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The kaon electromagnetic mass difference in AdS/QCD

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Abstract

In this thesis, a holographic model for quantum chromodynamics (QCD) is used to estimate the electromagnetic contribution to the kaon mass difference. The principal ideas of the model are inspired by the AdS/CFT correspondence, which is believed to be exact. The calculation is first performed theoretically, highlighting the expansion of the result into a diagrammatic structure referred to as Witten diagrams, and similar to that of Feynman diagrams of perturbative quantum field theory. To this end, several relations between the propagators are derived. An outline of the full theoretical calculation is given before proceeding to attempt to evaluate numerically the electromagnetic mass difference to first loop order. This calculation is done in Euclidean space, and the results are fitted to analytical formulae to extrapolate to Minkowski space. The final values are off by several orders of magnitude, which is believed to be in part due to an unidentified numerical glitch, but the overall expected physical behaviour, near the mass pole, is reproduced correctly by the model.

Populärvetenskapligt sammanfattning

Quantum field theory is the framework of the very successful Standard Model of particle physics, our best description yet of the behaviour of the elementary particles that we believe our world is made up of. It is therefore the language in which the subjacent ideas to this theory are univocally expressed, without the disputable interpretations that translating to our normal language requires. The Standard Model is split up into different sectors that each describe one or more of the four currently accepted fundamental interactions of nature. The two most important sectors are the electroweak sector and quantum chromodynamics (QCD). The electroweak sector is a unified theory of the electromagnetic and weak interactions, the first of which is responsible for cohesion of matter on our scale, and the latter can be used to explain radioactive reactions. Quantum chromodynamics, on the other hand, describes the strong-interaction responsible for the cohesion of the atomic nucleus.

Now, understanding how the world works on the scale of elementary particles may seem something of a curiosity at most, but it is exactly those theoretical advances that lead us to many of the electronic devices that are currently pullulating in our everyday lives. Besides, some of the large scale particle accelerators that we use to study this science have found applications in all sorts of unrelated domains, even in medicine!

Unfortunately quantum field theory is not perfect: there are practical issues that make some direct calculations extremely difficult; this is especially the case for QCD. In light of this, physicists often need to be creative in finding other ways to get at results. In particular, it is sometimes possible to find a way to map a difficult problem onto another, if possible, simpler one. When this is possible the problems are said to be dual; a simple example of this can be found in many optimisation problems, where minimising problems can be transformed into maximisation problems and vice versa.

Such an approach is adopted in this work, where we make use of an extension of the so-called AdS/CFT correspondence proposed by Juan Maldacena in 1997. In its original form, it relates a theory of gravity to a quantum field theory. In practice, this means that one can do calculations in the theory of gravity and deduce results in the quantum field theory or vice versa. As of yet, the correspondence is still formally at the stage of conjecture but given the empirical evidence for it, physicists would be extremely surprised if it turned out to be false.

Unfortunately however, the above correspondence cannot be exploited in its original form. This is because it postulates a relationship between two very particular theories possessing extremely stringent symmetry properties. Whilst symmetry often helps simplify the resolution of a problem, those in question here are not shared by realistic theories. For instance, the symmetry properties of these theories would forbid the existence of a mass scale; which is manifestly false. Nevertheless, some physicists hope that the correspondence still holds, at least in an approximate form, if some of the symmetries are removed in someway so that the resulting field theory displays characteristics of one of the more realistic theories, namely in the case of this thesis, quantum chromodynamics. Two kinds of approaches are possible at this point, the first, which is theoretically more satisfying, would be to propose a scheme describing explicitly how the symmetry should be broken and then show that the resulting theory has all the characteristics of quantum chromodynamics; this however is extremely difficult with our current understanding of the AdS/CFT correspondence. The second approach is much more phenomenological: it consists in starting from QCD, postulating that a correspondence holds by providing a so-called “dictionary”, and then fitting parameters in the model to reproduce known experimental/ theoretical results; the model used in the present work was obtained in this way. The aim of this thesis is to use a model, obtained using the phenomenological approach described above, to calculate a particular physical observable known as the ‘kaon electromagnetic mass difference’ in order to ascertain whether the prediction differs from other models. In turn, this could allow us to understand more about the workings of the strong interaction and provide further ways of testing the extremely successful Standard Model, hence pushing back in a tiny way the boundaries of human ignorance.

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Preface

This thesis is essentially organised into four main sections. The first is a brief introduction to the AdS/CFT correspondence, of which a large part is dedicated to the study of anti-de-Sitter (AdS) spacetime. The main aim was to understand, in more precise terms, what is meant by the boundary of anti-de-Sitter spacetime and in what sense this boundary could be thought of as flat Minkowski spacetime. It also seemed interesting to distinguish between the Poincaré Patch and global AdS spacetime; the model used in this thesis is restricted to the Poincaré Patch, but there are extended versions of the AdS/CFT correspondence on global AdS spacetime. Following this is a brief review of some of the arguments that establish the practical form of the AdS/CFT correspondence on which the philosophy of the AdS/QCD model is based.

The second section discusses the model used in the work, reproducing some of the main arguments to justify its postulated form and highlighting some of the freedom one has in the construction. It is then described how to use the model to calculate the kaon electromagnetic mass difference.

The third section outlines the theoretical calculation of 4-point functions using the holographic model and discusses the diagrammatic structure of the obtained expression.

Finally, the fourth section presents the numerical results of the calculation of the kaon electromagnetic mass difference.

1 Introduction

1.1 Overview

The present work is a continuation and extension of [1,2]. The main goal was to calculate the electromagnetic contribution to the kaon mass difference, in the holographic model for quantum chromodynamics (QCD) of [3–5], by generalising the methods used in [6] to calculate 4-point functions in the scalar case. The electromagnetic mass difference itself is interesting for precision calculations, where it is fruitful to separate out the different contributions having different origins. A large part of the problem of calculating the kaon electromagnetic mass difference, reduces to the effective computation of the non-perturbative diagram in figure 1.

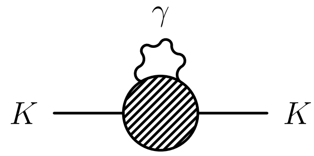


Figure 1: Kaon electromagnetic self-energy diagram

Currently, there is no known method for extracting the information contained in the shaded circle of diagram 1 directly from QCD alone, and it is therefore necessary to resort to different models, like Chiral perturbation theory, to obtain such results; this was done for example in [7]. In this thesis, a holographic or "AdS/QCD" model will be used to model the non-perturbative physics.

The principal theoretical idea of this work and such models stems from a conjecture formulated by J. Maldacena [8] in 1997 in the context of String Theory, known as the "AdS-CFT correspondence". In its strongest form, the conjecture postulates a full duality between string theory, in the background $AdS_5 \times S^5$, and a conformal field theory (CFT) in 4-dimensions. For this reason, it is considered to be a realisation of the holographic principle [9], as one can regard the 4-dimensional theory as "living" on the (conformal) boundary of the 5-dimensional anti-de-Sitter (AdS) spacetime. This type of phenomena is not totally foreign to mathematics and physics; one can notably cite Cauchy's integral formula for a holomorphic function f on a domain $\mathcal{D}$ of the complex plane:

$$\forall z_0 \in \mathcal{D}, f(z_0) = \frac{1}{2\pi i} \oint_{\partial\mathcal{D}} \frac{f(z)}{z - z_0} \mathrm{d}z \tag{1.1}$$

The formula shows that the information about the function inside the domain is actually contained on its contour, despite the difference in dimensions. From a practical point of view, the interesting feature of AdS/CFT is the fact that weak coupling calculations in one theory are strong coupling calculations in the other; on one hand, this makes the duality difficult to check, but, on the other, it means that if it holds then one can use perturbative methods in one theory to deduce non-perturbative results in the other. To date, however, there is no formal proof of Maldacena's conjecture, and it is beyond the scope of this text to attempt to explore that aspect of it. Nevertheless, one can note that recently numerical tests of the conjecture, like those in [10, 11], have provided compelling evidence for it. In these articles, calculations are done for quantum black holes and good agreement is found.

Maldacena's conjecture suggests that certain quantum field theories may be holographic duals to higher dimensional string theories, but it should be noted that the theories on both sides of the correspondence are highly symmetric. On one hand, anti-de-Sitter spacetime is a maximally symmetric solution to Einstein's field equations, and, on the other, conformal field theories can be thought of as fixed points of the renormalisation group and therefore possess, in particular, the very stringent property of scale invariance; a property that is shared by no realistic field theories. It is hence quite unclear whether such a principle could be generalised to less symmetric field theories. There are at least two ways to explore this possibility, one can either attempt to break some of the symmetries of the theories in the correspondence and attempt to derive more realistic field theories, or, one can start from a realistic field theory and attempt to guess the dual string theory. The second approach has, up to now, been the most fruitful and many calculations have been performed in QCD using so-called holographic models. It turns out that QCD is a particularly good candidate for testing this hypothesis as it is approximately conformal in the high energy limit; this gives an idea of what a dual theory may look like in this limit. Furthermore, from the point of view of QCD, exploring this hypothesis is particularly interesting as it provides alternative models for accessing non-perturbative results using well known methods.

This text begins with a brief review of the ideas of the AdS-CFT correspondence before moving on to describe the holographic model that will be used to effectively perform the calculation. This will be followed by an overview of the theoretical calculation carried out, and, finally, the numerical results shall be discussed.

1.2 *AdS* space-time

The main setting for the calculations that follow is the so-called anti-de-Sitter (AdS) space-time, which appears on the string theory side of the AdS/CFT correspondence. It can be defined in several ways, all of which give interesting insights into its nature; the first of which is to see it as a solution to Einstein's equations.

As a solution to Einstein's equations AdS_d can be characterised as the homogenous and isotropic solution to Einstein's equation (1.2) in d dimensions (the indices M, N run through $\{0, d-1\}$) with negative cosmological constant (Λ).

$$G_{MN} + \Lambda g_{MN} = 0 \tag{1.2}$$

From a purely geometric point of view, it is a maximally symmetric spacetime with Lorentzian signature and negative Ricci scalar or scalar curvature. Such spaces have constant sectional curvature and, as it can be shown that the curvature tensor takes the special form:

$$R_{\mu\nu\alpha\beta} = \kappa.(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}), \quad \kappa = \frac{R}{d(d-1)}$$

where d is the dimension of the manifold and R is the Ricci scalar. A proof of this result for manifolds with Euclidean signature $(+,+,+,+)$ can be found in [12][1]. Its relevance is that it shows that maximally symmetric spacetimes can be classified simply by their dimension and the sign of the Ricci scalar. Defining AdS space in this way puts the emphasis on the symmetry properties of AdS_d space, notably that it has a global $SO(2,d-1)$ symmetry; this proves to be an important feature that supports the plausibility of the AdS/CFT correspondence to be discussed in a future section. In this thesis, the convention for the Lorentzian metric signature will follow that of particle physicists, i.e. $(+,---)$.

As an embedded manifold A convenient way of studying and visualising the geometry of AdS space is to embed it into flat space with one extra *time-like* dimension; the main advantage is that many of its properties can be seen directly by calculation. It can, in that space, be defined as the set of points M that satisfy:

$$\|\overrightarrow{OM}\|^2 = L^2 \tag{1.3}$$

where O is an arbitrary origin for the affine spacetime and the constant L is related to the Ricci curvature tensor by[2]: $L^2 = -\frac{d(d-1)}{R}$. For example, the surface AdS_2 can be embedded into a 3-dimensional Minkowski spacetime with two time-like directions $(\vec{e_0}, \vec{e_2})$ and one space-like direction $(\vec{e_1})$. Choosing an orthonormal basis for the associated vector space $(\vec{e_0}, \vec{e_1}, \vec{e_2})$, and denoting by (X^0, X^1, X^2) the coordinates of a point M in the frame $(O, \vec{e_0}, \vec{e_1}, \vec{e_2})$, then equation (1.3) can be rewritten:

$$\frac{1}{L^2}((X^0)^2 - (X^1)^2 + (X^2)^2) = 1 \tag{1.4}$$

The reader will without a doubt recognise (1.4) as the equation for a one-sheeted hyperboloid in 3 dimensions; it is represented figure 2. This provides some geometric intuition about AdS spacetime: pictorially, the fields and objects under study will be constrained to live on a higher dimensional hyperboloid. More importantly, from (1.3) it can be seen directly that the isometry group is $O(2, n-1)$, as $O(2, n-1)$ is by definition the group of transformations of Minkowski spacetime that leave the Minkowski metric, and thus distances, invariant. The connected component containing the identity is $SO(2, n-1)$ and AdS space is therefore also invariant under that group.

1.2.1 Local coordinates, global parametrisations and metric

In order to write the metric of AdS_d space it is necessary to find solutions to (1.3). To do this it is convenient to first rewrite (1.3) with the canonical coordinates $(X_0, \dots, X_d)$ of the embedding Minkowski space:

$$\frac{1}{L^2}\left((X^0)^2 - \sum_{i=1}^{d-1}(X^i)^2 + (X^d)^2\right) = 1 \tag{1.5}$$

[1] The result differs only in the order of the indices: $R_{\rho\mu\nu\sigma} = \kappa.(g_{\nu\sigma}g_{\rho\mu} - g_{\mu\sigma}g_{\rho\nu})$

[2] In the following L will be referred to simply as the curvature, in virtue of this formula

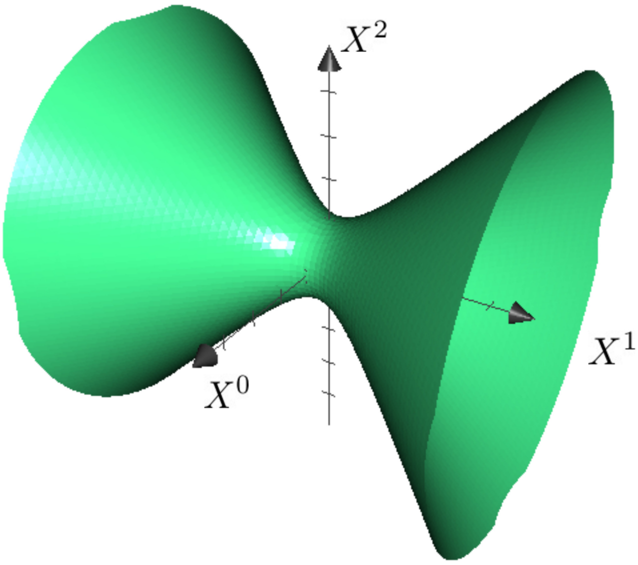


Figure 2: One-sheeted hyperboloid $\frac{1}{L^2}((X^0)^2 - (X^1)^2 + (X^2)^2) = 1$

In this work we will use the solution which define the so-called Poincaré coordinates:

$$
\begin{aligned}
X^0 &= \frac{L^2}{2r}\left(1 + \frac{r^2}{L^4}\left(\vec{x}^2 - t^2 + L^2\right)\right) \\
X^i &= \frac{rx^i}{L} \quad \text{for } i \in \{1, \ldots, d-2\} \\
X^{d-1} &= \frac{L^2}{2r}\left(1 + \frac{r^2}{L^4}\left(\vec{x}^2 - t^2 - L^2\right)\right) \\
X^d &= \frac{rt}{L} \\
r &> 0, t \in \mathbb{R}, \vec{x} = (x^1, \ldots, x^{d-2}) \in \mathbb{R}^{d-2}
\end{aligned}
\tag{1.6}
$$

In these coordinates the metric of AdS_d then reads:

$$
\mathrm{d}s^2 = -\frac{L^2}{r^2}\mathrm{d}r^2 + \frac{r^2}{L^2}(\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu)
$$

where $\eta_{\mu\nu}$ is the usual $(d-1)-$dimensional Minkowski metric with signature $(+, -, \ldots, -)$ Often, and as will be done in this work, one makes the change of variable $z = \frac{L^2}{r}$ so that the metric becomes:

$$
\mathrm{d}s^2 = \frac{L^2}{z^2}(\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu - dz^2) \tag{1.7}
$$

These coordinates do not, however, cover the whole of AdS spacetime but only an open set referred to as the Poincaré patch. This will nevertheless be sufficient in this work where an artificial cut-off of spacetime will be introduced. From (1.7) it can be seen that the Poincaré Patch is, topologically, "just" warped Minkowski spacetime with an extra spatial dimension denoted z. Moreover, in (1.7) the z-coordinate has been distinguished from the other spacetime dimensions. This notational convention will be maintained throughout this thesis work, more specifically: Greek indices will be used to denote the first $(d-1)$-coordinates and capital roman indices will be understood to contain the z coordinate. On many occasions, it will be preferable to introduce the short-hand, η_{NM} so that the metric can be written:

$$\mathrm{d}s^2 = \frac{L^2}{z^2}\eta_{NM}\mathrm{d}x^N\mathrm{d}x^M \tag{1.8}$$

i.e.

$$g_{MN} = \frac{L^2}{z^2}\eta_{MN}$$

where g_{MN} is the metric of AdS_d and η_{MN} is the metric of d-dimensional Minkowski spacetime.

Global parametrisation: Despite the Poincaré patch being sufficient for this work, in order to study the global structure of AdS-spacetime it is more convenient to use a more natural solution to (1.5) which is a generalisation to d dimensions of the standard parametrisation of a hyperbola $x = \cosh(t), y = \sinh(t), t \in \mathbb{R}$:

$$X^0 = L\cosh\rho\cos\tau \tag{1.9}$$
$$X^d = L\cosh\rho\sin\tau \tag{1.10}$$
$$X^i = L\Omega_{d,i}\sinh\rho \quad i \in \ \{1,\ldots,d-1\} \tag{1.11}$$
$$\tau \in [0,2\pi[, \rho \geq 0 \tag{1.12}$$

Where the $\Omega_{d,i}$ must satisfy $\displaystyle\sum_{i=1}^{d-1}\Omega_{d,i}^2 = 1$, or, in other words, parametrise a $(d-1)$-dimensional hypersphere. A solution can be defined by induction on the dimension d:

$$\Omega_{2,1} = \cos\theta_1; \quad \Omega_{2,2} = \sin\theta_1 \qquad \text{if } d = 2 \tag{1.13}$$
$$\Omega_{d+1,i} = \Omega_{d,i-1}\sin\theta_d, i \in \{1,\ldots,d-1\} \quad \Omega_{d+1,d} = \cos\theta_d \qquad \text{if } d \geq 2 \tag{1.14}$$

With this parametrisation, the metric of AdS space-time can be shown to take the following form:

$$\mathrm{d}s^2 = L^2(\cosh^2\rho\mathrm{d}\tau^2 - \mathrm{d}\rho^2 - \sinh^2\rho\mathrm{d}\Omega_{d-1}^2) \tag{1.15}$$

Where, $\mathrm{d}\Omega_d^2$ denotes the metric of the d-dimensional unit sphere, defined iteratively by the following equations $\mathrm{d}\Omega_1^2 = \mathrm{d}\theta_1$, $\mathrm{d}\Omega_{d+1}^2 = \mathrm{d}\theta_{d+1}^2\sin^2\theta_{d+1}\mathrm{d}\Omega_d^2$.

1.2.2 Conformal boundary

Given the above definition of AdS space the more mathematical reader may have realised that it is a manifold *without boundary* as it is locally homeomorphic to an open set of some $\mathbb{R}^n$. This would not be true if it had a boundary point in the usual (intuitive)[3] sense, as we would fail to find an open neighbourhood around that point that would be homeomorphic to $\mathbb{R}^n$. It is therefore natural to wonder what is meant by "boundary" in the case of the AdS/CFT correspondence.

In short, the term "boundary" will be used to denote the "infinitely far away", i.e. for example, in the global coordinates (τ, ρ, θ_i) defined above, when $\rho \to \infty$. Interestingly, it can be shown in these coordinates that in *AdS* Space it is possible for a light signal to reach this boundary in finite observer time (for a stationary observer at some $\rho = \rho_0$); this corroborates the idea that a theory on such a background might be holographic. In order to study this boundary, which, in the case of *AdS* space and in a sense to be defined in the following, is Minkowski spacetime, it is convenient to conformally compactify[4] spacetime; this procedure is in fact required to properly define global conformal transformations on Minkowski space-time.

Conformal compactification The main idea of conformal compactification is to attempt to study the asymptotic behaviour of a Lorentzian manifold by deforming it in such a way that angles are preserved and such that the "infinitely far away" is brought to a finite distance; in particular the topological and causal structures should be preserved. Mathematically, this amounts to mapping a semi-Riemannian manifold (M, g) into another (N, h) using a conformal map ϕ i.e. a map that has the property that there is a function $\Omega : M \longrightarrow \mathbb{R}_+$ such that:

$$\Omega(p) g_p(X_p, Y_p) = h_{\phi(p)}(\mathrm{d}\phi_p(X_p), \mathrm{d}\phi_p(Y_p))$$

for all vector fields X, Y and points $p \in M$. Having done this, the "points at infinity" can be adjoined to spacetime as they are now at a finite distance, the added points will be referred to as the conformal boundary. It is interesting to note that the above procedure is by no means uniquely defined and there are a whole class of conformal transformations that are possible to achieve this aim.

Minkowski spacetime In order to understand the above it is informative to look at the case of (1+1)-dimensional Minkowski space time for which the metric is:

$$ds^2 = dt^2 - dx^2 \tag{1.16}$$

The first step is to perform coordinate transformations so as to obtain coordinates that are finite in extent (this is always possible using the function arctan) and such that the

[3]Manifolds *with* boundaries are modelled on the half-space $\mathbb{H}^n = \{x \in \mathbb{R}^n, x_n \geq 0\}$

[4]The term compactification is a bit of a misnomer, as the obtained manifolds may not necessarily be compact in the usual topological sense

metric has an overall factor (which will be singular at the points that map to infinity) that can be removed. For Minkowski space time, the standard approach is to switch to "light cone coordinates", defined by:

$$u = t - x; v = t + x \tag{1.17}$$

after which the metric becomes:

$$ds^2 = dudv \tag{1.18}$$

The coordinates are then made finite in extent by the further coordinate transformation:

$$u = \tan \tilde{u}, v = \tan \tilde{v}; \quad \tilde{u}, \tilde{v} \in]-\frac{\pi}{2}, \frac{\pi}{2}[\tag{1.19}$$

after which the metric is:

$$ds^2 = \frac{1}{\cos^2 \tilde{u} \cos^2 \tilde{v}} d\tilde{u}d\tilde{v} \tag{1.20}$$

The overall factor can be removed by a conformal transformation and the resulting metric is: $\mathrm{d}\tilde{s}^2 = \mathrm{d}\tilde{u}\mathrm{d}\tilde{v}$. This new metric is regular at the boundary $+/-\frac{\pi}{2}$, where the physical metric would be singular.

To represent the above, one can draw a so-called "Penrose diagram" for Minkowski spacetime which is represented in figure 3, where we have introduced $\tilde{x}$ and $\tilde{t}$ such that: $\tilde{u} = \tilde{t} - \tilde{x}; \tilde{v} = \tilde{t} + \tilde{x}; \mathrm{d}\tilde{s} = \mathrm{d}\tilde{t}^2 - \mathrm{d}\tilde{x}^2$. The diagram represents the conformally deformed Minkowski spacetime and allows in particular to visualise infinity.

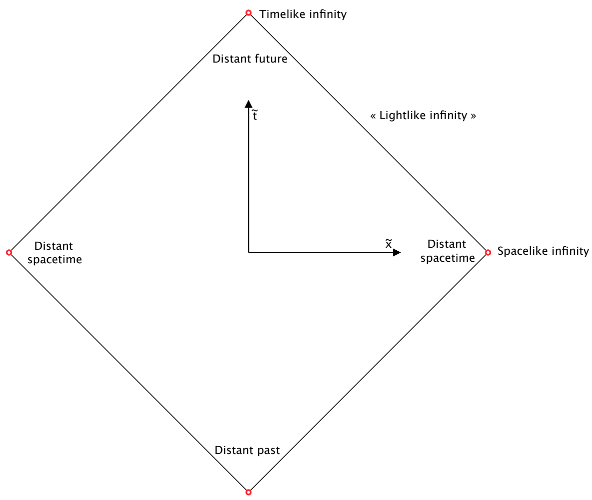


Figure 3: Penrose diagram for (1+1)-dimensional Minkowski space in the $\tilde{x} - \tilde{t}$ plane. The coloured dots and the edges of the diamond are the points at "infinity" that have been appended to the conformally deformed spacetime

For general d-dimensional Minkowski space-time, the procedure is quite similar with the exception that we use a positive r coordinate. The final result is essentially the same, except that only half of the diamond is retained (due to the restriction $r \geq 0$) and each point in the diamond in the 2D diagram represents a sphere. The reader can refer to [13] for a full treatment.

anti-de-Sitter spacetime Now we are familiar with the ideas of conformal compactification, we can look at what happens for AdS space-time. Recall that in global coordinates the metric takes the following form:

$$\mathrm{d}s^2 = \frac{L^2}{\cos^2\theta}(\mathrm{d}\tau^2 - \mathrm{d}\theta^2 - \sin^2\theta \mathrm{d}\Omega_{d-1}^2)$$

In these coordinates the boundary is situated at $\theta = \frac{\pi}{2}$. To conformally compactify, we perform a conformal transformation in order to remove the factor $\frac{L^2}{\cos^2\theta}$, and note that on the boundary the metric is:

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 - \mathrm{d}\Omega_{d-1}^2$$

which is conformally compactified d-dimensional Minkowski spacetime. It is in this sense that the boundary of AdS space will be considered to be Minkowski space-time. In Poincaré coordinates, the conformal boundary discussed here is situated at $z = 0$.

Physics in AdS space One final remark can be made regarding the AdS geometry as it has been introduced: as one can see from figure 2: it allows for closed time-like curves. Physically, this would lead us to run into causality issues and all of the paradoxes that follow. The issue is nevertheless easily circumvented: instead of considering AdS space as above, we can "unwind" the time axis by considering the universal cover of AdS space. This amounts to allowing τ to take values in all of $\mathbb{R}$, without identifying points, i.e. according to (1.9) the points of coordinates (τ, ρ, θ_i) and $(\tau + 2\pi, \rho, \theta_i)$ should be the same point, to "unwind" the time axis, one should consider these points to be distinct. Except from in this chapter, no further distinction will be made between AdS and its universal cover and it will always be understood that we are referring to its universal cover.

