first general result on the well-posedness of the boundary value problem for three-dimensional flows with vorticity is due to Alber [3]. The boundary conditions that he imposed were the normal component $n \cdot v$ of the velocity field on $\partial\Gamma$ as well as the normal component $n \cdot \operatorname{curl} v$ of the vorticity and the Bernoulli function H on the inflow set $\partial\Gamma_-$. He showed that, given a background ideal flow which satisfies certain technical conditions, and boundary data which lie sufficiently close to those of the background flow, it is possible to construct a new flow with the given boundary data. He also proved a local stability and uniqueness result for the solutions. By choosing H to be nonconstant, the new solution is guaranteed to have nonzero vorticity. Alber's results and methods will be reviewed in more detail in Section 2.

Using similar ideas, Tang and Xin [22] later proved a modification of Alber's result in which the background flow was not assumed to satisfy Euler's equations and the boundary conditions on $\partial\Gamma_-$ were replaced by $\operatorname{curl} v = av + b$, where a is a given scalar function and b a given vector-valued function satisfying $b \cdot n = 0$ and the compatibility condition $\operatorname{div}_{\partial\Gamma}(\phi b) = 0$. Note that $n \cdot \operatorname{curl} v = \phi a$, so that the normal component of the vorticity is determined also by these boundary conditions. Moreover, H and b are related by the identity $\nabla_T H = -(\phi b) \times n$ on $\partial\Gamma_-$, where ∇_T denotes the tangential gradient.

In this thesis we use Alber's methods to study flows with nonzero vorticity in the cylinder Ω. However, the fact that $\partial\Omega$ has sharp edges introduces new difficulties. To intuitively understand these methods it is useful to reformulate the problem above in what is known as the velocity-vorticity formulation. Taking the curl of equation (1.4) we find that

$$\operatorname{curl}(v \times \operatorname{curl} v) = 0. \tag{1.5}$$

Using $\operatorname{div} v$ together with the identity

$$\operatorname{curl}(v \times z) = v \operatorname{div} z + (z \cdot \nabla)v - z \operatorname{div} v - (v \cdot \nabla)z$$

we get that (1.5) is equivalent to

$$(v \cdot \nabla)\operatorname{curl} v = [(\operatorname{curl} v) \cdot \nabla)]v. \tag{1.6}$$

For simply connected domains equations (1.6), (1.2), (1.3) are equivalent to (1.1)-(1.3).

To find a solution with nonvanishing vorticity we will use an irrotational solution (v_0, p_0) solving the boundary value problem above. Existence and properties of such a solution will

[1] The terminology is not completely standardized. In Section 7 we will use the term 'nonlinear Beltrami field' when v and $\operatorname{curl} v$ are everywhere parallel and 'linear Beltrami fields' for flows in which $\operatorname{curl} v$ is a constant multiple of v, while others use 'force-free fields' for the former and 'Beltrami fields' for the latter.

be outlined in Appendix A. We prove that there exists a neighborhood of (v_0, p_0) in which we can find a unique flow which satisfies (1.1)-(1.3) and the two additional boundary conditions from Alber's paper [3] on the inflow set Ω_0. We find this solution through introducing a operator B on the neighborhood of (v_0, p_0) following the technique used by Alber. Then we show that this operator has a unique fixed point which corresponds to a solution of the problem (1.1)-(1.3) and the additional boundary conditions.

In Section 3 we introduce B and state Theorem 3.1, which is our main result. In sections 4 and 5 we show that B is well-defined and finally in Section 6 we prove that B is a contraction which we use to show that it has a unique fixed point. In this section we also show that the fixed point corresponds to the desired solution. We end with some examples and some open questions in Sections 7 and 8. However, first we introduce Sobolev spaces to those who are unfamiliar with the concept and establish some necessary notation below.

1.1 Sobolev Spaces

When first learning about differential equations it is natural to assume that the solution of an n:th order differential equation is n times continuously differentiable. However, in modern mathematics this is not generally the case and weaker solutions are found in a class of function spaces called Sobolev spaces. Here is a brief introduction to these spaces given together with some well known connected results. For a more comprehensive source see e.g. Adams [1].

Let M be a bounded open subset of $\mathbb{R}^n$. We begin by a definition concerning the boundary of M.

Definition 1.1. *(i)* Let $j \in \mathbb{N}$. We say that the boundary ∂M is C^j, or that M is of *class* C^j, if for each point $x' \in \partial M$ there exist neighborhood V and, if necessary, new orthogonal coordinates $\{y_1, \cdots, y_n\}$ such that

$$V := \{y = (y_1, \cdots, y_n) \in \mathbb{R}^n : -a_i < y_i < a_i, 1 \leq i \leq n\}$$

and C^j function, $\psi : V' \to \mathbb{R}^n$ where

$$V' := \{y' = (y'_1, \cdots, y'_n - 1) \in \mathbb{R}^{n-1} : -a_i < y'_i < a_i, 1 \leq i \leq n - 1\}$$

with the properties

$$|\psi(y')| \leq \frac{a_n}{2}, \qquad \forall\, y' \in V',$$
$$M \cap V = \{y = (y', y_n) \in V : \psi(y') > y_n\},$$
$$\partial M \cap V = \{y = (y', y_n) \in V : \psi(y') = y_n\},$$

(ii) Likewise, we say that ∂M is C^∞ if $\psi \in C^\infty(V'; \mathbb{R})$, ∂M is $C^{k,\lambda}$ if $\psi \in C^{k,\lambda}(V'; \mathbb{R})$ for $0 < \lambda \leq 1$, that is k times λ-Hölder continuously differentiable, and ∂M is analytic if ψ is analytic. The special case where $\psi \in C^{0,1}(V'; \mathbb{R})$ is often called a *Lipschitz boundary.*

To continue we let α be a multi-index, that is $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}^n$ with $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$, and let $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$. Also let $k \in \mathbb{N}$ and $p \in \mathbb{R}$ with $p \geq 1$ or

$p = \infty$. If f is a locally integrable function on M and $\varphi \in C_c^\infty(M)$ we can make sense of the expression

$$\int_M f D^\alpha \varphi \, dx.$$

We say that g is the weak α-th partial derivative of f if it is locally integrable and

$$\int_M f D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_M g \varphi \, dx$$

holds for all $\varphi \in C_c^\infty(M)$. If there exists such a function g it is uniquely defined almost everywhere and if $f \in C^{|\alpha|}(M)$ it is clear that $D^\alpha f = g$, therefore it make sense to use the notation $D^\alpha f := g$ even if the derivative doesn't exist in the classical sense. Weak derivatives allows us to make the following definition.

Definition 1.2. The *Sobolev space* $W^{k,p}(M)$ is the space of functions in $L^p(M)$, with weak derivatives up to order k in $L^p(M)$, that is

$$W^{k,p}(M) := \{ f \in L^p(M) : D^\alpha f \in L^p(M) \; \forall |\alpha| \leq k \},$$

equipped with the norm

$$\|f\|_{W^{k,p}(M)} = \begin{cases} \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p(M)}^p \right)^{1/p} & \text{for } 1 \leq p < \infty, \\ \max_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty(M)} & \text{for } p = \infty. \end{cases}$$

When $p = 2$ the Sobolev spaces are usually denoted by $H^k(M)$.

We note here that the norms in the definition are not the only possible norms on the Sobolev spaces. However all commonly used norms are equivalent in the sense of norms and we will use the ones above. It is also clear from the definition that $W^{0,p}(M) = L^p(M)$.

While all the Sobolev spaces are Banach spaces the spaces with $p = 2$ are Hilbert spaces with inner product

$$\langle f, g \rangle_{H^k(M)} = \sum_{|\alpha| \leq k} \langle D^\alpha f, D^\alpha g \rangle_{L^2(M)}.$$

Another way to define the space $W^{k,p}(M)$ is through the completion of $\{ f \in C^\infty(M) : \|f\|_{W^{k,p}(M)} < \infty \}$ with respect to the norm $\|\cdot\|_{W^{k,p}(M)}$. If we instead look at the completion of $C_c^\infty(M)$ we get another space. To characterize this space we need the trace operator introduced through Theorem 1.4 below (Theorem 7.53 in [1]). However, to state the theorem we first need to define Sobolev spaces on the boundary of M.

Definition 1.3. Let the boundary of M be $C^{j,1}$, and $V_1, ..., V_q$ be a covering of ∂M of hyper cubes, with corresponding $C^{k,1}$ functions $\psi_1, ..., \psi_q$ as in Definition 1.1. Let $\varphi_1, ..., \varphi_q \in C_c^\infty(\mathbb{R}^n)$ be a partition of unity subordinate to $V_1, ..., V_q$, that is for all $1 \leq i \leq q$

$$0 \leq \varphi_i \leq 1,$$

$$\mathrm{supp}\,(\varphi_i) \subset V_i,$$

and

$$\sum_{i=1}^{q} \varphi_i = 1$$

in a neighborhood of ∂M. Let $\varphi_i' = \varphi_i|_{V_i \cap \partial M}$ and define $\Psi_i : V_i' \to V_i \cap \partial M$ by

$$\Psi_i(y_1, ..., y_{q-1}) = (y_1, ..., y_{q-1}, \psi_i(y_1, ..., y_{q-1})).$$

If f is a distribution defined on ∂M and $s \in \mathbb{R}$, with $|s| \leq j+1$. We say that $f \in W^{s,p}(\partial M)$ if $(\varphi_i' f) \circ \Psi_i \in W^{s,p}(V_i')$ ($W^{s,p}(V_i')$ with $s \neq \mathbb{N}$ is defined in Definition 1.10 below.) for all $0 \leq i \leq q$. The space $W^{s,p}(\partial M)$ can be equipped with the norm

$$\|f\|_{W^{s,p}(\partial M)} = \sum_{i=1}^{q} \|(\varphi_i' f) \circ \Psi_i\|_{W^{s,p}(V_i')}.$$

Theorem 1.4 (Trace theorem 1)**.** *Assume M is bounded with Lipschitz boundary. Then there exists a bounded linear operator $T : W^{1,p}(M) \to L^p(\partial M)$ such that*

$$\begin{aligned} &Tf = f|_{\partial M} && \text{for } f \in W^{1,p}(M) \cap C(\bar{M}), \\ &\|Tf\|_{L^p(\partial M)} \leq c\|f\|_{W^{1,p}(M)}. \end{aligned}$$

Definition 1.5. The completion of $C_c^\infty(M)$ with respect to the norm $\|\cdot\|_{W^{k,p}(M)}$ is the space $W_0^{k,p}(M)$. This spaces can be characterized by

$$W_0^{k,p}(M) = \{f \in W^{k,p}(M) : TD^\alpha f = 0 \ \forall |\alpha| \leq k-1\}.$$

Analogously to before we denote $W_0^{k,2}(M)$ as $H_0^k(M)$.

The Sobolev spaces can easily be extended to vector valued functions $f : M \to \mathbb{R}^m$ by letting the components of f be functions in $W^{k,p}(M)$, i.e. $f \in (W^{k,p}(M))^m$. If it is clear from context we denote this simply as $f \in W^{k,p}(M)$, but where we want to make it explicitly clear that f is vector valued with m components we use the notation $f \in W^{k,p}(M; \mathbb{R}^m)$. A convention we also use for other function spaces.

We will also use the Sobolev spaces mapping an interval of the real line, $[0,T]$, into a Banach space, X with norm $\|\cdot\|_X$. However we begin by introducing three other types of spaces first (see [9]).

Definition 1.6. The space $L^p([0,T]; X)$ consists of the functions which are strongly measurable functions $f : [0,T] \to X$ such that the norm

$$\|f\|_{L^p([0,T];X)} = \begin{cases} \left(\int_0^T \|f(t)\|_X^p \, dt \right)^{1/p} & \text{for } 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{t \in [0,T]} \|f(t)\|_X & \text{for } p = \infty \end{cases}$$

is finite.

Definition 1.7. The space $C([0,T];X)$ consists of all continuous functions $f:[0,T]\to X$ and is equipped with the norm

$$\|f\|_{C([0,T];X)} = \max_{t\in[0,T]} \|f(t)\|_X.$$

Definition 1.8. The space $C_w([0,T];X)$ consists of all weakly continuous functions $f:[0,T]\to X$, that is, functions f such that $t\mapsto l(f(t))$ is continuous for any bounded linear functional $l\in X'$.

For a function in $f\in L^p([0,T];X)$ we say that it has a weak derivative of the k:th order if there exists a function $g\in L^p([0,T];X)$ such that

$$\int_0^T f(t)D^k\varphi(t)\,dt = (-1)^k\int_0^T g(t)\varphi(t)\,dt$$

for all test functions $\varphi(t)\in C_c^\infty([0,T])$. We denote these weak derivatives as $f', f'', f''', \cdots$ for $k=1,2,3,\cdots$, respectively, or by $D^k f$.

Definition 1.9. The space $W^{k,p}([0,T];X)$ is the space of functions $f\in L^p([0,T];X)$ such that the weak derivatives of f up to order k exists as functions in $L^p([0,T];X)$. This space is equipped with the norm

$$\|f\|_{W^{k,p}([0,T];X)} = \begin{cases} \left(\int_0^T \sum_{i=0}^k \|D^i f\|_X^p\,dt\right)^p & \text{for } 1\le p<\infty, \\ \operatorname{ess\,sup}_{t\in[0,T]} \sum_{i=0}^k \|D^i f\|_X & \text{for } p=\infty. \end{cases}$$

As before $W^{k,2}([0,T];X) = H^k([0,T];X)$.

We will also need an extension of these spaces so we can replace k with a real number. For $1<p<\infty$ and $M=\mathbb{R}^n$ an equivalent definition of the Sobolev spaces is

$$W^{k,p}(\mathbb{R}^n) := \{f\in L^p(\mathbb{R}^n) : \mathcal{F}^{-1}[(1+|\xi|^2)^{k/2}\mathcal{F}f]\in L^p(\mathbb{R}^n)\},$$

where $\mathcal{F}$ denotes the Fourier transform. Using this we can make the following definition

Definition 1.10. Let $s\in\mathbb{R}$ and $1<p<\infty$.
(i) For $M=\mathbb{R}^n$ the *Bessel potential space* $W^{s,p}(\mathbb{R}^n)$ is

$$W^{s,p}(\mathbb{R}^n) = \{f\in \mathcal{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}[(1+|\xi|^2)^{s/2}\mathcal{F}f]\in L^p(\mathbb{R}^n)\},$$

where $\mathcal{S}'(\mathbb{R}^n)$ is the tempered distributions on $\mathbb{R}^n$ [21]. The Bessel potential space is equipped with norm

$$\|f\|_{W^{s,p}(\mathbb{R}^n)} = \|\mathcal{F}^{-1}[(1+|\xi|^2)^{s/2}\mathcal{F}f]\|_{L^p(\mathbb{R}^n)}.$$

(ii) For $M \neq \mathbb{R}^n$ the *Bessel potential space* $W^{s,p}(M)$ is the restriction of functions in $W^{s,p}(\mathbb{R}^n)$ to M, that is

$$W^{s,p}(M) = \{f \in \mathcal{D}'(\mathbb{R}^n) : \exists\, g \in W^{s,p}(\mathbb{R}^n), g|_M = f\},$$

where $\mathcal{D}'(\mathbb{R}^n)$ is the distributions on $\mathbb{R}^n$ [21]. The Bessel potential space is equipped with norm

$$\|f\|_{W^{s,p}(M)} = \inf\{\|g\|_{W^{s,p}(\mathbb{R}^n)} : g \in W^{s,p}(\mathbb{R}^n), g|_M = f\}.$$

Again $W^{s,2}(M) = H^s(M)$.

By $\|\cdot\|_{H^s(M)}$ we mean $\|\cdot\|_{W^{s,2}(M)}$ if s is a natural number and $\|\cdot\|_{W^{s,2}(M)}$ otherwise. Additionally, we let $\|\cdot\|_{s,M} := \|\cdot\|_{H^s(M)}$ and if we have a function $f : \Omega \to \mathbb{R}$ such that $f(\cdot, x_3) \in H^m(U)$ for a given x_3 we use the notation $\|f\|_{H^m(\Omega_{x_3})} = \|f(\cdot, x_3)\|_{H^m(U)}$.

Some of the results we will use are summarized below. The first of which can be found as Theorem 5.4 in [1].

Theorem 1.11 (Sobolev's embedding theorem)**.** *Suppose M is a Lipschitz domain.*
(i) If $mp > n$, then

$$W^{j+m,p}(M) \hookrightarrow C_b^j(M),$$

where

$$C_b^j(M) = \{f \in C^j(M) : D^\alpha f \text{ is bounded for } |\alpha| \leq j\},$$

with norm

$$\|f\|_{C_b^j(M)} = \max_{|\alpha|\leq j} \sup_{x\in M} |f(x)|.$$

An embedding in the sense that any function in $W^{j+m,p}(M)$ is a function in $C_b^j(M)$ and that there exists a constant $c^{(1)} = c^{(1)}(M)$ such that

$$\|f\|_{C_b^j(M)} \leq c^{(1)} \|f\|_{W^{j+m,p}(M)}.$$

for all $f \in W^{j+m,p}(M)$.
(ii) If $mp > n > (m-1)p$, then

$$W^{j+m,p}(M) \hookrightarrow C^{j,\lambda}(\bar{M})$$

for $0 < \lambda \leq n - \frac{m}{p}$ in the same sense as in (i).

We also use a similar result (see e.g. [9], Theorem 2 in Section 5.9.2).

Theorem 1.12. *Let $1 \leq p \leq \infty$ then*

$$W^{1,p}([0,T]; X) \hookrightarrow C([0,T]; X),$$

which means that any function in $W^{1,p}([0,T]; X)$ is a function in $C([0,T]; X)$ and that there exists a constant $c^{(2)} = c^{(2)}(T)$ such that

$$\|f\|_{C([0,T];X)} \leq c^{(2)} \|f\|_{W^{1,p}([0,T];X)}$$

for all $f \in W^{1,p}([0,T]; X)$.

We will also frequently use the following result (see e.g. [13], Theorem 8.3.1).

Theorem 1.13. *Let $f \in H^\mu(M)$ and $g \in H^\nu(M)$, and let $\kappa \leq \min\{\mu, \nu\}$ and $\kappa < \mu+\nu-\frac{n}{2}$. Then $fg \in H^\kappa(M)$ and there exists a constant $c^{(3)} = c^{(3)}(U)$ such that*

$$\|fg\|_{\kappa,M} \leq c^{(3)} \|f\|_{\mu,M} \|g\|_{\nu,M}.$$

We will also use a refined version of the trace theorem (see [1], Theorem 7.53, [8] and Remark 1.16).

Theorem 1.14 (Trace theorem 2)**.** *(i) Assume M is bounded with C^k boundary and let $m \in (\frac{1}{2}, k]$. Then there exists a bounded linear operator $T : H^m(M) \to H^{m-\frac{1}{2}}(\partial M)$ such that*

$$Tf = f|_{\partial M} \qquad \text{for } f \in H^m(M) \cap C(\bar{M}),$$
$$\|Tf\|_{H^{m-\frac{1}{2}}(\partial M)} \leq c\|f\|_{H^m(M)}.$$

(ii) Assume M is bounded with Lipschitz boundary and let $m \in (\frac{1}{2}, \frac{3}{2})$. Then there exists a bounded linear operator $T : H^m(M) \to H^{m-\frac{1}{2}}(\partial M)$ such that

$$Tf = f|_{\partial M} \qquad \text{for } f \in H^m(M) \cap C(\bar{M}),$$
$$\|Tf\|_{H^{m-\frac{1}{2}}(\partial M)} \leq c\|f\|_{H^m(M)}.$$

(iii) Assume $M = \Omega$, $m \geq 1$ and let

$$\hat{H}^s(\partial\Omega) = H^s(\Omega_0) \times H^s(\Omega_L) \times H^s(\partial U \times (0, L)),$$

with compatibility conditions on $\partial\Omega_0 \cap \partial(\partial U \times (0, L))$ and $\partial\Omega_L \cap \partial(\partial U \times (0, L))$ (depending on s). Then there exists a bounded linear operator $T : H^m(\Omega) \to H^{m-\frac{1}{2}}(\partial\Omega)$ such that

$$Tf = f|_{\partial\Omega} \qquad \text{for } f \in H^m(\Omega) \cap C(\bar{\Omega}),$$
$$\|Tf\|_{\hat{H}^{m-\frac{1}{2}}(\partial\Omega)} \leq c\|f\|_{H^m(\Omega)}.$$

Remark 1.15. The trace is generally not a mapping between Bessel potential spaces, but another generalization of the Sobolev spaces called Besov spaces. However, for $p = 2$ the Bessel potential spaces and Besov spaces are the same. Hence, the formulation of the trace theorem above holds for Bessel potential spaces.

Remark 1.16. Part *(iii)* is true since Ω can locally be transformed onto a polyhedron by smooth maps. To get an idea of the proof see e.g. Theorem 1.5.2.3 in [11], where it is shown for domains in $\mathbb{R}^2$. The compatibility conditions on the edges are left unspecified since they are not needed for our purpose.

2 Earlier Result by Alber

In the introduction we noted that many ideas used in the proof comes from a paper by Alber [3] in which he shows a similar result. The difference between our result and Alber's comes from Alber studying flow in general bounded, simply connected domains in $\mathbb{R}^3$ of class C^∞, while we study the flow in a cylinder. To understand Alber's result we have to introduce some notation concerning the domain, which we will call $\Gamma \subset \mathbb{R}^3$. For an example of such a domain we can consider, as suggested by Alber, a cylinder with rounded top and bottom, see Figure 2.

The problem Alber studies is

$$(v \cdot \nabla)v + \nabla p = 0 \qquad \text{in } \Gamma, \tag{2.1}$$

$$\operatorname{div} v = 0 \qquad \text{in } \Gamma, \tag{2.2}$$

$$n \cdot v = \phi \qquad \text{on } \partial\Gamma. \tag{2.3}$$

Since Γ is of class C^∞ there exists open subsets $U_1, \cdots U_\mu$ of $\mathbb{R}^3$ with $\partial\Gamma \subset \cup_{i=1}^{\mu} U_i$ and diffeomorphisms $\Phi_i : D_3 \to U_i$, where

$$D_l = \{y \in \mathbb{R}^l : |y| < 1\},$$

such that

$$U_i \cap \partial\Omega = \Phi_i(D_3 \cap \{x_3 = 0\})$$

and

$$U_i \cap \Omega = \Phi_i(D_3 \cap \{x_3 > 0\}).$$

The following definitions are slightly different than the ones given in Section 1.1, but they are the ones used in [3] and are therefore included here to make the results stated in this section as clear as possible.

$H^k(\partial\Gamma)$ denotes the trace space. The functions $\psi_i : D_2 \to \partial\Gamma$ with

$$\psi_i(\xi_1, \xi_2) = \Phi_i(\xi_1, \xi_2, 0)$$

define coordinate systems on $\partial\Gamma$. If we let $\sigma_i : \partial\Gamma \to \mathbb{R}$, $i = 1, \cdots \mu$, be a partition of unity on $\partial\Gamma$ with $0 \le \sigma \le 1$, support$(\sigma_i) \subset \psi_i(D_2)$, and $\sigma_i \circ \psi_i \in C_c^\infty(D_2)$ we can define

$$\|q\|_{k,\partial\Gamma} = \sum_{i=1}^{\mu} \sum_{|\alpha| \le k} \|(\sigma_i \circ \psi_i) D^\alpha (q \circ \psi_i)\|_{0,D_2}$$

as norm on $H^k(\partial\Gamma)$. For $\phi \in H^2(\partial\Gamma)$ let

$$\partial\Gamma_- = \partial\Gamma_-(\phi) = \{x \in \partial\Gamma : \phi(x) < 0\}$$
$$\partial\Gamma_+ = \partial\Gamma_+(\phi) = \{x \in \partial\Gamma : \phi(x) > 0\}.$$

Note that due to the assumptions we put on $\phi \in H^2(\partial\Omega)$ for the problem in the cylinder Ω, the sets corresponding to $\partial\Gamma_-$ and $\partial\Gamma_+$ are Ω_0 and Ω_L respectively, which are subsets

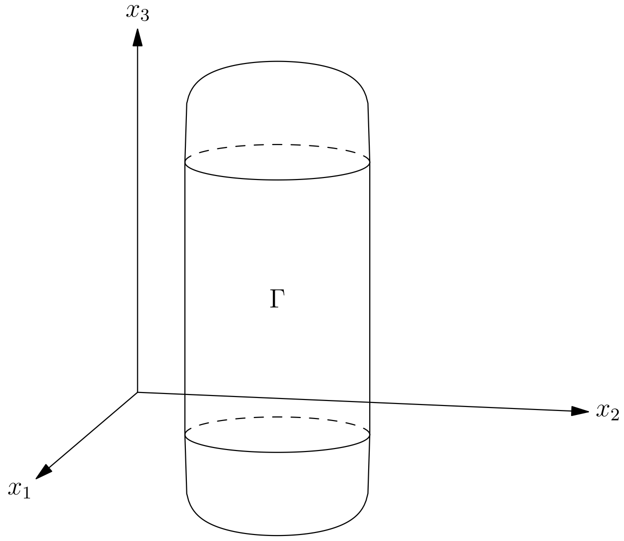


Figure 2: Example of a possible domain Γ.

of $\mathbb{R}^2$. This allows us to use the standard Sobolev space norms on these sets instead of the following norms used by Alber.

$\partial\Gamma_-, \partial\Gamma_+$ are open subsets of the C^∞-manifold $\partial\Gamma$ since ϕ is continuous and hence they themselves are C^∞-manifolds. The boundary of $\partial\Gamma_\pm$ relative to $\partial\Gamma$ is denoted by

$$\partial\partial\Gamma_\pm = \overline{\partial\Gamma_\pm} \cap (\overline{\partial\Gamma \setminus \partial\Gamma_\pm}).$$

We say that $\partial\Gamma_-$ has Lipschitz boundary if the functions $\Phi_i, \cdots \Phi_\mu$ can be chosen so that the domain

$$D_2^i = \psi_i^{-1}(\partial\Gamma_-)$$

is empty or has Lipschitz boundary for every $i = 1, \cdots, \mu$.

