

Black-Litterman Portfolio Allocation Stability and Financial Performance with MGARCH-M Derived Views

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Abstract

This paper deploys methodology typically utilized in financial econometrics, namely univariate and multivariate GARCH-M forecasting techniques, as inputs into the Black-Litterman asset allocation process. While previous works have examined the usefulness in deploying select GARCH specifications as a source for the required Black-Litterman views vector, to the best of our knowledge, this is the first such work comparing the effects of select GARCH specification on asset allocation volatility. This paper draws parallels with Beach and Orlov (2007) and Duqi, Franci, and Torluccio (2014) in finding improved portfolio financial performance after the incorporation of GARCH-derived views relative to market equilibrium weighting. Financial performance is further improved with the incorporation of the multivariate DCC models. While this increase in performance is accompanied by an increase in asset allocation instability, the multivariate portfolios provide a better return-to-risk relationship for the associated degree of allocation volatility. In both the univariate and multivariate specifications the more simple GARCH(1,1) provides superior performance relative to the asymmetric GJR model.

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Keywords:

Financial Econometrics, Black-Litterman, Asset Allocation Stability, MGARCH-M

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1 Introduction

Previous research confirms many desirable benefits associated with portfolio rebalancing. Financial markets are incredibly dynamic and a portfolio should be periodically adjusted to ensure its risk and return characteristics remain within desired limits as well as to ensure the portfolio has not become too concentrated. In reality these adjustments are costly and can quickly reduce the returns of a portfolio. As later included results will highlight, failing to include transaction costs in any allocation strategy, particularly with a volatile weight structure, can prove disastrous for the financial performance of the portfolio.

Traditional portfolio allocation has often been described in a Markowitz framework, where an investor seeks to maximize the relationship between risk and return in the assembled portfolio. While this approach is theoretically sound and provides the backbone for Modern Portfolio Theory, the model has been criticized outside academia. Typically passed through a numerical optimizer, the Markowitz results often present extreme corner solutions and highly unstable weights. In the context of the following investigation these properties may be responsible for large transaction costs, which when included in overall portfolio performance may heavily penalize the portfolio return.

The Black-Litterman approach, later presented in section 3.2, helps mitigate these above mentioned negative attributes. By relying heavily on an equilibrium reference model when blending information sets, the model mutes much of the heavy fluctuation in portfolio allocation. As the included literature demonstrates, establishing an appropriate set of supplied views has been a primary challenge in Black-Litterman research. Previous authors have investigated the use of volatility forecasting GARCH models for the formulation of investor supplied views. The models selected have demonstrably better financial performance when compared with solitary reliance on the reference model, indicating their potential financial usefulness. This paper furthers research into the incorporation of GARCH derived views through the inclusion of GARCH specifications not previously investigated. The inclusion of the DCC multivariate and GJR univariate specifications further distinguish this work from previous inquiries and contributes to the growing body of Black-Litterman research.

This paper examines asset allocation volatility and financial performance during the incorporation of increasingly complex GARCH structures in the context of the Black-Litterman asset allocation methodology. Of primary interest is the intersection between financial performance and allocation stability. While the simple GARCH(1,1) outperforms the asymmetric GJR specification in both the univariate and multivariate frameworks, there is a clear financial benefit associated with the inclusion of a full uncertainty matrix as observed in the Sharpe Ratio associated with the multivariate specifications. However, following the incorporation of ex-post transaction costs the univariate GARCH(1,1) provides the best financial performance relative to allocation volatility. The inability of the asymmetrical GARCH specification to outperform the simple GARCH(1,1) in either framework echoes previous research where forecasting benefits of incorporating asymmetry has been mixed; in some studies statistically insignificant or negligible.

As previous research has largely focused on US equities, this paper examines a subset of UK FTSE 100 assets. The portfolio assembled is designed so as to loosely represent the primary industry sectors within the index. The inclusion of one of the world's most commonly followed indices, valued in the fourth most traded currency, further broadens the available Black-Litterman literature.

2 Traditional Portfolio Formulation

The following segments in Section 2 introduce the foundations of modern portfolio allocation literature before presenting the more relevant associated drawbacks, spurring the search for alternative strategies. The influential Capital Asset Pricing Model is included for later use in the formation of the optimal Black-Litterman portfolios. Lastly, this section highlights the beneficial and negative effects associated with portfolio rebalancing, which will be influential in the results and analysis portion of this paper. The following section provides a background for the introduction of the Black-Litterman methodology presented in Section 3.

2.1 Modern Portfolio Theory

Introduced in 1952 by economist Harry Markowitz, *Modern Portfolio Theory* (or *mean-variance analysis*) suggests investors should assemble portfolios designed to maximize return for a given level of risk or to minimize risk for a specified level of return. In this context, risk is defined as the variance of returns. The distinguishing feature in the original Markowitz approach was the linkage between asset return and risk; neither should be considered in solitude and should be examined as to how the individual asset's risk-to-return characteristics match the assembled portfolio (Markowitz, 1952).

The Markowitz model assumes investors are risk averse and must be compensated for holding risky assets. Thus a rational agent would not hold an asset if another exists with an identical return and lower variance or equal variance and a higher return. The final assembled portfolio return is then a weighted combination of individual asset returns, while portfolio volatility is influenced by asset correlations and overall portfolio volatility can be reduced through diversification (Markowitz, 1952).

2.2 Markowitz Limitations

The Markowitz mean-variance optimization methodology is associated with highly unstable portfolio weights, leading to large and frequent adjustments in portfolio asset allocation, as is observed in Figure 1 (Kourtis, 2015). According to Kourtis, portfolio returns may be severely penalized by high allocation volatility stemming from the high transaction costs incurred by excessive rebalancing. Additionally, the typical mean-variance allocation methods are found lacking in regard to portfolio rebalancing as they do not account for the original portfolio composition, changes in funding, nor most influential for the inquiry at hand: transaction costs (Glen, 2011). Additional methods employed to reduce the frequency and costs associated with rebalancing have been proposed, including: portfolio drift constrained within bounds, diversification at country or sector level and lastly weight selection representing a trade-off between capitalization weighting and equal weighting (Bouchev, Nemtchinov, Paulsen and Stein, 2012).

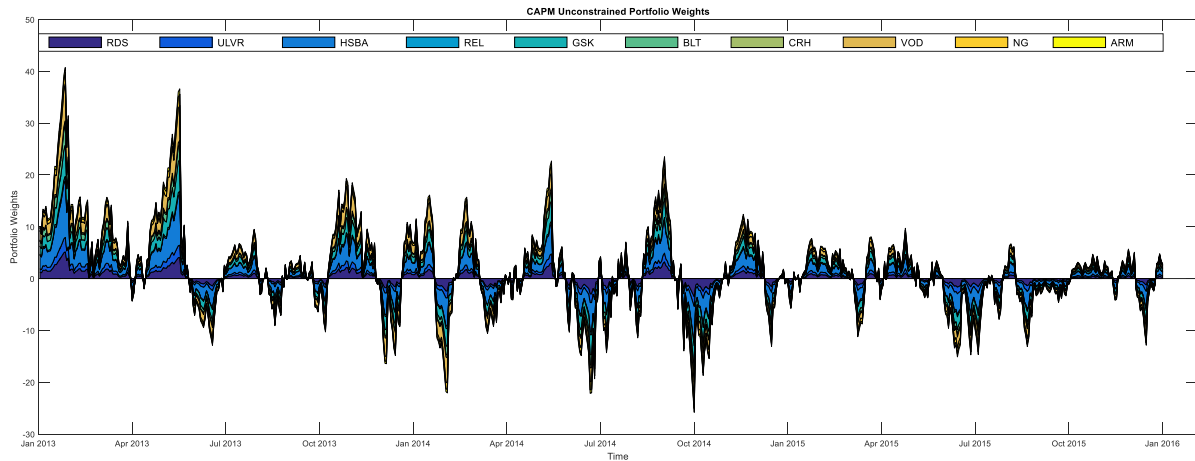


Figure 2.1. Evolution of portfolio allocation when no constraints are placed upon CAPM

Historically popular in academia, The Markowitz model has encountered implementation difficulties in practice (Zhou, 2009). He and Litterman (1999) highlight the previously introduced weaknesses associated with mean-variance portfolio allocation. The traditional Markowitz approach is associated with extreme swings in portfolio weights based on slight adjustments in expected returns as well as asset weighting contrary to even the most strongly maintained views (Bevan and Winkelmann, 1998). Fusai and Roncoroni (2008) highlight the tendency for mean-variance optimization to more heavily allocate towards assets which yield sizeable expected returns, a low degree of variance, and low correlation. DeMiguel and Nogales (2009) find the traditional mean-variance allocation based on sample mean and covariances to perform poorly out of sample. The Black-Litterman model helps alleviate these drawbacks, while the study presented will examine whether these benefits can be further strengthened through the inclusion of objective, computational return and volatility forecasts based on historical data.

The formation of large long and short positions as well as many zero positions with the introduction of zero short selling are two additional pitfalls in the common mean-variance optimization (Idzorek, 2004; Zhou, 2009). As a further drawback, portfolio managers in reality often specialize in a smaller subset of the overall market whereas the Markowitz approach requires expected returns for all assets in the market. Traditional asset allocation methods also have greater difficulty incorporating specific, profitable investment strategies. Lastly, portfolio managers tend to think of assets in terms of portfolio weightings as opposed to the explicit risk-return trade-off of mean-variance Markowitz optimization (He and Litterman, 1999). In summary, He and Litterman provide the following explanation for

the lack of broad mean-variance optimization support in a practical setting; *“In practice most managers find that the effort required to specify expected returns and constraints that lead to reasonable answers does not lead to a commensurate benefit.”* (He and Litterman, 1999 p.3).

2.3 Previous Research into Portfolio Allocation Stability

DeMiguel and Nogales (2009) highlight research into minimum-variance portfolios, those that ignore the mean altogether. Merton (1980) documented the difficulty in estimating mean returns. In general, this largely stems from the estimation error size of the sample mean being much larger than that of the sample covariance; Jagannathan and Ma (2003) argue the estimation error in the mean is so severe that the mean could be excluded in entirety. Such a methodology has typically outperformed the mean-variance out of sample. In their study, DeMiguel and Nogales suggest utilizing robust estimators in an attempt to alleviate this sensitivity in the mean-variance allocation. The paper demonstrates a reduction in weight sensitivity to fluctuations in asset-return distributions relative to standard mean-variance portfolios. The mean-variance however remains more volatile than the typical minimum-variance portfolio.

Further, DeMiguel and Nogales cite the efficiency lost in the maximum likelihood estimation, where the normality assumption is voided as the actual data distribution departs slightly from this assumption. As return distributions are typically not normal, they denote the particular importance of distributional assumptions in managing portfolios. The paper suggests robust estimators are able to provide useful information regarding return series when the sample distribution differs from the distribution assumed, typically a normal distribution.

Zhang and Maringer (2009) suggest a clustering technique to improve portfolio performance and increase allocation stability. The authors note the higher Sharpe Ratio¹ of the cluster portfolio relative to the non-clustering process. The procedure additionally reduces the issues associated with estimation errors on the assembled portfolios. Michaud (1989) on the other hand advances resampling in an effort to reduce portfolio sensitivity to estimation errors and increase asset weight stability. Later research by master's student Kohli (2005) echoes

¹ The Sharpe Ratio is a measure of reward relative to volatility. For further information, see Sharpe (1966).

Michaud's findings regarding weight stability but does not encounter a performance benefit. In summary, the resampling technique eliminates extreme and poorly diversified portfolios by means of an averaging process, inducing stability (Fusai and Roncoroni, 2008).

Perret-Gentil and Victoria-Feser (2003) advocate robust estimation in place of the standard sample mean and variance. The stability benefits are particularly evident in the presence of model misspecification, whereby an ill-suited distribution is chosen for the return process. The large deviation between robust and non-robust optimizer results often stem from a small number of outlier returns.

2.4 The Capital Asset Pricing Model

The well-known Capital Asset Pricing Model or CAPM, provides a theoretical model for determining the return on an asset. The model accounts for asset specific, non-diversifiable risk, the expected return on the market, as well as a risk-free asset. The included risk-free return compensates investors for the time-value of money while the risk measure β determines the degree of compensation considering the risk associated with the asset relative to that of the market portfolio. The investor will only be compensated for this additional, unsystematic risk that cannot be diversified away through portfolio composition. Provided one works with generally accepted concave utility functions, CAPM is theoretically sound in that investors must be compensated for holding additional risk.

CAPM makes a number of assumptions of varying degrees of plausibility. For use in the reverse optimization of equilibrium excess returns performed in Section 3.2, the following CAPM assumption is introduced: complete agreement. Given a vector of specified market clearing asset prices, agents must agree on the joint distribution of asset returns from this period to the next. This assumption entails that any market portfolio must be on the minimum-variance frontier if the market is to clear all positions (Fama and French, 2004).

Additionally, CAPM assumes investors are only concerned with the asset returns and variances, the first two moments. The model additionally assumes zero transaction costs, which in the context of this study will be of particular interest. CAPM then presents the individual asset return as singularly determined by the asset β or how the asset covaries with the variance of the market. One of the most commonly encountered formulations is presented below, providing an asset's appropriate required expected return.

$$E(R_i) = R_f + \beta_i(E(R_m) - R_f) \quad (2.1)$$

The above equation represents the relation between expected return and risk which must hold if the market is in equilibrium. While the above model is widely taught, it has been met with criticism in empirical studies. One such criticism is the impossibility of finding an actual market portfolio, as theoretically this should encompass all available assets. Additionally, according to CAPM, the only meaningful measure of risk is the above β , which captures relative asset volatility (Fama and French, 2004).

CAPM is based on equilibrium properties. In this study it has been assumed that the market is in equilibrium and the market portfolio is the sum of each weight in a stock divided by the total value of all stocks in the market portfolio. Provided all participants agree, a previously mentioned CAPM assumption, the market portfolio is the optimal portfolio. In Section 6.3 this study relies on CAPM theory to reverse optimize a series of implied equilibrium excess returns. In this instance, the risks and market capitalized weights are taken as given. From this information the excess return required for an investor to hold the market-capital weighted portfolio is determined. By doing this, it is assumed the portfolio lies on the SML (Shapiro, n.d.).

More recent research has cast additional doubt on basic CAPM results. Fama and French (1992) find the linear beta relationship may break down in short periods and during the period 1963 - 1990 variations in β do not fully explain divergences in asset performance. The inclusion of additional price ratio variables has been found significant in explaining asset returns, indicating a β which does not capture all risk effects. This result alone has been

argued to nullify CAPM. In general, CAPM's many restrictive assumptions are not representative of reality. However, few theoretically sound alternatives have been explored. Additionally, previous tests have largely been tests of proxy efficiency rather than CAPM itself, often referred to as the joint hypothesis problem (Fama and French, 2004).

2.5 Portfolio Rebalancing

While portfolio rebalancing does incur different types of fees associated with each change in relative asset allocation, as assets are bought and sold according to bid-ask spreads and other transaction costs, maintaining the original portfolio without the aforementioned adjustments is likely less desirable. These fees may be flat, fixed or proportional to the size of the trade (Holden and Holden, 2013). As expected future growth rates remain a mystery and must be predicted, concentrations may accumulate in the portfolio leading to poor future performance (Bouchev, Nemtchinov, Paulsen and Stein, 2012). The authors further state the diversification benefits of rebalancing. Upon inception, a buy-and-hold strategy may be well diversified, though through time the portfolio may become more concentrated owing to drifting asset values. The poor performance of capitalization weighted indices in comparison to other well-known diversification strategies such as equal weighting, minimum variance, mean-variance fundamental weighting, and others could stem from a lack of rebalancing (Bouchev et al., 2012).