1.3 Maldacena's Conjecture

The strongest form of Maldacena's conjecture stated in the introduction will not actually be required for the calculations in this work as the aim is to use it to determine a non-perturbative result of a given field theory. It will therefore only be necessary to work in the low-energy limit of the string theory where it reduces to supergravity. In this case, the AdS/CFT correspondence postulates equivalence between a strongly coupled conformal field theory and a weakly coupled supergravity theory. Moreover, as the calculations will only be done to tree order, the classical equations of motion for the fields will be sufficient. Before proceeding, a few words should be said on "conformal field theories", for a full introduction, the reader can refer to [14,15]. Conformal field theories are field theories on a flat space background with the property of being invariant under conformal transformations of spacetime as defined in 1.2.2. In order to determine the symmetry group of the theory it is actually necessary to conformally compactify Minkowski space-time in order to be able to define global conformal transformations [15]; the interesting point and a vital ingredient underlying the ideas presented in this chapter is that the symmetry group of a conformal field theory in d dimensions is the *same* as the isometry group of AdS_{d+1}; this helps to motivate some of the identifications made in the following. The reader will find in recent books such as [16, 17], a pedagogical review of some of the heuristic arguments used to "derive" the conjecture that will be not covered in this text. It will, however, be necessary to state the practical form of the correspondence, which is the object of this section.

1.3.1 Holographic dictionary

The set of rules that allow us to relate objects from the two different theories will be referred to as the "Holographic dictionary". In AdS/CFT, operators in the conformal field theory will be associated with fields in the supergravity theory with mass determined by the relation:

$$m^2 = \frac{1}{L^2}\left(\Delta(\Delta-(d-1)) + p(p-(d-1))\right) \tag{1.21}$$

where d is the dimension of the AdS space, L is the curvature of the AdS space, p is the tensor rank of the operator, which can be thought of as being related to its spin, and Δ is the conformal dimension of the operator, which, in turn, is related to its mass dimension through dimensional analysis. The conformal dimension of the operator describes how it behaves under scaling, for instance, for an eigenfunction of the scaling operator [16] we have the following transformation rules:

$$x \to \lambda x \tag{1.22}$$

$$\phi(x) \to \phi'(x) = \lambda^{\Delta}\phi(\lambda x) \tag{1.23}$$

In the scalar case, equation (1.21) can be derived by solving the Klein-Gordon equation $(\Box + m^2)\phi = 0$ and studying the solutions. In the Poincaré patch:

$$\Box = \frac{1}{\sqrt{g}}\partial_M(g^{MN}\sqrt{g}\partial_N) = \frac{z^d}{L^d}\eta^{NM}\partial_M(\frac{z^2}{L^2}\frac{L^d}{z^d}\partial_N) = \frac{1}{L^2}\left(z^2\eta^{\mu\nu}\partial_\mu\partial_\nu - z^d\partial_z z^{2-d}\partial_z\right)$$

So the Klein-Gordon equation becomes:

$$z^2\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - z^d\partial_z z^{2-d}\partial_z\phi + m^2L^2\ \phi = 0$$

Performing a partial Fourier transform in the first 4-coordinates:

$$(-z^2k^2 + m^2L^2)\hat{\phi} - z^d\partial_z z^{2-d}\partial_z\hat{\phi} = 0$$

or:

$$(-z^2k^2 + m^2L^2)\hat{\phi} - z(2-d)\partial_z\hat{\phi} - z^2\partial_z\partial_z\hat{\phi} = 0$$

This equation closely resembles Bessel's equation: $x^2y'' + xy' + (x^2 - \alpha^2)y = 0$, and can be brought to this form via two ad-hoc change of variables:

$$\tilde{z} = kz$$

which removes the k^2:

$$(-\tilde{z}^2 + m^2L^2)\ \hat{\phi} - \tilde{z}(2-d)\partial_{\tilde{z}}\hat{\phi} - \tilde{z}^2\partial_{\tilde{z}}\partial_{\tilde{z}}\hat{\phi} = 0$$

Followed by:

$$\hat{\phi} = z^{\frac{d-1}{2}}\psi$$

The equation for ψ is then the Bessel equation:

$$(\tilde{z}^2 - m^2L^2 - \frac{(d-1)^2}{4})\psi + \tilde{z}\ \partial_{\tilde{z}}\psi + \tilde{z}^2\partial_{\tilde{z}}\partial_{\tilde{z}}\psi$$

The two Bessel's functions (or modified Bessel's functions for $k^2 < 0$) J_α, K_α are two linearly independent solutions for ψ, thus $\hat{\phi}$ is some linear combination of $z^{\frac{d-1}{2}}J_\alpha((k^2)^{\frac{1}{2}}z)$ and $z^{\frac{d-1}{2}}K_\alpha((k^2)^{\frac{1}{2}}z)$. The asymptotic behaviour of ϕ is then: $\phi \sim c_2 z^{\Delta_+}f_1 + c_1 z^{\Delta_-}f_0$; where $\Delta_\pm = \frac{d-1}{2} \pm \sqrt{m^2L^2 + \frac{(d-1)^2}{4}}$. For large enough m^2, it can be shown that the $c_1 z^{\Delta_-}f_0$ part of the solution is non-normalisable[5] at the boundary[6]. Nevertheless, it should not be discarded as it defines a field on the boundary by $f_0(k) = \lim_{z\to 0}\phi(k,z)z^{-\Delta_-}$, or in position space $f_0(x) = \int \frac{\mathrm{d}^dk}{(2\pi)^d}f_0(k)e^{-ikx}$. The following reasoning shows how this operator should transform under scaling:

$$\begin{aligned} f_0(\lambda x) &= \lim_{z\to 0}\phi(\lambda x, z)z^{-\Delta_-} = \lim_{z\to 0}\lambda^{-\Delta_-}(z\lambda^{-1})^{-\Delta_-}\phi(\lambda x, z) \\ &= \lim_{z'\to 0}\lambda^{-\Delta_-}z'^{-\Delta_-}\ \underbrace{\phi(\lambda x, \lambda z')}_{\substack{=\phi(x,z') \\ \text{given the} \\ \text{symmetry group} \\ \text{of } AdS \text{ space}}}\ = \lambda^{-\Delta_-}f_0(x) \end{aligned}$$

f_0 can be interpreted as the source for the corresponding operator $\mathcal{O}$ in the field theory on the 4 dimensional boundary; it should be renormalised so as to make the integral finite. Lastly, Δ_+ is identified with the conformal dimension of $\mathcal{O}$. This final identification can be elucidated by the following heuristic argument [18]. First of all, to control the singularities in AdS space when $z \to 0$, the usual limiting procedure is applied, i.e. a small cut-off L_0 or

[5] A solution is said to be normalisable if the action is finite when evaluated on this field

[6] This is not always true. As discussed in [17], there is a mass regime, namely $-\dfrac{D^2}{4} < m^2L^2 \leq -\dfrac{D^2}{4} + 1; D = d-1$, where both solutions are normalisable and there are two consistent ways of imposing boundary conditions on the fields

ε is introduced and is understood to be sent to 0 in the final results. As the action contains terms of the type $\int_{z=\varepsilon} \mathrm{d}^d x \left(\frac{L}{\varepsilon}\right)^d \phi(\varepsilon, x)\mathcal{O}(\varepsilon, x) = \int \mathrm{d}^d x \left(\frac{L}{\varepsilon}\right)^d \varepsilon^{\Delta_-}\phi_0\mathcal{O}(x)$, these should be finite, which leads us to identify: $\mathcal{O}(\varepsilon, x) = \varepsilon^{\Delta_+}\mathcal{O}(x)$.

The above therefore shows how the theories are effectively linked together, and it leads to the further assumption that the partition functions are identical, i.e.

$$Z_{\mathcal{O}}[\phi_0]_{CFT} = Z_\phi[\phi_0]_{\text{string}}$$

In the classical limit of the string theory the partition function reduces to $e^{iS}[\phi[\phi_0]]$, from which it follows that the 4-dimensional n-point function of operators $\mathcal{O}_i$ can be calculated from the dual holographic theory via:

$$\boxed{\langle\Omega|\, \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\,|\Omega\rangle = \left.\frac{(-i)^n\delta^n\mathcal{S}}{\delta\phi_{0,1}(x_1)\dots\ \delta\phi_{0,n}(x_n)}\right|_{\phi_{0,1}=\dots=\phi_{0,n}=0}} \tag{1.24}$$

In (1.24), the state[7] $|\Omega\rangle$ is the interacting theory vacuum state and it is understood that all the source terms are set to 0 in the final result[8]. This formula is the basis for all the calculations that follow.

1.4 LSZ (Lehmann-Symanzik-Zimmerman) reduction formula

The AdS/CFT dictionary describes how to evaluate correlation functions of operators $\mathcal{O}_1, \dots \mathcal{O}_n$ in the quantum field theory using results from the dual higher dimensional theory. In practice however, the objects of interest are in fact S-matrix elements. The LSZ reduction formula is the tool from Quantum Field Theory that describes how to relate the correlation functions to S-matrix elements. When the operators $\mathcal{O}_i$ are field operators ϕ, the following relation can be derived from perturbation theory [19]:

$$\begin{aligned}\langle p_1 \dots p_n|\, S\,|q_1 \dots q_n\rangle = \left[i\int \frac{\mathrm{d}^4x_1 e^{-iq_1x_1}}{\sqrt{Z}}(\Box_1 + m^2)\right]\dots\left[i\int \frac{\mathrm{d}^4x_n e^{+ip_ny_n}}{\sqrt{Z}}(\Box_n + m^2)\right]\\ \times \langle\Omega|\, T\{\phi(x_1)\dots\phi(x_n)\phi(y_1)\dots\phi(y_n)\}\,|\Omega\rangle\end{aligned} \tag{1.25}$$

Interestingly enough, as remarked in [20], the only assumption about the fields that is required to derive that result is the fact that they can create one particle states, consequently, the formula holds for any sequence of such operators ($\mathcal{O}_i$):

$$\begin{aligned}\langle p_3 \dots p_n|\, S\,|p_1p_2\rangle = \left[i\int \frac{\mathrm{d}^4x_1 e^{-ip_1x_1}}{\sqrt{Z}}(\Box_1 + m^2)\right]\dots\left[i\int \frac{\mathrm{d}^4x_n e^{ip_nx_n}}{\sqrt{Z}}(\Box_n + m^2)\right]\\ \times \langle\Omega|\, T\{\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\dots\mathcal{O}_n(x_n)\}\,|\Omega\rangle\end{aligned} \tag{1.26}$$

Here, the constant Z is a field renormalisation factor.

[7]This notation will be used throughout this work

[8]i.e. The derivative is evaluated at zero

2 AdS/QCD model

In the previous discussion, the explicit form of the AdS/CFT correspondence, or so-called holographic dictionary, was given in the low energy string theory case. However, as previously suggested, it is not directly applicable to realistic phenomenological theories like QCD which do not possess the conformal symmetry present in the field theory of the correspondence (such an invariance forbids, for instance, the existence of any mass scale like that present in QCD). Nevertheless, under the assumption that such a duality may be extended to other less symmetric theories, it is possible to construct models, inspired by the above correspondence, to perform calculations to ascertain whether they can be used to predict known or new results in QCD. The case of QCD is particularly interesting as the theory is approximately conformal at high energy where the mass parameters become negligible. Hence, if there is a dual theory, it is natural to assume, by analogy with the conformal case, that the gravity background be of the form $AdS_5 \times X_5$ where X_5 is some compact manifold; in the following we will ignore expansions of the fields onto this compact manifold. The background should however be modified in some way when $z \to \infty$ to model low-energy QCD; the 5th coordinate z of AdS_5 being interpreted as an inverse energy scale. The simplest way to do this is to introduce a cut-off at some $z = L_1$; this is known as a hard-wall model and was first introduced in [21]. Different models have extended upon this idea and have introduced different types of particles. The model used in this work was first proposed in [3, 4].

With regards to the holographic dictionary of the theory, it is assumed that it takes the same form as in the conformal field theory case; any corrections, for example, to the conformal dimension Δ, which only really makes sense in the high-energy massless quark limit and is necessary in determining the mass of the corresponding field, are neglected.

Once this has been postulated, the model is then constructed by choosing operators from QCD and by writing the simplest possible Lagrangian for their corresponding fields. The operators chosen in [3] are listed in table 1; they are simply the quark bilinears. The model in this work will account for 3 quark flavours $q_{L,R} = \begin{pmatrix} u \\ d \\ s \end{pmatrix}_{L,R}$. The choice of the operators can be justified by the fact that they will have some non-zero overlap with the states of low-energy QCD, so it will be possible to extract information about low energy QCD processes using the LSZ theorem.

4D	5D	p	Δ	m^2L^2
$\overline{q}_L\gamma_\mu t^a q_L$	L^a_M	1	3	0
$\overline{q}_R\gamma_\mu t^a q_R$	R^a_M	1	3	0
$\overline{q}_L q_R$	$\frac{2}{z}X$	0	3	-3

Table 1: 4 dimensional operators and their 5D equivalents

In table 1 the t^a are infinitesimal generators of $SU(3)$ related to the Gell-Mann matrices by $t^a = \frac{\lambda^a}{2}$; the Gell-Mann matrices and their important properties are listed in appendix C. To write a Lagrangian, simple symmetry considerations are made: QCD has an approximate global $SU(3)_L \times SU(3)_R$ chiral flavour symmetry. In the 5-dimensional theory, this should be promoted to a spontaneously broken gauge symmetry. The simplest action, given in matrix form, encoding all of these components is then:

$$\mathcal{S} = \int \mathrm{d}^5x\sqrt{g}\mathrm{tr}\left((D^M X)^\dagger D_M X + \frac{3}{L^2}X^\dagger X - \frac{1}{4g_5^2}(F^{L,MN}F^L_{MN} + F^{R,MN}F^R_{MN})\right) \tag{2.27}$$

Where D is the gauge covariant derivative that acts on X according to:

$$D_M X = \partial_M X - iL_M X + iXR_M$$

and F^L_{MN}[9] is the field strength:

$$F^L_{MN} = \partial_M L_N - \partial_N L_M - i[L_M, L_N]$$

The matrices L_N and R_N are defined by: $L_N = L^a_N t^a$ and $R_N = R^a_N t^a$, they are Lie-algebra[10] valued 1-forms which, in the fundamental representation of $SU(3)$, is represented by the set of all traceless Hermitian matrices. This implies in particular that[11] $L^0_N = R^0_N = 0$

The model has now at least one parameter for fitting to QCD: the constant g_5. It can be determined by calculating the vector current two-point function [3] and comparing the result with the perturbation theory result in QCD. The result is:

$$g_5^2 = \frac{12\pi^2}{N_c}$$

where N_c is the number of colours and will be taken to be 3 in this work.

2.1 Symmetry breaking and vacuum

The matrix X is, a priori, a general complex matrix and, following [3], can be parametrised by $X = e^{i\pi}(X_0 + S)e^{i\pi}$. S and π are general Hermitian matrices representing respectively the scalar and pseudo-scalar degrees of freedom. A priori, the matrix π contains a $U(3)$ singlet field, i.e. its $0th$ component need not be zero. In this work, however, the singlet will be ignored. X_0, on the other hand, is the holographic equivalent of the vacuum[12]; it

[9] F^R_{MN} being defined similarly

[10] In mathematical literature, the Lie-algebra of SU(3) is often denoted $\mathfrak{su}(3)$

[11] It can be noted that the "low-energy limit" of the string theory in the AdS/CFT correspondence is related to a large N_c limit in the field theory [16]. In this limit the anomalous currents are approximately negligible and the flavour symmetry becomes an approximate U(3) symmetry; this means that we could in principle keep the $a = 0$ components.

[12] and is a function of z alone

corresponds to the solution to the equations of motion when all other excitation fields are set to zero, i.e. its components satisfy:

$$\partial_z\left(\frac{L^3}{z^3}\partial_z X_{0,ij}\right)+3\frac{L^3}{z^5}X_{0ij}=0 \tag{2.28}$$

The general solution can be shown to be of the form $A_{ij}z+B_{ij}z^3$, where the matrices A and B are determined by the boundary conditions at the IR and UV boundaries. From the holographic dictionary, the general interpretation of the parameters is as follows: A_{ij} is a source term for X_0 and B_{ij} is related to 1 point functions, like the quark condensate $\langle\overline{q}q\rangle$; in other words, A_{ij} introduces a mechanism to mimic the explicit chiral symmetry breaking in QCD by the quark masses and B_{ij} introduces a mechanism to recreate the spontaneous symmetry breaking. The interpretation of A_{ij} as a mass-matrix follows from the fact that, as noted in [16], adding $\frac{1}{2}X_0^{\alpha\beta}\overline{q}_\alpha q_\beta$ to the Lagrangian is equivalent to adding masses for the quarks. Several approaches are possible at this point, for instance, as in [4], one can introduce a potential on the IR boundary to fix the boundary condition at $z=L_1$; the parameters are then determined by comparing to QCD and fitting to experimental results. Alternatively, following [1], both A_{ij}, M_{ij} can be considered parameters to the model that can be fitted to experimental data. The latter approach will be used in this work and the approximation $m_u=m_d$[13] will made such that the solution can be written:

$$X_0(z)=\frac{1}{2}\begin{pmatrix} v_u(z) & 0 & 0\\ 0 & v_u(z) & 0\\ 0 & 0 & v_s(z)\end{pmatrix}$$

with $v_q(z)=\ \zeta m_q z+\sigma_q z^3\frac{1}{\zeta}, q\in\ \{u,s\}$.

The parameter $\zeta=\frac{\sqrt{Nc}}{2\pi}$ is introduced as suggested in [4, 22] so as to get the correct scaling with the number of colours N_c.

Alternatively, on the basis (t_a) defined in appendix C,

$$X_0(z)=\underbrace{\frac{1}{\sqrt{6}}(2v_u+v_s)}_{X_0^0}t_0+\underbrace{\frac{1}{\sqrt{3}}(v_u-v_s)}_{X_0^8}t_8 \tag{2.29}$$

Previously, in [1, 2], the scalar degree of freedom S was neglected.

[13]This simplifies things slightly, but it would not be adapted for a precise calculation of the kaon electromagnetic mass difference since it does play a role. Here however, our main aim is to ascertain whether or not AdS/QCD models could produce correct results

2.2 From AdS/QCD to the kaon electromagnetic mass difference

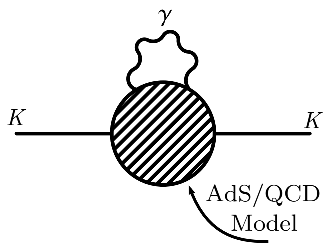


Figure 4: Kaon self-energy diagram

The AdS/QCD model described above provides the tools required to evaluate diagram 1 to first loop order. The diagram is reproduced in figure 4 for convenience. The expression that this diagram represents can be written in the form:

$$\boxed{\delta m^2 = \frac{ie^2}{2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{\eta^{\mu\nu}}{k^2+i0}\left\langle K\right| \mathcal{V}_\mu(k)\mathcal{V}_\nu(-k)\left|K\right\rangle} \tag{2.30}$$

where the operator $\mathcal{V}_\mu = \frac{2}{3}\overline{u}\gamma_\mu u - \frac{1}{3}\overline{d}\gamma_\mu d - \frac{1}{3}\overline{s}\gamma_\mu s$.

The amplitude $\left\langle K\right| \mathcal{V}_\mu(k)\mathcal{V}_\nu(-k)\left|K\right\rangle$ is related to the four-point function:

$$\left\langle \Omega\right| P(p)P^\dagger(-p)\mathcal{V}_\mu(k)\mathcal{V}_\nu(-k)\left|\Omega\right\rangle \tag{2.31}$$

via the LSZ theorem. The four-point function, on the other hand, can be evaluated using the holographic model by means of (1.24). Through the correspondence table 1 one can see that:

$$\mathcal{V}_\mu \longleftrightarrow \left(V^3_\mu + \frac{1}{\sqrt{3}}V^8_\mu\right)$$
$$P \longleftrightarrow \pi^a, a\in\{4,5,6,7\}$$

The "vector" field V^a_μ is defined by $V^a_\mu = \frac{1}{2}(L^a_\mu + R^a_\mu)$, and similarly an "axial" field is defined by $A^a_\mu = \frac{1}{2}(L^a_\mu - R^a_\mu)$.

This thesis will attempt to evaluate (2.30) through the study of the function:

$$\boxed{\Pi(p^2) = \frac{ie^2}{2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{\eta^{\mu\nu}}{k^2+i0}\left\langle \Omega\right| P(p)P^\dagger(-p)\mathcal{V}_\mu(k)\mathcal{V}_\nu(-k)\left|\Omega\right\rangle} \tag{2.32}$$

To determine Π, one must begin by determining the 4-point function (2.31). To this end the postulated holographic dictionary stipulates, through equation (1.24), that one must perform a functional derivative on the action of the 5D theory with respect to the sources fields $\pi^a_0, V^b_{0,\mu}$ of the 4-D operators $\mathcal{V}, P$. These source terms enter the problem as boundary conditions for the 5D fields when $z \to 0$, or, introducing a small cut-off L_0 to regulate the theory, at $z = L_0$. The action therefore needs to be rewritten as a function of these sources. This can be achieved through solving the equations of motion with the correct boundary conditions, expanding in powers of the source fields. As previously stated, calculations in the 5-dimensional theory will be performed to tree-order, which implies that the classical equations of motion are sufficient in this case. The method is therefore the following:

1. Solve classical equations of motion in 5D theory with the correct boundary conditions

2. Express the 5D action $\mathcal{S}$ as a function of the boundary sources

3. Evaluate the functional derivative as prescribed by (1.24)

3 Calculating 4-point functions

In this section, we will outline the theoretical calculation of momentum space 4-point functions of the form:

$$\langle\Omega|\, P^a(p_1)P^b(p_2)\mathcal{V}^{\mu,c}(p_3)\mathcal{V}^{\nu,d}(p_4)\,|\Omega\rangle \tag{3.33}$$

using the holographic model presented in the previous section. We introduce here the notation P^a and $\mathcal{V}^a_\mu$ to denote the 4D operators corresponding to the fields π^a and V^a_μ respectively. Since the AdS/QCD model action (2.27) contains an infinite amount of interactions when expanded in terms of the fields π, S, V and A, we must first determine which of those are relevant. In fact, it will only be necessary to expand to order 4 in these fields for the computation of four-point functions, as higher order terms will not contribute to the functional derivative.[14]

3.1 Equations of motion

The first step is to expand the action (2.27) to order 4 in the fields; as the overall expression is relatively long, the action will be separated into two parts: $\mathcal{S}_{\text{kinetic}}$ and $\mathcal{S}_{\text{inter}}$. $\mathcal{S}_{\text{kinetic}}$ contains all terms up to order 2 in the fields, whilst $\mathcal{S}_{\text{inter}}$ contains all interaction terms up to order 4. With this notation the equations of motion will take the form:

$$\frac{\delta\mathcal{S}_{\text{kinetic}}}{\delta X} = -\frac{\delta\mathcal{S}_{\text{inter}}}{\delta X}$$

where X is any one of the fields[15]. Due to the large number of terms in $\mathcal{S}_{\text{inter}}$, the two parts will be treated separately.

