

Exposure At Default During Financial Stress

- A Comparative Study

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Abstract

In recent years the capital requirements for banks have been updated which has complicated the pricing procedure for derivatives. Nordea has developed a proxy model that approximates the risk measure Exposure At Default, which is an important component in the recently updated requirements. The purpose of this thesis is to validate the accuracy of Nordea's model.

In order to fulfill this purpose, models for the Exposure At Default calculations are developed and implemented in both the risk neutral probability measure and the real world probability measure. Both models are based on time consuming Monte Carlo simulations. To improve speed a third, analytical solution in the risk neutral probability measure, is developed as well. The result shows that the two models in the risk neutral probability measure converge towards the same value, as the number of simulations in the Monte Carlo model increases. There is a difference in the results generated in the two measures which is assumed to depend on the absence of the risk premium in the real world probability measure.

The final conclusion is that Nordea's proxy model does not generate trustworthy results. However, considering this conclusion, Nordea was able to improve their proxy model to generate accurate results.

Keywords: Exposure At Default, *EAD*, Interest Rate Swap, Kalman Filter, Monte Carlo, Real World Probability Measure, Risk Neutral Probability Measure, Vasicek Model.

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Abbreviations

A summary of all abbreviations used throughout the thesis:

CEM	Current Exposure Method
EAD	Exposure At Default
EE	Expected Exposure
EEE	Effective Expected Exposure
EEPE	Effective Expected Positive Exposure
EURIBOR	Euro Interbank Offered Rate
KVA	Capital Valuation Adjustment
LIBOR	London Interbank Offered Rate
MLE	Maximum Likelihood Estimation
SM	Standardized Method
ZCB	Zero Coupon Bond
XVA	X-Value Adjustment

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Chapter 1

Introduction

“Mathematics is the most beautiful and most powerful creation of the human spirit.”

– Stefan Banach

1.1 Background

During the global financial crisis in 2008, several countries had to support banks financially to prevent them from collapsing. To decrease the probability of a similar situation to occur in the future, stricter capital requirements for banks have been prescribed by law in Europe, as well as in other regions. As a result banks need to allocate a higher amount of capital for each contract they form a part of, to insure against the risk of the trade. This buffer serves to prevent the banks from collapsing if counterparts default during a potential future financial crisis, or at least facilitate the handling of defaulted banks. Since the banks thereby face increased dividends to its shareholders, several banks have chosen to include the cost of this capital when pricing derivatives. This part of the price of the derivative is adjusted through different valuation adjustments, frequently referred to as the *X-Value Adjustment (XVA)* framework. One adjustment in this framework is called the *Capital Valuation Adjustment (KVA)* which accounts for the cost of the capital buffer over the lifetime of the trade. The capital buffer for this requirement serves to secure the bank from unexpected losses followed by sudden stress in the market. The capital buffer, and thereby also the *KVA*, is based on the risk measure *Exposure At Default (EAD)* and thus, to determine the size of the capital requirement, the *EAD* must be derived.

Often, Monte Carlo simulations are used to derive the *EAD* and several simulations are required in order to receive a correct result. Therefore, the calculations are time consuming and computationally burdensome. In the financial business, a potential trade is preferably priced within seconds. Consequently, there is not enough time to derive an *EAD* curve based on Monte Carlo simulations when the price of a derivative is requested by a customer. This results in a problem for the banks which want to include the cost of capital when pricing derivatives.

1.2 Subject Foundation

This work has been carried out in collaboration with the Pricing and Capital team at Nordea Markets, Stockholm. They have developed a proxy model which approximates the *EAD* to be included in the price of a derivative. This model fulfills the restricted speed criteria but it remains to test if the proxy model generates trustworthy results. Therefore, upon their request, the subject of this thesis was set to investigate the results generated by the proxy model. To do this, the time consuming calculations of the *EAD* will be performed and compared to the results obtained by Nordea's model. Maybe it will be too burdensome and time consuming to derive an accurate model and in that case the Monte Carlo simulations must be improved to increase speed. Another solution might be to implement an alternative model which solves the problem without the time consuming Monte Carlo simulations. Finally it is up to Nordea to conclude whether they trust their proxy model or not based on the results from this thesis.

1.3 Purpose

The main purpose of this thesis is to validate whether Nordea's proxy model for approximation of the risk measure *EAD* generates trustworthy results. Consequently the result of the proxy model must be compared to the results from a correct calculation of the *EAD*. This thesis will focus on the *EAD* generated by different types of interest rate swaps and consider extreme scenarios on financial markets.

1.4 Methodology

To find the *EAD* curve the first thing to do is to collect historical *EURIBOR* spot rates and then calibrate a short rate model to fit the historical data. Future *EURIBOR* forward rates are then generated by the short rate model using Monte Carlo simulation. Subsequently an interest rate swap is considered and to obtain the *EAD* curve the swap has to be valued in each time step. The valuation is done in both the risk neutral probability measure and in the real world probability measure and the different valuation methods result in different models. A third model is developed using an analytical solution instead of Monte Carlo simulations to eliminate the Monte Carlo error and to improve the speed of the calculations.

1.5 Contribution Statement

Different theories have been combined to develop models for calculations of *EAD* curves and this contributes to the existing literature in the field. An analytical model is developed which, to the best of our knowledge, is a thoroughly new approach to solving problems connected to the risk measure *EAD*. Furthermore, a comparison between the risk neutral probability measure and the real world probability measure is provided in an innovative aspect.

1.6 Outline

The outline for this thesis is:

Chapter 2: This chapter provides an introduction to arbitrage theory and interest rates. The importance of zero-coupon bonds is discussed and a description of different compounding types is given. Moreover, the basics of forward rates are provided.

Chapter 3: The underlying theory of risk neutral valuation is discussed in this chapter. Furthermore, the foundations of short rate models are covered and the Vasicek model is introduced. The parameter estimation method Maximum Likelihood Estimation is presented followed by a brief introduction to the Kalman filter. Finally, a computational algorithm for Monte Carlo simulation is provided.

Chapter 4: This chapter serves to give an understanding of interest rate swaps and how they can be valued in the risk neutral probability measure and in the real world probability measure respectively.

Chapter 5: The capital regulations from Basel III are presented in this chapter. Different methods to measure the capital requirements are discussed followed by the theory behind the *EAD* curve.

Chapter 6: The theory from the previous chapters is applied on interest rate swaps. Initially, the theory of Maximum Likelihood Estimation and the Kalman filter is used to fit the model to market data. Subsequently, the choice of measure is discussed and then three different models for the *EAD* calculations are developed.

Chapter 7: This chapter presents the results obtained from the previous chapter. Also, the results from this thesis are shown together with the results from Nordea's proxy model.

Chapter 8: The results and methodologies are discussed and a comparison between the models is provided in this chapter. Furthermore, the delimitations used in this thesis are reviewed and some directions on future research of the topic are given.

Chapter 9: A summary of the thesis together with a conclusion of the results is presented.

Chapter 2

Arbitrage Theory and Interest Rates

“An economist is an expert who will know tomorrow why the things he predicted yesterday didn’t happen today”

– Laurence J. Peter

In order to understand the methodology and the results in the upcoming chapters this chapter serves to give a brief explanation of the theory behind arbitrage and interest rates. Among others, the zero-coupon bond is introduced and to improve the understanding of this derivative, the time value of money is discussed. Moreover the importance of absence of arbitrage is emphasized since this is the foundation of the theory used in this thesis. The theory in this chapter is based on the book *Interest Rate Models - Theory and Practice* by Brigo and Mercurio [6], which gives a thorough introduction to the basics of arbitrage theory and interest rates. The theory utilizes the notation t for the current time, T for the maturity time and S for the contracting time.

2.1 The Bank Account

Money has a time value since it can be invested to either increase or decrease in value, depending on the interest rate. A negative interest rate will make the money decrease in value, while a positive interest rate will increase the value of the money over time. Thus, one unit of money received today might be more valuable than one unit of money received in the future. To express these concepts mathematically some definitions has to be introduced. The bank account is defined to show how money accrues in time.

Definition 2.1.1. *The Bank Account B_t is the value of a bank account at time $t \geq 0$. Assume $B(0) = 1$ and that the bank account evolves according to the following differential equation:*

$$\begin{aligned}dB_t &= r_t B_t dt, \\ B_0 &= 1,\end{aligned}$$

where r_t is a positive function of time. As a consequence,

$$B_t = e^{-\int_0^t r_s ds}. \tag{2.1}$$

Thus, investing one unit amount at time 0 will yield the value in (2.1) at time t . The bank account accrues with the *instantaneous rate*, r_t which will from now on be referred to as the *short rate*.

Many of the mathematical formulas in arbitrage theory and stochastic calculus are subject to one specific rule, there must be absence of *arbitrage*. An arbitrage portfolio is a portfolio h with the properties

$$\begin{aligned} V_0^h &= 0, \\ P(V_T^h \geq 0) &= 1, \\ P(V_T^h > 0) &> 0. \end{aligned}$$

This basically means that if arbitrage is present, it is possible to invest an amount of zero today and receive a non-negative amount tomorrow, without taking any risk. In the case of arbitrage the market is considered to be mispriced, [4].

2.2 Zero–Coupon Bonds

Zero – Coupon Bonds are used to express the time t value of a unit amount that will be received at $T > t$ and are defined in accordance with Björk in [4].

Definition 2.2.1. A **Zero-Coupon Bond (ZCB)** with maturity date T is a contract which guarantees the holder 1 unit to be paid on the date T . The price at time t of a bond with maturity date T is denoted by $p(t, T)$. The relation $p(s, s) = 1$ holds for all $s \in [t, T]$.

This means that a *ZCB* is a bond that is bought for a price which is not the same as the face value and the face value is repaid at the time of maturity, defined below.

Definition 2.2.2. The **Time to Maturity** ($T - t$) is the amount of time (in years) from the present time, t , to the maturity time, $T > t$.

ZCB prices are the basic quantities in interest rate theory and are often used to express prices of many types of contracts. However, these bond prices are theoretical instruments that are not directly observable on the market. Interest rates on the other hand are usually quoted in interbank financial markets and can thus be observed. The connection between interest rates and *ZCB* is given through the compounding frequency. To simplify the understanding of the upcoming theory, the continuously compounded spot interest rate and the simply compounded spot interest rate are defined.

Definition 2.2.3. The **Continuously Compounded Spot Interest Rate** prevailing at time t for the maturity T is denoted $R(t, T)$ and is the constant rate at which an investment of $p(t, T)$ units of currency at time t accrues continuously to yield a unit of currency at maturity T . In formulas,

$$R(t, T) := -\frac{\ln p(t, T)}{T - t}. \quad (2.2)$$

An interbank rate is the rate of interest charged when banks extend loans to each other. These loans always have a specified term. The most essential interbank rate, and often used as a reference, is the *London Interbank Offered Rate (LIBOR)*. There exist other interbank rates for other markets, such as the *Euro Interbank Offered Rate (EURIBOR)*. *EURIBOR* and *LIBOR* are simply compounded rates.

Definition 2.2.4. The **Simply Compounded Spot Interest Rate** prevailing at time t for the maturity T is denoted by $L(t,T)$ and is the constant rate at which an investment has to be made to produce an amount of one unit of currency at maturity, starting from $p(t,T)$ units of currency at time t , when accruing occurs proportionally to the investment time. In formulas,

$$L(t;S,T) = \frac{1 - p(t,T)}{(T-t)p(t,T)}. \quad (2.3)$$

Notice that the spot rate is the immediate rate on the market, hence the spot rate changes every time the market moves up or down. Using definition (2.2) and (2.3) the continuously compounded spot rate can easily be converted to a simply compounded spot rate, and vice versa.

2.3 Forward Rates

A forward rate is an interest rate based on the spot rate that is locked in today for an investment in a future time period. A *Forward Rate Agreement* (FRA) is a contract depending of three time variables, the current time t , the expiry time $S > t$ and the maturity time $T > S$. At the maturity time a payment based on a fixed rate R will be exchanged against a floating payment based on the spot rate $L(S,T)$ settled in S and with maturity T . The following definitions defines the continuously and simply compounded forward rates in the same way as Björk does in [4].

Definition 2.3.1. The **Continuously Compounded Forward Rate** (R) is the solution to the equation

$$e^{R(T-S)} = \frac{p(t,S)}{p(t,T)}. \quad (2.4)$$

Definition 2.3.2. The **Simple Forward Rate** for $[S,T]$ contracted at t , also referred to as the **LIBOR** forward rate, is defined as

$$L(t;S,T) = -\frac{p(t,T) - p(t,S)}{(T-S)p(t,T)}. \quad (2.5)$$

Chapter 3

Model and Calibration Methods

“All models are wrong, but some are useful.”

– George E. P. Box

A natural intuition would be to value derivatives in the real world, using historical data available on the market. However, to simplify the valuation of financial derivatives the risk neutral probability measure, introduced by mathematicians, can be used instead. In this thesis, valuations in both the real world probability measure and the risk neutral probability measure will be performed and therefore, this chapter explains the basics of this new probability measure. Furthermore, to be able to forecast the future exposure, short rate models are presented. Additionally, a calibration method of the short rate models is explained. Finally, a brief introduction to Monte Carlo methods is provided since two of the models developed in this thesis use this theory.

3.1 Risk Neutral Valuation

The risk of entering into a contract is carefully considered by investors. Most often they are risk averse and thus prepared to pay a premium to avoid risk or expect a higher profit when carrying risk. Accordingly, the fair price of a risky derivative is not only the discounted future payoffs under the real world probability measure, also called measure \mathbb{P} , because the risk premium must be considered as well. This means that the price first has to be calculated by discounting the future payoffs and then modified to consider the impact of the risk the investor is exposed to. This method is complex since it is difficult to collect and quantify all investors' risk preferences. Therefore an equivalent probability measure \mathbb{Q} was developed, here defined as in [6].

