Modeling Life Insurance Guarantees

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Abstract

Insurance contracts with guarantees have been issued by insurance companies for a long time. For example, in the 1970s and the 1980s, when the interest rate in the UK was as high as 15-20%, these contracts were issued with guaranteed rates of as high as 10%, thinking the interest rate levels would stay around the same. However, the interest rate decreased and, as of today, interest rate levels are around and even below zero, which has led to insolvency problems for many insurance companies. Currently, with the low interest rate environment, guaranteed rates are mostly offered at a zero guaranteed rate, but with a rising interest level in sight, these rates are likely to come back into the market because of competitiveness reasons. To avoid such insolvency issues again, every risk in the contracts has to be identified, and we have focused on the risk in the interest rate process.

This thesis highlights the risk involving the parameter estimation of the interest rate process. It also demonstrates the large price fluctuations that occur from different types of yield curves and how it differs for different maturities. In these contracts, put options are embedded and a Fourier-Gauss-Laguerre model is used to price the options and we include both stochastic volatility and stochastic interest rate.

Previous studies have shown that the parameter risk involving the mortality estimation cannot be ignored and the results in this thesis concludes that the uncertainty involved in the parameter estimation of the interest rate process canont be ignored either.

Acknowledgements

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Chapter 1

Introduction

1.1 Background

Insurance companies have for a long time issued different life insurance policies including a minimum guaranteed benefit payable to the policyholder. The policyholder pays a single or periodic premium(s) which is invested in a fund, and will receive a benefit upon death or maturity of contract. The benefit is based on the performance of the fund meaning that the policyholder will receive a higher benefit with a higher fund value. On the other hand, there is a guaranteed benefit the policyholder will receive no matter the outcome of the fund. So if the fund value at maturity(or death) is less than the guaranteed benefit, the insurance company would still have to pay the guaranteed amount to the policyholder, and this is a risk that needs to be priced into the contract. The insurance company takes on two main risks, interest rate risk and mortality risk.

There is a wide range of different guaranteed benefits available for customers to add on to their insurance contract. They exist as a hedge towards the downside market risk providing a safety for the customer knowing they will at least receive a guaranteed amount no matter how the market behaves. For insurance companies they exist as a tool for competitiveness. Below we list a couple of the most common guarantees offered by insurance companies.

Guaranteed Minimum Maturity Benefit(GMMB) is a benefit included in a life insurance guaranteeing a minimum benefit for the policyholder upon maturity of the contract. At maturity the policyholder will receive at least the benefit given by this guaranteed interest rate, or a higher benefit if the fund of the invested premiums performs better.

Guaranteed Minimum Death Benefit(GMDB) works in the same way as the GMMB except it is payable upon death of the policyholder instead of upon maturity.

Guaranteed Minimum Withdrawal Benefit(GMWB) gives the policyholder the right to withdraw a certain percentage of the investment value(from premiums or fund value, whichever is larger) every year until the initial payment(or fund value) is fully recovered.

Guaranteed Minimum Income Benefit(GMIB) gives the policyholder a guaranteed benefit payable as periodic benefits until death after the maturity of the contract. The benefits will be a percentage drawn from the greater value between the fund and a guaranteed value.

In the 1970's and 1980's in the UK, when the interest rates were between 15 and 20%, see [Bank of England, 2016], insurance companies issued long term insurance contracts with guaranteed interest rates of about as high as 10%. In the 90's when the interest rate had a huge downfall many insurance companies experienced their guaranteed rate all of a sudden was very close to being "in the money", and eventually falling below that limit. Insurers who had previously thought that their guaranteed rate was on the "safe side" suddenly faced insolvency problems as their high guaranteed rates still had to be paid out. To avoid such insolvency issues again, every risk in the contracts has to be identified, and we have focused on the risk in the interest rate process.

1.2 Problem Formulation

The aim of this thesis is to develop a pricing model for the two insurance products, the GMDB and the GMMB, using the Fourier-Gauss-Laguerre(FGL) method, and highlight the risk associated with the interest rate. The FGL method is a method for pricing European options and it is suggested in [Lindström et al., 2015a], that it is the best method for option pricing, weighing in both accuracy and speed. We will apply this to the embedded options in these contracts and study how different factors affect the market value(and to what extent), such as mortality and interest rate, and look at different scenarios generated by different types of yield curves. To the best of our knowledge this is the first time this study has been done in the context of life insurance.

A stochastic interest model will be used and we will go further and analyze the risk in the guarantees by including the risk associated with the parameter estimation, assuming parameter uncertainty. Parameter uncertainty for insurance contract is an unexplored area with barely any research available at all (As of May 24, 2016, there is an available research opportunity about the subject for the SOA, see [Society of Actuaries, 2016]). The subject was touched upon and discussed in [Cairns, 2000] and a few years later it was shown in [Cairns, 2006] that the parameter risk in mortality prediction for insurance policies cannot be overlooked. Instead of looking at the parameter risk for the mortality we will in this thesis look at the parameter risk involving the stochastic interest rate model. There is a lack of previous studies about this and this thesis explores the topic by investigating its effect on the products GMDB and GMMB. It should have a practical application for insurance companies that have been and will be issuing contracts with long-term maturities and guarantees. Since the subject is very little known about, it also opens doors for further research, which also will be discussed.

1.3 Outline

This thesis has the following outline:

Chapter 2 gives a theoretical background needed for the rest of the thesis, which includes some probability theory. The yield curve is explained and how it can be interpolated. The concept of mortality is introduced as well as two common risk measures.
Chapter 3 is an introduction to life insurance and annuities, and explains how options are embedded in some contracts. We then derive the expression for the specific contract including the GMDB and the GMMB that we are working with in this thesis.
Chapter 4 derives the Fourier-Gauss-Laguerre expression for pricing the options that arise from the insurance contract in Chapter 3. We introduce the Heston Hull-White model and give an expression for the characteristic function belonging to that model.
Chapter 5 consists of two parts. The first part is a simulation study of different scenarios for the evolution of the interest rate and the mortality. The second part deals with the parameter risk involving the interest rate model, which in our case is the Hull-White model.
Chapter 6 contains conclusions of the simulations and the results in Chapter 5.
Chapter 7 discusses and presents improvements of the model, and suggests further research opportunities.

Chapter 2

Theoretical Background

2.1 Probability, Fourier transform and Stochastic Processes

In this section we simply list some definitions needed for the rest of this thesis, where the material is mainly taken from [Lindström et al., 2015b]. We start by defining the keystone of financial modeling, the Brownian motion. It is a stochastic process that is defined to randomize the process of a stock price.

Definition 2.1.1 (**Brownian Motion**). *The stochastic process W is a brownian motion if*

i) $W_0 = 0,\ a.s.$

ii) The increments $W_{t_4} - W_{t_3}$ and $W_{t_2} - W_{t_1}$ are independent stochastic variables for $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4$.

iii) The increments of W are normally distributed.
i.e $(W_{t_2} - W_{t_1}) \sim \mathcal{N}(0, t_2 - t_1)$ for $0 \leq t_1 \leq t_2$.

iv) W is continuous.

We continue by defining the Characteristic function, which we will need to price options using the Fourier method, which is described later on.

Definition 2.1.2 (**Characteristic function**). *If X is a random variable, then the characteristic function of X is a function $\varphi_X : \mathbb{R} \to \mathbb{C}$ such that*

$$\varphi(u) = \mathbb{E}(e^{iuX}) = \int_{\mathbb{R}} e^{iux} f_X(x) dx, \quad u \in \mathbb{R} \tag{2.1}$$

Definition 2.1.3 (**Filtration**). *Given a stochastic process X on the time interval $[0, T]$. The filtration of X at time t is*

$$\mathcal{F}_t^X = \sigma\{X_s : s \leq t\} \tag{2.2}$$

where σ is the sigma algebra generated by the random process $\{X_s\}_{s<t}$

We will not go any deeper about explaining the concept of sigma algebras. In this thesis it is sufficient to think of $\mathcal{F}_t^X$ as the history of X, or that $\mathcal{F}_t^X$ consist of all historical events generated by X until time t.

Definition 2.1.4 (The Fourier Transform). *If f is an integrable function, then, for any real number ω, $\hat{f}$ is its Fourier transform defined as*

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\ e^{-2\pi i x\omega}\, dx, \tag{2.3}$$

Going back to function f from $\hat{f}$, the inverse Fourier transform is defined as

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\omega)\ e^{2\pi i\omega x}\, d\omega, \tag{2.4}$$

2.2 Valuation Theory and Affine Term Structure

In this section we introduce the risk neutral measure $\mathbb{Q}$ as well as the short rate model used to describe the interest rate, along with some important theorems. For more in-depth information see [Björk, 2004].

Definition 2.2.1 (Zero Coupon Bond). *A zero coupon bond is a contract that pays out 1 at maturity date T to the contract holder. The price of the zero coupon bond at time t with maturity T is denoted $d(t, T)$.*

The risk neutral measure $\mathbb{Q}$ is a probability measure such that the current value of an asset equals the future expected payoff discounted back by the risk free rate. From that we move forward and introduce the important risk neutral valuation formula.

Theorem 2.2.1 (Risk Neutral Valuation Formula). *Given a contingent claim with payoff function $\phi(\cdot)$ at maturity T, the arbitrage free price at time t is given by*

$$\pi_t = d(t, T)\mathbb{E}^{\mathbb{Q}}[\phi_T|\mathcal{F}_t] \tag{2.5}$$

where $d(t, T)$ is the price of a zero-coupon bond at time t.

Definition 2.2.2 (The Short Rate Model). *A short rate model is a model describing the future evolution of the interest rate $r(t)$. The short rate is defined as the instantaneous interest rate at a infinitesimally short time period. We assume the short rate dynamic under measure $\mathbb{Q}$ is given by*

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t) \tag{2.6}$$

where $\mu(t, r(t))$ and $\sigma(t, r(t))$ are deterministic functions.

The short rate can be related to a zero coupon bond by writing the zero coupon bond on an affine term structure. The affine term structure is defined as

Definition 2.2.3 (Affine Term Structure). *Let $r(t)$ be the instantaneous short rate and $d(t,T)$ be the zero-coupon bond maturing at time T. If $d(t,T)$ can be written as*

$$d(t,T) = F(t, r(t); T) \tag{2.7}$$

where F is

$$F(t, r; T) = e^{A(t,T) - B(t,T)r(t)} \tag{2.8}$$

*and $A(t,T)$ and $B(t,T)$ are deterministic functions, then the short rate model is said to have an **affine term structure**.*

To obtain the deterministic functions $A(t,T)$ and $B(t,T)$ in (2.8) we will use the following theorem.

Theorem 2.2.2 (Affine Term Structure). *Consider 2.6 and assume the drift and the volatility are on the form*

$$\begin{cases} \mu(t, r(t)) = \alpha(t)r + \beta(t) \\ \sigma(t, r(t)) = \sqrt{\gamma(t)r + \delta(t)} \end{cases} \tag{2.9}$$

Then the short rate model has an affine term structure of form (2.8) where $A(t,T)$ can be obtained by solving the system

$$\begin{cases} B_t(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) = -1 \\ B(T,T) = 0 \end{cases} \tag{2.10}$$

$$\begin{cases} A_t(t,T) = \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T) \\ A(T,T) = 0 \end{cases} \tag{2.11}$$

See [Björk, 2004] for proof.
$A(t,T)$ and $B(t,T)$ can be solved either numerically or analytically depending on how complex the interest model in (2.6) is.

2.3 Risk Measures

The regulation for capital requirement for financial institutions have increased significantly over the years in order to enhance the financial stability for both customers and the institution itself. For insurance companies, the Solvency II is a regulation that came into effect on January 1, 2016. It measures the stability of insurance companies and it's basically their version of the Basel regulation used for banks. The Solvency II is built based on the three pillars

1. Pillar I - Quantitative Requirements which says that an insurance company must be able to meet their obligations to their policyholder within a one year horizon with 99.5% certainty.
2. Pillar II - Sets requirement for governance of the insurer.
3. Pilar II - Covers reporting and disclosure.

We will introduce two of the most common risk metrics used by insurance companies and banks worldwide, which are Value at Risk(VaR) and Expected Shortfall (ES).

Value at Risk

The Value at Risk is a straight forward technique that measures the risk involved with an investment. It gives an idea on how much the loss would be in a worst case scenario for a given time period.