In the end, diversification and portfolio rebalancing is associated with a number of desirable outcomes: a reduction in concentration risk, volatility and downside risk; as well as an increase of the portfolio's long-term growth rate (Bouchev et al., 2012). The process of portfolio rebalancing, while beneficial, is not costless and investors should take into account these transaction cost when rebalancing their portfolio (Brodie, Daubechies, De Mol, Giannona and Loris, 2008). Bouchev et al. caution that in scenarios in which transaction costs are high, these benefits may be eroded away. In Section 7, the negative effects of transaction costs associated with rebalancing will be examined in accordance with the proposed specification of the Black-Litterman methodology presented in Equation 3.1.

2.6 Portfolio Transaction Costs

The above mentioned necessary portfolio adjustments may incur considerable transaction fees and their negative effects on portfolio returns may be significant (Feng, Medo, Zhang and Zhang, 2010). While in the real world these transaction fees produce very real factors mitigating investor profits, they are largely ignored in the literature or simplified as a proportion of the asset value which is then subtracted from the portfolio return (Glen, 2011). The incorporation of transaction costs is crucial in any meaningful portfolio optimization (Mitchell and Braun, 2013). Traditionally these costs have later been included in the mean-variance framework by including proportional transaction costs (Kourtis, 2015).

While neither of these methods is close to reality, the second is at best a poor approximation as in actuality, these incurred transaction costs lead to an attrition in portfolio value rather than limiting individual asset returns (Glen, 2011). Mitchell and Braun suggest a similar procedure and rescale funds accessible for investment after transaction costs have been accounted for, representing the reduction in principle associated with the transaction.

Barber and Odean (2000) specifically examines the effects of frequent trading on individual investor portfolio returns. They determine a severe penalty for actively altering portfolio composition in individual investors. The paper finds evidence supporting overconfidence to influence exaggerated trading. Barber and Odean note an annualized return difference between those households that trade frequently and those that do not of 7.1 percent. In the presence of transaction costs, the households studied significantly underperformed the included benchmark; as a result of frequent trading rather than portfolio composition. Further, an average transaction cost of roughly 1.5 percent is proposed.

While portfolio rebalancing in an effort to maintain a certain risk structure is advantageous according to Barber and Odean, the authors determine this not to be a primary reasoning for trading within the study. The degree of turnover encountered does not support a rebalancing approach based upon desired risk characteristics (Barber and Odean, 2000).

The above results the authors note closely mimic research performed in mutual funds. Compared with an index fund, the typical mutual fund tends to underperform in the same time period (Jensen, 1969; Malkiel, 1995). The paper further highlights the greater consequences inflicted upon individual investors, facing higher proportional commission costs as a result of a smaller trade size.

Alternative means of limiting transaction costs have been proposed. Clark and Mulready (2007) propose a turnover constraint. They describe turnover as the sum of the absolute value of position adjustments between time periods. The authors note the difficulty in modeling transaction costs, as they may be influenced by the magnitude of the trade, the type of the asset involved and other considerations. Clark and Mulready further emphasize that, despite the difficulty of incorporation, transaction costs should be included in the asset allocation decision process.

3 The Black-Litterman Model

The following Black-Litterman section of the paper begins with the rationale and logic behind the development of the Black-Litterman methodology. The Black-Litterman model is then presented in detail along with two of the more challenging aspects of its implementation. A detailed description of the necessary variable calculations is included before transitioning into supporting literature. Finally, the inclusion of proposed GARCH based volatility forecasts into the Black-Litterman framework is described.

3.1 Introduction & Rationale

In response to the above discussed shortcomings associated with the implementation of Modern Portfolio Theory, Fischer Black and Robert Litterman introduced their Black-Litterman model in 1990.

“The Black-Litterman asset allocation model, created by Fischer Black and Robert Litterman, is a sophisticated portfolio construction method that overcomes the problem of unintuitive, highly-concentrated portfolios, input-sensitivity, and estimation error maximization. These three related and well-documented problems with mean-variance optimization are the most likely reasons that more practitioners do not use the Markowitz paradigm, in which return is maximized for a given level of risk.” (Idzorek, 2004 p.1)

Since its introduction, the Black-Litterman methodology has become widely used and remains very flexible, providing asset returns and optimal weights (He and Litterman, 1999).

The Black-Litterman model incorporates a number of methods and theories from finance and asset allocation. For instance, the previously discussed CAPM from Section 2.4 is utilized in establishing the equilibrium reference model. The primary logic underlying the Black-Litterman model is as follows: investors should take risks where they hold positive views and the scale of these positions should be proportional to the certainty surrounding the supplied views (Bevan and Winkelmann, 1998). The Black-Litterman model is thus a model which merges a series of neutral reference returns and investor supplied views (Zhou, 2009).

The original methodology presented relied on Theil’s mixed estimation technique² rather than the Bayesian rule due to simplicity. Bayes methodology estimates parameters by mixing complete prior data and partial conditional data (Walters 2014). A primary strength of the Black-Litterman model is evident in the presence of a benchmark, beta or risk constraint, or other forms of constraints. In these situations the weights are less intuitive though the trade-off between risk and return is still operating in the presence of these imposed constraints (He and Litterman, 1999).

3.2 Black-Litterman Methodology Walkthrough

According to Walters, the Black-Litterman methodology follows the subsequent process: selection of the assets to be included, computation of the historical covariance matrix, determination of the market-capitalization weights of each asset, reverse optimization of

² Theil’s mixed estimation technique blends prior and sample information together using a scalar measure. For more information, see Theil (1963)

implied equilibrium excess returns, calculation of the Black-Litterman expected return, provision of the previous expected return vector to a mean-variance optimization routine, and lastly selection of the efficient portfolio allocation according to the desired level of risk. The Black-Litterman model thus melds historical data and those views supplied by the user. This amalgamation of the two information sets is then used to derive an ideal asset allocation within the portfolio.

The Black-Litterman expected returns are presented below according to Idzorek's presentation in his 2004 work, *"A Step-by-Step Guide to the Black-Litterman Model."*

$$E[R] = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\Pi + P'\Omega^{-1}Q] \quad (3.1)$$

$E[R]$ = Black-Litterman posterior combined return vector ($N \times 1$)

τ = a scalar often referred to as the weight on views when combined with the Ω helps determine the relative weighting between the equilibrium implied excess returns and the views supplied

Σ = covariance matrix of excess returns ($N \times N$)

P = pick matrix identifying which asset for which views will be supplied ($K \times N$)

Ω = matrix of uncertainties surrounding the views expressed as variances ($N \times N$)

Π = implied equilibrium excess return vector ($N \times 1$)

Q = vector of supplied views ($K \times 1$)

The above Black-Litterman expected return vector $E[R]$ can be thought of as a weighted sum of the equilibrium implied excess return vector and the views vector (Idzorek, 2004). The greater the return associated with the view vector, the greater the weighting shift towards the supplied view. The use of reverse optimization for the implied equilibrium excess returns

from historical data provides for a more stable return series (Beach and Orlov, 2007). This new return series, synthesized from investor supplied views and the reference model, is then provided to a mean-variance optimization procedure to determine the new asset allocation weights; incorporating the investor's outlook and the assets held (Meucci, 2010).

The model begins with a vector of implied equilibrium excess returns, typically represented as Π in most specifications of the Black-Litterman expected returns formula. The user then supplies the views vector representing individual views regarding particular assets included in the portfolio. Where views are held, the Black-Litterman portfolio allocation will deviate from the equilibrium reference weights (He and Litterman, 1999). This utilization of implied excess returns from an equilibrium model helps reduce portfolio allocation fluctuations due to shifts in expected return forecast (Satchell and Scowcroft, 2000). Chopra and Ziemba (1993) highlight the importance of mean returns, emphasizing the greater significance in mean estimation errors relative to those in covariance estimation. The authors further suggest a model not heavily reliant in individual mean forecasts, for example an equilibrium model. The equilibrium returns are formulated below through CAPM theory, representing all information available in the capital markets (Bevan and Winkelmann, 1998). In the Black-Litterman methodology, CAPM equilibrium will provide the prior distribution and the views the conditional distribution. The reference model contains assumptions about which variables are random and begins with the assumption of normally distributed returns (Walters, 2014).

$$\Pi = \lambda \Sigma w_{mkt} \tag{3.2}$$

Π = implied equilibrium excess return vector ($N \times 1$)

λ = a risk aversion parameter, according to He and Litterman (1999) a value of 2.5 would correspond to average world-wide risk tolerance

Σ = covariance matrix of excess returns ($N \times N$)

w_{mkt} = market capitalization weights of the assets ($N \times 1$)

Reverse optimization on the market portfolio yields the implied excess equilibrium returns. In order to work backwards towards these implied equilibrium returns one must make the following assumptions: markets are in equilibrium and a representative investor should hold some proportion of the global capitalization-weighted portfolio (Bevan and Winkelmann, 1998). As mentioned above, with these assumptions it is possible to move recursively from observed market-capitalization weights. Da and Jagannathan (2005) suggest calculating market capitalization weights from the market values of the selected assets, which this study adheres to.

The scalar τ , along with the uncertainty matrix Ω , provide a greater degree of difficulty in their determination. Adjustments to the value of τ can be used to calibrate the Black-Litterman results. Lower values indicate less confidence in the supplied views (Beach and Orlov, 2007).

The vector of supplied views, usually denoted by Q or V in the Black-Litterman expected returns formula, is where investors are able to supply their own views regarding the assets held. Were the manager to supply no views, the Black-Litterman result would exactly mimic the reference model portfolio weights (He and Litterman, 1999). The Black-Litterman methodology provides a new series of expected returns which are then used in the determination of an optimal portfolio allocation incorporating the investor's individual views weighted by volatilities and correlations across the assets included. These views may be subjective or determined from statistical data (Walters, 2014). This paper additionally assumes a value in the application of statistically derived views. This study employs absolute views for all assets provided by select GARCH-M processes, further described in Section 4.2. Where views are held, the new Black-Litterman portfolio will deviate from the reference model with the magnitude of the deviations tempered based upon the confidence in the supplied views. Through correlations in the included multivariate structures presented in Section 4.3, these views on asset returns will affect the returns of all assets in the portfolio. The Black-Litterman methodology is capable of incorporating subjective, relative, and conflicting views on future expected asset returns.

The uncertainty matrix Ω measures the degree of confidence in the supplied view or a measure of variance in the views (Zhou, 2009). Two primary assumptions in the original Black-Litterman methodology, independence and non-correlation in the supplied views, lead to the formation of a diagonal matrix. However, according to Duqi, Franci and Torluccio (2014) these assumptions are not mandatory. In this study, GARCH models used for forecasting one-step-ahead views and uncertainties are incorporated with increasing complexity. While two specifications in this paper, the standard GARCH(1,1) and GJR, adhere to the original model's assumption of diagonality. The last models included, those with dynamic conditional correlations, break this relationship. While not diagonal, the full variance-covariance matrix more closely mimics reality in that views should not be expected to be entirely uncorrelated.

This uncertainty matrix aids in the determination of the degree by which the Black-Litterman expected returns vector deviates from the equilibrium implied excess returns. The uncertainty matrix should be inversely proportional to the manager's confidence in the supplied views. A variety of methodologies for determining the uncertainty matrix have been employed: prior variance, confidence intervals, residual factor models, and the Idzorek methodology. The uncertainty matrix has been one of the more difficult and theoretically challenging components of the Black-Litterman model utilization (Walters, 2014).

Next, the covariance matrix of excess returns is discussed, estimated in order to determine the volatility of the capitalization-weighted portfolio. The matrix is most often represented by Σ . Typically the matrix is estimated from the most frequently available data and scaled up, often with exponential weighting (Walters, 2014). According to the methodology of Goldman-Sachs, volatilities and correlations are calculated with the weighted average of daily squared returns (Bevan and Winkelmann, 2005). The covariance matrix is combined with the market capitalization weights, determining the volatility of the capitalization-weighted portfolio. According to Idzorek, users should incorporate the best available estimate for the formation of the matrix of excess returns. As such, this paper selects an *Exponentially Weighted Moving Average* (EWMA) process for the estimation of the covariance matrix and market returns.

The so called *pick matrix*, denoted by P in the Idzorek depiction of the Black-Litterman expected returns formula and most others, is used to select those assets for which views will be supplied. Depending on the manner of the view supplied, the row sum will vary. A row sum of 1.0 indicates an asset for which an absolute view will be supplied while a sum of 0.0 indicates a relative view. This occurs when one asset is expected to outperform another. In this case, the outperforming asset would receive a +1.0 while the underperforming asset receives -1.0. In summation the two cancel. There is no requirement that a view must be specified for each asset nor, as specified previously, that views are not in conflict (Idzorek, 2004).

3.3 Black-Litterman Methodology Limitations

In the above described model found in Equation 3.1, the views vector (V or Q) and the uncertainty matrix Ω typically provide the most difficulty for researchers and practitioners. A number of studies have been performed in an attempt to provide some degree of tractability to these variables.

Beach and Orlov (2007) and Duqi et al. (2014) both investigate the applicability of statistical techniques in generating the utilized investor views. Both works suggest utilization of statistical results may be superior to employing subjective views as GARCH models are able to incorporate many of the stylized aspects of asset returns, namely: volatility clustering, excess kurtosis, asymmetry, mean reversion, autocorrelation in risk, time-varying volatility, leverage effects, and others. These stylistic attributes will be discussed in greater detail in Section 4.1. Both of the above-mentioned works conclude the asymmetric EGARCH-M³ specification to outperform models utilizing a symmetrical dynamic conditional variance process.

³ The *Exponential GARCH* is an asymmetrical GARCH model. For further information, see Nelson (1991).

3.4 Black-Litterman Existing Literature

As two primary sources of inspiration, Beach and Orlov and Duqi et al. investigate the benefits of incorporating a GARCH-M process in the formulation of the supplied views vector and uncertainty matrix in the Black-Litterman methodology. Beach and Orlov deployed an EGARCH-M specification with additional regressors and encountered portfolio performance surpassing portfolios solely reliant on the implied equilibrium excess return and the Markowitz allocation. The authors argue the potential advantage of relying on statistical models for the views supplied rather than subjective expectations.

In further investigation, Beach and Orlov find an increase in financial performance as measured by the Sharpe Ratio as the value of τ is increased. The adjustment in τ leads to a larger proportional weighting in the EGARCH-M derived inputs in the blending of the two information sets. They suggest the parameter could be used as a risk-adjustment factor whereby by τ is altered until a portfolio with the desired risk characteristics is formed. The authors argue this process is superior to optimization in the presence of constraints. Their result finds impressive performance for this risk-reduced Black-Litterman allocation.

Additionally, Duqi, et al. highlight their similar development of the Black-Litterman literature through the incorporation of similar EGARCH-M derived volatility forecasts with additional regressors as an input in determining investor views. They find the included additional regressors useful in explaining asset volatility dynamics. Duqi et al. concludes with findings supporting an increase in financial performance as the portfolio leans more heavily towards the EGARCH-M derived views vector. The authors similarly extoll the benefits of employing computational views, citing a lack of any model capable of explaining investor views. The research emphasizes the use of a diagonal structure on the uncertainty matrix though states the assumption is not mandatory.