Definition 3.1.1. *The Equivalent Martingale Measure \mathbb{Q} also referred to as the Risk Neutral Probability Measure \mathbb{Q} is a probability measure on the space (Ω, \mathcal{F}) such that*

- \mathbb{Q}_0 and \mathbb{Q} are equivalent measures, that is $\mathbb{Q}_0(A) = 0$ if and only if $\mathbb{Q}(A) = 0$, for every $A \in \mathcal{F}$,
- the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{Q}_0}$ belongs to $L^2(\Omega, \mathcal{F}, \mathbb{Q}_0)$ (i.e. it is square integrable with respect to \mathbb{Q}_0).

- the “discounted asset price” process $D(0, \cdot)S$ is an (F, \mathbb{Q}) -martingale, that is $E^{\mathbb{Q}}(D(0, t)S_t^k | \mathcal{F}_u) = D(0, u)S_u^k$, for all $k = 0, 1, \dots, K$ and all $0 \leq u \leq t \leq T$.

Instead of having to modify the expectation of the future cash flows afterwards, the risk neutral probability measure handles the risk premium simultaneously as the expectation is taken. In other words, all investors’ preferences are considered, and the risk premium becomes included in the price, when the risk neutral probabilities are used. Because of the possibility to use the risk neutral valuation formula which simplifies the calculations and gives a correct price under Definition 3.1.1, the measure \mathbb{Q} is frequently used in the financial industry when pricing derivatives. Björk [4] defines the risk neutral valuation as

Definition 3.1.2. Risk Neutral Valuation. Bond prices are given by the formula $p(t, T) = F(t, r_t; T)$, where

$$F(t, r_t; T) = E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} | \mathcal{F}_t].$$

Here the martingale measure \mathbb{Q} and the subscripts t, r denote that the expectation shall be taken given the dynamics for the short rate, defined in the next section.

3.2 Short Rate Models

To forecast the future of the interest rate market a model for the evolution of the interest rates is needed. For that purpose, short rate models are essential. The dynamics of a short rate model is in the risk neutral probability measure given by

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t^{\mathbb{Q}}, \quad (3.1)$$

where μ is the drift term, σ the diffusion term of the short rate and $W^{\mathbb{Q}}$ a Brownian motion under measure \mathbb{Q} . A Brownian motion is a Wiener process which in turn is a continuous stochastic process and accordingly, the last term in (3.1) is stochastic. The following applies to all Brownian motions

$$\Delta W^{\mathbb{Q}} = W_t^{\mathbb{Q}} - W_s^{\mathbb{Q}} \sim N(0, \sqrt{t-s}).$$

For more details of Brownian motions and stochastic calculus, see [4].

3.2.1 The Vasicek Model

The Vasicek model is one of the simplest short rate models and was developed 1977 by the mathematician Vasicek. He presents his results in [14] where he explains how the short rate can be assumed to evolve under the real world probability measure as an Ornstein-Uhlenbeck process. Brigo and Mercurio states in [6] that Vasicek’s reasoning can be applied to the risk neutral probability measure as well and therefore the model can be used in both measures. Here, it is defined in the risk neutral probability measure.

Definition 3.2.1. The Vasicek Model is a one factor model with dynamics

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^{\mathbb{Q}}. \quad (3.2)$$

When taking the expectation of expression (3.2),

$$E^{\mathbb{Q}}[dr_t | \mathcal{F}_t] = \kappa(\theta - r_t)dt + 0,$$

the following is obvious

$$\begin{aligned} \theta > r &\implies E^{\mathbb{Q}}[dr_t | \mathcal{F}_t] > 0, \\ \theta < r &\implies E^{\mathbb{Q}}[dr_t | \mathcal{F}_t] < 0, \\ \theta = r &\implies E^{\mathbb{Q}}[dr_t | \mathcal{F}_t] = 0. \end{aligned}$$

In other words, the Vasicek model is mean reverting, where θ is the long term mean and κ the speed of adjustment to the long term mean.

To simplify the use of the Vasicek model in future calculations, the model is discretized. From expression (3.2) it is possible to derive an exact discretization of the short rate model. By setting $Y_t = e^{\kappa t} r_t$ and with the use of Itô's Lemma (see [7] for details) the solution to (3.2) can be derived through the following steps.

$$dY_t = \frac{dY_t}{dt}dt + \frac{dY_t}{dr}dr = \kappa e^{\kappa t} r_t dt + e^{\kappa t} (\kappa(\theta - r_t)dt + \sigma dW_t) = \kappa \theta e^{\kappa t} dt + \sigma e^{\kappa t} dW_t$$

Integrating both sides yields

$$Y_t = Y_0 + \kappa \theta \int_0^t e^{\kappa s} ds + \sigma \int_0^t e^{\kappa s} dW_s.$$

By remembering that $Y_t = e^{\kappa t} r_t$ it can be seen that

$$r_t = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \sigma \int_0^t e^{-\kappa(t-s)} dW_s.$$

As the starting time is arbitrary the following conversion is possible

$$r_t = r_s e^{-\kappa(t-s)} + \theta(1 - e^{-\kappa(t-s)}) + \sigma \int_s^t e^{-\kappa(t-\tau)} dW_\tau. \quad (3.3)$$

This is the exact solution to (3.2) which also represents the exact discretization of the Vasicek model. The expectation and variance of (3.3) are

$$E[r_t | r_s] = e^{-\kappa(t-s)} r_s + \theta(1 - e^{-\kappa(t-s)}), \quad (3.4)$$

$$V[r_t | r_s] = V\left[\sigma \int_s^t e^{-\kappa(t-\tau)} dW_\tau\right] = \sigma^2 E\left[\int_s^t e^{-2\kappa(t-\tau)} d\tau\right] = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa(t-s)}), \quad (3.5)$$

where the Itô Isometry (see [7] for details) is used in the expression of the variance.

To simulate rates from a short rate model an approximation of the discretization is useful since it is difficult to simulate rates using (3.3). An Euler scheme can be used to provide a first order discrete approximation of the continuous Vasicek short rate model with dynamics defined in (3.2).

$$\begin{aligned} r_{t+1} - r_t &= \kappa(\theta - r_t)\Delta t + \sigma\sqrt{\Delta t}Z \\ &\iff \\ r_{t+1} &= (1 - \kappa\Delta t)r_t + \kappa\theta\Delta t + \sigma\sqrt{\Delta t}Z, \end{aligned} \quad (3.6)$$

where

$$Z \sim N(0, 1).$$

This concludes that the approximation of the short rates are normally distributed since the drift part is constant and the diffusion part is normally distributed. During volatile periods discretization of the otherwise continuous model can result in large jumps between each value of the short rate which can be problematic. This problem can be minimized by letting the length of the time steps Δt go towards zero and thereby letting the discretization go towards the continuous model, [6].

As discussed in Section 3.1 prices under the risk neutral probability measure can be derived from the Risk neutral valuation definition 3.1.2. When this formula is applied to the Vasicek model, the following results can be derived.

Proposition 3.2.1. *In the Vasicek model, bond prices are given by*

$$p(t, T) = e^{A(t, T) - B(t, T)r(t)}, \quad (3.7)$$

where

$$B(t, T) = \frac{1 - e^{-\kappa(T-t)}}{\kappa} \quad (3.8)$$

and

$$A(t, T) = \left(\theta - \frac{\sigma^2}{2\kappa^2} \right) (B(t, T) - T + t) - \frac{\sigma^2}{4\kappa} B^2(t, T). \quad (3.9)$$

For the derivation of Equation (3.8) and Equation (3.9), see [4].

3.3 Maximum Likelihood Estimation

When simulating rates using (3.2) the parameters κ , θ and σ must be estimated to fit some historical data. A commonly used method to estimate unknown parameters of a model when the data is known is the *Maximum Likelihood Estimation (MLE)*. By maximizing the likelihood function it is possible to calculate the estimates of the unknown parameters that are most likely to describe the data, given the probability density function of the data, which is discussed in [9] by Jakobsson. The argument that will maximize the likelihood will be the same for the log-likelihood, and since the log-likelihood is easier to handle, the log-likelihood is maximized instead, [10].

$$\hat{\mu}_{MLE} = \arg \max_{\mu} f_x(x_1, \dots, x_n | \mu) = \arg \max_{\mu} \ell(\mu, x), \quad (3.10)$$

where $f_x(x_1, \dots, x_n | \mu)$ is the probability density function of the data and $\ell(\mu, x)$ is the log-likelihood function.

3.4 The Kalman Filter

Some data is perfectly observable on the market, while some data is not. The unobservable data can be seen as a hidden state and one method to find these hidden states is to use a Kalman filter. The Kalman filter is an algorithm containing a transition equation that links two consecutive unobservable states and a measurement equation, relating the observed data to the hidden state. This recursive process consists of one prediction step and one updating step. In the prediction step the filter predicts the current state and their variances by using

all the information prior to that time step. Using the predicted value together with the next observed value the estimates can be updated using a weighted average, giving more weight to the predictions with higher certainty. The following brief introduction to the Kalman filter will follow Welch and Bishops' article, *An Introduction to the Kalman Filter* [15], which gives a detailed explanation of the theory behind the filter.

The Kalman filter estimates the state x of the discrete time process,

$$x_t = Fx_{t-1} + Gu_{t-1} + \eta_{t-1}, \quad (3.11)$$

with a measurement space $z \in \mathbb{R}^n$ that is the observed data,

$$z_t = Hx_t + v_t. \quad (3.12)$$

The random variables $\eta_t \sim N(0, Q)$ and $v_t \sim N(0, R)$, where Q and R are covariance matrices, represent the process and measurement noise respectively. F , in equation (3.11), relates the state x_{t-1} to the state x_t . G in the same expression relates the control input u_{t-1} to the state x_t . H in the measurement equation (3.12) relates the state to the measurement z_k . The equations used to predict the state and the variance is given as

$$\begin{aligned} \hat{x}_{t|t-1} &= F\hat{x}_{t-1|t-1} + Gu_{t-1}, \\ V_{t|t-1} &= FV_{t-1|t-1}F^T + Q. \end{aligned}$$

The following expressions show the error term, the Kalman gain and how to update x and the variance respectively.

$$\begin{aligned} \varepsilon_{t|t-1} &= z_t - H\hat{x}_{t|t-1} \\ K_t &= V_{t|t-1}H^T(HV_{t|t-1}H^T + R)^{-1} \\ \hat{x}_{t|t} &= \hat{x}_{t|t-1} + K\varepsilon_{t|t-1} \\ V_{t|t} &= (1 - KH)V_{t|t-1} \end{aligned}$$

Here $\varepsilon_{t|t-1}$ is the *innovation* or *measurement residual* which represents the difference between the observed data at t and the data that was expected to be observed in the next time step at $t - 1$. The Kalman gain K_t is the relative importance of ε with respect to the prior estimate $x_{t|t-1}$.

3.5 Monte Carlo Methods

This section will follow the notations in Wiktorsson's notes in [16]. The principal aim of Monte Carlo methods is to calculate the expectation

$$\tau = E[\phi(X)] = \int_A \phi(x)f(x)dx,$$

where

- X is a random variable taking values in $A \subseteq \mathbb{R}^d$,
- $f: A \rightarrow \mathbb{R}_+$ is the probability density of X and

- $\phi : A \rightarrow \mathbb{R}$ is a function such that the above expectation is finite.

The Monte Carlo method calculates the same thing several times, each time with a different set of random values. Subsequently, the mean of all calculations is assumed to be the most likely outcome. Accordingly, it enables a way of solving complex problems even though an analytic solution is not easily found. Thanks to the result of the law of large numbers the method ensures accuracy when a sufficiently large number of random variables is used. Thus an unbiased estimator of the solution to the problem can be assured when Monte Carlo methods are used. The following algorithm is used in the Monte Carlo method.

Algorithm 1 Monte Carlo algorithm

Require: A sufficiently large number of random variables.

```
for  $i = 1 \rightarrow N$  do  
    draw  $X_i \sim f$   
end for  
set  $\tau_n = \frac{1}{N} \sum_{i=1}^N \phi(X_i)$   
return  $\tau_n$ 
```

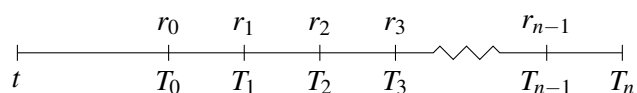
Chapter 4

Interest Rate Swaps

“Economics is all about consumption. People either spend money now or they use financial instruments – like bonds, stocks and savings accounts – so they can spend more later.”

– Adam Davidson

An interest rate swap is a generalization of the *FRA* defined in Section 2.3 and can be seen as an agreement between two parties where one stream of future payments is exchanged against another stream of future payments, based on a principal amount, K . An interest rate swap is an arrangement where one party pays a fixed rate while the other party pays a floating rate. The counterparty who pays the fixed rate is referred to as the “payer” and the counterparty receiving the fixed rate is the “receiver”. To improve the understanding of the valuation of an interest rate swap, the following timeline is provided.



Studying this timeline, it can be seen that the dates T_0, \dots, T_n are equally spaced. Payments occur at T_1, \dots, T_n and the rate that specifies a floating payment at T_i is determined at T_{i-1} . This foundation is used in this chapter, where two different methodologies of swap valuation are explained.