Definition 2.3.1 (Value at Risk). *For a given confidence interval $\alpha \in (0, 1)$, the value at risk is given by the smallest number l such that the probability that a loss L exceeds l is less or equal to $1 - \alpha$.*

$$\mathrm{VaR}_\alpha(L) = \inf\{l \in \mathbb{R} : P(L > l) \leq 1 - \alpha\}.$$

Expected Shortfall

Expected shortfall (ES) is another risk measure which describes the expected value of the loss, given that the loss is greater than the value at risk. The advantage of the ES compared to the VaR is that it takes into account the tail properties of the distribution.

Definition 2.3.2 (Expected Shortfall). *For a given confidence interval $\alpha \in (0, 1)$ the expected shortfall is defined as*

$$ES_\alpha = \frac{1}{\alpha} \int_0^\alpha VaR_\gamma(X) d\gamma$$

Value at Risk and Expected Shortfall will later be used when we investigate the worst case scenarios just by considering the parameter uncertainty.

2.4 Yield Curve

The yield curve is a curve showing different yields (returns) for different maturities of a security, i.e it shows the return an investor would receive annually if he/she chooses to invest in the security. It also gives information about the general market expectation of the evolution of interest rates. For example if the market expects the interest rates to rise then longer maturities will yield a higher interest rate (generating a normal yield curve) while if the market expects the interest rates to decrease then longer maturities will yield a lower interest rate (generating an inverted yield curve).

One of the most important securities traded most frequently in the world is the US treasury rate controlled by the Federal Reserve (FED). The yield curve of government bonds have usually had a normal yield curve, but in some historical periods it has also been inverted, where the yield curve takes a downward sloping feature. In figure 2.1 we see an example of a normal and an inverted yield curve. There is also a so called "flat" yield curve where the yield curve flattens out and is somewhat in between the two curves in Figure 2.1.

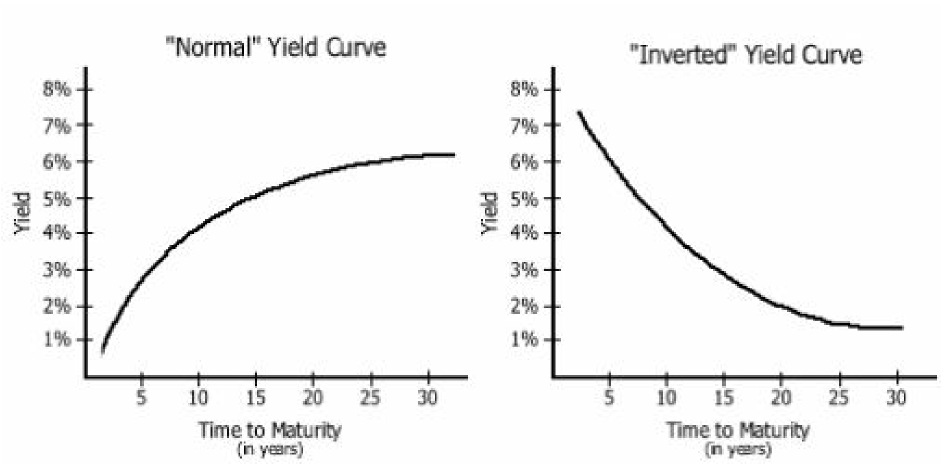


Figure 2.1: Two different kinds of yield curve

Historical events (seven times since 1970)[1] have shown that an inverted yield curve predicts an economic recession and the relation has also been shown theoretically by [Harvey, 1986].
Only some yields are given for certain maturities, typically 1, 3, 5, 10, 30 years, and to get the whole yield curve one needs to interpolate it to get the whole yield curve like the ones in 2.1.

2.4.1 Nelson-Siegel

A common way of fitting the yield curve is to use the Nelson-Siegel function and it is used worldwide by central banks, see [Bank for International Settlements, 2005]. The function is of the form

$$f(0,t) = \beta_0 + \beta_1 \frac{1-\exp(-\lambda t)}{\lambda t} + \beta_2 \left(\frac{1-\exp(-\lambda t)}{\lambda t} - \exp(-\lambda t) \right) \qquad (2.12)$$

where $f(0,t)$ is the current yield for maturity t. The different parameters can be interpreted as

- β_0 is the long term mean level of interest rate.
- β_1 dictates the shape of the curve. For example, a largely positive β_1 indicates an inverted yield curve.
- β_2 can produce so called "humps" in the yield curve and affects β_0 in the medium term.
- λ is the speed factor determining the speed of the mean reversion.

We will later on study how much the parameters in the Nelson-Siegel function affects the price of an insurance contract with guarantees.

[1] [Harvey, 1970]

2.5 Mortality

In this section we give an introduction to mortality and some actuarial notations for probabilities of survival and death for a life, which of course is a crucial part when it comes to life insurances.
Let (x) denote a policyholder aged x where $x \geq 0$. At any time $x+t$ there is a risk that (x) will die, and we denote the future lifetime for (x) as K_x, where K_x is a random variable. The probability of (x) surviving t years is denoted as

$$\Pr(K_x > t) = {}_tp_x \tag{2.13}$$

Then the probability of (x) **not** surviving t years, i.e dying within t years is

$$\Pr(K_x \leq t) = 1 - {}_tp_x = {}_tq_x \tag{2.14}$$

The notation q_x is known as the mortality rate and when insurance companies deal with life insurance contracts they have a mortality table consisting of mortality rates for different ages. In the case of $t = 1$, the probabilities are simply written as p_x and q_x.

We will now derive an important relationship. We consider K_x and K_{x+t} for a policyholder aged $x+t$. That means K_x represents the future lifetime for $x+t$ when it was aged x. Since the policyholder is aged $x+t$ we know that it survived t years from when he was x years old, i.e $K_x > t$. If the policyholder dies within u years from $x+t$ then we have $K_x < t+u$ and $K_{x+t} < u$. Therefore, given survival to age $x+t$, we require the events $K_x < t+u$ and $K_{x+t} < u$ to be equivalent, and make the following assumption:

$$\Pr(K_{x+t} \leq u) = \Pr(K_x \leq t+u | K_x > t) \tag{2.15}$$

Now given Bayes theorem for events A and B we know that

$$\Pr(A|B) = \frac{\Pr(A \text{ and } B)}{\Pr(B)} \tag{2.16}$$

we can rewrite the expression in (2.15) and get

$$\underbrace{\Pr(K_{x+t} \leq u)}_{{}_uq_{x+t}} = \Pr(K_x \leq t+u | K_x > t) = \frac{\Pr(t < K_x \leq t+u)}{\Pr(K_x < t)} \tag{2.17}$$

$$= \frac{{}_{t+u}q_x - {}_tq_x}{{}_tp_x} \tag{2.18}$$

$$= \frac{{}_tp_x - {}_{t+u}p_x}{{}_tp_x} \tag{2.19}$$

Using (2.14) we get

$${}_up_{x+t} = 1 - {}_uq_{x+t} \tag{2.20}$$

$$= \frac{{}_{t+u}p_x}{{}_tp_x} \tag{2.21}$$

which can be rearranged as

$${}_{t+u}p_x = {}_tp_x \, {}_up_{x+t} \tag{2.22}$$

In this thesis we will work in discrete time assuming that benefits will be paid out to the policyholder at the end of the year if it dies during that year. Therefore we want an expression for the probability that the policyholder dies within a specific year. The probability of (x) surviving t years and then dies within 1 year is written as

$$\Pr(t \leq K_x < t+1) = {}_{t+1}q_x - {}_tq_x \tag{2.23}$$
$$= {}_tp_x - {}_{t+1}p_x \tag{2.24}$$

Using our result in (2.22) we can rewrite it as

$${}_tp_x - {}_{t+1}p_x = {}_tp_x - {}_tp_x\, p_{x+t} \tag{2.25}$$
$$= {}_tp_x(1 - p_{x+t}) \tag{2.26}$$
$$= {}_tp_x\, q_{x+t} = {}_{t|}q_x \tag{2.27}$$

The notation ${}_{t|}q_x$ in (2.27) is called the one-year deferred mortality rate.

Over the years, life expectancy has increased significantly and in the following figure we see the mortality rate (probability of dying within one year) for different years.

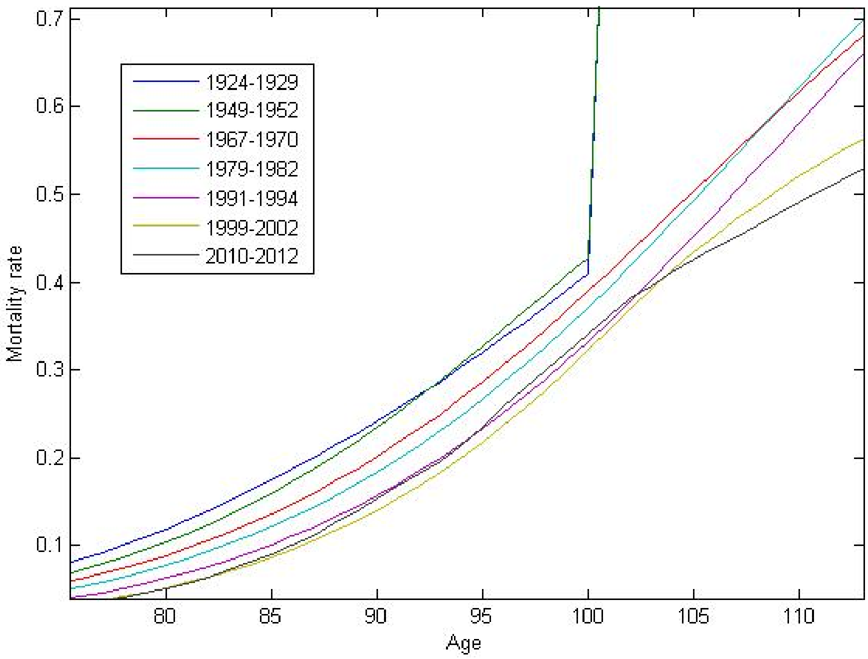


Figure 2.2: Plot of mortality rates throughout different years

The mortality rates in Figure 2.2 have been used for the same type of group which is "males permanent assurance" all collected from [Institute and Faculty of Actuaries, 2016], except for the mortality table for 2010-2012 which is a different group from [Office for National Statistics, 2015] including both men and women. The life table from 2010-2012 will be the base of the mortality used later when calculating the guarantees.

In Figure 2.2 we see a significant shift to the right of the mortality meaning the probability of dying from year to year has decreased over the years. The mortality table created in 1924-1929 as well as the table from 1949-1952 only had mortality rates up to 100 years compared to today's mortality tables, which often includes rates up to 120 years. Looking at the two latest mortalities i.e from 1999-2002 and 2010-2012 we see a decay of the increase of the mortality, which is the effect of taking the future increase of life expectancy into account. We will later investigate the impact of the price of different products when using different mortality tables and see how it has changed over the years.

Chapter 3

Life Insurance and Annuities

A life insurance contract is an agreement between the insurer and the policyholder, where the insurer agrees to pay a single or periodic benefit(s) upon maturity, death or sickness of the policyholder. In return they will receive a single or periodic payments from the policyholder. In this chapter we will describe a life insurance and an annuity contract along with an introduction to the equivalence principle. Finally we will derive the expression for the two products we will work with in this thesis, the GMDB and the GMMB.

3.1 Insurance Benefits

A life insurance contract pays out a benefit to the insured upon death. The benefit will be paid out at the end of the year of death. So if (x) dies between year $x+t$ and $x+t+1$, the benefit will be paid out at time $x+t+1$.
The insurance benefit can be modeled as a random variable Z. Given a life (x), we let the future lifetime of (x) be a variable denoted K_x as in Section 2.5. The present value of the random variable is then

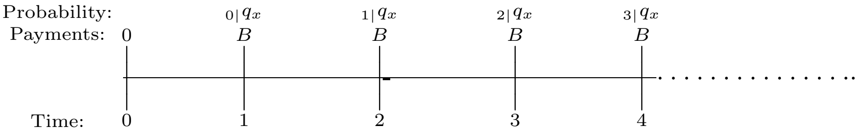
$$Z = v(K_x) \tag{3.1}$$

where $v(t)$ is the discount factor at time t. We denote the discount factor as $v(t)$ to emphasize that we discount assuming a constant interest rate. However, later when we let the interest rate be stochastic, the discount factor is represented by a zero-coupon bond and then instead denoted by $d(0,t)$ as before. We are interested in the expected value of Z, i.e the expected present value of the benefit the insurance company has to pay out, or the expected present value of future premiums.