Duqi et al. (2014) additionally cite the subjectiveness of τ . The paper, as above mentioned, mirrors results from Beach and Orlov (2007), finding additional financial performance as weight is shifted towards the supplied views, through the inclusion of a larger τ value. More confidence in the implied equilibrium excess return, accomplished with a smaller value of τ is associated with a decrease in the Sharpe Ratio. The authors argue for a τ -value lower than 1.0

as the uncertainty around average returns is lower than that of a single stock. The study calls attention to the large difference between the historical covariance matrix and the estimated.

In additional Black-Litterman research, Fernandes, Ornelas, and Cusicanqui (2011), find strong support for the use of resampling techniques in improving standard Black-Litterman results. The authors cite two primary benefits stemming from the incorporation of resampling: greater diversification with smoother transitions and fewer sudden swings in allocations. A result this paper concurrently finds. The methodology they advise is capable of blending any views source. Their investigation on three dimensions: financial performance, allocation stability, and diversification provided a foundation for the analysis in Section 7.

Zhou (2009) in “*Letting the Data Speak*” further developed the Black-Litterman methodology by incorporating Bayesian learning; exploiting all available information. The innovation provided lies in the linkage between the Black-Litterman model, quantitative asset return models and the wealth of Bayesian decision making findings. The paper melds equilibrium literature with asset research to form a Bayesian prior which is then combined with a prior reflecting the future evolution of the market.

Simonian and Davis (2011) and Giacometti, Bertochi, Rachel, and Fabozzi (2007) investigate robust methods for model misspecifications and more realistic asset return distributions respectively. Simonian and Davis recommend a robust approach, deploying multiple priors around a selected mean excess return and covariance matrix. The presented methodology incorporates the possibility of model misspecification and builds upon standard bootstrap⁴ approaches. Meanwhile, Giacometti et al. further develop the Black-Litterman methodology by incorporating more realistic asset return models: normal, t-student, and stable distributions. In addition to these distribution inquiries the authors include additional risk measures.

Additionally, Palomba (2008) follows a similar procedure involving a VAR process employing the multivariate FDCC⁵. The proposed procedure provides a model for tactical asset allocation utilizing multivariate GARCH estimates.

⁴ Bootstrapping is a method of resampling. For further information, see Efron (1979).

⁵ *Flexible Dynamic Conditional Correlations* model. For additional information, see Billio, Caporin and Gobbo (2006).

While investigating *Markov-Regime-Switching* GARCH⁶ (MRS-GARCH) models, Wang (2010) demonstrates the specification's usefulness when supplied as the Black-Litterman model views vector. During the study, the applied specification provides exceptional returns based on the data analyzed and the portfolio management techniques employed. The paper finds support for the superiority of the MRS-GARCH derived views over the equilibrium implied excess returns and EGARCH-M supplied views.

The above review is presented to highlight recent additional research into the Black-Litterman methodology. As the model continues to increase in popularity, additional research is required to ensure optimal usage.

3.5 Incorporating MGARCH-M Derived Views

As highlighted in Section 3.2, investor supplied views are of critical importance in determining the benefits of the Black-Litterman asset allocation methodology. Any examination of Black-Litterman results is also an examination of the supplied views (Beach and Orlov, 2007). The authors determine the utilized EGARCH-M specification allows for significant advantages, with greater weight on the supplied views providing greater returns (Beach and Orlov, 2007). These benefits, in the form of a higher Sharpe Ratio, are diminished as the value of τ is reduced; indicating a lower degree of confidence around the supplied views. Duqi et al. also extol the expository benefits of deploying statistically derived investor views. Schulmerich (2015) emphasizes the supplied views may stem from quantitative models, fundamental analysis or blind belief.

This paper differs from previous research in its use of the GARCH specification in examining the relationship between the financial performance and allocation stability of the assembled portfolios. Rather than a direct interest in accurate forecasting, this study examines the effects of GARCH supplied views and uncertainties on the risk-to-return relationship relative to the degree of asset weight stability. As such, the accuracy of the GARCH forecast is not a central concern, as a very accurate forecast could lead to greater weight fluctuation which would not necessarily be offset by the increase in return. These asset weight fluctuations must be

⁶ MRS-GARCH, a multivariate model, allows for volatility shifts in discrete time. See Dueker (1997).

considered alongside the risk-to-return properties of the portfolio. The reduction in portfolio allocation fluctuations must not come at the price of significantly lower returns. Figure 3.1 presented below demonstrates the stabilizing effect provided by the Black-Litterman methodology where no views are provided. This contrasts sharply with the graph presented in Figure 2.1.

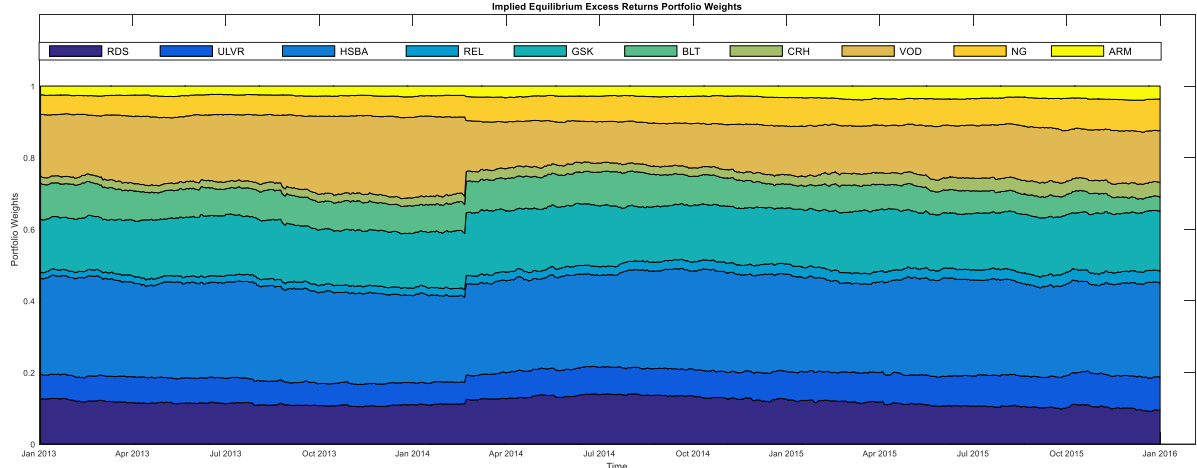


Figure 3.1. Evolution of asset allocations where no views are supplied

4 Econometric Volatility Models

As described in Section 3.5, the presented study will meld the discussed Black-Litterman methodology with investor views and uncertainties supplied by univariate and multivariate GARCH-M specifications. The following section presents the evolution of the necessary volatility forecasting techniques deployed in the study. A summary of the stylized attributes associated with asset return volatility is first presented.

4.1 Stylized Volatility Attributes

In order to be suitable for forecasting, a desired model must be able to integrate a variety of stylistic patterns exhibited by financial data (Engle and Patton, 2000; Poon and Granger, 2003). These behaviors include: asymmetrical innovations, mean-reversion tendencies, persistence in volatility, and the ability of letting exogenous variables influence volatility. As volatility models are largely utilized in the forecasting of future volatilities, the ability to incorporate these tendencies is paramount (Engle and Patton, 2000).

Traditional volatility models had previously assumed conditional volatility to be equally affected by positive and negative errors, that is; shocks impact the volatility forecast symmetrically. However, a leverage effect originally examined by Black (1976), has been documented in subsequent research; Christie (1982), Nelson (1991), Glosten et al (1993) and Engle and Ng (1993) for example.

This suggests evidence supporting the inclusion of asymmetric effects, it is however possible that no leverage effect exists and that positive and negative shocks equally affect the conditional forecast. The popularity and support of the GARCH(1,1), which is incapable of incorporating a leverage effect, lends credence to this line of reasoning (Engle and Patton, 2000). The GARCH(1,1) is included in this study due to its simplicity and popularity. In 1987, Bollerslev argued the GARCH(1,1) is often sufficient as it adequately fits time series data.

Mean-reversion in the volatility process implies that volatility has some long-term mean from which the forecast may stray but will return to in time. This ensures that long-term volatility forecasts converge to some specific level (Engle and Patton, 2000).

Volatility clustering in the return series, or persistence, was first posited by Mandelbrot (1963). Such clustering implies that large swings in volatility are followed by additional large swings while smaller movements in volatility are succeeded by additional smaller movements (Engle and Patton, 2000).

Additionally, the ability of a model deployed in forecasting volatility to incorporate exogenous variables is a desirable attribute, as these exogenous variables may influence time-series volatility. Engle and Patton emphasize that it would be most unusual for financial assets to be unaffected by movements in the markets.

The existence of excess kurtosis, or heavy tails, is a well-known attribute of financial asset returns. With empirical findings supporting a kurtosis from four to fifty, the normality assumption regarding the innovations process may be weak (Engle and Patton, 2000; Poon and Granger, 2003).

In their work, Poon and Granger hypothesize that a GARCH model will forecast well in some periods and in others perform poorly. The determining component in the performance is the perturbations in the underlying variables.

4.2 Univariate Volatility Models

4.2.1 Autoregressive Conditional Heteroscedasticity

The ARCH or “*Autoregressive Conditional Heteroscedasticity*” model was introduced in an attempt to address the dubious assumption of a constant one-period-ahead forecast of conditional variance, which had been employed in previous econometric work (Engle, 1982).

An ARCH process is serially uncorrelated with dynamic conditional variance, constant unconditional variance, and an expected value of zero. Information regarding the one-period forecast variance is provided in the recent past.

$$\varepsilon_t = \sigma_t z_t$$

In the above formulation, z_t is a white noise process.

The ARCH(q) specification is demonstrated subsequently:

$$\sigma_t^2 = \alpha_0 + a_1 \varepsilon_{t-1}^2 + \dots + a_q \varepsilon_{t-q}^2 = \alpha_0 + \sum_{i=1}^q a_i \varepsilon_{t-i}^2 \quad (4.1)$$

Where $\alpha_0 > 0$ and $a_1 \geq 0, i > 0$.

The ability of the ARCH process to provide accurate volatility forecasts does vary between periods, though for econometric use it does exhibit attractive features (Engle, 1982). According to Bollerslev (1986), one such feature is the model's ability to permit the conditional variance to adjust through time as a function of past error terms. Further advancements in the field of volatility forecasting such as GARCH are discussed below.

4.2.2 ARCH-In-Mean

Engle, Lilien, and Robins (1987) further developed the ARCH literature with the inclusion of the conditional variance as a determinant in the mean equation, introducing the ARCH-M. In this specification, fluctuations in conditional variance directly affect the expected asset return.

The ARCH-M model is given by:

$$y_t | \Psi_{t-1} \sim N(x_t' \beta + \delta \sigma_t^2, \sigma_t^2) \quad (4.2)$$

Where y_t represents the conditional return, Ψ denotes the information available up until time $t - 1$ and x_t' is a vector of explanatory variables.

The above structure breaches the block-diagonality between the conditional variance parameters and the conditional mean, requiring the two sets to be estimated in unison to attain asymptotic efficiency. In a similar manner, non-linear functions of the conditional variance may also be incorporated into the conditional mean equation (Bollerslev, 2008).

4.2.3 Generalized Autoregressive Conditional Heteroscedasticity

Bollerslev (1986) extended the ARCH framework into a model of “*Generalized Autoregressive Conditional Heteroscedasticity*” (GARCH), where a more suitable lag structure is proposed.

The GARCH(p, q) is presented below.

$$\varepsilon|\Psi_{t-1} \sim N(0, \sigma_t)$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^q a_i \varepsilon_{t-i}^2 + \sum_{i=1}^p \beta_i \sigma_{t-i}^2 \quad (4.3)$$

Where $p \geq 0$, $q > 0$, $\alpha_0 > 0$, $a_i \geq 0, i = 1, \dots, p$ and $\beta_i \geq 0, i = 1, \dots, q$.

The innovation process ε_t is dependent on the information set Ψ which contains all information of volatility up to time $t - 1$.

The primary distinguishing feature moving from the ARCH to the GARCH specification is the ability of the GARCH model to assimilate lagged conditional variance into the model. In contrast, the ARCH model assumes the conditional variance to be a linear function of past variances.

This study includes the GARCH(1,1) specification as one method for forecasting the diagonal elements in the correlation family of multivariate GARCH models which will be presented in Section 4.3. The GARCH(1,1) is provided below.

$$\sigma_t^2 = \alpha_0 + a_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \quad (4.4)$$

Where $\alpha_0 > 0$, $a_1 \geq 0$, $\beta_1 \geq 0$ (Bollerslev, 1986).

4.2.4 GARCH-In-Mean

Engle and Bollerslev (1986) further evolved the GARCH framework, developing a model similar to the ARCH-M. The introduced “*GARCH-in-mean*” (GARCH-M) includes a simultaneously estimated mean equation as was included in the ARCH-M presented in 4.2.2.

$$R_t = a + b\sigma_t^2 + \varepsilon_t \quad (4.5)$$

In the above equation the conditional variance σ_t^2 is introduced into the formulation of the conditional mean, R_t (Nelson, 1991). The connection between the conditional variance and conditional mean is crucial for the selected methodology, as this equation will yield the expected returns supplied as investor held views to the Black-Litterman model. A similar conditional mean specification will be included for each GARCH model deployed.

The GARCH-M model has been used previously under varying conditions in an attempt to evaluate return volatility and its effect upon the expected return. Below a selection of GARCH-M research literature is presented.

Brewer, Carson, Elyasiani, Mansur and Scott (2007) use the GARCH-M model on monthly stock returns of life insurers while investigating interest rate sensitivity. They apply the model in assessing how insurer stock returns depend on the feedback effect between returns and

volatility. The paper suggests a negative parameter value for the variance is theoretically hard to justify, while a value of zero is often found in the banking sector as well as others. This finding reduces the GARCH-M to an ordinary GARCH(1,1), implying a lack of evidence supporting a feedback effect between asset return and volatility.

Devaney (2001) applies the GARCH-M to real estate investment trusts data in an attempt to assess the return generating process. The study finds the conditional variance parameter may differ heavily between periods, dependent upon the period characteristics, be they tranquil or volatile.

Elyasiani and Mansur (1998), like Brewer et al., investigate the relation between volatility of bank returns and the interest rate, employing the same GARCH-M model. Elyasiani and Mansur argue that the GARCH-M model satisfies important criteria and aids in the mitigation of important problems, including: estimation errors and bias, heteroscedasticity, as well as incorporating time variations in the volatility. Furthermore, they stipulate the importance of incorporation of the mean equation. By doing so, the GARCH-M model portrays the vital connection between volatility and expected returns. Neuberger (1994) highlights the appeal of the inclusion of volatility into the mean equation. As investors are not indifferent towards volatility, fluctuations in the dispersion of stock returns should influence the risk premium required by investors (Elyasiani and Mansur, 1998). As return volatility fluctuates more heavily today than previously, the rationale behind the inclusion of a volatility term in the mean equation is strengthened. This argument is even more apparent in those sectors which are highly leveraged (Elyasiani and Mansur, 1998).