For $k \leq 2$ the norms on $\partial\Gamma_-$ are defined as

$$\|q\|_{k,\partial\Gamma_-} = \sum_{i=1}^{\mu} \sum_{|\alpha|\leq k} \|(\sigma_i \circ \psi_i) D^\alpha (q \circ \psi_i)\|_{0,D_2^i},$$

$$|q|_{k,\partial\Gamma_-} = \sum_{i=1}^{\mu} \sum_{|\alpha|+|\beta|\leq k} \left\|(\sigma_i \circ \psi_i) D^\alpha \left(\frac{1}{\phi \circ \psi_i}\right) D^\beta (q \circ \psi_i)\right\|_{0,D_2^i},$$

$$|||q|||_{k,\partial\Gamma_-} = \sum_{i=1}^{\mu} \sum_{|\alpha|+|\alpha'|+|\beta|\leq k} \left\|(\sigma_i \circ \psi_i) D^\alpha \left(\frac{1}{\phi \circ \psi_i}\right) D^{\alpha'} \left(\frac{1}{\phi \circ \psi_i}\right) D^\beta (q \circ \psi_i)\right\|_{0,D_2^i}$$

if the expressions are finite. The last two norms require q to vanish sufficiently fast at the boundary $\partial\partial\Gamma_-$ to be finite.

This allows us to express the main result in Alber's paper [3], which can be compared to our main result Theorem 3.1.

Theorem 2.1 (Theorem 1.1 in [3]). *Let Γ be a bounded simply connected domain of class C^∞. Assume that $\phi \in H^2(\partial\Gamma)$ satisfies*

$$\int_{\partial\Gamma} \phi(x)\, dS_x = 0$$

and is such that $\partial\Gamma_-(\phi)$ is a manifold with Lipschitz boundary.

Let $(v_0, p_0) \in H^3(\Gamma)$ be a solution of (2.1)-(2.3) *satisfying* $\operatorname{curl} v_0 \in H^3(\Gamma)$ *and*

$$\underline{v}_0 = \inf_{x\in\Gamma} |v_0(x)| > 0.$$

Moreover, assume that v_0 does not have closed integral curves and that the least upper bound L_0 of the length of all integral curves of v_0 is finite. Finally assume that there exist constants $\hat{c} > 0$, $\hat{t} > 0$ such that

$$\operatorname{dist}(\partial\Gamma_-(\phi), x + tv_0(x)) \geq \hat{c}t$$

for all $x \in \partial\partial\Gamma_-(\phi)$ and for all $0 \leq t \leq \hat{t}$, and

$$\operatorname{dist}(\partial\Gamma_+(\phi), x - tv_0(x)) \geq \hat{c}t$$

for all $x \in \partial\partial\Gamma_+(\phi)$ and for all $0 \leq t \leq \hat{t}$.

Then there exist constants

$$\bar{\gamma} = \bar{\gamma}(v_0, \Gamma) > 0,$$
$$\hat{K}_i = \hat{K}_i(L_0, \underline{v}_0, |v_0|_{3,\Gamma}, \phi, \bar{\gamma}, \Gamma) > 0,\ i = 1, 2, 3$$

with the following properties:

Let $g \in H^3(\partial\Gamma_-)$, $h \in H^2(\partial\Gamma_-)$ and v_0 satisfy

$$I(g, h, \operatorname{curl} v_0) \leq \hat{K}_1 \tag{2.4}$$

with

$$I(g,h,\operatorname{curl} v_0) = \left\|\frac{h}{\phi}\right\|_{2,\partial\Gamma_-} + \left|\left|\left|\frac{1}{\phi}\nabla_T g\right|\right|\right|_{2,\partial\Gamma_-} + |D^2 \operatorname{curl} v_0|_{0,\partial\Gamma_-}$$
$$+ \sum_{m=0}^{1} |||D^m \operatorname{curl} v_0|||_{2-m,\partial\Gamma_-}$$
$$+ \left|\left|\left|\frac{1}{\phi}(n\cdot \operatorname{curl} v_0)\right|\right|\right|_{2,\partial\Gamma_-} + |\operatorname{curl} v_0\|_3, \Omega.$$

Here $D^m \operatorname{curl} v_0$ denotes the vector

$$D^m \operatorname{curl} v_0 = (D^\alpha (\operatorname{curl} v_0)_j)_{\substack{j = 1,2,3 \\ |\alpha| \leq m}}.$$

$(\operatorname{curl} v_0)_j$ *are the components of* $\operatorname{curl} v_0$, *and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index. Then there exists a solution $(v, p) \in H^3(\Gamma)$ of* (2.1)-(2.3) *with*

$$n(x)\cdot \operatorname{curl} v(x) = h(x) + n(x)\cdot \operatorname{curl} v_0(x) \tag{2.5}$$

$$\frac{1}{2}|v(x)|^2 + p(x) = g(x) + \frac{1}{2}|v_0(x)|^2 + p_0(x) \tag{2.6}$$

for all $x \in \partial\Gamma_-$.

v satisfies

$$\|v - v_0\|_{3,\Gamma} \leq \bar{\gamma}, \tag{2.7}$$

and (v, p) is the only solution of (2.1)-(2.3), (2.5), (2.6) from $H^3(\Gamma)$ satisfying this estimate.

If $(g^{(1)}, h^{(1)})$ and $(g^{(2)}, h^{(2)})$ are two sets of boundary data on $\partial\Gamma_-(\phi)$ both satisfying (2.4), and if $(v^{(1)}, p^{(1)})$, $(v^{(2)}, p^{(2)})$ are solutions of (2.1)-(2.3), (2.5), (2.6) to the boundary data $(g^{(1)}, h^{(1)})$ and $(g^{(2)}, h^{(2)})$, respectively, both satisfying (2.7), then

$$\|v^{(1)} - v^{(2)}\|_{1,\Gamma} \leq \hat{K}_2(|h^{(1)} - h^{(2)}|_{0,\partial\Gamma_-} + |\nabla_T(g^{(1)} - g^{(2)})|_{0,\partial\Gamma-}),$$

$$\|p^{(1)} - p^{(2)}\|_{1,\Gamma} \leq \hat{K}_3(|h^{(1)} - h^{(2)}|_{0,\partial\Gamma_-} + |\nabla_T(g^{(1)} - g^{(2)})|_{0,\partial\Gamma-} + \|g^{(1)} - g^{(2)}\|_{0,\partial\Gamma_-}).$$

To prove this, an operator Λ is constructed on a subspace V of $H^3(\Gamma)$, which has a fixed point u such that $v = v_0 + u$ is the velocity field of a solution. The subspace V is defined by functions $w \in H^3(\Gamma)$ such that

$$\begin{aligned} \operatorname{div} w = 0 \quad & \text{in } \Gamma, \\ n \cdot w = 0 \quad & \text{on } \partial\Gamma. \end{aligned}$$

For $\gamma > 0$ we let V_γ be the closed ball in V of radius γ, that is, V_γ consists of functions $w \in V$ such that $\|w\|_{3,\Gamma} \leq \gamma$. Now let $W \in H^3(\Gamma)$ with $\operatorname{div} W = 0$ and let z be the solution of

$$\begin{aligned} [(v_0 + u) \cdot \nabla]\, z &= (z \cdot \nabla)(v_0 + u) - (u \cdot \nabla)W + (W \cdot \nabla)u \quad && \text{in } \Gamma, \\ z &= \eta && \text{on } \partial\Gamma_-, \end{aligned} \tag{2.8}$$

where η is given by

$$n \cdot \eta = h$$

and

$$\eta_T = \frac{h}{\phi}(v_0 + u)_T + \frac{1}{\phi}(n \cdot W)u_T - \frac{1}{\phi} n \times \nabla_T g$$

with ϕ, g and h from (2.3), (2.5) and (2.6). The operator Λ is defined as

$$\Lambda(u) = w$$

where

$$\operatorname{curl} w = z$$

To understand why the operator is defined this way compare equations (1.6) and (2.8) with W replaced by $\operatorname{curl} v_0$.

The solution z depends on g, h, W, v_0 and W, i.e. $z = z[g, h, W, v_0, u]$. The vector field W is to be replaced by $\operatorname{curl} v_0$ and the notation $z[g, h, v_0, u] = z[g, h, \operatorname{curl} v_0, v_0, u]$ is used. As a consequence Λ also depends on g, h, W and v_0 so we have $\Lambda = \Lambda[g, h, W, v_0]$. In a similar way as for z we also use the notation $\Lambda[g, h, v_0] = \Lambda[g, h, \operatorname{curl} v_0, v_0]$.

To show that this is a well-defined operator with a fixed point Alber [3] shows a sequence of lemmas and theorems, which we state here without proof.

Lemma 2.2 (Lemma 2.1 in [3])**.** *Let* $v_0 \in H^3(\Gamma)$ *satisfy the hypothesis of Theorem 2.1. Then there exist constants* $\hat{C}$ *and* $\gamma_0 > 0$ *with the following three properties*

(i) The vector field $v = v_0 + u$ *with* $u \in V_{\gamma_0}$ *satisfies*

$$\underline{v} = \inf_{x \in \Gamma} |v(x)| \geq \underline{v}_0 - \hat{C}\|u\|_{3,\Gamma} \geq \underline{v}_0 - \hat{C}\gamma_0 > 0.$$

(ii) No vector field $v \in v_0 + V_{\gamma_0}$ *has closed integral curves. For* $0 < \gamma \leq \gamma_0$ *let* L_γ *denote the least upper bound of the length of all integral curves of all the vector fields* $v \in v_0 + V_\gamma$*. Then* $L_\gamma < \infty$ *and*

$$\lim_{\gamma \to 0} L_\gamma = L_0$$

(iii) If an integral curve of $v \in v_0 + V_{\gamma_0}$ *is tangential to the boundary* $\partial\Gamma$ *at one point then it is completely contained in the boundary (see Remark 3.4, but substitute* Ω *with* Γ*).*

This lemma implies that every integral curve of a function $v \in v_0 + V_{\gamma_0}$ that passes over a point $x \in \Gamma$ meets the boundary once in $\partial\Gamma_-$, the starting point of the integral curve, once in $\partial\Gamma_+$, the endpoint of the integral curve, and in no other point of the boundary. It follows that Γ is completely covered by integral curves that start in $\partial\Gamma_-$ and end in $\partial\Gamma_+$. Together with the fact that (2.8) is an inhomogeneous linear system of ordinary differential equations for z along the integral curves of $v_0 + u$ with initial values at $\partial\Gamma_-$ this can be used to prove the following lemma.

Lemma 2.3 (Lemma 2.2 [3])**.** *For every* $u \in V_\gamma$ *with* $g \leq \gamma_0$ *and every* $W \in H^3(\Gamma)$ *with* $\operatorname{div} W = 0$ *the unique solution* z *of* (2.8) *exists in all of* Γ *and satisfies* $\operatorname{div} z = 0$.

These two lemmas above from Alber are comparable to our Lemma 3.3, Lemma 3.5, and Theorem 5.7. To show that Λ is well-defined one more result is needed.

Theorem 2.4 (Theorem 2.4 in [3])**.** *Let* $z \in H^2(\Gamma)$ *satisfy* $\operatorname{div} z = 0$ *and let* Γ *be a bounded, simply connected domain of class* C^∞*. Then there exists a unique function* $w \in H^3(\Gamma)$ *with*

$$\operatorname{curl} w = z \qquad \text{in } \Gamma,$$

$$\operatorname{div} w = 0 \qquad \text{in } \Gamma,$$

$$n \cdot w = 0 \qquad \text{on } \partial\Gamma.$$

Moreover, there exists a constant $\tilde{M}$*, only depending on* Γ*, such that*

$$\|w\|_{3,\Gamma} \leq \tilde{M}\|z\|_{2,\Omega}$$

We dedicate Section 4 to find a similar result which holds for our cylindrical domain and it can be found in Theorem 4.19.

It follows from the lemmas and theorem above that Λ is well-defined if the solution to (2.8) is a function in $H^2(\Gamma)$. This is shown through a series of estimates, which take up the major part of Alber's paper [3]. To show that Λ is a contraction the following estimates are proven.

Theorem 2.5 (Theorem 2.3 in [3]). *There exists a constant* $\tilde{M} = \tilde{M}(\Gamma) > 0$, *and to any* $\gamma \leq \gamma_0$ *constants* $\tilde{K}_i = \tilde{K}_i(L_\gamma, \underline{v}_0, \|v_0\|_{3,\Gamma}, \phi, \gamma, \Gamma) > 0$, $i = 1, 2, 3$, *which remain bounded for* $L_\gamma \to 0$, *such that for all* $u, w \in V_\gamma$

$$\|z[g,h,W,v_0,u]\|_{0,\Gamma} \leq L_\gamma^{1/2}\tilde{K}_1 \left(|h|_{0,\partial\Gamma_-} + |n \cdot W|_{0,\partial\Gamma_-} + |\nabla_T g|_{0,\partial\Gamma_-} + \|W\|_{3,\Gamma}\right),$$

$$\|z[g,h,v_0,u]\|_{2,\Gamma} \leq L_\gamma^{1/2}\tilde{K}_2 I(g,h,\operatorname{curl} v_0),$$

$$\|z[g,h,v_0,u] - z[g,h,v_0,w]\|_{0,\Gamma} \leq L_\gamma^{1/2}\tilde{K}_3 I(g,h,\operatorname{curl} v_0)\|u-w\|_{1,\Gamma},$$

and

$$\|B[g,h,W,v_0](u)\|_{1,\Gamma} \leq \tilde{M}L_\gamma^{1/2}\tilde{K}_1 \left(|h|_{0,\partial\Gamma_-} + |n \cdot W|_{0,\partial\Gamma_-} + |\nabla_T g|_{0,\partial\Gamma_-} + \|W\|_{3,\Gamma}\right),$$

$$\|B[g,h,v_0](u)\|_{3,\Gamma} \leq \tilde{M}L_\gamma^{1/2}\tilde{K}_2 I(g,h,\operatorname{curl} v_0),$$

$$\|B[g,h,v_0](u) - B[g,h,v_0](w)\|_{1,\Gamma} \leq \tilde{M}L_\gamma^{1/2}\tilde{K}_3 I(g,h,\operatorname{curl} v_0)\|u-w\|_{1,\Gamma},$$

with

$$\begin{aligned} I(g,h,\operatorname{curl} v_0) = {} & \left\|\frac{h}{\phi}\right\|_{2,\partial\Gamma_-} + \left|\left|\left|\frac{1}{\phi}\nabla_T g\right|\right|\right|_{2,\partial\Gamma_-} + |D^2\operatorname{curl} v_0|_{0,\partial\Gamma_-} \\ & + \sum_{m=0}^{1} |||D^m \operatorname{curl} v_0|||_{2-m,\partial\Gamma_-} \\ & + \left|\left|\left|\frac{1}{\phi}(n \cdot \operatorname{curl} v_0)\right|\right|\right|_{2,\partial\Gamma_-} + |\operatorname{curl} v_0\|_3, \Omega. \end{aligned}$$

Here $D^m\operatorname{curl} v_0$ *denotes the vector*

$$D^m \operatorname{curl} v_0 = \left(D^\alpha(\operatorname{curl} v_0)_j\right)_{\substack{j = 1,2,3 \\ |\alpha| \leq m}}.$$

$(\operatorname{curl} v_0)_j$ *are the components of* $\operatorname{curl} v_0$, *and* $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ *is a multi-index.*

Theorems 2.4 and 2.5 is then used to prove corollary below.

Corollary 2.6 (Corollary 2.5 in [3]). *For every* γ *with* $0 < \gamma \leq \gamma_0(v_0)$ *the operator* $\Lambda[g,h,v_0]$ *maps* V_γ *into itself if*

$$I(g,h,\operatorname{curl} v_0) \leq \frac{\gamma}{\tilde{M}L_\gamma^{1/2}\tilde{K}_2}. \tag{2.9}$$

The operator $\Lambda[g,h,v_0]$ *has a unique fixed point in* V_γ *if* (2.9) *is satisfied and if*

$$I(g,h,\operatorname{curl} v_0) \leq \frac{1}{\tilde{M}L_\gamma^{1/2}\tilde{K}_3}. \tag{2.10}$$

If $g^{(1)}$, $h^{(1)}$ *and* $g^{(2)}$, $h^{(2)}$ *are two sets of boundary data on* $\partial\Gamma_-(\phi)$, *both satisfying* (2.10), *and if* $u^{(1)}$, $u^{(2)} \in V_\gamma$ *are fixed points of* $B[g^{(1)}, h^{(1)}, v_0]$ *and* $B[g^{(2)}, h^{(2)}, v_0]$, *respectively, then*

$$\|u^{(1)} - u^{(2)}\|_{1,\Gamma} \leq \frac{\tilde{M}L_\gamma^{1/2}\tilde{K}_1 \left(|h^{(1)} - h^{(2)}|_{0,\partial\Gamma_-} + |\nabla_T(g^{(1)} - g^{(2)})|_{0,\partial\Gamma_-}\right)}{1 - \tilde{M}L_\gamma^{1/2}\tilde{K}_3 I(g^{(1)}, h^{(1)}, \operatorname{curl} v_0)}.$$

The final important piece in the proof of Theorem 2.1 is the following lemma.

Lemma 2.7 (Lemma 2.6 in [3])**.** *(i) Let* $u \in V_\gamma$ *with* $0 < \gamma \leq \gamma_0$. *Then* u *is a fixed point of* $B[g, h, v_0] : V_\gamma \to V$ *if and only if* $v = v_0 + u$ *is the velocity field of a solution* $(v, p) \in H^3(\Gamma)$ *of* (2.1)-(2.3), (2.5), (2.6)*.*

(ii) If (v, p),$(\tilde{v}, \tilde{p}) \in H^3(\Gamma)$ *are solutions of* (2.1)-(2.3), (2.5), (2.6) *with* $v = \tilde{v}$, *then also* $p = \tilde{p}$.

Also to these last three results we have comparable results of ours. Theorem 2.5 can be compared with Proposition 6.1 and Corollary 6.2, Corollary 2.6 with Proposition 6.3 and Lemma 2.7 with Lemma 6.4.

3 Main Result

What we ultimately want to prove is the following theorem.

Theorem 3.1. *Let $\Omega = U \times (0, L) \subset \mathbb{R}^2 \times \mathbb{R}$ where U is open, bounded and simply connected with a C^∞ boundary. Assume that $(v_0, p_0) \in H^3(\Omega)$ is a solution of* (1.1)-(1.3) *with* $\operatorname{curl} v_0(x) = 0$*, where ϕ satisfies*

$$\int_{\partial\Omega} \phi(x)\, dS_x = 0,$$

and there exists a constant $b > 0$ such that $\phi(x) \geq b$ for all $x \in \Omega_L$, $\phi(x) \leq -b$ for all $x \in \Omega_0$, and $\phi(x) = 0$ for all $x \in \partial U \times (0, L)$.

Then there exist constants

$$\hat{\gamma} = \hat{\gamma}(v_0, \Omega) > 0,$$
$$K_i = K_i(L_0, b, \|v_0\|_{3,\Omega}, \phi, \hat{\gamma}, \Omega) > 0, \qquad i \in \{1, 2, 3\}$$

with the following properties:

Let $g \in H_0^3(\Omega_0)$, $h \in H_0^2(\Omega_0)$ satisfy

$$\|h\|_{2,U} + \|\nabla_T g\|_{2,U} \leq K_1. \tag{3.1}$$

Then there exists a solution $(v, p) \in H^3(\Omega)$ of (1.1)-(1.3) *with ϕ as defined above, which also satisfies*

$$n \cdot \operatorname{curl} v = h \qquad \text{on } \Omega_0, \tag{3.2}$$
$$\frac{1}{2}|v|^2 + p = g + \frac{1}{2}|v_0|^2 + p_0 \qquad \text{on } \Omega_0. \tag{3.3}$$

v also satisfies

$$\|v - v_0\|_{3,\Omega} \leq \hat{\gamma}, \tag{3.4}$$

and (v, p) is the only solution of (1.1)-(1.3), (3.2) *and* (3.3) *in $H^3(\Omega)$ satisfying this estimate.*

Additionally, if $(g^{(1)}, h^{(1)})$ and $(g^{(2)}, h^{(2)})$ are two sets of boundary data on Ω_0 both satisfying (3.1) *and if $(v^{(1)}, p^{(1)})$, $(v^{(2)}, p^{(2)})$ are solutions of* (1.1)-(1.3), (3.2) *and* (3.3) *with boundary data $(g^{(1)}, h^{(1)})$ and $(g^{(2)}, h^{(2)})$, respectively, both satisfying* (3.4)*, then*

$$\|v^{(1)} - v^{(2)}\|_{1,\Omega} \leq K_2(\|h^{(1)} - h^{(2)}\|_{0,\Omega_0} + \|\nabla_T(g^{(1)} - g^{(2)})\|_{0,\Omega_0}) \tag{3.5}$$

and

$$\|p^{(1)} - p^{(2)}\|_{1,\Omega} \leq K_3(\|h^{(1)} - h^{(2)}\|_{0,\Omega_0} + \|\nabla_T(g^{(1)} - g^{(2)})\|_{0,\Omega_0} + \|g^{(1)} - g^{(2)}\|_{0,\Omega_0}). \tag{3.6}$$

Remark 3.2. That a solution (v_0, p_0) exists and satisfies $v_{03} \geq b$ in Ω is shown in Appendix A. It follows that v_0 has no closed integral curves and L_0, the least upper bound of the length of all integral curves, is finite.

As seen in the previous section our proof of this theorem have roughly the same outline as the proof of Theorem 2.1 summarized there. Thus we need to construct an operator B analogous to the operator Λ.

We want to construct an operator B on a subspace V of $H^3(\Omega)$, which has a fixed point u such that $v = v_0 + u$ is the velocity field of a solution to (1.1)-(1.3), (3.2) and (3.3).

The subspace V is defined as the space of functions $w \in H^3(\Omega)$ which satisfy

$$\begin{aligned} \operatorname{div} w &= 0 \quad \text{in } \Omega, \\ n \cdot w &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

V is closed in $H^3(\Omega)$ and hence a Hilbert space, with the scalar product of $H^3(\Omega)$. For $\gamma > 0$ let V_γ be the closed ball in V with radius γ, i.e. $w \in V$ such that $\|w\|_{3,\Omega} \leq \gamma$. The operator $B : V_\gamma \to V$ is defined as

$$B(u) = w, \tag{3.7}$$

where

$$\operatorname{curl} w = f \tag{3.8}$$

and f is the solution to

$$\begin{aligned} &[(v_0 + u) \cdot \nabla] f = f \cdot \nabla(v_0 + u) && \text{in } \Omega, \\ &f = \eta && \text{on } \Omega_0, \end{aligned} \tag{3.9}$$

with η defined by

$$\begin{aligned} &n \cdot \eta = h && \text{on } \Omega_0, \\ &\eta_T = \frac{h}{v \cdot n} v_T - \frac{1}{v \cdot n} n \times \nabla_T g && \text{on } \Omega_0. \end{aligned} \tag{3.10}$$

The definition of B is very close to that of Λ. However, the terms with W in equation (2.8) have no counterpart in equation (3.10). This is due to the fact that we assume that $\operatorname{curl} v_0 \equiv 0$.

We note that, if well-defined, the operator B depends on g, h, and v_0, i.e. $B = B[g, h, v_0]$ and that the mapping $(g, h) \to B[g, h, v_0]$ is linear.

For B to be well-defined we need to show that for every $u \in V_\gamma$ we get a unique $w \in V$. This requires us to put some additional conditions on g and h. We do this in two parts. First we assume f is any function in $H^2(\Omega)$ and find which additional conditions we need to impose on f to show that there exists a unique $\omega \in V$ satisfying (3.8). This is done in Section 4. After that we investigate which conditions we have to put on η, and hence g and h, to get a unique solution to (3.9) which satisfies the required conditions found in Section 4. This is done in Section 5. To show these results we need the following two lemmas.

Lemma 3.3. *Let $v_0 \in H^3(\Omega)$ satisfy the hypothesis of Theorem 3.1. Then for any fixed $0 < \gamma_0 < \frac{b}{c^{(1)}}$ the following three properties are satisfied:*

(i) For any vector field $v = v_0 + u$ with $u \in V_{\gamma_0}$

$$v_3 = v_{03} - u_3 \geq b - c^{(1)}\gamma_0 > 0.$$

(ii) No vector field $v \in v_0 + V_{\gamma_0}$ has closed integral curves. For $0 < \gamma \leq \gamma_0$ let L_γ denote the least upper bound of the length of all integral curves of all the vector fields $v \in v_0 + V_\gamma$. Then $L_\gamma < \infty$.

(iii) If an integral curve contains some point in $\partial U \times (0, L)$, then it is completely contained in the boundary.

Remark 3.4. The function $v \in H^3(\Omega)$ can be extended to $H^3_{loc}(\mathbb{R}^3) \subset C^1(\mathbb{R}^3)$. Restricting the integral curves of this extension to $\bar{\Omega}$ gives integral curves on $\bar{\Omega}$ independent of the extension. Part *(iii)* means that the condition holds for these integral curves on $\bar{\Omega}$.