There are different kinds of insurance contracts and we will give an introduction to the three build blocks of life insurance; whole life insurance, term insurance and pure endowment insurance. Note that we only consider life insurances where the benefit is payable at the end of year of death.

3.1.1 Whole life insurance

A whole-life insurance pays out a benefit to the insured in case of death, and is valid from the issuing date throughout the whole lifetime of (x) according to



The present value Z is the sum of all *fixed* future discounted payments B

$$Z = B \sum_{k=0}^{\infty} v(k+1) \tag{3.2}$$

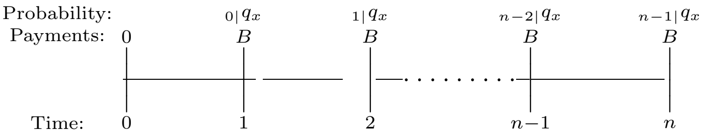
Taking the expected value of (3.2) we get

$$\mathbb{E}[v(K_x)] = B \sum_{k=0}^{\infty} \mathbb{E}[v(k+1)]\, _{k|}q_x \tag{3.3}$$

where $_{k|}q_x$ is the probability of dying between year k and $k+1$ described in (2.27).

3.1.2 Term insurance

A term insurance pays out a benefit if the insured dies at any time from the issuing date to the end of the contract. If the insured dies after the contract, no benefit will be paid, according to



It is the same as a whole life insurance with the difference that it is only valid for a certain amount of years. Given an n-year term insurance, and using the same procedure as with the whole life insurance, the expected value of future benefits is

$$\mathbb{E}[v(K_x)] = B \sum_{k=0}^{n-1} \mathbb{E}[v(k+1)]\, _{k|}q_x \tag{3.4}$$

3.1.3 Pure endowment

A pure endowment contract pays a benefit if the insured survives to time n, but does not pay anything if the insured dies before n. Given an n-year pure endowment contract, the expected value of the present value is

$$\mathbb{E}[v(K_x)] = \mathbb{E}[v(n)]\, _{n}p_x \tag{3.5}$$

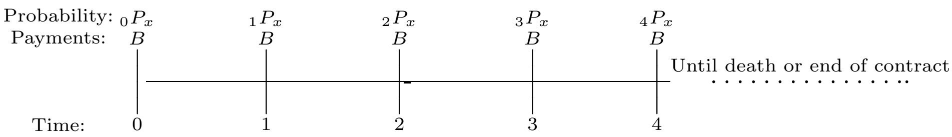
It is common to combine an n-year pure endowment contract with an n-year term insurance. This is called an endowment contract and either pays out a benefit if the insured dies within n years, or a benefit if it survives the n years.

3.2 Life Annuities

Life annuities are regular payments made to the policyholder(annuitant) as long as it is alive on the day of payment. As opposed to insurance benefits, annuities make payments upon survival instead of upon death. It could work as a safety to the annuitant in case it would outlive its income, so it could be thought of as a pension. There are various forms of life annuities such as whole life annuities, term annuities and deferred annuities, and we will give a short brief about them here. We note that payments are made on a yearly basis, but it can also be modeled continuously and for other discrete time intervals.

3.2.1 Whole life annuity

A whole life annuity is an annuity where regular payments will be made to the insured on a yearly basis from the date of issue until death. It is illustrated in the following figure with a fixed benefit B.



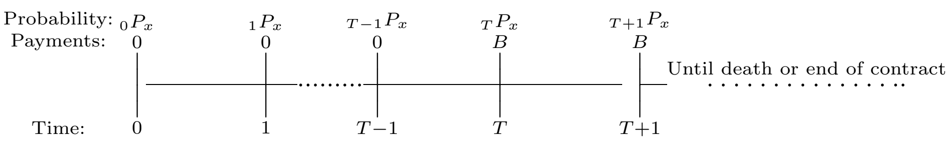
The present value Z is again the sum of future discounted payments as in (3.2). Since the benefit is payable upon survival instead of death, the expected present value of future benefits becomes

$$\mathbb{E}[v(K_x)] = B\sum_{k=0}^{\infty}\mathbb{E}[v(k)]\,{}_kp_x \tag{3.6}$$

where ${}_kp_x$ is the probability of a life (x) surviving k years described in (2.13).

3.2.2 Deferred life annuity

A deferred annuity can either be a deferred term annuity or a deferred whole life annuity, where the term annuity only runs for a limited amount of years. We will later work with a deferred whole life annuity which is a whole life annuity that does not start until T years according to



If the annuitant dies before the deferred period, no benefits will be paid out. The present value of the future benefits is given by

$$Z = B\sum_{k=T}^{\infty} v(k) \tag{3.7}$$

Since it is an annuity, payments are made upon survival and we get the expected present value of future benefits given as

$$\mathbb{E}[v(K_x)] = B\sum_{k=T}^{\infty} \mathbb{E}[v(k)]\,{}_kp_x \tag{3.8}$$

3.3 Equivalence Principle

The setup for pricing life insurance contracts are based on the so called equivalence principle. The idea of the equivalence principle is to set up a contract so that the expected present value of the loss equals zero, in other words, the liabilities should equal the assets. We define the loss as the difference between benefits and premiums:

$$\mathbf{L} = \underbrace{\textbf{Present Value of Future Benefits}}_{PVFB} - \underbrace{\textbf{Present Value of Future Premiums}}_{PVFP} \tag{3.9}$$

Taking the expectation with an optional probability measure $\mathbb{N}$ of (3.9) and set it to zero we get

$$\mathbb{E}^{\mathbb{N}}[\mathbf{L}] = \mathbb{E}[\textbf{PVFB-PVFP}] \tag{3.10}$$
$$= \textbf{EPVFB} - \textbf{EPVFP} = 0 \tag{3.11}$$

or

$$\textbf{EPVFB=EPVFP} \tag{3.12}$$

where **EPVFB** is the expected present value of future benefits and **EPVFP** is the expected present value of future premiums received. In reality, insurance companies have expenses related to the insurance policy so to include that in the equivalence principle we write the loss as

$$\mathbf{L} = \textbf{PVFB} + \textbf{PVFE} - \textbf{PVFP} \tag{3.13}$$

where **PVFE** is the present value of future expenses. Taking the expected value of the loss we end up with

$$\textbf{EPVFB + EPVFE = EPVFP} \tag{3.14}$$

For simplicity in this thesis we will set **EPVFE**$= 0$, i.e we exclude expenses and work with (3.12). The structure of **EPVFP** and **EPVFB** depends on what kind of life insurance contract we are working with but the overall goal is to charge to policyholder such that the equivalence principle in (3.12) is fulfilled, after which a profit can be added to the price.

3.4 Contract Description

The contract we will model and analyze further combines the GMMB and the GMDB, described in Chapter 1, where the guaranteed rate tied to the two guarantees is denoted G_t. The obvious problem with these contracts is that the insurer doesn't know what benefit they will pay out until either death or maturity, where the benefit will be at least the guaranteed rate no matter how the market performs. The risk has to be priced into the contract and using the idea of the equivalence principle we will study what the price of the premium should be, or rather, how much does the guaranteed rate have to be, given a premium. The premium is how much the insurance company charge the policyholder for the contract and for simplicity we will model the premium as a single payment S_0 at time 0. Using the notation in the previous section we then have $\mathbf{EPVFP} = S_0$. From now on we assume the interest rate to follow a stochastic process. Then the discount factor $v(t)$ explained in (3.1) is the price of a zero coupon bond at time zero with maturity t denoted $d(0, t)$.

3.4.1 GMDB

The guaranteed minimum death benefit is one of the most common guarantees that has been issued and we model it in the following way:

- The contract is a term life insurance with maturity T.
- The policyholder receives a benefit at the end of the year of death. So if the policyholder dies between year 1 and year 2, it will receive the benefit at time 2.
- The benefit at time t payable upon death will be the greater value of G_t and S_t.

The benefit payable upon death will be the maximum value between a predetermined guaranteed benefit G_t and S_t, where S_t is the fund value at time t and G_t is given by a guaranteed interest rate g such that

$$G_t = S_0 e^{gt} \tag{3.15}$$

We denote the benefit at time t by B_t and it is given by

$$B_t = \begin{cases} S_t & \text{if} \quad S_t > G_t \\ G_t & \text{if} \quad S_t < G_t \end{cases} \tag{3.16}$$

$$= \max(G_t, S_t) \tag{3.17}$$

$$= \max(G_t - S_t, 0) + S_t \tag{3.18}$$

$$= (G_t - S_t)^+ + S_t \tag{3.19}$$

Note that $(G_t - S_t)^+$ in (3.19) is the payoff function for a European put option with maturity t, strike G_t and underlying asset S_t. Since this is a term insurance we know the present value of future benefits equals

$$\mathbf{PVFB}_{\text{GMDB}} = B_1 d(0,1) + B_2 d(0,2) + ... + B_{T-1} d(0,T-1) + B_T d(0,T) \tag{3.20}$$

$$= \sum_{k=1}^{T} B_k d(0,k) \tag{3.21}$$

with B_k given by (3.19). Taking the expectation of (3.21) under the risk neutral measure $\mathbb{Q}$ gives us the EPVFB at time zero.

$$\mathbf{EPVFB}_{\text{GMDB}} = \sum_{k=1}^{T} \mathbb{E}^{\mathbb{Q}}[B_k\, d(0,k)]\ {}_{k-1|}q_x \tag{3.22}$$

$$= \sum_{k=1}^{T} \mathbb{E}^{\mathbb{Q}}\Big[\big((G_k - S_k)^+ + S_k\big) d(0,k)\Big]\ {}_{k-1|}q_x \tag{3.23}$$

$$= \sum_{k=1}^{T} \Bigg(\mathbb{E}^{\mathbb{Q}}\big[(G_k - S_k)^+ d(0,k)\big] + \mathbb{E}^{\mathbb{Q}}\big[S_k\, d(0,k)\big] \Bigg)\ {}_{k-1|}q_x \tag{3.24}$$

Assuming independence between interest rate and equity we can apply the risk neutral valuation formula in (2.2.1) on (3.24) and obtain the price of the GMDB.

$$\mathbf{EPVFB}_{\text{GMDB}} = \sum_{k=1}^{T} \{C_0^{put}(G_k, S_0, k) + S_0\}\ {}_{k-1|}q_x \tag{3.25}$$

where $C_0^{put}(G_k, S_0, k)$ is a put option with strike G_k, maturity k and stock price S_0. We will later refer to (3.25) as a GMDB with maturity T.

3.4.2 GMMB

The guaranteed minimum maturity benefit is another common guarantee issued by insurance companies. However, in our model, the GMMB works a little different and will start working after the contract length of which the GMDB is valid and we will include *several* GMMBs. Our version of the GMMB is modeled as:

- A T-year deferred whole life annuity payable yearly.
- The benefit at time T will be the greater value of pG_T and pS_T, where p is a predetermined rate, and G_T is given as before.

The difference in the benefit for the GMMB compared to the one in the GMDB, is the rate p. In other words, a predetermined constant rate is withdrawn from the fund value or the guaranteed benefit on a yearly basis until death. Therefore, our version of GMMB is a product almost working as a GMWB and/or GMIB. The benefit at time t will thus be, using the same derivation as we used to arrive at (3.19)

$$B_t = p\left((G_t - S_t)^+ + S_t\right) \tag{3.26}$$

From previous section we know that the present value of a T-year deferred whole life annuity is given by

$$\mathbf{PVFB}_{\text{GMMB}} = B_T v(T) + B_{T+1} v(T+1) + \text{..........} \tag{3.27}$$

$$= \sum_{k=T}^{\infty} B_k \, d(0,k) \tag{3.28}$$

where B_t is given by (3.26). As with the GMDB, we take the expectation under $\mathbb{Q}$ and get

$$\mathbf{EPVFB}_{\text{GMMB}} = \sum_{k=T}^{\infty} \mathbb{E}^{\mathbb{Q}}[B_k \, d(0,k)] \, {}_kp_x \tag{3.29}$$

$$= \sum_{k=T}^{\infty} \mathbb{E}^{\mathbb{Q}}\Big[p\big((G_k - S_k)^+ + S_k\,\big) d(0,k)\Big] \, {}_kp_x \tag{3.30}$$

$$= p \sum_{k=T}^{\infty} \Bigg(\mathbb{E}^{\mathbb{Q}}\big[(G_k - S_k)^+ d(0,k)\big] + \mathbb{E}^{\mathbb{Q}}\big[S_k \, d(0,k)\big] \Bigg) \, {}_kp_x \tag{3.31}$$

$$= p \sum_{k=T}^{\infty} \{C_0^{put}(G_k, S_0, k) + S_0\} \, {}_kp_x \tag{3.32}$$

which is a similar formula to the one obtained in (3.25), but with longer maturities, higher strikes and conditioned on survival instead of death. We will later refer to (3.32) as a GMMB with maturity T.