Elyasiniani and Mansur argue the utilization of the GARCH-M over the standard GARCH specification provides two primary advantages. Firstly, the GARCH-M process nests both ARCH and GARCH and is thus able to test for both the ARCH and GARCH effects as well as the additional influence in the mean equation. Additionally, the GARCH-M allows for the feedback effect of volatility to be captured in the expected return equation. This is opposed to the standard GARCH in which it is implicitly held constant during the sample period. Should the volatility parameter be significant, both the volatility and the mean equation may differ between calm and turbulent periods. This intuitive feature is essential in forecasting accuracy, as the arbitrary assumption that volatility does not influence the return series may be distortive. The results of Elyasiani & Mansur (1998) lend support to suitability

of the GARCH-M, as they indicate the importance of the volatility process in determining the expected stock return in the bank sector.

Deploying the GARCH(2,2)-M model, Galido and Khanser (2013) investigate how market returns are affected by natural disasters. While they fail to discover such a link, the authors argue that the GARCH(2,2)-M model does appear a good fit for similar research.

In their 2000 work, Grier and Perry employ a GARCH-M specification in a nested hypothesis test. The authors examine four hypotheses; estimating the conditional means, variances, and covariances in inflation and output growth, investigating effects of real and nominal uncertainty on average inflations and output growth. They conclude the GARCH-M to be a good fit for the included tests.

Panait and Slavescu (2012) make use of the GARCH-M model on the Romanian stock market, comparing the volatility structure of different trading frequencies. In their research, the GARCH-M was unable to confirm a link between increasing volatility and higher returns. This lack of confirmation may lie in the statistical insignificance of the volatility parameter during the sample period. Additionally, their results show the GARCH-M to better fit the weekly and monthly data as opposed to the more frequent daily data. The authors argue that the use of the GARCH-M model is appropriate as expected return is correlated with market risk and thus variance should be incorporated in the mean equation.

Lastly, while investigating the Istanbul stock market for evidence of a risk-return-volume relationship, Salman (2002) utilizes the now familiar GARCH-M. His findings indicate a tenacious daily return volatility, varying over time.

4.2.5 GJR

Developed through the work of Glosten, Jagannathan and Runkle (1993), the GJR model was designed to incorporate seasonal volatility patterns and asymmetry. While these aspects allow for a model similar to the also popular EGARCH, the GJR model does differ in its construction, with a GJR(1,1) given by

$$\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \gamma \varepsilon_{t-1}^2 I(\varepsilon_{t-1} < 0) + \beta \sigma_{t-1}^2 \quad (4.6)$$

With $\omega > 0, \alpha \geq 0, \beta \geq 0$ and $I(\cdot)$ denoting the indicator function which equals one when ε_{t-1} is negative and zero otherwise.

The GJR model stresses the importance of the leverage effect by including a leverage parameter in its specification; implying in theory a suitability for forecasting (Ramasamy and Munisamy, 2012). In the above specification, the parameter γ is the determinant of the leverage effect. The time t conditional variance will be higher (lower) following a positive (negative) shock if the γ parameter is positive (negative) (Fornari and Mele, 1996). In a typical study, γ is found to be positive when estimating the GJR model. This finding would indicate that volatility is increasing to a lower degree following positive innovations relative to negative, indicating the presence of a leverage effect (Bollerslev, 2008).

While research into the model has provided mixed results, Engle and Ng (1993) find the GJR superior in explain asymmetries in stock returns. Fornari and Mele find the performance of the GJR specification to vary greatly as a function of the specific country examined.

In graduate studies work, Jiang (2012) also examines the performance of a selection of GARCH models: GARCH, EGARCH, and GJR. While the results are mixed, the paper concludes that different varieties of the GJR specification outperform the others included the study. While forecasting Value-At-Risk, Su, Huang and Lin (2011) find the GJR

specification to be well suited. However, the GARCH-M was also included and performed well, suggesting an absence of a leverage effect in their study.

Additional research has uncovered even weaker support for asymmetric GARCH models. Ramasamy and Munisamy (2012) examine the performance of the GARCH, EGARCH, and GJR models, and were unable to determine the most effective model in estimating exchange rate dynamics. The study concludes that the incorporation of the leverage effect, found in the EGARCH and GJR specifications, does not substantially improve the results. Although they do conclude the GJR does outperform the EGARCH specification.

Villar (2010) also explores the GJR specification. He argues the parameter restrictions, necessary to satisfy finite kurtosis, too heavily constrain the ability of the GJR to capture the leverage effects. In this regard, the EGARCH provides more flexibility. Despite the popularity of the GJR model in empirical studies, the specification lacks the flexibility required to capture the financial return asymmetry.

In forecasting volatility, Brownlees, Engle, and Kelly (2011) find one-step-ahead volatility forecasts to function accurately during the recent financial crisis. The paper cited the strengths of a simple asymmetric GARCH specification, the TGARCH⁷, and suggested frequent re-estimation with long-sample sizes.

4.3 Multivariate Volatility Models

4.3.1 Multivariate Rationale

The late 1980s ushered in the development of the multivariate GARCH (MGARCH) specifications, with significant progress made from the late 1990s onwards (Bauwens, Laurent and Rombouts, 2006).

Bauwens et al. argue for the deployment of multivariate volatility models. The authors cite the more or less established fact that financial asset volatility does not fluctuate independent

⁷ *Threshold* GARCH. See Zakoian (1994).

of other assets. This points to the usefulness of the multivariate approach of explicitly incorporating a multivariate framework when considering volatility rather than employing separate univariate volatility specifications for individual assets. The integration of the multivariate structure should improve portfolio decision making and composition.

The use of a multivariate volatility specification can provide the user with rich information sets when investigating the dynamics of multiple assets. The multivariate model demonstrates how volatility in one asset or asset class moves in relation to the volatilities in the other assets and if the effect is direct (by conditional variances) or indirect (through conditional covariances). The manner and magnitude by which shocks in one market affect other markets can be witnessed. Additionally, the static or dynamic structure of the covariances can be captured. The above benefits are gained through the utilization of a multivariate framework (Bauwens, Laurent and Rombouts, 2006).

There are currently three primary approaches to the incorporation of a multivariate volatility framework: direct generalization of univariate models, such as Diagonal VEC and BEKK, linear combinations of univariate models, such as; Factor GARCH and Principal Component GARCH, and lastly; nonlinear combinations of univariate models, such as CCC (*Constant Conditional Correlation*) and DCC (*Dynamic Conditional Correlation*) (Lin, 2006).

The primary drawback to the deployment of the multivariate GARCH framework lies in its complexity. The models require the estimation of a large number of parameters which become increasingly difficult to estimate as the number of assets is increased. While this is a common issue in handling multivariate time-series, the process becomes even more difficult when including GARCH models as the conditional variance matrix must be inverted in the Gaussian likelihood-based estimation procedures. The primary means of circumventing this problem has been to constrain the model structure or other estimation criteria or to forcibly reduce the number of parameters required for estimation (Francq and Zakoian, 2014).

In 1990, Bollerslev proposed the CCC model to help alleviate this estimation problem without constraining or hampering the model. This specification and its extension, the DCC model by Engle (2002), are estimated through a two-step procedure. The first stage consists of univariate GARCH estimation, obtaining the conditional variances for the diagonal, while the

second stage utilizes the standardized residuals from the first stage in the estimation of the off-diagonal correlations (Francq and Zakoian, 2014).

Clements, Scott, and Silvennoinen (2015) find the properties of the CCC model well suited for periods of market tranquility, particularly when operating with large portfolios. Although the CCC model is intuitively appealing, the restriction of the conditional correlations to a constant has been found empirically to be too restrictive (Silvennoinen and Teräsvirta, 2009), shifting this study's focus towards the dynamic conditional correlations model

4.3.2 Constant Conditional Correlation Model

While this study employs the DCC framework developed by Engle (2002), itself an extension on the earlier work of Bollerslev (1990), the foundational CCC is presented below for completeness. In this formulation of the MGARCH model the diagonal of the covariance matrix follows a univariate GARCH while the off-diagonal elements are obtained by multiplying the conditional standard deviations of the asset returns by a constant correlation coefficient (Bollerslev, 1990).

The CCC model is given by:

$$H_t = D_t R D_t = (\rho_{ij} \sqrt{\sigma_{iit}^2 \sigma_{jjt}^2}) \quad (4.7)$$

with

$$D_t = \text{diag}\{\sigma_{11t} \dots \sigma_{NNt}\}$$

and σ_{iit}^2 is specified by any univariate GARCH process. And finally,

$$R = (\rho_{ij})$$

is a positive definite matrix with $\rho_{ii} = 1, \forall i$.

In the CCC model, R is the matrix containing the constant conditional correlations, ρ_{ij} (Bollerslev, 1990; Bauwens et al., 2006).

4.3.3 Dynamic Conditional Correlation Model

Engle (2002) further developed upon the CCC model by allowing the conditional covariances to be time-varying. The DCC is defined as follows:

$$H_t = D_t R_t D_t \quad (4.8)$$

where

$$R_t = \text{diag}(q_{11,t}^{-\frac{1}{2}} \dots q_{NN,t}^{-\frac{1}{2}}) Q_t \text{diag}(q_{11,t}^{-\frac{1}{2}} \dots q_{NN,t}^{-\frac{1}{2}})$$

And the symmetric positive definite matrix $Q_t = (q_{ij,t})$ process is:

$$Q_t = (1 - \alpha - \beta) \bar{Q} + \alpha u_{t-1} u'_{t-1} + \beta Q_{t-1} \quad (4.9)$$

with $u_{it} = \varepsilon_{it} / \sigma_{iit}$ (Bauwens et al., 2006).

The CCC of Bollerslev and the DCC of Engle differ only in the behavior of R_t . In the DCC, the matrix of conditional correlations is permitted to vary over time (Engle, 2002).

The matrix Q_t is converted to a matrix of correlations. An unfortunate necessity, in order to ensure the positive definiteness of R_t , is the restriction that all conditional correlations follow the same dynamics through the scalars α and β (Bauwens et al., 2006).

The primary advantage of deploying the DCC stems from its estimation flexibility. In contrast to other MGARCH models, correlations models, such as the DCC, can be estimated through a two-stage procedure. Bauwens et al. argue that the flexibility in the conditional correlation structure allows for the use of more advanced univariate GARCH models in the estimation of conditional volatility. This way, the number of parameters needed to be estimated is greatly reduced (Engle, 2002). Engle shows that the DCC usually performs well despite its relative simplicity, beating the other models of interest in estimation precision. This method allows for the estimation and forecasting of very large covariance matrices.

The two-step estimator is found to be both consistent and asymptotically normal. Additionally, ensuring the positive definiteness of the covariance matrix is trivial as it only requires the same restrictions as the univariate GARCH (Engle and Sheppard, 2001).

Laurent, Rombouts and Violante (2009) rank selected multivariate models based on their prediction of out-of-sample conditional covariance matrices. The authors conclude the DCC-GARCH(1,1) and DCC-GJR to outperform almost all other models, with the exception of the CCC counterparts which in their study yielded similar results.

5 Data

In the following analysis, ten stocks from the FTSE 100 are selected based on industry sector and market capitalization. Datastream was utilized to collect the price series beginning in January 2006 and ending December 2015. Where the largest asset by market capitalization did not have a sufficiently large price series the next largest security based on market capitalization was selected. Thus, ten years of price and market capitalization observations are obtained for the following securities: RDS, ULVR, HSBA, REL, GSK, BLT, CRH, VOD, NG, ARM. These assets were then combined to form the market portfolio.

Name	RDS	ULVR	HSBA	REL	GSK
Sector	Oil & Gas	Consumer Goods	Financials	Consumer Services	Healthcare
Name	BLT	CRH	VOD	NG	ARM
Sector	Basic Materials	Industrials	Telecommunications	Utilities	Technology

Table 5.1. Assets included in the portfolio sorted by industrial category

Before starting the analysis, dates corresponding to UK banking holidays were first removed. If included, these observations would indicate a higher number of zero returns than is actually observed as these zero returns indicate a lack of trading rather than an actual zero return. Excess returns are then calculated from the price series where the short term UK three-month Treasury bill is used as a proxy for the risk-free rate.

The included securities were chosen to approximately represent the primary industrial sectors of the FTSE 100. Originally this study intended to utilize these sectors in the formation of the market portfolio rather than the individual assets, however insufficient was available. In

response, the largest assets in each sector with sufficient data were selected as a representative. The inclusion of a larger number of assets should not have a significant effect as each asset would be included in one of the ten FTSE 100 industrial sectors. By including the largest security in each industry the work should capture the largest portion of asset dynamics.

The investigation makes use of a rolling window forecast consisting of 756 days. Thus the window used for estimation contains 1,771 observations. This corresponds to a roughly seven year estimation period and three year forecast. The relative proportion of estimation period and the one-step-ahead forecast length mirrors that of Clements, Scott, and Silvennoinen (2015) in their analysis of equicorrelations in portfolio allocation. While distant observation may influence and bias estimates, Brownlees et al. (2011) recommend using the longest estimation window possible.

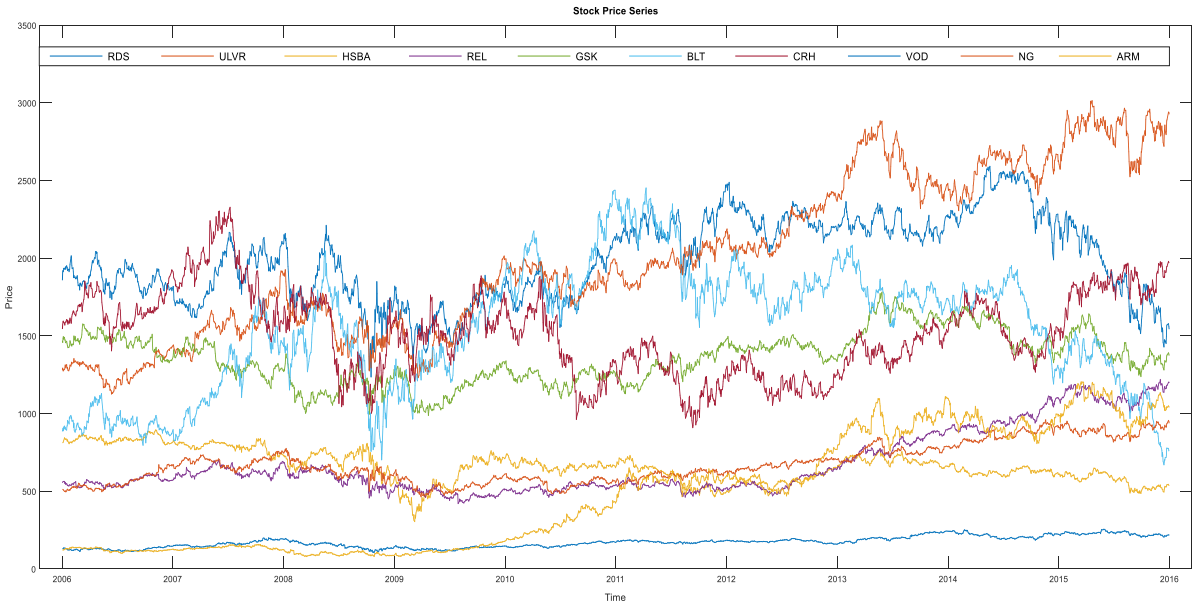


Figure 5.1. Evolution of asset price series during the sample period

6 Methodology

6.1 Maximum Likelihood Estimation

As maximum likelihood is the means by which the previously discussed GARCH models will be estimated, a brief introduction to maximum likelihood is necessary. First introduced by R. A. Fisher in 1922, maximum likelihood has become one of the primary tools in statistical analysis and will be utilized in the estimation of the necessary GARCH parameters used in the rolling forecast (Aldrich, 1997).