Proof. (i) If $u \in V_{\gamma_0}$ then

$$u_3 \leq \|u\|_{C_b(\Omega)} \leq c^{(1)} \|u\|_{3,\Omega} \leq c^{(1)} \gamma_0$$

so

$$v_3 = v_{03} - u_3 \geq b - c^{(1)} \gamma_0 > 0.$$

(ii) From the property in (i) it follows that for any integral curve $\omega(t)$ to any vector field $v \in v_0 + V_{\gamma_0}$ satisfies

$$\frac{\partial \omega_3}{\partial t} = v_3 \geq b - c^{(1)} \gamma_0.$$

For $t_2 \geq t_1$ this means

$$\omega_3(t_2) - \omega_3(t_1) \geq (b - c^{(1)} \gamma_0)(t_2 - t_1).$$

It follows that $\omega_3(t_2) > \omega_3(t_1)$ if $t_2 > t_1$ and hence no integral curves are closed. Let I be the interval on which ω is defined. Since $\omega_3(t) \in [0, L]$ for all $t \in I$ we also get that

$$L \geq \omega_3(t_2) - \omega_3(t_1),$$

which means that

$$\frac{L}{(b - c^{(1)} \gamma_0)} \geq (t_2 - t_1)$$

for any $t_2, t_1 \in I$, $t_2 \geq t_1$. It follows that $|I| \leq \frac{L}{(b - c^{(1)} \gamma_0)}$ so the length of the integral curve can be estimated by

$$\begin{aligned}
&\int_I |v(\omega(t))| \, dt \\
&\quad \leq \int_I (\|v_0\|_{C_b(\Omega)} + \|u\|_{C_b(\Omega)}) \, dt \\
&\quad \leq |I| c^{(1)} (\|v_0\|_{3,\Omega} + \|u\|_{3,\Omega}) \\
&\quad \leq \frac{L}{(b - c^{(1)} \gamma_0)} c^{(1)} (\|v_0\|_{3,\Omega} + \gamma_0)
\end{aligned}$$

Since this is true for any integral curve to any vector field in $v_0 + V_{\gamma_0}$, we find that

$$L_{\gamma_0} \leq \frac{L}{(b - c^{(1)} \gamma_0)} c^{(1)} (\|v_0\|_{3,\Omega} + \gamma_0),$$

and since $v_0 + V_\gamma \subset v_0 + V_{\gamma_0}$ for all $0 < \gamma \leq \gamma_0$ we get that $L_\gamma \leq L_{\gamma_0} < \infty$.

(iii) Let $x_0 \in \partial U \times (0, L)$. Then there exists a C^∞ function $\Psi : W_{x_0} \to W_{y_0} \subset \mathbb{R}^3$ with C^∞ inverse and $\Psi_3(x) = 0$ if and only if $x \in W_{x_0} \cap \partial U \times (0, L)$, where W_{x_0} is an open neighborhood of x_0 and W_{y_0} is an open neighborhood of $y_0 = \Psi(x_0)$. Let

$$D\Psi = \begin{pmatrix} \frac{\partial \Psi_1}{\partial x_1} & \frac{\partial \Psi_1}{\partial x_2} & \frac{\partial \Psi_1}{\partial x_3} \\ \frac{\partial \Psi_2}{\partial x_1} & \frac{\partial \Psi_2}{\partial x_2} & \frac{\partial \Psi_2}{\partial x_3} \\ \frac{\partial \Psi_3}{\partial x_1} & \frac{\partial \Psi_3}{\partial x_2} & \frac{\partial \Psi_3}{\partial x_3} \end{pmatrix},$$

then for any $x \in W_{x_0} \cap \partial U \times (0, L)$ we have $D\Psi(x) \cdot v(x) = \bar{v}(x) = (\bar{v}_1(x), \bar{v}_1(x), 0)$ since $\left(\frac{\partial \Psi_3}{\partial x_1}, \frac{\partial \Psi_3}{\partial x_2}, \frac{\partial \Psi_3}{\partial x_3}\right)$ is normal to $\partial\Omega$ at x and v is tangential to $\partial\Omega$ at x. Now we can define $\bar{v}(y) = \bar{v}(\Psi^{-1}(y))$ as a vector field with $\bar{v}_3 = 0$ on $\{y \in W_{y_0} : y_3 = 0\}$. This means that solving

$$\frac{\partial \bar{\omega}(t)}{\partial t} = \bar{v}(\bar{\omega}(t))$$
$$\bar{\omega}(0) = y_0$$

gives an integral curve to $\bar{v}$ in $\{y \in W_{y_0} : y_3 = 0\}$. This in turn gives a curve $\omega(t) = \Psi^{-1}(\bar{\omega}(t))$ contained in $W_{x_0} \cap \partial U \times (0, L)$ with the properties

$$\frac{\partial \omega(t)}{\partial t} = D\Psi^{-1}(\bar{\omega}(t)) D\Psi(\omega(t)) v(\omega(t)) = v(\omega(t))$$

and

$$\omega(0) = \Psi^{-1}(\bar{\omega}(0)) = \Psi^{-1}(y_0) = x_0,$$

an integral curve to v. Due to the results from (ii) this curve will leave $W_{x_0} \cap \partial U \times (0, L)$ at some point $\partial U \times (0, L]$ for some $t_{x_0} > 0$. If the point is on $\partial U \times (0, L)$ we can just repeat the above argument to extend ω in $\partial\Omega$. If it is a point on $\partial\Omega_L$, then ω is not defined for any $t > t_{x_0}$ since if n is the normal to Ω_L, then $\frac{\partial \omega}{\partial t} \cdot n > 0$, which would imply $\omega_3 > L$ for such t. Similarly the curve can be extended for $t < 0$ in $\partial\Omega$ until it reaches $\partial\Omega_0$, where it no longer can be extended with $\omega_3 > 0$. It now follows from the theory from ordinary differential equations that this is the only integral curve passing through x_0, so any integral curve passing through this point is contained in $\partial\Omega$. $\square$

In the following we will assume that γ_0 is a fixed constant satisfying assumption in the previous lemma.

Lemma 3.5. *Assume $f \in H^2(\Omega)$ is a solution to* (3.9), *then* $\operatorname{div} f = 0$ *holds as an equality in $H^1(\Omega)$.*

Proof. We differentiate (3.9) to obtain

$$(v \cdot \nabla) \operatorname{div} f + \sum_{i=1}^{3} \left(\frac{\partial v}{\partial x_i} \cdot \nabla\right) f_i = (f \cdot \nabla) \operatorname{div} v + \sum_{j=1}^{3} \left(\frac{\partial f}{\partial x_j} \cdot \nabla\right) v_j$$

and note that $\operatorname{div} v = 0$ and

$$\sum_{i=1}^{3} \left(\frac{\partial v}{\partial x_i} \cdot \nabla\right) f_i = \sum_{i=1}^{3} \left(\sum_{j=1}^{3} \frac{\partial v_j}{\partial x_i} \frac{\partial f_i}{\partial x_j}\right) = \sum_{j=1}^{3} \left(\sum_{i=1}^{3} \frac{\partial f_i}{\partial x_j} \frac{\partial v_j}{\partial x_i}\right) = \sum_{j=1}^{3} \left(\frac{\partial f}{\partial x_j} \cdot \nabla\right) v_j,$$

which means that

$$(v \cdot \nabla) \operatorname{div} f = 0. \tag{3.11}$$

The ACL characterization of Sobolev spaces (see e.g. Section 1.1.3 in [15]), after a change of coordinates to initial values of and parameter along the integral curves of v, implies that $\operatorname{div} f$ is absolutely continuous along almost every integral curve of v. Hence, if we let $\omega(t)$ be such an integral curve with $\omega(0) = x_0 \in \Omega_0$ and $\omega(s) = x \in \Omega$, we get

$$\begin{aligned}
\operatorname{div} f(x) &= \operatorname{div} f(x_0) + \int_0^s \frac{d}{dt} \operatorname{div} f(\omega(t))\, dt \\
&= \operatorname{div} f(x_0) + \int_0^s \nabla \operatorname{div} f(\omega(t)) \cdot \frac{d}{dt}\omega(t)\, dt \\
&= \operatorname{div} f(x_0) + \int_0^s (v(\omega(t)) \cdot \nabla) \operatorname{div} f(\omega(t))\, dt.
\end{aligned}$$

Combining this with equation (3.11) we find that $\operatorname{div} f(x)$ is constant along almost every integral curve of v. If $\operatorname{div} f = 0$ on Ω_0 it follows that $\operatorname{div} f = 0$ almost everywhere in Ω. Since $f \in H^2(\Omega)$ and thus $\operatorname{div} f \in H^1(\Omega)$ it would imply that $\operatorname{div} f = 0$ holds in $H^1(\Omega)$.

From vector calculus we have

$$\operatorname{curl}(v \times f) = v \operatorname{div} f + (f \cdot \nabla)v - f \operatorname{div} v - (v \cdot \nabla) f.$$

Using (3.9) and $\operatorname{div} v = 0$ it follows that

$$\operatorname{curl}(v \times f) = v \operatorname{div} f. \tag{3.12}$$

From (3.9) and (3.10) we get

$$(v \cdot n) f_T = (n \cdot f) v_T - n \times \nabla_T g,$$

on Ω_0, which is equivalent to

$$n \times (v \times f) = n \times \nabla_T g$$

or

$$(v \times f)_T = \nabla_T g.$$

Applying Stokes' theorem gives

$$\begin{aligned}
\int_W \operatorname{curl}(v \times f) \cdot n\, dx\, dy &= \int_W \operatorname{curl}((v \times f)_T) \cdot n\, dx\, dy \\
&= \oint_{\partial W} (v \times f)_T \cdot ds \\
&= \oint_{\partial W} \nabla_T g \cdot ds \\
&= 0
\end{aligned}$$

for any $W \subset \Omega_0$ that is simply connected and has smooth boundary, hence

$$\operatorname{curl}(v \times f) \cdot n = 0 \tag{3.13}$$

on Ω_0. Combining (3.4), (3.12) and (3.13) gives

$$\phi \operatorname{div} f = n \cdot v \operatorname{div} f = n \cdot \operatorname{curl}(v \times f) = 0,$$

on Ω_0 and and hence $\operatorname{div} f = 0$ on Ω_0. □

Lemma 3.5 is true under the assumption that a solution f exists and that its derivatives is defined almost everywhere in Ω. That this is true is proven in Section 5.

4 Div-Curl Problem

As noted in the previous section, the first step we take to show that the operator B is well-defined is to show the existence of a unique solution in $H^3(\Omega)$ to the problem

$$\begin{aligned} \operatorname{curl} v &= f \quad \text{in } \Omega, \\ \operatorname{div} v &= 0 \quad \text{in } \Omega, \\ n \cdot v &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

where f is a function in $H^2(\Omega)$ with $\operatorname{div} f = 0$ and n is the unit normal to $\partial\Omega$. Since the domain Ω is a cylinder n is not defined on all of $\partial\Omega$, so the boundary condition needs to be interpreted with some care. Since we seek a solution in $H^3(\Omega) \subset C^1(\bar{\Omega})$ the the meaning of the condition is clear on Ω_0, Ω_L and $\partial U \times (0, L)$. For the edges we need the boundary conditions to be compatible so we interpret $v \cdot n$ as

$$\begin{aligned} n' \cdot v' &= 0 \\ v_3 &= 0 \end{aligned}$$

on $\partial\Omega_0$ and $\partial\Omega_L$, where $v' = (v_1, v_2)$ and n' is the normal to ∂U.

There exist previous results showing the existence of a unique weak solution in Lipschitz domains (e.g. see [25]), however the solution does not have the desired regularity. To show such regularity for the solution we impose additional conditions on f. In this section we find suitable conditions to impose on f and show the result under the assumption of these conditions through a method based on separation of variables taking the spacial geometry of Ω into account. After this section was completed we found an earlier regularity result by Zajączkowski [26] for general domains with edges based on standard techniques for elliptic problems in polyhedral domains. The discussion in [26] about the compatibility conditions is however somewhat vague and we believe the approach here is of independent interest.

We begin by introducing a vector potential u satisfying $\operatorname{curl} u = -v$ and $\operatorname{div} u = 0$ (see [4]). This turns equation (4.2) into

$$\begin{aligned} \Delta u &= f \quad \text{in } \Omega, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega, \\ n \times u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.2}$$

The boundary condition, $n \times u = 0$, is interpreted similarly to the one above for v giving $u = 0$ on $\partial\Omega_0$ and $\partial\Omega_L$. Where the normal is defined it implies that the solution u is parallel to the normal on the boundary. On Ω_0 and Ω_L this means that $u_1 = u_2 = 0$. On $\partial\Omega_{x_3}$, for all $x_3 \in (0, L)$, this means that $u_3 = 0$ and that $u_1 n_2 = n_1 u_2$. If we combine these conditions with the fact that u is divergence free we can find additional boundary conditions. On Ω_0 and Ω_L this means that $\frac{\partial u_3}{\partial x_3} = 0$. On $\partial\Omega_{x_3}$, for all $x_3 \in (0, L)$, this means that $\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0$. To solve problem (4.2) we separate it into two parts, with the first part only involving u_3:

$$\begin{aligned}
&\Delta u_3 = f_3 && \text{in } \Omega, \\
&\frac{\partial u_3}{\partial x_3} = 0 && \text{on } \Omega_0 \text{ and } \Omega_L, \\
&u_3 = 0 && \text{on } \partial\Omega_{x_3}\ \forall x_3 \in (0, L),
\end{aligned} \tag{4.3}$$

and the second part only involving u_1 and u_2:

$$\begin{aligned}
&\Delta(u_1, u_2) = (f_1, f_2) && \text{in } \Omega, \\
&(u_1, u_2) = 0 && \text{on } \Omega_0 \text{ and } \Omega_L, \\
&n_1 u_2 - n_2 u_1 = 0 && \text{on } \partial\Omega_{x_3}\ \forall x_3 \in (0, L). \\
&\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0 && \text{on } \partial\Omega_{x_3}\ \forall x_3 \in (0, L).
\end{aligned} \tag{4.4}$$

These equations are equivalent to (4.2). Indeed, It is clear through the reasoning after (4.2) that (4.2) implies (4.3) and (4.4). Following the same reasoning backwards we get that (4.3) and (4.4) implies (4.2). Using $\operatorname{div} f = 0$ and the fact that the divergence commutes with the laplacian we get

$$\begin{aligned}
&\Delta \operatorname{div} u = \operatorname{div} f = 0 && \text{in } \Omega, \\
&\operatorname{div} u = 0 && \text{on } \partial\Omega,
\end{aligned}$$

which implies $\operatorname{div} u = 0$ in Ω.

The equations (4.3) and (4.4) can be rewritten in a more general form as

$$\begin{aligned}
&\frac{\partial^2 w}{\partial x_3^2} = Aw + g, && 0 < x_3 < L, \\
&\frac{\partial w}{\partial x_3} = 0, && x_3 = 0, \\
&\frac{\partial w}{\partial x_3} = 0, && x_3 = L,
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
&\frac{\partial^2 w}{\partial x_3^2} = Aw + g, && 0 < x_3 < L, \\
&w = 0, && x_3 = 0, \\
&w = 0, && x_3 = L,
\end{aligned} \tag{4.6}$$

respectively, given that in the first case we let $w = u_3$ and $g = f_3$ and in the second case we let $w = (u_1, u_2)$ and $g = (f_1, f_2)$. We treat these as boundary value problems for an abstract ODE in a Hilbert space X, where A is an unbounded, densely defined, closed operator on X. In the first case $X = L^2(U)$ and A is the Dirichlet realization of $-\Delta$ on U with domain $H^2(U) \cap H_0^1(U)$. In the second case $X = L^2(U; \mathbb{R}^2)$ and

$$A = \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix}$$

with domain

$$\left\{(u_1,u_2)\in (H^2(U))^2 : (n_1u_2-n_2u_1)|_{\partial U}=0,\ \left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)\bigg|_{\partial U}=0\right\}.$$

In the first case we call the operator S and in the second case we call the operator T.

4.1 The operator S

It is well-known that S is a positive, self-adjoint operator with discrete spectrum [17]. It has a complete ON-basis $\{e_n(x,y)\}_{n=1}^{\infty}$ with corresponding eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ that satisfy $0<\lambda_1\le\lambda_2\le\cdots$ and $\lambda_n\to\infty$ as $n\to\infty$. From elliptic regularity it follows that the eigenfunctions are smooth up to the boundary. Using this we can define S^α for any α by using the spectral theorem. If we write $u\in L^2(U)$ as $u=\sum_{n=1}^{\infty}\hat{u}_ne_n$ then

$$S^\alpha u=\sum_{n=1}^{\infty}\lambda_n^\alpha\hat{u}_ne_n,$$

with domain

$$D(S^\alpha):=\{u: u\in L^2(U),\ \sum_{n=1}^{\infty}\lambda_n^{2\alpha}|\hat{u}_n|^2<\infty\}$$

for $0<\alpha<1$. For the future we require a more useful characterization of $D(S^\alpha)$. To express the characterization succinctly we introduce the space $H_{00}^{1/2}(U)$. If ν is a function defined on U let

$$\operatorname{ext}_0[\nu](x)=\begin{cases}\nu(x) & \text{if } x\in U,\\ 0 & \text{if } x\notin U.\end{cases}$$

This allows us to define the space

$$H_{00}^{1/2}(U)=\{\nu\in H^{1/2}(U):\operatorname{ext}_0[\nu]\in H^{1/2}(\mathbb{R}^2)\}.$$

For the future note that for any $\nu\in H_0^1(U)$ we have that

$$\|\operatorname{ext}_0[\nu]\|_{1/2,\mathbb{R}^2}\lesssim\|\operatorname{ext}_0[\nu]\|_{1,\mathbb{R}^2}\lesssim\|\nu\|_{1,U},$$

which means that $H_0^1(U)\subset H_{00}^{1/2}(U)$. In fact, $H_{00}^{1/2}(U)$ arises as an interpolation space between $L^2(U)$ and $H_0^1(U)$. Domains like $D(S^\alpha)$ have been characterized by Grisvard [10] in the scalar case and Seeley [19, 18, 20] more generally. Using their results we get

$$D(S^\alpha)=\begin{cases}\{w\in H^{2\alpha}(U): w|_{\partial U=0}\}, & 1/4<\alpha<1,\\ H_{00}^{1/2}(U), & \alpha=1/4,\\ H^{2\alpha}(U), & 0<\alpha<1/4,\end{cases}$$

with equivalence between the graph norm

$$\|w\|_{D(S^\alpha)}=\|w\|_{0,U}+\|S^\alpha w\|_{0,U}$$

and

$$\|w\|_{2\alpha,U},$$

that is, there exists constants $c^{(4)}$ and $c^{(5)}$ such that

$$c^{(4)}\|w\|_{D(S^\alpha)} \leq \|w\|_{2\alpha,U} \leq c^{(5)}\|w\|_{D(S^\alpha)}$$

for all $w \in D(S^\alpha)$ and $\alpha \neq 1/4$. For $\alpha = 1/4$ we instead have equivalence between the graph norm and

$$\|w\|_{H^{1/2}_{00}(\Omega)} = \|\mathrm{ext}_0[w]\|_{1/2,\mathbb{R}^3},$$

that is

$$c^{(4)}\|w\|_{D(S^{1/4})} \leq \|w\|_{H^{1/2}_{00}(\Omega)} \leq c^{(5)}\|w\|_{D(S^{1/4})},$$

for all $w \in D(S^{1/4})$.

4.2 The operator T

Recall that T was defined as

$$T = \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix}$$

with domain

$$\left\{(u_1, u_2) \in H^2(U;\mathbb{R}^2) : (n_1u_2 - n_2u_1)|_{\partial U} = 0,\ \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)\bigg|_{\partial U} = 0\right\}.$$

The operator T has principal symbol

$$\sigma_\xi(T) = \begin{pmatrix} |\xi|^2 & 0 \\ 0 & |\xi|^2 \end{pmatrix}.$$

This is a positive definite matrix for $\xi \neq 0$ and hence T is a strongly elliptic operator [12].

For later use we must show that T and its boundary conditions is parameter-elliptic on every ray except the positive real axis. We define this property below. However we start with defining the somewhat weaker property of complementing condition.

Definition 4.1. For $D = (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$ let $L(x, D)$ be a strongly elliptic partial differential operator of order $2m$ acting on q-tuples of functions on an open set $W \subset \mathbb{R}^n$ with smooth boundary ∂W and let $B_j(x, D)$ be mq boundary operators of order less than $2m$. For a point $x_0 \in \partial W$, with outer normal n, define the halfspace $H = \{x : (x - x_0) \cdot n < 0\}$ with boundary $\partial H = \{x : (x - x_0) \cdot n = 0\}$. Now consider the problem

$$\begin{aligned} &L^P(x_0, D)u = 0, \quad \text{in } H, \\ &B^P_j(x_0, D)u = 0, \quad \text{on } \partial H, \text{ for } j = 1, \cdots, mq \end{aligned} \tag{4.7}$$

where $L^P(x, D)$ and $B^P_j(x, D)$ denotes the principal part of $L(x, D)$ and $B_j(x, D)$ respectively. We say that the *complementing condition* holds at x_0 for $\{L, B_1, \cdots, B_{mq}\}$ if there are no nontrivial solutions to (4.7) of the form

$$u(x) = \exp(i\xi \cdot (x - x_0))v(\eta), \tag{4.8}$$

where ξ is a real nonzero vector perpendicular to n, $\eta = n \cdot (x - x_0)$ and $v(\eta) \to 0$ exponentially as $\eta \to -\infty$.

In light of this we can define parameter-ellipticity along a ray.

Definition 4.2. For a given $-\pi < \theta \leq \pi$ define a ray in the complex plane as $R_\theta = \{z \in \mathbb{C} : \arg z = \theta\}$. Under the same assumptions as in Definition 4.1 consider the problem

$$\begin{aligned} &L^P(x_0, D)u = \lambda u, \quad \text{in } H, \\ &B_j^P(x_0, D)u = 0, \qquad \text{on } \partial H, \text{ for } j = 1, \cdots, mq. \end{aligned} \tag{4.9}$$

We say that $\{L, B_1, \cdots, B_{mq}\}$ is *parameter-elliptic* on the ray R_θ if:

(i) There are no nontrivial solutions to (4.9) of the form

$$u(x) = \exp(i\xi \cdot (x - x_0))v(\eta), \tag{4.10}$$

for all $x_0 \in \partial W$ and $(\lambda, \xi) \neq 0$, where $\lambda \in R_\theta$, ξ is a real vector with $\xi \cdot n = 0$, $\eta = n \cdot (x - x_0)$ and $v(\eta) \to 0$ exponentially as $\eta \to -\infty$.

(ii) The matrix

$$L^P(x_0, i\xi') - \lambda I$$

is invertible for all $x_0 \in W$ and $(\lambda, \xi') \neq 0$ with $\lambda \in R_\theta$, $\xi' \in \mathbb{R}^n$.

Remark 4.3. Note that if $\{L, B_1, \cdots, B_{mq}\}$ is parameter-elliptic on some ray in the complex plane then the complementing condition holds for all x_0.

Lemma 4.4. *T and its boundary conditions is parameter-elliptic on R_θ for every $\theta \neq 0$.*

Proof. For any point $x_0 \in \partial U$ we can define x' by

$$\begin{aligned} x_1' &= n_1(x_1 - x_{0,1}) + n_2(x_2 - x_{0,2}) \\ x_2' &= n_2(x_1 - x_{0,1}) - n_1(x_2 - x_{0,2}). \end{aligned}$$

In these coordinates equation (4.10) becomes

$$u(x') = \exp(\pm i|\xi|x_2')v(x_1'). \tag{4.11}$$

By noting that

$$\begin{aligned} \frac{\partial}{\partial x_1} &= \frac{\partial x_1'}{\partial x_1}\frac{\partial}{\partial x_1'} + \frac{\partial x_2'}{\partial x_1}\frac{\partial}{\partial x_2'} = n_1\frac{\partial}{\partial x_1'} + n_2\frac{\partial}{\partial x_2'} \\ \frac{\partial}{\partial x_2} &= \frac{\partial x_1'}{\partial x_2}\frac{\partial}{\partial x_1'} + \frac{\partial x_2'}{\partial x_2}\frac{\partial}{\partial x_2'} = n_2\frac{\partial}{\partial x_1'} - n_1\frac{\partial}{\partial x_2'}, \end{aligned}$$

we can calculate

$$\begin{aligned} \Delta &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \\ &= n_1^2\frac{\partial^2}{\partial {x_1'}^2} + n_2^2\frac{\partial^2}{\partial {x_2'}^2} + 2n_1n_2\frac{\partial^2}{\partial x_1 x_2} + n_2^2\frac{\partial^2}{\partial {x_1'}^2} + n_1^2\frac{\partial^2}{\partial {x_2'}^2} - 2n_1n_2\frac{\partial^2}{\partial x_1 x_2} \\ &= \frac{\partial^2}{\partial {x_1'}^2} + \frac{\partial^2}{\partial {x_2'}^2}. \end{aligned}$$

This allows us to express the problem corresponding to (4.9) for T with its boundary conditions as

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial {x_1'}^2}+\frac{\partial^2}{\partial {x_2'}^2}\right)u=-\lambda u, && x_1'<0,\\
&\left(n_1\frac{\partial}{\partial x_1'}+n_2\frac{\partial}{\partial x_2'}\right)u_1+\left(n_2\frac{\partial}{\partial x_1'}-n_1\frac{\partial}{\partial x_2'}\right)u_2=0, && x_1'=0,\\
&n_2u_1-n_1u_2=0, && x_1'=0,
\end{aligned}$$

where we let λ be any complex number in any R_θ with $\theta\neq 0$ such that $(\lambda,\xi)\neq 0$. Assuming u a solution of the form given in equation (4.10) turns the equations above into

$$\begin{aligned}
&v''(x_1')-(|\xi|^2-\lambda)v(x_1')=0, && x_1'<0,\\
&n_1v_1'(0)\pm i|\xi|n_2v_1(0)+n_2v_2'(0)\mp i|\xi|n_1v_2(0)=0,\\
&n_2v_1(0)-n_1v_2(0)=0.
\end{aligned}$$

Under these conditions $(|\xi|^2-\lambda)\notin(-\infty,0]$ since it would imply $\lambda\in(0,\infty)\subset R_0$ or $(\lambda,\xi)=0$. Also note that this gives $-\pi<\arg(|\xi|^2-\lambda)<\pi$, which means that $-\frac{\pi}{2}<\arg(|\xi|^2-\lambda)^{1/2}<\frac{\pi}{2}$ and hence $\operatorname{Re}\left((|\xi|^2-\lambda)^{1/2}\right)>0$. The ODE above has the general solution

$$(v_1(x_1'),v_2(x_1'))=(a_+e^{(|\xi|^2-\lambda)^{1/2}x_1'}+a_-e^{-(|\xi|^2-\lambda)^{1/2}x_1'},b_+e^{(|\xi|^2-\lambda)^{1/2}x_1'}+b_-e^{-(|\xi|^2-\lambda)^{1/2}x_1'}).$$

Using $\operatorname{Re}\left((|\xi|^2-\lambda)^{1/2}\right)>0$ together with the condition that $v(x_1')\to 0$ as $x_1'\to-\infty$ leaves us with with

$$(v_1(x_1'),v_2(x_1'))=(a_+e^{(|\xi|^2-\lambda)^{1/2}x_1'},b_+e^{(|\xi|^2-\lambda)^{1/2}x_1'}).$$

Substituting this into the boundary conditions gives

$$\begin{pmatrix}(|\xi|^2-\lambda)^{1/2}n_1\pm i|\xi|n_2) & (|\xi|^2-\lambda)^{1/2}n_2\mp i|\xi|n_1\\ n_2 & n_1\end{pmatrix}\begin{pmatrix}a_+\\ b_+\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},$$

since the matrix has determinant $-(|\xi|^2-\lambda)^{1/2}\neq 0$ the solution $a_+=b_+=0$ to this system of equations is unique and it follows that $u\equiv 0$. This means that part (i) of the definition of parameter-ellipticity is satisfied for T and its boundary conditions for all R_θ with $\theta\neq 0$.