Since both the GMDB and the GMMB are included in one contract we simply add the expected benefits of the both guarantees to get the total EPVFB of the contract.

$$\mathbf{EPVFB} = \mathbf{EPVFB}_{\text{GMDB}} + \mathbf{EPVFB}_{\text{GMMB}} \tag{3.33}$$

$$= \sum_{k=1}^{T} \{C_0^{put}(G_k, S_0, k) + S_0\} \, {}_{k-1|}q_x + p \sum_{k=T}^{+\infty} \{C_0^{put}(G_k, S_0, k) + S_0\} {}_kp_x \tag{3.34}$$

The goal for the insurance company is to find the rate g and p according to the equivalence principle. This is easily done by different minimizing functions such as goal seek in excel or lsqnonlin in matlab etc. In this thesis we will hold the rate p constant and find the g that meets the equivalence principle.
Looking at the expression in (3.34) above we have an indecisive amount of option contracts to value, depending on how long the mortality table is. Additionally if the death benefit and the pension payments were payable monthly instead of yearly, which is the most common type of contract, the calculations would be even heavier, which means Monte-Carlo simulations would require too much computational effort. Therefore we will introduce the fast FGL-method in the next chapter to price the options.

Chapter 4

Fourier-Gauss-Laguerre Pricing

In order to determine the price of the contract in the last chapter we have to choose a model for the fund value S_t. In the previous chapter we mentioned that we will model the interest rate as a stochastic variable, which immediately excludes the famous Black-Scholes model, which would imply a convenient way of pricing the options. Another drawback of the Black-Scholes model is its inability to capture the volatility smile, meaning that different maturity and strike prices generate different volatilities. To be able to model both the volatility and the interest rate as stochastic variables we will combine the well-known Heston model with the one factor Hull-White model. Since both the volatility and the interest rate are stochastic we are not able to use Black-Scholes formula to price the option. This leaves us with PDE, Monte-Carlo and Fourier based methods. Because of the excessive amount of options we need to price just for one single contract in (3.34), Monte-Carlo and PDE methods are not practical in our case, and we will instead derive an expression for the price of a European call option using the Fourier transform, from which we can use the put-call parity to get the put price.

4.1 Fourier Transform of a European Call Option

This section is based on [Wiktorsson, 2015] and we will derive the expression for the price of a European call option, using the Fourier transform, where we will end up with an integral we will approximate with the Gauss-Laguerre quadrature. With strike $K = e^k$, maturity T and stock price $S = e^s$, the payoff function for a European call option is given by

$$\begin{aligned} \phi(T) &= \max(S(T) - K, 0) && (4.1) \\ &= \max(e^{\log S(T)} - e^{\log K}, 0) && (4.2) \\ &= \max(e^s - e^k, 0) && (4.3) \end{aligned}$$

Taking the Fourier transform of (4.3) we get

$$h(z,T) = \int_{-\infty}^{\infty} e^{zk} \max(e^s - e^k, 0) dk \tag{4.4}$$
$$= \int_{-\infty}^{s} e^{zk}(e^s - e^k) dk \tag{4.5}$$
$$= \int_{-\infty}^{s} (e^{zk+s} - e^{k(z+1)}) dk \tag{4.6}$$
$$= \left[\frac{e^{zk+s}}{z} - \frac{e^{k(z+1)}}{z+1} \right]_{-\infty}^{s} \tag{4.7}$$
$$= \frac{e^{s(z+1)}}{z(z+1)} \tag{4.8}$$

To get back to the payoff $\phi(T)$ of the option we need to use the inverse Fourier transform on $h(z,T)$. Since the option price is a real number, we are only interested of the real part of the Fourier transform, so the payoff is

$$\phi(T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Re\left[e^{-(\gamma+i\omega)\log(K)} h(\gamma+i\omega, T) \right] d\omega \tag{4.9}$$

To obtain the option price we discount with the zero coupon bond $d(0,T)$ and take the conditional expectation of the expression in (4.9).

$$C_0(K,T) = \frac{d(0,T)}{2\pi} \int_{-\infty}^{\infty} \Re\left[e^{-(\gamma+i\omega)\log(K)} \mathbb{E}^{\mathbb{Q}}[h(\gamma+i\omega,T)] \right] d\omega$$
$$= \frac{d(0,T)}{2\pi} \int_{-\infty}^{\infty} \Re\left[\frac{e^{-(\gamma+i\omega)\log(K)}}{(\gamma+i\omega)(\gamma+i\omega+1)} \mathbb{E}^{\mathbb{Q}}[e^{\log(S_T)(\gamma+i\omega+1)}] \right] d\omega$$
$$= \frac{d(0,T)}{2\pi} \int_{-\infty}^{\infty} \Re\left[\frac{e^{-(\gamma+i\omega)\log(K)}}{(\gamma+i\omega)(\gamma+i\omega+1)} M^{\mathbb{Q}}_{\ln S_T}(\gamma+i\omega+1) \right] d\omega$$
$$= \frac{d(0,T)}{\pi} \int_{0}^{\infty} \Re\left[\frac{e^{-(\gamma+i\omega)\log(K)}}{(\gamma+i\omega)(\gamma+i\omega+1)} M^{\mathbb{Q}}_{\ln S_T}(\gamma+i\omega+1) \right] d\omega \tag{4.10}$$
$$= \frac{d(0,T)}{\pi} \int_{0}^{\infty} \Re\left[g(\gamma+i\omega) \right] d\omega \tag{4.11}$$

where the last equality holds since the integrand is an even function of ω, and $M^{\mathbb{Q}}_{\ln S_T}$ is the conditional moment generating function for $\ln S_T$ under $\mathbb{Q}$, depending on what model we use for S_T. Parameter γ should be chosen such that $\gamma = \gamma_{min}$ according to

$$\gamma_{min} = \underset{\gamma \in A^+_{S_T}}{\arg} \min g(\gamma) \tag{4.12}$$

where $A^+_{S_T}$ is defined as

$$A^+_{S_T} = \{\gamma > 0 : \mathbb{E}^{\mathbb{Q}}[S_T^{1+\gamma}] < \infty\} \tag{4.13}$$

The minimum γ can be found fast with standard numerical algorithms such as the golden section search.

4.1.1 Gauss-Laguerre approximation

We want to make the integral in (4.11) numerically computable and we do so by approximating it using the Gauss-Laguerre quadrature formula according to

$$\int_0^\infty e^{-x} f(x) dx \approx \sum_{i=1}^{n} w_i f(x_i) \tag{4.14}$$

where x_i is the i:th root of the Laguerre polynomial $L_n(x_i)$ and w_i is its weight given by

$$w_i = \frac{x_i}{(n+1)^2 [L_{n+1}(x_i)]^2} \tag{4.15}$$

Applying (4.14) on (4.11) we get the final option call price as

$$C_0(K,T) = \frac{d(0,T)}{\pi} \int_0^\infty \Re\left[g(\gamma + i\omega) \right] d\omega \tag{4.16}$$

$$\approx \frac{d(0,T)}{\pi} \sum_{j=1}^{n} w_j^{(n)} \Re\left[g(\gamma_{min} + i x_j^{(n)}) \right] e^{x_j^{(n)}} \tag{4.17}$$

where the parameter γ_{min} should be chosen according to (4.12). The weights w_j and the nodes x_j can be obtained for every n in a eigenvalue/vector problem of a tridiagonal symmetric $n \times n$ matrix. See [Wiktorsson, 2015] for further details. In our simulations we have used $n = 50$.

4.2 Heston Model

The moment generating function $M^{\mathbb{Q}}_{\ln S_T}$ in (4.10) depends on what model to use and as mentioned before we are going to use the Heston model. The Heston model models the stock price S_t with the following setup under the risk-neutral measure $\mathbb{Q}$

$$dS_t = rS_t\, dt + \sqrt{\nu_t} S_t\, dW_t^S, \tag{4.18}$$

$$d\nu_t = \kappa(\theta - \nu_t)\, dt + \xi \sqrt{\nu_t}\, dW_t^\nu, \tag{4.19}$$

where r is the risk-free rate, ν_t the volatility which follows a CIR-process, θ is the long-tern mean of the volatility process, κ is the rate of reversion and ξ is the volatility of the volatility. W^S and W^v are two correlated Brownian motions with constant correlation $dW_t^S \cdot dW_t^\nu = \rho dt$.

Its moment generating function can be shown to follow

$$M^{\mathbb{Q}}_{\ln S_T}(z) = \exp\left(z(\log(S_0) + rT) + C(z) + D(z)\,v_0\right), \tag{4.20}$$

$$C(z) = \frac{\kappa\theta}{\xi^2}\left((\kappa - \rho\xi z - d(z))T - 2\log\left(\frac{(\kappa - \rho\xi z)(1 - e^{-d(z)T}) + d(z)(e^{-d(z)T} + 1)}{2d(z)}\right)\right), \tag{4.21}$$

$$D(z) = \frac{(z^2 - z)(1 - e^{-d(z)T})}{(\kappa - \rho\xi z)(1 - e^{-d(z)T}) + d(z)(e^{-d(z)T} + 1)}, \tag{4.22}$$

$$d(z) = \sqrt{(\rho\xi z - \kappa)^2 + \xi^2(z - z^2)}, \tag{4.23}$$

See [Heston, 1993] for details.

4.3 Hull-White Model

Instead of holding the interest rate constant in the Heston model we will let it follow a stochastic process. In this thesis we choose to do it with a Hull-White model, which is an extension of the well-known Vasicek model and includes a link to the yield curve described in Section 2.4. The dynamics of the short-rate for Hull-White is given by

$$dr_t = \kappa_r\left[\theta_t - r_t\right]dt + \sigma_r\,dW_t \tag{4.24}$$

where θ_t is a time-varying deterministic function, chosen to fit the initial term structure of interest rate observed in the market. It can be shown, see [Brigo and Mercurio, 2007], that θ_t is given by

$$\theta_t = \frac{1}{\kappa_r}\left[\frac{d}{dt}f(0,t) + \kappa_r f(0,t) + \frac{\sigma_r^2}{2\kappa_r}(1 - e^{-2\kappa_r t})\right] \tag{4.25}$$

where $f(0,t)$ describes the yield curve at time 0 for maturity t. The advantage of this compared to the Vasicek model is that it includes the market's expectation of the evolution of the interest rate. In our simulations later on we will investigate the impacts of the parameters included in the θ_t. We will fit $f(0,t)$ using the Nelson-Siegel function

$$f(0,t) = \beta_0 + \beta_1\frac{1 - \exp(-\lambda t)}{\lambda t} + \beta_2\left(\frac{1 - \exp(-\lambda t)}{\lambda t} - \exp(-\lambda t)\right) \tag{4.26}$$

To calibrate the Hull-White interest rate model later on we need to write it on an affine term structure, see [Björk, 2004], according to

$$d(t,T) = e^{A(t,T) - B(t,T)r(t)} \tag{4.27}$$

We then obtain $A(t,T)$ and $B(t,T)$ by solving the system of equations in theorem 2.2.2 using the Euler method.