As the GARCH models in question are no longer in a traditional linear form, estimation is typically carried out by maximum likelihood. Conceptually, maximum likelihood estimation is a means by which to establish those distributional parameters which maximize the likelihood of encountering the observed sample. This amounts to finding those parameter values which are most likely considering the data at hand. Maximum likelihood estimation is typically done by numerical methods rather than by maximization of the function analytically as the latter approach becomes more difficult when the model proposed is overparameterized or complex.

The estimation procedure begins by specifying a likelihood function which represents the joint probability density of the sample under investigation. This joint probability density function can typically be written as a product of the individual likelihood functions. As the resulting maximization problem would be difficult to solve with respect to the included parameters, the logarithm typically is used to transform the previously multiplicative model into an additive formulation. As the transformation is monotonic, the two functions will be optimized at the same values. Below an example of the joint probability density or probability mass function of the independent observation sample y conditional on X is presented.

$$f(y_1, \dots, y_N | X; \theta) = \prod_{i=1}^N f(y_i | x_i; \theta)$$

With the corresponding likelihood function for the observations being the inverse of the above expression.

$$L(\theta|y, X) = \prod_{i=1}^N L_i(\theta|y_i; x_i) = \prod_{i=1}^N f(y_i|x_i; \theta)$$

The maximum likelihood estimator for the parameter vector is thus found by the solution to the expression below.

$$\max_{\theta} \log L(\theta) = \max_{\theta} \sum_{i=1}^N \log L_i(\theta)$$

In theory, the new log-likelihood function is differentiated with respect to the model parameters. In all but the most simple and known cases, this will be done numerically whereby an algorithm searches for an optimization. In the estimation of GARCH models by maximum likelihood the possibility of local optima or a solution where the log-likelihood is flat around the optimum are known pitfalls.

As an estimation technique, maximum likelihood is associated with a number of asymptotic properties. Assuming the likelihood function is properly specified, the maximum likelihood estimator can be shown under weak regularity conditions to be consistent, asymptotically normal, asymptotically efficient, and asymptotically unbiased.

The primary drawback associated with maximum likelihood estimation is the necessary specification of the functional form of the distribution generating the sample observations. While it is conceivable to challenge the normality assumption regarding the errors terms, this study will nonetheless employ a normal distribution in the estimation procedure as is commonly done. This decision is supported through Brownlees et al. (2011) who finds no additional benefit in utilizing more complex distributional assumptions in place of the typical Gaussian in forecasting volatility.

6.2 GARCH-M Forecasting

The calculation of the user supplied views vector Q and the uncertainty matrix Ω is next examined. In the univariate GARCH specifications these inputs are jointly estimated by maximum likelihood in R-Studio through the use of the “rugarch” package, while the estimation of the full covariance matrix supplied as the uncertainty matrix in the Black-Litterman multivariate formulation is handled in the “rmgarch” package. Both packages were constructed by Alexios Ghalanos. In each case, a GARCH specification is created and supplied to a wrapping function incorporating estimation and the one-step ahead rolling forecast. The views vector in the MGARCH specification is taken from the corresponding univariate forecast. As mentioned in Section 5, Brownlees et al. suggest incorporating the longest estimation period possible to yields the best results.

As specified previously, the univariate inquiry includes the standard bearer GARCH(1,1) as well as an asymmetric GJR, similar to Beach and Orlov (2007) and Duqi et al. (2014). The variance forecasts are placed into the diagonal uncertainty matrix while the mean forecast is inserted into the views column vector.

The above process is replicated in the multivariate investigation. The primary difference between the univariate and multivariate implementations occurs in the uncertainty matrix. While the aforementioned univariate models more closely mimic the original assumption of the Black-Litterman model; that views are independent of one another, this study aims to investigate the effects of the inclusion of greater information in the full variance-covariance uncertainty matrix.

The multivariate analysis includes the DCC-GARCH(1,1) and the DCC-GJR. These two specifications are calculated in a two-stage process whereby first a univariate GARCH process is estimated for the diagonal and subsequently correlations are modeled on the off-diagonals. As mentioned in Section 4.3, the resulting specification is much easier to estimate relative to earlier more complex multivariate models. Though the variance-covariance matrix is estimated by a multivariate GARCH process, the supplied views vector remains identical to that of the univariate specification.

6.3 Black-Litterman Portfolio Allocation

The following section walks through the calculation of the required elements in the Black-Litterman expected return formula which will later be optimized in the presence of commonly imposed constraints. The analysis begins with the calculation of the market capitalization weights. In this example, each of the ten securities selected to represent the market is taken from the FTSE 100; as each asset is presented in a market capitalized form and equally scaled, the new market portfolio capitalization will be the sum of the individual asset market capitalizations. With this information, the weighting of each asset can be calculated simply as the market capitalization of each individual asset divided by the sum of the individual market capitalization.

The previously presented time series of returns is used to calculate the historical covariance matrix, as well as the expected market returns utilized by the CAPM formulation. The calculations are done by means of an EWMA model specification formulated in MATLAB⁸ in accordance with RiskMetrics, with the optimal decay factor for daily observations set to $\lambda = 0.94$, as per extensive testing by RiskMetrics (Longerstaey and Spencer, 1996).

$$\mu_t = \lambda\mu_{t-1} + (1 - \lambda)R_{t-1} \quad (6.1)$$

$$\sigma_t^2 = \lambda\sigma_{t-1}^2 + (1 - \lambda)\varepsilon_{t-1}^2 \quad (6.2)$$

$$\sigma_{xyt} = \lambda\sigma_{xyt-1} + (1 - \lambda)R_{xt-1}R_{yt-1} \quad (6.3)$$

In the above, Equation (6.1) is used to calculate the expected return on the market, (6.2) is used to calculate the historical variance, and (6.3) determines the covariances.

⁸ All MATLAB code available upon request.

The EWMA process ensures that more recent observations hold the most influence in the estimation procedure. This is an appealing property, as those variance-covariances and market returns occurring more recently are likely to have greater influence on short term projections. The EWMA formulation can also be thought of as a special case of the GARCH(1,1) with $\omega = 0$, $\alpha = 1 - \lambda$ and $\beta = \lambda$ (Bollerslev, 2008). Thus the model is not only in line with financial theory but also with the investigation of GARCH processes in the Black-Litterman methodology.

The CAPM portfolio is then calculated using the previously obtained EWMA covariance matrix and market return, along with the risk-free rate. This portfolio will provide the benchmark with which to compare the different Black-Litterman portfolios provided by the selected GARCH-M forecast models. The CAPM benchmark portfolio then undergoes a mean-variance optimization routine created in MATLAB along with the recommended constraints. Additionally, the routine includes those restrictions most common in portfolio allocation methodologies: restricting short and leverage positions, and full investment.

With the EWMA covariance matrix, the implied equilibrium excess returns can be ascertained. This term is found as an analytical solution to the reverse optimization. It is this return vector that provides the large stabilizing benefits associated with the Black-Litterman model. The subsequent investigation will determine whether these benefits may be further increased in accordance with a number of GARCH specifications.

With all required Black-Litterman variables determined, Equation 3.1 is utilized. The resulting synthesis, reliant on the two information sets, then yields a vector of Black-Litterman expected returns.

The resulting expected return vector is then placed into the above described mean-variance optimization routine, along with the recommended constraints, resulting in an optimal Black-Litterman portfolio allocation. The resulting portfolios are then analyzed and compared against one another and the CAPM benchmark portfolio. In this body of research, special focus will remain on weight fluctuations across time in each GARCH specification; both the frequency and magnitude of the deviation will be examined. Assumptions are made on the transaction cost structure which allows for consideration of the long-term financial effects of

portfolio rebalancing. Additionally, these allocation fluctuations are compared with the financial performance of the portfolio in question.

7 Results & Analysis

7.1 Portfolio Allocation and Stability Analysis

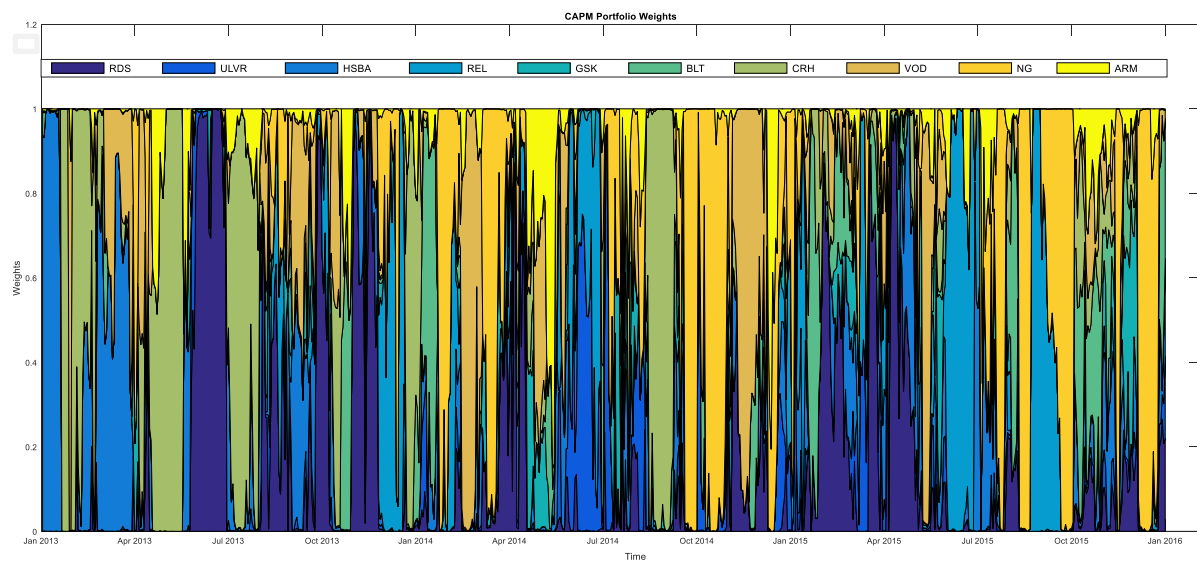


Figure 7.1. Evolution of CAPM asset allocation within the portfolio after imposing commonly utilized constraints

The above figure demonstrates the evolution of the optimal portfolio allocation over the 756 day rolling forecast period in accordance with CAPM. Common limitations of Markowitz mean-variance optimization can be observed. The recommended allocations are typically non-diversified, exhibiting large weights in a single asset; a corner solution. In addition, as expected, asset weights are highly volatile.

After visually demonstrating two of the more common drawbacks associated with mean-variance optimization, the selected GARCH models are next presented. The following section moves through each GARCH specification in increasing complexity.

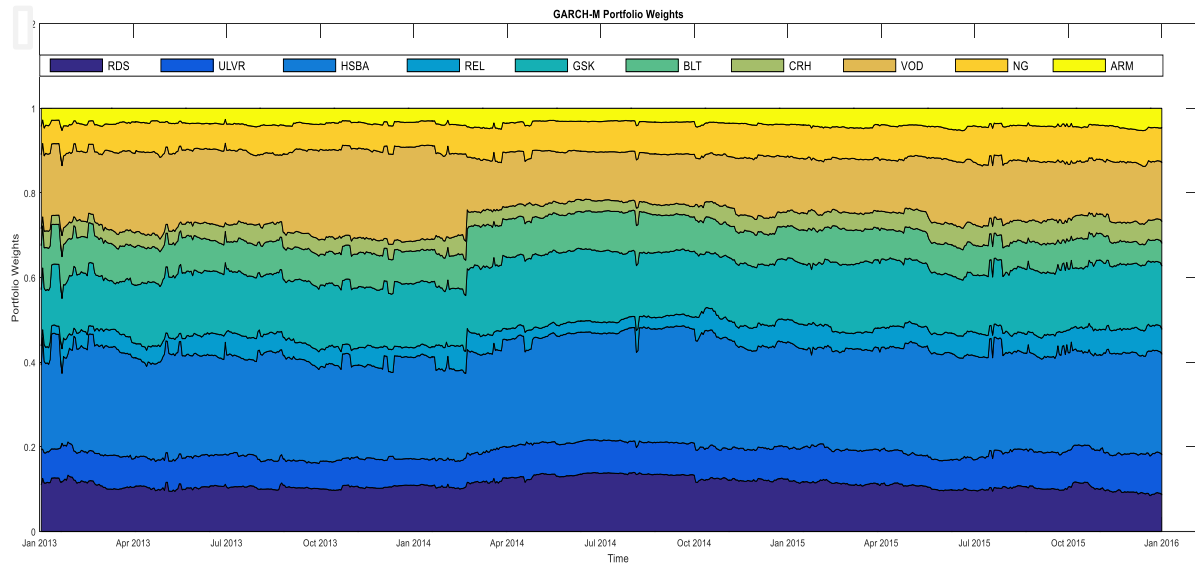


Figure 7.2. Evolution of Black-Litterman allocation with GARCH(1,1)-derived views

The GARCH-M is first presented. As can be observed in the area graph above, the Black-Litterman methodology with GARCH-M derived views provides a significant reduction in weight volatility relative to the previously presented CAPM allocation. As can be seen in each of the following area graphs; when compared with the previously presented evolution of the portfolio allocation in the absence of views, presented in Figure 2.1. The great stabilizing effect stems from the inclusion of the equilibrium implied excess returns.

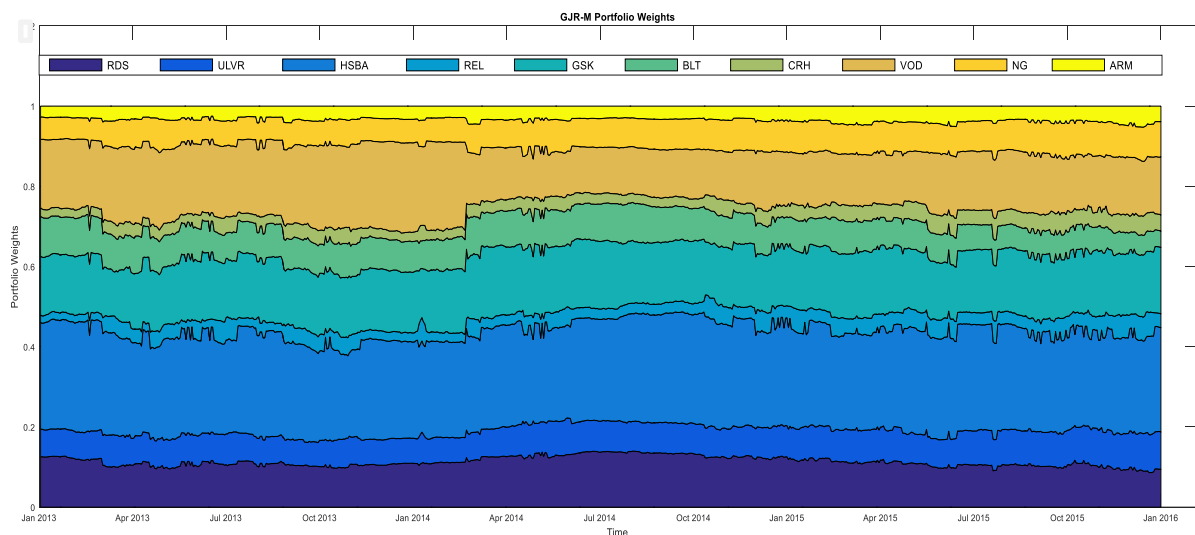


Figure 7.3. Evolution of Black-Litterman allocation with GJR-derived views

Next the asymmetric GJR-M is shown. A slight increase in allocation volatility can be observed. While the GJR provides a more complex GARCH structure, these effects are largely muted by the heavy anchoring effects of the implied equilibrium excess return.