Now part (ii) is satisfied if the matrix

$$\begin{pmatrix}|\xi'|^2-\lambda & 0\\ 0 & |\xi'|^2-\lambda\end{pmatrix}$$

is invertible for all $(\lambda,\xi')\neq 0$ with $\lambda\in R_\theta$, $\xi'\in\mathbb{R}^n$. The matrix is singular only if $|\xi'|^2-\lambda=0$, which either implies $\lambda\in(0,\infty)\subset R_0$ or $(\lambda,\xi')=0$. It follows that part (ii) of the definition of parameter-ellipticity is satisfied for T and its boundary conditions for all R_θ with $\theta\neq 0$. □

According to Agmon, Douglis, Nirenberg [2], we find that any solution of the homogeneous problem in $H^2(U;\mathbb{R}^2)$ is actually smooth. Moreover, we have the estimates

$$\|(u_1,u_2)\|_{s+2,U} \le C(\|(f_1,f_2)\|_{s,U} + \|(u_1,u_2)\|_{0,U})$$

for any $s \ge 0$. This implies in particular that T is a closed operator. Here we also note that if T is bijective, as an operator on $L^2(U;\mathbb{R}^2)$ with domain $D(T)$, the inequality above holds without the term $\|(u_1,u_2)\|_{0,U}$. Additionally we get that $T - \lambda I : D(T) \to L^2(U;\mathbb{R}^2)$ is invertible for all negative λ with large enough absolute value.

Lemma 4.5. *T is a symmetric operator.*

Proof. First assume that

$$u,v \in D^\infty(T) = \left\{ (u_1,u_2) \in C^\infty(\bar{U};\mathbb{R}^2) : n_1u_2 - n_2u_1|_{\partial U} = 0,\ \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\bigg|_{\partial U} = 0 \right\}.$$

Then

$$\begin{aligned}
&\int_U (\Delta u_1 v_1 + \Delta u_2 v_2 - u_1 \Delta v_1 - u_2 \Delta v_2)\, dx_1\, dx_2 \\
=& \int_{\partial U} \left(\frac{\partial u_1}{\partial x_1} v_1 n_1 + \frac{\partial u_1}{\partial x_2} v_1 n_2 + \frac{\partial u_2}{\partial x_1} v_2 n_1 \right. \\
&\qquad \left. + \frac{\partial u_2}{\partial x_2} v_2 n_2 - \frac{\partial v_1}{\partial x_1} u_1 n_1 - \frac{\partial v_1}{\partial x_2} u_1 n_2 - \frac{\partial v_2}{\partial x_1} u_2 n_1 - \frac{\partial v_2}{\partial x_2} u_2 n_2 \right) ds \\
=& \int_{\partial U} \left(-\frac{\partial u_2}{\partial x_2} v_1 n_1 + \frac{\partial u_1}{\partial x_2} v_2 n_1 + \frac{\partial u_2}{\partial x_1} v_1 n_2 \right. \\
&\qquad \left. - \frac{\partial u_1}{\partial x_1} v_2 n_2 + \frac{\partial v_2}{\partial x_2} u_1 n_1 - \frac{\partial v_1}{\partial x_2} u_2 n_1 - \frac{\partial v_2}{\partial x_1} u_1 n_2 + \frac{\partial v_1}{\partial x_1} u_2 n_2 \right) ds \\
=& \int_{\partial U} ((n_2, -n_1) \cdot \nabla)(u_2 v_1 - u_1 v_2)\, ds \\
=& \, 0,
\end{aligned}$$

where the last integral vanishes since $(n_2, -n_1)$ is a tangent to ∂U and ∂U is a closed curve. Next, consider $u, v \in D(T)$. Due to elliptic regularity $D^\infty(T)$ is dense in $D(T)$ so we can find sequences $\{u^n\}$, $\{v^n\}$ in $D^\infty(T)$ such that $u^n \to u$, $v^n \to v$ in $H^2(U)$. Repeating the above computations with u^n and v^n instead of u and v, we find that the relation above holds for all u^n, v^n and hence also for $u, v \in D(T)$. □

Proposition 4.6. *The operator T is self-adjoint.*

Proof. This follows from the fact that the operator is closed, densely defined, symmetric and $T - \lambda I$ is invertible for all λ in the complex plane not on the positive real axis with large enough absolute value. This means that no value in the complex plane outside of the real axis is in the spectrum of T and it is well-known that this implies for a closed, densely defined, symmetric operator that it is self-adjoint (see e.g. Theorem 8.68 in [17]). □

Proposition 4.7. *The operator T has compact resolvent.*

Proof. This follows immediately from the fact that $H^2(U)$ is compactly embedded in $L^2(U)$. $\square$

From this it follows that T has a discrete spectrum, that is, the spectrum consists of isolated eigenvalues and they can only accumulate at $\pm\infty$.

Proposition 4.8. *The spectrum of T is contained in* $(0, \infty)$

Proof. We assume that we have a eigenvalue $\lambda \leq 0$ with corresponding eigenvector $(u_1, u_2) \in C^\infty(\bar{U}; \mathbb{R}^2)$. Set $w = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}$. Then

$$\Delta w = \lambda w.$$

Since $w = 0$ on ∂U we get $w = 0$ in U (see Section 4.1). From this it follows that there exists a function ψ such that $\left(\frac{\partial \psi}{\partial x_2}, -\frac{\partial \psi}{\partial x_1}\right) = (u_1, u_2)$. Now we can calculate

$$\nabla(\Delta\psi - \lambda\psi) = (-\Delta u_2 + \lambda u_2, \Delta u_1 - \lambda u_1) = (0, 0).$$

Hence $\Delta\psi - \lambda\psi = C$ for some constant C. Using the other boundary condition we get $\frac{\partial \psi}{\partial n} = n \cdot \nabla\psi = n_2 u_1 - n_1 u_2 = 0$ on ∂U. For $\lambda < 0$ and a given C this problem has the unique solution $\psi = -C/\lambda$ [17], yielding $(u_1, u_2) = 0$. If $\lambda = 0$ on the other hand we get

$$\int_U C \, dx_1 dx_2 = \int_U \Delta\psi \, dx_1 dx_2 = \int_{\partial U} \frac{\partial \psi}{\partial n} \, ds = 0,$$

implying that $C = 0$. However this means that $\Delta\psi = 0$, which again implies that ψ is a constant and $(u_1, u_2) = 0$ $\square$

The previous results shows that T has a ON-basis of smooth eigenfunctions $\{e_n(x, y)\}_{n=1}^\infty$ with corresponding eigenvalues $\{\lambda_n\}_{n=1}^\infty$, such that $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ and $\lambda_n \to \infty$ as $n \to \infty$. Once again we can use the spectral theorem to define T^α for $0 < \alpha < 1$ and the results of Seeley [19, 18, 20] to characterize the domains for this family of operators.

$$D(T^\alpha) = \begin{cases} \left\{u \in H^{2\alpha}(U; \mathbb{R}^2) \colon n_2 u_1 - n_1 u_2|_{\partial U} = 0, \left(\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2}\right)_{\partial U} = 0\}\right\}, & 3/4 < \alpha < 1, \\ \left\{u \in H^{3/2}(U; \mathbb{R}^2) \colon n_2 u_1 - n_1 u_2|_{\partial U} = 0, \dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} \in H_{00}^{1/2}(U)\right\}, & \alpha = 3/4, \\ \{u \in H^{2\alpha}(U; \mathbb{R}^2) \colon n_2 u_1 - n_1 u_2|_{\partial U} = 0\}, & 1/4 < \alpha < 3/4, \\ \{u \in H^{1/2}(U; \mathbb{R}^2) \colon n_2 u_1 - n_1 u_2 \in H_{00}^{1/2}(U)\}, & \alpha = 1/4, \\ \{u \in H^{2\alpha}(U; \mathbb{R}^2)\}, & 0 < \alpha < 1/4. \end{cases}$$

Also here we have equivalence between the graph norm

$$\|w\|_{D(T^\alpha)} = \|w\|_{0,U} + \|T^\alpha w\|_{0,U}$$

and

$$\|w\|_{2\alpha,U},$$

that is, there exists constants $c^{(6)}$ and $c^{(7)}$ such that

$$c^{(6)}\|w\|_{D(T^\alpha)} \leq \|w\|_{2\alpha,U} \leq c^{(7)}\|w\|_{D(T^\alpha)}$$

for all $w \in D(T^\alpha)$ and $\alpha \neq 1/4, 3/4$. For $\alpha = 1/4$ we instead get

$$c^{(6)}\|w\|_{D(T^{1/4})} \leq \|w\|_{1/2,U} + \|n_2 w_1 - n_1 w_2\|_{H_{00}^{1/2}} \leq c^{(7)}\|w\|_{D(T^{1/4})}$$

for all $w \in D(T^{1/4})$, and for $\alpha = 3/4$

$$c^{(6)}\|w\|_{D(T^{3/4})} \leq \|w\|_{3/2,U} + \left\|\frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2}\right\|_{H_{00}^{1/2}} \leq c^{(7)}\|w\|_{D(T^{3/4})}$$

for all $w \in D(T^{3/4})$.

4.3 The BVPs

Now we can return to the boundary value problems we introduced in the beginning of this section. We begin by considering

$$\begin{aligned} w'' &= Aw + g, & 0 < x_3 < L, \\ w' &= 0, & x_3 = 0, \\ w' &= 0, & x_3 = L, \end{aligned} \tag{4.12}$$

where A is assumed to be an unbounded, densely defined operator on a Hilbert space X. A is also self-adjoint with discrete, positive spectrum. We denote the eigenvalues by $\{\lambda_n\}_{n=1}^\infty$ with $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ and the corresponding ON basis of eigenfunctions by $\{e_n\}_{n=1}^\infty$. we also assume that $g \in L^2([0, L]; X)$. By the assumptions on A the graph norm of $D(A^\alpha)$ is equivalent to the norm $\|A^\alpha w\|_X$, that is there exists a constant $c^{(8)}$ such that

$$\|A^\alpha w\|_X \leq \|w\|_{D(A^\alpha)} \leq c^{(8)}\|A^\alpha w\|_X$$

for all $w \in D(A^\alpha)$.

Proposition 4.9. *Under the assumptions given above, there exists a unique solution of* (4.12) *in* $L^2([0, L]; D(A)) \cap H^1([0, L]; D(A^{1/2})) \cap H^2([0, L]; X)$.

Remark 4.10. A solution is assumed to mean a function satisfying $w'' = Aw + g$ as an equality in $L^2([0, L]; X)$ and the boundary conditions as equalities in X.

Proof. We express $w(x_3)$ and $g(x_3)$ as generalized Fourier series

$$w(x_3) = \sum_{n=1}^{\infty} \hat{w}_n(x_3) e_n,$$

and

$$g(x_3) = \sum_{n=1}^{\infty} \hat{g}_n(x_3) e_n,$$

yielding the equations

$$\begin{aligned} \hat{w}_n'' &= \lambda_n \hat{w}_n + \hat{g}_n, && 0 < x_3 < L, \\ \hat{w}_n' &= 0, && x_3 = 0, \\ \hat{w}_n' &= 0, && x_3 = L, \end{aligned}$$

for every n. According to the theory of ODEs these equation have unique solutions. From the ODEs we get the identity

$$\int_0^L \left(|\hat{w}_n''(x_3)|^2 - 2\lambda_n \hat{w}_n''(x_3)\hat{w}_n(x_3) + \lambda_n^2 |\hat{w}_n(x_3)|^2 \right) dx_3 = \int_0^L |\hat{g}_n(x_3)|^2 \, dx_3$$

for $0 < x_3 < L$. Integration by parts

$$-\int_0^L \hat{w}_n''(x_3)\hat{w}_n(x_3) \, dx_3 = \int_0^L |\hat{w}_n'(x_3)|^2 \, dx_3,$$

gives

$$\int_0^L |\hat{w}_n''(x_3)|^2 + 2\lambda_n |\hat{w}_n'(x_3)|^2 + \lambda_n^2 |\hat{w}_n(x_3)|^2 \, dx_3 = \int_0^L |\hat{g}_n(x_3)|^2 \, dx_3.$$

By summing over n it follows that

$$\|w''\|_{L^2([0,L];X)} \leq \|g\|_{L^2([0,L];X)},$$

$$\|w'\|_{L^2([0,L];D(A^{1/2}))} \leq c^{(8)} \|g\|_{L^2([0,L];X)},$$

and

$$\|w\|_{L^2([0,L];D(A))} \leq c^{(8)} \|g\|_{L^2([0,L];X)},$$

hence $w \in L^2([0,L]; D(A)) \cap H^1([0,L]; D(A^{1/2})) \cap H^2([0,L]; X)$ □

Proposition 4.11. *Assume that $g \in H^2([0,L]; X)$ with $g'|_{x_3=0}, g'|_{x_3=L} \in D(A^{1/4})$, then w satisfies*

$$\|w''''\|_{L^2([0,L];X)} \leq c^{(9)} (\|g''\|_{L^2([0,L];X)} + \|g'|_{x_3=0}\|_{D(A^{1/4})} + \|g'|_{x_3=L}\|_{D(A^{1/4})}), \tag{4.13}$$

$$\|w'''\|_{L^2([0,L];D(A^{1/2}))} \leq c^{(10)} (\|g''\|_{L^2([0,L];X)} + \|g'|_{x_3=0}\|_{D(A^{1/4})} + \|g'|_{x_3=L}\|_{D(A^{1/4})}), \tag{4.14}$$

and

$$\|w''\|_{L^2([0,L];D(A))} \leq c^{(11)} (\|g''\|_{L^2([0,L];X)} + \|g'|_{x_3=0}\|_{D(A^{1/4})} + \|g'|_{x_3=L}\|_{D(A^{1/4})}), \tag{4.15}$$

where $c^{(9)} = c^{(9)}(L)$, $c^{(10)} = c^{(10)}(L)$ and $c^{(11)} = c^{(11)}(L)$.

Proof. To prove this we use that w'' satisfies

$$\begin{aligned} (w'')'' &= Aw'' + g'', && 0 < x_3 < L, \\ (w'')' &= g', && x_3 = 0, \\ (w'')' &= g', && x_3 = L. \end{aligned}$$

This problem can be solved by splitting it into three different parts by keeping either the source term in the equation or in either of the boundary values. The first part is the same problem as in Proposition 4.9. The second part and third part are very similar and we illustrate by solving the second part, which is a problem of the form

$$\begin{aligned} v'' &= Av, && 0 < x_3 < L, \\ v' &= h, && x_3 = 0, \\ v' &= 0, && x_3 = L, \end{aligned}$$

In Fourier variables this becomes

$$\begin{aligned} \hat{v}_n'' &= \lambda_n \hat{v}_n, && 0 < x_3 < L, \\ \hat{v}_n' &= \hat{h}_n, && x_3 = 0, \\ \hat{v}_n' &= 0, && x_3 = L, \end{aligned}$$

which has the solution

$$\hat{v}_n(x_3) = -\frac{\cosh(\sqrt{\lambda_n}(x_3 - L))}{\sqrt{\lambda_n}\sinh(\sqrt{\lambda_n}L)}\hat{h}_n.$$

This allows us to compute

$$\begin{aligned} \int_0^L \lambda_n^2 |\hat{v}_n(x_3)|^2 \, dx_3 &= \frac{\lambda_n |\hat{h}_n|^2}{\sinh^2(\sqrt{\lambda_n}L)} \int_0^L \cosh^2(\sqrt{\lambda_n}(x_3 - L)) \, dx_3 \\ &= \frac{\lambda_n |\hat{h}_n|^2}{\sinh^2(\sqrt{\lambda_n}L)} \int_0^L \frac{1 + \cosh(2\sqrt{\lambda_n}(x_3 - L))}{2} \, dx_3 \\ &= \frac{\lambda_n |\hat{h}_n|^2}{\sinh^2(\sqrt{\lambda_n}L)} \left(\frac{L}{2} + \frac{\sinh(2\sqrt{\lambda_n}L)}{4\sqrt{\lambda_n}} \right) \\ &\leq c^{(12)} \lambda_n^{1/2} |\hat{h}_n|^2. \end{aligned}$$

Similarly we get

$$\int_0^L \lambda_n |\hat{v}_n'(x_3)|^2 \, dx_3 \leq c^{(13)} \lambda_n^{1/2} |\hat{h}_n|^2$$

and

$$\int_0^L |\hat{v}_n''(x_3)|^2 \, dx_3 \leq c^{(12)} \lambda_n^{1/2} |\hat{h}_n|^2,$$

for some constants $c^{(12)} = c^{(12)}(L)$ and $c^{(13)} = c^{(13)}(L)$. Using the estimates for the other parts of w'' and summing over n gives us the desired inequalities. □

We have taken a different approach in the proofs of Propositions 4.9 and 4.11. However, we can note that Proposition 4.9 could have been proven in a similar way to Proposition 4.11 by writing the solution to the boundary value problem with the help of a Green's function.

To prove further regularity we have to make some additional assumptions about $X = X_0$. We assume that for $s \geq 0$ there is a scale of Hilbert spaces X_s such that $X_r \subseteq X_s$ if $s \leq r$, with $\|\cdot\|_{X_s} \leq \|\cdot\|_{X_r}$, and that A^{-1} is a bounded operator from X_s into X_{s+2}. Additionally we assume that $D(A^s) \subseteq X_{2s}$, with $\|\cdot\|_{X_{2s}} \leq c^{(14)}\|\cdot\|_{D(A^s)}$. In our particular application we will have $X_s = H^s(U;\mathbb{R}^d)$ for $d = 1, 2$.

Proposition 4.12. *Assume in addition that* $g \in L^2([0,L];X_2) \cap H^1([0,L];X_1)$, *then*

$$w \in \cap_{k=0}^{4} H^k([0,L];X_{4-k})$$

and w *satisfies*

$$\|w\|_{\cap_{k=0}^{2} H^k([0,L];X_{2-k})} \leq c^{(15)}\|g\|_{L^2([0,L];X)} \tag{4.16}$$

and

$$\|w\|_{\cap_{k=0}^{4} H^k([0,L];X_{4-k})} \leq c^{(16)}(\|g\|_{\cap_{i=0}^{2} H^i([0,L];X_{2-i})} + \|g'|_{x_3=0}\|_{D(A^{1/4})} + \|g'|_{x_3=L}\|_{D(A^{1/4})}), \tag{4.17}$$

where $c^{(15)} = c^{(15)}(L)$ *and* $c^{(16)} = c^{(16)}(L)$.

Remark 4.13. If Y and Z are two normed spaces then we let the norm of $Y \cap Z$ be

$$\|\cdot\|_{Y\cap Z} = \|\cdot\|_Y + \|\cdot\|_Z.$$

Proof. The inequalities shown in Proposition 4.11 together with the result of Proposition 4.9 immediately gives us (4.16) and

$$\|w\|_{\cap_{k=3}^{4} H^k([0,L];X_{4-k})} \leq c^{(17)} \left(\|g\|_{H^2([0,L];X_0)} + \|g'|_{x_3=0}\|_{D(A^{1/4})} + \|g'|_{x_3=L}\|_{D(A^{1/4})}\right), \tag{4.18}$$

for some constant $c^{(17)} = c^{(17)}(L)$. By using $w = A^{-1}(w'' - g)$ we get

$$\begin{aligned}
\|w\|_{L^2([0,L];X_4)} &= \|A^{-1}(w'' - g)\|_{L^2([0,L];X_4)} \\
&\leq \|A^{-1}\|_{\mathcal{L}(X_2,X_4)}(\|g\|_{L^2([0,L];X_2)} + \|w''\|_{L^2([0,L];X_2)}) \\
&\leq \|A^{-1}\|_{\mathcal{L}(X_2,X_4)}(\|g\|_{L^2([0,L];X_2)} + c^{(14)}\|w''\|_{L^2([0,L];D(A))}) \\
&\leq c^{(18)}\|g\|_{L^2([0,L];X_2)} + \|g\|_{H^2([0,L];X_0)} + \|g'|_{x_3=0}\|_{D(A^{1/4})} + \|g'|_{x_3=L}\|_{D(A^{1/4})},
\end{aligned}$$

for some constant $c^{(18)} = c^{(18)}(L)$, where we have used the continuity of $A^{-1} : X_2 \to X_4$ and equation (4.15). Similarly using equation (4.14) together with $w' = A^{-1}(w''' - g')$ we get

$$\begin{aligned}
\|w'\|_{L^2([0,L];X_3)} &= \|A^{-1}(w''' - g')\|_{L^2([0,L];X_3)} \\
&\leq \|A^{-1}\|_{\mathcal{L}(X_1,X_3)}(\|g'\|_{L^2([0,L];X_1)} + \|w'''\|_{L^2([0,L];X_1)}) \\
&\leq \|A^{-1}\|_{\mathcal{L}(X_1,X_3)}(\|g'\|_{L^2([0,L];X_1)} + c^{(14)}\|w'''\|_{L^2([0,L];D(A^{1/2}))}) \\
&\leq c^{(19)}\|g\|_{H^1([0,L];X_1)} + \|g\|_{H^2([0,L];X_0)} + \|g'|_{x_3=0}\|_{D(A^{1/4})} + \|g'|_{x_3=L}\|_{D(A^{1/4})},
\end{aligned}$$

for some constant $c^{(19)} = c^{(19)}(L)$, and from (4.15) we get

$$\|w''\|_{L^2([0,L];X_2)} \leq c^{(11)} c^{(14)} (\|g\|_{H^2([0,L];X_0)} + \|g'|_{x_3=0}\|_{D(A^{1/4})} + \|g'|_{x_3=L}\|_{D(A^{1/4})}).$$

Combining these three inequalities gives

$$\|w\|_{\cap_{k=0}^{2} H^k([0,L];X_{4-k})} \leq c^{(20)} (\|g\|_{\cap_{i=0}^{2} H^i([0,L];X_{i-2})} + \|g'|_{x_3=0}\|_{D(A^{1/4})} + \|g'|_{x_3=L}\|_{D(A^{1/4})}),$$

for some constant $c^{(20)} = c^{(20)}(L)$, which together with equation (4.18) gives equation (4.17). From this it follows that

$$w \in \cap_{k=0}^{4} H^k([0,L]; X_{4-k})$$

since all the terms of the right hand side of equation (4.17) are finite. □

Next we consider the other BVP

$$\begin{aligned} w'' &= Aw + g, && 0 < x_3 < L, \\ w &= 0, && x_3 = 0, \\ w &= 0, && x_3 = L, \end{aligned} \tag{4.19}$$

where A satisfies the same assumptions as in the previous BVP. The proof of the following proposition is almost identical to the proof of Proposition 4.9 and is therefore omitted.

Proposition 4.14. *Under the given assumptions with the additional assumption that* $g \in L^2([0,L]; X)$*, there is a unique solution to* (4.19) *in* $H^2([0,L]; X) \cap H^1([0,L]; D(A^{1/2})) \cap L^2([0,L]; D(A))$.

Proposition 4.15. *Assume that* $g \in H^2([0,L]; X)$ *with* $g|_{x_3=0}, g|_{x_3=L} \in D(A^{3/4})$*, then* w *satisfies*

$$\|w''''\|_{L^2([0,L];X)} \leq c^{(21)} \|g''\|_{L^2([0,L];X)} + \|g|_{x_3=0}\|_{D(A^{3/4})} + \|g|_{x_3=L}\|_{D(A^{3/4})},$$

$$\|w'''\|_{L^2([0,L];D(A^{1/2}))} \leq c^{(22)} \|g''\|_{L^2([0,L];X)} + \|g|_{x_3=0}\|_{D(A^{3/4})} + \|g|_{x_3=L}\|_{D(A^{3/4})},$$

and

$$\|w''\|_{L^2([0,L];D(A))} \leq c^{(23)} \|g''\|_{L^2([0,L];X)} + \|g|_{x_3=0}\|_{D(A^{3/4})} + \|g|_{x_3=L}\|_{D(A^{3/4})},$$

where $c^{(21)} = c^{(21)}(L)$*,* $c^{(22)} = c^{(22)}(L)$ *and* $c^{(22)} = c^{(22)}(L)$.