4.1 Swap Valuation in the Risk Neutral Probability Measure \mathbb{Q}

To value a swap at the current time t all future cash flows must be discounted to the current time. First, the cash flows at the payment dates, $T_i, i = 1, \dots, n$, are derived. When a cash flow occurs the payer of the swap gains the amount

$$K\delta L[T_{i-1}, T_i],$$

where $\delta = T_i - T_{i-1}$ and L is the floating rate, and pays the amount

$$K\delta R,$$

where R is the fixed rate. Consequently the payer's cash flow at T_i is

$$c_i(T_i) = K\delta(L[T_{i-1}, T_i] - R). \quad (4.1)$$

Using expression (2.5) for *LIBOR* forward rates, (4.1) can be rewritten as

$$\begin{aligned} c_i(T_i) &= K\delta \frac{p(t, T_{i-1}) - p(t, T_i)}{(T_i - T_{i-1})p(t, T_i)} - K\delta R \\ &= K \frac{p(t, T_{i-1})}{p(t, T_i)} - K(1 + \delta R) \\ &= \frac{K}{p(T_{i-1}, T_i)} - K(1 + \delta R). \end{aligned} \quad (4.2)$$

where $p(t, T_i)$ is the price of a ZCB with maturity T_i at time t . Next, the terms in Expression (4.2) are discounted separately. The present value of the second term, $-K(1 + \delta R)$, is easily obtained by using the definition of a ZCB, defined in 2.2.1, to discount the term to its present value, $-K(1 + \delta R)p(t, T_i)$. The first term, $K \frac{p(t, T_{i-1})}{p(t, T_i)}$, is more complex to value at time t and therefore the following procedure is performed. At time t , K units of ZCB with maturity time $T_{i-1} > t$ are bought to a cost of $Kp(t, T_{i-1})$. This will result in a gain of K at time T_{i-1} and then this amount is reinvested in ZCB with a maturity time T_i . This results in $\frac{K}{p(T_{i-1}, T_i)}$ bonds and each of them will be worth 1 at time T_i and consequently the holder of the bonds will receive the amount $\frac{K}{p(T_{i-1}, T_i)}$ at time T_i . In summary, the value of the first term is $Kp(t, T_{i-1})$ at time t and the total value of the coupon received at T_i is equal to

$$c_i(t) = Kp(t, T_{i-1}) - K(1 + \delta R)p(t, T_i)$$

at time t .

Finally, the total price of the swap at time t can be derived by discounting all future coupons to time t . Notice that this formula is only valid when $t < T_0$.

$$\begin{aligned} \Pi(t) &= \sum_{i=1}^n c_i(t) = K \sum_{i=1}^n (p(t, T_{i-1}) - (1 + \delta R)p(t, T_i)) \\ &= K \sum_{i=1}^n p(t, T_{i-1}) - p(t, T_i) - K \sum_{i=1}^n \delta R p(t, T_i) \\ &= Kp(t, T_0) - Kp(t, T_n) - K \sum_{i=1}^n \delta R p(t, T_i) \\ &= Kp(t, T_0) - K \sum_{i=1}^n d_i p(t, T_i), \end{aligned}$$

where

$$d_i = R\delta, i = 1, \dots, n-1$$

and

$$d_n = 1 + R\delta.$$

If $T_0 = t$, definition (2.2.1) induces that $p(t, T_0) = p(t, t) = 1$.

It remains to calculate the fixed rate, R , and the value of the swap is by choice set to be zero at the contracting day. When the contract is written at $t = 0$ the fixed rate becomes

$$R = \frac{p(0, T_0) - p(0, T_n)}{\delta \sum_{i=1}^n p(0, T_i)}.$$

The formula can be reduced to

$$R = \frac{1 - p(0, T_n)}{\delta \sum_{i=1}^n p(0, T_i)}, \quad (4.3)$$

in the case when $T_0 = 0$.

4.2 Swap Valuation in the Real World Probability Measure \mathbb{P}

As mentioned before, an interest rate swap is basically a series of cash flows at predefined dates in the future. Thus, the value of the swap can be calculated by summarizing the present value of all future cash flows. The future cash flows depend on the future spot rates, therefore this valuation method requires that future spot rates are forecasted. The future spot rates are simulated at t and can thus be seen as forward rates contracted at t and prevailing in the future. To get the present value of the future cash flows it is necessary to estimate the discount factor that will discount the future cash flows to their present value. It is calculated using the simulated future spot rates.

For example, if a 3 month *LIBOR* swap with maturity time one year is considered, the 3, 6, 9 and 12 month spot rates are used in the discounting at the contracting time. In this example, only 3 month rates are available and therefore the simulated rates must be converted to fit the other maturity times as well. This is possible since the simulated future *LIBOR* spot rates can be combined to estimate other *LIBOR* spot rates. For example, the 6 month *LIBOR* spot rate can be approximated by the 3 month *LIBOR* spot rate specified at t_0 and the 3 month *LIBOR* spot rate specified 3 months from t_0 , at t_1 , as follows

$$1 + L_{t_0}^{6m} = (1 + L_{t_0}^{3m})(1 + L_{t_1}^{3m}).$$

A general expression for the approximation of the *LIBOR* spot rate becomes

$$1 + L_{t_0}^{kp} = \prod_{i=0}^{k-1} (1 + L_{t_i}^p), \quad (4.4)$$

where p stands for the length of each period in the swap (i.e. a 3 month swap means that $p = 3$) and k specifies the number of periods. Consequently, the *LIBOR* forward rates used in the discounting are

$$L = [L_{t_0}^p, L_{t_0}^{2p}, \dots, L_{t_0}^{np}],$$

where n is number of periods in the swap. These rates are in annualized form and must be converted to the actual time period of the coupon when used in the discounting. Finally, the discount vector at t_0 is derived to be

$$DF = \left[\frac{1}{1 + \frac{p}{12} L_{t_0}^p}, \frac{1}{1 + \frac{2p}{12} L_{t_0}^{2p}}, \dots, \frac{1}{1 + \frac{np}{12} L_{t_0}^{np}} \right], \quad (4.5)$$

where the first factor discounts the cash flow that occurs in T_1 , the second discounts the cash flow in T_2 and so on. See the timeline in the beginning of this chapter for better understanding.

The value of a swap can be divided into two parts, the value of the fixed leg and the value of the floating leg. The total value of the swap is then

$$\Pi_{swap} = \Pi_{flt} - \Pi_{fix} \quad (4.6)$$

for the party who pays a fixed rate and receives a floating rate. The values of the different legs are at the initiation date

$$\Pi_{flt} = K \sum_{i=1}^N L_{t_i}^p DF_i, \quad (4.7)$$

$$\Pi_{fix} = KR \sum_{i=1}^N DF_i, \quad (4.8)$$

where K is the nominal amount and R is the fixed rate. To set the swap to be at the money at the initiation date, thus giving the swap a value of zero, the fixed leg is equated to the floating leg which results in the following expression for the fixed rate

$$R = \frac{\sum_{i=1}^N L_{t_i}^p DF_i}{\sum_{i=1}^N DF_i}. \quad (4.9)$$

Once the fixed rate is known, it is possible to value the swap in every time step.

Chapter 5

Regulations

“Learn the rules like a pro, so you can break them like an artist.”

– Pablo Picasso

The Basel committee has agreed upon new regulations to promote more resilient banks and banking systems. These regulations are collected in a global regulatory framework, named Basel III [2] and consists of stricter capital requirements than earlier regulations. This chapter presents and explains the approach for deriving the risk measure *EAD*. Furthermore its fields of application is highlighted.

5.1 Capital Requirements

A lender, in this case a bank, is exposed to counterparty risk. Counterparty risk is the risk that the borrower defaults on it's loan due to bankruptcy. In recent years a number of different price value adjustments have been introduced to handle this type of risk, among other the *KVA*. This, together with some other adjustments make up the *XVA* framework, which is a framework often used to value derivatives [13]. The *KVA* is a new valuation adjustment for the cost of the capital buffer. The capital buffer captures the risk of unexpected losses and considers the tails of the historical distribution. Therefore it focuses on extreme scenarios and the regulatory framework requires that data from a stressed period is used in the calculations of the *EAD*. A stressed period is considered to be a period with some kind of extreme scenarios, which results in that the data from this period differ from the usual pattern of the market. [2] [8]

The capital requirement of a derivative is not constant and it is therefore essential to forecast the capital requirement and the cost of capital over the entire time frame of the trade. The future capital requirement for the default risk of a derivative transaction or a portfolio of derivatives is based on the *EAD* at each time point in the future.

5.2 Internal Models Method

There are different methods available to measure the capital requirements of banks for various counterparty credit risk scenarios. One method is the *Internal Models Method (IMM)*, which was developed under the Basel II regulations, [1]. Two other methods available to calculate the required reserves are the *Standardised Method (SM)* and the *Current Exposure Method (CEM)*. As the Basel committee discusses in [3], the *SM* and the *CEM* have been subject to a lot of criticism and are significantly simpler than the *IMM*, but using the *IMM* requires approval by the national supervisory authorities. This method accepts calculations in both the real world probability measure and the risk neutral probability measure, see [1].

5.3 Exposure At Default

The *EAD* is the total value that the bank is exposed to in case of default of the counterparty. The following methodology to calculate the *EAD* follows the regulations for the *IMM* and the formulas follow the notation in Green, Kenyon and Dennis' article *KVA: Capital Valuation Adjustment* [8]. First the *Expected Exposure (EE)* is calculated as

$$EE_{t_k}^i = \mathbf{E}^{\mathbb{Q}}[\max(\Pi_{t_k}^i, 0) | \mathcal{F}_{t_k}], \quad (5.1)$$

where t_k is the time point of interest and i is initial time point of the *EE* curve. In other words, this value is an expectation of the value of the concerned derivative, conditional on that the value of the derivative is larger than zero. This is due to the fact that the *EE* values the risk that the bank is exposed to and since a negative value of the derivative means that the bank owes its counterpart money instead of the other way around, the bank is not exposed to a credit loss. Consequently, the exposure is expected to be zero in these cases.

From the *EE*, the *Effective Expected Exposure (EEE)* can be derived as

$$EEE_{t_k}^i = \max(EEE_{t_{k-1}}^i, EE_{t_k}^i). \quad (5.2)$$

In words, the *EEE* is the maximum of the *EEE* in the previous time step and the *EE* in the current time step and with induction the *EEE* can easily be derived to be the maximum of the *EE* in all previous time steps,

$$EEE_{t_k}^i = \max(EEE_{t_{k-1}}^i, EE_{t_k}^i) = \max(EEE_{t_{k-2}}^i, EE_{t_{k-1}}^i, EE_{t_k}^i) = \dots = \max(EE_{t_k}^i, EE_{t_{k-2}}^i, \dots, EE_{t_0}^i).$$

Accordingly, the *EEE* can never decrease over time and the *EE* curve will never be above the *EEE* curve. Thereafter, the *Effective Expected Positive Exposure (EEPE)* is calculated by

$$EEPE_{t_k} = \sum_k^{\min(1 \text{ year, maturity})} EEE_{t_k}^i \cdot \delta_{t_k}, \quad (5.3)$$

where $\delta_{t_k} = t_k - t_{k-1}$. Finally the *EAD* can be received by

$$EAD = \alpha \cdot EEPE, \quad (5.4)$$

where α is set to 1.4 according to the requirements by the Basel committee, [1].

Chapter 6

Model Design

“Statisticians, like artists, have the bad habit of falling in love with their models.”

– George E. P. Box

This thesis has developed three different models which all derive the *EAD* curve. The approaches of these models are presented in this chapter using the theory covered in the previous chapters. Nordea has, as one of few Nordic banks, received an approval to implement the *IMM* when calculating the *EAD* and therefore the calculations in this thesis follow the *IMM* framework. To begin with, the parameters are calibrated through *MLE* and a Kalman filter and then future spot rates are simulated under the real world probability measure. As mentioned in Section 5.2 the regulatory framework [2] allow calculation of the *EAD* in both the real world probability measure \mathbb{P} and in the risk neutral probability measure \mathbb{Q} , when using the *IMM*. Since it is easier and less time consuming to calculate the *EAD* in the risk neutral probability measure, the *EAD* will primary be calculated in \mathbb{Q} . One Monte Carlo solution, Model 1, and one analytical solution, Model 2, is provided in this measure. Finally, the *EAD* is calculated in the real world probability measure as well and this method will be referred to as Model 3.

This thesis has chosen to use interest rate swaps in the calculations of the *EAD*, performed in the the software program MATLAB. Unless otherwise stated, a 3 month *EURIBOR* swap with a lifetime of 3 years is considered. For simplicity this thesis assumes that there are 30 days in a month and 360 days per year.

6.1 Data Selection

The historical data used is the same throughout the whole project. Figure 6.1 shows how the *EURIBOR* spot rates has evolved during the past 12 years ¹. The regulatory framework [2] requires that three years of historical data is used to estimate the parameters of the model. For the stress calibration the parameters must be estimated using three years of data that

¹The data was retrieved on 2016-02-29 from https://www.quandl.com/data/BOF/QS_D_IEUTI03M-EURIBOR-3-Months-Daily

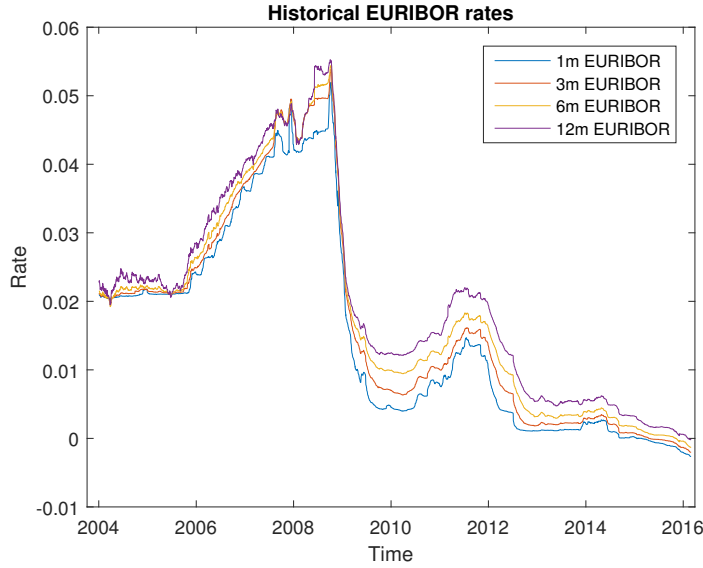


Figure 6.1: Historical *EURIBOR* forward rates for the period 2004-01-02 – 2016-02-29

include a period of stress. The model to calibrate to historical data is chosen to be the one factor Vasicek model, which is a simple model with some drawbacks. The dependence of one single factor limits the possible shapes of the curve generated by Vasicek, which makes it hard to fit the model to complex market data. Hence, instead of choosing the extremely stressed period between 2007 and 2009, the period 2005-01-01 to 2008-01-01 is selected for the stressed calibration. Even though this is not the most stressed period in this data series, it still contains a stressed period and is thus allowed as input in the stressed calibration. The period that is considered to represent normal conditions is 2010-03-29 to 2016-02-29.