The following multivariate GARCH specifications exhibit drastically different evolutions. While the weights do still maintain a similar proportion as those in the univariate specifications, the increase in allocation volatility is stark. With the introduction of a multivariate framework, covariances are now influential in asset allocation. The incorporation of these multivariate specifications deviates from much of previous research where views are assumed to be independent and the uncertainty matrix is diagonal. This paper explores the multivariate framework as in reality views are not likely to be independent. Previously discussed literature has highlighted the intuition that block-diagonality in the uncertainty matrix is not a mandatory requirement but does simplify the implementation.

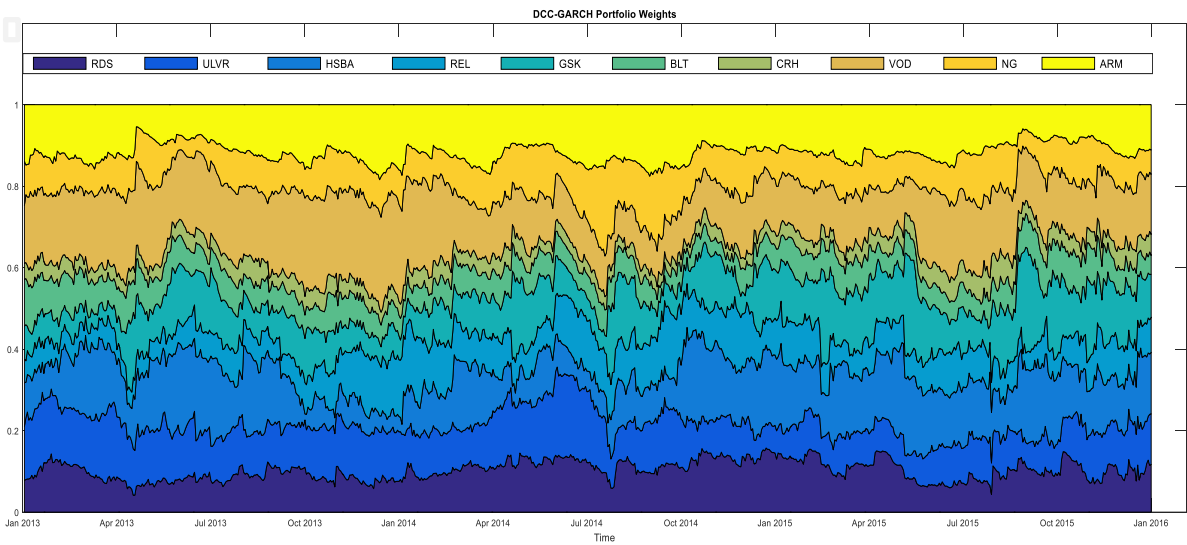


Figure 7.4. Evolution of Black-Litterman allocation with DCC-GARCH-derived views

The above graph of the DCC-GARCH exhibits notable divergences from those of the previous univariate models; a dramatic increase in weight volatility. In accordance with previous discussions in Section 2.5, this increase in weight fluctuations would appear a drawback when compared with the univariate specifications, especially in the face of very real transaction costs. Before any conclusion is to be made, the financial performance of the portfolios put forth must be considered. The increase in weight volatility need be compensated for through increased financial performance.

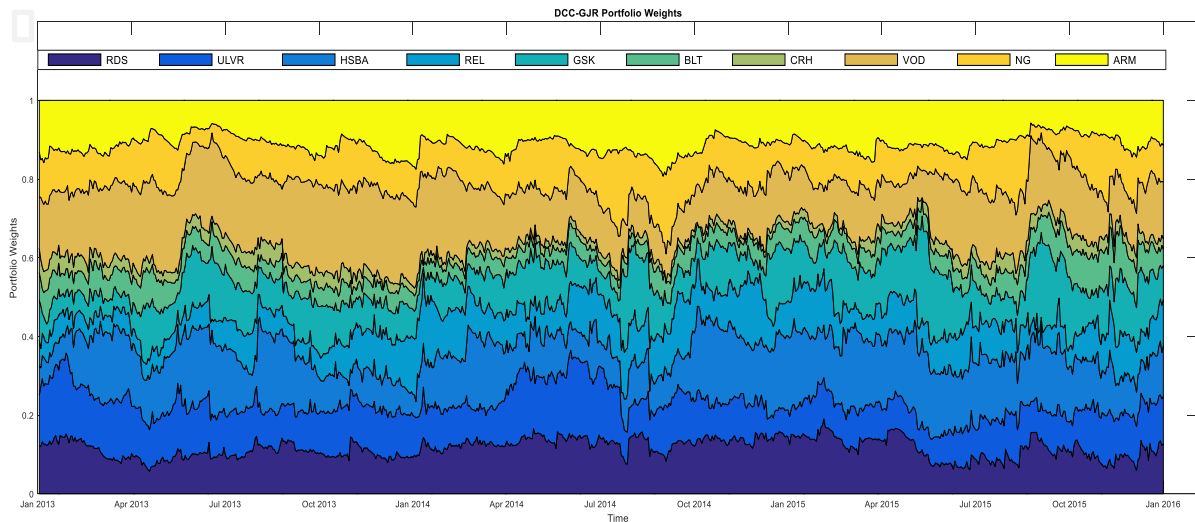


Figure 7.5. Evolution of Black-Litterman allocation with DCC-GJR-derived views

As anticipated, the DCC-GJR provides an even more volatile allocation evolution. The increased proportion held in VOD is an interesting divergence from the DCC-GARCH.

In accordance with previous research, the Black-Litterman model has been shown to provide a significant increase in weight stability when compared to the CAPM allocation. The large portion of this stability, in accordance with theory, stems from the incorporation of the equilibrium implied excess returns as an anchor.

This paper expands upon previous Black-Litterman research, through the incorporation of computational inputs into the Black-Litterman methodology. While the ability to incorporate any information source into the Black-Litterman expected return is advantageous, a search for beneficial computation sources adds tractability to the methodology.

While the findings presented do support previous research by Beach and Orlov (2007), where GARCH specified views offer an improvement upon sole reliance on the implied equilibrium excess returns, in the sample analyzed the more simple GARCH specification outperforms the asymmetric GJR; in both the univariate and multivariate framework.

7.2 Financial Performance Analysis

The financial performance of each portfolio is next examined. As can be seen in the table presented below, the daily Sharpe Ratio in the multivariate specifications exceeds those of the univariate. If one assumes that the multivariate specification is more closely linked to reality, it would be expected that a better understanding of asset dynamics would lead to better financial performance. While the assets themselves do appear much more volatile, the relative weighting between assets appears quite stable. This trend likely stems from the anchoring of the equilibrium implied excess return binding the assets in relation to one another.

Measure	CAPM	GARCH	GJR	DCC-GARCH	DCC-GJR
Sharpe	0.0319	0.0059	0.0027	0.0153	0.0140
Total returns	25.85%	0.91%	-1.35%	7.78%	6.76%
WeightΔ	168.6522	4.5264	6.4950	7.3126	8.7540
(Sharpe/WeightΔ)*100	0.0189	0.1303	0.0416	0.2092	0.1599

Table 7.1. Comparison of financial performance; observe the increase in performance as the GARCH specifications move from the univariate to the multivariate

Where in the above, WeightΔ is calculated as:

$$Weight\Delta = \sum |(w_t - w_{t-1})/w_{t-1}| \quad (7.1)$$

While the above results do not explicitly consider transaction costs, the following section hints at the effects of incorporating the loss associated with transaction costs.

In order to compare a portfolio's performance relative to the weight fluctuation required to achieve said performance, a measure was required. This paper examines the sum of the absolute value of the percentage weight change in each asset per day. This measure should capture how much the portfolio allocation adjusts from day to day. The sum ensures the total fluctuation across the lifetime of the portfolio is captured. This figure then serves as the denominator in a simple ratio with the Sharpe Ratio as the numerator. The result is then multiplied by a factor of 100, solely to aid in relative comparison. The measure proposed then allows for the investigation of the trade-off between financial performance and the degree of weight fluctuation required. By examining the portfolio evolution graphs in tandem with the portfolio composition evolution, a high weight fluctuation is required for the outperformance of the CAPM as indicated by the Sharpe Ratio. The inclusion of the above proposed metric adds richness to the comparison. The CAPM allocation requires a much more unstable weighting in return for its financial performance than each of the GARCH models included in the study.

When transaction costs are considered, a reverse result is noted; whereby by the univariate volatility specifications outperform their multivariate counterparts. These results are only displayed for demonstrative benefit. In reality these costs should be specifically modeled and included in the optimization routine. The following assumes that investors do not at all consider transaction costs in the formation and management of the included portfolios, an assumption not likely to hold in reality. These transaction costs are only handled ex-post. Additionally, it is assumed these costs operate by reducing the total value of the portfolio. As not a primary inquiry in this research, the actual, direct effects of including transaction costs into the optimization routine are left for future research.

The evolution of portfolio value according to each specification is now presented. In each figure the blue line represents the value of the portfolio excluding transaction costs, the raw performance, while the red line includes the financial consequences of transaction costs associated with the portfolio allocation evolution. It should again be emphasized that these transaction costs are ex-post, where each trade is made regardless of the transaction costs and are included merely to show the consequences of excessively fluctuating portfolio allocations.

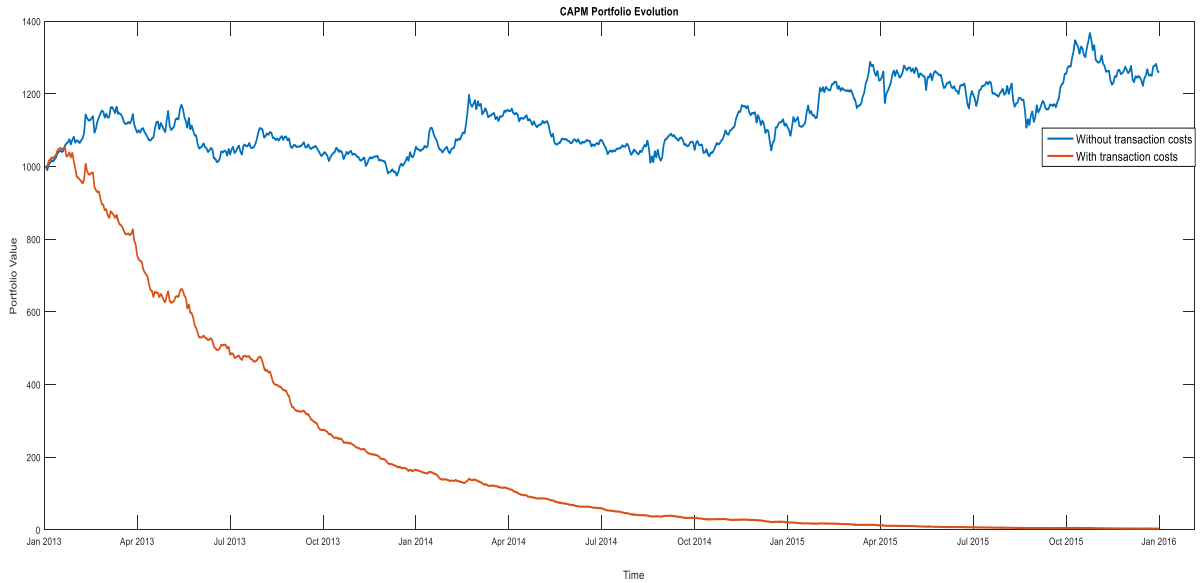


Figure 7.6. CAPM portfolio value evolution; note the drastic decline in value as transaction costs are considered ex-post

The CAPM portfolio value evolution is particularly telling. While the CAPM portfolio value does exceed that of each GARCH specification at the end of the time period for a τ value of 0.05, the difference when considering transaction costs is striking. The CAPM portfolio returns are quickly eroded by excess trading costs; providing a terminal portfolio value close to zero.