Proof. To prove this we proceed in an almost identical fashion as in the proof of Proposition 4.11. We use that w'' satisfies

$$\begin{aligned} (w'')'' &= Aw'' + g'', && 0 < x_3 < L, \\ w'' &= g, && x_3 = 0, \\ w'' &= g, && x_3 = L. \end{aligned}$$

Again, we split the problem into three different parts by keeping either the source term in the equation or in either of the boundary values. The first part is the same problem as in Proposition 4.14. The second part and third part are very similar and we illustrate by solving the second part, which is a problem of the form

$$\begin{aligned} v'' &= Av, \quad 0 < x_3 < L, \\ v &= h, \qquad x_3 = 0, \\ v &= 0, \qquad x_3 = L, \end{aligned}$$

In Fourier variables this becomes

$$\begin{aligned} \hat{v}_n'' &= \lambda_n \hat{v}_n, \quad 0 < x_3 < L, \\ \hat{v}_n &= \hat{h}_n, \qquad x_3 = 0, \\ \hat{v}_n &= 0, \qquad x_3 = L, \end{aligned}$$

which has the solution

$$\hat{v}_n(x_3) = -\frac{\sinh(\sqrt{\lambda_n}(x_3 - L))}{\sinh(\sqrt{\lambda_n}L)} \hat{h}_n.$$

This allows us to compute

$$\begin{aligned} \int_0^L \lambda_n^2 |\hat{v}_n(x_3)|^2 \, dx_3 &= \frac{\lambda_n^2 |\hat{h}_n|^2}{\sinh^2(\sqrt{\lambda_n}L)} \int_0^L \sinh^2(\sqrt{\lambda_n}(x_3 - L)) \, dx_3 \\ &= \frac{\lambda_n^2 |\hat{h}_n|^2}{\sinh^2(\sqrt{\lambda_n}L)} \int_0^L \frac{\cosh(2\sqrt{\lambda_n}(x_3 - L)) - 1}{2} \, dx_3 \\ &= \frac{\lambda_n^2 |\hat{h}_n|^2}{\sinh^2(\sqrt{\lambda_n}L)} \left(\frac{\sinh(2\sqrt{\lambda_n}L)}{4\sqrt{\lambda_n}} - \frac{L}{2} \right) \\ &\le c^{(24)} \lambda_n^{3/2} |\hat{h}_n|^2. \end{aligned}$$

Similarly we get

$$\int_0^L \lambda_n |\hat{v}_n'(x_3)|^2 \, dx_3 \le c^{(25)} \lambda_n^{3/2} |\hat{h}_n|^2$$

and

$$\int_0^L |\hat{v}_n''(x_3)|^2 \, dx_3 \le c^{(24)} \lambda_n^{3/2} |\hat{h}_n|^2,$$

for some constants $c^{(24)} = c^{(24)}(L)$ and $c^{(25)} = c^{(25)}(L)$. Using the estimates for the other parts of w'' and summing over n gives us the desired inequalities. □

The proof of the following proposition is very similar to the proof of Proposition 4.12 and it is therefore omitted.

Proposition 4.16. *Assume in addition that* $g \in L^2([0, L]; X_2) \cap H^1([0, L]; X_1)$, *then*

$$w \in \cap_{k=0}^4 H^k([0, L]; X_{4-k})$$

and w *satisfies*

$$\|w\|_{\cap_{k=0}^2 H^k([0,L];X_{2-k})} \le c^{(26)} \|g\|_{L^2([0,L];X)}$$

and

$$\|w\|_{\cap_{k=0}^4 H^k([0,L];X_{4-k})} \le c^{(27)} (\|g\|_{\cap_{i=0}^2 H^i([0,L];X_{i-2})} + \|g|_{x_3=0}\|_{D(A^{3/4})} + \|g|_{x_3=L}\|_{D(A^{3/4})}),$$

where $c^{(26)} = c^{(26)}(L)$ *and* $c^{(27)} = c^{(27)}(L)$.

4.4 Application to the div-curl problem

As stated in the beginning of this section we want a solution, v, to (4.2) in $H^3(\Omega)$. To accomplish this we introduced a vector potential, $u = (u_1, u_2, u_3)$, which should be a function in $H^4(\Omega)$ for v to be a function in $H^3(\Omega)$. The last component, u_3, solves (4.3) which can be written

$$\begin{aligned} \frac{\partial^2 u_3}{\partial x_3^2} &= S u_3 + f_3, && \text{for } 0 < x_3 < L, \\ \frac{\partial u_3}{\partial x_3} &= 0, && \text{for } x_3 = 0 \text{ and } x_3 = L, \end{aligned} \tag{4.20}$$

where S is the operator introduced above.

Proposition 4.17. *Assume that $f_3 \in H^2(\Omega)$ and $\frac{\partial f_3}{\partial x_3}|_{x_3=0}, \frac{\partial f_3}{\partial x_3}|_{x_3=L} \in H_{00}^{1/2}$. Then there exists a unique solution, u_3, to (4.20) in $H^4(\Omega)$ which satisfies*

$$\|u_3\|_{2,\Omega} \leq c^{(28)} \|f_3\|_{0,\Omega} \tag{4.21}$$

and

$$\|u_3\|_{4,\Omega} \leq c^{(29)} \left(\|f_3\|_{2,\Omega} + \left\| \frac{\partial f_3}{\partial x_3}\bigg|_{x_3=0} \right\|_{D(S^{1/4})} + \left\| \frac{\partial f_3}{\partial x_3}\bigg|_{x_3=L} \right\|_{D(S^{1/4})} \right), \tag{4.22}$$

where $c^{(28)} = c^{(28)}(L)$ and $c^{(29)} = c^{(29)}(L)$.

Proof. The problem (4.20) is of the form of (4.12), where S satisfies the assumptions on A given in the paragraph following (4.12) if we let $X = L^2(U)$. Now according to Proposition 4.9 we have a unique solution $f_3 \in H^2([0,L]; L^2(U)) \cap H^1([0,L]; D(S^{1/2})) \cap L^2([0,L]; D(S))$. Additionally it satisfies the assumptions made in the paragraph following Proposition 4.11 given that we let $X_s = H^s(U)$. The assumption that $f_3 \in H^2(\Omega)$ means that $f_3 \in H^2([0,L]; L^2(U)) \cap H^1([0,L]; H^1(U)) \cap L^2([0,L]; H^2(U))$ and thus satisfies the conditions assumed for g in propositions 4.11 and 4.12 except for the assumption $g'|_{x_3=0}, g'|_{x_3=L} \in D(A^{1/4})$. However in this particular case it means that we want $\frac{\partial f_3}{\partial x_3}|_{x_3=0} = \frac{\partial f_3}{\partial x_3}|_{x_3=L} \in D(S^{1/4}) = H_{00}^{1/2}$, which is exactly the other assumption on f_3. Hence the assumptions for both propositions are satisfied and it follows from their conclusions that

$$u_3 \in \cap_{k=0}^4 H^k([0,L]; H^{4-k}(U)) \subseteq H^4(\Omega)$$

and that u_3 satisfies (4.21) and (4.22). □

The other components (u_1, u_2) solve (5.9) which can be written

$$\begin{aligned} \frac{\partial^2}{\partial x_3^2}(u_1, u_2) &= T(u_1, u_2) + (f_1, f_2), && \text{for } 0 < x_3 < L, \\ (u_1, u_2) &= 0, && \text{for } x_3 = 0 \text{ and } x_3 = L, \end{aligned} \tag{4.23}$$

where T is the operator defined above.

Proposition 4.18. *Assume that* $(f_1, f_2) \in H^2(\Omega; \mathbb{R}^2)$*,* $n_2 f_1 - n_1 f_2 = 0$ on $\partial\Omega_0$ and $\partial\Omega_L$ *and* $\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial x_2}|_{x_3=0}, \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial x_2}|_{x_3=0} \in H^{1/2}_{00}(U)$*. Then there exists a unique solution,* (u_1, u_2)*, to* (4.20) *in* $H^4(\Omega)$ *which satisfies*

$$\|(u_1, u_2)\|_{2,\Omega} \leq c^{(30)} \|(f_1, f_2)\|_{0,\Omega} \tag{4.24}$$

and

$$\|(u_1, u_2)\|_{4,\Omega} \leq c^{(31)} \left(\|(f_1, f_2)\|_{2,\Omega} + \|(f_1, f_2)|_{x_3=0}\|_{D(T^{3/4})} + \|(f_1, f_2)|_{x_3=0}\|_{D(T^{3/4})} \right), \tag{4.25}$$

where $c^{(30)} = c^{(30)}(L)$ *and* $c^{(31)} = c^{(31)}(L)$*.*

Proof. The problem (4.23) is of the form of (4.19). It follows from propositions 4.6, 4.7 and 4.8 that T satisfies the assumptions on A given in the paragraph following (4.19) if we let $X = L^2(U)$. Now according to Proposition 4.14 we have a unique solution $(f_1, f_2) \in H^2([0, L]; L^2(U; \mathbb{R}^2)) \cap H^1([0, L]; D(T^{1/2})) \cap L^2([0, L]; D(T))$ Additionally it satisfies the assumptions made in the paragraph following Proposition 4.11 given that we let $X_s = H^s(U; \mathbb{R}^2)$. The assumption that $(f_1, f_2) \in H^2(\Omega; \mathbb{R}^2)$ means that $(f_1, f_2) \in H^2([0, L]; L^2(U; \mathbb{R}^2)) \cap H^1([0, L]; H^1(U; \mathbb{R}^2)) \cap L^2([0, L]; H^2(U; \mathbb{R}^2))$ and thus satisfies the conditions assumed for g in propositions 4.15 and 4.16 except that we assume $g|_{x_3=0}, g|_{x_3=L} \in D(A^{3/4})$. However in this particular case it means that we want $(f_1, f_2)|_{x_3=0} = (f_1, f_2)|_{x_3=L}$ to be in

$$D(A^{3/4}) = \left\{ u \in H^{3/2}(U; \mathbb{R}^2) \colon n_2 u_1 - n_1 u_2|_{\partial U} = 0,\ \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \in H^{1/2}_{00}(U) \right\}.$$

That $(f_1, f_2)|_{x_3=0} = (f_1, f_2)|_{x_3=L} \in H^{3/2}(U; \mathbb{R}^2)$ follows from $(f_1, f_2) \in H^2(\Omega; \mathbb{R}^2)$, and the conditions $n_2 f_1 - n_1 f_2 = 0$ on $\partial\Omega_0$ and $\partial\Omega_L$ and $\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial x_2}|_{x_3=0}, \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial x_2}|_{x_3=0} \in H^{1/2}_{00}(U)$ are exactly the other assumption on (f_1, f_2). Hence the assumptions for both propositions are satisfied and it follows from their conclusions that $(u_1, u_2) \in \cap_{k=0}^4 H^k([0, L]; (H^{4-k}(U))^2) \subseteq H^4(\Omega)$ and that (u_1, u_2) satisfies (4.24) and (4.25). □

Theorem 4.19. *Assume that* $f \in H^2(\Omega)$*,* $\frac{\partial f_3}{\partial x_3}|_{x_3=0}, \frac{\partial f_3}{\partial x_3}|_{x_3=L} \in H^{1/2}_{00}$*,* $n_2 f_1 - n_1 f_2 = 0$ on $\partial\Omega_0$ and $\partial\Omega_L$ *and* $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}|_{x_3=0}, \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}|_{x_3=0} \in H^{1/2}_{00}(U)$*, then there exists a unique solution,* v*, to* (4.1) *in* $H^3(\Omega)$*. The solution satisfies*

$$\|v\|_{1,\Omega} \leq M \|f\|_{0,\Omega}$$

and

$$\|v\|_{3,\Omega} \leq M \left(\|f\|_{2,\Omega} + \left\| \frac{\partial f_3}{\partial x_3} \right\|_{H^{1/2}_{00}(\Omega_0)} + \left\| \frac{\partial f_3}{\partial x_3} \right\|_{H^{1/2}_{00}(\Omega_L)} + \left\| \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right\|_{H^{1/2}_{00}(\Omega_0)} + \left\| \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right\|_{H^{1/2}_{00}(\Omega_L)} \right),$$

where $M = M(\Omega)$*.*

Proof. This follows from Propositions 4.17 and 4.18, the characterizations of $D(S^{1/4})$ and $D(T^{3/4})$, Theorem 1.14 and the fact that $\|v\|_{i,\Omega} = \|\operatorname{curl} u\|_{i,\Omega} \leq c^{(32)} \|u\|_{i+1,\Omega}$, where $c^{(32)} = c^{(32)}(\Omega)$. □

5 The Transport Equation

The next part in showing that B is a well-defined operator is making sure that f satisfies the conditions assumed in Theorem 4.19. Recall that f was defined as the solution to

$$\begin{aligned} &[(v_0 + u) \cdot \nabla] f = f \cdot \nabla(v_0 + u) && \text{in } \Omega, \\ &f = \eta && \text{on } \Omega_0. \end{aligned} \tag{5.1}$$

Now we want to put sufficient conditions on η to get a solution $f \in H^2(\Omega)$ that satisfies the following conditions

$$\left.\frac{\partial f_3}{\partial x_3}\right|_{x_3=0}, \left.\frac{\partial f_3}{\partial x_3}\right|_{x_3=L} \in H_{00}^{1/2}(U) \tag{5.2}$$

$$n_2 f_1 - n_1 f_2 = 0 \text{ on } \partial\Omega_0 \text{ and } \partial\Omega_L \tag{5.3}$$

and

$$\left.\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)\right|_{x_3=0}, \left.\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)\right|_{x_3=L} \in H_{00}^{1/2}(U). \tag{5.4}$$

This is done through first finding approximating $\eta \in C_c^\infty(\Omega_0)$ and $v = v_0 + u \in C^\infty(\Omega)$. Under these assumptions we can solve (5.1) along the integral curves of v. Due to Lemma 3.3 we know that the integral curves cover Ω so we get a solution f defined in all of Ω. It also gives a solution $f \in C^\infty(\Omega)$ and, if v is a good enough approximation, with the property $f|_{x_3=x_3'} \in C_c^\infty(U)$ for all $x_3' \in [0, L]$. For this solution we make a series of estimates. These estimates are then used to construct a solution satisfying the desired conditions as the limit of a sequence of solutions with the smooth data.

5.1 Estimates of f, $\frac{\partial f}{\partial x_3}$ and $\frac{\partial^2 f}{\partial x_3^2}$

We let $x' = (x_1, x_2)$ and $\nabla_{x'} = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$. Equation (5.1) can be written in the form of a transport equation

$$\begin{aligned} &\frac{\partial f}{\partial x_3} + (\mathbf{v} \cdot \nabla_{x'}) f + A f = 0 && \text{in } \Omega, \\ &f = \eta && \text{on } \Omega_0. \end{aligned} \tag{5.5}$$

This is done by dividing equation (5.1) by v_3 and moving all terms to the left hand side. We can divide by v_3 since Lemma 3.3 shows that $v_3 \geq b - c^{(1)}\gamma_0 > 0$. We get

$$\begin{aligned} &\frac{\partial f}{\partial x_3} + \frac{1}{v_3}\left((v_1, v_2) \cdot \nabla_{x'}\right) f - \frac{1}{v_3}(f \cdot \nabla) v = 0 && \text{in } \Omega, \\ &f = \eta && \text{on } \Omega_0. \end{aligned} \tag{5.6}$$

This is of the form of equation (5.5) if we take

$$\mathbf{v} = \frac{(v_1, v_2)}{v_3},$$

and

$$A = -\frac{1}{v_3}\begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{pmatrix}.$$

We note that

$$\left\| D^\alpha \left(\frac{1}{v_3}\right)\right\|_{0,\Omega}$$

for $|\alpha| \leq 3$ can be bounded by a constant depending on b, γ_0 and $\|v\|_{3,\Omega}$ if we estimate factors of the form $\frac{1}{v_3^k}$ with $\frac{1}{(b-c^{(1)}\gamma_0)^k}$ and use Theorem 1.13 to estimate the remaining factors. Hence we can also estimate $\left\|\frac{1}{v_3}\right\|_{3,\Omega}$ by a constant depending on b, γ_0 and $\|v\|_{3,\Omega}$. Using Theorem 1.13 again gives us that $\mathbf{v}$ is a function in $H^3(\Omega)$ and the components of A are functions in $H^2(\Omega)$ both bounded in norm by constants depending on b, γ_0 and $\|v\|_{3,\Omega}$.

This reformulation allows us to estimate the solution of equation (5.1) by estimating the solution of equation (5.5), which is done below.

Theorem 5.1. *Assuming $v \in C^\infty(\Omega)$ and $f \in C^\infty(\Omega)$ with $f|_{x_3=x_3'} \in C_c^\infty(U)$ for all $x_3' \in [0, L]$ the solution f of (5.5) satisfies*

$$\sup_{x_3\in[0,L]} \|f\|_{2,\Omega_{x_3}} \leq c^{(33)}\|\eta\|_{2,U} \tag{5.7}$$

and

$$\|f\|_{L^2([0,L];H^2(U))} \leq c^{(34)}\|\eta\|_{2,U}, \tag{5.8}$$

where $c^{(33)} = c^{(33)}(\Omega, b, \gamma_0, \|v\|_{3,\Omega})$ and $c^{(34)} = c^{(34)}(\Omega, b, \gamma_0, \|v\|_{3,\Omega})$.

To prove Theorem 5.1 we first need to prove a number of lemmas.

Lemma 5.2. *Under the assumptions given in Theorem 5.1 the solution to (5.5) satisfies the inequalities*

$$\frac{d}{dx_3}\|f\|^2_{0,\Omega_{x_3}} \leq c^{(35)}(\|\mathbf{v}\|_{3,\Omega_{x_3}} + \|A\|_{2,\Omega_{x_3}})\|f\|^2_{0,\Omega_{x_3}}, \tag{5.9}$$

$$\sup_{x_3\in[0,L]} \|f\|_{0,\Omega_{x_3}} \leq c^{(36)}\|\eta\|_{0,U} \tag{5.10}$$

and

$$\|f\|_{0,\Omega} \leq c^{(37)}\|\eta\|_{0,U}, \tag{5.11}$$

where $c^{(35)} = c^{(35)}(U)$, $c^{(36)} = c^{(36)}(\Omega, b, \gamma_0, \|v\|_{3,\Omega})$ and $c^{(37)} = c^{(37)}(\Omega, b, \gamma_0, \|v\|_{3,\Omega})$.

Proof. Using the notation $\int_{\Omega_{x_3}} h\,dx' = \int_U h(\cdot, x_3)\,dx'$ for a function $h : \Omega \to \mathbb{R}$ we begin with the expression

$$\begin{aligned}\frac{d}{dx_3}\int_{\Omega_{x_3}} |f|^2\,dx' &= \int_{\Omega_{x_3}} 2f \cdot \frac{d}{dx_3} f\,dx' \\ &= -2\int_{\Omega_{x_3}} f \cdot (\mathbf{v} \cdot \nabla_{x'})f\,dx' - 2\int_{\Omega_{x_3}} f \cdot Af\,dx'.\end{aligned}$$

Integration by parts gives us

$$-2\int_{\Omega_{x_3}} f\cdot(\mathbf{v}\cdot\nabla_{x'})f\,dx' = \int_{\Omega_{x_3}} f^2\nabla_{x'}\cdot u\,dx',$$

where the boundary term disappears since $f|_{x_3=x_3'}\in C_c^\infty(U)$. We also note that

$$|f\cdot Af|\le |A||f|^2$$

where $|A|$ is the induced 2-matrix norm of A. The matrix norm can be estimated by

$$|A|\le\left(\sum_{i,j}a_{ij}^2\right)^{1/2} = |A|_f,$$

where $|\cdot|_f$ is the Frobenius norm. Hence we get the estimate

$$\begin{aligned}\frac{d}{dx_3}\|f\|^2_{0,\Omega_{x_3}} &\le \left|\int_{\Omega_{x_3}} f^2\nabla_{x'}\cdot\mathbf{v}\,dx' - 2\int_{\Omega_{x_3}} f\cdot Af\,dx'\right|\\ &\le \int_{\Omega_{x_3}} f^2|\nabla_{x'}\cdot\mathbf{v}|\,dx' + 2\int_{\Omega_{x_3}}|A|_f f^2\,dx'\\ &\le (\|\nabla_{x'}\cdot\mathbf{v}\|_{C_b(\Omega_{x_3})} + 2\|A\|_{C_b(\Omega_{x_3})})\|f\|^2_{0,\Omega_{x_3}}.\end{aligned} \tag{5.12}$$

By Sobolev's embedding theorem we get (5.9). The inequality in (5.10) follows from applying Grönwall's inequality to (5.9) which gives

$$\sup_{x_3\in[0,L]}\|f\|^2_{0,\Omega_{x_3}}\le\|\eta\|^2_{0,U}\exp\left(\int_0^L c^{(35)}(\|\mathbf{v}\|_{3,\Omega_{x_3}}+\|A\|_{2,\Omega_{x_3}})\,dx_3\right),$$

were $\int_0^L c^{(35)}(\|\mathbf{v}\|_{3,\Omega_{x_3}}+\|A\|_{2,\Omega_{x_3}})\,dx_3\le c^{(35)}L^{1/2}(\|\mathbf{v}\|_{3,\Omega}+\|A\|_{2,\Omega})$.

Finally we note that from this it follows that

$$\|f\|^2_{0,\Omega}=\int_0^L\|f\|^2_{0,\Omega_{x_3}}\,dx_3\le c^{(36)}\int_0^L\|\eta\|^2_{0,U}\,dx_3\le c^{(36)}L\|\eta\|^2_{0,U},$$

which is the inequality in (5.11). □

Lemma 5.3. *Under the assumptions given in Theorem 5.1 the solution to* (5.5) *satisfies the inequality*

$$\frac{d}{dx_3}\left\|\frac{\partial f}{\partial x_1}\right\|^2_{0,\Omega_{x_3}}\le c^{(38)}(\|\mathbf{v}\|_{3,\Omega_{x_3}}+\|A\|_{2,\Omega_{x_3}})\|f\|^2_{1,\Omega_{x_3}}, \tag{5.13}$$

and

$$\frac{d}{dx_3}\left\|\frac{\partial f}{\partial x_2}\right\|^2_{0,\Omega_{x_3}}\le c^{(39)}(\|\mathbf{v}\|_{3,\Omega_{x_3}}+\|A\|_{2,\Omega_{x_3}})\|f\|^2_{1,\Omega_{x_3}}, \tag{5.14}$$

where $c^{(38)}=c^{(38)}(U)$ *and* $c^{(39)}=c^{(39)}(U)$.

Proof. Since the inequalities can be proven almost identically we only show the proof of the first one. Differentiating equation (5.5) with respect to x_1 gives us

$$\frac{\partial}{\partial x_3}\frac{\partial f}{\partial x_1} + (\mathbf{v}\cdot\nabla_{x'})\frac{\partial f}{\partial x_1} + \left(\frac{\partial \mathbf{v}}{\partial x_1}\cdot\nabla_{x'}\right) f + A\frac{\partial f}{\partial x_1} + \frac{\partial A}{\partial x_1} f = 0,$$

allowing us to get the estimate

$$\begin{aligned}\frac{d}{dx_3}\int_{\Omega_{x_3}}\left(\frac{\partial f}{\partial x_1}\right)^2 dx' \leq \int_{\Omega_{x_3}}\left|\frac{\partial f}{\partial x_1}\right|^2 |\nabla_{x'}\cdot\mathbf{v}|\,dx' + 2\int_{\Omega_{x_3}}\left|\frac{\partial f}{\partial x_1}\right|^2 |A|_f\,dx' \\ + 2\int_{\Omega_{x_3}}\left|\frac{\partial \mathbf{v}_1}{\partial x_1}\right|\left|\frac{\partial f}{\partial x_1}\right|^2 dx' + 2\int_{\Omega_{x_3}}\left|\frac{\partial \mathbf{v}_2}{\partial x_1}\right|\left|\frac{\partial f}{\partial x_1}\cdot\frac{\partial f}{\partial x_2}\right| dx' \\ + 2\int_{\Omega_{x_3}}\left|\frac{\partial A}{\partial x_1}\right|_f\left|\frac{\partial f}{\partial x_1}\right||f|dx'.\end{aligned}$$

This yields

$$\begin{aligned}\frac{d}{dx_3}\left\|\frac{\partial f}{\partial x_1}\right\|^2_{0,\Omega_{x_3}} \leq (\|\nabla_{x'}\cdot\mathbf{v}\|_{C_b(\Omega_{x_3})} + 2\|A\|_{C_b(\Omega_{x_3})})\left\|\frac{\partial f}{\partial x_1}\right\|^2_{0,\Omega_{x_3}} \\ + 2\left\|\frac{\partial \mathbf{v}_1}{\partial x_1}\right\|_{C_b(\Omega_{x_3})}\left\|\frac{\partial f}{\partial x_1}\right\|^2_{0,\Omega_{x_3}} + 2\left\|\frac{\partial \mathbf{v}_2}{\partial x_1}\right\|_{C_b(\Omega_{x_3})}\left\|\frac{\partial f}{\partial x_1}\right\|_{0,\Omega_{x_3}}\left\|\frac{\partial f}{\partial x_2}\right\|_{0,\Omega_{x_3}} \\ + c^{(3)}\left\|\frac{\partial A}{\partial x_1}\right\|_{1,\Omega_{x_3}}\|f\|_{1,\Omega_{x_3}}\left\|\frac{\partial f}{\partial x_1}\right\|_{0,\Omega_{x_3}}.\end{aligned} \tag{5.15}$$

By Sobolev's embedding theorem and straightforward estimates of the norms we get the desired inequality from (5.15) □

Lemma 5.4. *Under the assumptions given in Theorem 5.1 the solution to* (5.5) *satisfies the inequality*

$$\frac{d}{dx_3}\left\|\frac{\partial^2 f}{\partial x_1^2}\right\|^2_{0,\Omega_{x_3}} \leq c^{(40)}(\|\mathbf{v}\|_{3,\Omega_{x_3}} + \|A\|_{2,\Omega_{x_3}})\|f\|^2_{2,\Omega_{x_3}}, \tag{5.16}$$

$$\frac{d}{dx_3}\left\|\frac{\partial^2 f}{\partial x_2^2}\right\|^2_{0,\Omega_{x_3}} \leq c^{(41)}(\|\mathbf{v}\|_{3,\Omega_{x_3}} + \|A\|_{2,\Omega_{x_3}})\|f\|^2_{2,\Omega_{x_3}}, \tag{5.17}$$

and

$$\frac{d}{dx_3}\left\|\frac{\partial^2 f}{\partial x_1\partial x_2}\right\|^2_{0,\Omega_{x_3}} \leq c^{(42)}(\|\mathbf{v}\|_{3,\Omega_{x_3}} + \|A\|_{2,\Omega_{x_3}})\|f\|^2_{2,\Omega_{x_3}}, \tag{5.18}$$

where $c^{(40)} = c^{(40)}(U)$, $c^{(41)} = c^{(41)}(U)$ *and* $c^{(42)} = c^{(42)}(U)$.