4.4 Heston + Hull-White

We will here extend the Heston model by letting the interest rate be stochastic instead of constant, and we will model the short rate with the Hull-White model described above. The model will then look like

$$dS_t = r_t S_t\, dt + \sqrt{\nu_t} S_t\, dW_t^S \tag{4.28}$$

$$d\nu_t = \kappa_v(\theta - \nu_t)\, dt + \xi\sqrt{\nu_t}\, dW_t^\nu \tag{4.29}$$

$$dr_t = \kappa_r\,[\theta_t - r_t]\; dt + \sigma_r\, dW_t^r \tag{4.30}$$

$$dW_t^\nu \cdot dW_t^S = \rho dt,\; dW_t^r \cdot dW_t^S = 0,\; dW_t^r \cdot dW_t^\nu = 0 \tag{4.31}$$

In reality there is a relation between equity and interest rate but we make the assumptions in (4.31) that there is no correlation between W_t^r and W_t^S nor between W_t^r and W_t^ν. It has been done in order to facilitate the expression for the characteristic function of the Heston+HW model.

The conditional moment generating function $M^{\mathbb{Q}}_{\ln S_T}(z)$ of the Heston+Hull-White model can be shown to follow

$$M^{\mathbb{Q}}_{\ln S_T}(z) = \exp\left(z\log(S_0) + C(z) + D(z)\,v_0 + F(z) + H(z)r_0 - \log(P(0,T))\right) \tag{4.32}$$

$$F(z) = \theta_r zT - \theta_r + \frac{\theta_r}{\kappa_r}(z-1)(e^{-\kappa_r T} - 1) + \frac{\sigma_r^2}{\kappa_r^2}(z-1)^2\left[\frac{-3 + 2\kappa_r T + 4e^{-\kappa_r T} - e^{-2\kappa_r T}}{4\kappa_r}\right] \tag{4.33}$$

$$H(z) = \frac{(1 - e^{-\kappa_r T})(z-1)}{\kappa_r} \tag{4.34}$$

where $C(z)$, $D(z)$ are given as for the Heston model, see [Marques, 2008] and [Grzelak and Oosterlee, 2011] for details. This is the final model we will use to simulate prices for the contract.

Chapter 5

Simulation and Results

In this chapter we calculate and present prices for the GMDB, GMMB and the combination of the two, GMDB+GMMB seen in (3.34), by using the Heston Hull-White model described in the previous chapter. The chapter consists of three parts. In Section 5.1 we generate different interest scenarios by assuming and calibrating towards three different types of yield curves; the normal yield curve, the inverted yield curve and a (almost) flat yield curve. We look at two different inverted yield curves, one where the short term yield increases drastically and one where the long term yield decreases drastically. In Section 5.2 we simply present prices using different mortality tables that have been used over the years in the UK, and study how prices have changed. In the final part, in Section 5.3, we add uncertainty to the parameter estimation of the interest rate model, from which we get an insight on how risky it is to enter these kind of contracts, as well as what parameters affect the price most/least. Since we are studying and focusing mainly on the interest rate rather than the volatility, the parameters of the Heston model has not been calibrated and therefore only been arbitrarily chosen. The FGL method described in the last chapter is used for pricing the embedded options in the contracts.

For the simulations we assume the following:

- A \$100 premium payment is made at issue by the policyholder, so we have to find the guaranteed rate g such that the price of the GMDB+GMMB equals \$100 and thereby fulfils the equivalence principle.
- The contract consists of a combination of a GMDB and a GMMB as described in (3.34).
- For every scenario, the contract is issued to policyholders aged x=15, 35 and 55.
- The retirement age is assumed to be 65 why the contracts will be of maturity lengths T=50, 30, 10 for ages x=15, 35 and 55. So the GMMB will kick in at age 65 and the GMDB will end at 65.
- The rate p is hold constant at $p = 0.02$ through every scenario.

- The mortality rate follows the mortality table from [Office for National Statistics, 2015] except for in Section 5.2 where different mortality tables from [Institute and Faculty of Actuaries, 2016] are compared.

5.1 Interest Rate Scenarios

In this section we generate and compare four different scenarios, each one calibrated to a specific yield curve; normal, inverted and flat, where we will have two different kinds of inverted yield curves. The yield curves are seen in Figure 5.1, and in Figure 5.5. The inverted yield curve can be interpreted as a scenario where an economic crisis is expected according to [Harvey, 1986] and [Harvey, 1970].

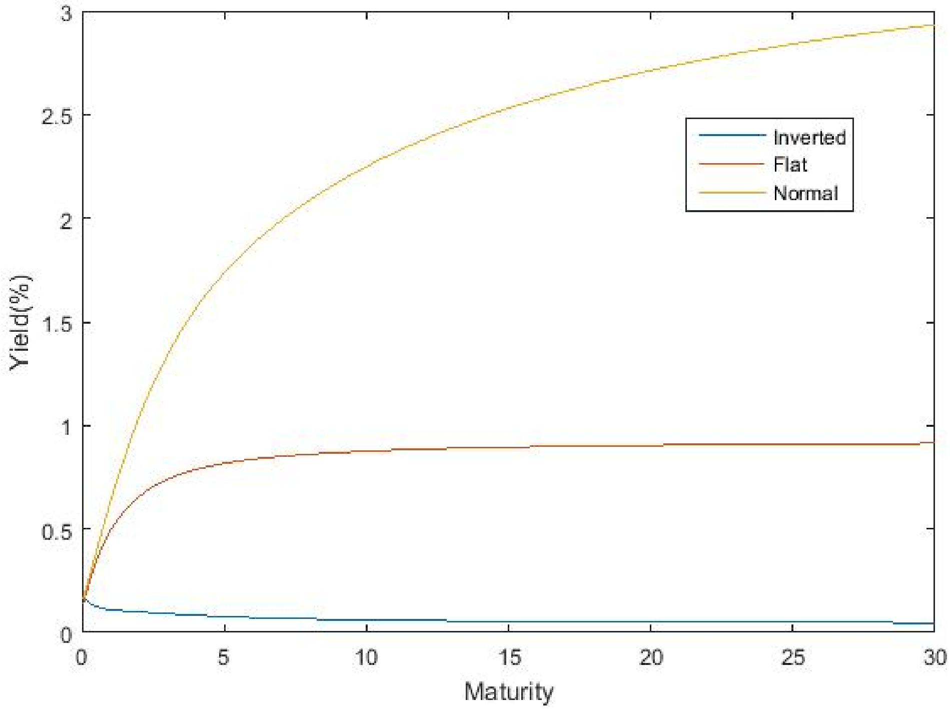


Figure 5.1: Yield curves used for different scenarios.

Calibration of the Hull-White model to the yield curve is done by using the relation between the yield curve $Y(t,T)$ and zero coupon bonds $d(t,T)$ given by

$$Y(t,T) = -\frac{\log(d(t,T))}{T-t} \tag{5.1}$$

The error term we want to minimize is given by

$$\epsilon = Y(t,T) + \frac{\log(d(t,T))}{T-t} \tag{5.2}$$

which is the difference between the actual observed yield curve $Y(t,T)$ and the yield curve predicted based on the parameters for the interest rate model. By minimizing the sum of the squared errors

$$S = \sum_{i=1}^{n} {\epsilon_i}^2 = \sum_{i=1}^{n} \left(Y(t_i, T) + \frac{\log(d(t_i, T))}{T - t_i} \right)^2 \tag{5.3}$$

we get an estimate of the parameters in the interest rate model. To accomplish this, the function nlinfit in matlab was used. We note that n in (5.3) is the amount of days the yield curve has been calibrated against.

5.1.1 Normal Yield Curve

In this scenario we assume the yield curve to follow the normal yield curve in Figure 5.1 which is calibrated for US treasury yields from April 1, 2016 and 60 days back resulting in the parameters in Table 5.1. Variances for the parameters can be seen in the covariance matrix in (5.6).

HW parameters						
κ_r	β_0	β_1	β_2	λ	σ_r	r_0
0.135	0.044	-0.012	-0.005	0.98	0.02	0.0025

Table 5.1: Estimated parameters for Hull-White model

This normal yield curve will serve as a benchmark to the other yield curves so we can study the change of the guaranteed interest rate g. The guaranteed rates obtained to fulfil the equivalence principle for the normal yield curve are given in Table 5.2.

T	**g (%)**
50	3.94
30	4.63
10	6.70

Table 5.2: The required rate g for the normal yield curve in Figure 5.1

Given those rates we run through the model and get the prices in Figure 5.2.

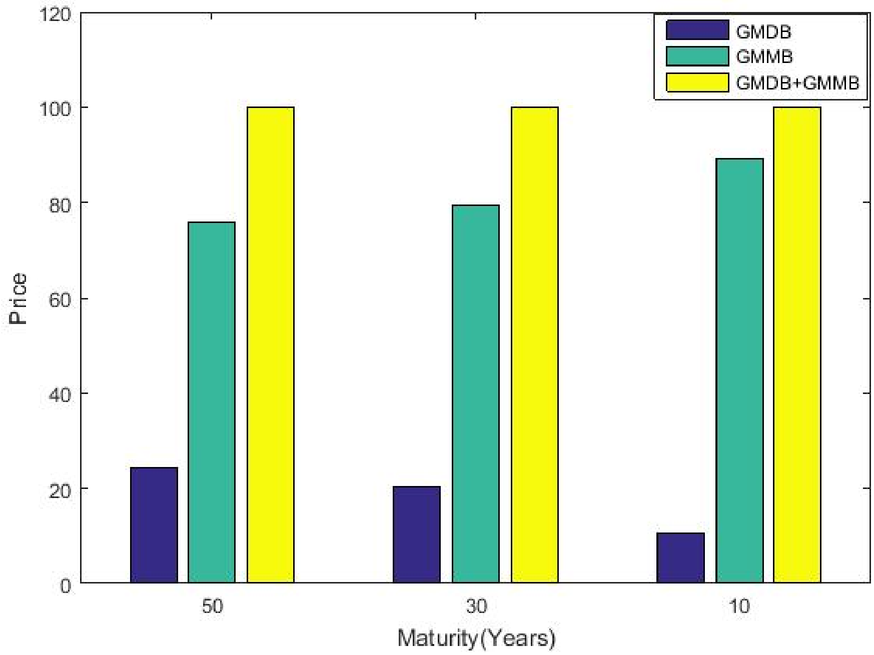


Figure 5.2: Prices of guarantees for different maturities with a normal yield curve

5.1.2 Inverted Yield Curve

In this scenario we study the inverted yield curve seen in Figure 5.1. It is self-constructed with the sole purpose of demonstrating the specific scenario of an inverted yield curve. As mentioned before, there are real world examples of inverted yield curves, but never has the interest rate been this low and in order to make a better comparison we chose to construct it ourselves, and let the long term yield decrease heavily while the short term rates starts from the same point as for the normal yield curve.

Later on in Section 5.1.4, we study another inverted yield curve where we instead let the short term rates increase heavily and then converge to the same mean as the normal yield curve. For now we are using the inverted yield curve in Figure 5.1 and calculating prices under this scenario with the guaranteed rates g in Table 5.2. The prices obtained are presented in Figure 5.3.

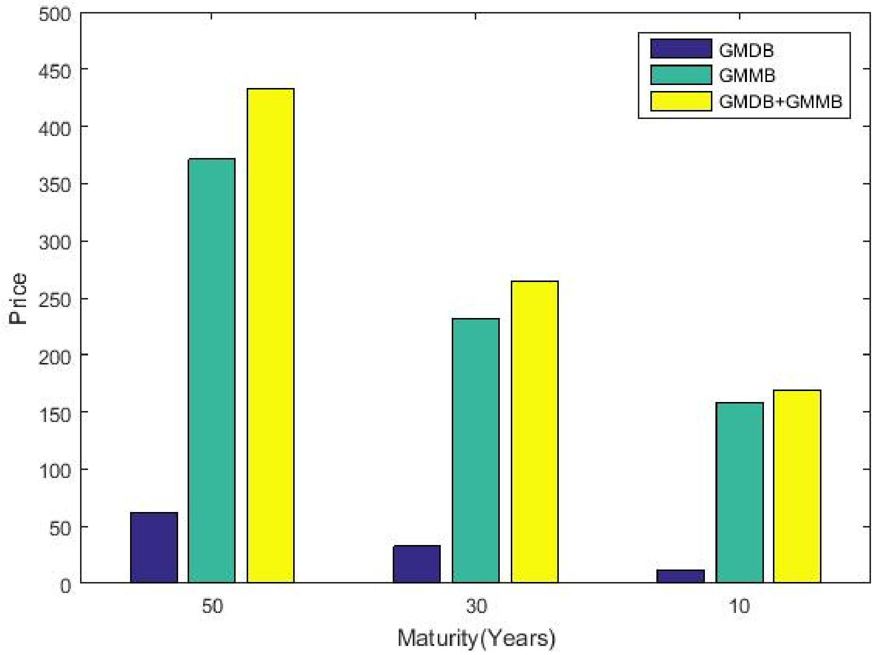


Figure 5.3: Prices of guarantees for different maturities with an inverted yield curve

To fulfill the equivalence principle we search for the guaranteed rate which gives us a $100 price for the combined GMDB+GMDB product. The guaranteed rates required for the inverted yield curve are presented in Table 5.3, along with the change from the rates used for the normal yield curve.