6.2 Calibration with the Kalman Filter

As mentioned in Section 3.4 the short rate can not be directly observed in the market but *EURIBOR* spot rates on the other hand are observable. The Vasicek model is a short rate model, hence the parameters κ , θ and σ in (3.2) should be calibrated to historical short rates and thus a Kalman filter is used to estimate the short rates that in reality are unobservable. To get a good and reliable estimation of the short rate it is important to use as much information from the market as possible. Therefore historical *EURIBOR* spot rates with maturities 1, 3, 6 and 12 months are used as input in the filter, see Figure 6.1. The Kalman filter uses these four series of data to construct one series of short rates. The filter only finds the short rate given historical data and the initial guess of the parameters, it does not optimize the parameters. Therefore *MLE* is used in the filter to get an estimation of the parameters that will minimize the error between the predicted and the real values of the short rate. For that purpose the log-likelihood is needed,

$$\ell(\boldsymbol{\mu}) = -\frac{n}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^N \ln [\det(V_{t|t})] - \frac{1}{2} \boldsymbol{\varepsilon}_t (V_{t|t})^{-1} \boldsymbol{\varepsilon}_t^T,$$

where n is the dimension of the data and N is the length of the data.

For simplicity, the simply compounded *EURIBOR* short rates are converted to continuously compounded rates using (2.2) and (2.3). In turn the value of the short rate can be derived from the continuously compounded rate. By combining the expression for the price of a *ZCB* (3.7) and the expression for a continuously compounded rate (2.2) the measure space equation (3.12) for this Kalman filter can finally be derived as

$$R(t, T) = -\frac{1}{\tau}(A(t, T) - B(t, T)r_t) + v_t = C(\bar{\mu})r_t + D(\bar{\mu}) + v_t, \quad (6.1)$$

where $v_t \sim N(0, c)$ is the measurement noise with c as covariance matrix, chosen to be constant. For simplicity the parameters C and D are introduced as $C(\bar{\mu}) = \frac{B(\tau_i)}{\tau_i}$ and $D(\bar{\mu}) = \frac{-A(\tau_i)}{\tau_i}$, where $\bar{\mu} = [\kappa, \theta, \sigma, t, T]$. In the expression above τ represents the different maturities of the rates, i.e. $\tau = [\frac{1}{12}, \frac{1}{4}, \frac{1}{2}, 1]$.

The state space equation is given by the expected value and the variance of the exact discretization of the Vasicek model given in Expression (3.4). Thus, the state space equation is

$$r_t = Fr_{t-1} + G + \eta_t, \quad (6.2)$$

where $\eta_t \sim N(0, Q_t)$ and

$$\begin{aligned} F &= e^{-\kappa\Delta t}, \\ G &= \theta(1 - e^{-\kappa\Delta t}), \\ Q_t &= \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa\Delta t}). \end{aligned}$$

The first step in the Kalman filter is to choose the initial mean as $E[r_1 | r_0] = \theta$ and the initial variance as $V[r_1 | r_0] = \frac{\sigma^2}{2\kappa}$. Below the algorithm for the Kalman filter is outlined, including the log-likelihood function.

Algorithm 2 The Kalman Filter

Require: Historical data R_t of continuously compounded rates.

for each time step t **do**

Prediction step:

$$r_{t|t-1} = G + Fr_{t-1|t-1}$$

$$V_{t|t-1} = FV_{t-1|t-1}F^T + Q$$

Update step:

$$\varepsilon_{t|t-1} = R_t - (Cr_{t|t-1} + D)$$

$$K = V_{t|t-1}C^T(CV_{t|t-1}C^T + c)^{-1}$$

$$r_{t|t} = r_{t|t-1} + K\varepsilon_{t|t-1}$$

$$V_{t|t} = (1 - KC)V_{t|t-1}$$

$$\ell_t = -\frac{n}{2} \ln(2\pi) - \frac{1}{2} \ln[\det(V_{t|t})] - \frac{1}{2} \varepsilon_t(V_{t|t})^{-1} \varepsilon_{t|t-1}^T$$

end for

return short rates $r_{t|t}$ and the log likelihood ℓ

The MATLAB function `fminsearch` is used to optimize the parameters that will generate the best fitting curve for the short rate by minimizing the error. This calibration is done once for a period representing normal conditions on the market and once when a period of stress is included in the historical data. Unfortunately, the optimized parameters are unrealistic

which will be discussed in Section 8. Therefore, the short rates generated from the Kalman filter are used to manually estimate the parameters used in the simulation. In other words, the *MLE* optimization is still used to generate parameters that result in a short rate that fits the historical data, but the parameters used in the future simulations are estimated manually. Since $\hat{\theta}$ is the long term mean, see Section 3.2.1, it is set to be equal to the mean of the generated short rates. The variance of the Vasicek model is given by Expression (3.5) and the standard deviation $\hat{\sigma}$ can therefore be estimated as

$$\hat{\sigma} = \lim_{\Delta t \rightarrow \infty} \sqrt{\frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa\Delta t})} = \sqrt{\frac{\sigma^2}{2\kappa}}. \quad (6.3)$$

As shown in Section 3.2.1, κ stands for the speed of reversion. Studying other articles discussing the Vasicek model, the conclusion that κ usually is within the interval $[0.5, 3]$ can be drawn. Different combinations of stressed calibrated and normally calibrated $\hat{\kappa}$ are tested to find a proper combination with the other parameters.

6.3 Choice of Measure for Simulation

To value an interest rate swap at future time instants, short rates must be simulated. The future rates can be obtained by either risk neutral simulations or by real world simulations. However, using risk neutral simulations the risk free interest rate will be simulated, making the unrealistic assumption that the value will increase with the risk free interest rate, as discussed in [5] by de Boer. A more realistic assumption of the evolution of the short rate is that it evolves as the actual expected rate given the market conditions right now. Therefore, the rates will in this thesis be obtained by real world simulations for all methods while the valuations are made in both measures.

6.4 Model 1 - Swap Valuation in the Risk Neutral Probability Measure \mathbb{Q}

As mentioned in Section 5.3 the main task is to find the stressed expected exposure curves which in turn can be used to calculate the *EAD* curve throughout time. To succeed with this, the Vasicek model with parameters calibrated with the Kalman filter in Section 6.2 is used to simulate future short rate scenarios. First, the short rate model is calibrated for normal conditions and secondly for stressed conditions. To improve speed only the short rates in the current time step are stored.

6.4.1 Simulation of Interest Rates

Monte Carlo simulations, see Section 3.5, are performed to generate short rates and the number of time-steps, the number of simulated paths and the approximated parameters of the Vasicek model are used as input parameters. First, a normal and non-stressed Monte Carlo simulation of N scenarios is performed for each time step, based on the non-stressed simulation from the previous time step. Consequently, there are N paths of normally simulated short rates at the first time point and these will continue until maturity, T , of the interest rate swap, resulting in $N \cdot T$ simulations. Secondly, since the market can become

stressed at any point in time, a new Monte Carlo simulation of S scenarios is performed to generate stressed future short rates, starting at the normal calibrated rates in each time step. This results in S estimations of stressed short rates for each of the N non-stressed estimations. These stressed short rates also continue until the derivative reaches maturity and consequently, there will be $N \cdot S$ new stressed short rates for each future time step. Hence, there will exist $N \cdot S \cdot T + N$ short rates in time step T . An example is illustrated in Figure 6.2 where there is one normally calibrated short rate and two stressed calibrated short rates. In each time step there are $N \cdot S = 1 \cdot 2 = 2$ new stressed short rates and at time step 6, there are $N \cdot S \cdot T + N = 1 \cdot 2 \cdot 6 + 1 = 13$ short rates. Another illustration of the evolution of the short rates is shown in Figure 6.3 where the paths of the short rates starting in the first time step are shown. Here, it is also possible to compare the short rates with the fixed rate. The fact that the stressed rates evolve towards a higher mean indicates that they have a higher long term mean than the normally calibrated short rates.

Since a generator simulates scenarios conditional on simulations of another generator, the used method is called a nested Monte Carlo method. In this case the exterior generator is the normally calibrated Vasicek model and the interior generator is the stressed calibrated Vasicek model.

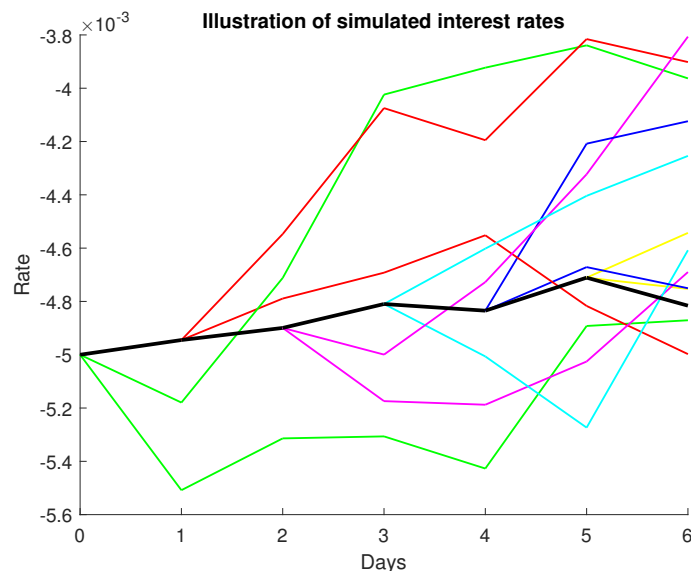


Figure 6.2: The black line is a normally calibrated interest rate and the other lines are stressed calibrated interest rates, colored after in which time step they become stressed. Here, 2 stressed and 1 normal scenario are used and the first 6 time steps are shown.

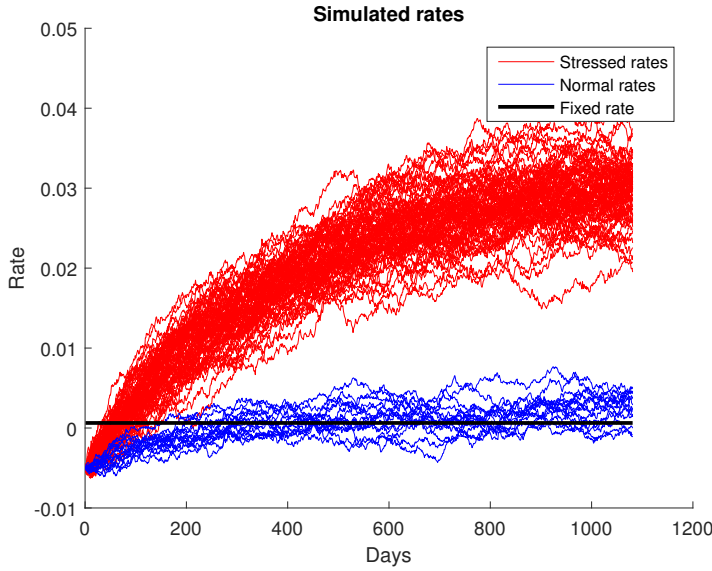
Algorithm 3 Nested Monte Carlo simulation**Require:** Two random generators.**for** each time step t **do** **for** $i = 1 \rightarrow N$ **do** draw $r_i^N(t) | r_i^N(t-1)$ **for** $j = 1 \rightarrow S$ **do** draw $r_j^S(t) | r_i^N(t-1)$ **end for** **end for** **for** $k = 1 \rightarrow t \cdot N \cdot S$ **do** draw $r_k^S(t) | r_k^S(t-1)$ **end for****end for****return** normally calibrated short rates $r^N(t)$ and stressed calibrated short rates $r^S(t)$ 

Figure 6.3: The blue lines are normally calibrated interest rate and red lines are stressed calibrated interest rates, all starting in the first time step. The maturity time of the swap is 3 years and 20 stressed and 20 normal scenarios are used.

6.4.2 Valuation of the Interest Rate Swap

As well as being a foundation of the simulation of the interest rates in the next time step, the rates contribute to the valuation of the interest rate swap. Every rate in each time point originate a value of the swap, derived in Section 4.1 to be

$$\Pi(t) = Kp(t, T_0) - K \sum_{i=1}^n d_i p(t, T_i), \quad (6.4)$$

where

$$d_i = R\delta, i = 1, \dots, n-1$$

and

$$d_n = 1 + R\delta.$$

Unfortunately, this formula is only applicable when $t < T_0$. To be able to form an *EAD* curve, the value of the *EAD* must be calculated in each time step until maturity of the interest rate swap and thus the restriction on t is not fulfilled. To obtain a valuation formula without this restriction, some modifications must be made. When $T_{i-1} < t < T_i$ the above formula can, without loss of generality, be used for the time period $T_i \rightarrow T_n$. The formula then manages the cash flows that occur at $T_{i+1}, T_{i+2}, \dots, T_n$ but the cash flow that occurs at T_i needs to be handled separately. Since the floating rate for this coupon is set at $T_{i-1} < t$ and the fixed rate is constant, the value of the coupon is fixed and known at time t . Thus, this cash flow can be seen as a *ZCB* with time to maturity $T_i - t$. The cash flow at time T_i is

$$c_i(T_i) = K\delta[L(T_{i-1}, T_i) - R].$$

Reasoning as above, the value of the *ZCB* with maturity $T_i - t$ is, at time t is

$$c_i(t) = K\delta(L[T_{i-1}, T_i] - R) \cdot p(t, T_i).$$

Adding this term to equation (6.4) results in the following expression for the value of the swap

$$\Pi(t) = K(\delta(L[T_{i-1}, T_i] - R) + 1)p(t, T_i) - K \sum_{j=i+1}^n d_j p(t, T_j), \quad (6.5)$$

where

$$d_j = R\delta, j = 1, \dots, n-1$$

and

$$d_n = 1 + R\delta.$$

Using Expression (6.5) it is possible to value a swap under the risk neutral probability measure without any restrictions on t .