3.1.1 $\mathcal{S}_{\textbf{kinetic}}$

It is shown in appendix A, by direct expansion of (2.27), that the kinetic term takes the following form :

$$\begin{aligned}\mathcal{S}_{kinetic} = \int \mathrm{d}^5x\sqrt{g}\,\Bigg(&\frac{1}{2}M_V^{a2}V^{M,a}V^a_M - \frac{1}{4g_5^2}(\nabla_M V^a_N - \nabla_N V^a_M)(\nabla^M V^{N,a} - \nabla^N V^{M,a})\\ &+\frac{1}{2}g^{MM'}\partial_M S^a\partial_{M'}S^a + 2g^{MM'}\mathrm{tr}(\partial_M X_0 t^a)\partial_{M'}S^a - \frac{1}{2}f^{abc}X_0^cV^{M,a}\partial_M S^b\\ &-\frac{1}{2}f^{abc}g^{MM'}\partial_M X_0^c S^a V^b_{M'} + \frac{3}{L^2}X_0^aS^a + \frac{3}{2L^2}S^aS^a\\ &+\frac{1}{2}M_A^{a\,2}(\partial_M\pi^a - A^a_M)(\partial^M\pi^a - A^{M,a}) - \frac{1}{4g_5^2}(\nabla_M A^a_N - \nabla_N A^a_M)(\nabla^M A^{N,a} - \nabla^N A^{M,a})\Bigg)\end{aligned} \tag{3.34}$$

[14] They will cancel when the derivative is evaluated at $\pi_0 = V_0 = A_0 = S_0 = 0$

[15] or field components

where f^{abc} are the $SU(3)$ structure constants defined in appendix C. The result of $\frac{\delta\mathcal{S}_{\text{kinetic}}}{\delta X}$ for the different fields is given in appendix B.1. It will also be practical to treat the components of the fields $V = V_z\vec{e}^z + V_\mu\vec{e}^\mu$ and $A = A_z\vec{e}^z + A_\mu\vec{e}^\mu$ separately and to decompose the projections of these fields onto $(\vec{e}^\mu)$ into longitudinal and transverse parts, (cf. the *Hodge/Helmholtz Decomposition Theorem* [23]) i.e.:

$$V^a_\mu = V^a_{\perp,\mu} + \partial_\mu \xi^a \qquad \eta^{\mu\nu}\partial_\nu V^a_{\perp,\mu} = 0 \tag{3.35}$$
$$A^a_\mu = A^a_{\perp,\mu} + \partial_\mu \phi^a \qquad \eta^{\mu\nu}\partial_\nu A^a_{\perp,\mu} = 0 \tag{3.36}$$

Furthermore, the boundary conditions for the first four coordinates (x_0, x_1, x_2, x_3) will be assumed such that it is possible to perform a partial Fourier transform of the fields:

$$f(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{-ik\cdot x}\hat{f}(k) \,\Bigg|\, k\cdot x = \eta_{\mu\nu}k^\mu x^\nu = k_\mu x^\nu \tag{3.37}$$

Introducing the above, and:

$$\alpha^f(z) = \frac{g_5^2 L^2 M_V^{f\,2}}{z^2} \qquad \beta^f(z) = \frac{g_5^2 L^2 M_A^{f\,2}}{z^2}$$

it can be shown that the equations of motion take the form:

$$\partial_z\left(\frac{1}{z}\partial_z\hat{\xi}^f\right) - \frac{\alpha^f(z)}{z}\hat{\xi}^f + \frac{L^2 g_5^2}{2z^3} f^{fbc}\hat{S}^b X_0^c = -(2\pi)^4 \frac{k_\sigma}{ik^2}\frac{\delta\mathcal{S}_{\text{int}}}{\delta\hat{V}^{*f}_\sigma}\frac{g_5^2}{L} \tag{3.38}$$

$$\frac{\alpha^f - k^2}{z}\hat{V}^f_{\mu,\perp} - \partial_z\left(\frac{1}{z}\partial_z\hat{V}^f_{\mu,\perp}\right) = -(2\pi)^4\frac{g_5^2}{L}\frac{\delta\mathcal{S}_{\text{int}}}{\delta\hat{V}^{*f}_\sigma}\left(\eta_{\mu\sigma} - \frac{k_\mu k_\sigma}{k^2}\right) \tag{3.39}$$

$$-\frac{L^2 g_5^2}{2z^3} f^{fbc}\partial_z\hat{S}^b X_0^c - \frac{L^2 g_5^2}{z^3}\frac{f^{afc}}{2}\partial_z X_0^c\hat{S}^a + \frac{k^2}{z}\partial_z\hat{\xi}^f = +(2\pi)^4\frac{g_5^2}{L}\frac{\delta\mathcal{S}_{\text{int}}}{\delta\hat{V}^{*f}_z} \tag{3.40}$$

$$\partial_z\left(\frac{1}{z}\partial_z\hat{\phi}^f\right) - \frac{\beta^f(z)}{z}(\hat{\phi}^f - \hat{\pi}^f) = (2\pi)^4 i\frac{g_5^2}{L}\frac{k_\sigma}{k^2}\frac{\delta\mathcal{S}_{\text{int}}}{\delta\hat{A}^{*f}_\sigma} \tag{3.41}$$

$$\frac{\beta^f - k^2}{z}\hat{A}^f_{\mu,\perp} - \partial_z\left(\frac{1}{z}\partial_z\hat{A}^f_{\mu,\perp}\right) = -(2\pi)^4\frac{g_5^2}{L}\frac{\delta\mathcal{S}_{\text{int}}}{\delta\hat{A}^{*f}_\sigma}\left(\eta_{\mu\sigma} - \frac{k_\mu k_\sigma}{k^2}\right) \tag{3.42}$$

$$\partial_z\left(\frac{\beta^f}{z}\partial_z\hat{\pi}^f\right) - \frac{k^2\beta^f}{z}(\hat{\phi}^f - \hat{\pi}^f) = -(2\pi)^4\frac{g_5^2}{L}\frac{\delta\mathcal{S}_{\text{int}}}{\delta\hat{\pi}^{*f}} \tag{3.43}$$

$$\frac{-k^2}{z}\partial_z\hat{\phi}^f + \frac{\beta^f}{z}\partial_z\hat{\pi}^f = +(2\pi)^4\frac{g_5^2}{L}\frac{\delta\mathcal{S}_{\text{int}}}{\delta\hat{A}^{*f}_z} \tag{3.44}$$

$$\partial_z\left(\frac{L^3}{z^3}\partial_z\hat{S}^f\right) + L^3\left(\frac{k^2}{z^3} + \frac{3}{z^5}\right)\hat{S}^f - \frac{f^{afc}}{2}k^2\frac{L^3}{z^3}X_0^c\hat{\xi}^a = -(2\pi)^4\frac{\delta\mathcal{S}_{\text{int}}}{\delta\hat{S}^{*f}} \tag{3.45}$$

where it has been used that, for any *real*-valued field X:

$$\frac{\delta\mathcal{S}}{\delta\hat{X}^*(p,z)} = \int \frac{\mathrm{d}^4x}{(2\pi)^4} e^{ip\cdot x} \frac{\delta\mathcal{S}}{\delta X(x,z)}$$

The * denotes complex conjugation. Equations (3.41),(3.44),(3.43) and (3.38),(3.40),(3.39) respectively can be shown to be linearly dependant and, in the following, as the chosen gauge is such that $A_z = V_z = 0$, equations (3.44) and (3.40) will be discarded. Whilst it is relatively clear that the lefthand sides (LHS) of equations (3.41),(3.43) and (3.44) are linearly dependant, more work is required to show it for the LHS of (3.38),(3.40) and (3.39).

Starting from the LHS of (3.40), one begins by taking a further z-derivative:

$$-\partial_z\left(\frac{L^2g_5^2}{2z^3}f^{fbc}\partial_z\hat{S}^bX_0^c\right) - \partial_z\left(\frac{L^2g_5^2}{z^3}\frac{f^{afc}}{2}\partial_zX_0^c\hat{S}^a\right) + \partial_z\left(\frac{k^2}{z}\partial_z\hat{\xi}^f\right)$$

Or, after expanding out the derivatives:

$$-\partial_z\left(\frac{L^2g_5^2}{2z^3}f^{fbc}\partial_z\hat{S}^b\right)X_0^c - \frac{L^2g_5^2}{2z^3}f^{fbc}\partial_z\hat{S}^b\partial_zX_0^c - \partial_z\left(\frac{L^2g_5^2}{z^3}\frac{f^{afc}}{2}\partial_zX_0^c\right)\hat{S}^a$$
$$-\frac{L^2g_5^2}{z^3}\frac{f^{afc}}{2}\partial_zX_0^c\partial_z\hat{S}^a + \partial_z\left(\frac{k^2}{z}\partial_z\hat{\xi}^f\right)$$

Using (2.28) and the antisymmetry of the symbols f^{abc}, the expression becomes:

$$-\partial_z\left(\frac{L^2g_5^2}{2z^3}f^{fbc}\partial_z\hat{S}^b\right)X_0^c + \frac{3L^2g_5^2}{z^5}\frac{f^{afc}}{2}X_0^c\hat{S}^a + \partial_z\left(\frac{k^2}{z}\partial_z\hat{\xi}^f\right)$$

Multiplying (3.45) by $\frac{f^{fbc}g_5^2X_0^c}{2L}$ and adding it to the above then yields:

$$\frac{3L^2g_5^2}{z^5}\frac{f^{afc}}{2}X_0^c\hat{S}^a + \partial_z\left(\frac{k^2}{z}\partial_z\hat{\xi}^f\right) + \frac{L^2g_5^2f^{fbc}X_0^c}{2}\left(\frac{k^2}{z^3}+\frac{3}{z^5}\right)\hat{S}^b - \frac{f^{fbc}f^{dbe}}{4}k^2\frac{L^2g_5^2}{z^3}X_0^eX_0^c\hat{\xi}^d$$

Again, using the antisymmetry of f^{abc}, the above simplifies to:

$$\partial_z\left(\frac{k^2}{z}\partial_z\hat{\xi}^f\right) + \frac{L^2g_5^2f^{fbc}X_0^c}{2}\frac{k^2}{z^3}\hat{S}^b - \frac{f^{fbc}f^{dbe}}{4}k^2\frac{L^2g_5^2}{z^3}X_0^eX_0^c\hat{\xi}^d$$

Furthermore as: $-\frac{1}{8}X_0^cX_0^df^{cak}f^{dbk} = [t^a,X_0][t^b,X_0]$ and is only non-zero when $a=b$ in which case: $[t^a,X_0][t^a,X_0] = -\frac{1}{2}M_V^a$, the above expression is in fact:

$$\partial_z\left(\frac{k^2}{z}\partial_z\hat{\xi}^f\right) + \frac{L^2g_5^2f^{fbc}X_0^c}{2}\frac{k^2}{z^3}\hat{S}^b - \frac{k^2\alpha^f(z)}{z}\xi^f$$

which is linearly dependent on (3.38).

It can be checked explicitly using FORM and follows from gauge invariance that the interaction terms satisfy:

$$ik_\sigma\frac{\delta\mathcal{S}_{\text{int}}}{\delta\hat{V}_\sigma^{*f}} = \partial_z\frac{\delta\mathcal{S}_{\text{int}}}{\delta\hat{V}_z^{*f}} - \frac{f^{fbc}}{2}\frac{\delta\mathcal{S}_{\text{int}}}{\delta\hat{S}^{*b}}$$

3.1.2 Propagators

Solving the above equations exactly is a difficult problem due to the non-linear terms coming from the interaction Lagrangian. Furthermore, for the purposes of this work, it is also a necessary requirement that the dependence of the solution on the boundary source fields appear explicitly in the solution so as to facilitate the computation of the functional derivative. If the interaction terms were independent of the fields, then the problem would be to solve a non-homogenous linear equation, which can be done using a Green's function technique. This leads to the idea of attempting to solve iteratively the above equations, which is in very many ways similar to the perturbative expansions of usual quantum field theory. To illustrate the method it is informative to study the more simple scalar case studied in [2, 6]. The action for this toy scalar model is:

$$\mathcal{S} = \int \mathrm{d}^5x\sqrt{g}\ \frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{24}\phi^4$$

from which we deduce the equation of motion:

$$(\Box + m^2)\phi = -\frac{\lambda}{6}\phi^3 \tag{3.46}$$

where $\Box = \frac{1}{\sqrt{g}}\partial_M(g^{MN}\sqrt{g}\partial_N\phi)$.

To implement correctly the asymptotic behaviour near the boundary a small cut off L_0 is introduced and is understood to be sent to 0 in the final results. The boundary condition at L_0 is then $\phi(x, L_0) = L_0\phi_0(x)$. In order to completely determine the solution a second boundary condition is required; the simplest being $\partial_z\phi(x, L_1) = 0$, which will be used throughout this work. Dependence of the results on this second boundary condition was explored in [3] where it was found that it had little effect on the final results.

For the iterative resolution two Green's functions are defined: K and G, which will be referred to respectively as the bulk to boundary and bulk to bulk propagators. They are the solutions to the following systems of equations:

$$\begin{cases} (\Box + m^2)K(x, x', z) = 0 \\ K(x, x', L_0) = \delta(x - x') \\ \partial_z K(x, x', L_1) = 0 \end{cases} \tag{3.47}$$

$$\begin{cases} (\Box + m^2)G(x, x', z, z') = \frac{\delta(z-z')\delta(x-x')}{\sqrt{g}} \\ G(x, x', L_0, z') = 0 \\ \partial_z G(x, x', L_1, z') = 0 \end{cases} \tag{3.48}$$

The bulk to boundary propagator will be used to set the correct boundary condition, and the bulk to bulk propagator will be used to incorporate the interactions. The iterative procedure is then defined as follows:

$$\phi_1(x,z) = \int K(x,x',z) L_0 \phi_0(x) \mathrm{d}^4x' \tag{3.49}$$

$$n \geq 1: \quad \phi_{n+1}(x,z) = \phi_1 - \frac{\lambda}{6} \int \sqrt{g} G(x,x',z,z')(\phi_n(x',z'))^3 \mathrm{d}z' \mathrm{d}^4x' \tag{3.50}$$

It is left to the reader to check that if the sequence converges then the limit will be a solution to (3.46).

A few more remarks are in order: firstly, as $\Box$ does not depend explicitly on x, $K(x,x',z) = K(x-x',0,z)$ and $G(x,x',z,z') = G(x-x',0,z,z')$; in other words the propagators only depend on the difference $x-x'$. Thus, the integrals over the variable x' can be seen as convolutions with the functions $\tilde{K}(x,z) = K(x,0,z)$ and $\tilde{G}(x,z,z') = G(x,0,z,z')$. In particular this means that these convolutions will become products when performing the Fourier transform on the variable x^μ. In the rest of this report, the $\tilde{}$ will be dropped and the notations $G(x,x',z,z'); G(x-x',z,z')$ will be used interchangeably. Lastly, the presence of the factor $\frac{1}{\sqrt{g}}$ in the definition of the bulk to bulk propagator is a convention that is particularly useful for obtaining coordinate independent expressions and integrals. As in this work only the Poincaré coordinates are used, this precaution will not be taken.

In the vein of the above example, the Fourier space bulk to boundary and bulk to bulk propagators for the fields $V_\perp^a$, are naturally defined by the equations:

$$\begin{cases} \frac{\alpha^a - k^2}{z} K^a_{V_\perp}(k,z) - \partial_z \left(\frac{1}{z} \partial_z K^a_{V_\perp}(k,z) \right) = 0 \\ K^a_{V_\perp}(k,L_0) = 1 \\ \partial_z K^a_{V_\perp}(k,L_1) = 0 \end{cases} \tag{3.51}$$

$$\begin{cases} \frac{\alpha^a - k^2}{z} G^a_{V_\perp}(k,z,z') - \partial_z \left(\frac{1}{z} \partial_z G^a_{V_\perp}(k,z,z') \right) = \delta(z-z') \\ G^a(k,L_0,z') = 0 \\ \partial_z G^a(k,L_1,z') = 0 \end{cases} \tag{3.52}$$

and similarly for $K^a_{A_\perp}, G^a_{A_\perp}$. In some cases, an analytic solution in terms of Bessel's functions can be found [1,2], but generally it will be necessary to resort to numerics.

The situation is slightly more complicated for the fields ξ^a, S^a, π^a and ϕ^a, as in the chosen basis their equations are coupled. However, viewing the system of equations as a matrix equation allows us to see that the Green's "function", if it exists, would be a matrix of functions. This is best illustrated by the equations involving π and ϕ, since there is no mixing between components with different values of a, which is the case for S and ξ.

Starting with the bulk to boundary propagators, the aim is to find functions $K^a_{\phi\phi}$, $K^a_{\phi\pi}$, $K^a_{\pi\phi}$, $K^a_{\pi\pi}$ such that: π_1, ϕ_1 defined by:

$$\begin{pmatrix} \phi_1^a \\ \pi_1^a \end{pmatrix} = \underbrace{\begin{pmatrix} K^a_{\phi\phi} & K^a_{\phi\pi} \\ K^a_{\pi\phi} & K^a_{\pi\pi} \end{pmatrix}}_{\mathcal{K}} \begin{pmatrix} \phi_0^a \\ \frac{L_0 \zeta}{2} \pi_0^a \end{pmatrix}$$

solve the homogenous equations associated with (3.43),(3.41) with the correct boundary conditions. The off-diagonal elements of $\mathcal{K}$ will be referred to as "mixed propagators". The correct boundary condition for the field π is in fact $\frac{L_0\zeta}{2}\pi_0$ as, according to table 1, $\frac{2}{z}X$ corresponds to $\overline{q}_L q_R$ and the boundary condition was modified in 2.1 as advocated by [22].

It can then be seen that $K^a_{\phi\phi}, K^a_{\phi\pi}, K^a_{\pi\phi}, K^a_{\pi\pi}$ should be solutions to the following boundary value problems:

$$\begin{cases} \partial_z\left(\frac{1}{z}\partial_z K^a_{\phi\phi}\right) - \frac{\beta^a(z)}{z}(K^a_{\phi\phi} - K^a_{\pi\phi}) = 0 \\ \partial_z\left(\frac{\beta^a}{z}\partial_z K^a_{\pi\phi}\right) - \frac{k^2\beta^a}{z}(K^a_{\phi\phi} - K^a_{\pi\phi}) = 0 \\ K^a_{\phi\phi}(k, L_0) = 1 \\ K^a_{\pi\phi}(k, L_0) = 0 \\ \partial_z K^a_{\phi\phi}(k, L_1) = \partial_z K^a_{\pi\phi}(k, L_1) = 0 \end{cases} \tag{3.53}$$

$$\begin{cases} \partial_z\left(\frac{1}{z}\partial_z K^a_{\phi\pi}\right) - \frac{\beta^a(z)}{z}(K^a_{\phi\pi} - K^a_{\pi\pi}) = 0 \\ \partial_z\left(\frac{\beta^a}{z}\partial_z K^a_{\pi\pi}\right) - \frac{k^2\beta^a}{z}(K^a_{\phi\pi} - K^a_{\pi\pi}) = 0 \\ K^a_{\pi\pi}(k, L_0) = 1 \\ K^a_{\phi\pi}(k, L_0) = 0 \\ \partial_z K^a_{\pi\pi}(k, L_1) = \partial_z K^a_{\phi\pi}(k, L_1) = 0 \end{cases} \tag{3.54}$$

Analogously, one finds that $G^a_{\phi\phi}, G^a_{\phi\pi}, G^a_{\pi\phi}$ and $G^a_{\pi\pi}$ are solutions to the boundary value problems given by:

$$\begin{cases} \partial_z\left(\frac{1}{z}\partial_z G^a_{\phi\phi}(k,z,z')\right) - \frac{\beta^a(z)}{z}(G^a_{\phi\phi}(k,z,z') - G^a_{\pi\phi}(k,z,z')) = \delta(z-z') \\ \partial_z\left(\frac{\beta^a}{z}\partial_z G^a_{\pi\phi}(k,z,z')\right) - \frac{k^2\beta^a}{z}(G^a_{\phi\phi}(k,z,z') - G^a_{\pi\phi}(k,z,z')) = 0 \\ G^a_{\phi\phi}(k, L_0, z') = 0 = G^a_{\pi\phi}(k, L_0, z') \\ \partial_z G^a_{\phi\phi}(k, L_1, z') = \partial_z G^a_{\pi\phi}(k, L_1, z') = 0 \end{cases} \tag{3.55}$$

$$\begin{cases} \partial_z\left(\frac{1}{z}\partial_z G(k,z,z')^a_{\phi\pi}\right) - \frac{\beta^a(z)}{z}(G^a_{\phi\pi}(k,z,z') - G^a_{\pi\pi}(k,z,z')) = \delta(z-z') \\ \partial_z\left(\frac{\beta^a}{z}\partial_z G^a_{\pi\pi}(k,z,z')\right) - \frac{k^2\beta^a}{z}(G^a_{\phi\pi}(k,z,z') - G^a_{\pi\pi}(k,z,z')) = 0 \\ G^a_{\pi\pi}(k, L_0, z') = G^a_{\phi\pi}(k, L_0, z') = 0 \\ \partial_z G^a_{\pi\pi}(k, L_1, z') = \partial_z G^a_{\phi\pi}(k, L_1, z') = 0 \end{cases} \tag{3.56}$$

The iterative procedure then generalises in the following fashion:

$$n \geq 1: \begin{pmatrix} \phi^a_{n+1}(k,z) \\ \pi^a_{n+1}(k,z) \end{pmatrix} = \begin{pmatrix} \phi^a_1(k,z) \\ \pi^a_1(k,z) \end{pmatrix} + \int \mathrm{d}z' \underbrace{\begin{pmatrix} G^a_{\phi\phi}(k,z,z') & G^a_{\phi\pi}(k,z,z') \\ G^a_{\pi\phi}(k,z,z') & G^a_{\pi\pi}(k,z,z') \end{pmatrix}}_{\mathcal{G}} \begin{pmatrix} (2\pi)^4 i \frac{g_5^2}{L}\frac{k_\sigma}{k^2}\frac{\delta \mathcal{S}_{\text{int}}}{\delta \hat{A}^{*a}_\sigma}\Big|_{\phi^a_n,\pi^a_n,k,z'} \\ -(2\pi)^4\frac{g_5^2}{L}\frac{\delta \mathcal{S}_{\text{int}}}{\delta \hat{\pi}^{*a}}\Big|_{\phi^a_n,\pi^a_n,k,z'} \end{pmatrix}$$

where the vertical bar notation $|_{\phi^a_n,\pi^a_n,k,z'}$ indicates that these expressions should be evaluated at the specified fields and values of k and z. Again, the off-diagonal elements $\mathcal{G}$ will be referred to as mixed propagators.

Finally, regarding S^a and ξ^a, it can be seen from equations (3.38),(3.45) that one must look at the values of f^{alc} for a given $l \in \{0, \ldots, 8\}$ to ascertain how many equations are indeed coupled. One finds that:

- S^4 couples with ξ^5
- S^5 couples with ξ^4
- S^6 couples with ξ^7
- S^7 couples with ξ^6
- All ξ^a and S^b, $a, b \in \{0, 1, 2, 3, 8\}$ are mutually independent

The propagators can then be defined analogously to the previous cases, except that now any "off-diagonal" propagators will mix different values of the gauge index a. No more details will be given here as for the particular problem studied here it will be shown that they will not be required.

3.1.3 $\mathcal{S}_{\text{inter}}$

The interaction terms in the action can be determined by continuing the expansion of (2.27) to order 4. Terms remain manageable up to order 3 and the full calculation, although tedious, can be performed by hand. For verification purposes, the calculations were also performed using FORM. Not all of the terms are in fact required for the computation of the process in diagram 1 and therefore, in order to keep the number of terms manageable, the irrelevant terms should be identified and discarded. The four-point functions of interest should contain two pseudo-scalar fields and two vector fields. Diagrammatically, to tree order, there are only a small number of types of process (ignoring physical considerations) that one can envisage, represented figure 5.

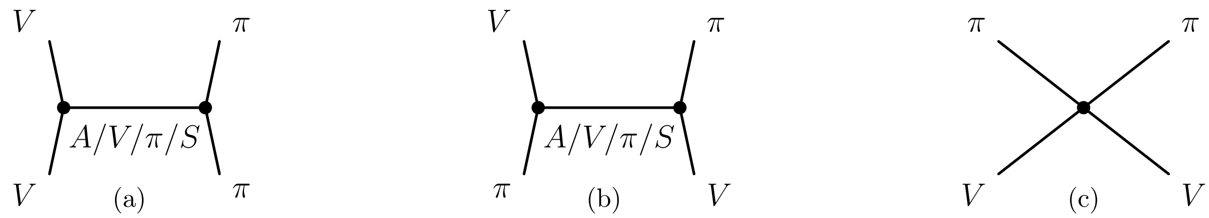


Figure 5: Interactions expected heuristically, no physical considerations taken into account. The fields can be either the field or their derivatives

This suggests the types of vertices that should be kept in our expansion. However, due to the fact that the equations for ϕ and π and ξ and S are coupled, there are, a priori, more 3-4 point vertices to consider than that which the naive analysis suggests. Indeed, the expansion of A will contain factors of π_0 and the expansion of S will contain factors of ξ_0.

This means that, for instance, the two three-point vertices AAS, VVS, or a 4-point vertex $AASS$ could actually produce valid terms. Nevertheless, for the calculation of diagram 1, only the components $a \in \{3, 8\}$ are of any relevance for the external vector fields and for these values of a there is no ξ, S mixing.

Overall, the interaction terms contained in the theory are given in table 2. This table should also be understood to define the coefficients λ_i.

Order 3	Order 4	Order 4
$\lambda_1^{abc}\partial_M V_N^a V^M V^N$	$\lambda_{16}^{abcd} V_M^a V_N^b A^{M,c} A^{N,d}$	$V_M V^M SS$
$\lambda_2^{abc}\partial_M V_N^a A^{b,M} A^{c,N}$	$\lambda_{17}^{abcd} V_M^a V^{M,b} A_N^c A^{N,d}$	$\partial_M \pi \partial^M \pi\pi\pi$
$\lambda_3^{abc}\partial_M A_N V^M A^N$	$\lambda_{18}^{abcd} V_M^a V^{M,b} \pi^c \pi^d$	$\partial_M \pi A^M \pi\pi$
$\lambda_4^{abc}\partial_M A_N V^N A^M$	$V_M V_N V^M V^N$	$A_M A^M \pi\pi$
$\lambda_5^{abc} V_M^a A^{b,M} \pi^c$	$A_M A_N A^M A^N$	$\partial_z \pi\pi\pi\pi$
$\lambda_6^{abc,z} \partial_z \pi^a \pi^b S^c$	$\partial_M \pi \partial^M \pi SS$	$A_z \pi\pi\pi$
$\lambda_7^{abc}\partial_M \pi^a \, \partial^M S^b \pi^c$	$\partial_M \pi \partial^M S \pi S$	
$\lambda_8^{abc,z} S\pi A_z$	$\partial_M S A^M \pi S$	
$\lambda_9^{abc}\partial_M S^a A^{M,b} \pi^c$	$\partial_M \pi A^M SS$	
$\lambda_{10}^{abc}\partial_M \pi^a A^{M,b} S^c$	$V_z \pi\pi S$	
$\lambda_{11}^{abc}\partial_M S^a V^{M,b} S^c$	$\partial_M \pi V^M \pi S$	
$\lambda_{12}^{abc}\partial_M \pi^a \, \partial^M \pi^a$	$\partial_M S V^M \pi\pi$	
$\lambda_{13}^{abc}\partial_M \pi^a V^{M,b} \pi^c$	$V_M A^M \pi S$	
$\lambda_{14}^{abc} V_M^a V^{M,b} S^c$	$A_M A^M SS$	
$\lambda_{15}^{abc} A_M^a A^{M,b} S^c$	$V_M V^M SS$	

Table 2: Interaction terms : When specified, the coefficient λ_i is an expression involving combinations of X_0, f^{ijk} and d^{ijk}; as not all terms were relevant not all of the interactions were given labels. Their expressions are given in appendix B.2

The interaction Lagrangian density $\mathcal{L}_{\text{inter}}$ defined by $\mathcal{S}_{\text{inter}} = \int \mathrm{d}^4 x \sqrt{g} \mathcal{L}_{\text{inter}}$ is then simply the sum of all of these terms. After performing a Fourier transform on the first four spacetime coordinates, multiplications become convolutions according to the transformation rule:

$$A(x,z)B(x,z) \mapsto \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \hat{A}(k',z)\hat{A}(k-k',z)$$

Physically, this can be interpreted as momentum conservation since the above can be rewritten:

$$\int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \hat{A}(k',z)\hat{A}(k-k',z) = \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \frac{\mathrm{d}^4 k''}{(2\pi)^4} (2\pi)^4 \delta(k''+k'-k) \hat{A}(k',z)\hat{A}(k'',z)$$

3.2 Iteration

As all fields have been determined after a partial Fourier transform on the first 4 space-time coordinates, it is more convenient to express the action $\mathcal{S}$ as a function of the Fourier transformed fields. The momentum-space 4-point correlation function for operators $\mathcal{O}_i$ is given by:

$$\langle\mathcal{O}_1(p_1)\mathcal{O}_2(p_2)\mathcal{O}_3(p_3)\mathcal{O}_4(p_4)\rangle = \int \mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{d}^4x_3\mathrm{d}^4x_4 e^{i(p_1x_1+\cdots+p_4x_4)}\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle \tag{3.57}$$

Just like the position-space correlation function can be obtained from the action $\mathcal{S}$ of the 5-dimensional theory by taking the functional derivative with respect to the operator source, it follows that the momentum-space correlation function can be obtained from $\mathcal{S}$ by taking functional derivatives with respect to the Fourier transform of the source field. The relationship that follows from (3.57) is:

$$\langle\Omega|O_1(p_1)O_2(p_2)O_3(p_3)O_4(p_4)|\Omega\rangle = (2\pi)^{16} \left.\frac{\delta^4\mathcal{S}}{\delta\hat{\phi}^*_{0,1}(p_1)\delta\hat{\phi}^*_{0,2}(p_2)\delta\hat{\phi}^*_{0,3}(p_3)\delta\hat{\phi}^*_{0,4}(p_4)}\right|_{\hat{\phi}_{0,1}=\cdots=\hat{\phi}_{0,4}=0}$$

where, * indicates complex conjugation. Again, this derivative should be evaluated at 0 (all the source terms are set to 0 in the final result).