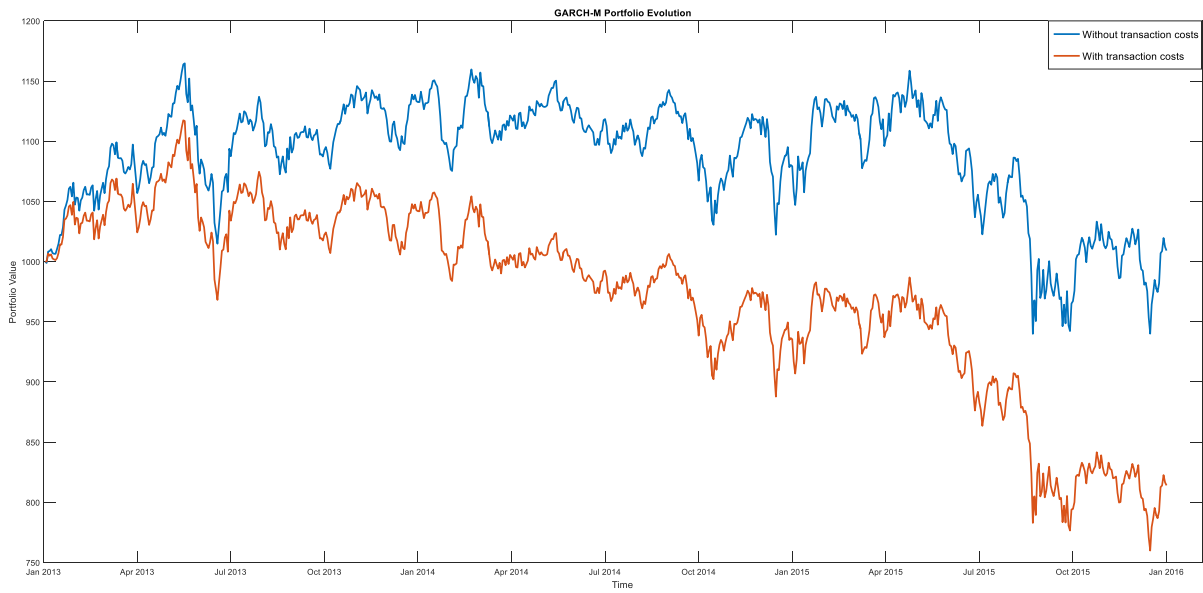


Figure 7.7. GARCH(1,1) portfolio value evolution

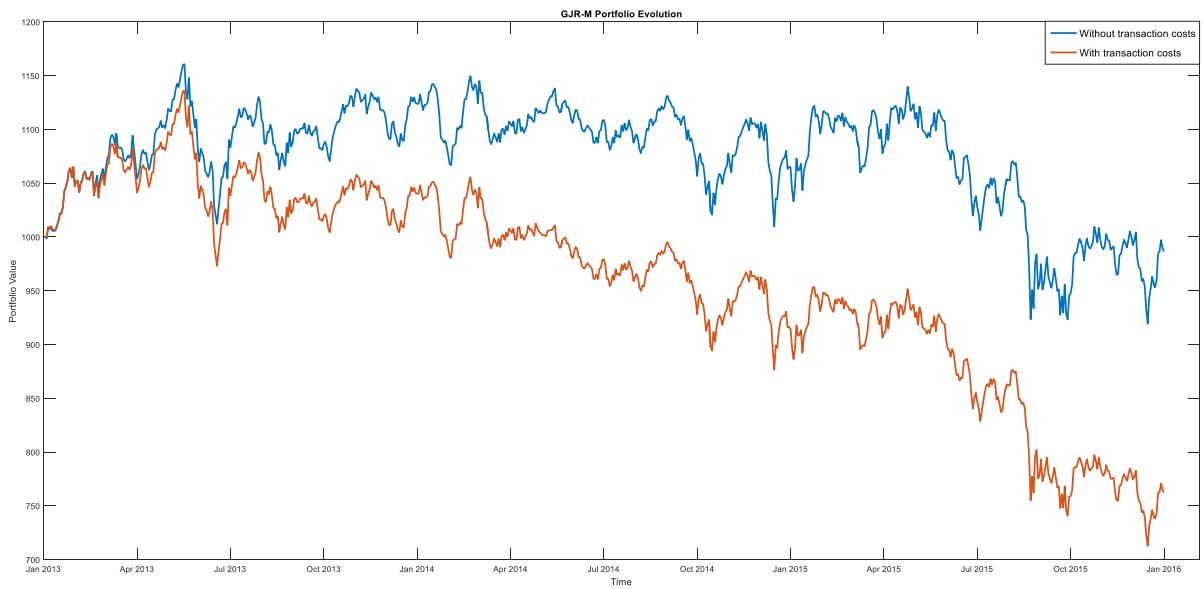


Figure 7.8. GJR portfolio value evolution

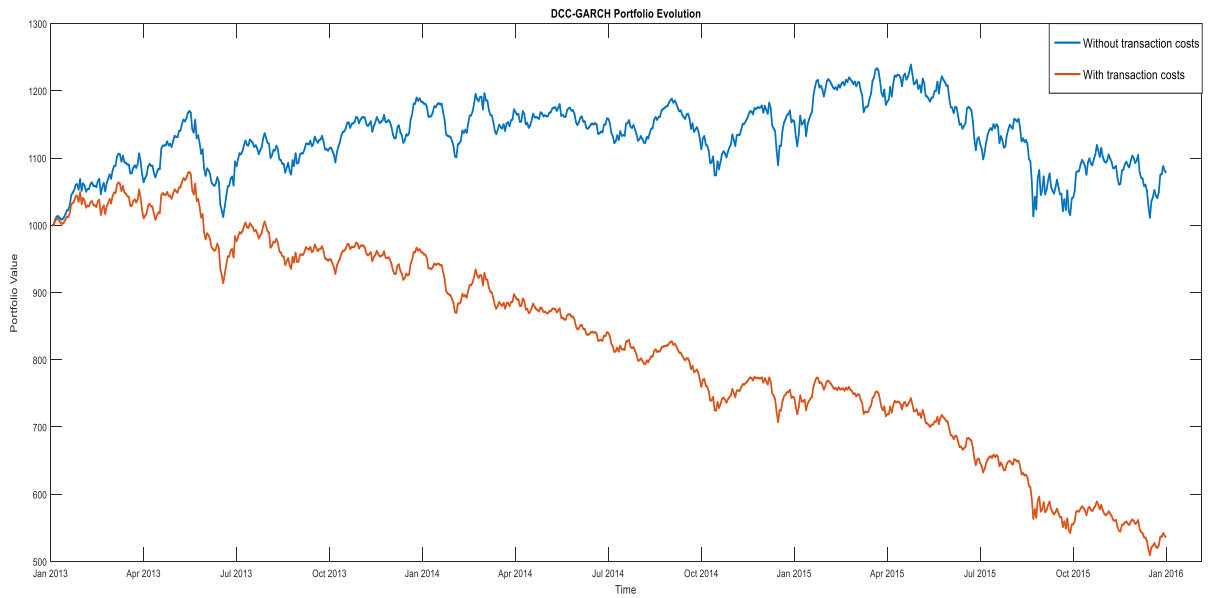


Figure 7.9. DCC-GARCH portfolio value evolution

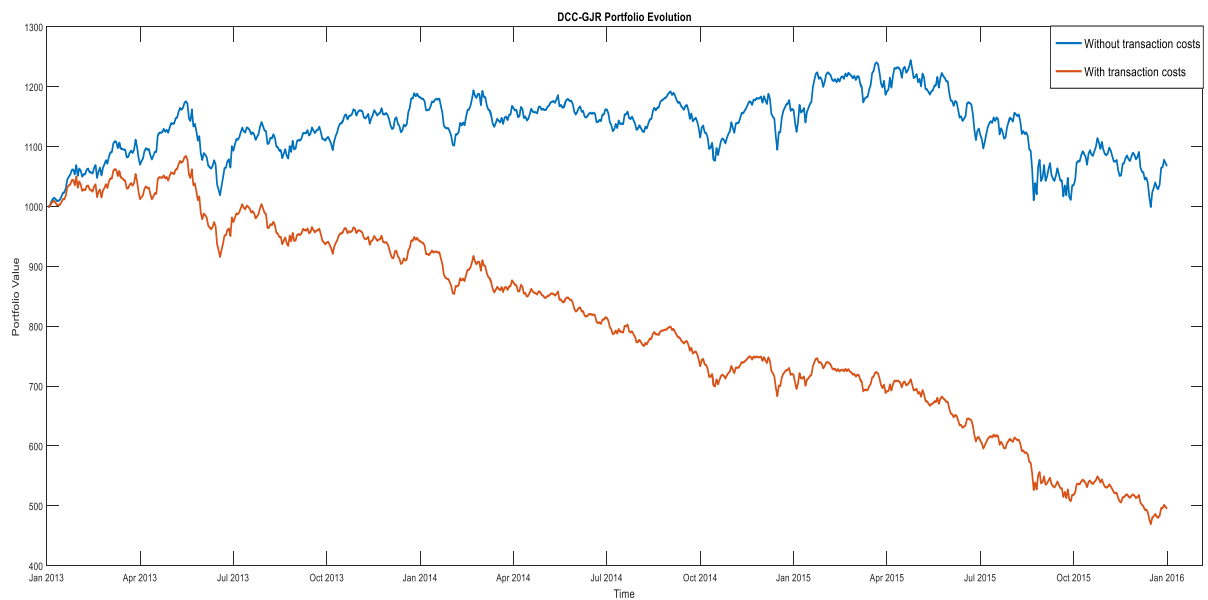


Figure 7.10. DCC-GJR portfolio value evolution

As can be seen in each GARCH specification, the portfolio values reliant on GARCH derived views perform poorly in the last third of the forecasted period, subduing overall financial performance. A potential cause of this pattern may be considered upon inclusion of the asset price series located in Figure 5.1. The implied equilibrium excess returns have each portfolio heavily weighted in HSBA, BLT, and VOD. While HSBA performs reasonably well, VOD remains highly stable throughout the period and BLT performs poorly in the final third of the period. Additionally, RDS, along with BLT, is also weighted relatively heavily. The two assets, both commodity firms, move almost in tandem.

In contrast, CRH and ULVR, both lightly weighted based upon the implied equilibrium perform well in the last third of the forecast time period. During this study the implied equilibrium provides a strong stabilizing effect; not allowing the portfolios to alter the relative weighting to a large degree.

In further investigation the dramatic effects of changing the value of the parameter τ is observed. These effects will be further discussed in Section 8.

The increase in rapidity by which the effects of transaction costs diminish total portfolio value moving from the univariate GARCH specifications to the multivariate is further observed.

The above results and analysis lead to the following observation. There is an obvious tradeoff between the financial performance of an asset allocation strategy and the degree of weight instability. Neither specification for the standard included value of τ surpasses the financial performance of CAPM barring transaction costs.

8 Discussion & Limitations

This section of the paper will include a discussion of the variable τ , suggestions for future research, as well as some limitations in the study presented. While no specific conclusions will be drawn from this section of the paper, the inclusion of the following discussion increases the breadth of the study at hand.

As previously mentioned, the precise manner by which to specify the parameter τ has not been solidified. Some specifications such as Meucci (2010) remove the parameter altogether, while He and Litterman (1999) recommend a value of 0.05. Still others argue only for a value between 0.0 and 1.0 or a value proportional to the sample size employed. Walters (2014) highlights this disagreement. In the included specifications there appears to be a decreasing marginal benefit for the increase of τ , with only a slight improvement in financial performance moving from a τ -value of 7.0 to a τ of 48.0.

By increasing the value of τ , an improvement in the financial performance of the portfolios under evaluation is noted, as measured by an improved Sharpe Ratio relative to the *Weight* Δ -measure; though this increased performance comes at the cost of a decrease in weight stability. Previously presented results are obtained by implementing a τ equal to 0.05 in line with previous research and in accordance with He and Litterman.

In the Black-Litterman model, the parameter τ represents a scalar influencing the degree of certainty in the supplied views. The lower the value of τ , the lower the level of confidence in the user supplied views; in this study mean and variance forecasts from the selected GARCH specifications. Therefore, a small value of τ indicates a lack of confidence in the GARCH supplied views and a greater reliance on the implied equilibrium excess returns, limiting the dynamic influence of the GARCH specifications.

Future research may include additional regressors in the GARCH specifications. As specified in Duqi et al., (2014) these regressors may include information on the performance of currency markets and commodity markets or interest rates. Those variables which may be useful in determining the performance of the market as a whole, such as oil price, may also be useful. Duqi et al., specifically includes the Euro/Dollar exchange rate, WTI (West Texas Intermediate) oil price, and the term spread on US Treasuries in their examination of the US blue chip assets on the Dow Jones Industrial Index.

Beach and Orlov (2007) also include additional regressors in their EGARCH-M specification. Their study includes macroeconomic factors that mimic previous multifactor inquiries. These factors include: production growth, inflation, US Dollar index return, corporate bond yield premiums, Eurodollar and US T-bill spreads, bond yields, and percentage change in the price of oil. While the inclusion of these variables may aid in the

accuracy of forecasted returns, this was never the expressed intent of this paper. While one may expect the results to be largely similar, future research may seek to incorporate similar variables relevant to the UK stock market.

As previously expressed, the accuracy of the specific GARCH forecast was never considered, merely their ability to provide a reasonable approximation for future values and to provide a financial benefit relative to the increase in asset allocation instability. Prior research has highlighted the difficulty in accurately forecasting financial performance; one of the factors influencing the implementation of the Black-Litterman methodology. For this reason, a search for the most accurate means of forecasting was not of specific interest.

9 Conclusion

This paper supports the notion that more complex volatility models, able to more closely mimic reality, may offer potential benefits in portfolio allocation as improvements in financial performance moving from the univariate to the multivariate GARCH specifications are observed. The results of this study also agree with previous research highlighting the persistent relevance of the simple GARCH(1,1). While the incorporation of asymmetries is theoretically attractive, in agreement with Ramasamy and Munisamy (2012) this paper finds no substantial improvement with the inclusion of the GJR model.

However, the inclusion of ex-post transaction costs yields a reverse result, whereby the simpler univariate specifications outperform their multivariate counterparts. As previously extracted from Barber and Odean (2000); increased allocation instability, observed in the multivariate uncertainty matrix portfolios, yields a higher transaction cost penalty. In both the univariate and the multivariate frameworks, portfolio performance with the inclusion of transaction costs in the manner described in Section 7.2 yields results exceeding the CAPM allocation. These findings should further be tested in a method whereby some model of transaction costs is explicitly included in the optimization routine to present a more definitive conclusion.

This study utilized daily rebalancing and parameter re-estimation. While this approach may more closely resemble the modern tactical portfolio allocation processes, Panait and Slavescu

(2012) show the GARCH-M process better suited to weekly or monthly financial data. The use of a less frequent interval would likely influence allocation volatility and portfolio performance.

In the above research it has been shown that the use of the equilibrium implied excess return provides the major anchoring force in the stability of the Black-Litterman methodology; in line with theory. As the largest increase in stability stems from the inclusion of the implied equilibrium excess return as a reference model, the incorporation of the multivariate specification and an explicit cost function may yield better performance compared with the univariate specifications.

As significant previous research has pointed to the ability of various GARCH specifications to have some descriptive use in forecasting asset returns and volatilities, this study advocates for continued research investigating the GARCH framework and other forecasting techniques incorporated into the Black-Litterman methodology. The strong mediating force of the implied equilibrium excess return may alleviate the concern for exact forecasting techniques. As previously mentioned, research has highlighted the extreme difficulty in formulating accurate forecasting models. Future additional research should place particular emphasis on the value of τ and how the selection may interact with the forecasting techniques used to supply investor views.

References

Articles

1. Aldrich, John. (1997). R.A. Fisher and the Making of Maximum Likelihood 1912–1922. *Statistical Science*. 12 (3), pp. 162-176.
2. Appiah-Kusi, Joe. Menyah, Kojo. (2003). Return predictability in African stock markets. *Review of Financial Economics*. 12 (3), pp. 247-270.
3. Barber, Brad M. Odean, Terrance. (2000). Trading Is Hazardous to Your Wealth: The Common Stock Investment Performance of Individual Investors. *The Journal of Finance*. 55 (2), pp. 773-806.
4. Bauwens, Luc. Laurent, Sébastien. Rombouts, Jeroen V.K. (2006). MULTIVARIATE GARCH MODELS: A SURVEY. *Journal of Applied Econometrics*. 21 (1), pp. 79-109.
5. Beach, L Steven. Orlov, G Alexei. (2007). An application of the Black–Litterman model with EGARCH-M-derived views for international portfolio management. *Financial Markets and Portfolio Management*. 21 (21), pp. 147-166.
6. Bevan, Andrew. Winkelmann, Kurt. (1998). Using the Black-Litterman Global Asset Allocation Model: Three Years of Practical Experience. *Fixed Income Research*.
7. Billio, Monica. Caporin, Massimiliano. Gobbo, Michele.. (2006). Flexible Dynamic Conditional Correlation Multivariate GARCH Models for Asset Allocation. *Applied Financial Economics Letters*. 2 (2), pp. 123-130.
8. Black, F. (1976). Studies of stock price volatility changes. In: *Proceedings of the 1976 Meeting of the Business and Economic Statistics Section*, American Statistical Association, Washington, D.C.
9. Bollerslev, Tim. (1986). Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics*. 31, pp. 307-327.
10. Bollerslev, Tim. (1987). A Conditionally Heteroskedastic Time Series Model for Speculative Prices and Rates of Return. *The Review of Economics and Statistics*. 69 (3), pp. 542-547.
11. Bollerslev, Tim. (1990). Modeling the Coherence in Short-Run Nominal Exchange Rates: A Multivariate Generalized ARCH Model. *Review of Economics and Statistics*. 72 (2), pp. 498-505.