T	g (%)	Required g (%)	Diff
50	3.94	1.39	2.56
30	4.63	2.16	2.47
10	6.70	4.33	2.37

Table 5.3: The required rate g for the inverted yield curve in Figure 5.1

5.1.3 Flat Yield Curve

In this section we model the interest rate based on the flat yield curve seen in Figure 5.1. As with the inverted curve, this is also a ”made-up” curve used for demonstrating purposes only. We calculate the prices using the guaranteed rates in Table 5.2 for the normal yield curve and present those prices in Figure 5.4.

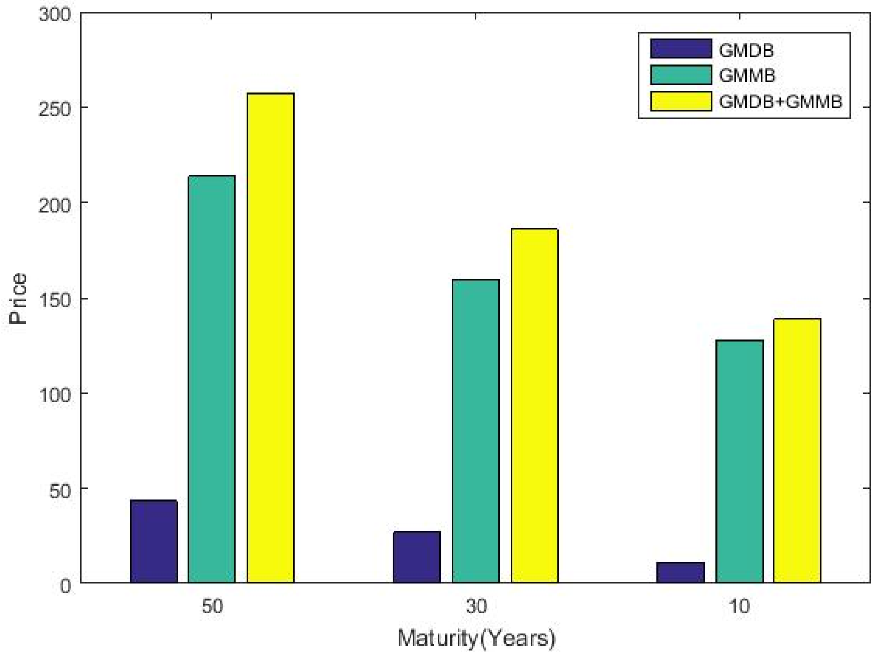


Figure 5.4: Prices of guarantees for different maturities with a flat yield curve

To adjust for the yield curve we do the same thing as with the inverted yield curve and arrive at the following required rates for different maturities

T	**g (%)**	**Required g (%)**	**Diff**
50	3.94	2.26	1.68
30	4.63	3.03	1.60
10	6.70	5.18	1.51

Table 5.4: The required rate g for the flat yield curve in Figure 5.1

5.1.4 Another Inverted Yield Curve

In this section we study another inverted yield curve, which is shown in Figure 5.5. The normal yield curve in Figure 5.5 is the same curve as we studied before. The difference between this yield curve and the inverted one we studied in Section 5.1.2 is that we now change the short term yields instead of the long term yields. It can be compared to the crisis in Greece in 2011 when the short term interest rates rose quickly and peaked around 23%, while the long term yield stayed around the same. As before, we calculate the prices using the guaranteed rates obtained for the normal yield curve, and present them in Figure 5.6.

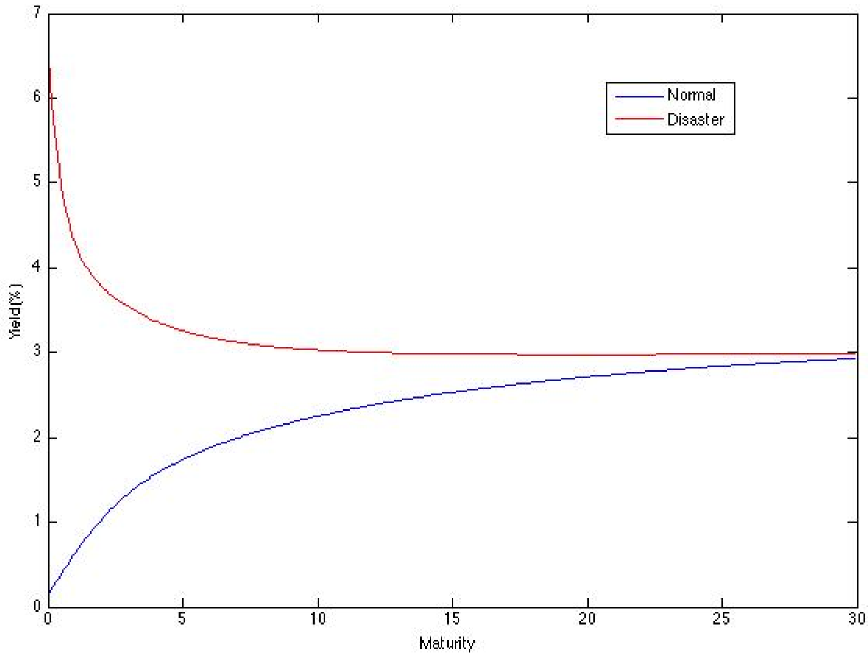


Figure 5.5: Showing the second inverted yield curve along with the same normal yield curve as before

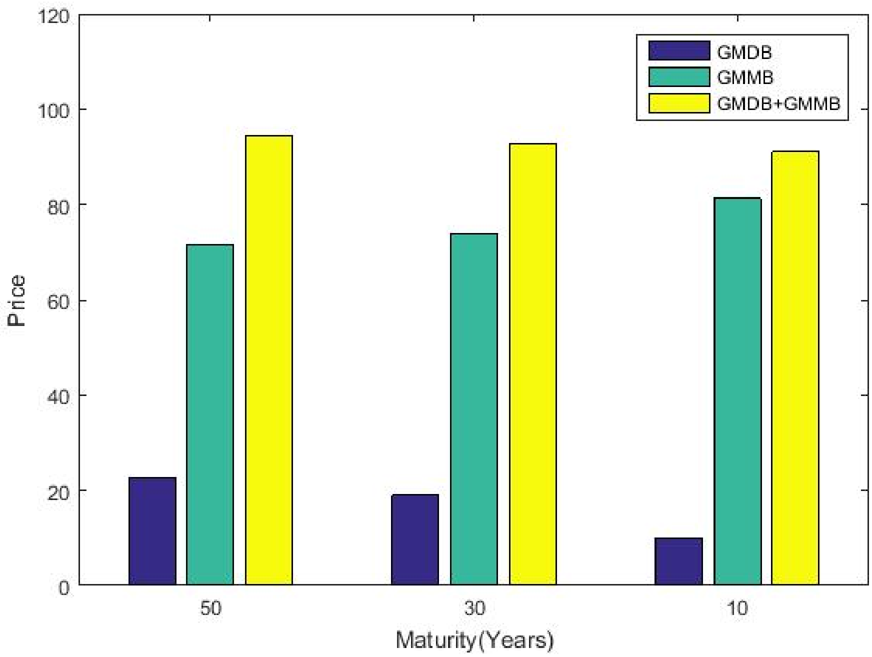


Figure 5.6: Prices of guarantees for different maturities with the second inverted yield curve

5.1.4.1 Analysis of Interest Rate Scenarios

The prices change as expected. A normal yield curve is better for the insurance company (and the policyholder) in the sense that it enables them to offer a higher guaranteed rate. The difference in the rates they can offer for different yield curves is significant.

Looking at the prices for the inverted yield curve using the rate used for the normal yield curve, we see an expected price that is approximately 4.5 times larger because of the yield curve for the contract issued to a policyholder aged 15. For the contract issued to $x = 35$ and $x = 55$, the price is approximately 2.5 and 1.5 times larger respectively. For the maturities it needed a decrease of the guaranteed rate g of 2.56, 2.47 and 2.35%, for maturity 50, 30 and 10, so the change increases a lot with the maturity as expected.

The flat yield curve changes a lot as well. Even though they are not as large as for the inverted yield curve they are still significant. The price before adjusting to the correct guaranteed rate is 2.5, 1.75 and 1.5 times larger. For 10 years of maturity it is almost the same difference as for the inverted yield curve. Looking at Table 5.5, which reflects the graphs, we notice that the impact of the yield curve does not affect the GMDB as much as one might think for 10 years of maturity. The low risk of dying within 10 years for a 55 year old policyholder is on the level such that it cancels out the impact from the yield curve. There is a 9% difference between the GMBD price with 10 years of maturity using the inverted and the normal yield curve, and a 4% difference between the flat and the normal yield curve. Thinking of the actual error in using a completely wrong yield curve, these changes are not so bad, considering that an investor does not make that bad projections of the future interest rate in reality over 10 years, unless a huge economic crisis spanning over 10 years occurs.

	GMDB (10 years)	GMDB (30 years)	GMDB (50 years)
Inverted#1	9%	60%	156%
Flat	4%	32%	78%
Inverted#2	-8%	-8%	-7%
	GMMB (10 years)	GMMB (30 years)	GMMB (50 years)
Inverted#1	77%	192%	390%
Flat	43%	100%	182%
Inverted#2	-10%	-8%	-6%

Table 5.5: Changes in prices of the inverted and the flat yield curve compared to the normal yield curve for different maturities.

Going forward and looking at the second inverted yield curve we studied, we see a different change of the prices. First of all the prices are much lower, and the other thing is that the price fluctuation decrease with the maturity as opposed to the other curves. This can be explained by the fact that the yield curve in the second inverted yield curve and the normal yield curve converge to the same long term yield. Since all the contracts are working for a very long time, it is reasonable to think that the long term mean of the interest rate affects the price the most. What is noticeable is that this, which is supposed to be a crisis, does actually not look that bad if you consider the prices in Figure 5.6. So even though we know that an inverted yield curve is bad for the future it does not look bad on paper. Instead this "crisis" looks like it would be good thing for the insurance company. What this implies is that the pricing model does not take into account factors from a macro-perspective point of view, since inverted yield curve implies a crisis where the risk of default increases.

5.2 Comparison with Different Mortality Tables

In this section we study the sensitivity of the prices if we compare the prices of the products using different mortality tables throughout the history. As we mentioned earlier, life expectancy has increased so we expect a lower price of the GMDB today compared to if we used the mortality tables from 50 years ago, as well as we expect a higher price of the GMMB today compared to back then.