The action should now be rewritten in terms of the momentum space fields, that in turn, should then be expanded in powers of the source fields by the iterative procedure discussed in 3.1.2. Beginning with $\mathcal{S}_{\text{kinetic}}$, that is itself split up into a vector/scalar and pseudo-scalar sectors, the result is:

$$\begin{aligned}\mathcal{S}^{\hat{A}/\hat{\pi}}_{\text{kinetic}} = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\int \mathrm{d}z \frac{L}{2g_5^2}\left(\frac{\beta^a-k^2}{z}\hat{A}^a_{\mu,\perp}(k,z) - \partial_z\left(\frac{1}{z}\partial_z\hat{A}^a_{\mu,\perp}(k,z)\right)\right)\eta^{\mu\nu}\hat{A}^{a,*}_{\nu,\perp}(k,z) \\ +\frac{L}{2g_5^2}\left(\partial_z\left(\frac{\beta^a\partial_z\hat{\pi}^a(k,z)}{z}\right) + \frac{\beta^a k^2}{z}(\hat{\pi}^a(k,z)-\hat{\phi}^a(k,z))\right)\hat{\pi}^{a,*}(k,z) \\ +\frac{L}{2g_5^2}k^2\left(\frac{\beta^a}{z}(\hat{\phi}^a(k,z)-\hat{\pi}^a(k,z)) - \partial_z\left(\frac{1}{z}\partial_z\hat{\phi}^a(k,z)\right)\right)\hat{\phi}^{a,*}(k,z) \\ +\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{L}{2g_5^2}\left[\frac{k^2}{z}\partial_z\hat{\phi}^a\hat{\phi}^{a,*}\right]^{L_1}_{L_0} + \int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{L}{2g_5^2}\left[\frac{1}{z}\eta^{\rho\tau}\partial_z A^a_{\rho,\perp}A^{a,*}_{\tau,\perp}\right]^{L_1}_{L_0} \\ -\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{L}{2g_5^2}\left[\frac{\beta^a}{z}\partial_z\pi^a(-k,z)\pi^a(k,z)\right]^{L_1}_{L_0}\end{aligned}$$

$$
\begin{aligned}
\mathcal{S}^{\hat{V}/\hat{S}}_{\text{kinetic}} = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \int \mathrm{d}z \frac{L}{2g_5^2} \left(\frac{\alpha^a - k^2}{z} \hat{V}^a_{\mu,\perp}(k,z) - \partial_z \left(\frac{1}{z} \partial_z \hat{V}^a_{\mu,\perp}(k,z) \right) \right) \eta^{\mu\nu} \hat{V}^{a,*}_{\nu,\perp}(k,z) \\
+ \frac{L}{2g_5^2} k^2 \left(\frac{\alpha^a}{z} \hat{\xi}^a(k,z) - \partial_z \left(\frac{1}{z} \partial_z \hat{\xi}^a(k,z) \right) - \frac{g_5^2 L^2}{2z^3} f^{abc} X_0^c \hat{S}^b \right) \hat{\xi}^{a,*}(k,z) \\
+ \frac{1}{2} \left(\partial_z \left(\frac{L^3}{z^3} \partial_z \hat{S}^a \right) + \left(\frac{k^2 L^3}{z^3} + 3 \frac{L^3}{z^5} \right) \hat{S}^a - f^{bac} \frac{k^2 X_0^c L^3}{2z^3} \hat{\xi}^b \right) \hat{S}^{a,*} \\
+ \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\eta^{\mu\nu} L}{2g_5^2} \left[\frac{1}{z} \partial_z \hat{V}^a_{\mu,\perp}(-k,z) \hat{V}^a_{\nu,\perp}(k,z) \right]^{L_1}_{L_0} + \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^2 L}{2g_5^2} \left[\frac{1}{z} \partial_z \hat{\xi}^a(-k,z) \hat{\xi}^a(k) \right]^{L_1}_{L_0} \\
- \left[\frac{L^3}{z^3} \partial_z X_0^a \hat{S}^a(0,z) \right]^{L_1}_{L_0} - \int \frac{\mathrm{d}^4k}{(2\pi)^4} \left[\frac{1}{2} \frac{L^3}{z^3} \partial_z \hat{S}^a(-k,z) \hat{S}^a(k,z) \right]^{L_1}_{L_0}
\end{aligned}
$$

The boundary terms come from an integration by parts performed in order to make the equations of motion appear explicitly in the expression. In this way, the final result only contains interaction terms and boundary terms. It can be seen that the bulk kinetic terms essentially reproduce all of the interaction terms after the use of eqs. (3.38), (3.39), (3.41) to (3.43) and (3.45) and the change of variable $k \to -k$. The notable exception to this are interactions containing z-derivatives like λ_{12} or λ_7, where a further integration by parts is sometimes required to bring it to that form; this will produce extra boundary terms. As before, when expressing $\mathcal{S}_{\text{inter}}$ as a function of the Fourier transformed fields, all multiplications give rise to momentum conservation integrals such that it is a sum of terms of the form:

$$
\begin{cases}
\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathrm{d}^4k'}{(2\pi)^4} \frac{\mathrm{d}^4k''}{(2\pi)^4} \mathrm{d}z \sqrt{g} (2\pi)^4 \delta(k+k'+k'') \lambda A(k) B(k') C(k'') & \text{3-point vertices} \\
\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathrm{d}^4k'}{(2\pi)^4} \frac{\mathrm{d}^4k''}{(2\pi)^4} \frac{\mathrm{d}^4k'''}{(2\pi)^4} \mathrm{d}z \sqrt{g} (2\pi)^4 \delta(k+k'+k''+k''') \lambda A(k) B(k') C(k'') D(k''') & \text{4-point vertices}
\end{cases}
$$

In order to calculate a 4-point function of the form: $\langle \Omega | \pi^a \pi^b V^c_\mu V^d_\nu \mid \Omega \rangle$; formula (3.2) implies that relevant terms should be exactly of order 2 in the source fields $V_{0,\perp}, \xi_0$ and order 2 in π_0. This indicates how many times the equations of motion should be used to generate all of the required terms. Under the assumption that all the equations of motion for the different fields are used simultaneously once per field and per iteration, then a total of 4-5 iterations are required.

Finally, out of the above boundary terms the only ones that will be able to produce relevant terms, under the above criterion for a 4-point function of the form (3.33), are a priori :

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\eta^{\mu\nu}L}{2g_5^2}\left[\frac{1}{z}\partial_z\hat{V}^a_{\mu,\perp}(-k,z)\hat{V}^a_{\nu,\perp}(k,z)\right]_{L_0}^{L_1} = -\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\eta^{\mu\nu}L}{2g_5^2}\frac{1}{L_0}\partial_z\hat{V}^a_{\mu,\perp}(-k,L_0)\hat{V}^a_{0,\nu,\perp}(k)$$

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k^2L}{2g_5^2}\left[\frac{1}{z}\partial_z\hat{\xi}^a(-k,z)\hat{\xi}^a(k)\right]_{L_0}^{L_1} = -\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k^2L}{2g_5^2}\frac{1}{L_0}\partial_z\hat{\xi}^a(-k,L_0)\hat{\xi}^a_0(k)$$

$$-\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{L}{2g_5^2}\left[\frac{\beta^a}{z}\partial_z\pi^a(-k,z)\pi^a(k,z)\right]_{L_0}^{L_1} = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{L}{2g_5^2}\frac{\beta^a(L_0)}{L_0}\partial_z\pi^a(-k,L_0)\pi^a_0(k)\frac{\zeta L_0}{2}$$

The IR ($z = L_1$) boundary conditions on the fields $V_\perp, \xi$ and π imply that only the contributions at $z = L_0$ contribute to the final expression.

3.2.1 Intermediate considerations

The iterative expansion of the action $\mathcal{S}$ as a function of the fields $\hat{\xi}_0, \hat{V}_{\perp,0}$ and $\hat{\pi}_0$ was performed using the computer algebra software FORM. More than 2000 terms were generated. This highlighted a defect in the approach presented here: despite the fact that they simplify and make expressions more readable, the constants λ_i (cf. table 2) also hide many of the possible cancellations that can occur. The simplest example of this, for instance, is the fact that $\lambda_1, \ldots, \lambda_4$ are equal up to sign. Nevertheless, an attempt to perform the whole calculation using FORM without introducing these constants produced even more terms (~ 100000 !). Whilst it is not necessarily a problem that cancellations happen at a later point in the calculation (when expanding the coefficients λ_i), there are some cases where it is important to know that the they occur before, for example, using a momentum conservation Dirac delta function. In particular, this is the case for the terms containing factors of $\frac{1}{k_1^2}\delta(k_1+p_1+p_2)$, in the case $p_2 = -p_1$. When calculating the 4-point function: $\langle\Omega|\,\pi^a(p_1)\pi^b(p_2)V^{\mu,c}(p_3)V^{\nu,d}(p_4)\,|\Omega\rangle$ this can only occur for terms containing λ_1 or λ_{14} as can be seen from table 2. This is because, to produce such a momentum conservation integral the two vector fields must be on the same vertex. The terms containing λ_{14} are quickly removed as it can be shown that: $\forall b, d, \lambda_{14}^{3bd} = \lambda_{14}^{8bd} = 0$. Using FORM, it was verified that the λ_1 terms of this form also vanish when $p_2 = -p_1$ in virtue of, firstly, the $z \leftrightarrow z'$ symmetry in our expressions (as they are dummy integration variables and the expression is symmetric in these variables) and, secondly, the antisymmetry properties of λ_1. Due to these cancellations, it also transpires that the mixed propagators for S^a and ξ^b are not required for this 4-point function. The expression is thus well defined in the case $p_2 = -p_1$.

3.2.2 Functional derivative

In order to obtain the desired 4-point function, the functional derivative required is with respect to the full vector field source V_0, so the following relations should be used:

$$\hat{V}_{\perp 0,\mu} = \left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\eta^{\nu\sigma}\hat{V}_{0,\sigma}; \quad \hat{\xi}_0 = \frac{ik_\nu}{k^2}\eta^{\upsilon\sigma}\hat{V}_{0,\sigma} \tag{3.58}$$

Up to this point, our expansion has been done in the general case where we assume that the vector field has both longitudinal and transverse components. However, in the 4-dimensional theory and, at the very least for $a \in 3, 8$, despite our gauge choice in the 5 dimensional theory, there should still be a $U(1)$ gauge freedom such that we can choose to work in the Lorentz gauge where $\hat{V}_0^a = \hat{V}_{0,\perp}^a$. This choice is made in the following and thus no further distinction will be made between V and $V_\perp$. This can be justified further by the absence of physical particles contained in the longitudinal modes $\xi^a, a \in \{3, 8\}$, as can be seen from equation (3.38). Indeed, in the case $a \in \{3, 8\}$ the mixing term and α^a vanish making the equation reduce to:

$$\partial_z \left(\frac{1}{z} \partial_z \hat{\xi}^f \right) = -(2\pi)^4 \frac{k_\sigma}{ik^2} \frac{\delta \mathcal{S}_{\text{int}}}{\delta \hat{V}_\sigma^{*f}} \frac{g_5^2}{L}$$

The bulk to boundary propagator is then trivial: $K_\xi = 1$, and has no poles.

3.2.3 Diagrammatic structure - Witten diagrams

The final expansion of the 4-point function $\langle \Omega | P^a(p_1) P^a(p_2) \mathcal{V}_\mu^b(p_3) \mathcal{V}_\nu^c(p_4) | \Omega \rangle$ has a diagrammatic structure similar to that of Feynman diagrams in QED/QCD. These diagrams are known as Witten diagrams and, in this expansion, are essentially of two types:

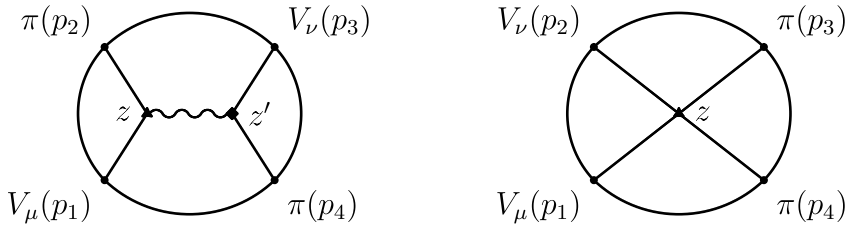


Figure 6: Types of Witten diagram : the diagram on the left will be referred to as Type I and the one to the right Type II .

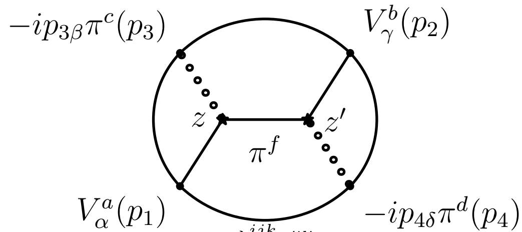
The momentum space Witten rules[16] are:

- The outer ellipse represents the boundary; the points on it are the boundary operators
- A line originating from the boundary and terminating in the bulk is a bulk to boundary propagator.
  - External lines with fields ϕ or ξ should be multiplied by $-ik_\mu$
  - Multiply by $(-ik_\mu)$ for each derivative $\partial_\mu X(k)$

[16] In these rules, the factors of L have been omitted. In our model its actual value is of no importance and so the simplest choice ($L = 1$) was made. It is however relatively straightforward to determine what they should be. They are produced by the metric related terms $\sqrt{g}, g^{\mu\nu}$. The only subtlety is that one must multiply by $\frac{g_5^2}{L}$ instead of just g_5^2 to account for the asymmetry between terms coming from the iteration and those sitting directly in the action.

- Any line between two inner vertices (like the wiggly line in 6) is a bulk to bulk propagator.
  - Transverse propagators must by multiplied by a factor $\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$ where k is the momentum exchanged
  - Longitudinal propagators should be multiplied by a factor $\frac{k_\mu k_\nu}{k^2}$
- The inner vertices are interactions from table 2. They carry a coefficient λ_i and factors of z coming from metric related terms like $\sqrt{g}g^{\mu\nu}g^{\rho\tau}$ or $\sqrt{g}g^{\mu\nu}$
- Multiply by $\int dz$ for each inner vertex
- Momentum is conserved at each vertex
- Multiply by the infamous symmetry factor, determined just as for Feynman diagrams.
- Multiply by $\frac{\zeta L_0}{2}$ for each external pion line. (This is due to the holographic dictionary: $\overline{q}q \leftrightarrow \frac{X}{2z}$ and the boundary conditions for X_0)
- For diagrams with two vertices, multiply by g_5^2 (This comes from an asymmetry between terms coming from the iteration — the equations of motion are multiplied by $\frac{g_5^2}{L}$ — and those in the action. The asymmetry can be resolved by absorbing a factor of g_5 into the λ_i corresponding to 3-vertices.)
- Multiply by the overall momentum conservation factor $(2\pi)^4\delta(p_1 + p_2 + p_3 + p_4)$ (usually hidden in the normalisations)

To illustrate the above rules, we consider an example. In particular, it is insightful to look at an example that involves mixed propagators. For instance, one can draw the following diagram:



Where the pentagram is, for example, $\frac{\lambda_5^{ijk}\eta^{\mu\nu}}{z^3}$ and the dotted lines indicate mixed bulk to boundary propagators. The corresponding expression is $\propto$

$$\frac{g_5^2(\zeta L_0)^2\eta^{\alpha\beta}\eta^{\gamma\delta}}{4}\sum_{f=1}^{8}\int\frac{\mathrm{d}z\mathrm{d}z'}{z^3z'^3}G^f_{\pi\pi}(p_1+p_3,z,z')$$
$$\times K^a_{V_\perp}(p_1,z)K^b_{V_\perp}(p_2,z')(-ip_{3\gamma})K^c_{\phi\pi}(p_3,z)(-ip_{4\gamma})K^d_{\phi\pi}(p_4,z')\lambda_5^{acf}\lambda_5^{bdf}$$

Whilst this structure transpires quite naturally for the bulk terms, further relationships between the bulk to boundary and boundary to boundary propagators K and G are required to show that it is also the case for the boundary terms. For the non-mixed equations, these relations can be derived in a classical manner in position space using Green's second identity, or, in momentum space, by a simple integration by parts. For instance:

$$\begin{aligned}
K^a_{V_\perp}(k,z') &= \int \mathrm{d}z K^a_{V_\perp}(k,z')\delta(z-z') \\
&= \int \mathrm{d}z K^a_{V_\perp}(k,z')\left(-\partial_z\left(\frac{1}{z}G^a_{V_\perp}(k,z,z')\right)+\frac{\alpha^a-k^2}{z}G^a_{V_\perp}(k,z,z')\right) \\
&= \int \mathrm{d}z G^a_{V_\perp}(k,z,z')\underbrace{\left(-\partial_z\left(\frac{1}{z}K^a_{V_\perp}(k,z)\right)+\frac{\alpha^a-k^2}{z}K^a_{V_\perp}(k,z)\right)}_{=0} \\
&- \left[\frac{1}{z}K^a_{V_\perp}(k,z)\partial_z G^a_{V_\perp}(k,z,z')\right]^{L_1}_{L_0} + \left[\frac{1}{z}\partial_z K^a_{V_\perp}(k,z)G^a_{V_\perp}(k,z,z')\right]^{L_1}_{L_0} \\
&= \frac{1}{L_0}\partial_z G^a_{V_\perp}(k,L_0,z')
\end{aligned}$$

Similar results are obtained for the other propagators:

$$\begin{aligned}
K^a_{\phi\phi}(k,z') &= -\frac{1}{L_0}\partial_z G^a_{\phi\phi}(k,L_0,z') \\
K^a_{\pi\phi}(k,z') &= \frac{k^2}{L_0}\partial_z G^a_{\phi\pi}(k,L_0,z') \\
k^2 K_{\phi\pi}(k,z') &= \frac{\beta^a(L_0)}{L_0}\partial_z G^a_{\pi\phi}(k,L_0,z') \\
K_{\pi\pi}(k,z') &= -\frac{\beta^a(L_0)}{L_0}\partial_z G^a_{\pi\pi}(k,L_0,z')
\end{aligned}$$

A short derivation of some of these results is given in appendix B.3; analogous results can also be obtained for S and ξ.

The full expression of $\langle\Omega|\,\pi^a(p_1)\pi^b(p_2)V^{\mu,c}(p_3)V^{\nu,d}(p_4)\,|\Omega\rangle$ is the sum of all possible Witten diagrams and, in the general case, there are a large number of terms. However, in order to calculate the function Π defined by equation (2.32), one must specialise to the case $p_1=-p_2=p; p_3=-p_4=k; a=b; c,d\in\{3,8\}$. Here, the sum of all Witten diagrams of type I[17] contains near 800 terms which reduces to around 200 when the open indices are contracted to calculate $\Pi(p^2)$ (2.31). This long expression will not be given in this text.

[17]cf. fig 6

On the contrary, there are a very small number of diagrams of type II which are:

$$
\begin{aligned}
\int \mathrm{d}z \left(\frac{\zeta L_0}{2}\right)^2 \Bigg(& \frac{2}{z}\lambda_{16}^{cda}(z) p^\mu p^\nu K_V^c(k,z) K_V^d(k,z) K_{\phi\pi}^a(p,z)^2 \\
& + \frac{2}{z}\lambda_{16}^{dca}(z) p^\mu p^\nu K_V^c(k,z) K_V^d(k,z) K_{\phi\pi}^a(p,z)^2 \\
& + \frac{2}{z}\lambda_{17}^{cda}(z) \eta^{\mu\nu} K_V^c(k,z) K_V^d(k,z) K_{\phi\pi}^a(p,z)^2 p^2 \\
& + \frac{2}{z}\lambda_{17}^{dca}(z) \eta^{\mu\nu} K_V^c(k,z) K_V^d(k,z) K_{\phi\pi}^a(p,z)^2 p^2 \\
& + \frac{2}{z^3}\lambda_{18}^{cda}(z) \eta^{\mu\nu} K_V^c(k,z) K_V^d(k,z) K_{\pi\pi}^a(p,z)^2 \\
& + \frac{2}{z^3}\lambda_{18}^{dca}(z) \eta^{\mu\nu} K_V^c(k,z) K_V^d(k,z) K_{\pi\pi}^a(p,z)^2 \Bigg)
\end{aligned}
$$

This expression serves as a good, simple illustration of the Witten rules described above[18]

[18]The overall momentum conservation Dirac delta function $(2\pi)^4\delta(p_1 + p_2 + p_3 + p_4)$ has been omitted

4 Numerical results

4.1 Introduction

The numerical study here will use the parameters of model AI in [1], given in table 3.

Parameter	Value
L_1	$(322.47\ \text{MeV})^{-1}$
m_u	8.291 MeV
m_s	188.48 MeV
$\sigma_s = \sigma_u$	$(213.66\ \text{MeV})^3$

Table 3: AI Model parameters

These values were determined by viewing m_u, m_s and σ_s as fitting parameters. Whilst they play the same roles as the quark masses and quark condensate respectively, their values need not be the same and indeed, as can be seen from table 3, are not in general. Model AI was matched to the experimental values of m_π, f_π, m_K and M_ρ. The value used for the kaon mass m_K in [1] was the average[19] of that of the charged kaons, 493.7 MeV, and the neutral kaons, 497.7 MeV[20], which is $m_K = 495.7$ MeV.

Given electromagnetic interactions, the fact that the neutral kaon is "heavier" than the charged kaon is rather counter-intuitive. It can partly be understood by the fact that $m_u < m_d$. This explanation is supported further by other examples. In the case of the pion, where the averaged mass content of u and d is approximately the same, one finds that: $m(\pi^\pm) = 139.5\ \text{MeV} > m(\pi^0) = 135.0\ \text{MeV}$, whereas like in the kaon case, $m(B^+ = \bar{b}u) = 5279.2\ \text{MeV} < m(B^0 = \bar{b}d) = 5279.5\ \text{MeV}$. A brief estimate of the expected electromagnetic contribution can be obtained through classical considerations: the 0.56 fm charge radius of the K^+, if inserted into Coulombs law, corresponds to a mass split of the order of a MeV.

The value of $1 \times 10^{-3}\ \text{GeV}^{-1}$ was chosen for the UV cut-off L_0. As represented in figure 7, the dependence on the choice of L_0 of the first pole — which corresponds to the kaon state — in the pseudo-scalar bulk to boundary propagators is approximately negligible for values of L_0 below $\frac{1}{100}$. Finally, as in [1,2], L was fixed to 1 since it has no importance.