6.4.3 Calculation of the Exposure At Default

When the valuation of all the simulated short rates is done, there are $S \cdot N$ predicted stressed values of the derivative that originate from the normally calibrated short rates in each time step. Following the theory in Section 5.3, and specifically Equation (5.1), the average of the $S \cdot N$ values that start in a certain time point is calculated, conditional on that the value of the derivative is larger than zero. These calculations are done in each succeeding time step until maturity of the interest rate swap to receive the *EE* curve for the current time point. Here it is important to separate the short rates starting in different time steps from each other since they will be a part of different mean calculations. Consequently, the number of values in the *EE* curves increases as the interest rates continue their walk. Furthermore the number of *EE* curves increases simultaneously as interest rates get stressed and deviate from the normal path. When they do, a new *EE* curve starts in this time point and the starting value for the curve is the average of the valuations of the normal interest rates in this time point.

To simplify the understanding of this complex procedure, an example is provided. For that purpose Figure 6.4, which is an illustration over the values of the derivative arising from the interest rates in Figure 6.2, is helpful. The black line shows the value of the derivative in each time step, based on the normally calibrated short rate. The green lines are the values

of the derivative arising from the short rates that become stressed in the starting point. In $t = 0$ the green lines both have value zero and therefore the mean of them and thus the EE , become zero. In the next time step the green short rates from Figure 6.2 both give rise to a negative value and therefore the EE , which is the positive mean of the values of the derivative is zero once more. These green lines continue until maturity of the interest rate swap and the EE in each time step, derived from these values, together form the EE curve that starts in $t = 0$. When $t = 1$, the short rates that give rise to the values of the derivative that form the red lines get stressed. The first value of the EE curve that starts in $t = 1$ is the value arising from the normally calibrated short rate in this time step but if there had been additional normally calibrated short rates, the mean of these would have been considered instead. In the next step both values on the red line are positive and thus the mean of these is calculated to generate the value of the EE in this time step. As time goes by, new short rates get stressed in each time step and by that a new EE curve starts which will continue until maturity of the swap. In other words, after the green and the red lines that form the first two EE curves, the values of the purple, then the turquoise and so on will all form a specific EE curve that starts in the same time step as the short rates get stressed.

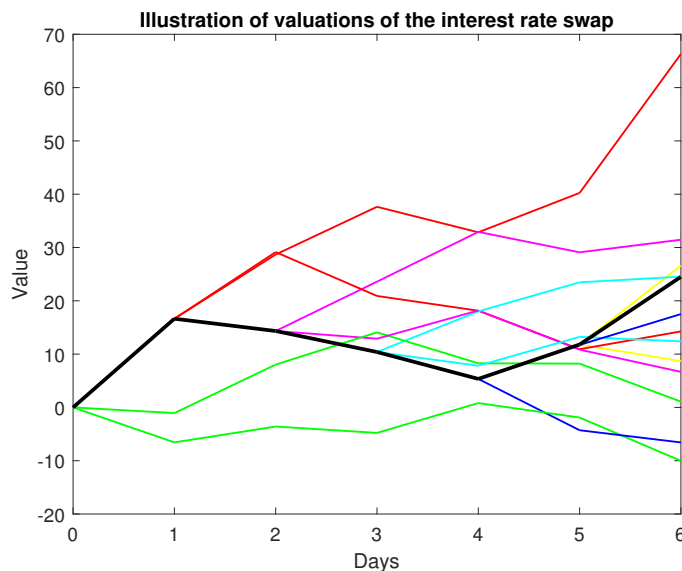


Figure 6.4: The black line shows the values of the interest rate swap that arise from the normally calibrated interest rate, illustrated in Figure 6.2. The other lines are values of the interest rate swap that arise from the stressed calibrated interest rates in the same figure, colored after in which time step they become stressed. Here, 2 stressed and 1 normal scenario are used and the first 6 time steps are shown.

Now it is possible to calculate the expected exposure using Equation (5.2) in Section 5.3. After that the $EEPE$ can be derived from Equation (5.3) in the same section. Since δ is equally spaced, this is equal to the average of the EEE of the rates starting in one time step. If the maturity time is more than one year ahead, the average over one year is taken and otherwise the average over the time until maturity is calculated. Accordingly, there will be one value of the $EEPE$ in each time point, together forming an $EEPE$ curve for the interest rate swap. Finally the EAD can be computed by Equation (5.4) and now there exists an EAD curve which can be used in the calculations of the KVA .

6.5 Model 2 - Analytic Swap Valuation in the Risk Neutral Probability Measure \mathbb{Q}

The Monte Carlo simulations in Model 1, described in Section 6.4, require many calculations, especially since nested Monte Carlo methods are used. Another drawback of the large number of simulations required by Model 1 is that numerical errors are accumulated. The reason for this is that MATLAB solves problems numerically which means that the program rounds every answer off before using it again in the next calculation. This results in the problem that for every time step, every short rate is calculated with a numerical error which is increased when the short rates continue their way forward. Because of these two adverse qualities of Model 1 an alternative method, Model 2, is developed in an attempt to improve speed and accuracy and decrease the computational burden.

6.5.1 Simulation of Interest Rates

To begin with, the expected path of the normally calibrated short rate is derived using the exact discretization of the Vasicek model in Equation (3.3). In every point in time the expected value of the short rate, (3.4), is calculated conditional on the value in the previous time step. Together, these values form an expected curve for the normally calibrated short rates. Due to the fact that the market might become stressed at any point in time, there must be one expected stressed calibrated rate that deviate from the normal path in every time step and play the role as the average of all rates that become stressed in this specific time point, see Figure 6.5. This expected stressed rate gets a specific path until the time of maturity of the swap. Consequently, the above calculations need to be done for the normally calibrated expected rate as well as for all the stressed calibrated rates in every time point until maturity of the swap. Algorithm 4 shows the procedure for the simulation of the short rates in this model. This is the most probable way of the short rates to evolve during the active time

Algorithm 4 Model 2

Require: Expected paths for normal and stressed rates.

```

for each time step  $t$  do
  calculate  $E[r^N(t)|r^N(t-1)]$ 
  calculate  $E[r_t^S(t)|r^N(t-1)]$ 
  for  $i = 1 \rightarrow t - 1$  do
    calculate  $E[r_i^S(t)|r_i^S(t-1)]$ 
  end for
end for
return expected normally calibrated short rates  $r^N$  and expected stressed calibrated short rates  $r^S$ 

```

of the swap but since there is a stochastic variable in the structure of the interest rates, they will in reality choose different paths and deviate from their expected path. This method analytically calculates how much the interest rates are expected to differ from the expected paths instead of generating a large number of rates.

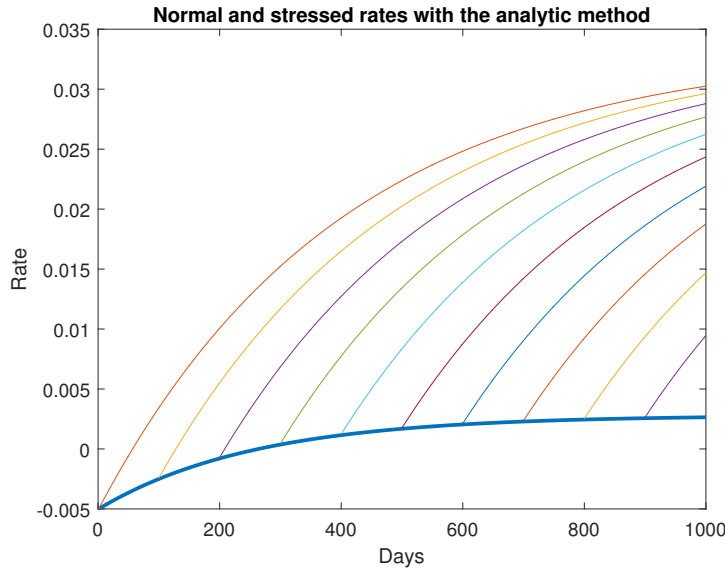


Figure 6.5: Illustration of how the rates develop over time. The bold blue line shows the expected rate and a new stressed rate starts in every time step. To facilitate understanding, only every hundredth stressed rate is printed. In reality there starts one stressed rate in each day.

6.5.2 Approximation of the Distribution of the Value of the Interest Rate Swap

To do this, it is essential to find the distribution as well as the expected value and the variance of the value of the interest rate swap. Studying Equation (6.5) it can be seen that the value of the swap consists of a difference of prices of *ZCB*. In turn Proposition (3.2.1) shows that the price of a *ZCB* in the Vasicek model is lognormally distributed due to the fact that the short rates are normally distributed as shown in Expression (3.6).

Definition 6.5.1. A random variable X is lognormally distributed if

$$X = e^Y$$

and

$$Y \sim N(\mu, \sigma).$$

Then

$$E(X) = e^{\mu + \sigma^2/2}$$

and

$$V(X) = e^{2\mu + 2\sigma^2}(e^{\sigma^2} - 1).$$

Consequently the value of the swap is a difference of lognormally distributed variables and the distribution of this is not easily found. A usual approach is to approximate the distribution of the difference to a shifted lognormal distribution as Lo does in *Research Article – The Sum and Difference of Two Lognormal Random Variables*, [11]. On the other hand, the interest rates are close to zero which induces that the exponent term $e^{-B(t,T)r(t)}$ in the expression for the price of a *ZCB* under the Vasicek model (3.7), is close to one and so

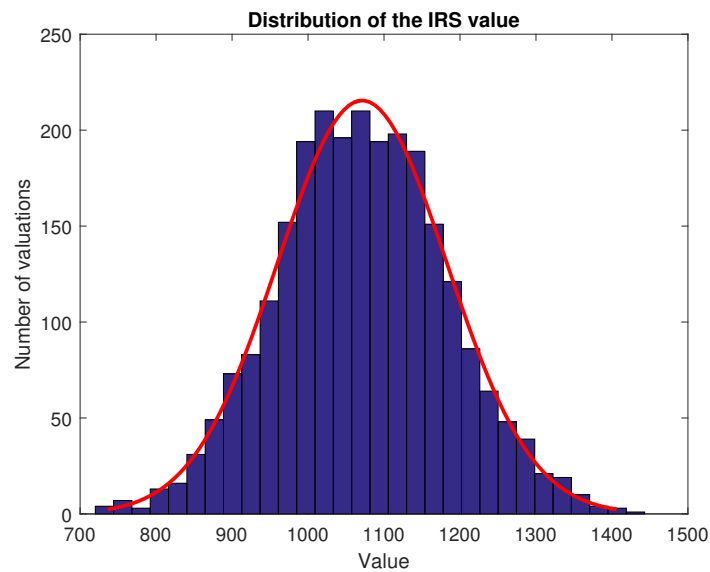


Figure 6.6: The distribution of the value of an interest rate swap at $t = 850$. The rates are stressed calibrated from $t = 0$, the maturity time of the swap is 3 years and 50 stressed and 50 normal scenarios are used.

the price is close to $e^{A(t,T)}$ which is not stochastic. This indicates that the price might be close to normally distributed. To be able to make the approximation of the distribution as good as possible, interest rates are generated and for every rate in each time step the swap is valued as in Section 6.4. In every time step the different values of the swap are collected and their formation studied. The result is shown in Figure 6.6, where the red line corresponds to a normal distribution. The data seems to fit the normal distribution well, thus a hypothesis that the value of the swap is normally distributed is tested with the χ^2 goodness of fit test. The result is that the hypothesis can not be rejected for a p-value of 0.01 and thus the data seems to be normally distributed. An explanation to this result can be that the difference of the lognormally distributed terms might cancel out the skewness from each other and thereby affect the value of the interest rate swap to become normally distributed. When there is just one coupon left, the price is, as explained in Section 4.1, considered as a fixed payment discounted to the current time and so there is no difference of lognormally random variables in this case. Therefore this case is studied extra carefully but even then, the tests give the same result and the formation of the values can be seen in Figure 6.7. Consequently, the distribution of the value of the swap is approximated to a normal distribution at every point in time.

It remains to find the expected value and the variance of the distribution and a first guess is that the expected value can be derived as the sum of the expected values of the *ZCB*, just as if they were independent and normally distributed. Definition 6.5.2 defines the sum of independent normally distributed random variables as discussed in [12] by Ross.

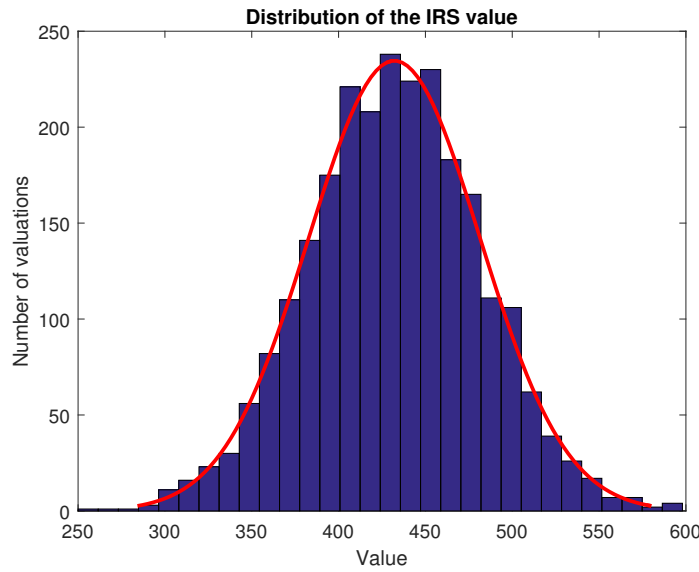


Figure 6.7: The distribution of the value of an interest rate swap at $t = 1000$. The rates are stressed calibrated from $t = 0$, the maturity time of the swap is 3 years and 50 stressed and 50 normal scenarios are used.