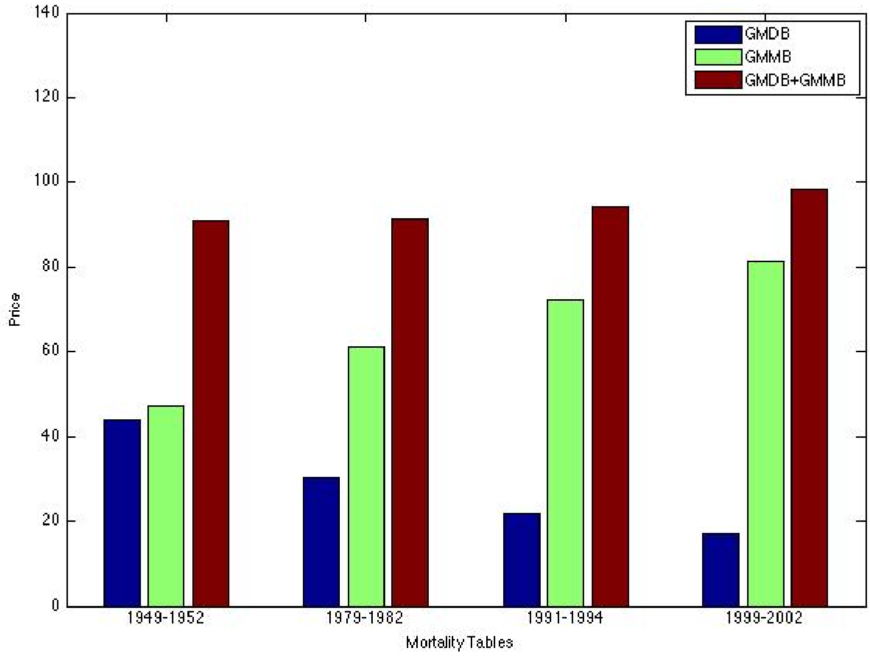


Figure 5.7: Prices of guarantees for different mortality tables with 50 years to maturity

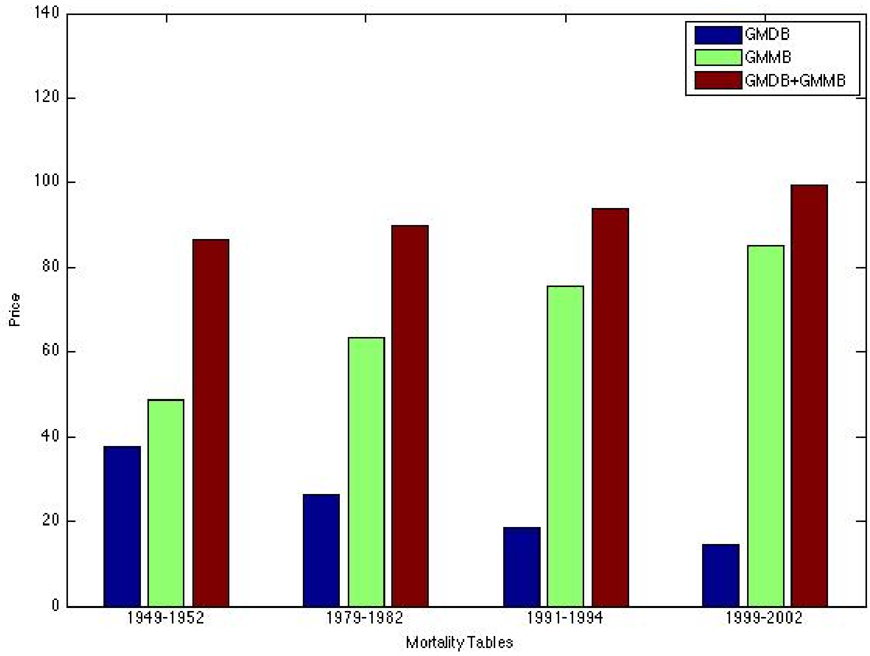


Figure 5.8: Prices of guarantees for different mortality tables with 30 years to maturity

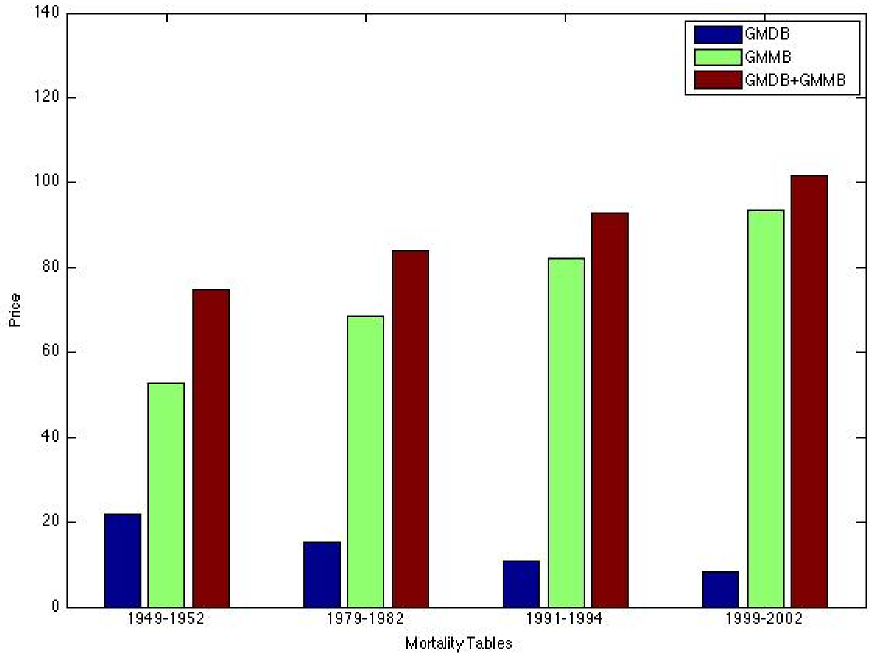


Figure 5.9: Prices of guarantees for different mortality tables with 10 years to maturity

5.2.1 Analysis of the Mortality Rate

The prices of the GMDB and the GMMB have moved the opposite direction over the years as expected, making the total price a slow increase. A lower mortality rate has implied a higher price of the GMDB and a lower price of the GMMB. This means that the need of accurately forecasting the interest rate on a long term basis has increased over the years, because of the decrease of the mortality rate.

5.3 Expected Price with Parameter Uncertainty

Looking at contracts with long term maturities, one thing to consider and study is the uncertainty of the parameter estimation in the interest rate model, since it is fair to question whether the parameter chosen are valid 30 years or more ahead of time. We will study the parameters in the Nelson-Siegel function $\beta_0, \beta_1, \beta_2, \lambda$ as well as the volatility of the interest rate model σ_r, when it is calibrated towards US treasury data for 60 days up to and including April 1, 2016, which represents a normal yield curve. As mentioned before, the parameters were estimated using the nlinfit function in matlab, giving us the corresponding variances to each parameter estimation. If we assume the parameters $\hat{\beta} = (\beta_0, \beta_1, \beta_2, \sigma_r, \lambda)$ in Table 5.1 to follow a normal distribution we get

$$(\beta_0, \beta_1, \beta_2, \sigma_r, \lambda) \sim \mathcal{N}(\mu, \Sigma) \tag{5.4}$$

where mean μ and covariance matrix Σ are estimated to

$$\mu = (0.044, -0.012, -0.005, 0.02, 0.98)^T \tag{5.5}$$

$$\Sigma = 10^{-5} \cdot \begin{pmatrix} 0.39 & 0.76 & 0.69 & 0.47 & -2 \\ 0.76 & 1.63 & 1.57 & 0.9 & -4.52 \\ 0.69 & 1.57 & 1.77 & 0.82 & -5.64 \\ 0.47 & 0.9 & 0.82 & 0.56 & -2.41 \\ -2 & -4.52 & -5.64 & -2.41 & 21.62 \end{pmatrix} \tag{5.6}$$

The other parameters not given in 5.4 are assumed to be known and simply given by the values in Table 5.1.

Algorithm 1 Find variance of price π

1. Fit $\hat{\beta} \sim \mathcal{N}(\mu, \sigma^2)$
2. Draw N samples from $\pi(\hat{\beta})$
3. $\hat{E}[\pi] = \frac{1}{N} \sum \pi(\beta_i)$
4. $\hat{V}[\pi] = \frac{1}{N} \sum (\pi(\beta_i) - \hat{E}[\pi])^2$

We apply **Algorithm 1** on $\beta_0, \beta_1, \beta_2, \sigma_r, \lambda$ separately and study which one of them gives rise to the most change of the price by looking at the standard deviation (std) and the coefficient of variance (cv). We also present the 95%

confidence interval. The function mvnrnd in matlab is used to generate the multivariate random numbers. The results for parameters $(\beta_0, \beta_1, \beta_2, \sigma_r)$ are seen in Table 5.6, 5.7, 5.8 and 5.9 respectively. The effect from λ to the price was so minimal that we do not have a table for that parameter.

$\hat{\beta}_0 \sim \mathcal{N}(\mu, \sigma^2)$ for T=50					
	Lower Bound	**Mean**	**Upper Bound**	**std**	**cv**
GMDB+GMMB	100.26	100.68	101.09	6.73	0.0668
GMDB	24.26	24.32	24.39	1.06	0.0435
GMMB	76	76.35	76.7	5.67	0.0742
$\hat{\beta}_0 \sim \mathcal{N}(\mu, \sigma^2)$ for T=30					
$\hat{\beta}_0 \sim \mathcal{N}(\mu, \sigma^2)$	**Lower Bound**	**Mean**	**Upper Bound**	**std**	**cv**
GMDB+GMMB	99.94	100.22	100.51	4.55	0.0454
GMDB	20.46	20.48	20.51	0.46	0.0223
GMMB	79.49	79.74	79.99	4.09	0.0513
$\hat{\beta}_0 \sim \mathcal{N}(\mu, \sigma^2)$ for T=10					
$\hat{\beta}_0 \sim \mathcal{N}(\mu, \sigma^2)$	**Lower Bound**	**Mean**	**Upper Bound**	**std**	**cv**
GMDB+GMMB	100.08	100.25	100.42	2.69	0.0268
GMDB	10.66	10.66	10.67	0.04	0.0041
GMMB	89.42	89.59	89.75	2.64	0.0295

Table 5.6: Expected price, standard deviation and coefficient of variance of the price of the guarantees for different maturities assuming β_0 is unknown.

$\hat{\beta}_1 \sim \mathcal{N}(\mu, \sigma^2)$ for T=50					
$\hat{\beta}_1 \sim \mathcal{N}(\mu, \sigma^2)$	**Lower Bound**	**Mean**	**Upper Bound**	**std**	**cv**
GMDB+GMMB	100	100.01	100.01	0.13	0.0013
GMDB	24.23	24.23	24.24	0.01	0.0003
GMMB	75.76	75.77	75.78	0.12	0.0016
$\hat{\beta}_1 \sim \mathcal{N}(\mu, \sigma^2)$ for T=30					
$\hat{\beta}_1 \sim \mathcal{N}(\mu, \sigma^2)$	**Lower Bound**	**Mean**	**Upper Bound**	**std**	**cv**
GMDB+GMMB	100	100	100	0.03	0.0003
GMDB	20.47	20.47	20.47	0.02	0.001
GMMB	79.53	79.53	79.53	0.05	0.0006
$\hat{\beta}_1 \sim \mathcal{N}(\mu, \sigma^2)$ for T=10					
$\hat{\beta}_1 \sim \mathcal{N}(\mu, \sigma^2)$	**Lower Bound**	**Mean**	**Upper Bound**	**std**	**cv**
GMDB+GMMB	100	100	100.01	0.1	0.001
GMDB	10.66	10.66	10.66	0.02	0.002
GMMB	89.34	89.34	89.35	0.08	0.0008

Table 5.7: Expected price, standard deviation and coefficient of variance of the price of the guarantees for different maturities assuming β_1 is unknown.

$\hat{\beta_2} \sim \mathcal{N}(\mu, \sigma^2)$ for T=50					
$\hat{\beta_2} \sim \mathcal{N}(\mu, \sigma^2)$	**Lower Bound**	**Mean**	**Upper Bound**	**std**	**cv**
GMDB+GMMB	99.98	100	100.03	0.37	0.0037
GMDB	24.23	24.24	24.24	0.07	0.0028
GMMB	75.75	75.77	75.79	0.3	0.004
$\hat{\beta_2} \sim \mathcal{N}(\mu, \sigma^2)$ for T=30					
$\hat{\beta_2} \sim \mathcal{N}(\mu, \sigma^2)$	**Lower Bound**	**Mean**	**Upper Bound**	**std**	**cv**
GMDB+GMMB	99.99	100	100.02	0.31	0.0031
GMDB	20.47	20.47	20.47	0.04	0.0018
GMMB	79.52	79.53	79.55	0.27	0.0034
$\hat{\beta_2} \sim \mathcal{N}(\mu, \sigma^2)$ for T=10					
$\hat{\beta_2} \sim \mathcal{N}(\mu, \sigma^2)$	**Lower Bound**	**Mean**	**Upper Bound**	**std**	**cv**
GMDB+GMMB	99.99	100	100.01	0.2	0.002
GMDB	10.66	10.66	10.66	0.01	0.0008
GMMB	89.33	89.34	89.35	0.19	0.0022

Table 5.8: Expected price, standard deviation and coefficient of variance of the price of the guarantees for different maturities assuming β_2 is unknown.