The aim is now to obtain a numerical estimate of (2.32):

$$\Pi(p^2) = \frac{ie^2}{2} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\eta^{\mu\nu}}{k^2 + i0} \langle\Omega| P(p)P^\dagger(-p)\mathcal{V}_\mu(k)\mathcal{V}_\nu(-k) |\Omega\rangle$$

The pseudo-scalar current operator P that has a non-zero overlap with the charged kaon

[19] For mass considerations, as we have assumed $m_u = m_d$, we cannot really distinguish between K_0 and $K^\pm$

[20] Values from [24]

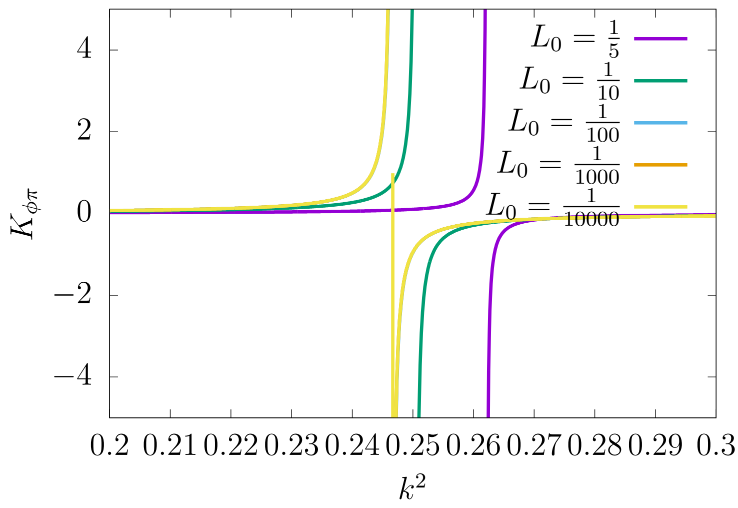


Figure 7: Position of pole in $K_{\phi\pi}$ for different values of L_0

states $K^{+/-}$ corresponds, in the holographic model, to: $\frac{1}{2}(\pi^4(p,z) - \; i\pi^5(p,z))$, therefore:

$$\begin{aligned}
\langle\Omega|\, P(p)P^{\dagger}(-p)\mathcal{V}_{\mu}(k)\mathcal{V}_{\nu}(-k)\,|\Omega\rangle &= \frac{1}{4}\,\langle\Omega|\, P^4(p)P^4(-p)(\mathcal{V}^3_{\mu}(k) + \frac{1}{\sqrt{3}}\mathcal{V}^8_{\mu}(k))(\mathcal{V}^3_{\nu}(-k) + \frac{1}{\sqrt{3}}\mathcal{V}^8_{\nu}(-k)\,|\Omega\rangle \\
&+ \frac{1}{4}\,\langle\Omega|\, P^5(p)P^5(-p)(\mathcal{V}^3_{\mu}(k) + \frac{1}{\sqrt{3}}\mathcal{V}^8_{\mu}(k))(\mathcal{V}^3_{\nu}(-k) + \frac{1}{\sqrt{3}}\mathcal{V}^8_{\nu}(-k)\,|\Omega\rangle \\
&= \frac{1}{2}\,\langle\Omega|\, P^4(p)P^4(-p)(\mathcal{V}^3_{\mu}(k) + \frac{1}{\sqrt{3}}\mathcal{V}^8_{\mu}(k))(\mathcal{V}^3_{\nu}(-k) + \frac{1}{\sqrt{3}}\mathcal{V}^8_{\nu}(-k)\,|\Omega\rangle
\end{aligned} \tag{4.59}$$

Where charge conservation and conjugation, which are both symmetries of our 5D action and QCD, have been used to conclude that:

$$\begin{aligned}
\langle\Omega|\, P^4(p)P^4(-p)\mathcal{V}_{\mu}(k)\mathcal{V}_{\nu}(-k)\,|\Omega\rangle &= \langle\Omega|\, P^5(p)P^5(-p)\mathcal{V}_{\mu}(k)\mathcal{V}_{\nu}(-k)\,|\Omega\rangle \\
\langle\Omega|\, P^4(p)P^5(-p)\mathcal{V}_{\mu}(k)\mathcal{V}_{\nu}(-k)\,|\Omega\rangle &= \langle\Omega|\, P^5(p)P^4(-p)\mathcal{V}_{\mu}(k)\mathcal{V}_{\nu}(-k)\,|\Omega\rangle = 0
\end{aligned}$$

Thes relations follow from :

$$\begin{aligned}
\langle\Omega|\, P(p)P^{\dagger}(-p)\mathcal{V}_{\mu}(k)\mathcal{V}_{\nu}(-k)\,|\Omega\rangle &= \langle\Omega|\, P^{\dagger}(p)P(-p)\mathcal{V}_{\mu}(k)\mathcal{V}_{\nu}(-k)\,|\Omega\rangle \\
\langle\Omega|\, P(p)P(-p)\mathcal{V}_{\mu}(k)\mathcal{V}_{\nu}(-k)\,|\Omega\rangle &= 0 \\
\langle\Omega|\, P^{\dagger}(p)P^{\dagger}(-p)\mathcal{V}_{\mu}(k)\mathcal{V}_{\nu}(-k)\,|\Omega\rangle &= 0
\end{aligned}$$

Determining the general expression of the required quantities was the objective of the previous section. The dependence of that result on the variables k, z, z', p is such that:

$$\frac{\eta^{\mu\nu}}{k^2+i0}\langle\Omega|\,P^a(p)P^a(-p)\mathcal{V}^{c,\mu}(k)\mathcal{V}^{d,\mu}(-k)\,|\Omega\rangle = \underbrace{\int \mathrm{d}z\mathrm{d}z' f_1^{acd}(k^2,k.p,p^2,z,z')}_{\text{Type I diagrams}} + \underbrace{\int \mathrm{d}z f_2^{acd}(k^2,p^2,z)}_{\text{Type II diagrams}} \tag{4.60}$$

In general, momentum integrals with Minkowski signature, $(+,---)$, are ill-defined and difficult to evaluate due to their oscillatory behaviour. However, sometimes it is possible to perform a change of variables, referred to as a Wick rotation, to go to Euclidean signature, $(+,+,+,+)$, where things are generally better defined. A Wick rotation amounts to the change of variable $k_0 \to -ik_0$. In an integral, this is the same as calculating the integral along the contour in figure 8, assuming that the contribution on the circle segments vanishes when the radius tends to infinity and that the poles are contained in the upper left and lower right quadrants of the complex plane. We will suppose that our expressions satisfy these hypotheses. Furthermore, to avoid having to deal numerically with complex numbers, we will also make the change of variable $p_0 = -ip_0$ and study Π for Euclidean $p_E^2 = \sum(p_i)^2$. To evaluate the momentum space integral with Euclidean signature, one can

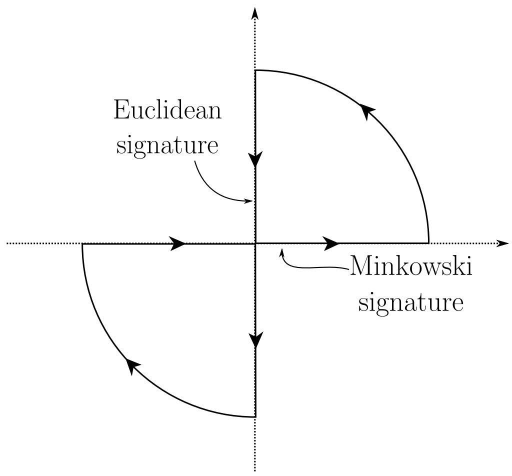


Figure 8: Wick rotation integration contour

use 4-dimensional spherical coordinates (k_E,ϕ,ϕ_2,ϕ_3). If we choose the axes such that p is on one them:

$$\tilde{\Pi}(p_E^2) = \Pi(-p_E^2) = \frac{e^2}{(2\pi)^4}\left(4\pi\int \mathrm{d}k_E \sin^2\phi \mathrm{d}\phi \mathrm{d}z\mathrm{d}z' k_E^3 f_1^{acd}(-k_E^2,\phi,-p_E^2,z,z') + \int 2\pi^2 \mathrm{d}k_E \mathrm{d}z k_E^3 f_2^{acd}(-k_E^2,-p_E^2,z)\right)$$

We will now concentrate on the numerical evaluation of $\tilde{\Pi}(p_E^2)$.

4.2 Calculating $\tilde{\Pi}(p_E^2)$

In order to compute $\tilde{\Pi}(p_E^2)$[21] one must evaluate numerically two multidimensional integrals. It was anticipated that the integral with respect to k_E be the most difficult to evaluate numerically, and thus it was treated separately from the others. For the Type II diagrams, with the exception of the z integral the others are trivial and can be done analytically; the z integral behaves very well numerically. On the contrary, for the Type I diagrams, whilst integration with respect to z and z' at fixed k_E and ϕ does not seem to pose any numerical difficulties, the 3-dimensional integral z, z', ϕ proved to be more troublesome. For this reason, the ϕ integral was also done separately. In figures 9, and 10, the typical dependancy of the integrand on ϕ , after integration with respect to z and z', is represented for several values of p_E^2 and $u = \frac{k}{1+k}$. The variables f_1, f_2 and f_3 correspond respectively to the $SU(3)$ indices c, d and a in (4.60).

Despite their slightly oscillatory behaviour, these curves are reasonably well approximated by a sequence of monomials[22]. They were fitted to polynomials of degree 10 by the least square method using Gnuplot [25]. To illustrate the "goodness" of the fit, figure 11 represents one of the curves and its polynomial approximation; since we are only interested in approximating the integral this is more than sufficient.

[21] All the curves and results presented here correspond in reality to $\tilde{\Pi}(p_E^2)\frac{8}{L_0^2\zeta^2 e^2}$. The constant $\frac{L_0^2\zeta^2}{4}$ will in fact cancel

[22] Fitting to Chebyshev polynomials may have been more appropriate

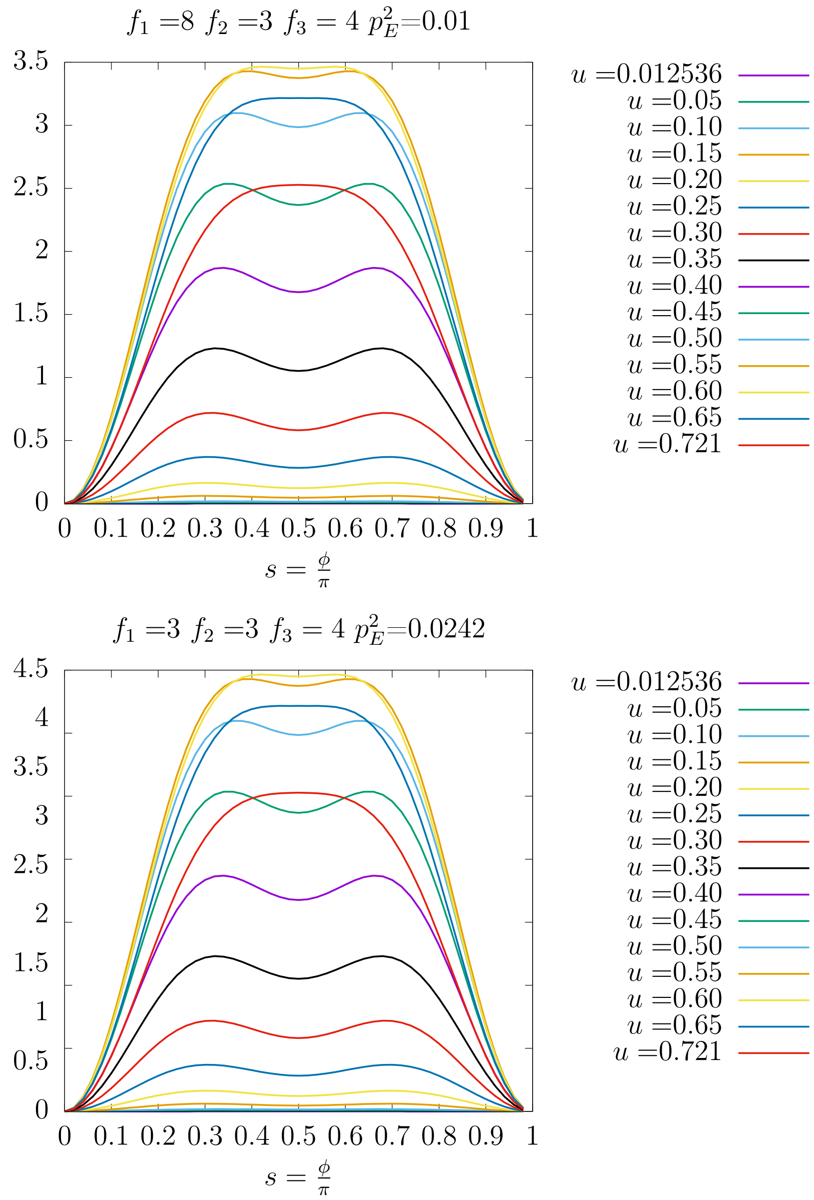


Figure 9: Integrand as a function of $s = \frac{\phi}{\pi}$; the variable u is defined by $k = \frac{u}{1-u}$

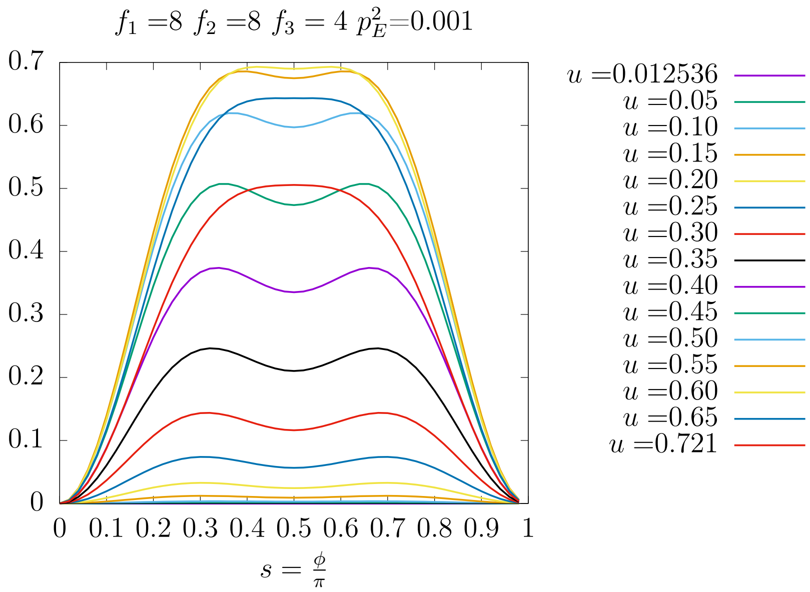


Figure 10: Integrand as a function of $s = \frac{\phi}{\pi}$; the variable u is defined by $k = \frac{u}{1-u}$

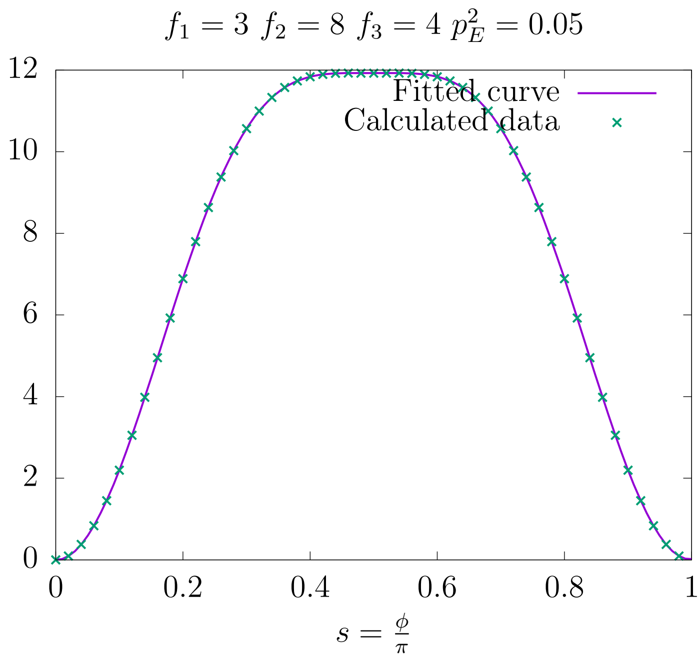


Figure 11: Example of least square fit to polynomial of degree 10

From the fitted curves, we obtain an estimate of the function of k_E that is left to be integrated. The results are represented graphically in figures 12,13,14 and 15. For the Type I diagram contribution, it became very difficult to obtain numerical values for points $u \geq 0.721$, and similarly for the Type II diagrams for $u \geq 0.9$. In these respective ranges of u, the numerical evaluation of the bulk-to-boundary and bulk-to-bulk propagators (which are obtained by solving numerically the differential equations using the shooting method, as in [2]) had difficulties converging. This hindered significantly the execution time of the numerical integration routine. Nevertheless, the calculated curve sections are again relatively well approximated by polynomials of degree 10. In this case, fitting to Chebyshev polynomials was rendered impossible because of the practical difficulties in the neighbourhood of 1.

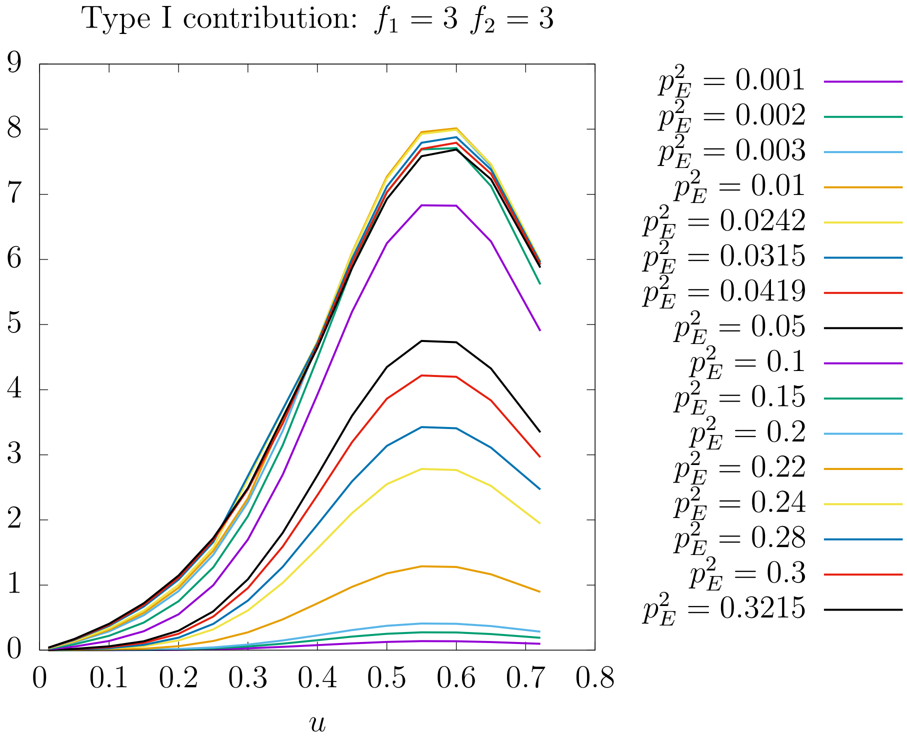
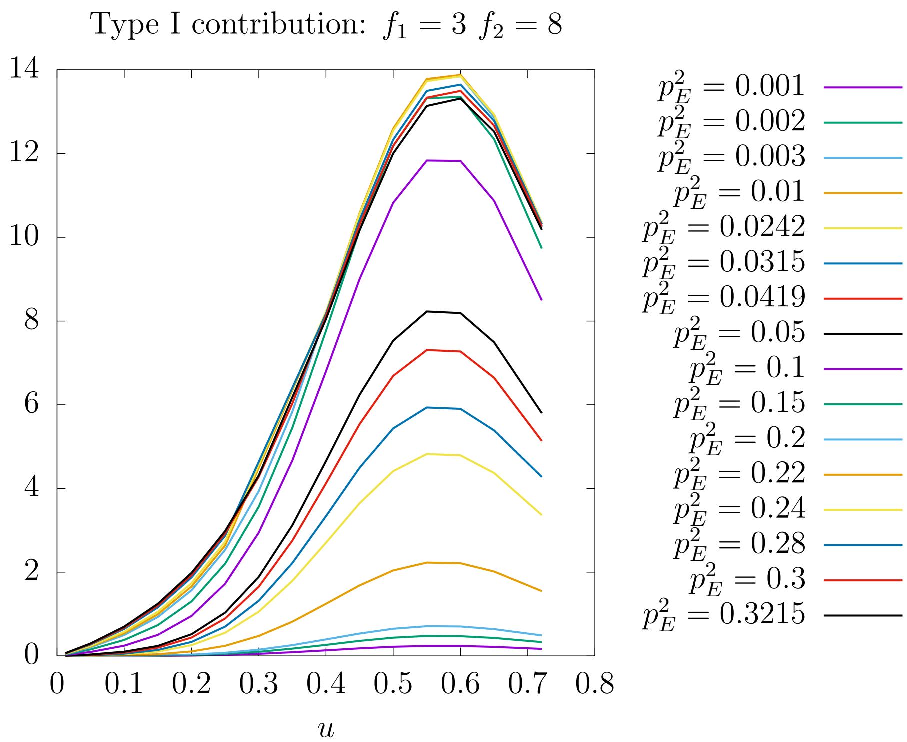


Figure 12


Type I contribution: $f_1 = 3$ $f_2 = 8$
14
12
10
8
6
4
2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
u
$p_E^2 = 0.001$
$p_E^2 = 0.002$
$p_E^2 = 0.003$
$p_E^2 = 0.01$
$p_E^2 = 0.0242$
$p_E^2 = 0.0315$
$p_E^2 = 0.0419$
$p_E^2 = 0.05$
$p_E^2 = 0.1$
$p_E^2 = 0.15$
$p_E^2 = 0.2$
$p_E^2 = 0.22$
$p_E^2 = 0.24$
$p_E^2 = 0.28$
$p_E^2 = 0.3$
$p_E^2 = 0.3215$

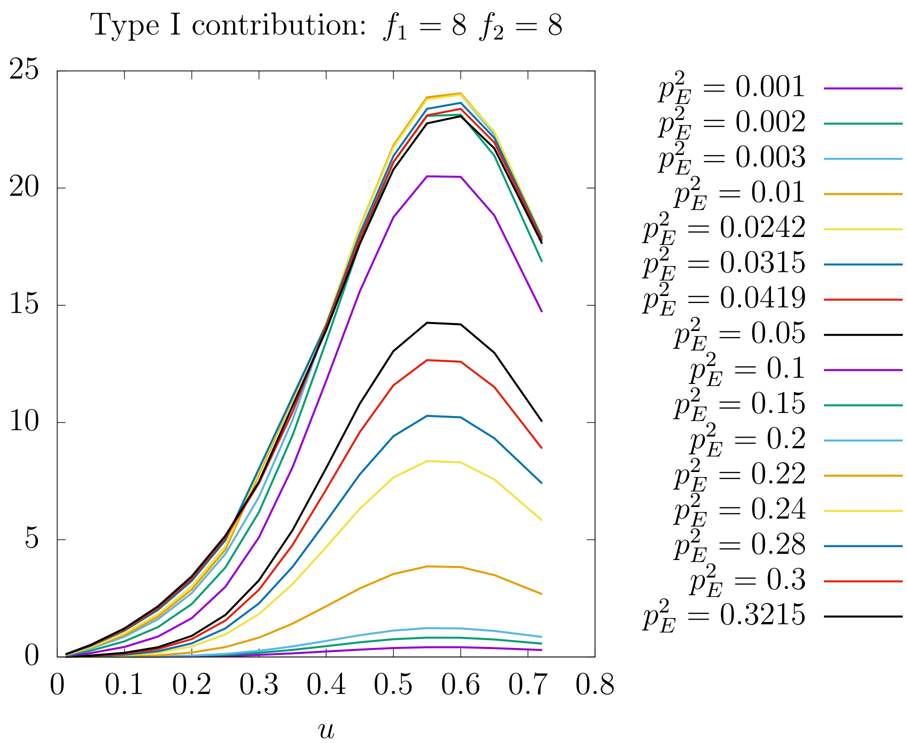
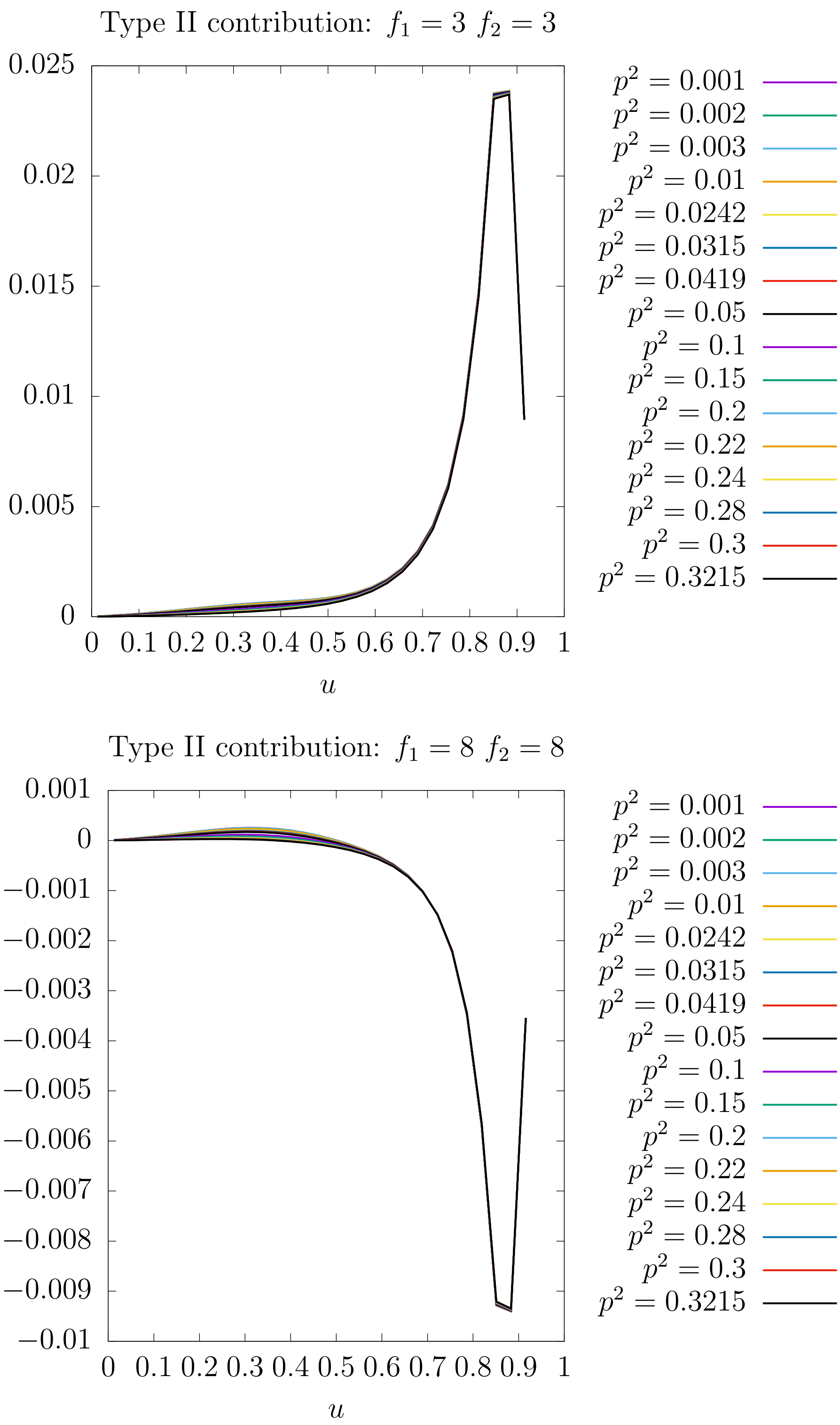