Proof. Since the proof of the three inequalities are almost identical we only show the proof of the first one. By a taking a second derivative of equation (5.5) with respect to x_1 we get

$$\begin{aligned}\frac{\partial}{\partial x_3}\frac{\partial^2 f}{\partial x_1^2} + (\mathbf{v}\cdot\nabla_{x'})\frac{\partial^2 f}{\partial x_1^2} + 2\left(\frac{\partial \mathbf{v}}{\partial x_1}\cdot\nabla_{x'}\right)\frac{\partial f}{\partial x_1} + \left(\frac{\partial^2 \mathbf{v}}{\partial x_1^2}\cdot\nabla_{x'}\right) f \\ + A\frac{\partial^2 f}{\partial x_1^2} + 2\frac{\partial A}{\partial x_1}\frac{\partial f}{\partial x_1} + \frac{\partial^2 A}{\partial x_1^2} f = 0.\end{aligned}$$

From this we can, as before, get the estimate

$$
\begin{aligned}
\frac{d}{dx_3}\left\|\frac{\partial^2 f}{\partial x_1^2}\right\|_{0,\Omega_{x_3}}^2 \leq{}& (\|\nabla_{x'}\cdot\mathbf{v}\|_{C_b(\Omega_{x_3})} + 2\|A\|_{C_b(\Omega_{x_3})})\left\|\frac{\partial^2 f}{\partial x_1^2}\right\|_{0,\Omega_{x_3}}^2 \\
&+ 4\left\|\frac{\partial \mathbf{v}_1}{\partial x_1}\right\|_{C_b(\Omega_{x_3})}\left\|\frac{\partial^2 f}{\partial x_1^2}\right\|_{0,\Omega_{x_3}}^2 + 4\left\|\frac{\partial \mathbf{v}_2}{\partial x_1}\right\|_{C_b(\Omega_{x_3})}\left\|\frac{\partial^2 f}{\partial x_1\partial x_2}\right\|_{0,\Omega_{x_3}}\left\|\frac{\partial^2 f}{\partial x_1^2}\right\|_{0,\Omega_{x_3}} \\
&+ c^{(3)}\left\|\frac{\partial^2 \mathbf{v}_1}{\partial x_1^2}\right\|_{1,\Omega_{x_3}}\left\|\frac{\partial f}{\partial x_1}\right\|_{1,\Omega_{x_3}}\left\|\frac{\partial^2 f}{\partial x_1^2}\right\|_{0,\Omega_{x_3}} \\
&+ c^{(3)}\left\|\frac{\partial^2 \mathbf{v}_2}{\partial x_1^2}\right\|_{1,\Omega_{x_3}}\left\|\frac{\partial f}{\partial x_2}\right\|_{1,\Omega_{x_3}}\left\|\frac{\partial^2 f}{\partial x_1^2}\right\|_{0,\Omega_{x_3}} \\
&+ c^{(3)}\left\|\frac{\partial A}{\partial x_1}\right\|_{1,\Omega_{x_3}}\left\|\frac{\partial f}{\partial x_1}\right\|_{1,\Omega_{x_3}}\left\|\frac{\partial^2 f}{\partial x_1^2}\right\|_{0,\Omega_{x_3}} \\
&+ \left\|\frac{\partial^2 A}{\partial x_1^2}\right\|_{0,\Omega_{x_3}}\left\|\frac{\partial^2 f}{\partial x_1^2}\right\|_{0,\Omega_{x_3}}\|f\|_{C_b(\Omega_{x_3})}
\end{aligned}
$$

which, again by Sobolev's embedding theorem and straightforward estimates of the norms, gives the desired inequality. □

Proof of Theorem 5.1. Summing the inequalities proven in Lemmas 5.2-5.4 gives

$$\frac{d}{dx_3}\|f\|_{2,\Omega_{x_3}}^2 \leq c^{(43)}(\|\mathbf{v}\|_{3,\Omega_{x_3}} + \|A\|_{2,\Omega_{x_3}})\|f\|_{2,\Omega_{x_3}}^2,$$

where $c^{(43)} = c^{(43)}(U)$. By applying Grönwall's inequality we get

$$\sup_{x_3\in[0,L]}\|f\|_{2,\Omega_{x_3}}^2 \leq \|\eta\|_{2,U}^2 \exp\left(\int_0^L c^{(43)}(\|\mathbf{v}\|_{3,\Omega_{x_3}} + \|A\|_{2,\Omega_{x_3}})\,dx_3\right).$$

By estimating

$$\exp\left(\int_0^L c^{(43)}(\|\mathbf{v}\|_{3,\Omega_{x_3}} + \|A\|_{2,\Omega_{x_3}})\,dx_3\right) \leq \exp\left(c^{(43)}L^{1/2}(\|\mathbf{v}\|_{3,\Omega} + \|A\|_{2,\Omega})\right)$$

we get the inequality in (5.7). This also implies that

$$
\begin{aligned}
\|f\|_{L^2([0,L];H^2(U))} &= \left(\int_0^L \|f\|_{2,\Omega_{x_3}}^2\,dx_3\right)^{1/2} \\
&\leq \left(\int_0^L \sup_{x_3\in[0,L]}\|f\|_{2,\Omega_{x_3}}^2\,dx_3\right) \\
&\leq \left(\int_0^L (c^{(33)})^2\|\eta\|_{2,U}^2\,dx_3\right)^{1/2} \\
&\leq c^{(33)}L^{1/2}\|\eta\|_{2,U}
\end{aligned}
\tag{5.19}
$$

which gives the inequality (5.8) □

Theorem 5.5. *Assuming $v \in C^\infty(\Omega)$ and $f \in C^\infty(\Omega)$ with $f|_{x_3=x_3'} \in C_c^\infty(U)$ for all $x_3' \in [0,L]$ the solution f of (5.5) satisfies*

$$\sup_{x_3\in[0,L]} \left\| \frac{\partial f}{\partial x_3} \right\|_{1,\Omega_{x_3}} \le c^{(44)} \|\eta\|_{2,U}. \tag{5.20}$$

and

$$\|f\|_{H^1([0,L];H^1(U))} \le c^{(45)} \|\eta\|_{2,U}, \tag{5.21}$$

were $c^{(44)} = c^{(44)}(\Omega, b, \gamma_0, \|v\|_{3,\Omega})$ and $c^{(45)} = c^{(45)}(\Omega, b, \gamma_0, \|v\|_{3,\Omega})$.

Proof. By using equation (5.5) we get the estimate

$$\begin{aligned}
\left\| \frac{\partial f}{\partial x_3} \right\|_{1,\Omega_{x_3}} &\le \|((\mathbf{v}_1,\mathbf{v}_2)\cdot\nabla_{x'})\, f\|_{1,\Omega_{x_3}} + \|Af\|_{1,\Omega_{x_3}} \\
&\le \left\| \mathbf{v}_1 \frac{\partial f}{\partial x_1} \right\|_{1,\Omega_{x_3}} + \left\| \mathbf{v}_2 \frac{\partial f}{\partial x_2} \right\|_{1,\Omega_{x_3}} + \|Af\|_{1,\Omega_{x_3}} \\
&\le c^{(3)} \|\mathbf{v}_1\|_{2,\Omega_{x_3}} \left\| \frac{\partial f}{\partial x_1} \right\|_{1,\Omega_{x_3}} + c^{(3)} \|\mathbf{v}_2\|_{2,\Omega_{x_3}} \left\| \frac{\partial f}{\partial x_2} \right\|_{1,\Omega_{x_3}} \\
&\quad + c^{(3)} \|A\|_{1,\Omega_{x_3}} \|f\|_{2,\Omega_{x_3}} \\
&\le 3c^{(3)} \left(\|\mathbf{v}\|_{2,\Omega_{x_3}} + \|A\|_{1,\Omega_{x_3}} \right) \|f\|_{2,\Omega_{x_3}},
\end{aligned} \tag{5.22}$$

This implies

$$\sup_{x_3\in[0,L]} \left\| \frac{\partial f}{\partial x_3} \right\|_{1,\Omega_{x_3}} \le 3c^{(3)} c^{(2)} (\|\mathbf{v}\|_{3,\Omega} + \|A\|_{2,\Omega}) \|\eta\|_{2,U} \le c^{(44)} \|\eta\|_{2,U},$$

by using $H^n(\Omega) \subset H^1([0,L]; H^{n-1}(U))$ and Theorem 1.12. This is the inequality (5.20). It also allows us to estimate

$$\int_0^L \left\| \frac{\partial f}{\partial x_3} \right\|_{1,\Omega_{x_3}}^2 dx_3 \le \int_0^L (c^{(44)})^2 \|\eta\|_{2,U}^2 \, dx_3 \le (c^{(44)})^2 L \|\eta\|_{2,U}^2$$

which gives

$$\begin{aligned}
\|f\|_{H^1([0,L];H^1(U))} &= \left(\int_0^L \left(\|f\|_{1,\Omega_{x_3}}^2 + \left\| \frac{\partial f}{\partial x_3} \right\|_{1,\Omega_{x_3}}^2 \right) dx_3 \right)^{1/2} \\
&\le \left(\int_0^L \left(\|f\|_{2,\Omega_{x_3}}^2 + \left\| \frac{\partial f}{\partial x_3} \right\|_{1,\Omega_{x_3}}^2 \right) dx_3 \right)^{1/2} \\
&\le (c^{(33)} + c^{(44)}) L^{1/2} \|\eta\|_{2,U}
\end{aligned} \tag{5.23}$$

by using equation (5.8), which proves the inequality (5.21). □

Theorem 5.6. *Assuming $v \in C^\infty(\Omega)$ and $f \in C^\infty(\Omega)$ with $f|_{x_3=x_3'} \in C_c^\infty(U)$ for all $x_3' \in [0,L]$ the solution f of (5.5) satisfies*

$$\sup_{x_3\in[0,L]} \left\| \frac{\partial^2 f}{\partial x_3^2} \right\|_{0,\Omega_{x_3}} \le c^{(46)} \|\eta\|_{2,U}. \tag{5.24}$$

and

$$\|f\|_{H^2([0,L];L^2(U))} \le c^{(47)} \|\eta\|_{2,U} \tag{5.25}$$

were $c^{(46)} = c^{(46)}(\Omega, b, \gamma_0, \|v\|_{3,\Omega})$ and $c^{(47)} = c^{(47)}(\Omega, b, \gamma_0, \|v\|_{3,\Omega})$.

Proof. Differentiating equation (5.5) with respect to x_3 gives

$$\frac{\partial^2 f}{\partial x_3^2} = \frac{\partial \mathbf{v}}{\partial x_3} \cdot \nabla_{x'} f + \mathbf{v} \cdot \nabla_{x'} \frac{\partial f}{\partial x_3} + \frac{\partial A}{\partial x_3} f + A \frac{\partial f}{\partial x_3}$$

With this we get the estimate

$$\begin{aligned}
\left\| \frac{\partial^2 f}{\partial x_3^2} \right\|_{0,\Omega_{x_3}} &\le \left\| \frac{\partial \mathbf{v}}{\partial x_3} \cdot \nabla_{x'} f \right\|_{0,\Omega_{x_3}} + \left\| \mathbf{v} \cdot \nabla_{x'} \frac{\partial f}{\partial x_3} \right\|_{0,\Omega_{x_3}} \\
&\quad + \left\| \frac{\partial A}{\partial x_3} f \right\|_{0,\Omega_{x_3}} + \left\| A \frac{\partial f}{\partial x_3} \right\|_{0,\Omega_{x_3}} \\
&\le \max\{c^{(3)}, c^{(3)}\} \Bigg(\left\| \frac{\partial \mathbf{v}_1}{\partial x_3} \right\|_{1,\Omega_{x_3}} \left\| \frac{\partial f}{\partial x_1} \right\|_{1,\Omega_{x_3}} + \left\| \frac{\partial \mathbf{v}_2}{\partial x_3} \right\|_{1,\Omega_{x_3}} \left\| \frac{\partial f}{\partial x_2} \right\|_{1,\Omega_{x_3}} \\
&\qquad + \|\mathbf{v}_1\|_{2,\Omega_{x_3}} \left\| \frac{\partial^2 f}{\partial x_1 \partial x_3} \right\|_{0,\Omega_{x_3}} + \|\mathbf{v}_2\|_{2,\Omega_{x_3}} \left\| \frac{\partial^2 f}{\partial x_2 \partial x_3} \right\|_{0,\Omega_{x_3}} \\
&\qquad + \left\| \frac{\partial A}{\partial x_3} \right\|_{0,\Omega_{x_3}} \|f\|_{2,\Omega_{x_3}} + \|A\|_{1,\Omega_{x_3}} \left\| \frac{\partial f}{\partial x_3} \right\|_{1,\Omega_{x_3}} \Bigg) \\
&\le 2\max\{c^{(3)}, c^{(3)}\} \Bigg(\left(\left\| \frac{\partial \mathbf{v}}{\partial x_3} \right\|_{1,\Omega_{x_3}} + \left\| \frac{\partial A}{\partial x_3} \right\|_{0,\Omega_{x_3}} \right) \|f\|_{2,\Omega_{x_3}} \\
&\qquad + \left(\|\mathbf{v}\|_{2,\Omega_{x_3}} + \|A\|_{1,\Omega_{x_3}} \right) \left\| \frac{\partial f}{\partial x_3} \right\|_{1,\Omega_{x_3}} \Bigg),
\end{aligned}$$

This implies

$$\begin{aligned}
\sup_{x_3\in[0,L]} \left\| \frac{\partial^2 f}{\partial x_3^2} \right\|_{0,\Omega_{x_3}} &\le 2\max\{c^{(3)}, c^{(3)}\} c^{(2)} \Bigg(\left(\left\| \frac{\partial \mathbf{v}}{\partial x_3} \right\|_{2,\Omega} + \left\| \frac{\partial A}{\partial x_3} \right\|_{1,\Omega} \right) \\
&\qquad + \left(\|\mathbf{v}\|_{3,\Omega} + \|A\|_{2,\Omega} \right) \Bigg) \|\eta\|_{2,U} \\
&\le c^{(46)} \|\eta\|_{2,U},
\end{aligned}$$

which is the inequality (5.24), by the same argument that gives (5.20). From this we get the estimate

$$\int_0^L \left\| \frac{\partial^2 f}{\partial x_3^2} \right\|_{0,\Omega_{x_3}}^2 dx_3 \le \int_0^L (c^{(46)})^2 \|\eta\|_{2,U}^2 \, dx_3 \le (c^{(46)})^2 L \|\eta\|_{2,U}^2$$

which allows us to estimate

$$\begin{aligned} \|f\|_{H^2([0,L];L^2(U))} &= \left(\int_0^L \left(\|f\|_{0,\Omega_{x_3}}^2 + \left\| \frac{\partial f}{\partial x_3} \right\|_{0,\Omega_{x_3}}^2 + \left\| \frac{\partial^2 f}{\partial x_3^2} \right\|_{0,\Omega_{x_3}}^2 \right) dx_3 \right)^{1/2} \\ &\le \left(\int_0^L \left(\|f\|_{2,\Omega_{x_3}}^2 + \left\| \frac{\partial f}{\partial x_3} \right\|_{1,\Omega_{x_3}}^2 + \left\| \frac{\partial^2 f}{\partial x_3^2} \right\|_{0,\Omega_{x_3}}^2 \right) dx_3 \right)^{1/2} \\ &\le (c^{(33)} + c^{(44)} + c^{(46)}) L^{1/2} \|\eta\|_{2,U} \end{aligned} \tag{5.26}$$

by using (5.8) and (5.21), this proves the inequality (5.25). □

5.2 Sufficient conditions on η

We want to show the following result.

Theorem 5.7. *For $v \in H^3(\Omega)$ and $\eta \in H_0^2(U)$ there exists a unique function $f \in W^{1,\infty}([0,L]; H_0^1(U)) \cap L^\infty([0,L]; H_0^2(U))$, which is a solution to equation* (5.5) *in the sense that for almost every $x_3 \in [0,L]$ the first equation in* (5.5) *holds as an equality in $H^1(U)$ and $f|_{x_3=0} = \eta$ in $H^1(U)$. Furthermore the conditions* (5.2), (5.3) *and* (5.4) *are satisfied for this solution.*

Remark 5.8. Note that $f \in C([0,L]; H^{3/2}(U))$ and $\frac{\partial f}{\partial x_3} \in C([0,L]; H^{1/2}(U))$ (cf. [14, Chapter 1, Theorem 3.1]) so that the conditions (5.2)–(5.4) make sense. In fact, we show below that $f \in C_w([0,L]; H_0^2(U))$ and $\frac{\partial f}{\partial x_3} \in C_w([0,L]; H_0^1(U))$. With a little more work it is also possible to show that $f \in C([0,L]; H_0^2(U))$ with $\frac{\partial f}{\partial x_3} \in C([0,L]; H_0^1(U))$ (see e.g. [23, Chapter 16]), but since this result is not needed for the purpose of this thesis the proof is not included.

Proof. The condition $\eta \in H_0^2(U)$ means that η can be approximated by a sequence of functions, $\{\eta_i\}_{i=1}^\infty$ in $C_c^\infty(U)$. If we also approximate v by a sequence of functions, $\{v_k\}_{k=1}^\infty$ in $C^\infty(\Omega)$ we get, as noted in the beginning of this section, corresponding solutions to equation (5.1), $f_{i,k}$, in $C^\infty(\Omega)$. For any i there is a large enough k_i such that $f_{i,k_i}|_{x_3=x_3'} \in C_c^\infty(U)$ for all $x_3' \in [0,L]$, which can be chosen in such a way that $k_i \to \infty$ as $i \to \infty$. The sequence $\{f_{i,k_i}\}_{i=1}^\infty$ is bounded in $H^2([0,L]; L^2(U)) \cap H^1([0,L]; H^1(U)) \cap L^2([0,L]; H^2(U))$ so we can extract a subsequence which is weakly convergent to some function f. We note that

$$f \in H^2([0,L]; L^2(U)) \cap H^1([0,L]; H^1(U)) \cap L^2([0,L]; H^2(U)) \subset H^2(\Omega).$$

Due to equation (5.7) we can bound the sequence $\{f_{i,k_i}|_{x_3=x_3'}\}_{i=1}^\infty$ in $H^2(U)$ for a given $x_3' \in [0,L]$. This means that we can extract a weakly convergent subsequence with limit

$\hat{f}_{x_3'} \in H^2(U)$. For any linear functional l on $H^1(U)$ we can define $l_{x_3'}(g) = l(g|_{x_3=x_3'})$ as a linear functional on $H^1([0,L];H^1(U))$ since $C([0,L];H^1(U)) \subset H^1([0,L];H^1(U))$. Now

$$l(f|_{x_3=x_3'}) = l_{x_3'}(f) = \lim_{i\to\infty} l_{x_3'}(f_{i,k_i}) = \lim_{i\to\infty} l(f_{i,k_i}|_{x_3=x_3'}) = l(\hat{f}_{x_3'})$$

for all linear functionals on $H^1(U)$ and hence $f|_{x_3=x_3'} = \hat{f}_{x_3'}$ in $H^1(U)$. However, since $\hat{f}_{x_3'} \in H^2(U)$ the equality also holds in $H^2(U)$ and thus $f|_{x_3=x_3'} \in H^2(U)$. Using that the trace operator T is bounded and linear we get that

$$Tf|_{x_3=x_3'} = T\hat{f}_{x_3'} = w - \lim_{i\to\infty} Tf_{i,k_i}|_{x_3=x_3'} = 0.$$

Similarly we get that $T\frac{\partial f|_{x_3=x_3'}}{\partial x_1} = T\frac{\partial f|_{x_3=x_3'}}{\partial x_2} = 0$. It follows that $f|_{x_3=x_3'} \in H_0^2(U)$ and hence $f \in L^\infty([0,L];H_0^2(U))$. By the same reasoning we get $\left.\frac{\partial f}{\partial x_3}\right|_{x_3=x_3'} \in H_0^1(U)$, which together with $f|_{x_3=x_3'} \in H_0^2(U)$ gives $f \in W^{1,\infty}([0,L];H_0^1(U))$.

Moreover, from [14, Chapter 1, Theorem 3.1]) it follows that $f \in C([0,L];H^{3/2}(U))$ and $\frac{\partial f}{\partial x_3} \in C([0,L];H^{1/2}(U))$ and hence

$$f \in C_w([0,L];H_0^2(U)), \quad \frac{\partial f}{\partial x_3} \in C_w([0,L];H_0^1(U))$$

by [14, Chapter 3, Lemma 8.1]). It is therefore clear that f satisfies the conditions (5.2)–(5.4).

To show that f is a solution to (5.1) we use

$$f_{i,k_i}|_{x_3=x_3'} \rightharpoonup f|_{x_3=x_3'} \text{ in } H^2(U),$$

$$\left.\frac{\partial f_{i,k_i}}{\partial x_3}\right|_{x_3=x_3'} \rightharpoonup \left.\frac{\partial f}{\partial x_3}\right|_{x_3=x_3'} \quad \text{in } H^1(U),$$

as $i \to \infty$, which was shown above and

$$\mathbf{v}_{k_i}|_{x_3=x_3'} \to \mathbf{v}|_{x_3=x_3'} \text{ in } H^2(U)$$

$$A_{k_i}|_{x_3=x_3'} \to A|_{x_3=x_3'} \text{ in } H^1(U)$$

as $i \to \infty$, which follows from $H^1([0,L];H^n(U)) \subset C([0,L];H^n(U))$. Together these limits imply

$$\left.\left(\frac{\partial f_{i,k_i}}{\partial x_3} + (\mathbf{v}_{k_i}\cdot\nabla)f_{i,k_i} + A_{k_i}f_{i,k_i}\right)\right|_{x_3=x_3'} \rightharpoonup \left.\left(\frac{\partial f}{\partial x_3} + (\mathbf{v}\cdot\nabla)f + Af\right)\right|_{x_3=x_3'} \text{ in } H^1(U)$$

as $i \to \infty$. But since

$$\left\|\frac{\partial f_{i,k_i}}{\partial x_3} + (\mathbf{v}_{k_i}\cdot\nabla)f_{i,k_i} + A_{k_i}f_{i,k_i}\right\|_{1,\Omega_{x_3'}} = 0 \ \forall i$$

we get that

$$\begin{aligned}&\left\|\frac{\partial f}{\partial x_3}+(\mathbf{v}\cdot\nabla)f+Af\right\|_{1,\Omega_{x_3'}}\\&\leq\liminf_{i\to\infty}\left\|\frac{\partial f_{i,k_i}}{\partial x_3}+(\mathbf{v}_{k_i}\cdot\nabla)f_{i,k_i}+A_{k_i}f_{i,k_i}\right\|_{1,\Omega_{x_3'}}\\&=0.\end{aligned}$$

Hence the first equation in (5.5) holds for f as an equality in $H^1(U)$ for almost every $x_3\in[0,L]$. That $f|_{x_3=0}=\eta$ follows immediately from

$$f_{i,k_i}|_{x_3=0}\rightharpoonup f|_{x_3=0}\ \text{ in }\ H^2(U)$$

and

$$f_{i,k_i}|_{x_3=0}=\eta_i\to\eta\ \text{ in }\ H^2(U).$$

It remains to prove uniqueness. For any solution to equation (5.5), $f\in L^\infty([0,L];H^2(U))\cap W^{1,\infty}([0,L];H^1(U))$, we can repeat the steps in Lemma 5.2 (the integration by parts gives no boundary term here since $\mathbf{v}\cdot n=0$) and get an inequality like (5.12), that is

$$\frac{d}{dx_3}\|f\|^2_{0,\Omega_{x_3}}\leq(\|\nabla_{x'}\cdot\mathbf{v}\|_{C_b(\Omega_{x_3})}+2\|A\|_{C_b(\Omega_{x_3})})\|f\|^2_{0,\Omega_{x_3}}.$$

We estimate

$$\begin{aligned}(\|\nabla_{x'}\cdot\mathbf{v}\|_{C_b(\Omega_{x_3})}+2\|A\|_{C_b(\Omega_{x_3})})&\leq(\|\nabla_{x'}\cdot\mathbf{v}\|_{C_b(\Omega)}+2\|A\|_{C_b(\Omega)})\\&\leq c^{(1)}(\|\nabla_{x'}\cdot\mathbf{v}\|_{2,\Omega}+2\|A\|_{2,\Omega})\end{aligned}$$

by using Sobolev's embedding theorem and apply Grönwall's lemma to get

$$\|f\|^2_{0,\Omega_{x_3}}\leq\|\eta\|^2_{0,U}\exp\left(\int_0^z c^{(1)}(\|\nabla_{x'}\cdot\mathbf{v}\|_{2,\Omega}+2\|A\|_{2,\Omega})\,dx_3\right).$$

This inequality shows that $\eta=0$ implies $f=0$ and hence the solution is unique. □

6 Proof of the Main Result

It remains to prove that B is a contraction. To do this we will need some additional estimates of the solutions to (5.1). However we begin by introducing some notation for the difference of two solutions to (5.1) for two different v, which we will denote $v^{(1)}$ and $v^{(2)}$. Similarly we let all related functions use the same superscript, i.e. $f^{(1)}$ solves (5.1) for $v = v^{(1)}$ and $f^{(2)}$ the same equation for $v = v^{(2)}$. Additionally we introduce $[\cdot]$ to denote the difference of two such functions, i.e. $[v] = v^{(1)} - v^{(2)}$ and $[f] = f^{(1)} - f^{(2)}$. The estimates we want to show are summarized in the following proposition.