12. Bollerslev, Tim. (2008). Glossary to ARCH (GARCH). *Center for Research in Econometric Analysis of Time Series. School of Economics and Management, University of Aarhus. Denmark.*
13. Booth, G. Geoffrey. Martikainen, Teppo. Tse, Yiuman. (1997). Price and volatility spillovers in Scandinavian stock markets. *Journal of Banking & Finance.* 21 (6), pp. 811-823.
14. Bouchev, Paul. Nemtchinov, Vassilii. Paulsen, Alex. Stein, David M. (2012). Volatility Harvesting: Why Does Diversifying and Rebalancing Create Portfolio Growth? *Journal Of Wealth Management.* 15, pp. 26-35.
15. Brandt, Michael W. Jones, Christopher S. (2006). Volatility Forecasting With Range-Based EGARCH Models. *Journal of Business & Economic Statistics.* 24 (4), pp. 470-486.
16. Brewer III, Elijah. Carson, James M. Elyasiani, Elyas. Mansur, Iqbal. Scott, William L. (2007). Interest Rate Risk and Equity Values of Life Insurance Companies: A GARCH–M Model. *The Journal of Risk and Insurance.* 74 (2), pp. 401-423.
17. Brodie, Joshua. Daubechies, Ingrid. De Mol, Christine. Giannone, Domenico. Loris, Ignace. (2008). Sparse And Stable Markowitz Portfolios. *European Central Bank, Working Paper Series.* 936
18. Brownlees, Christian. Engle, Robert F. Kelly, Bryan. (2011). A Practical Guide to Volatility Forecasting through Calm and Storm. *The Journal of Risk.* 14 (2), pp. 3-22.
19. Cheung, Yin-Wong. Ng, Lilian K. (1992). Stock Price Dynamics and Firm Size: An Empirical Investigation. *The Journal of Finance.* 48 (5), pp. 1985-1997.
20. Chopra, Vijay K. Ziemba, William T. (1993). The Effect of Errors in Means, Variances and Covariances on Optimal Portfolio Choice. *Journal of Portfolio Management.* 19 (2), pp. 6-11.
21. Christie, Andrew A. (1982). The stochastic behavior of common stock variances: Value, leverage and interest rate effects. *Journal of Financial Economics.* 10, pp. 407-432.
22. Clements, Adam. Scott, Ayesha. Silvennoinen, Annastiina. (2015). On the Benefits of Equicorrelation for Portfolio Allocation. *Journal of Forecasting.* 34 (6), pp. 507-522.
23. Da, Zhi. Jagannathan, Ravi. (2005). Teaching Note on Black-Litterman Model. *Northwestern University, Illinois*

24. DeMiguel, Victor. Nogales, Francisco J. (2009). Portfolio Selection with Robust Estimation. *Informs*. 57 (3), pp. 560-577.
25. Devaney, Michael. (2001). Time varying risk premia for real estate investment trusts: A GARCH-M model. *The Quarterly Review of Economics and Finance*. 41 (3), pp. 335-346.
26. Dueker, Michael J.. (1997). Markov Switching in GARCH Processes and Mean-Reverting Stock-Market Volatility. *Journal of Business & Economic Statistics*. 15 (1), pp. 26-34.
27. Duqi, Andi. Franci, Leonardo. Torluccio, Giuseppe. (2014). The Black–Litterman model: the definition of views based on volatility forecasts. *Applied Financial Economics*. 24 (19), pp. 1285-1296.
28. Efron, Bradley. (1979). Bootstrap Methods: Another Look at the Jackknife. *The Annals of Statistics*. 7 (1), pp. 1-26.
29. Elyasiani, Elyas. Mansur, Iqbal. (1998). Sensitivity of the bank stock returns distribution to changes in the level and volatility of interest rate: A GARCH-M model. *Journal of Banking & Finance*. 22 (5), pp. 535-563.
30. Engle, Robert F. (1982). Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation. *Econometrica*. 50 (4), pp. 987-1007.
31. Engle, Robert F. Bollerslev, Tim. (1986). Modelling the persistence of conditional variances. *Econometric Reviews*. 5 (1), pp. 1-50.
32. Engle, Robert F. Lilien, David M. Robins, Russell P. (1987). Estimating Time Varying Risk Premia in the Term Structure: The Arch-M Model. *Econometrica*. 55 (2), pp. 391-407.
33. Engle, Robert F. Ng, Victor K. (1993). Measuring and Testing the Impact of News on Volatility. *The Journal of Finance*. 48 (5), pp. 1749-1778.
34. Engle, Robert F. Patton, Andrew J. (2001). What good is a volatility model? *Quantitative Finance*. 1, pp. 237-245.
35. Engle, Robert. (2002). Dynamic Conditional Correlation: A Simple Class of Multivariate Generalized Autoregressive Conditional Heteroskedasticity Models. *Journal of Business & Economic Statistics*. 20 (3), pp. 339-350.

36. Engle, Robert. Sheppard, Kevin. (2001). THEORETICAL AND EMPIRICAL PROPERTIES OF DYNAMIC CONDITIONAL CORRELATION MULTIVARIATE GARCH. *NBER Working Paper*. (8554).
37. Fama, Eugene F. French, Kenneth R. (1992). The Cross-Section of Expected Returns. *The Journal of Finance*. 48 (2), pp. 427-465.
38. Fama, Eugene F. French, Kenneth R. (2004). The Capital Asset Pricing Model: Theory and Evidence. *Journal of Economic Perspectives*. 18 (6), pp. 25-46.
39. Feng, Yu. Medo, Matúš. Zhang, Liang. Zhang, Yi-Cheng. (2011). Transaction fees and optimal rebalancing in the growth-optimal portfolio. *Physica A*. 390, pp. 1635-1645.
40. Fernandes, José Luiz Barros. Ornelas, José Renato Haas. Cusicanqui, Oscar Augusto Martinez. (2012). Combining equilibrium, resampling, and analyst's views in portfolio optimization. *Journal of Banking & Finance*. 36 (5), pp. 1354-1361.
41. Fornari, Fabio. Mele, Antonio. (1996). Modeling the changing asymmetry of conditional variances. *Economic Letters*. 50 (2), pp. 197-203.
42. Francq, Christian. Zakoian, Jean-Michel. (2014). Estimating multivariate GARCH and Stochastic Correlation models equation by equation.
43. Galido, Adrian P. Khanser, Marites A. (2013). Natural Disasters and the Philippine Stock Exchange Index: A Garch-M Analysis. *DLSU Business & Economics Review*. 22 (2), pp. 54-63.
44. Giacometti, Rosella. Bertocchi, Marida. Rachev, Svetlozar T. Fabozzi, Frank J. (2007). Stable distributions in the Black–Litterman approach to asset allocation. *Quantitative Finance*. 7 (4), pp. 423-433.
45. Glen, JJ. (2011). Mean-variance portfolio rebalancing with transaction costs and funding changes. *The Journal of the Operational Research Society*. 62 (4), pp. 667-676.
46. Glosten, Lawrence R. Jagannathan, Ravi. Runkle, David E. (1993). On the Relation between the Expected Value and the Volatility of the Nominal Excess Return on Stocks. *Journal of Finance*. 48 (5), pp. 1779-1801.
47. Grier, Kevin B. Perry, Mark J. (2000). The effects of real and nominal uncertainty on inflation and output growth: some GARCH-M evidence. *Journal of Applied Econometrics*. 15 (1), pp. 45-58.

48. He, Guangliang. Litterman, Robert. (1999). The Intuition Behind Black-Litterman Model Portfolios. *Investment Management Research*.
49. Holden, Helge. Holden, Lars. (2013). Optimal rebalancing of portfolios with transaction costs. *Stochastics*. 85 (3), pp. 371-394.
50. Idzorek, Thomas M. (2004). A STEP-BY-STEP GUIDE TO THE BLACK-LITTERMAN MODEL: Incorporating user-specified confidence levels. *Zephyr Associates, Inc.* California.
51. Jagannathan, Ravi. Ma, Tongshu. (2003). Risk Reduction in Large Portfolios: Why Imposing the Wrong Constraints Helps. *The Journal of Finance*. 58 (4), pp. 1651-1683.
52. Jensen, Michael C. (1969). Risk, The Pricing of Capital Assets, and The Evaluation of Investment Portfolios. *The Journal of Business*. 42 (2), pp. 167-247.
53. Kourtis, Apostolos. (2015). A Stability Approach to Mean-Variance Optimization.
54. Laurent, Sébastien. Rombouts, Jeroen V.K. Violante, Francesco. (2009). Consistent ranking of multivariate volatility models.
55. Lin, Jin-Lung. (2006). Notes on Multivariate Volatility Models. Department of Finance, National DongHwa University
56. Longestaey, Jacques. Spencer, Martin. (1996). *RiskMetrics - Technical Document*. Available: http://pascal.iseg.utl.pt/~aafonso/eif/rm/TD4ePt_2.pdf. Last accessed 25th May 2016.
57. Malkiel, Burton G. (1995). Returns from Investing in Equity Mutual Funds 1971 to 1991. *Journal of Finance*. 50 (2), pp. 549-572.
58. Mandelbrot, Benoit. (1963). The Variation of Certain Speculative Prices. *The Journal of Business*. 36 (4), pp. 394-419.
59. Markowitz, Harry. (1952). Portfolio Selection. *The Journal of Finance*. 7 (1), pp. 77-91.
60. Merton, Robert C. (1980). On estimating the expected return on the market: An exploratory investigation. *Journal of Financial Economics*. 8 (4), pp. 323-361.
61. Meucci, Attilio. (2010). The Black-Litterman Approach: Original Model and Extensions. *The Encyclopedia of Quantitative Finance*.
62. Michaud, Richard O. (1989). The Markowitz Optimization Enigma: Is 'Optimized' Optimal? *Financial Analysts Journal*. 45 (1), pp. 31-42.

63. Mitchell, John E. Braun, Stephen. (2013). Rebalancing an investment portfolio in the presence of convex transaction costs, including market impact costs. *Optimization Methods and Software*. 28 (3), pp. 523-542.
64. Nelson, Daniel B. (1991). Conditional Heteroskedasticity in Asset Returns: A New Approach. *Econometrica*. 59 (2), pp. 347-370.
65. Neuberger, Jonathan A. (1994). Interest rate risk at US commercial banks, *Working paper*. Federal Reserve Bank of San Francisco.
66. Pagan, Adrian R. Schwert, G. William. (1990). Alternative models for conditional stock volatility. *Journal of Econometrics*. 45 (1-2), pp. 267-290.
67. Palomba, Guilio. (2008). Multivariate GARCH models and the Black-Litterman approach for tracking error constrained portfolios: an empirical analysis. *Global Business and Economics Review*. 10 (4), pp. 379-413.
68. Panait, Iulian. Slavescu, Ecaterina Oana. (2012). Using GARCH-IN-mean model to investigate volatility and persistence at different frequencies for Bucharest Stock Exchange during 1997-2012. *Theoretical & Applied Economics*. 19 (5), pp. 55-76.
69. Perret-Gentil, Cédric. Victoria-Feser, Maria-Pia. (2005). Robust Mean-Variance Portfolio Selection. *FAME International Center for Financial Asset Management and Engineering*. Research Paper 140
70. Poon, Ser-Huang. Granger, Clive W.J. (2003). Forecasting Volatility in Financial Markets: A Review. *Journal of Economic Literature*. 41 (1), pp. 478-539.
71. Ramasamy, Ravindran. Munisamy, Shanmugam. (2012). Predictive Accuracy of GARCH, GJR and EGARCH Models Select Exchange Rates Application. *Global Journal of Management and Business Research*. 12 (15), pp. 88-100.
72. Salman, Ferhan. (2002). Risk-return-volume relationship in an emerging stock market. *Applied Economics Letters*. 9 (1), pp. 549-552.
73. Satchell, Stephen. Scowcroft, Alan. (2000). A demystification of the Black–Litterman model: Managing quantitative and traditional portfolio construction. *Journal of Asset Management*. 1, pp. 138-150.
74. Schulmerich, Dr. Marcus. (2015). Can the Black-Litterman Framework Improve Asset Management Outcomes? *IQ Insights: State Street Global Advisors*.
75. Shapiro, Alex. (n.d.) The Capital Asset Pricing Model: Lecture Notes. *Foundations of Finance*. NYU Stern.

76. Sharpe, William F. (1966). Mutual Fund Performance. *The Journal of Business*. 39 (1), pp. 119-138.
77. Simonian, Joseph. Davis, Josh. (2011). Incorporating uncertainty into the Black-Litterman portfolio selection model. *Applied Economics Letters*. 18 (1), pp. 1719-1722.
78. Su, Y.C. Huang, H.C. Lin, Y.J. (2011). GJR-GARCH model in value-at-risk of financial holdings. *Applied Financial Economics*. 21 (24), pp. 1819-1829.
79. Theil, Henri. (1963). On the Use of Incomplete Prior Information in Regression Analysis. *Journall of the American Statistical Association*. 58 (302), pp. 401-414.
80. Walters, Jay. (2014). *The Black-Litterman Model In Detail*. Available: <http://ssrn.com/abstract=1314585>. Last accessed 15th May 2016.
81. Wang, Wei. (2010). Strategic Asset Allocation and Markov Regime Switch with GARCH Model. *Proceedings of the Northeast Business & Economics Association*. 1 (1), pp. 391-399.
82. Zakoian, Jean-Michel.. (1994). Threshold heteroskedastic models. *Journal of Economic Dynamics and Control*. 18 (5), pp. 931-955.
83. Zhang, Jin and Maringer, Dietmar. (2009). Improving Sharpe Ratios and Stability of Portfolios by Using a Clustering Technique. *Proceedings of the World Congress on Engineering*. (1), London
84. Zhou, Guofu. (2009). Beyond Black-Litterman: Letting the Data Speak. *The Journal of Portfolio Management*. 36 (1), pp. 36-45.

Books

1. Fusai, Gianluca. Roncoroni, Andrea. (2008). *Implementing Models in Quantitative Finance: Methods and Cases*. New York: John Wiley and Sons inc.
2. Silvennoinen, Annastiina. Teräsvirta, Timo. (2009). Multivariate GARCH models. In: Anderson, T.G. Davis, R.A. Kreiss, J.-P. Mikosch, T. *Handbook of Financial Time Series*. Berlin: Springer. pp. 201-229.
3. Verbeek, Marno. (2012). *A Guide to Modern Econometrics*. New Jersey: Wiley. 514.

Theses

1. Clark, Jessica M. Mulready, Sean E. (2007). Portfolio Optimization with Transaction Costs. (BSc thesis), *Worcester Polytechnic Institute*. Retrieved from <https://www.wpi.edu/Pubs/E-project/Available/E-project-042707-112035/unrestricted/TurnoverConstraintsMQP.pdf>
2. Jiang, Wei. (2012). Using the GARCH model to analyze and predict the different stock markets. (MSc thesis). *Uppsala Universitet*. Retrieved from <http://www.diva-portal.org/smash/get/diva2:576215/FULLTEXT02>
3. Kohli, Jasraj. (2005). An Empirical Analysis of Resampled Efficiency. (MSc thesis), *Worcester Polytechnic Institute*. Retrieved from https://www.wpi.edu/Pubs/ETD/Available/etd-042605-143547/unrestricted/Raj_Project.pdf
4. Villar, M José Rodriguez. (2010). Volatility Models with Leverage Effect. (PhD thesis). *Universidad Carlos III De Madrid*. Retrieved from http://e-archivo.uc3m.es/bitstream/handle/10016/10411/tesis_maria_jose_rodriguez.pdf?sequence=1