Type I contribution: $f_1 = 8$ $f_2 = 8$
25
20
15
10
5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
u
$p_E^2 = 0.001$
$p_E^2 = 0.002$
$p_E^2 = 0.003$
$p_E^2 = 0.01$
$p_E^2 = 0.0242$
$p_E^2 = 0.0315$
$p_E^2 = 0.0419$
$p_E^2 = 0.05$
$p_E^2 = 0.1$
$p_E^2 = 0.15$
$p_E^2 = 0.2$
$p_E^2 = 0.22$
$p_E^2 = 0.24$
$p_E^2 = 0.28$
$p_E^2 = 0.3$
$p_E^2 = 0.3215$

Figure 13



Type II contribution: $f_1 = 3$ $f_2 = 3$
0.025
0.02
0.015
0.01
0.005
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
u
$p^2 = 0.001$
$p^2 = 0.002$
$p^2 = 0.003$
$p^2 = 0.01$
$p^2 = 0.0242$
$p^2 = 0.0315$
$p^2 = 0.0419$
$p^2 = 0.05$
$p^2 = 0.1$
$p^2 = 0.15$
$p^2 = 0.2$
$p^2 = 0.22$
$p^2 = 0.24$
$p^2 = 0.28$
$p^2 = 0.3$
$p^2 = 0.3215$
Type II contribution: $f_1 = 8$ $f_2 = 8$
0.001
0
−0.001
−0.002
−0.003
−0.004
−0.005
−0.006
−0.007
−0.008
−0.009
−0.01
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
u
$p^2 = 0.001$
$p^2 = 0.002$
$p^2 = 0.003$
$p^2 = 0.01$
$p^2 = 0.0242$
$p^2 = 0.0315$
$p^2 = 0.0419$
$p^2 = 0.05$
$p^2 = 0.1$
$p^2 = 0.15$
$p^2 = 0.2$
$p^2 = 0.22$
$p^2 = 0.24$
$p^2 = 0.28$
$p^2 = 0.3$
$p^2 = 0.3215$

Figure 14

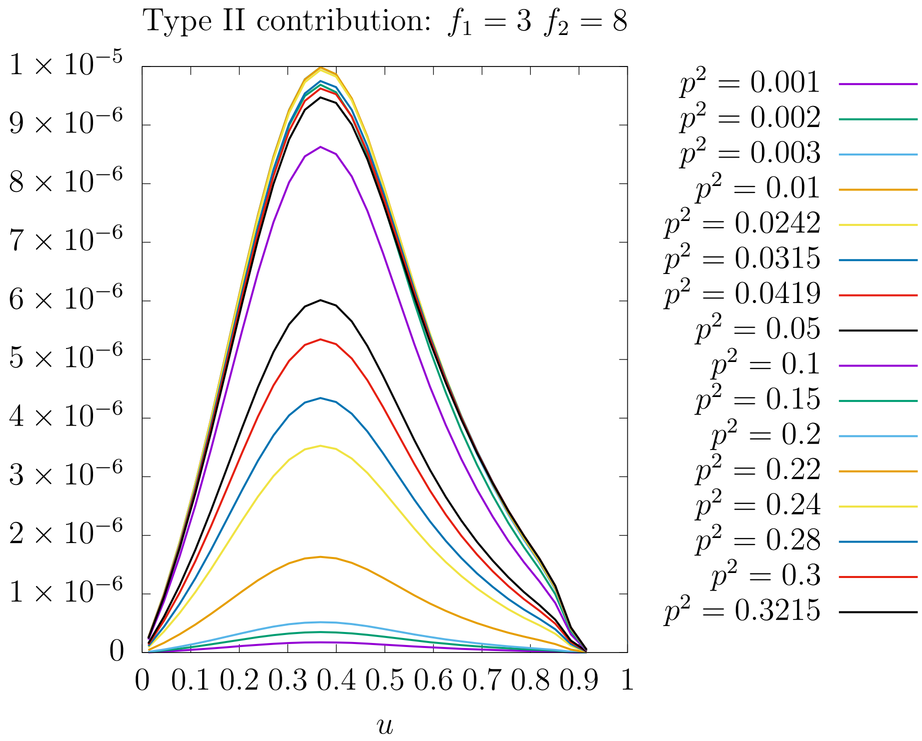
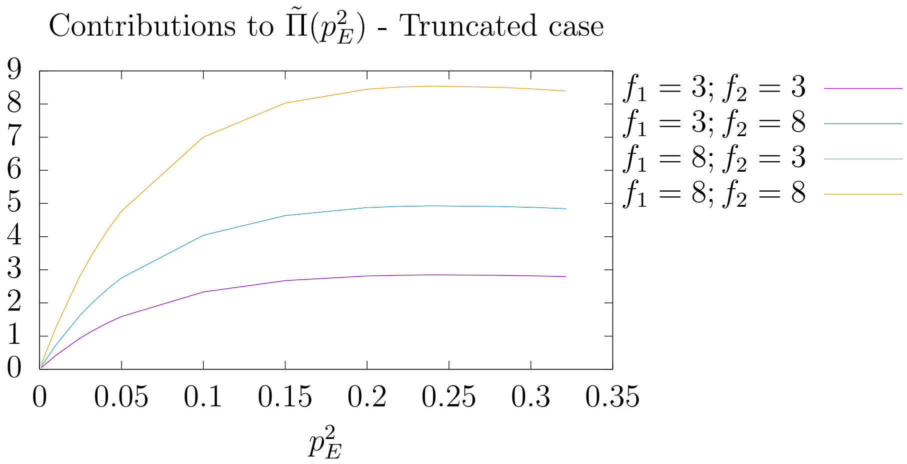


Figure 15

In order to proceed to estimate the full momentum integral, one can either truncate the integral at the last calculated value or, alternatively, attempt a guess at the behaviour of the function beyond that point; both possibilities were considered in the following.[23] When the curves was completed with an analytical guess, a simple logarithm $A \ln u + B$ was chosen in the case of the Type I diagrams, and an exponential $C(1 - \exp(b(1-u)))$ in that of the Type II diagrams. These choices were founded on the steepness of the observed final decrease and were fitted under the assumption that the functions are integrable and hence, at the very least, vanish when $k \to \infty$ or $u \to 1$. In particular, this implies that $B = 0$ in the anstaz $A \ln u + B$. The parameter b was intended to be chosen so as to match the derivatives, but a value of $b = -50$ was retained globally, as it gave a reasonable match for most of the curves. Furthermore, the contribution of Type II diagrams being, in amplitude, much smaller than that of type I diagrams, a perfect fit was not necessary.

After combining these results according to (4.59), we obtain the curves in figure 17. The values of p_E^2 were chosen near 0, where the effect of the pole at the kaon mass for Minkowski p^2 should be the dominant behaviour.

[23] A third possibility is of course also possible: to complete the calculation with data from high-energy perturbative QCD as in [7]. Lack of time prevented us from pursuing this possibility



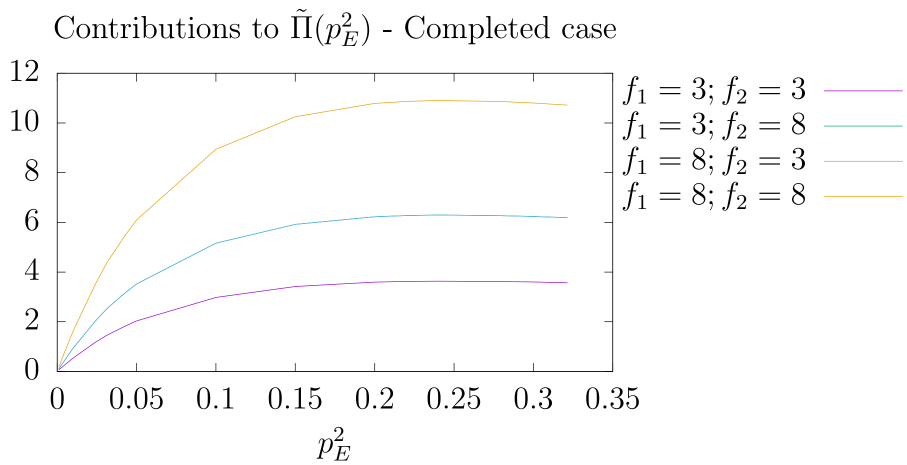


Figure 16: Different contributions to $\tilde{\Pi}(p_E^2)$

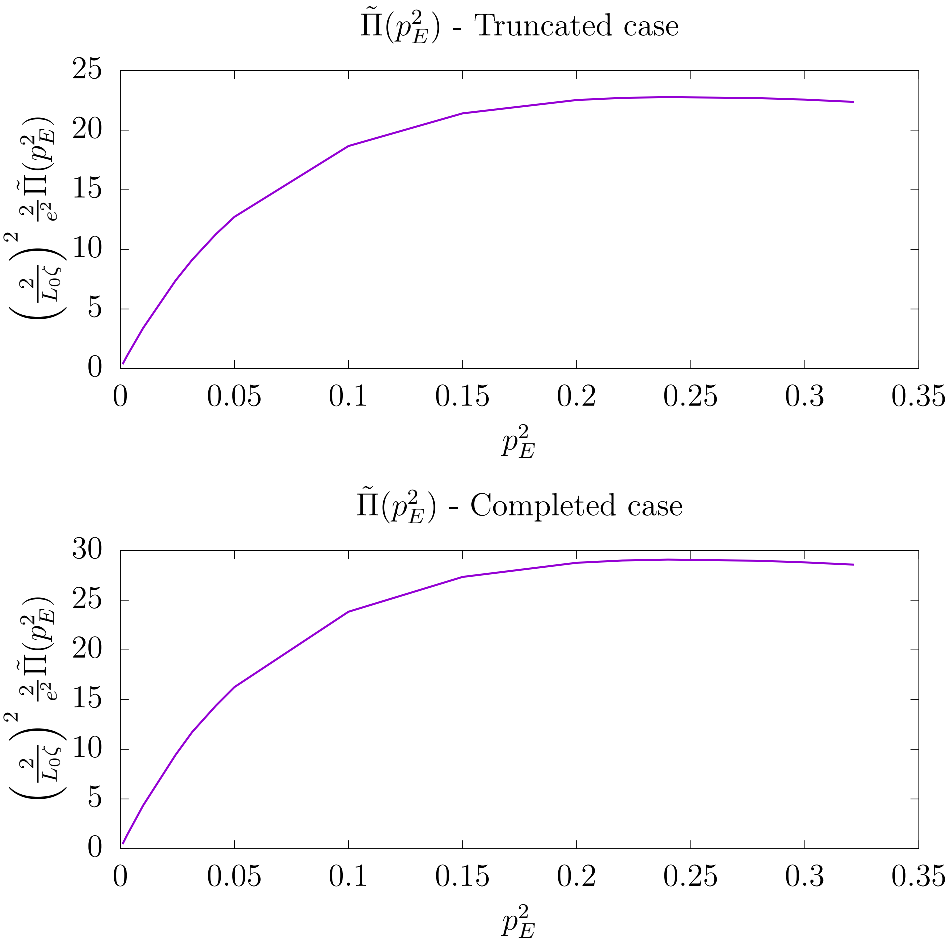


Figure 17: $\tilde{\Pi}(p_E^2)$

4.3 Kaon electromagnetic mass difference

In order to obtain the kaon-electromagnetic mass difference from the curves in figure 17; they were fitted to a function of the type $\frac{Ap_E^2+B}{(p_E^2+m^2)^2}$. This function accounts for the double pole behaviour of the 4-point function; the LSZ reduction formula was then applied, after using the fitted curve to extrapolate to the pole. Firstly, however, we must determine the field renormalisation constant Z that appears in the LSZ formula. This can be determined from the kaon two-point function through it's asymptotic behaviour near the pole:

$$\langle\Omega|\, P(p)P^\dagger(-p)\,|\Omega\rangle = \langle\Omega|\, P^4(p)P^4(-p)\,|\Omega\rangle \sim \frac{Z}{p^2-m_K^2}$$

Fortunately, the two-point function can be retrieved straight-forwardly from the AdS/QCD model. In the expansion of section 3, it can only be produced by the boundary term:

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{L}{2g_5^2}\frac{\beta^a(L_0)}{L_0}\partial_z\pi^a(-k,L_0)\pi_0^a(k)\frac{\zeta L_0}{2}$$

This yields, after iterating to second order in π_0^a and applying the Fourier space version of equation (1.24):

$$\langle\Omega|\, P^4(p)P^4(-p)\,|\Omega\rangle = (2\pi)^4\delta(p+(-p))\frac{L}{g_5^2}\frac{\beta^4(L_0)}{L_0}\partial_z K^4_{\pi\pi}(p,L_0)\left(\frac{\zeta L_0}{2}\right)^2$$

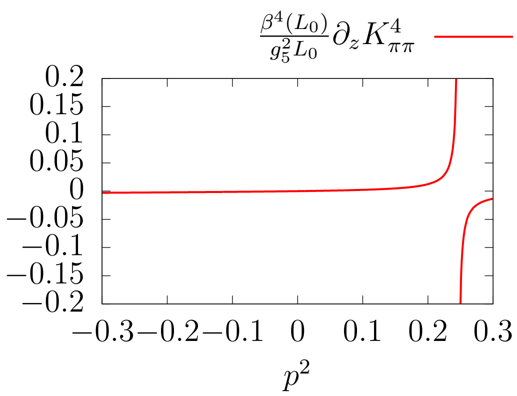


Figure 18: 2-point function

The general behaviour of the 2-point function is illustrated in figure 18. To extract the constant Z, we fit the curve to $\frac{Z_0+Z_1p^2+Z_2p^4}{p^2-m_K^2}$ in the range $[-0.3, 0]$ where we obtained values for the function Π. An estimation of the kaon electromagnetic mass difference will then be given by:

$$\delta m^2 = \frac{e^2}{2}\frac{B-Am_K^2}{Z_0+Z_1m_K^2+Z_2m_K^4}$$

	A	B
Truncated	5.60 ± 0.01	$-3 \cdot 10^{-3} \pm 1 \cdot 10^{-3}$
Completed	7.15 ± 0.01	$-3 \cdot 10^{-3} \pm 1 \cdot 10^{-3}$

Table 4: Values of fitting parameters for $\tilde{\Pi}(p_E^2)$

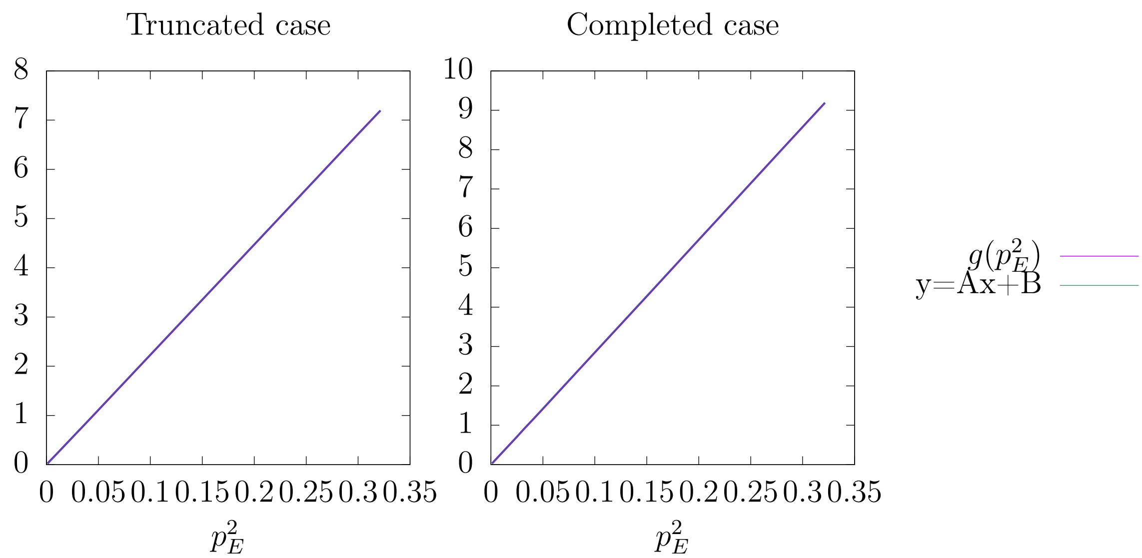


Figure 19: Plot of $g(p_E^2) = (p_E^2 + m_K^2)^2 \left(\frac{2}{\zeta L_0}\right)^2 \tilde{\Pi}(p_E^2)$ and its linear fit . The curves overlap so well that we can hardly distinguish them.

The obtained values for A and B in our two cases are given in table 4; they were obtained by fitting a line to $(p_E^2 + m_K^2)^2 \left(\frac{2}{\zeta L_0}\right)^2 \tilde{\Pi}(p_E^2)$. The overall constant $\frac{\zeta L_0}{2}$ appears in both the expression for $\tilde{\Pi}(p_E^2)$ and the two-point function and so has been factored out of both expressions. In figure 19, the curve $g(p_E^2) = (p_E^2 + m_K^2)^2 \left(\frac{2}{\zeta L_0}\right)^2 \tilde{\Pi}(p_E^2)$ has been represented with its linear fit to illustrate graphically the goodness of the fit. There are slight discrepancies to the linear model for larger values of p_E^2 where higher order corrections($\sim \ln \frac{m_K^2 + p_E^2}{m_K^2}$ for instance) begin to contribute.

The renormalisation constant was estimated to be

$$Z_0 + Z_1 m_K^2 + Z_2 m_K^4 = -7.07 \times 10^{-4} \text{ GeV}^2 / \frac{\mathrm{L}_0^2 \zeta^2}{4}$$

which leads to the estimations of the electromagnetic mass difference given in table 5.

	δm_K^2
Truncated	89.25 GeV^2
Completed	114 GeV^2

Table 5: Model estimates of the kaon electromagnetic mass difference

The order of magnitude is very different from the value obtained in [7] which was:

$$\delta m_{K^+}^2 = (2.32 \pm 0.70) \cdot 10^{-3}\ \text{GeV}^2$$

However, if we divide the results by $(2\pi)^6$ we would obtain $1.45 \cdot 10^{-3}$ GeV^2 and $1.85 \cdot 10^{-3}$ GeV^2 respectively for the truncated and completed cases; these values are closer to the value of [7], although this may simply be coincidental. This suggests that there is an incorrect normalisation factor in the program written for this thesis; unfortunately it was not identified. Similarly, multiplying by L_0^2 would give the correct order of magnitude which further supports the hypothesis of an overall mistake in some constant. Moreover, the two-point function is surprisingly small which could be a defect of the model employed. It can also be noted that the large k_E behaviour of the curves in figures 12,13,14 and 15 is essentially unknown. Inspired by the form of the curves obtained for the the Type II contribution, where a sharp dip is observed in the $u \sim 0.8 - 0.9$ region, one can imagine that similar behaviour could occur for the Type I curves. This could introduce further cancellations, although it is not believed that they will be sufficient to account for the several orders of magnitude difference with the expected result. Nevertheless, despite this disappointing result, one can note that the AdS/QCD model has correctly reproduced the expected physical behaviour of the 4-point function near the pole; notably the double mass pole singularity.

5 Conclusions

In this thesis, an attempt was made to calculate the electromagnetic contribution to the kaon electromagnetic mass difference using a 5-dimension holographic model for QCD. The full calculation of 4-point functions of the type $\langle\Omega| P^a(p_1)P^b(p_2)V^c_\mu(p_3)V^c_\nu(p_4)|\Omega\rangle$ was performed using scripts in FORM, that, although contains sections of code that are specialised to this particular problem, should be straight-forward to generalise to any n-point function. Much time was spent establishing that the calculation could be recast into a diagrammatic formalism similar to perturbation theory Feynman diagrams, known as Witten diagrams, and reducing the large number of terms in the full expansion. For the latter, certain sections of the expansion were studied in detail in order to verify that they vanish and symmetry properties of the coupling constants was studied in detail.

After the theoretical expansion was completed, the 1-loop momentum integral was evaluated numerically in several steps. Certain intermediate functions were fitted to polynomials to accelerate the calculation which, unfortunately, was rather time consuming. The "goodness" of the fit was studied briefly, but, an overall estimation of errors was not attempted and generally the error is not yet under control. This is mostly due to numerical difficulties encountered in evaluating the expression of the 4-point for large values of k. In this region, the execution time of the integration routine was significantly increased ($>$11 days without convergence) and the numerical determination of the bulk-to-boundary (K) and bulk-to-bulk propagators (G) had difficulties attaining a relative accuracy of 10^{-3}. Furthermore, in terms of execution time, the computation of K and G is the limiting factor. Time is to be gained here by improving the treatment of these functions. One possibility could be to attempt to fit them to analytic functions at large k^2, or alternatively, one could attempt to incorporate the integration routine into the main program in order to limit the number of calls to the functions that evaluate K and G by storing the values. It can also be noted that a significant[24] difference in execution time was observed between different compilers. The GNU GCC compiler produced an executable that was approximately 8-10 times slower than the executable produced by the LLVM compiler; this phenomenon is not understood.

The actual numerical value obtained for the kaon electromagnetic mass difference was several orders of magnitude off from the value calculated in [7]. It is believed that this is due to an unidentified error in a normalisation constant in the numerical program. Nevertheless, the model correctly reproduced the expected physical behaviour near the kaon mass, which is an encouraging result for the model. Moreover, all of these considerations were performed using a so-called "Hard-Wall model", that, despite being the simplest way of modifying the AdS geometry to account for the non-conformal behaviour of QCD at low energies, is known to be rather crude. Whilst such a model can produce good results for the ground states, it does not, for instance, produce the correct Regge trajectories: (Hard-wall models predict: $m_n \sim n$, where it should be $m_n \sim n^2$). An interesting extension to the work here

[24] The difference was surprisingly significant: the Intel Core i5 2.5Ghz (2 cores) processor in the laptop using the LLVM compiler executed the program faster than the Intel i7 3.5Ghz (8 cores) processor in the desktop using the GNU compiler!

would be to reproduce these calculations, after correcting the numerical errors, using other numerical fitting schemes or in a “Soft-wall model" like in [1], where a background dilaton field is introduced to smoothly break the conformal symmetry.

Finally a short remark can be made about the AdS/QCD model used here. Although it is built by analogy with the conformal case, there is still some amount of freedom in the choice of parameters and the way to fit it to experimental and theoretical results from QCD; this is particularly apparent in the choice of the boundary conditions for the field X_0 which are treated differently across the literature. The generalisation of quantities that are specifically defined in conformal theories, like the conformal dimension, is also an open question.

Acknowledgements

I would like to thank Prof. Johan Bijnens for supervising this project and introducing me to this fascinating domain of theoretical physics. I would also like to thank him for his patience and support throughout. It goes without saying that I am very grateful for the convivial atmosphere that I have enjoyed in company of my “office mates” Astrid Ordell, Benjamin Alvarez, Nils Truedsson and Erik Kofoed. A special mention is in order for Benjamin Alvarez for our discussions and walks in the park and, of course, for his help and advice with writing this thesis. I am also beholden to Cassandra Ygorra, who proof read my thesis. Thank you to Louis Dussarps and Hadrien Godard, for their advice on numerical optimisation that enabled me to significantly reduce the run-time of my program. Lastly, special thanks to my brother Dominic Borthwick and my good friends Eric Bergvall and Francis Cullinan, for allowing me to mistreat their computers and thus enabling me to successfully complete the numerical part of this thesis on time.

A Explicit calculation of the kinetic part of the action

The aim of this appendix is to perform the rather cumbersome calculation that consists in expanding (A.1) to second order in the fields π, A, V and S. The overall expansion should be done to order 4, but, the full calculation will only be shown explicitly to order 2 in order to illustrate the methods. Furthermore, although this calculation is already present in [1], it is presented again here with slightly different notation in order to highlight some hidden assumptions that were made in [1]. The full calculation to order 4 was performed at first by hand, and then checked using FORM [26].

$$\mathcal{S} = \int \mathrm{d}^5x\sqrt{g}\mathrm{Tr}\left((D_MX)^\dagger(D_MX) + \frac{3}{L^2}X^\dagger X - \frac{1}{4g_5^2}(F^L_{MN}F_L^{MN} + F^R_{MN}F_R^{MN})\right) \tag{A.1}$$

where L and R are not indices but simply denote the left-handed and right-handed parts.

A.1 Notations

For convenience, the notation is recalled here:

$$\begin{aligned}
F^L_{MN} &= \partial_M L_N - \partial_N L_M - i[L_M, L_N] \\
F^R_{MN} &= \partial_M R_N - \partial_N R_M - i[R_M, R_N] \\
L_M &= V_M + A_M \\
R_M &= V_M - A_M \\
D_M X &= \partial_M X - iL_M X + iXR_M \\
X &= e^{i\pi}(X_0 + S)e^{i\pi} \text{ with } X_0^\dagger = X_0
\end{aligned}$$

For further clarity, the following notation will be used throughout this section:

- '$=$' will always denote an exact equality in the usual sense
- '$\sim$' will denote that two expressions are related by the equivalence relation defined by $\mathrm{Tr}(A) = \mathrm{Tr}(B)$.

Recall that the trace is linear form on the vector space $\mathcal{M}_n(\mathbb{C})$ that is invariant under transposition and has the following important property that will be used extensively in the following:

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA) \tag{A.2}$$

or with our notations $AB \sim BA$. This means that products of several matrices that are related by circular permutations of the factors will be equal in the sense of $\sim$.

Finally, using the linearity of the trace, the three terms of the sum in (A.1) will be treated separately.

A.2 Quadratic term $X^\dagger X$

This is the easiest term because, in virtue of (A.2): $X^\dagger X = e^{-i\pi}(X_0+S)^2 e^{i\pi} \underset{tr}{=} X_0^2 + 2X_0 S + S^2$.