Proposition 6.1. *f satisfies the inequalities*

$$\|f\|_{0,\Omega} \leq c^{(48)}(\|h\|_{0,U} + \|\nabla_T g\|_{0,U})$$

and

$$\|f\|_{2,\Omega} + \left\|\frac{\partial f_3}{\partial x_3}\right\|_{H^{1/2}_{00}(\Omega_0)} + \left\|\frac{\partial f_3}{\partial x_3}\right\|_{H^{1/2}_{00}(\Omega_L)} + \left\|\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right\|_{H^{1/2}_{00}(\Omega_0)} + \left\|\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right\|_{H^{1/2}_{00}(\Omega_L)}$$
$$\leq c^{(49)}(\|h\|_{2,U} + \|\nabla_T g\|_{2,U}),$$

and $[f]$, as defined above, satisfies the inequality

$$\|[f]\|_{0,\Omega} \leq c^{(50)}(\|h\|_{2,U} + \|\nabla_T g\|_{2,U})\|[v]\|_{1,\Omega},$$

where $c^{(48)} = c^{(48)}(\Omega, b, \gamma_0, \|v\|_{3,\Omega})$, $c^{(49)} = c^{(49)}(\Omega, b, \gamma_0, \|v\|_{3,\Omega})$ and $c^{(50)} = c^{(50)}(\Omega, b, \gamma_0, \|v^{(1)}\|_{3,\Omega}, \|v^{(2)}\|_{3,\Omega})$.

Proof. To prove the first inequality we note that due to the way the solution was constructed the inequality in (5.11) holds and hence

$$\|f\|_{0,\Omega} \leq c^{(37)}\|\eta\|_{0,U}.$$

Now recall that η is defined by

$$\begin{aligned} n \cdot \eta &= h && \text{on } \Omega_0, \\ \eta_T &= \frac{h}{v \cdot n} v_T - \frac{1}{v \cdot n} n \times \nabla_T g && \text{on } \Omega_0. \end{aligned}$$

This gives

$$\begin{aligned} \|\eta\|_{0,U} &\leq \|h\|_{0,U} + \left\|\frac{h}{v \cdot n} v_T\right\|_{0,\Omega_0} + \left\|\frac{1}{v \cdot n} n \times \nabla_T g\right\|_{0,\Omega_0} \\ &\leq \|h\|_{0,U} + c^{(3)} \left\|\frac{v_T}{v \cdot n}\right\|_{2,\Omega_0} \|h\|_{0,U} + \left\|\frac{1}{v \cdot n} n \times \nabla_T g\right\|_{0,\Omega_0} \\ &\leq \|h\|_{0,U} + c^{(3)}\|\mathbf{v}\|_{2,\Omega_0} \|h\|_{0,U} + \frac{1}{b} \|n \times \nabla_T g\|_{0,U} \\ &\leq \|h\|_{0,U} + c^{(3)}c^{(2)}\|\mathbf{v}\|_{3,\Omega} \|h\|_{0,U} + \frac{1}{b} \|\nabla_T g\|_{0,U} \\ &\leq c^{(51)}(\|h\|_{0,U} + \|\nabla_T g\|_{0,U}), \end{aligned}$$

where $c^{(51)} = c^{(51)}(\Omega, b, \gamma_0, \|v\|_{3,\Omega})$. From this it follows that

$$\|f\|_{0,\Omega} \leq c^{(37)} c^{(51)} (\|h\|_{0,\Omega_0} + \|\nabla_T g\|_{0,\Omega_0}).$$

To prove the second inequality we note that

$$\left\| \frac{\partial f_3}{\partial x_3} \right\|_{H^{1/2}_{00}(\Omega_0)} \lesssim \left\| \frac{\partial f_3}{\partial x_3} \right\|_{1,\Omega_0}$$

and

$$\left\| \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right\|_{H^{1/2}_{00}(\Omega_0)} \lesssim \left\| \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right\|_{1,\Omega_0} \leq 2\|f\|_{2,\Omega_0}.$$

Since the same inequalities hold on Ω_L it is sufficient to prove

$$\|f\|_{2,\Omega} + \|f\|_{2,\Omega_0} + \|f\|_{2,\Omega_L} + \left\| \frac{\partial f}{\partial x_3} \right\|_{1,\Omega_0} + \left\| \frac{\partial f}{\partial x_3} \right\|_{1,\Omega_L} \leq c^{(49)} (\|h\|_{2,U} + \|\nabla_T g\|_{2,U}).$$

As a consequence to the way the solution was constructed it satisfies equations (5.7), (5.8), (5.21), (5.20) and (5.25), hence

$$\begin{aligned}
&\|f\|_{2,\Omega} + \|f\|_{2,\Omega_0} + \|f\|_{2,\Omega_L} + \left\| \frac{\partial f}{\partial x_3} \right\|_{1,\Omega_0} + \left\| \frac{\partial f}{\partial x_3} \right\|_{1,\Omega_L} \\
&\quad = (\|f\|^2_{H^2([0,L],L^2(U))} + \|f\|^2_{H^1([0,L],H^1(U))} + \|f\|^2_{H^1([0,L],H^2(U))})^{1/2} \\
&\qquad + \|f\|_{2,\Omega_0} + \|f\|_{2,\Omega_L} + \left\| \frac{\partial f}{\partial x_3} \right\|_{1,\Omega_0} + \left\| \frac{\partial f}{\partial x_3} \right\|_{1,\Omega_L} \\
&\quad \leq (2c^{(33)} + c^{(34)} + c^{(45)} + 2c^{(44)} + c^{(47)}) \|\eta\|_{2,U}.
\end{aligned}$$

Now

$$\begin{aligned}
\|\eta\|_{2,U} &\leq \|h\|_{2,U} + \left\| \frac{h}{v \cdot n} v_T \right\|_{2,\Omega_0} + \left\| \frac{1}{v \cdot n} n \times \nabla_T g \right\|_{2,\Omega_0} \\
&\leq \|h\|_{2,U} + c^{(3)} \left\| \frac{v_T}{v \cdot n} \right\|_{2,\Omega_0} \|h\|_{2,U} + c^{(3)} \left\| \frac{1}{v \cdot n} \right\|_{2,\Omega_0} \|n \times \nabla_T g\|_{2,U} \\
&\leq \|h\|_{2,U} + c^{(3)} c^{(2)} \|\mathbf{v}\|_{3,\Omega} \|h\|_{2,U} + c^{(3)} c^{(2)} \left\| \frac{1}{v_3} \right\|_{3,\Omega} \|\nabla_T g\|_{2,U} \\
&\leq c^{(52)} (\|h\|_{2,U} + \|\nabla_T g\|_{2,U}),
\end{aligned}$$

where $c^{(52)} = c^{(52)}(\Omega, b, \|v\|_{3,\Omega})$. From these estimates it follows that

$$\begin{aligned}
&\|f\|_{2,\Omega} + \|f\|_{2,\Omega_0} + \|f\|_{2,\Omega_L} + \left\| \frac{\partial f}{\partial x_3} \right\|_{1,\Omega_0} + \left\| \frac{\partial f}{\partial x_3} \right\|_{1,\Omega_L} \\
&\quad \leq (2c^{(33)} + c^{(34)} + c^{(45)} + 2c^{(44)} + c^{(47)}) c^{(52)} (\|h\|_{2,U} + \|\nabla_T g\|_{2,U}).
\end{aligned}$$

For the third inequality we begin by writing equation (5.1) in the form of equation (5.5) and taking the difference of the equation with $v = v^{(1)}$ and the one with $v = v^{(2)}$ gives us an equation for $[f]$

$$\begin{aligned}
&\frac{\partial[f]}{\partial x_3} + (\mathbf{v}^{(1)} \cdot \nabla_{x'})f^{(1)} - (\mathbf{v}^{(2)} \cdot \nabla_{x'})f^{(2)} + A^{(1)}f^{(1)} - A^{(2)}f^{(2)} = 0, \quad \text{in } \Omega \\
&[f] = [\eta], \qquad \text{on } \Omega_0.
\end{aligned}$$

Through some algebra it is not hard to see that this is equivalent to

$$\begin{aligned}
&\frac{\partial[f]}{\partial x_3} + (\mathbf{v}^{(1)} \cdot \nabla_{x'})[f] + A^{(1)}[f] = -([\mathbf{v}] \cdot \nabla_{x'})f^{(2)} - [A]f^{(2)}, \quad \text{in } \Omega \\
&[f] = [\eta], \qquad \text{on } \Omega_0.
\end{aligned}$$

That is, $[f]$ satisfies a very similar equation to (5.5). If we let $\rho = -([\mathbf{v}]\cdot\nabla_{x'})f^{(2)} - [A]f^{(2)}$ we can redo the steps in Lemma 5.2 and get a similar inequality to (5.12). but with an additional term involving ρ

$$\frac{d}{dx_3}\|[f]\|^2_{0,\Omega_{x_3}} \leq (\|\nabla_{x'} \cdot \mathbf{v}^{(1)}\|_{C_b(\Omega_{x_3})} + 2\|A^{(1)}\|_{C_b(\Omega_{x_3})})\|[f]\|^2_{0,\Omega_{x_3}} + 2\int_{\Omega_{x_3}} |[f]\cdot\rho|\,dx'.$$

By estimating

$$2\int_{\Omega_{x_3}} |[f]\cdot\rho|\,dx' \leq \int_{\Omega_{x_3}} |[f]|^2\,dx' + \int_{\Omega_{x_3}} |\rho|^2\,dx' = \|[f]\|^2_{0,\Omega_{x_3}} + \|\rho\|^2_{0,\Omega_{x_3}}$$

we get

$$\frac{d}{dx_3}\|[f]\|^2_{0,\Omega_{x_3}} \leq (\|\nabla_{x'} \cdot \mathbf{v}^{(1)}\|_{C_b(\Omega_{x_3})} + 2\|A^{(1)}\|_{C_b(\Omega_{x_3})} + 1)\|[f]\|^2_{0,\Omega_{x_3}} + \|\rho\|^2_{0,\Omega_{x_3}}$$

By applying Grönwall's inequality and Sobolev's embedding theorem, as before, we get

$$\begin{aligned}
\sup_{x_3\in[0,L]} \|[f]\|^2_{0,\Omega_{x_3}} &\leq \exp\left(L + c^{(35)}L^{1/2}(\|\mathbf{v}^{(1)}\|_{3,\Omega} + \|A^{(1)}\|_{2,\Omega})\right)\left(\|[\eta]\|^2_{0,U} + \int_0^L \|\rho\|^2_{0,\Omega_{x_3}}\,dx_3\right) \\
&= c^{(53)}(\|[\eta]\|^2_{0,U} + \|\rho\|^2_{0,\Omega}),
\end{aligned}$$

where $c^{(53)} = c^{(53)}(\Omega, b, \gamma_0, \|v^{(1)}\|_{3,\Omega})$ hence

$$\|[f]\|^2_{0,\Omega} \leq L\sup_{x_3\in[0,L]} \|[f]\|^2_{0,\Omega_{x_3}} \leq c^{(53)}L(\|[\eta]\|^2_{0,U} + \|\rho\|^2_{0,\Omega}).$$

Through the definition of η we find that

$$[\eta] = [\eta_T] = \frac{h}{v\cdot n}[v_T],$$

since $v\cdot n = v_0\cdot n + u\cdot n = v_0\cdot n$. Hence

$$\|[\eta]\|_{0,U} = \left\|\frac{h}{v\cdot n}[v_T]\right\|_{0,\Omega_0} \leq c^{(3)}\frac{1}{b}\|h\|_{2,U}\|[v]\|_{0,\Omega_0}.$$

Also by definition

$$\|\rho\|_{0,\Omega} \leq \|([\mathbf{v}] \cdot \nabla_{x'}) f^{(2)}\|_{0,\Omega} + \|[A] f^{(2)}\|_{0,\Omega},$$

hence

$$\|\rho\|_{0,\Omega} \leq 2c^{(3)} \|[\mathbf{v}]\|_{1,\Omega} \|f^{(2)}\|_{2,\Omega} + c^{(3)} \|[A]\|_{0,\Omega} \|f^{(2)}\|_{2,\Omega}.$$

Both $\|[\mathbf{v}]\|_{1,\Omega}$ and $\|[A]\|_{0,\Omega}$ can be estimated by $c^{(54)} \|[v]\|_{1,\Omega}$, where $c^{(54)} = c^{(54)}(b, \gamma_0, \|v^{(1)}\|_{3,\Omega}, \|v^{(2)}\|_{3,\Omega})$. Through the second inequality of this proposition $\|f^{(2)}\|_{2,\Omega}$ can be estimated by $c^{(49)}(\|h\|_{2,U} + \|\nabla_T g\|_{2,U})$, which completes the proof of the third inequality. □

Corollary 6.2. *The operator B satisfies*

$$\|B[g,h,v_0](u)\|_{1,\Omega} \leq Mc^{(48)}(\|h\|_{0,U} + \|\nabla_T g\|_{0,U}) \tag{6.1}$$

$$\|B[g,h,v_0](u)\|_{3,\Omega} \leq Mc^{(49)}(\|h\|_{2,U} + \|\nabla_T g\|_{2,U}) \tag{6.2}$$

$$\|B[g,h,v_0](u) - B[g,h,v_0](w)\|_{1,\Omega} \leq Mc^{(50)}(\|h\|_{2,U} + \|\nabla_T g\|_{2,U})\|u - w\|_{1,\Omega}, \tag{6.3}$$

Proof. This follows immediately from the previous proposition and the inequalities in Theorem 4.19. □

Now we are ready to prove that B is a contraction and that it has a unique fixed point u.

Proposition 6.3. *For every γ with $0 < \gamma \leq \gamma_0$ the operator B maps V_γ into itself if*

$$\|h\|_{0,U} + \|\nabla_T g\|_{0,U} \leq \frac{\gamma}{Mc^{(49)}}. \tag{6.4}$$

The operator B has a unique fixed point in V_γ if (6.4) *is satisfied and if*

$$\|h\|_{0,U} + \|\nabla_T g\|_{0,U} < \frac{1}{Mc^{(50)}}. \tag{6.5}$$

Additionally, if $g^{(1)}$, $h^{(1)}$ and $g^{(2)}$, $h^{(2)}$ are two sets of boundary data both satisfying (6.5) *and if $u^{(1)}$, $u^{(2)}$ are the fixed points of $B[g^{(1)}, h^{(1)}, v_0]$ and $B[g^{(2)}, h^{(2)}, v_0]$ respectively, then*

$$\|u^{(1)} - u^{(2)}\|_{1,\Omega} \leq \frac{Mc^{(48)}(\|h^{(1)} - h^{(2)}\|_{0,U} + \|\nabla_T(g^{(1)} - g^{(2)})\|_{0,U})}{1 - Mc^{(50)}(\|h^{(1)}\|_{0,U} + \|\nabla_T g^{(1)}\|_{0,U})}. \tag{6.6}$$

Proof. Combining inequalities (6.2) and (6.4) gives

$$\|B(u)\|_{3,\Omega} \leq \gamma,$$

that is B maps V_γ into itself. To show that B has a unique fixed point we will apply Banach's fixed point theorem. The inequalities (6.3) and (6.5) gives

$$\|B(u) - B(w)\|_{1,\Omega} < \|u - w\|_{1,\Omega},$$

which means that $B\colon V_\gamma \subset H^1(\Omega) \to H^1(\Omega)$ is a contraction. To apply Banach's fixed point theorem it remains to show that V_γ is a closed subset of $H^1(\Omega)$. Now assume that we have a sequence $\{u_i\}_{i=1}^{\infty} \subset V_\gamma$, which converges to u in the norm of $H^1(\Omega)$. Since the sequence is bounded in $H^3(\Omega)$ norm it has a subsequence which converges weakly in $H^3(\Omega)$ to w. Since V_γ is closed and convex it is weakly closed so $w \in V_\gamma$. Due to $H^3(\Omega) \subset H^1(\Omega)$ we have that the space of continuous linear functionals on $H^1(\Omega)$ is a subspace to the space of continuous linear functionals on $H^3(\Omega)$. This implies that $\{u_i\}_{i=1}^{\infty}$ also converges weakly to w in $H^1(\Omega)$. Since weak limits are equal to limits in norm $u = w \in V_\gamma$, hence V_γ is a closed subset of $H^1(\Omega)$ and we can apply Banach's fixed point theorem to B.

To prove (6.6) we use (6.1) and (6.3) to get

$$\begin{aligned}
\|u^{(1)} - u^{(2)}\|_{1,\Omega} &= \|B[g^{(1)}, h^{(1)}, v_0](u^{(1)}) - B[g^{(2)}, h^{(2)}, v_0](u^{(2)})\|_{1,\Omega} \\
&\leq \|B[g^{(1)}, h^{(1)}, v_0](u^{(1)}) - B[g^{(1)}, h^{(1)}, v_0](u^{(2)})\|_{1,\Omega} \\
&\quad + \|B[g^{(1)}, h^{(1)}, v_0](u^{(2)}) - B[g^{(2)}, h^{(2)}, v_0](u^{(2)})\|_{1,\Omega} \\
&\leq Mc^{(50)}(\|h^{(1)}\|_{2,U} + \|\nabla_T g^{(1)}\|_{2,U})\|u^{(1)} - u^{(2)}\|_{1,\Omega} \\
&\quad + \|B[g^{(1)} - g^{(2)}, h^{(1)} - h^{(2)}, v_0](u^{(2)})\|_{1,\Omega} \\
&\leq Mc^{(50)}(\|h^{(1)}\|_{2,U} + \|\nabla_T g^{(1)}\|_{2,U})\|u^{(1)} - u^{(2)}\|_{1,\Omega} \\
&\quad + Mc^{(48)}(\|h^{(1)} - h^{(2)}\|_{0,U} + \|\nabla_T g^{(1)} - g^{(2)}\|_{0,U}),
\end{aligned}$$

where we have used the linearity of $(g, h) \to B[g, h, v_0](u)$. Rearranging the terms gives the desired inequality. □

Lemma 6.4. *Let $u \in V_\gamma$ with $0 < \gamma \leq \gamma_0$. Then u is a fixed point of the operator $B[g, h, v_0]$ if and only if $v = u + v_0$ is the velocity field of a solution $(v, p) \in H^3(\Omega)$ of* (1.1)-(1.3), (3.2) *and* (3.3).

Additionally, if both $(v, p) \in H^3(\Omega)$ and $(\tilde{v}, \tilde{p}) \in H^3(\Omega)$ are solutions of (1.1)-(1.3), (3.2) *and* (3.3) *with $v = \tilde{v}$, then $p = \tilde{p}$.*

Proof. First we assume that u is a fixed point of B and let $v = v_0 + u$, then

$$\operatorname{div} v = \operatorname{div} v_0 + \operatorname{div} u = 0$$

and

$$n \cdot v|_{\partial\Omega} = n \cdot v_0|_{\partial\Omega} + n \cdot u|_{\partial\Omega} = \phi$$

so equations (1.2) and (1.3) are satisfied. We also get

$$\operatorname{curl} v = \operatorname{curl} v_0 + \operatorname{curl} u = \operatorname{curl} B(u) = f.$$

Since f satisfies (5.1) we get

$$\operatorname{curl} v|_{\Omega_0} = \eta, \tag{6.7}$$

which means that (3.2) is satisfied and

$$(v \cdot \nabla)\operatorname{curl} v = (\operatorname{curl} v \cdot \nabla)v,$$

which is equivalent to

$$\operatorname{curl}(v \times \operatorname{curl} v) = 0.$$

Now we just need to construct p in such a way that (1.1) and (3.3) are satisfied. This is done by letting p be defined on Ω_0 by (3.3) and setting $\frac{1}{2}|v|^2 + p$ equal to a constant along the integral curves of v. Through this definition p will be defined in all of Ω since the integral curves of v cover Ω. If we let $x(y) \in \Omega_0$ be the starting point of an integral curve of v passing through y we get

$$p(y) = \frac{1}{2}|v(x(y))|^2 + p(x(y)) - \frac{1}{2}|v(y)|^2.$$

This means that $p(y)$ is continuously differentiable since x and v are continuously differentiable and p is continuously differentiable on Ω_0. From (3.10) and (6.7) we get

$$(n \cdot \operatorname{curl} v)v_T - v \cdot n(\operatorname{curl} v)_T = n \times \nabla_T g,$$

on Ω_0. Through some algebra it is easy to show that this is equivalent to

$$n \times (v \times \operatorname{curl} v) = n \times \nabla_T g$$

which is the same as

$$(v \times \operatorname{curl} v)_T = \nabla_T g.$$

Since $\frac{1}{2}|v_0|^2 + p_0$ is equal to a constant we can use our definition of p to get

$$(v \times \operatorname{curl} v)_T = \nabla_T \left(\frac{1}{2}|v|^2 + p \right).$$

This means that

$$\tau \cdot (v \times \operatorname{curl} v) = \tau \cdot \nabla \left(\frac{1}{2}|v|^2 + p \right)$$

for any unit vector τ tangential to Ω_0, hence we have

$$\begin{aligned} \frac{1}{2}|v(x)|^2 + p(x) &= \int_\omega \tau(y) \cdot \nabla \left(\frac{1}{2}|v(y)|^2 + p(y) \right) ds_y + \frac{1}{2}|v(x_0)|^2 + p(x_0) \\ &= \int_\omega \tau(y) \cdot (v(y) \times \operatorname{curl} v(y))\, ds_y + \frac{1}{2}|v(x_0)|^2 + p(x_0), \end{aligned}$$

for any $x \in \Omega_0$ connected to some fixed $x_0 \in \Omega_0$ by some arc $\omega \subset \Omega_0$ with tangent vector τ. If we instead let τ be a unit vector tangential to the integral curves of v then

$$\tau(x) \cdot (v(x) \times \operatorname{curl} v(x)) = 0$$

for all $x \in \Omega$. This implies that

$$\frac{1}{2}|v(x)|^2 + p(x) = \int_\omega \tau(y) \cdot (v(y) \times \operatorname{curl} v(y))\, ds_y + \frac{1}{2}|v(x_0)|^2 + p(x_0),$$

for all $x \in \Omega$ given that ω is an arc consisting of segments in Ω_0 and integral curves of v. Since $\operatorname{curl}(v \times \operatorname{curl} v) = 0$ we can apply Stokes' theorem to get

$$\frac{1}{2}|v(x)|^2 + p(x) = \int_{\omega'} \tau(y) \cdot (v(y) \times \operatorname{curl} v(y))\, ds_y + \frac{1}{2}|v(x_0)|^2 + p(x_0),$$

for any arc ω' in Ω connecting x_0 with x. This means that

$$v(x) \times \operatorname{curl} v(x) = \nabla \left(\frac{1}{2}|v(x)|^2 + p(x) \right), \tag{6.8}$$

for all $x \in \Omega$, which is equivalent to (1.1). From Theorem 1.13 it follows that $\nabla p \in H^2(\Omega)$ since $v \in H^3(\Omega)$, hence $p \in H^3(\Omega)$. Altogether this means that $(v, p) \in H^3(\Omega)$ is a solution to (1.1)-(1.3), (3.2) and (3.3).

On the other hand if we assume $u \in V_\gamma$ and that $v = v_0 + u$ is the velocity field for a solution $(v, p) \in H^3(\Omega)$ to (1.1)-(1.3), (3.2) and (3.3), then

$$((v_0 + u) \cdot \nabla)\operatorname{curl} u = (\operatorname{curl} u \cdot \nabla)(v_0 + v). \tag{6.9}$$

Combining (3.3) with (6.8) we get

$$(v \times \operatorname{curl} v)_T = \nabla_T g,$$

on Ω_0. Following the argument above backwards we see that this implies

$$(\operatorname{curl} u)_T = \frac{n \cdot \operatorname{curl} u}{v \cdot n} v_T - \frac{1}{v \cdot n} n \times \nabla_T g, \tag{6.10}$$

on Ω_0. (3.2), (6.9) and (6.10) together show that $\operatorname{curl} u$ satisfies (5.1), which means $\operatorname{curl} u = f$. By definition $B(u) \in V$ and using that together with (3.7) and (3.8) we get

$$\begin{aligned} \operatorname{curl}(B(u) - u) = \operatorname{curl} B(u) - \operatorname{curl} u = 0 \quad &\text{in } \Omega \\ \operatorname{div}(B(u) - u) = \operatorname{div} B(u) - \operatorname{div} u = 0 \quad &\text{in } \Omega \\ n \cdot (B(u) - u) = n \cdot B(u) - n \cdot u = 0 \quad &\text{on } \partial\Omega \end{aligned}$$

for $(B(u) - u) \in H^3(\Omega)$. Due to Theorem 4.19 we know that a function satisfying these equations is unique and hence $B(u) - u = 0$, which means u is a fixed point of B.