$\hat{\sigma_r} \sim \mathcal{N}(\mu, \sigma^2)$ for T=50					
$\hat{\beta} \sim \mathcal{N}(\mu, \sigma^2)$	**Lower Bound**	**Mean**	**Upper Bound**	**std**	**cv**
GMDB+GMMB	100.76	101.44	102.12	10.95	0.108
GMDB	24.33	24.44	24.55	1.84	0.075
GMMB	76.44	77	77.57	9.11	0.118
$\hat{\sigma_r} \sim \mathcal{N}(\mu, \sigma^2)$ for T=30					
$\hat{\beta} \sim \mathcal{N}(\mu, \sigma^2)$	**Lower Bound**	**Mean**	**Upper Bound**	**std**	**cv**
GMDB+GMMB	100.11	100.55	101	7.19	0.071
GMDB	20.47	20.51	20.56	0.77	0.038
GMMB	79.64	80.04	80.43	6.42	0.08
$\hat{\sigma_r} \sim \mathcal{N}(\mu, \sigma^2)$ for T=10					
$\hat{\beta} \sim \mathcal{N}(\mu, \sigma^2)$	**Lower Bound**	**Mean**	**Upper Bound**	**std**	**cv**
GMDB+GMMB	99.95	100.17	100.39	3.5	0.035
GMDB	10.66	10.66	10.66	0.05	0.005
GMMB	89.29	89.51	89.72	3.45	0.039

Table 5.9: Expected price, standard deviation and coefficient of variance of the price of the guarantees for different maturities assuming σ_r is unknown.

Now we let all the parameters $\beta_0, \beta_1, \beta_2, \lambda, \sigma_r$ be stochastic at the same time and study the price for the different maturities.

$(\beta_0, \beta_1, \beta_2, \sigma_r, \lambda) \sim \mathcal{N}(\mu, \Sigma)$ for T=50					
$\hat{\beta} \sim \mathcal{N}(\mu, \sigma^2)$	**Lower Bound**	**Mean**	**Upper Bound**	**std**	**cv**
GMDB+GMMB	99.99	100.22	100.45	3.76	0.0375
GMDB	24.22	24.27	24.31	0.72	0.0299
GMMB	75.77	75.95	76.14	3.04	0.04
$(\beta_0, \beta_1, \beta_2, \sigma_r, \lambda) \sim \mathcal{N}(\mu, \Sigma)$ for T=30					
$\hat{\beta} \sim \mathcal{N}(\mu, \sigma^2)$	**Lower Bound**	**Mean**	**Upper Bound**	**std**	**cv**
GMDB+GMMB	100.2	100.36	100.52	2.59	0.0259
GMDB	20.49	20.51	20.53	0.33	0.0162
GMMB	79.71	79.85	79.99	2.26	0.0283
$(\beta_0, \beta_1, \beta_2, \sigma_r, \lambda) \sim \mathcal{N}(\mu, \Sigma)$ for T=10					
$\hat{\beta} \sim \mathcal{N}(\mu, \sigma^2)$	**Lower Bound**	**Mean**	**Upper Bound**	**std**	**cv**
GMDB+GMMB	100.19	100.24	100.29	0.83	0.0083
GMDB	10.66	10.66	10.66	0.02	0.0022
GMMB	89.52	89.57	89.62	0.81	0.009

Table 5.10: Expected price, standard deviation and coefficient of variance of the price of the guarantees for different maturities assuming (5.4)

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5.3.1 Risk Analysis

First we analyzed the impact every parameter in the Nelson-Siegel function has for the price. We notice that the uncertainty involving β_1 and β_2 is very low and even for maturities up to 50 years, the impact is minimal. Especially for β_1, the effect cancels out as maturity grows which makes sense looking at the Nelson-Siegel function. The parameters mainly impacting the price are the long term mean, and the volatility of the interest rate, where the volatility has the highest impact. It is noticeable that only a little variance for both the long term mean and the interest rate variance impacts the price uncertainty significantly.

After looking at each parameter separately, we then let all the variables in the Nelson-Siegel function plus the volatility of the interest rate be unknown at the same time. In Figure 5.10 we see that the effect cancels out a bit because of the covariance between the variables. Corresponding to that table is Figure 5.10, where we see the distributions for the products and how it changes for different maturities, with 10,000 simulations.

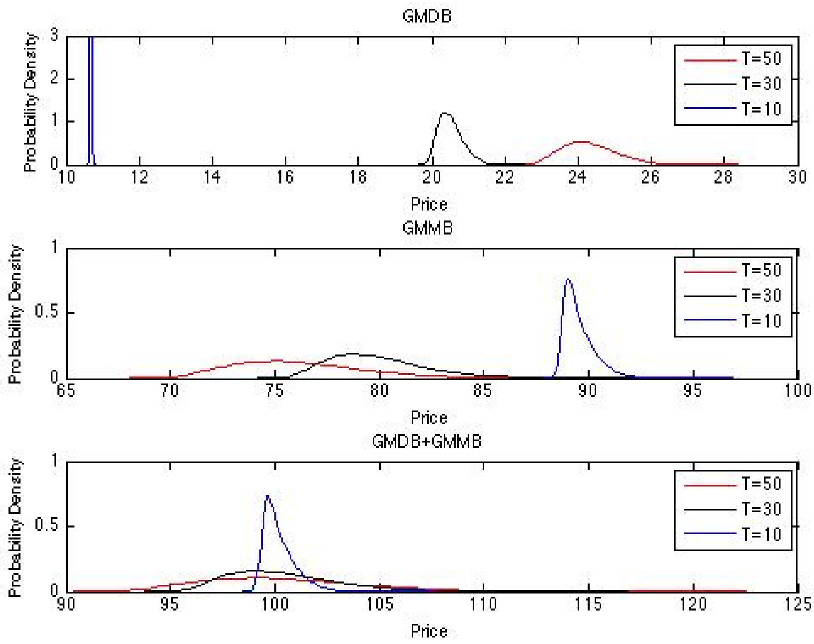

Figure 5.10: Distributions of the prices for the specific products and for different maturities.

We see how the standard deviation and the uncertainty increases fast with the maturities, for both products. As we are adjusting the guaranteed rate g because of the different maturities to fulfil the equivalence principle for the total price, we see how the prices for the GMDB and the GMMB moves in opposite directions, meaning that the longer until retirement, the more impact the GMDB has on the contract. To get a better feeling for the simulations and what they actually mean we calculate the 95% Value at Risk and the 95% expected shortfall explained in Chapter 1. They are presented in Table 5.11.

GMDB				
	VaR$_{.95}$	**% from mean**	**ES$_{.95}$**	**% from mean**
T=50	25.68	5.63 %	27.5330	13.2%
T=30	21.15	3.08 %	22.4684	9.5%
T=10	10.70	0.38 %	11.2801	5.8%
GMMB				
	VaR$_{.95}$	**% from mean**	**ES$_{.95}$**	**% from mean**
T=50	81.99	7.66 %	88.5467	16.3%
T=30	84.29	5.49 %	90.2241	12.9%
T=10	91.06	1.69 %	96.5739	7.8%
GMDB+GMMB				
	VaR$_{.95}$	**% from mean**	**ES$_{.95}$**	**% from mean**
T=50	107.7	7.18 %	116.0793	15.5%
T=30	105.4	4.99 %	112.6919	12.2%
T=10	101.8	1.54 %	107.8531	7.6%

Table 5.11: Risk measures for the products

The values in Table 5.11 show clearly how much uncertainty there is involved in the parameter estimation in long term contracts. For example, the total price of the contract with 50 years to maturity has a 5% chance of being at least 7.18% higher than expected, just because of the parameter uncertainty of the interest rate model. For the maturity of 10 years, the risk is not that large for any of the products but as soon as it reaches 30 years or more, the risk really has to be considered when determining the price.

Now looking at the covariance matrix in (5.6) we realize that the uncertainty in our actual parameters are not that big, as they are supposed to fit a certain normal yield curve that is assumed to reflect the future. However, even though this assumption is made we still see a rather large uncertainty in the prices. To summarize it, the price change we see is significant and should not be ignored, when working with maturities larger than 10 years.

Chapter 6

Conclusion

We started off by generating four different scenarios of the yield curve; one normal, two inverted and one flat yield curve. The results were intuitive and expected. A normal yield curve serves in favor of the insurance company, compared to the other curves, and the guaranteed rate can be significantly higher compared to the rates for the inverted and the flat curve, looking at tables 5.3 and 5.4. We also notice that the higher maturity the more the rate needs to be adjusted, and a higher maturity requires a lower guaranteed rate. So the longer maturity the more changes are needed in the rate g to adjust for different yields. We also compared to a second inverted yield curve (Figure 5.5), that differed to the other inverted yield curve in the sense that the short term yields were assumed to increase but the long term yield stayed around the same. The changes in the prices for the second yield curve were in favor of the insurance company, which actually is a false alarm. From a macroeconomic perspective this is bad news since the risk of being insolvent increases significantly but this is not taking into account in the model. So the final conclusion of the interest rate scenarios is that the impact on the prices are huge, why it is a correct choice to consider the interest rate to be stochastic. We also conclude that one should not focus too narrowly on the result without thinking on the macro-perspective meaning of the yield curve. As we saw, one of the inverted yield curves serves in favor of the insurance company when in fact this could imply an economic catastrophe.

We continued by comparing different mortality tables over the years, where the tables have been used for the same group of people namely males with permanent assurance. As expected, the cost of the GMMB has increased a lot over the years, and the price of the GMDB has decreased with almost the same speed. The price of the specific contract in this thesis is mainly consisting of the GMMB but looking at the old mortality tables, the distribution between the two products were more even in the past. The impact is substantial and it is obvious that insurance companies has to project the future increase of life expectancy into their calculations. This is in line with what [Cairns, 2006] concluded.

In the final study we added uncertainty to the parameters in the Nelson-Siegel function and to the volatility parameter in the Hull-White model. We assumed that they followed a normal distribution given the covariance matrix from the parameter estimation using the least squared error method. As expected, the uncertainty of the long term mean, i.e the parameter β_0, has the highest impact on the price compared to the other parameters in the Nelson-Siegel function. The volatility σ_r has a very high impact on the end result, even higher than β_0. Just by a little uncertainty in the parameters we have shown that it gives rise to a relative large variance of the price, and from an insurance company point of view it is a very risky business entering contracts with long term maturities over 10 years with a guaranteed rate. The final conclusion is that the parameter uncertainty cannot be overlooked in these type of contracts, and the higher maturity, the more it will matter.

Chapter 7

Discussion

The objective of this thesis was to develop a pricing model for insurance contracts with guarantees and study the price due to changes in the interest rate and the mortality. Using the risk metrics Value at Risk and Expected Shortfall we concluded that the price is very sensitive to the parameters in an interest rate model, and that the uncertainty increases along with the maturity. We have seen the uncertainty involving the estimation of the interest rate process and realized the complexity of it. Some further research would be to study the mortality and the interest rate risk at the same time, i.e a combination of this thesis and [Cairns, 2006], instead of doing it separately.

More opportunities for research would be to include the parameter uncertainty involved in the volatility process of the stock price, and to extend the Heston-Hull-White model where there is a correlation between the interest rate process and the stock and volatility. The Nelson-Siegel could also be extended to the Nelson-Siegel-Svensson function where one more factor is added to the function, and it would be interesting to see how much that factor affects the price in the long term. However, both the β_1 and β_2 affected the price very little so the guess is that another factor would not have any big impact on the parameter risk.

In terms of interest rate models, the Hull-White has its advantages by taken into account the current expectation of the market, but there is also disadvantages that the *change* of the expectation is not included. Today's expectation of the market 50 years ahead will not be the same as the expectation in 5 years, which is the main problem when you have to forecast interest rate that long into the future. Things that affects the economy on a larger scale such as unemployment rate is not taken into account and a model where things like that would be included would be a step in the right direction. This disadvantage became obvious when we studied the second inverted yield curve.

Another thing that could be included in the model is the surrender risk. The surrender risk is the risk that a policyholder choose to stop the contract and collect the premiums that has been invested in the fund. That risk was completely ignored in this thesis.

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