A.3 The field strength term: $F^L_{MN}F_L^{MN} + F^R_{MN}F_R^{MN}$

To simplify further this short calculation the following extra notations will be used:

$$V_{MN} = \partial_M V_N - \partial_N V_M \tag{A.3a}$$
$$A_{MN} = \partial_M A_N - \partial_N A_M \tag{A.3b}$$
$$C_{MN} = [V_M, V_N] + [A_M, A_N] \tag{A.3c}$$
$$D_{MN} = [V_M, A_N] + [A_M, V_N] \tag{A.3d}$$

By direct calculation it can be shown that, in terms of the fields V and A, F^L_{MN} and F^R_{MN} take the form:

$$\begin{aligned} F^L_{MN} &= V_{MN} + A_{MN} - i(C_{MN} + D_{MN}) \\ F^R_{MN} &= V_{MN} - A_{MN} - i(C_{MN} - D_{MN}) \end{aligned}$$

C_{MN} and D_{MN} are already terms of order 2 in the fields and and therefore will only produce terms of higher order, thus:

$$\begin{aligned} F^L_{MN}F_L^{MN} + F^R_{MN}F_R^{MN} \sim \quad & V_{MN}V^{MN} + V_{MN}A^{MN} + A_{MN}V^{MN} + A_{MN}A^{MN} \\ & + V_{MN}V^{MN} - V_{MN}A^{MN} - A_{MN}V^{MN} + A_{MN}A^{MN} \\ \sim \quad & 2V_{MN}V^{MN} + 2A_{MN}A^{MN} \end{aligned}$$

In this work, the Hermitian matrices will be systematically decomposed onto the basis (t_a) as it is an orthogonal basis with respect to the inner product defined by $(A, B) = Tr(AB)$. In fact, $Tr(t^a t^b) = \frac{1}{2}\delta^{ab}$.

Decomposing onto this basis then yields:

$$V_{MN} = \partial_M V_N - \partial_N V_M = \sum_{a=1}^{8} (\nabla_M V_N^a - \nabla_N V_M^a) t^a$$

$$\begin{aligned} V_{MN}V^{MN} &= \left(\sum_{a=1}^{8} (\nabla_M V_N^a - \nabla_N V_M^a) t^a \right) \left(\sum_{b=1}^{8} (\nabla^M V^{N,b} - \nabla^N V^{M,b}) t^b \right) \\ &\underset{tr}{=} \frac{1}{2} \sum_{a=1}^{8} (\nabla_M V_N^a - \nabla_N V_M^a)(\nabla^M V^{N,a} - \nabla^N V^{M,a}) \end{aligned}$$

Reproducing the above steps for $A_{MN}A^{MN}$, the final result is then:

$$F^L_{MN}F^{MN}_L + F^R_{MN}F^{MN}_R \underset{tr}{\sim} \sum_{a=1}^{8}(\nabla_M V^a_N - \nabla_N V^a_M)(\nabla^M V^{N,a} - \nabla^N V^{M,a})$$
$$+ \sum_{a=1}^{8}(\nabla_M A^a_N - \nabla_N A^a_M)(\nabla^M A^{N,a} - \nabla^N A^{M,a})$$

A.4 The kinetic term $(D_M X)^\dagger (D_M X)$

Expanding the exponential terms in X, it follows that:

$$X = X_0 + S + i\{X_0,\pi\} - \frac{1}{2}\{\pi,\{\pi,X_0\}\} + i\{\pi,S\} - \frac{i}{6}\{\pi^3,X_0\} - \frac{i}{2}\pi\{X_0,\pi\}\pi$$
$$-\frac{1}{2}\{\pi,\{S,\pi\}\} + \frac{1}{24}\{\pi^4,X_0\} - \frac{i}{6}\{S,\pi^3\} - \frac{i}{2}\pi\{S,\pi\}\pi + \frac{1}{6}\pi\{\pi\{X_0,\pi\}\}\pi - \frac{1}{12}\pi^2X_0\pi^2 + \mathcal{O}(5) \tag{A.4}$$

Where $\{,\}$ denotes an anti-commutator.

A.4.1 Calculations with the matrices π, V_M, A_M

The first step is to calculate: $D_M X = \partial_M X - iL_M X + iXR_M$. From (A.4) it follows that:

$$\partial_M X = \partial_M X_0 + \partial_M S + i\{\partial_M\pi, X_0\} + i\{\pi,\partial_M X_0\} - \frac{1}{2}\{\partial_M\pi,\{X_0,\pi\}\} - \frac{1}{2}\{\pi,\{\partial_M X_0,\pi\}\}$$
$$-\frac{1}{2}\{\pi,\{X_0,\partial_M\pi\}\} + i\{\partial_M\pi,S\} + i\{\pi,\partial_M S\} + \mathcal{O}(3)$$

As $\{,\}$ is a bilinear map and therefore the derivative will act on it in the same way as a product.

Replacing L_M and R_M by their expressions in terms of V_M and A_M:

$$-iL_M X + iXR_M = i[X_0,V_M] - i\{A_M,X_0\} + [V_M,\{\pi,X_0\}] + \{A_M,\{\pi,X_0\}\} + i[S,V_M] - i\{A_M,S\}$$

where $[,]$ denotes a commutator.

The covariant derivative is then:

$$D_M X = \partial_M X_0 + \partial_M S + i\{\partial_M\pi, X_0\} + i\{\pi,\partial_M X_0\} - \frac{1}{2}\{\partial_M\pi,\{X_0,\pi\}\} - \frac{1}{2}\{\pi,\{\partial_M X_0,\pi\}\}$$
$$-\frac{1}{2}\{\pi,\{X_0,\partial_M\pi\}\} + i\{\partial_M\pi,S\} + i\{\pi,\partial_M S\} + i[X_0,V_M] - i\{A_M,X_0\}$$
$$+[V_M,\{\pi,X_0\}] + \{A_M,\{\pi,X_0\}\} + i[S,V_M] - i\{A_M,S\} + \mathcal{O}(3)$$

From which we deduce its hermitian conjugate:

$$\begin{aligned}(D_M X)^\dagger =&\partial_M X_0 + \partial_M S - i\{\partial_M\pi, X_0\} - i\{\pi, \partial_M X_0\} - \frac{1}{2}\{\partial_M\pi, \{X_0, \pi\}\} - \frac{1}{2}\{\pi, \{\partial_M X_0, \pi\}\}\\ &- \frac{1}{2}\{\pi, \{X_0, \partial_M\pi\}\} - i\{\partial_M\pi, S\} - i\{\pi, \partial_M S\} + i[X_0, V_M] + i\{A_M, X_0\}\\ &- [V_M, \{\pi, X_0\}] + \{A_M, \{\pi, X_0\}\} + i[S, V_M] + i\{A_M, S\} + \mathcal{O}(3)\end{aligned}$$

This follows from the fact that $\frac{\mathrm{d}(M^\dagger)}{\mathrm{d}t} = (\frac{\mathrm{d}M}{\mathrm{d}t})^\dagger$ and from the following basic facts about Hermitian matrices A, B:

- $\{A, B\}^\dagger = \{A^\dagger, B^\dagger\} = \{A, B\}$
- $[A, B] = [B^\dagger, A^\dagger] = -[A^\dagger, B^\dagger] = -[A, B]$

The next step is to perform the product of the above two expressions (contracting the open M index):

0-th order

The only term to 0-th order is clearly: $\partial_M X_0 \partial^M X_0$

1st order

To first order:

$$\begin{aligned}\{\text{First order terms}\} = &\,\partial^M X_0\,(\partial_M S + i\{\partial_M\pi, X_0\} + i\{\pi, \partial_M X_0\} + i[X_0, V_M] - i\{A_M, X_0\})\\ &+(\partial^M S - i\{\partial^M\pi, X_0\} - i\{\pi, \partial^M X_0\} + i[X_0, V^M] + i\{A^M, X_0\})\partial_M X_0\end{aligned}$$

where for a scalar field $\partial^M = g^{MN}\partial_M = g^{MN}\nabla_M$

As only the trace of the above expression is interesting, it can be simplified greatly using (A.2):

$$\{\text{First order terms}\} \sim 2\partial_M X_0 \partial^M S + 2i[\partial^M X_0, X_0]V_M$$

The term $2i[\partial^M X_0, X_0]V_M$ cancels because $[\partial^M X_0, X_0] = 0$ follows from the choice of vacuum made in section 2.1.

2nd order

Applying the same techniques as above the second order terms are:

$$
\begin{aligned}
\{\text{Second order terms}\} = & -\partial^M X_0 \left(\{\partial_M \pi, \{\pi, X_0\}\} + \{\pi, \{\partial_M \pi, X_0\}\} + \{\pi, \{\pi, \partial_M X_0\}\}\right) \\
& + 2\partial_M X_0 (i[S, V_M] + \{A_M, \{\pi, X_0\}\}) + \partial_M S \partial^M S + 2i\partial_M S[X_0, V^M] \\
& + \{\partial^M \pi, X_0\}\{\partial_M \pi, X_0\} + 2\{\partial^M \pi, X_0\}\{\pi, \partial_M X_0\} + \{\pi, \partial_M X_0\}\{\pi, \partial^M X_0\} \\
& - 2\{\partial_M \pi, X_0\}\{A_M, X_0\} - [X_0, V_M][X_0, V^M] - 2\{\pi, \partial_M X_0\}\{A^M, X_0\} \\
& + \{A_M, X_0\}\{A^M, X_0\}
\end{aligned} \tag{A.5}
$$

Several further simplifications are possible after expansion of the double anti-commutators. The green terms can be simplified as follows:

$$
\begin{aligned}
& -\partial^M X_0 \left(\{\partial_M \pi, \{\pi, X_0\}\} + \{\pi, \{\partial_M \pi, X_0\}\} + \{\pi, \{\pi, \partial_M X_0\}\}\right) + 2\{\partial^M \pi, X_0\}\{\pi, \partial_M X_0\} \\
& \qquad + \{\pi, \partial_M X_0\}\{\pi, \partial^M X_0\} \\
\sim & 2(\partial_M \pi X_0 \pi \partial_M X_0 + \partial_M \pi X_0 \partial_M X_0 \pi + X_0 \partial_M \pi \pi \partial_M X_0 + X_0 \partial_M \pi \partial X_0 \pi) \\
& + \underbrace{(\pi \partial_M X_0 \pi \partial_M X_0 + \pi (\partial_M X_0)^2 \pi + \partial_M X_0 \pi^2 \partial_M X_0 + \partial_M X_0 \pi \partial_M X_0 \pi)}_{2(\pi \partial_M X_0 \pi \partial_M X_0 + \pi^2 (\partial_M X_0)^2)} \\
& - \Big(\partial_M X_0 \partial_M \pi X_0 \pi + \partial_M X_0 \partial_M \pi \pi X_0 + \partial_M X_0 X_0 \pi \partial_M \pi + \partial_M X_0 \pi X_0 \partial_M \pi + \partial_M X_0 \pi \partial_M X_0 \pi \\
& + \partial_M X_0 \pi^2 \partial_M X_0 + (\partial_M X_0)^2 \pi^2 + \partial_M X_0 \pi \partial_M X_0 \pi + \partial_M X_0 \pi X_0 \partial_M \pi + \partial_M X_0 \pi \partial_M \pi X_0 \\
& + \partial_M X_0 X_0 \partial_M \pi \pi + \partial_M X_0 \partial_M \pi X_0 \pi \Big)
\end{aligned}
$$

Terms of the same colour (that is not black) in the above expression are equal, so when a term appears twice in the negative part it cancels its counterpart in the positive part. Unfortunately, not all terms cancel immediately and the remaining ones are:

$$
\begin{aligned}
& X_0 \partial_M \pi \pi \partial_M X_0 - \partial_M X_0 \partial_M \pi \pi X_0 + \partial_M \pi X_0 \partial_M X_0 \pi - \partial_M X_0 X_0 \pi \partial_M \pi \\
\sim & \partial_M X_0 X_0 \partial_M \pi \pi - X_0 \partial_M X_0 \partial_M \pi \pi + \pi \partial_M \pi X_0 \partial_M X_0 - \pi \partial_M \pi \partial_M X_0 X_0 \\
\sim & [\partial_M X_0, X_0] \partial_M \pi \pi + \pi \partial_M \pi [X_0, \partial_M X_0] \\
\sim & [\partial_M X_0, X_0][\partial_M \pi, \pi]
\end{aligned}
$$

The term therefore cancels under the assumption that $[X_0, \partial_M X_0] = 0$.
The red terms in (A.5) simplify as follows:

$$
\begin{aligned}
&- 2\{\pi, \partial^M X_0\}\{A_M, X_0\} + 2\partial^M X_0\{A_M, \{\pi, X_0\}\} \\
=&2\left(-(\pi\partial^M X_0 + \partial^M X_0\pi)(A_M X_0 + X_0 A_M) + \partial^M X_0(A_M(\pi X_0 + X_0\pi) + (\pi X_0 + X_0\pi)A_M)\right) \\
&= 2\left(-(\pi\partial^M X_0 A_M X_0 + \pi\partial^M X_0 X_0 A_M + \partial^M X_0\pi A_M X_0 + \partial^M X_0\pi X_0 A_M) + \partial^M X_0 A_M\pi X_0\right. \\
&\qquad\left. + \partial^M X_0 A_M X_0\pi + \partial^M X_0\pi X_0 A_M + \partial^M X_0 X_0\pi A_M\right) \\
&\sim 2(\partial^M X_0 A_M\pi X_0 - \pi\partial^M X_0 X_0 A_M + \partial^M X_0 X_0\pi A_M - \partial^M X_0\pi A_M X_0) \\
&\sim 2([X_0, \partial^M X_0]A_M\pi + [\partial^M X_0, X_0]\pi A_M) \\
&\sim 2[\partial^M X_0, X_0][\pi, A_M]
\end{aligned}
$$

The remaining terms in (A.5) can be reshuffled so that the above shows that, under the assumption $[\partial_z X_0, X_0] = 0$:

$$
\begin{aligned}
(D_M X)^\dagger(D_M X) \sim\ & \partial_M X_0\partial^M X_0 + 2i[\partial_M S, X_0]V^M + 2i[\partial_M X_0, S]V^M + \{\partial^M\pi, X_0\}\{\partial_M\pi, X_0\} \\
& -[X_0, V^M][X_0, V_M] + \{A^M, X_0\}\{A_M, X_0\} - 2\{\partial^M\pi, X_0\}\{A_M, X_0\}
\end{aligned} \tag{A.6}
$$

A.4.2 Introducing explicitly the SU(3) index

As before the matrices π, S, V and A should be decomposed onto the basis (t_a). (A.6) then becomes:

$$
\begin{aligned}
(D_M X)^\dagger(D_M X) \sim\ & \partial_M X_0^a\partial^M X_0^b t^a t^b + 2i\partial_M S^a X_0^b V^{M,c}[t^a, t^b]t^c + 2i\partial_M X_0^a S^b V^{c,M}[t^a, t^b]t^c \\
& + \partial^M\pi^a\partial_M\pi^b\{t^a, X_0\}\{t^b, X_0\} - V^{a,M}V^b_M[X_0, t^a][X_0, t^b] \\
& + A^{a,M}A^b_M\{t^a, X_0\}\{t^b, X_0\} - 2\partial^M\pi^a A^b_M\{t^a, X_0\}\{t^b, X_0\}
\end{aligned} \tag{A.7}
$$

All of the above traces can be calculated as functions of the structure coefficients f^{ijk} and d^{ijk} defined in C.

It is important to evaluate $[X_0, t^a][X_0, t^b]$ and $\{X_0, t^a\}\{X_0, t^b\}$ for all $a, b \in \{0, \ldots, 8\}$. This can be done by a direct calculation:

$$
\begin{aligned}
[X_0, t^a][X_0, t^b] &= X_0^c X_0^d [t^c, t^a][t^d, t^b] \\
&= \frac{1}{4} X_0^c X_0^d f^{cak} f^{dbl} t^k t^l \\
\therefore \quad \mathrm{tr}([X_0, t^a][X_0, t^b]) &= -\frac{1}{8} X_0^c X_0^d f^{cak} f^{dbk} \\
&= -\frac{1}{8}\left((X_0^0)^2 f^{0ak} f^{0bk} + X_0^0 X_0^8(f^{0ak} f^{8bk} + f^{8ak} f^{0bk}) + (X_0^8)^2 f^{8ak} f^{8bk}\right)
\end{aligned}
$$

Clearly: $f^{0ak} = 0$ for all a, hence:

$$\mathrm{tr}[X_0, t^a][X_0, t^b] = -\frac{1}{8}(X_0^8)^2 f^{8ak} f^{8bk} \tag{A.8}$$

It is easily seen from (C.25) that $f^{8ak} f^{8bk} = 3\delta^{ab}$, so that :

$$\mathrm{tr}[X_0, t^a][X_0, t^b] = -\frac{3}{8}(X_0^8)^2 \delta^{ab} \tag{A.9}$$

Similarly for $\mathrm{tr}\{X_0, t^a\}\{X_0, t^a\}$:

$$\begin{aligned}
\{X_0, t^a\}\{X_0, t^a\} &= X_0^c X_0^d d^{cak} d^{dbl} t^k t^l \\
\therefore \quad \mathrm{tr}(\{X_0, t^a\}\{X_0, t^a\}) &= \frac{1}{2} X_0^c X_0^d d^{cak} d^{dbl} \\
&= \frac{(X_0^0)^2}{2} d^{a0k} d^{b0k} + \frac{X_0^0 X_0^8}{2}(d^{a8k} d^{b0k} + d^{a0k} d^{b8k}) + \frac{(X_0^8)^2}{2} d^{a8k} d^{b8k}
\end{aligned}$$

The only combinations of the indices $\{a, k, 0\}$ and $\{a, k, 8\}$, such that $d^{ak8} \neq 0$ or $d^{ak0} \neq$ are:

$$\forall a \in \{0, \dots, 8\}, d^{aa0} = \frac{\sqrt{6}}{3}; d^{118} = d^{228} = d^{338} = \frac{1}{\sqrt{3}}; d^{448} = d^{558} = d^{668} = d^{788} = -\frac{\sqrt{3}}{6}; d^{888} = -\frac{1}{\sqrt{3}} \tag{A.10}$$

Thus:

$$\mathrm{tr}(\{X_0, t^a\}\{X_0, t^b\}) = \begin{cases} 0 & \text{if } a \neq b \\ \frac{(X_0^0)^2}{2}(d^{aa0})^2 + X_0^0 X_0^8 \sqrt{\frac{2}{3}} d^{aa8} + \frac{(X_0^8)^2}{2}(d^{aa8})^2 & \text{if } a = b \notin \{0, 8\} \\ \frac{(X_0^0)^2}{2}(d^{000})^2 + \frac{(X_0^8)^2}{2}(d^{088})^2 & \text{if } a = b = 0 \\ \frac{(X_0^0)^2}{2}(d^{808})^2 + X_0^0 X_0^8 (d^{888} d^{880}) + \frac{(X_0^8)^2}{2}(d^{880} d^{880} + d^{888} d^{888}) & \text{if } a = b = 8 \\ \frac{X_0^0 X_0^8}{2}((d^{088})^2 + d^{000} d^{880}) + \frac{(X_0^8)^2}{2} d^{088} d^{888} & \text{if } \{a, b\} = \{0, 8\} \end{cases} \tag{A.11}$$

Finally, using (A.10) and the expressions for X_0^0 and X_0^8 given in (2.29) :

$$\mathrm{tr}[X_0, t^a][X_0, t^b] = \begin{cases} -\frac{1}{8}(v_u - v_s)^2 \delta^{ab} & \text{if } a \in 4, 5, 6, 7 \\ 0 & \text{otherwise} \end{cases} \tag{A.12}$$

$$\{X_0, t^a\}\{X_0, t^b\} = \begin{cases} \frac{1}{6}(2v_u^2 + v_s^2) & a = b = 0 \\ \frac{1}{2} v_u^2 & a = b \in \{1, 2, 3\} \\ \frac{1}{8}(v_u + v_s)^2 & a = b \in \{4, 5, 6, 7\} \\ \frac{1}{6}(v_u^2 + 2v_s^2) & a = b = 8 \\ \frac{\sqrt{2}}{6}(v_u^2 - v_s^2) & \{a, b\} = \{0, 8\} \end{cases} \tag{A.13}$$

So that one can define the quantities M_V^a and M_A^a, following [1,2]:

$$\frac{1}{2}M_V^{a\,2} = -\mathrm{Tr}([t^a, X_0]^2) \tag{A.14}$$
$$\frac{1}{2}M_A^{a\,2} = \mathrm{Tr}(\{t^a, X_0\}^2) \tag{A.15}$$

For completeness, the mixed quantity $M_A^{80\,2}$ can be defined by:

$$\frac{1}{4}M_A^{80\,2} = \mathrm{tr}(\{t^a, X_0\}\{t^b, X_0\}) \tag{A.16}$$

It is however not needed unless a pseudo-scalar flavour singlet is included in the theory. Overall:

$$\begin{aligned}(D_M X)^\dagger(D_M X) \sim \frac{1}{2}\partial_M X_0^a \partial^M X_0^a - \frac{1}{2}f^{abc}\partial_M S^a X_0^b V^{M,c} - \frac{1}{2}f^{abc}\partial_M X_0^a S^b V^{M,c} \\ +\frac{1}{2}M_V^{a\,2} V^{M,a} V_M^a + \frac{1}{2}M_A^{a\,2}(\partial_M \pi^a - A_M^a)(\partial^M \pi^a - A^{M,a})\end{aligned} \tag{A.17}$$

Combining all of the previous results yields equation (3.34).

B Intermediate results for the calculation of the 4-point function

B.1 Kinetic part functional derivative

$$\begin{aligned}\frac{\delta \mathcal{S}_{\text{kinetic}}}{\delta V_L^f(x')} = \frac{L}{g_5^2}\Bigg(\frac{\alpha^f}{z}\eta^{LM'}V_{M'}^f - \frac{\eta^{LM'}}{2}(f^{fbc}X_0^c\partial_{M'}S^b + f^{afc}\partial_{M'}X_0^c S^a) \\ +\eta^{MM'}\eta^{LN'}\partial_M(\frac{1}{z}(\partial_{M'}V_{N'}^f - \partial_{N'}V_{M'}^f))\Bigg)\end{aligned} \tag{B.18}$$

$$\frac{\delta \mathcal{S}_{\text{kinetic}}}{\delta A_L^f(x')} = \frac{L}{g_5^2}\left(\frac{\beta^f}{z}\eta^{LM'}(A_{M'} - \partial_{M'}\pi^f) + \eta^{MM'}\eta^{LN'}\partial_M(\frac{1}{z}(\partial_{M'}A_{N'}^f - \partial_{M'}A_{M'}^f))\right) \tag{B.19}$$

$$\frac{\delta \mathcal{S}_{\text{kinetic}}}{\delta \pi^f(x')} = -\frac{L}{g_5^2}\eta^{MM'}\partial_{M'}\left(\frac{\beta^f}{z}(\partial_M \pi^f - A_M^f)\right) \tag{B.20}$$

$$\begin{aligned}\frac{\delta \mathcal{S}_{\text{kinetic}}}{\delta S^f(x')} = -\eta^{MM'}\partial_M(\frac{L^3}{z^3}\partial_{M'}S^f) + \underbrace{\left(-\eta^{MM'}\partial_M(\frac{L^3}{z^3}\partial_{M'}X_0^f) + \frac{3X_0^f L^3}{z^5}\right)}_{=0 \text{ EOM for vacuum}} \\ -\eta^{MM'}\frac{L^3}{2z^3}f^{afc}\partial_M X_0^a V_{M'}^c + \frac{1}{2}f^{fbc}\eta^{MM'}\partial_M(\frac{L^3}{z^3}X_0^b V_{M'}^c) + \frac{3L^3}{z^5}S^f\end{aligned} \tag{B.21}$$

Where the following notations have been introduced:

$$\alpha^f(z) = \frac{g_5^2 L^2 M_V^{f\,2}}{z^2} \quad \beta^f(z) = \frac{g_5^2 L^2 M_A^{f\,2}}{z^2} \tag{B.22}$$

B.2 Coupling coefficients

In this section, the coupling coefficients λ_i are given in terms of the SU(3) structure constants d and f, the Einstein summation convention is used in that repeated indices in a product are summed over. Lowercase roman letters are gauge indices and vary from 0 to 8. The expressions in table 6 were calculated by hand and checked with FORM; all other expressions were obtained using FORM.