As for the second part of the lemma we note that (1.1) implies that $p - \tilde{p}$ is constant and (3.3) shows that $p - \tilde{p} = 0$ on Ω_0 and hence in all of Ω. □

Finally we are ready to prove our main result.

Proof of Theorem 3.1. By choosing $g \in H_0^3(\Omega_0)$, $h \in H_0^2(\Omega_0)$ we know there exists sequences of functions $g^{(i)} \in C_c^\infty(\Omega_0)$, $h^{(i)} \in C_c^\infty(\Omega_0)$ which converge to g and h in respective norm. Using $g^{(i)}$ and $h^{(i)}$ gives us a sequence of functions $\eta^{(i)}$ in $H_0^2(U)$, which converges to η in $H^2(U)$. Since $H_0^2(U)$ is closed this means that $\eta \in H_0^2(U)$. Hence B is a well-defined operator.

Now pick $\hat{\gamma}$ as any constant such that $0 < \hat{\gamma} \leq \gamma_0$. For K_1 choose any constant that satisfies

$$K_1 \leq \frac{\hat{\gamma}}{Mc^{(49)}}$$

and

$$K_1 \leq \frac{1}{Mc^{(50)}}.$$

From (3.1) it then follows that the assumptions of Proposition 6.3 are satisfied. This means that the operator $B[g,h,v_0]$ has a unique fixed point $u \in V_{\hat{\gamma}}$. Now from Lemma 6.4 it follows that there exists a solution $(v,p) \in H^3(\Omega)$ of (1.1)-(1.3), (3.2) and (3.3) with $v = v_0 + u$, hence

$$\|v - v_0\|_{3,\Omega} = \|u\|_{3,\Omega} \leq \hat{\gamma}.$$

Now if $(\tilde{v},\tilde{p}) \in H^3(\Omega)$ is any solution of (1.1)-(1.3), (3.2) and (3.3) satisfying (3.4), then from Lemma 6.4 it follows that $\tilde{v} - v_0 \in V_{\hat{\gamma}}$ is the unique fixed point of $B[g,h,v_0]$. This means that $\tilde{v} = v$ and from the second part of Lemma (6.4) we get that $\tilde{p} = p$, which shows that (v,p) is the unique solution of (1.1)-(1.3), (3.2) and (3.3) satisfying (3.4).

To prove (3.5) we note that since both $v^{(1)}$, $v^{(2)}$ satisfy (3.4), $u^{(1)} = v^{(1)} - v_0$, $u^{(2)} = v^{(2)} - v_0$ are the fixed points of $B[g^{(1)},h^{(1)},v_0]$, $B[g^{(2)},h^{(2)},v_0]$ respectively. Since both $(g^{(1)},h^{(1)})$, $(g^{(2)},h^{(2)})$ satisfy (3.1) the assumptions of Proposition 6.3 are satisfied and equation (6.6) gives

$$\|v^{(1)} - v^{(2)}\|_{1,\Omega} = \|u^{(1)} - u^{(2)}\|_{1,\Omega} \leq K_2(\|h^{(1)} - h^{(2)}\|_{0,U} + \|\nabla_T(g^{(1)} - g^{(2)})\|_{0,U}),$$

where

$$K_2 = \frac{Mc^{(48)}}{1 - Mc^{(50)}K_1}.$$

To prove (3.6) we use (1.1) to obtain

$$\begin{aligned}
\|\nabla p^{(1)} - \nabla p^{(2)}\|_{0,\Omega} &= \|(v^{(2)} \cdot \nabla)v^{(2)} - (v^{(1)} \cdot \nabla)v^{(1)}\|_{0,\Omega} \\
&\leq \|[(v^{(2)} - v^{(1)}) \cdot \nabla]v^{(1)}\|_{0,\Omega} + \|(v^{(2)} \cdot \nabla)(v^{(2)} - v^{(1)})\|_{0,\Omega} \\
&\leq \|\nabla v^{(1)}\|_{C_b(\Omega)}\|v^{(2)} - v^{(1)}\|_{0,\Omega} + \|v^{(2)}\|_{C_b(\Omega)}\|v^{(2)} - v^{(1)}\|_{1,\Omega} \qquad (6.11) \\
&\leq c^{(1)}(\|v^{(1)}\|_{3,\Omega} + \|v^{(2)}\|_{3,\Omega})\|v^{(2)} - v^{(1)}\|_{1,\Omega} \\
&\leq 2c^{(1)}(\|v_0\|_{3,\Omega} + \hat{\gamma})\|v^{(2)} - v^{(1)}\|_{1,\Omega}
\end{aligned}$$

and (3.3) to obtain

$$\begin{aligned}
\|p^{(1)} - p^{(2)}\|_{0,\Omega_0} &= \left\|g^{(1)} - \frac{1}{2}|v^{(1)}|^2 - g^{(2)} + \frac{1}{2}|v^{(2)}|^2\right\|_{0,\Omega_0} \\
&\leq \|g^{(1)} - g^{(2)}\|_{0,\Omega_0} + \frac{1}{2}(\|v^{(1)}\|_{C_b(\Omega_0)} + \|v^{(2)}\|_{C_b(\Omega_0)})\|v^{(2)} - v^{(1)}\|_{0,\Omega_0} \\
&\leq \|g^{(1)} - g^{(2)}\|_{0,\Omega_0} + \frac{c^{(1)}}{2}(\|v^{(1)}\|_{3,\Omega} + \|v^{(2)}\|_{3,\Omega})\|v^{(2)} - v^{(1)}\|_{0,\Omega_0} \\
&\leq \|g^{(1)} - g^{(2)}\|_{0,\Omega_0} + c^{(2)}c^{(1)}(\|v_0\|_{3,\Omega} + \hat{\gamma})\|v^{(2)} - v^{(1)}\|_{1,\Omega}.
\end{aligned} \qquad (6.12)$$

Next we note that for any function $q \in H^1(\Omega)$.

$$\|q\|_{0,\Omega} \leq L^{1/2}\|q\|_{0,\Omega_0} + L\|\nabla q\|_{0,\Omega}. \qquad (6.13)$$

This follows from the fact that we can write

$$q(x_1,x_2,x_3) = q(x_1,x_2,0) + \int_0^{x_3} \frac{\partial}{\partial x_3'} q(x_1,x_2,x_3')\,dx_3',$$

which implies

$$
\begin{aligned}
\|q\|_{0,\Omega} &= \left(\int_\Omega |q(x)|^2\,dx\right)^{1/2} \\
&\le \left(\int_\Omega |q(x_1,x_2,0)|^2\,dx\right)^{1/2} + \left(\int_\Omega \left|\int_0^{x_3} \frac{\partial}{\partial x_3'} q(x_1,x_2,x_3')\,dx_3'\right|^2\,dx\right)^{1/2} \\
&\le L^{1/2}\left(\int_U |q(x_1,x_2,0)|^2\,dx_1\,dx_2\right)^{1/2} + \left(\int_\Omega x_3 \int_0^{x_3} \left|\frac{\partial}{\partial x_3'} q(x_1,x_2,x_3')\right|^2\,dx_3'\,dx\right)^{1/2} \\
&\le L^{1/2}\|q\|_{0,\Omega_0} + \left(\int_0^L x_3\,dx_3 \int_U \int_0^L \left|\frac{\partial}{\partial x_3'} q(x_1,x_2,x_3')\right|^2\,dx_3'\,dx_1\,dx_2\right)^{1/2} \\
&\le L^{1/2}\|q\|_{0,\Omega_0} + L\|\nabla q\|_{0,\Omega}.
\end{aligned}
$$

Combining (6.11)-(6.13) gives

$$
\begin{aligned}
\|p^{(1)} - p^{(2)}\|_{1,\Omega} &\le \|p^{(1)} - p^{(2)}\|_{0,\Omega} + \|\nabla p^{(1)} - \nabla p^{(2)}\|_{0,\Omega} \\
&\le L^{1/2}\|p^{(1)} - p^{(2)}\|_{0,\Omega_0} + (1+L)\|\nabla p^{(1)} - \nabla p^{(2)}\|_{0,\Omega} \\
&\le L^{1/2}\|g^{(1)} - g^{(2)}\|_{0,\Omega_0} \\
&\qquad + (L^{1/2}c^{(2)}c^{(1)} + (1+L)2c^{(1)})(\|v_0\|_{3,\Omega} + \hat{\gamma})\|v^{(2)} - v^{(1)}\|_{1,\Omega},
\end{aligned}
$$

and using (3.5) to estimate $\|v^{(2)} - v^{(1)}\|_{1,\Omega}$ gives (3.6) with

$$
K_3 = \max\{L^{1/2}, K_2(L^{1/2}c^{(2)}c^{(1)} + (1+L)2c^{(1)})(\|v_0\|_{3,\Omega} + \hat{\gamma})\}.
$$

□

7 Examples

In this section we look at examples relating to our main result. The first example is a flow in with no x_2-component. It is not so closely related to the main result as we consider this in a slab that extends infinitely in the x_2 direction. However, it is related to the second example, where we consider axisymmetric flow in a cylinder. The third example solutions that are Beltrami fields, which means that the flow is parallel to the vorticity.

7.1 Two-Dimensional Flow

We consider a domain Γ, which is defined by

$$\Gamma := \{x : 0 \leq x_1 \leq R, 0 \leq x_3 \leq L\}$$

and consider a flow with no x_2 component and ϕ independent of x_2. This means it has the form $v = (v_1(x_1, x_3), 0, v_3(x_1, x_3))$. We note that the since the flow is divergence free it can be written in terms of a stream function ψ, which is defined by $v = \left(-\frac{\partial \psi}{\partial x_3}, 0, \frac{\partial \psi}{\partial x_1}\right)$. Under the condition $v_3 \geq b > 0$ it solves $\Delta\psi = F(\psi)$ for some function F [5]. with $v \cdot n = \phi$ on the boundary we assume that

$$\begin{aligned} \phi &< 0, & x_3 &= 0, \\ \phi &= 0, & 0 < x_3 &< L, \\ \phi &> 0, & x_3 &= L, \end{aligned}$$

Using this we find that $n \cdot \left(-\frac{\partial \psi}{\partial x_3}, 0, \frac{\partial \psi}{\partial x_1}\right) = 0$ for $0 < x_3 < L$, but in this case $n = \pm(1, 0, 0)$ giving

$$\frac{\partial \psi}{\partial x_3} = 0, \qquad 0 < x_3 < L.$$

Hence ψ is constant along both $x_1 = 0$ and $x_1 = R$, which we will call ψ_0 and ψ_R respectively. For $x_3 = 0$ we have $n = -(0, 0, 1)$ which gives

$$\frac{\partial \psi}{\partial x_1} = -\phi, \qquad x_3 = 0.$$

Through this we can get the values of ψ at $x_3 = 0$ as

$$\underline{\psi}(x_1) := \psi_0 - \int_0^{x_1} \phi(y_1, 0) dy_1,$$

where we require $\psi_P = \psi_0 - \int_0^P \phi(y_1, 0) dy_1$. Similarly we get that ψ at $x_3 = L$ is

$$\overline{\psi}(x_1) := \psi_0 + \int_0^{x_1} \phi(y_1, L) dy_1.$$

This gives that ψ solves

$$\Delta\psi = F(\psi)$$

with boundary conditions

$$\begin{aligned}
\psi &= \underline{\psi}, && 0 < x_1 < R, x_3 = 0, \\
\psi &= \psi_0, && x_1 = 0, 0 < x_3 < L, \\
\psi &= \psi_P, && x_1 = R, 0 < x_3 < L, \\
\psi &= \overline{\psi}, && 0 < x_1 < R, x_3 = L.
\end{aligned}$$

If $\operatorname{curl} v$ is known at $x_3 = 0$ we can also determine F. Since

$$\operatorname{curl} v = \left(0, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, 0\right) = (0, -\Delta\psi, 0)$$

we get

$$-\Delta\psi = (\operatorname{curl} v)_{x_2}(x_1(\psi))$$

at $x_3 = 0$, where we have used that $\underline{\psi}$ is strictly increasing as a function of x_1 to express x_1 as a function of ψ. That this holds in the whole domain follows from the fact that ψ and $-\Delta\psi$ are constant along the integral curves of v. To see this we note that from definition we have

$$(v \cdot \nabla)\psi = -\frac{\partial\psi}{\partial x_3}\frac{\partial\psi}{\partial x_1} + \frac{\partial\psi}{\partial x_1}\frac{\partial\psi}{\partial x_3} = 0.$$

and from

$$(v \cdot \nabla)\operatorname{curl} v = (\operatorname{curl} v \cdot \nabla)v = (\operatorname{curl} v)_{x_2}\frac{\partial v}{\partial x_2} = 0$$

it follows that

$$(v \cdot \nabla)(-\Delta\psi) = 0.$$

An equivalent condition is to specify the Bernoulli function, H, at $x_3 = 0$. since

$$\nabla H \times n = (v \times \operatorname{curl} v) \times n = (n \cdot v)\operatorname{curl} v - (n \cdot \operatorname{curl} v)v = \phi \operatorname{curl} v$$

because $n \cdot \operatorname{curl} v = 0$. This means we can determine $\operatorname{curl} v$ if H is known. We can compare this to our main result where we required the boundary conditions (3.2) and (3.3). We see that the condition that v is independent of x_2 gives us a condition similar to (3.2) and that the other condition we have to impose is comparable to (3.3).

7.2 Axisymmetric Flow Without Swirl

The domain in the two dimensional example can be seen as a section of a circular cylinder of radius R and height L instead of a section in a straight infinite Slab. Where instead of the condition that there is no flow in the x_2 direction we require the flow to be axisymmetric and without swirl, by which we mean that the flow is invariant under rotations in the axis of the cylinder and nonzero only in the radial and vertical directions. In this case the equations to solve become a somewhat different than in the previous example. To show these differences we define the unit vectors for cylindrical coordinates (r, θ, x_3) as

$$\begin{aligned}
e_r &= (\cos\theta, \sin\theta, 0) \\
e_\theta &= (-\sin\theta, \cos\theta, 0) \\
e_{x_3} &= (0, 0, 1).
\end{aligned}$$

That the flow is without swirl means $v_\theta = 0$, hence

$$v = v_r e_r + v_{x_3} e_{x_3},$$

where v_r and v_{x_3} are independent of θ since the flow is axisymmetric. Again we also assume $v_{x_3} \geq b > 0$. Since our field is divergence free we can write it in terms of a stream function $\psi = \psi(r, x_3)$, where

$$v_r = -\frac{1}{r}\frac{\partial \psi}{\partial x_3}, \qquad v_{x_3} = \frac{1}{r}\frac{\partial \psi}{\partial r}.$$

The velocity field also satisfies

$$(\operatorname{curl} v \cdot \nabla) v = (v \cdot \nabla) \operatorname{curl} v.$$

Using

$$\operatorname{curl} v = \left(\frac{\partial v_r}{\partial x_3} - \frac{\partial v_{x_3}}{\partial r} \right) e_\theta$$

gives

$$\begin{aligned}(v \cdot \nabla) \operatorname{curl} v &= \left(v_r \frac{\partial}{\partial r} + v_{x_3} \frac{\partial}{\partial x_3} \right) ((\operatorname{curl} v)_\theta e_\theta) \\ &= \left(v_r \frac{\partial (\operatorname{curl} v)_\theta}{\partial r} + v_{x_3} \frac{\partial (\operatorname{curl} v)_\theta}{\partial x_3} \right) e_\theta\end{aligned}$$

and

$$\begin{aligned}(\operatorname{curl} v \cdot \nabla) &= \frac{(\operatorname{curl} v)_\theta}{r} \frac{\partial}{\partial \theta} (v_r e_r + v_{x_3} e_{x_3}) \\ &= \frac{(\operatorname{curl} v)_\theta v_r}{r} e_\theta,\end{aligned}$$

so

$$v_r \frac{\partial (\operatorname{curl} v)_\theta}{\partial r} - v_r \frac{(\operatorname{curl} v)_\theta}{r} + v_{x_3} \frac{\partial (\operatorname{curl} v)_\theta}{\partial x_3} = 0,$$

which is equivalent to

$$(v \cdot \nabla) \frac{(\operatorname{curl} v)_\theta}{r} = v_r \frac{\partial}{\partial r} \frac{(\operatorname{curl} v)_\theta}{r} + v_{x_3} \frac{\partial}{\partial x_3} \frac{(\operatorname{curl} v)_\theta}{r} = 0.$$

Hence $\frac{(\operatorname{curl} v)_\theta}{r}$ is constant along the integral curves of v, but

$$\begin{aligned}(\operatorname{curl} v)_\theta &= \frac{\partial v_r}{\partial x_3} - \frac{\partial v_{x_3}}{\partial r} \\ &= -\frac{1}{r}\frac{\partial^2 \psi}{\partial x_3^2} - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r} \right) \\ &= -\frac{1}{r}\left(\frac{\partial^2 \psi}{\partial x_3^2} - r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r} \right) \right),\end{aligned}$$

so, instead of $-\Delta \psi = F(\psi)$ as we got in the previous example, we get

$$-\frac{1}{r^2}\left(\frac{\partial^2 \psi}{\partial x_3^2} - r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r} \right) \right) = F(\psi).$$

The boundary conditions can be treated in a similarly to the previous example. There are two differences though. The first difference is that we have no boundary condition that gives us ψ constant at $r = 0$, however, this follows from that the flow is axisymmetric. The second difference is that we get a factor r in the integrals when defining $\underline{\psi}$ and $\overline{\psi}$ as the stream function is defined slightly different in this example. Hence in this case too, we need $\phi = v \cdot n$ on the boundary and H at $x_3 = 0$ to determine the flow.

7.3 Beltrami Fields

The Beltrami fields defined by $\operatorname{curl} v = \alpha v$ are separated into the linear case (treated in [7]), where α is a constant, and the nonlinear case (treated in [6]), where α is a scalar function $\alpha = \alpha(x)$. This gives $v \times \operatorname{curl} v = 0$. By equation (1.4) we get that v satisfies (1.1) if $\nabla H = 0$, that is H constant. So a divergence free Beltrami field gives a flow satisfying (1.1)-(1.2) if we let

$$p = -\frac{1}{2}|v|^2.$$

On the other hand

$$v \cdot \nabla H = v \cdot (v \times \operatorname{curl} v) = 0,$$

that is, H is constant along the integral curves so if H is constant on Ω_0 and the integral curves of v cover Ω we get that H is constant in all of Ω. This in turn implies that

$$v \times \operatorname{curl} v = \nabla H = 0,$$

hence v is a Beltrami field.

So requiring a solution to be a Beltrami field essentially determines the boundary condition in equation (3.3) as opposed to requiring the solution to be axisymmetric which determines the boundary condition in (3.2). However, if we require the solution to be a linear Beltrami field we find that

$$n \cdot \operatorname{curl} v = \alpha_0 n \cdot v = \alpha_0 \phi.$$

for some constant α_0, so in this case we only need to know this constant for both boundary conditions to be determined.

The nonlinear case is different in this regard as such a boundary condition is not restricted. In [6] an existence result is proven for the problem

$$\begin{aligned} &\operatorname{curl} v = \alpha v && \text{in } \Gamma, \\ &\operatorname{div} v && \text{in } \Gamma, \\ &n \cdot v = \phi && \text{on } \partial\Gamma, \\ &n \cdot \operatorname{curl} v = \alpha_0 \phi && \text{on } \partial\Gamma_-, \end{aligned}$$

for smooth domains Γ where $\partial\Gamma_-$, as in section 2, is the subset of $\partial\Gamma$ with $\phi < 0$, and α and α_0 are functions in Γ and on $\partial\Gamma_-$, respectively. Hence, for nonlinear Beltrami fields a condition similar to (3.2) (or maybe rather (2.5) since the domain is smooth) is required. The reason that the last condition is only needed on $\partial\Gamma_-$ comes from the fact that

$$0 = \operatorname{div} \operatorname{curl} v = \operatorname{div}(\alpha v) = \alpha \operatorname{div} v + (v \cdot \nabla)\alpha$$

combined with $\operatorname{div} v = 0$ implies that α is constant along the integral curves of v and if they cover Γ this is sufficient.

8 Discussion

The main result in this thesis is proven under numerous restrictions on the boundary data. Some of these restrictions are possibly artificially put in place and can hopefully be reduced or removed. For example the requirement that we have no vorticity on the vertical sides seems like a good candidate to be removed as it is easy to construct a flow satisfying (1.1)-(1.3) which has constant vorticity in the x_3-direction in all of Ω.

Another possible thing to look into is the bound on g and h given by equation (3.1). A possibility for this was proposed by Alber as he found that the corresponding constant bounding g and h behaves like $\sim \frac{1}{L_0}$, where L_0 is the least upper bound of all the integral curves of v_0. This would mean that for 'short' domains the bound practically disappears and the condition could be removed if 'longer' domains could be treated as several 'short' domains the condition could be removed. This approach seems more feasible for cylindrical domains than for the general domains Alber worked with. The reason is that the constant also depends on the length of the domain and to estimate this dependence seems more straightforward in the cylindrical case.

The most obvious way, though, to continue the work done in this thesis would be to consider more general domains with edges and corners. Maybe the most natural would be to allow the the boundary of U to have corners.

A Irrotational Solutions

For our main result to be meaningful we also need to know that there exists an irrotational solution to (1.1)-(1.3) with enough regularity, that is, a function $v \in H^3(\Omega)$ with $\operatorname{div} v = 0$, $\operatorname{curl} v = 0$ and

$$n \cdot v = \phi$$

on $\partial\Omega$. Sufficient conditions for this are that

$$\phi = 0 \qquad \text{on } \partial U \times (0, L),$$

$$\phi_0 = \phi|_{\Omega_0}, \phi_L = \phi|_{\Omega_L} \in H^{5/2}(U), \tag{A.1}$$

$$\int_{\Omega_0} \phi \, dS = -\int_{\Omega_L} \phi \, dS, \tag{A.2}$$

and

$$n \cdot \nabla\phi_0 = n \cdot \nabla\phi_L = 0 \qquad \text{on } \partial U. \tag{A.3}$$

This can be seen by introducing a potential Φ, which satisfies $\Phi = \nabla v$ and $\Delta\Phi = 0$ in Ω and $n \cdot \nabla\Phi = \phi$ on $\partial\Omega$ and treating this problem in a similar way to the one in Section 4. The difference is that here the operator S has Neumann boundary conditions. Due to this 0 is an eigenvalue, which gives us the condition (A.2). In Fourier variables (as in Propositions 4.9 and 4.11) the problem reads

$$\begin{aligned} \hat{\Phi}_n'' &= \lambda_n \hat{\Phi}_n, \quad 0 < x_3 < L \\ \hat{\Phi}_n' &= -\hat{\phi}_{0,n}, \quad x_3 = 0, \\ \hat{\Phi}_n' &= \hat{\phi}_{L,n}, \qquad x_3 = L, \end{aligned}$$

with solution

$$\hat{\Phi}_0 = -\hat{\phi}_{0,0} x_3 + a = \hat{\phi}_{L,0} x_3 + a,$$

for $n = 0$, where a is an arbitrary constant and

$$\hat{\Phi}_n = \hat{\phi}_{0,n} \frac{\cosh(\sqrt{\lambda_n}(x_3 - L))}{\sqrt{\lambda_n}\sinh(\sqrt{\lambda_n}L)} + \hat{\phi}_{L,n} \frac{\cosh(\sqrt{\lambda_n}x_3)}{\sqrt{\lambda_n}\sinh(\sqrt{\lambda_n}L)}$$

for $n \neq 0$ given that $\hat{\phi}_{0,0} = -\hat{\phi}_{L,0}$. Squaring and integrating gives

$$\int_0^L (\lambda_n^4|\hat{\Phi}_n|^2 + \lambda_n^3|\hat{\Phi}_n'|^2 + \lambda_n^2|\hat{\Phi}_n''|^2 + \lambda_n|\hat{\Phi}_n'''|^2 + |\hat{\Phi}_n''''|^2) \, dx_3 \lesssim \lambda_n^{5/2}(|\hat{\phi}_{0,n}|^2 + |\hat{\phi}_{L,n}|^2),$$

which gives $\Phi \in H^4(\Omega)$ given that ϕ_0, $\phi_L \in D(S^{5/4})$. Through the characterization of $D(S^{5/4})$ this means that (A.1) and (A.3) has to be satisfied.

Additionally it follows that $v_3 = \frac{\partial}{\partial x_3}\Phi \geq b > 0$ in $\bar{\Omega}$ given that the inequality holds at Ω_0 and Ω_L. Through elliptic regularity and Sobolev's embedding theorem we get $\frac{\partial}{\partial x_3}\Phi \in C^\infty(\Omega) \cap C^1(\bar{\Omega})$. The function is also harmonic and it follows that it takes its least value at the boundary. To eliminate the possibility that this happens at the boundary $\partial U \times (0, L)$

we use Hopf's lemma (see e.g. maximum principles in [9]). It states that if $\frac{\partial}{\partial x_3}\Phi$ has a minimum at some $x \in \partial U \times (0, L)$ then either $n \cdot \nabla \frac{\partial}{\partial x_3}\Phi(x) < 0$ or $\frac{\partial}{\partial x_3}\Phi$ is a constant in $\bar{\Omega}$, which would immediately imply that $\frac{\partial}{\partial x_3}\Phi \geq b > 0$ since this is true at Ω_0 and Ω_L. As for the case if $\frac{\partial}{\partial x_3}\Phi$ is not constant we already know $n \cdot \nabla \frac{\partial}{\partial x_3}\Phi(x) = 0$ for all $x \in \partial U \times (0, L)$ so $\frac{\partial}{\partial x_3}\Phi$ takes its minimum at Ω_0 or Ω_L, which also means that $\frac{\partial}{\partial x_3}\Phi \geq b > 0$ in $\bar{\Omega}$.

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Lund University
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