Definition 6.5.2. *The sum of independent normally distributed random variables is normally distributed with the following expected value and variance*

$$X_i \sim N(\mu_i, \sigma_i^2),$$

$$\sum_{i=1}^N a_i X_i \sim N\left(\sum_{i=1}^n a_i \mu_i, \sum_{i=1}^n (a_i \sigma_i)^2\right).$$

Using the formula for the expected value of a lognormal distribution in Definition 6.5.1, the Euler discretization, (3.6), and the structure of the price of a ZCB in the Vasicek model, (3.7), the expected value of a ZCB in the Vasicek dynamics is calculated to be

$$E[p(t, T) | \mathcal{F}_t] = e^{A(t, T) + B(t, T)E[r(t)] + B(t, T)^2 \sigma^2 \Delta t / 2}.$$

Now it is possible to calculate the expected value of every term in the equation of the interest rate swap, Equation (6.5). For comparison a large number of short rates are generated and they all give rise to a valuation of the interest rate swap in the same way as in Model 1. Comparing the formation of these values with the sum of the expected values of all terms in the price formula, the analytic calculation of the expected value turned out to be a good fit to the mean of the values from the Monte Carlo simulation. Consequently, the expected value is approximated to be the difference of the expected values of the separate terms in the formula of the price of an interest rate swap.

Due to the successful result of the approximation of the expected value, the same assumption is made in the calculations of the variance. This means that the terms in the expression for the price are assumed to be independent of each other and normally distributed. Consequently, the variance is calculated as the sum of the variances of the separate terms of the ZCB, calculated with the expression for the variance in Definition 6.5.1. This turned out to be bad match, probably since the different terms in the formula of the price of a ZCB, (6.5),

are in reality correlated. Therefore the correlations between the different terms are added to the variance. This resulted in a significant improvement of the result and so, the variance was approximated to be the summarized variances of the separated terms with the correlations added. See Appendix A for details on how the variance and the covariance are derived. Only one problem emerges, namely that in a few cases when the variance of the price of the interest rate swap is approximated in this manner, the result is negative. Of course, this is wrong and it also cause an issue in the succeeding calculations since the standard deviation is needed which is the square root of the variance. Consequently, an additional approximation is needed and therefore in those cases, the variance is set to be very small instead of the negative value.

6.5.3 Calculation of the Exposure At Default

Now it is possible to calculate the EE , which according to formula (5.1) is the mean of the values of the swap after all negative values have been set to be equal to zero. Because of the fact that just one interest rate is generated in each case, in each time step, it is not possible to calculate the mean of several rates as done in Model 1. Instead, the approximated distribution of the price of the swap is used to calculate the EE .

$$\begin{aligned} EE_{t_k} &= \mathbf{E}^{\mathbb{Q}}[\max(x, 0) | \mathcal{F}_{t_k}] = \int_0^{\infty} xf(x)dx = \int_0^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= \frac{\sigma^2}{\sqrt{2\pi}\sigma^2} e^{-\frac{\mu^2}{2\sigma^2}} + \mu(1 - F(0)), \end{aligned}$$

where x is the value of the swap at time point t_k , $f(x)$ the probability density function of x , approximated as derived above, and F the normal cumulative distribution function of x .² Finally, the EEE , the $EEPE$ and the EAD can be calculated in the same way as in Model 1.

6.6 Model 3 - Swap Valuation in the Real World Probability Measure \mathbb{P}

A third model which applies the theory from Section 4.2 is implemented. To begin with, the fixed rate is derived from an expected curve of future rates. The expected curve is given by the expected values of the Vasicek model, in the same way as in Model 2. The discount factors and the expression of the fixed rate are calculated as in Equation (4.5) and (4.9).

6.6.1 Simulation of Interest Rates

The next step is to simulate the rates and as in Model 1, a nested Monte Carlo simulation is used to estimate the value of the future short rates. When a swap is valued in the real world probability measure \mathbb{P} , all rates must be simulated, or forecasted, before any calculations can be made. This means that all rates must be stored which requires a tremendous amount of memory which is a major difference from the valuation in the risk neutral probability measure. Another difference is that the valuation in \mathbb{P} requires simply compounded *LIBOR*

²Note that μ and σ are the mean and the variance of the price respectively and not parameters of the Vasicek model as in the rest of this thesis.

spot rates instead of short rates. Short rate models generate short rates and when this type of model is used for simulation, a complication arises. However, the following procedure shows that the *LIBOR* spot rate can be approximated by the short rate, if the maturity of the *LIBOR* spot rate is short enough. First, setting $S = t$ in the definition of the continuously compounded forward rate R , Definition 2.3.1, yields

$$p(t, T) = e^{-R(t, T)(T-t)}. \quad (6.6)$$

A Taylor expansion of first order of $p(t, T)$ around T results in

$$p(t, T) = e^{-R(t, T)(T-t)} - e^{-R(t, T)(T-t)}(R'_T(t, T)(T-t) + R(t, T))(T-t). \quad (6.7)$$

The following definition from [6] says that when the maturity time goes towards the current time, the continuously compounded forward rate R equals the short short rate,

$$\lim_{T \rightarrow t} R(t, T) = r(t). \quad (6.8)$$

With use of the definition of the *LIBOR* forward rate, Definition 2.5, the Taylor expansion of $p(t, T)$ (6.7) and Expression (6.8) the limit of the *LIBOR* forward rate can be derived as

$$\begin{aligned} \lim_{T \rightarrow t} L(t, T) &= \lim_{T \rightarrow t} -\frac{p(t, T) - p(t, t)}{(T-t)p(t, T)} \\ &= \lim_{T \rightarrow t} -\frac{e^{-R(t, T)(T-t)} - e^{-R(t, T)(T-t)}(R'_T(t, T)(T-t) + R(t, T))(T-t) - 1}{(T-t)p(t, T)} \\ &= \lim_{T \rightarrow t} -\frac{e^{-R(t, T)(T-t)} - 1}{(T-t)p(t, T)} + \frac{e^{-R(t, T)(T-t)}(R'_T(t, T)(T-t) + R(t, T))}{p(t, T)} \end{aligned} \quad (6.9)$$

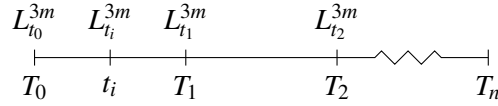
$$= r(t). \quad (6.10)$$

Here, the first term in (6.9) is equal to zero since the exponential term converges faster towards one than $T - t$ does towards zero in the denominator. In summary, this shows that if the frequency of the cash flows in a swap is high enough, the short rates simulated by Vasicek can be assumed to follow the *LIBOR* forward rates.

6.6.2 Valuation of the Interest Rate Swap

Once the rates are forecasted, it is time to value the swap today and in each time step in the future, until maturity of the swap. To do so, a discount curve is needed and derived as in Equation (4.4). Each path of rates has its own discount curve, which means that a future cash flow will be discounted differently depending of the path of the rate. In theory this is not realistic, since two equal future cash flows should have the same present value independently of how the rate is expected to move, otherwise there is an arbitrage possibility. This will be discussed further in Section 8. Once the discount curve for the first time step is calculated, the value of the swap can be determined from Equation (4.6), (4.7) and (4.8). Notice that L_t^p in Equation (4.7) should in this case be the 3 month spot rate determined three months ahead of the actual cash flow (i.e. $p=3$). To be able to value the swap in every future time step, the discount factor must be updated in each time step. The reason for this is that the first term in the Expression (4.4) changes when the payment dates get closer.

Consider the following timeline.



To obtain the present value of all future cash flows at time t_i the rate that represents the time period $t_i \rightarrow T_1$ days is needed. The rate $L_{t_i}^{3m}$ settled in time t_i is a 3 month rate (since 3 month rates are simulated) which for the discounting must be converted to the $T_1 - t_i$ days rate. This rate can be approximated by taking the one day rate to the power of $T_1 - t_i$. The one day rate can in a similar way be approximated by the 3 month rate. This argumentation is shown in the following formulas

$$(1 + L_{t_i}^{1d})^{90} = 1 + L_{t_i}^{3m} \iff 1 + L_{t_i}^{1d} = (1 + L_{t_i}^{3m})^{\frac{1}{90}}$$

$$1 + L_{t_i}^{(T_1-t_i)d} = (1 + L_{t_i}^{1d})^{T_1-t_i} \iff 1 + L_{t_i}^{(T_1-t_i)d} = (1 + L_{t_i}^{3m})^{\frac{T_1-t_i}{90}}.$$

Consequently a way of approximating the rate that is settled today and prevailing in the next cash flow is derived and this new rate is used in the first term when calculating the discount factors, see Expression (4.4). When the discount factor in a certain time step is known it is possible to calculate the present value of the swap in that time step. When the value has been calculated in each time step the *EAD* is obtained in the same way as in Model 1.

6.7 Analytic Swap Valuation in the Real World Probability Measure \mathbb{P}

To improve the speed of the calculations in Model 3, the intention is to derive an analytic solution in the real world probability measure, like Model 2 is in the risk neutral probability measure in Section 6.5. Exactly as in the development of Model 2, the distribution of the value of the swap must be found. The value of an interest rate swap in the real world probability measure is in Section 4.2 derived to be

$$\Pi_{swap} = \Pi_{flt} - \Pi_{fix} = K \sum_{i=1}^N L_{t_i}^p DF_i - KR \sum_{i=1}^N DF_i. \quad (6.11)$$

As discussed in Section 6.5, the rates generated by the Vasicek model are normally distributed. This means that L in Expression (6.11) is normally distributed and consequently the price is a sum of stochastic terms in which the denominators consist of a product of one plus normally distributed random variables, see Section 4.2. It is not likely that such a formation can be approximated to be Gaussian, as in Model 2. In an attempt to find a more reasonable approximation of the distribution of the value, a first order Maclaurin expansion of Expression (6.11) is performed. Unfortunately, the result was not successful when comparing this value to the mean of the simulations in Model 3. Further effort was not added to this problem since other parts of the analysis in this thesis were prioritized.

6.8 Weighting of the Stress in the Models

The *KVA* considers extreme scenarios and banks must take this adjustment into consideration to be prepared for the risk followed by sudden financial stress. Therefore the stressed

EAD is studied in this thesis and thus normally calibrated short rates are generated, which in turn give rise to stressed calibrated short rates at each time step. This means that the rates jump from being in a calm market to being in a stressed market over one night. In the real world a stressed period often arises during a longer period than just one night. Therefore it was settled during a discussion with Nordea that the normal and stressed values should be combined with a weight, w , to make the result as realistic as possible. In this way, the market gets gradually more stressed as time goes by and is finally completely stressed. The weight is implemented to increase linearly and to start at 0% stress and 100% normality in the market and to be the other way around after five years. Subsequently, the market is assumed to be completely stressed until maturity of the swap. Consequently, the swaps with a life time less than five years will never be completely stressed. Note that the normal values are not affected by the weighting procedure. Equation (6.12) shows the stressed values after the weighting.

$$\Pi_t^S = \Pi_t^S wt + \Pi_t^N (1 - wt) \quad (6.12)$$

Here the weight is set to be $w = \frac{1}{360.5}$ and $0 \leq t \leq T$. In other words, the short rates that become stressed at the starting point will be valued with no weight on the stressed parameters despite that they should get stressed in this time point. In the succeeding time steps the stressed weight of the parameters increase linearly as described above. Although, for the short rates that become stressed in a future time step, they will get weights equal to the weights of the short rates starting in the starting point. Accordingly, they will go from being in a normal market to being in a $t \cdot w$ stressed market over one night and after that get gradually more stressed, like the ones starting in the starting point.

Chapter 7

Results

“As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality.”

– Albert Einstein

This chapter presents the results obtained from the calibration and the different valuation methodologies. It also presents a comparison between the *EAD* curves obtained from the models used in this thesis and the curve created by Nordea’s proxy model. Unless anything else is mentioned, the upcoming results consider a payer swap in Nordea’s perspective with a principal amount of $K = 100\,000\text{ EUR}$.

7.1 Calibration of the Parameters

The Kalman filter in combination with MLE of the parameters yields an estimate of the short rate corresponding to the historical data which is described in Section 6.2. The normally calibrated short rate is shown in Figure 7.1 and the short rate calibrated to the stressed data is shown in Figure 7.2. The parameters corresponding to the short rates given by the optimization can be seen in Table 7.1.