The first 4 coefficients are not independent and are related by:

$$\lambda_1^{abc} = \lambda_2^{abc} = \lambda_3^{abc} = -\lambda_4^{abc} \tag{B.23}$$

λ_1^{abc}	$-\frac{1}{2g_5^2} f^{abc}$
$\lambda_6^{z,abc}$	$-\frac{f^{cdl} f^{abl}}{8} \partial^z X_0^d$
λ_7^{abc}	$\frac{1}{8} f^{acl} f^{bdl} X_0^d$
λ_9^{abc}	$-\frac{1}{4} f^{adl} f^{cbl} X_0^d$
λ_{10}^{abc}	$-X_0^d (d^{acl} d^{bdl} + d^{adl} d^{bcl})$
λ_{11}^{abc}	$\frac{1}{2} f^{abc}$
λ_{12}^{abc}	$d^{acl} d^{bdl} X_0^d$
λ_{14}^{abc}	$\frac{1}{4} X_0^d f^{dal} f^{cbl}$
λ_{15}^{abc}	$X_0^d d^{cbl} d^{adl}$
λ_{16}^{abc}	$-\frac{1}{8g_5^2} (f^{abl} f^{cdl} + f^{cbl} f^{adl})$
λ_{17}^{abc}	$-\frac{1}{8g_5^2} f^{ack} f^{bdk}$

Table 6: Coupling coefficients

$$\begin{aligned}
\lambda_5^{abc} = &-\frac{1}{32} X_0^i X_0^j f^{ack} f^{bik} d^{jkk} i - \frac{1}{16} X_0^i X_0^j f^{ack} f^{bki} d^{jkk} i - \frac{3}{16} X_0^i X_0^j f^{ack} d^{bki} d^{jkk} - \frac{1}{32} X_0^i X_0^j f^{aik} f^{bck} d^{jkk} i \\
&-\frac{1}{16} X_0^i X_0^j f^{aik} f^{ckj} d^{bkk} i + \frac{3}{16} X_0^i X_0^j f^{aik} d^{bck} d^{jkk} + \frac{3}{16} X_0^i X_0^j f^{aik} d^{bkk} d^{ckj} + \frac{1}{16} X_0^i X_0^j f^{aki} f^{bck} d^{jkk} i \\
&-\frac{1}{32} X_0^i X_0^j f^{aki} f^{cjk} d^{bkk} i + \frac{1}{16} X_0^i X_0^j f^{bck} d^{aki} d^{jkk} + \frac{1}{16} X_0^i X_0^j f^{bik} f^{ckj} d^{akk} i - \frac{1}{16} X_0^i X_0^j f^{bik} d^{ack} d^{jkk} \\
&-\frac{1}{16} X_0^i X_0^j f^{bik} d^{akk} d^{ckj} - \frac{1}{32} X_0^i X_0^j f^{bki} f^{cjk} d^{akk} i + \frac{1}{16} X_0^i X_0^j f^{cik} d^{akj} d^{bkk} - \frac{3}{16} X_0^i X_0^j f^{cik} d^{akk} d^{bkj} \\
&-\frac{1}{8} X_0^i X_0^j d^{ack} d^{bik} d^{jkk} i - \frac{1}{4} X_0^i X_0^j d^{ack} d^{bki} d^{jkk} i - \frac{1}{8} X_0^i X_0^j d^{aik} d^{bck} d^{jkk} i - \frac{1}{4} X_0^i X_0^j d^{aik} d^{bkk} d^{ckj} i \\
&+\frac{1}{4} X_0^i X_0^j d^{aki} d^{bck} d^{jkk} i + \frac{1}{8} X_0^i X_0^j d^{aki} d^{bkk} d^{cjk} i + \frac{1}{4} X_0^i X_0^j d^{akk} d^{bik} d^{ckj} i + \frac{1}{8} X_0^i X_0^j d^{akk} d^{bki} d^{cjk} i
\end{aligned}$$

$$
\begin{aligned}
\lambda_{13}^{abc} = {} & \frac{1}{128}X_0^iX_0^jf^{abl}f^{cik}f^{jkl} + \frac{1}{64}X_0^iX_0^jf^{abl}f^{cik}d^{jlk}i + \frac{1}{64}X_0^iX_0^jf^{abl}f^{j,l,k}d^{cik}i + \frac{1}{32}X_0^iX_0^jf^{abl}d^{cik}d^{jlk} \\
& -\frac{1}{128}X_0^iX_0^jf^{acl}f^{bjk}f^{ilk} - \frac{1}{64}X_0^iX_0^jf^{acl}f^{bjk}d^{ilk}i + \frac{1}{128}X_0^iX_0^jf^{acl}f^{blk}f^{ijk} - \frac{1}{64}X_0^iX_0^jf^{acl}f^{blk}d^{ijk}i \\
& +\frac{1}{64}X_0^iX_0^jf^{acl}f^{ijk}d^{blk}i + \frac{3}{64}X_0^iX_0^jf^{acl}f^{ilk}d^{bjk}i - \frac{3}{32}X_0^iX_0^jf^{acl}d^{bjk}d^{ilk} + \frac{1}{32}X_0^iX_0^jf^{acl}d^{blk}d^{ijk} \\
& +\frac{3}{128}X_0^iX_0^jf^{ajl}f^{bck}f^{ikl} + \frac{3}{64}X_0^iX_0^jf^{ajl}f^{bck}d^{ilk}i + \frac{1}{32}X_0^iX_0^jf^{ajl}f^{blk}d^{cik}i + \frac{1}{64}X_0^iX_0^jf^{ajl}f^{bkl}f^{cik} \\
& +\frac{1}{32}X_0^iX_0^jf^{ajl}f^{cik}d^{blk}i - \frac{3}{64}X_0^iX_0^jf^{ajl}f^{ilk}d^{bck}i - \frac{3}{32}X_0^iX_0^jf^{ajl}d^{bck}d^{ilk} + \frac{1}{16}X_0^iX_0^jf^{ajl}d^{blk}d^{cik} \\
& +\frac{1}{64}X_0^iX_0^jf^{alk}f^{bcl}f^{ijk} - \frac{1}{32}X_0^iX_0^jf^{alk}f^{bcl}d^{ijk}i - \frac{1}{32}X_0^iX_0^jf^{alk}f^{bjl}d^{cik}i - \frac{1}{64}X_0^iX_0^jf^{alk}f^{bjk}f^{cil} \\
& -\frac{1}{32}X_0^iX_0^jf^{alk}f^{cil}d^{bjk}i - \frac{1}{32}X_0^iX_0^jf^{alk}f^{ijl}d^{bck}i + \frac{1}{16}X_0^iX_0^jf^{alk}d^{bcl}d^{ijk} - \frac{1}{16}X_0^iX_0^jf^{alk}d^{bjk}d^{cil} \\
& +\frac{1}{32}X_0^iX_0^jf^{bcl}f^{ijk}d^{alk}i - \frac{5}{64}X_0^iX_0^jf^{bcl}f^{ilk}d^{ajk}i + \frac{5}{32}X_0^iX_0^jf^{bcl}d^{ajk}d^{ilk} + \frac{1}{16}X_0^iX_0^jf^{bcl}d^{alk}d^{ijk} \\
& -\frac{1}{32}X_0^iX_0^jf^{bjl}f^{cik}d^{alk}i + \frac{1}{64}X_0^iX_0^jf^{bjl}f^{ilk}d^{ack}i + \frac{1}{32}X_0^iX_0^jf^{bjl}d^{ack}d^{ilk} - \frac{1}{16}X_0^iX_0^jf^{bjl}d^{alk}d^{cik} \\
& +\frac{1}{32}X_0^iX_0^jf^{blk}f^{cil}d^{ajk}i - \frac{1}{64}X_0^iX_0^jf^{blk}f^{ijl}d^{ack}i + \frac{1}{32}X_0^iX_0^jf^{blk}d^{acl}d^{ijk} + \frac{1}{16}X_0^iX_0^jf^{blk}d^{ajk}d^{cil} \\
& -\frac{3}{64}X_0^iX_0^jf^{cil}f^{j,l,k}d^{abk}i + \frac{3}{32}X_0^iX_0^jf^{cil}d^{abk}d^{jlk} - \frac{1}{16}X_0^iX_0^jf^{cil}d^{ajk}d^{blk} + \frac{1}{16}X_0^iX_0^jf^{cil}d^{alk}d^{bjk} \\
& -\frac{1}{32}X_0^iX_0^jf^{ijl}d^{ack}d^{blk} - \frac{1}{16}X_0^iX_0^jf^{ijl}d^{alk}d^{bck} - \frac{3}{32}X_0^iX_0^jf^{ilk}d^{acl}d^{bjk} + \frac{5}{32}X_0^iX_0^jf^{ilk}d^{ajk}d^{bcl} \\
& -\frac{3}{32}X_0^iX_0^jf^{j,l,k}d^{abk}d^{cil} - \frac{3}{16}X_0^iX_0^jd^{abl}d^{cik}d^{jkl}i - \frac{3}{16}X_0^iX_0^jd^{acl}d^{bjk}d^{ilk}i + \frac{1}{16}X_0^iX_0^jd^{acl}d^{blk}d^{ijk}i \\
& +\frac{5}{16}X_0^iX_0^jd^{ajl}d^{bck}d^{ikl}i + \frac{1}{8}X_0^iX_0^jd^{ajl}d^{bkl}d^{cik}i + \frac{1}{8}X_0^iX_0^jd^{alk}d^{bcl}d^{ijk}i - \frac{1}{8}X_0^iX_0^jd^{alk}d^{bjk}d^{cil}i
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{18}^{abcd} = {} & \frac{1}{128} X_0^i X_0^j f^{abk} f^{cdk} f^{kij} d^{kkk} i + \frac{1}{64} X_0^i X_0^j f^{abk} f^{cdk} d^{kij} d^{kkk} + \frac{1}{128} X_0^i X_0^j f^{abk} f^{cki} f^{dkj} d^{kkk} i \\
& + \frac{1}{64} X_0^i X_0^j f^{abk} f^{cki} d^{dkj} d^{kkk} - \frac{1}{64} X_0^i X_0^j f^{abk} f^{dki} d^{ckj} d^{kkk} + \frac{1}{64} X_0^i X_0^j f^{abk} f^{kij} d^{cdk} d^{kkk} \\
& - \frac{1}{32} X_0^i X_0^j f^{abk} d^{cdk} d^{kij} d^{kkk} i + \frac{1}{32} X_0^i X_0^j f^{abk} d^{cki} d^{dkj} d^{kkk} i - \frac{1}{64} X_0^i X_0^j f^{ack} f^{bik} d^{dkj} d^{kkk} \\
& + \frac{1}{128} X_0^i X_0^j f^{ack} f^{bki} f^{dkj} d^{kkk} i - \frac{1}{64} X_0^i X_0^j f^{ack} f^{dki} d^{bkj} d^{kkk} + \frac{1}{32} X_0^i X_0^j f^{ack} d^{bki} d^{dkj} d^{kkk} i \\
& - \frac{3}{128} X_0^i X_0^j f^{adk} f^{bki} f^{ckj} d^{kkk} i + \frac{3}{64} X_0^i X_0^j f^{adk} f^{bki} d^{ckj} d^{kkk} + \frac{1}{32} X_0^i X_0^j f^{adk} f^{cik} d^{bkj} d^{kkk} \\
& - \frac{1}{64} X_0^i X_0^j f^{adk} f^{cki} d^{bkj} d^{kkk} - \frac{3}{32} X_0^i X_0^j f^{adk} d^{bki} d^{ckj} d^{kkk} i - \frac{1}{32} X_0^i X_0^j f^{aik} f^{bdk} d^{ckj} d^{kkk} \\
& - \frac{1}{64} X_0^i X_0^j f^{aik} f^{cdk} d^{bkj} d^{kkk} - \frac{1}{32} X_0^i X_0^j f^{aik} f^{ckj} d^{bdk} d^{kkk} - \frac{1}{16} X_0^i X_0^j f^{aik} d^{bdk} d^{ckj} d^{kkk} i \\
& - \frac{1}{64} X_0^i X_0^j f^{aki} f^{bdk} f^{ckj} d^{kkk} i - \frac{1}{64} X_0^i X_0^j f^{aki} f^{bkj} f^{cdk} d^{kkk} i - \frac{1}{32} X_0^i X_0^j f^{aki} f^{bkj} d^{cdk} d^{kkk} \\
& - \frac{1}{64} X_0^i X_0^j f^{aki} f^{cdk} d^{bkj} d^{kkk} - \frac{1}{32} X_0^i X_0^j f^{bdk} f^{cik} d^{akj} d^{kkk} + \frac{1}{16} X_0^i X_0^j f^{bdk} d^{aki} d^{ckj} d^{kkk} i \\
& - \frac{1}{64} X_0^i X_0^j f^{bik} f^{cdk} d^{akj} d^{kkk} + \frac{1}{64} X_0^i X_0^j f^{bik} f^{dkj} d^{ack} d^{kkk} - \frac{1}{32} X_0^i X_0^j f^{bik} d^{ack} d^{dkj} d^{kkk} i \\
& - \frac{1}{32} X_0^i X_0^j f^{bik} d^{adk} d^{ckj} d^{kkk} i - \frac{1}{64} X_0^i X_0^j f^{bki} f^{cdk} d^{akj} d^{kkk} - \frac{1}{32} X_0^i X_0^j f^{bki} f^{cjk} d^{adk} d^{kkk} \\
& - \frac{1}{64} X_0^i X_0^j f^{bki} f^{ckj} d^{adk} d^{kkk} + \frac{1}{64} X_0^i X_0^j f^{cdk} f^{kij} d^{abk} d^{kkk} - \frac{1}{32} X_0^i X_0^j f^{cdk} d^{abk} d^{kij} d^{kkk} i \\
& + \frac{1}{16} X_0^i X_0^j f^{cdk} d^{aki} d^{bkj} d^{kkk} i + \frac{1}{32} X_0^i X_0^j f^{cik} d^{abk} d^{dkj} d^{kkk} i + \frac{1}{32} X_0^i X_0^j f^{cik} d^{adk} d^{bkj} d^{kkk} i \\
& - \frac{1}{16} X_0^i X_0^j f^{cik} d^{akj} d^{bdk} d^{kkk} i + \frac{1}{64} X_0^i X_0^j f^{cki} f^{dkj} d^{abk} d^{kkk} - \frac{1}{32} X_0^i X_0^j f^{dik} d^{abk} d^{ckj} d^{kkk} i \\
& + \frac{1}{32} X_0^i X_0^j f^{dik} d^{ack} d^{bkj} d^{kkk} i - \frac{1}{32} X_0^i X_0^j f^{kij} d^{abk} d^{cdk} d^{kkk} i - \frac{1}{16} X_0^i X_0^j d^{abk} d^{cdk} d^{kij} d^{kkk} \\
& + \frac{1}{16} X_0^i X_0^j d^{abk} d^{cki} d^{dkj} d^{kkk} - \frac{1}{16} X_0^i X_0^j d^{ack} d^{bki} d^{dkj} d^{kkk} + \frac{1}{16} X_0^i X_0^j d^{adk} d^{bki} d^{ckj} d^{kkk} \\
& - \frac{1}{8} X_0^i X_0^j d^{aki} d^{bdk} d^{ckj} d^{kkk} + \frac{1}{8} X_0^i X_0^j d^{aki} d^{bkj} d^{cdk} d^{kkk}
\end{aligned}
$$

B.3 Short derivation of relations between K and G

The relations between the mixed propagators given in the main text were:

$$
\begin{aligned}
K^a_{\phi\phi}(k,z') &= -\frac{1}{L_0}\partial_z G^a_{\phi\phi}(k,L_0,z')\\
K^a_{\pi\phi}(k,z') &= \frac{k^2}{L_0}\partial_z G^a_{\phi\pi}(k,L_0,z')\\
k^2 K_{\phi\pi}(k,z') &= \frac{\beta^a(L_0)}{L_0}\partial_z G^a_{\pi\phi}(k,L_0,z')\\
K_{\pi\pi}(k,z') &= -\frac{\beta^a(L_0)}{L_0}\partial_z G^a_{\pi\pi}(k,L_0,z')
\end{aligned}
$$

These relations can be derived by performing a double integration by parts to specific linear combinations of eqs. (3.53) to (3.56). For example, the first two follow from:

$$
\begin{aligned}
-k^2 K_{\phi\phi}(k,z') &= -k^2 K_{\phi\phi}(k,z') + 0\times K_{\pi\phi}(k,z)\\
&= -k^2\int \mathrm{d}z K_{\phi\phi}(k,z)\delta(z-z')\\
&+\int \mathrm{d}z K_{\pi\phi}(k,z)\left(\partial_z\left(\frac{\beta^a}{z}\partial_z G_{\pi\phi}(k,z,z')\right)-\frac{k^2\beta^a}{z}(G_{\phi\phi}(k,z,z')-G_{\pi\phi}(k,z,z'))\right)\\
&= -k^2\int \mathrm{d}z K_{\phi\phi}(k,z)\left(\partial_z\left(\frac{1}{z}\partial_z G_{\pi\phi}(k,z,z')\right)-\frac{\beta^a}{z}(G_{\phi\phi}(k,z,z')-G_{\pi\phi}(k,z,z'))\right)\\
&+\int \mathrm{d}z K_{\pi\phi}(k,z)\left(\partial_z\left(\frac{\beta^a}{z}\partial_z G_{\pi\phi}(k,z,z')\right)-\frac{k^2\beta^a}{z}(G_{\phi\phi}(k,z,z')-G_{\pi\phi}(k,z,z'))\right)\\
&= -k^2\left[\frac{1}{z}K_{\phi\phi}\partial_z G_{\phi\phi}\right]_{L_0}^{L_1} + k^2\underbrace{\left[\frac{1}{z}\partial_z K_{\phi\phi}G_{\phi\phi}\right]_{L_0}^{L_1}}_{=0} + \underbrace{\left[\frac{\beta^a}{z}K_{\pi\phi}\partial_z G_{\pi\phi}\right]_{L_0}^{L_1}}_{=0} - \underbrace{\left[\frac{\beta^a}{z}\partial_z K_{\pi\phi}G_{\pi\phi}\right]_{L_0}^{L_1}}_{=0}\\
&- k^2\int \mathrm{d}z G_{\phi\phi}\underbrace{\left(\partial_z\left(\frac{1}{z}\partial_z K_{\phi\phi}(k,z)\right)-\frac{\beta^a}{z}(K_{\phi\phi}(k,z)-K_{\pi\phi}(k,z))\right)}_{=0}\\
&+\int \mathrm{d}z G_{\pi\phi}\underbrace{\left(\partial_z\left(\frac{\beta^a}{z}\partial_z K_{\pi\phi}(k,z)\right)-\frac{k^2\beta^a}{z}(K_{\phi\phi}(k,z)-K_{\pi\phi}(k,z))\right)}_{=0}\\
&= \frac{k^2}{L_0}\partial_z G_{\phi\phi}(k,L_0,z')
\end{aligned}
$$

$$
\begin{aligned}
K_{\pi\phi} &= \int \mathrm{d}z K_{\pi\phi}(k,z)\delta(z-z') - \int \mathrm{d}z 0 \times k^2 K_{\phi\phi}(k,z) \\
&= \int \mathrm{d}z K_{\pi\phi}(k,z) \left(\partial_z \left(\frac{\beta^a}{z}\partial_z G_{\pi\pi}(k,z,z')\right) - \frac{k^2\beta^a}{z}(G_{\phi\pi}(k,z,z') - G_{\pi\pi}(k,z,z'))\right) \\
&- k^2 \int \mathrm{d}z K_{\phi\phi}(k,z) \left(\partial_z \left(\frac{1}{z}\partial_z G_{\phi\pi}(k,z,z')\right) - \frac{\beta^a}{z}(G_{\phi\pi}(k,z,z') - G_{\pi\pi}(k,z,z'))\right) \\
&= -k^2 \left[\frac{1}{z}K_{\phi\phi}\partial_z G_{\phi\pi}\right]_{L_0}^{L_1} + k^2 \underbrace{\left[\frac{1}{z}\partial_z K_{\phi\phi} G_{\phi\pi}\right]_{L_0}^{L_1}}_{=0} + \underbrace{\left[\frac{\beta^a}{z}K_{\pi\phi}\partial_z G_{\pi\pi}\right]_{L_0}^{L_1}}_{=0} - \underbrace{\left[\frac{\beta^a}{z}\partial_z K_{\pi\phi} G_{\pi\pi}\right]_{L_0}^{L_1}}_{=0} \\
&- k^2 \int \mathrm{d}z G_{\phi\pi} \underbrace{\left(\partial_z \left(\frac{1}{z}\partial_z K_{\phi\phi}(k,z)\right) - \frac{\beta^a}{z}(K_{\phi\phi}(k,z) - K_{\pi\phi}(k,z))\right)}_{=0} \\
&+ \int \mathrm{d}z G_{\pi\pi} \underbrace{\left(\partial_z \left(\frac{\beta^a}{z}\partial_z K_{\pi\phi}(k,z)\right) - \frac{k^2\beta^a}{z}(K_{\phi\phi}(k,z) - K_{\pi\phi}(k,z))\right)}_{=0} \\
&= \frac{k^2}{L_0}\partial_z G_{\phi\pi}(k,L_0,z')
\end{aligned}
$$

The two remaining relations are obtained in a similar fashion.

C $SU(3)$

The defining commutation relations of infinitesimal generators g_i are given by:

$$[g_i, g_j] = if^{ijk}g_k \tag{C.24}$$

In the fundamental representation of the group an interesting orthogonal basis for the Lie algebra $\mathfrak{su}(3)$ is given by the so-called Gell-Mann matrices, they are:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

These matrices are Hermitian, traceless and such that $\mathrm{tr}(\lambda_i \lambda_j) = 2\delta_{ij}$.

The structure constants f^{ijk} can be obtained by orthogonally projecting $[\lambda_i, \lambda_j]$ onto λ_k:

$$f^{ijk} = \frac{\mathrm{tr}([\lambda_i, \lambda_j]\lambda_k)}{i\mathrm{tr}(\lambda_k \lambda_k)} = \frac{\mathrm{tr}([\lambda_i, \lambda_j]\lambda_k)}{2i}$$

From this formula it can be seen that the structure constants are completely antisymmetric in the three indices and:

$$f^{123} = 2; f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = 1; f^{458} = f^{678} = \sqrt{3} \tag{C.25}$$

The subset of Hermitian matrices forms a subspace of the real vector space of complex valued matrices $\mathcal{M}_n(\mathbb{C})$ and an orthogonal basis can be constructed from the Gell-Mann matrices by appending the matrice $\lambda_0 = \sqrt{\frac{2}{3}}I$ to the set $\{\lambda_i\}_{i\in\{1,\dots,8\}}$. The structure constants can be extended to include the value 0 for its indices and remain totally antisymmetric.

In this thesis, Hermitian matrices are decomposed onto an orthogonal basis t_i defined by $t_i = \frac{\lambda_i}{2}$ such that $\mathrm{tr}(t_i t_j) = \frac{1}{2}\delta_{ij}$. In this basis a general Hermitian matrix $A = \begin{pmatrix} a & d+ie & f+ig \\ d-ie & b & h+ij \\ f-ig & h-ij & c \end{pmatrix}$ can be decomposed in the following fashion:

$$A = \sqrt{\frac{2}{3}}(a+b+c)t_0 + 2dt_1 - 2et_2 + (a-b)t_3 + 2ft_4 - 2gt_5 + 2ht_6 - 2jt_7 + \frac{1}{\sqrt{3}}(a+b-2c)t_8 \tag{C.26}$$

Furthermore in this extended basis (t_a), one can define the coefficients d^{ijk} by:

$$d^{ijk} = 2\mathrm{Tr}(\{t^i, t^j\}t^k) \tag{C.27}$$

In this way:

$$\{t^i, t^j\} = d^{ijk} t^k \tag{C.28}$$

It should be noted that equation (C.28) holds only in the extended basis $\{t^a : a \in \{0, \ldots, 8\}\}$, this is not the case is most other sources on the topic where the t^0 component is separated from the others so that:

$$\{t^a, t^b\} = \frac{\delta^{ab}}{3} I + \sum_{a=1}^{8} \sum_{b=1}^{8} d^{abk} t_k$$

In this thesis however, unless stated explicitly otherwise, it will always be understood that t_0 is included in the sum, so that equation (C.28) is valid.

References

[1] S. Möller, "Exploring the AdS/QCD Correspondence," Bachelor's thesis, Lund University, 2010.

[2] D. S. Seth, "Vector Meson Masses in AdS/QCD," Bachelor's thesis, Lund University, 2013.

[3] J. Erlich, E. Katz, D. T. Son, and M. A. Stephanov, "QCD and a Holographic Model of Hadrons," *Phys. Rev. Lett.*, vol. 95, p. 261602, 2005, hep-ph/0501128.

[4] L. D. Rold and A. Pomarol, "The scalar and pseudoscalar sector in a five-dimensional approach to chiral symmetry breaking," *JHEP*, vol. 01, p. 157, 2006, hep-ph/0510268v1.

[5] L. D. Rold and A. Pomarol, "Chiral symmetry breaking from five dimensional spaces," *PoS*, vol. HEP2005, p. 355, 2006, hep-ph/0501218v3.

[6] E. Gustafsson, "Scalar Theory as an Example of AdS/QCD," bachelor's thesis, Lund University, 2014.

[7] J. Bijnens and J. Prades, "Electromagnetic Corrections for Pions and Kaons: Masses and Polarizabilities," *Nucl. Phys.*, vol. B490, pp. 239–271, 1997, hep-ph/9610360.

[8] J. Maldacena, "The Large N limit of Superconformal field theories and supergravity," *International Journal of Theoretical Physics*, vol. 38, pp. 1113–1133, 1999, hep-th/9711200v3.

[9] G. 't Hooft, "The Holographic Principle: Opening lecture," *Subnucl. Ser*, vol. 37, pp. 72–100, 2001, hep-th/0003004v2.

[10] Y. Hyakutake, "Quantum near horizon geometry of black 0-brane," *PTEP*, p. 033B04, 2014, hep-th/1311.7526v3.

[11] M. Hanada, Y. Hyakutake, G. Ishiki, and J. Nishimura, "Holographic description of quantum black hole on a computer," *Science*, vol. 344, pp. 882–885, 2013, hep-th/1311.5607v1.

[12] S. Gudmunsson, "An Introduction to Riemannian Geometry."

[13] V. D. Carlo, "Conformal compactification and Anti-de-Sitter space," Master's thesis, Stockholms Universitet, 2007.

[14] R. Blumenhagen and E. Plauschinn, *Introduction to Conformal Field Theory With Applications to String Theory*, vol. 779 of *Lecture Notes in Physics*. Springer-Verlag Berlin Heidelberg, 2009.

[15] M. Schottenloher, *A Mathematical Introduction to Conformal Field Theory*. No. 759 in Lecture Notes in Physics, Springer, Berlin Heidelberg, Second Edition ed., 2008.

[16] H. Nastase, *Introduction to the AdS/CFT Correspondence*. Cambridge University Press, 2015.

[17] M. Ammon and J. Erdmenger, *Gauge/Gravity Duality: Foundations and Applications*. Cambridge University Press, 2015.

[18] J. McGreevy, "8.821 f2008 lecture 13: Masses of fields and dimenions of operators." `http://ocw.mit.edu/courses/physics/8-821-string-theory-fall-2008/lecture-notes/lecture13.pdf`, 2008.

[19] G. B. Folland, *Quantum Field Theory: A Tourist Guide For Mathematicians*, vol. 149 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2008.

[20] M. D. Schwartz, *Quantum Field Theory and the Standard Model*. Cambridge University Press, first edition ed., 2014.

[21] J. Polchinski and M. J. Strassler, "Hard scattering and gauge/string duality," *Phys. Rev. Lett.*, vol. 88, p. 031601, 2002, hep-th/0109174v1.

[22] A. Cherman, T. D. Cohen, and E. S. Werbos, "The Chiral condensate in holographic models of QCD," *Phys. Rev.*, vol. C79, p. 045203, 2009, hep-ph/0804.1096v3.

[23] J. M. Lee, *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics, Springer, second edition ed., 2013.

[24] K. Olive *et al.*, "Particle Data Group," *Chin. Phys. C*, vol. 38, no. 090001, 2015.

[25] T. Williams, C. Kelley, and many others, "Gnuplot 5: an interactive plotting program." `http://gnuplot.sourceforge.net/`, February 2016.

[26] J. Vermaseren, "New features of FORM," 2000, math-ph/0010025.