Table 7.1: Parameters calibrated through maximum likelihood estimation

	Normal parameter	Stressed parameter
κ	0.0797	0.3447
θ	0.2533	0.0868
σ	0.1503	0.1719

Despite the fact that the short rates in Figure 7.1 and 7.2 look realistic, the parameters are not, which means that the parameter estimation failed. This will be discussed in Section 8. Since the Kalman filter generates short rates that fits the data, these can be used to manually estimate the parameters instead of using the unrealistic results from the *MLE* optimization, which is discussed Section 6.2. Therefore $\hat{\theta}$ is assumed to be the mean of the short rate,

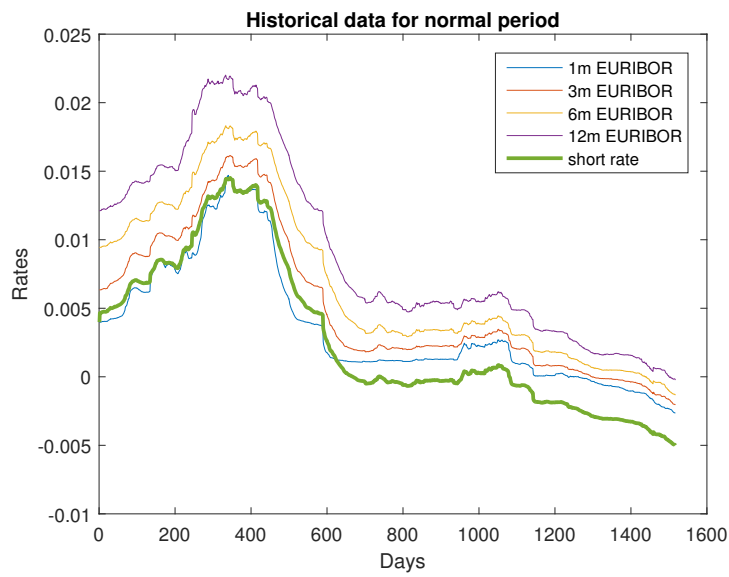


Figure 7.1: The short rate fitted to historical data from the normal period 2010-03-29 to 2016-02-29

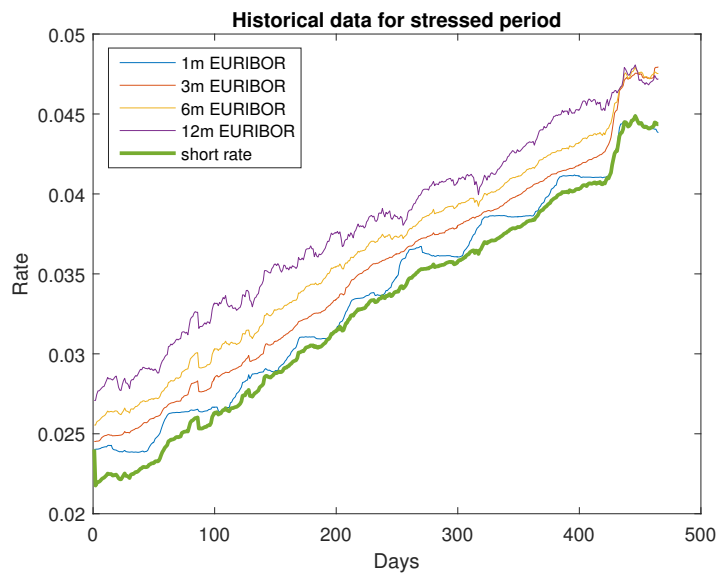


Figure 7.2: The short rate fitted to historical data from the stressed period 2005-12-07 to 2007-11-12

show in Figure 7.1 and 7.2, given by the optimization. The $\hat{\sigma}$ -parameter stands for the standard deviation and is estimated by $\hat{\sigma} = \sqrt{\frac{\sigma^2}{2\hat{\kappa}}}$, where σ is the standard deviation of the short given by the optimization. Since the stressed data, shown in Figure 7.2, drifts away from its previous mean, see Figure 6.1, it is not a mean reverting process during the stressed period which means that the mean reversion, $\hat{\kappa}$, is weak. Therefore $\hat{\kappa}$ can, in the stressed case, be assumed to be lower than in the normal case. Through this reasoning the new parameters, that will be used in all future simulations, are shown in Table 7.2.

Table 7.2: Parameters for the Vasicek model

	Normal parameter	Stressed parameter
$\hat{\kappa}$	1.4	0.9
$\hat{\theta}$	0.0028	0.0334
$\hat{\sigma}$	0.0033	0.0051

The fixed rates, calculated in the different models, using Equation (4.3) and (4.9) respectively, are shown in Table 7.3.

Table 7.3: Fixed rate

	Model 1	Model 2	Model 3
Fixed rate	0.0985%	0.0968%	0.0633%

7.2 Exposure At Default for the Different Models

Different *EAD* curves obtained from the three different models are shown in Figure 7.3. Here, Model 1 and Model 3 considers 25 normal and 25 stressed scenarios. Model 2 does not use any Monte Carlo simulations and is thus substantially faster than the other models. Following the argumentation in Section 6.4 and studying Algorithm 3, the number of short rates that must be generated in the last time step in Model 1 and Model 3 is $N \cdot S \cdot T + N$, where N is the number of normally calibrated short rates, S the number of stressed calibrated short rates and T the maturity time. From the first to the last time step,

$$\sum_{t=1}^T (N \cdot S \cdot t + N) = \frac{T(N \cdot S \cdot T + N \cdot S)}{2} + N \cdot T$$

simulations are made in these models. For example when $N = 50$, $S = 50$, $T = 3$ years and the length of each time step is one day, the number of interest rates that must be simulated is 1 459 404 000. Of course this means that the computational burden to calculate the *EAD* with Model 1 and 3 is large and the algorithm is slow for large N , S or T . Studying Algorithm 4, it can be seen that the number of simulated short rates in the last time step in Model 2 is reduced from $T \cdot N \cdot S + N$ to $T + 1$, because there are T stressed and one normally calibrated short rate. Furthermore the total amount of simulations are

$$\sum_{i=1}^T (1 + i) = \frac{T(1 + T + 1 + 1)}{2} = \frac{T(T + 3)}{2} \quad (7.1)$$

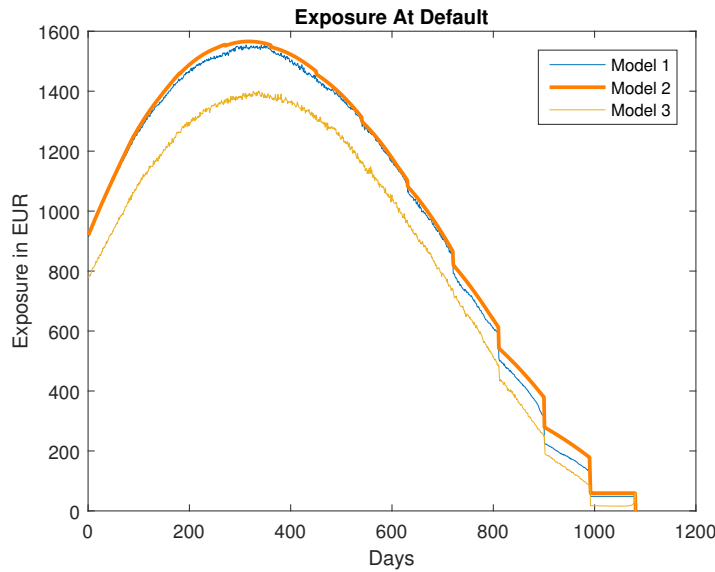


Figure 7.3: *EAD* curves from the different models

and the number of simulations would instead be 584 820 and thereby be reduced by 1 458 819 180 simulations or equivalently a reduction by 2 496 times. There is an even larger difference when N , S or T is increased.

The time required to obtain the different curves in Figure 7.3 is presented in Table 7.4.

Table 7.4: Time requirement of the models

	Model 1	Model 2	Model 3
3y 25 normal 25 stressed	32min	30s	1h59min
3y 50 normal 50 stressed	2h30min	30s	–
7y 20 normal 20 stressed	18h13min	5min2s	–

Model 3 requires a lot of memory and therefore it is not possible to test that model for more than 25 normal and 25 stressed scenarios. It is however interesting to see how the results from Model 1 are improved when additional scenarios are used. Figure 7.4 and Figure 7.5 show that the result from Model 1 converges towards the result from Model 2 when sufficiently many scenarios are used. It is hard to distinguish between the lines in Figure 7.5, therefore a zoomed plot of the same lines is presented in Figure 7.6

To be able to present a fair comparison between Model 1 and Model 2, the simulations in Model 2 are performed several times, to decrease the Monte Carlo error. As shown earlier, there is a relation between the number of scenarios and the accuracy of the result, but increasing the number of scenarios also increases the time requirement of the model. Thus, the number of scenarios must be chosen in a way that trustworthy result are generated whilst the time requirement is kept at a reasonable level. The combination of 30 normal and 30 stressed scenarios meet the criteria. The mean of 50 simulations of the *EAD* curve from Model 1 is plotted together with the *EAD* from Model 2 in figure 7.7, to show how Model 1 converges towards Model 2 when a large number of simulations is used. To simplify the

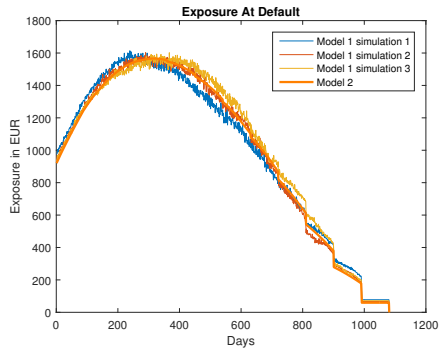


Figure 7.4: EAD curves from Model 1 and 2. In Model 1, 5 normal and 5 stressed scenarios are used.

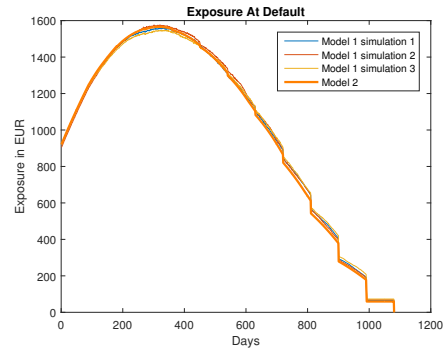


Figure 7.5: EAD curves from Model 1 and 2. In Model 1, 50 normal and 50 stressed scenarios are used.

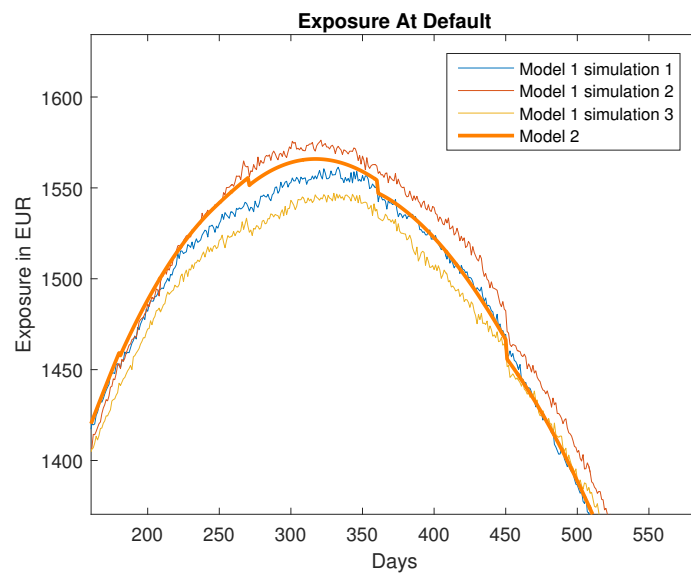


Figure 7.6: A zoomed plot of the EAD curves from Model 1 and Model 2 in Figure 7.5. In Model 1, 50 normal and 50 stressed scenarios are used.

comparison between the *EAD* curves in Figure 7.7, a zoomed plot of the same figure is provided in Figure 7.8.

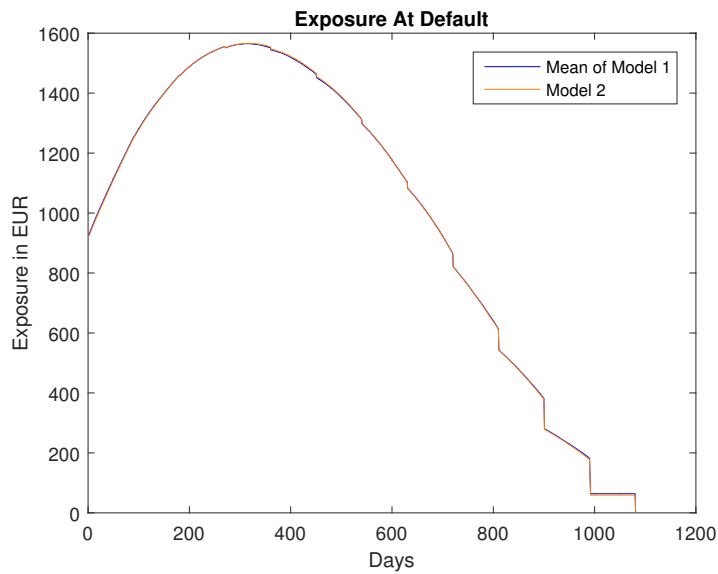


Figure 7.7: The mean of 50 simulations of the *EAD* curve with Model 1 with 30 normal and 30 stressed scenarios, together with the *EAD* from Model 2.

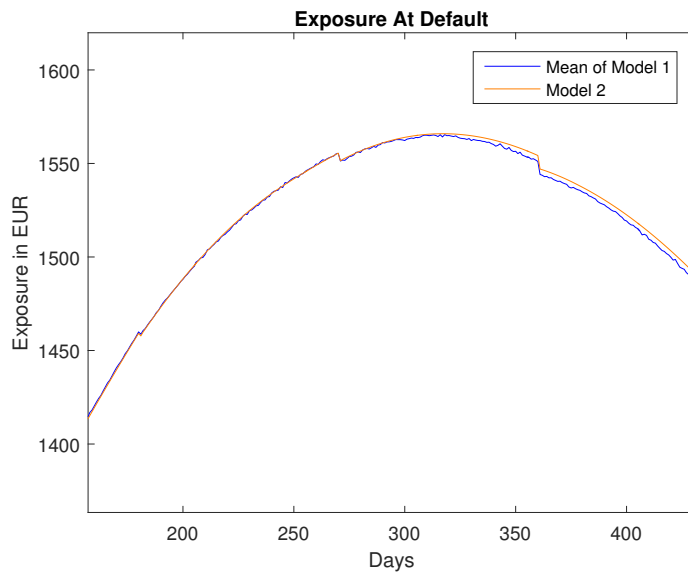


Figure 7.8: A zoomed plot of Figure 7.7

A swap with a longer maturity time is considered as well, to see how or if the result changes when the swap has a lifetime which allows the stressed period to have full weight. A swap with life time 7 years and 20 normal and 20 stressed scenarios in Model 1 is compared to the results from Model 2 for a swap with the same life time in Figure 7.9. In Model 1, it is too time consuming to consider more than 20 normal and 20 stressed scenarios for a life

time of this length. The amount of time required by the different models is presented in Table 7.4 above.

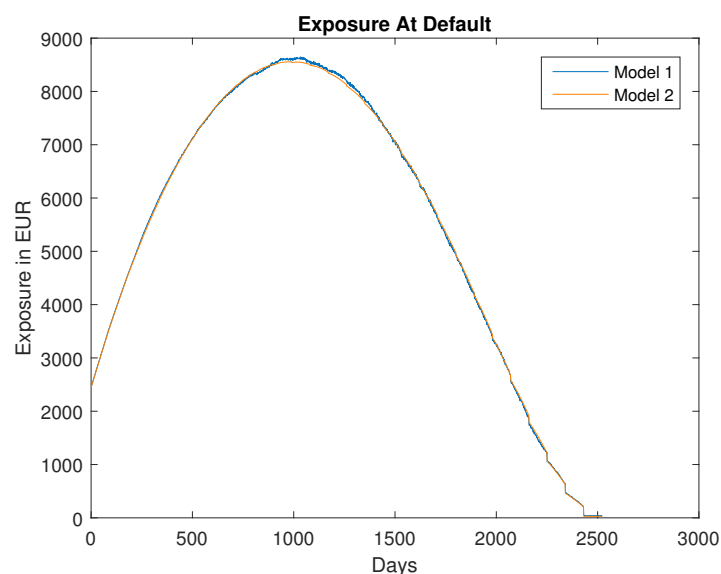


Figure 7.9: The *EAD* curves obtained from the valuation with Model 1 plotted together with the *EAD* curve obtained from the valuation in Model 2. Here 20 normal and 20 stressed scenarios are used for Model 1, the life time of the swap is 7 years and the fixed rate is 0.2%.

7.3 Further Results

Since Model 2 is free from Monte Carlo errors and only one new short rate is generated in each time step, see Section 6.5, the graphs of this model are easier to analyze than the graphs of the other models. Furthermore the results from Model 1 are similar to the results from Model 2 and therefore the results from Model 2, shown in this section, can be seen as representative of the results from Model 1 as well. Figure 7.10 shows how the stressed and normally calibrated short rates starting at $t = 0$ evolve and Figure 7.11 shows the expected evolution of the 3 month *EURIBOR* spot rates. In reality the *EURIBOR* spot rates vary continuously but in this figure only the rates which will be used for the cash flows are shown and updated at each payment date.

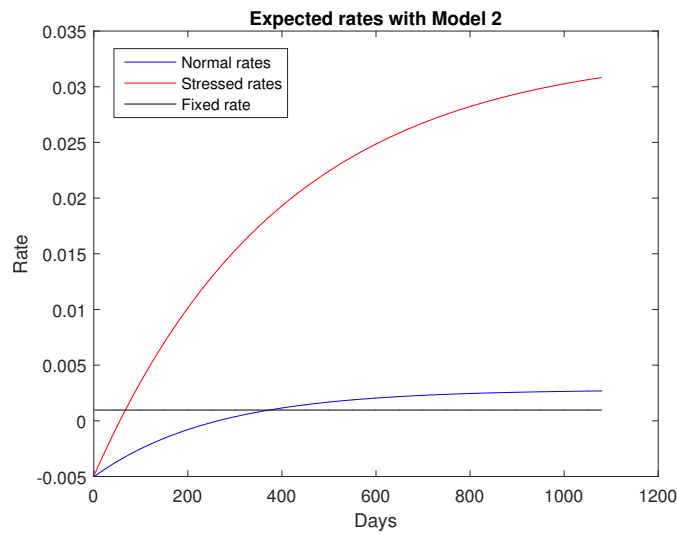


Figure 7.10: The expected paths of the normally and the stressed calibrated short rates starting at $t = 0$, plotted together with the fixed rate. The path with stressed rates intersect the fixed rate at $t = 68$ days, and the normal rate intersects at $t = 373$ days.

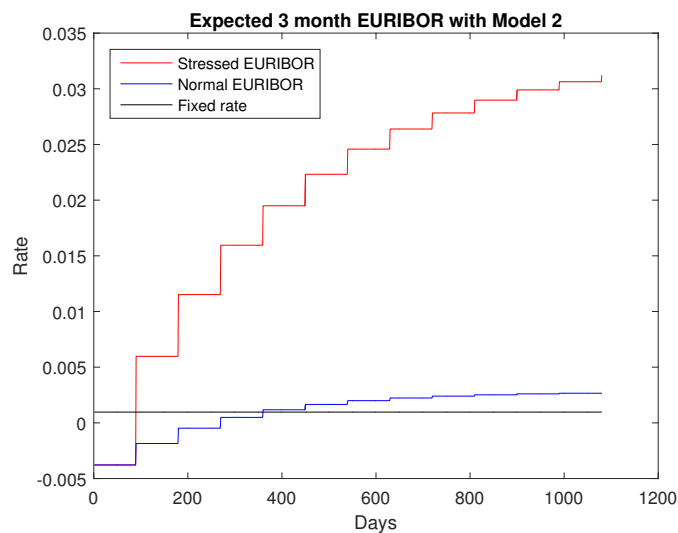
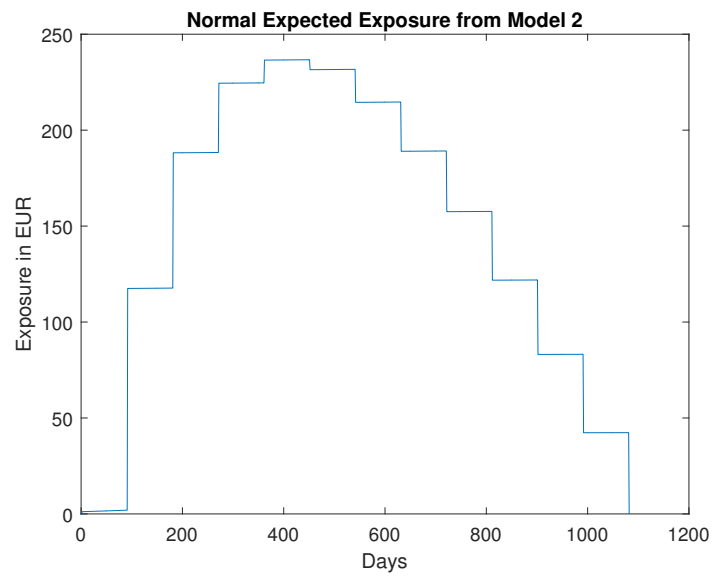
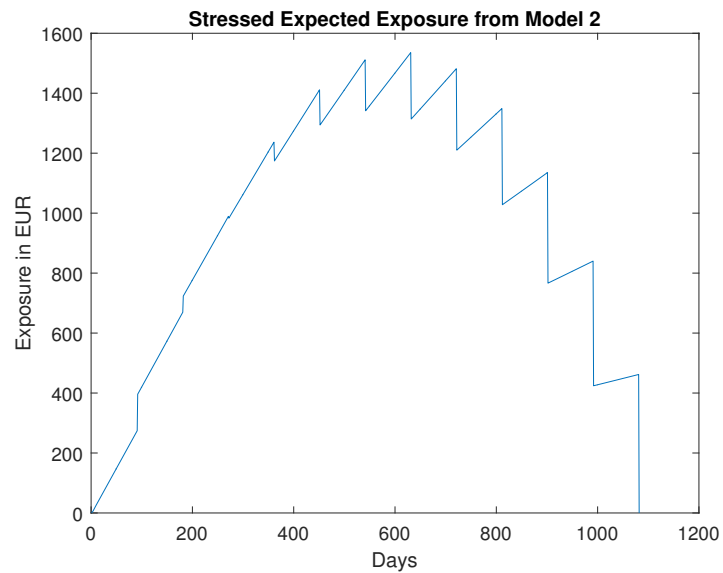


Figure 7.11: The evolution of the stressed and normal *EURIBOR* forward rates plotted together with the fixed rate. The stressed *EURIBOR* forward rate curve intersect the fixed rate at $t = 90$ days and the normal curve intersects at $t = 360$ days.

Figure 7.12 shows how the normal *EE* evolves through time and Figure 7.13 shows the evolution of the stressed *EE*.

Figure 7.12: The normally calibrated EE Figure 7.13: The stressed calibrated EE .

As explained in the previous chapters, an EEE curve is generated in each time step and all these are shown in Figure 7.14. This result is not essential for the further discussion in Section 8 but it is provided for the interested reader.

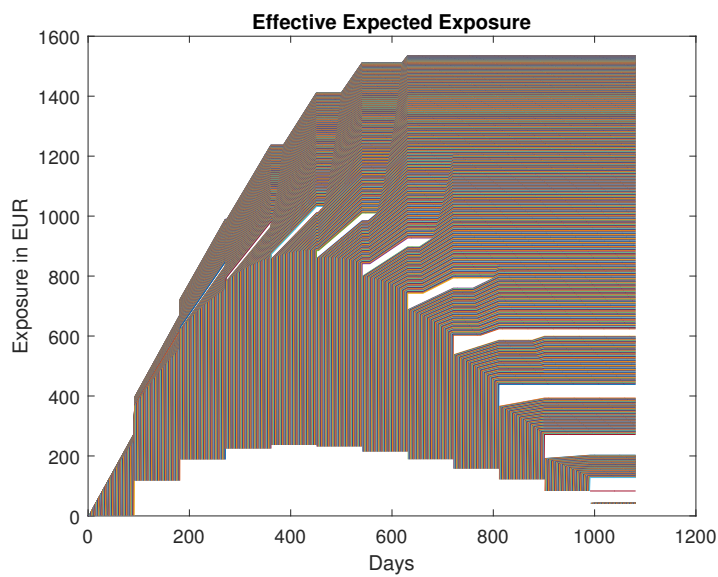


Figure 7.14: The EEE curve starting at each time point calculated by Model 2.

7.4 Comparison Between the Measures

The comparison between the *EAD* curves in the two measures is already shown in Figure 7.3. This result is derived from the *EE* curves and therefore they are essential in the analysis of the appearance of the *EAD* curves. Figure 7.15 shows the normally calibrated *EE* while Figure 7.16 shows the stressed calibrated *EE*.

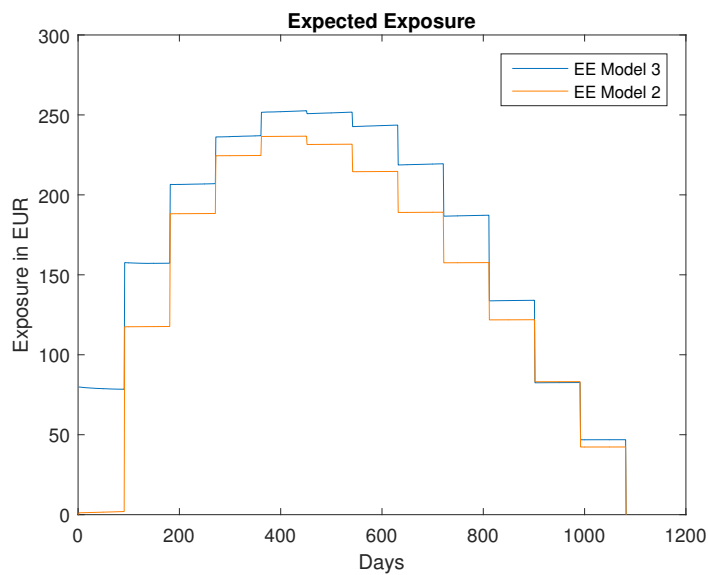


Figure 7.15: The normally calibrated *EE* from Model 2 and Model 3. 20 normal and 20 stressed scenarios are used in Model 3.

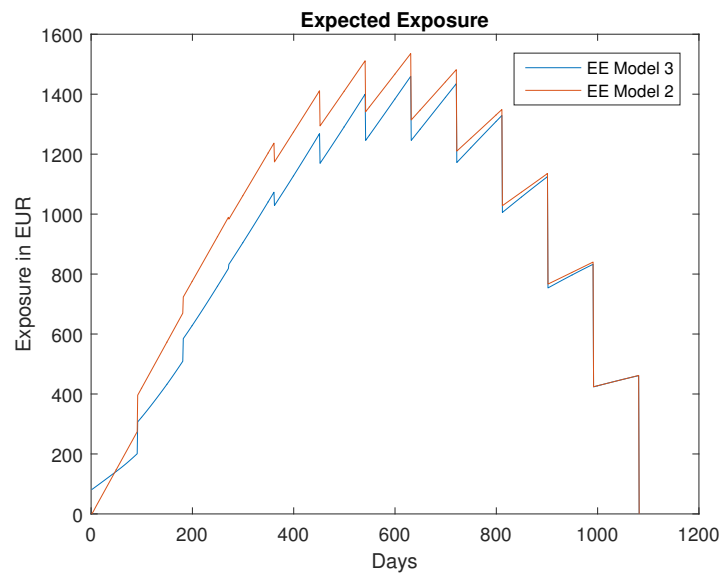


Figure 7.16: The stressed calibrated *EE* from Model 2 and Model 3. 20 normal and 20 stressed scenarios are used in Model 3.

7.5 Comparison with Nordea's Proxy Model

To compare Nordea's model with the results in this thesis a confidence interval is created. The 50 simulations of the *EAD* curve that were performed to obtain the mean in Figure 7.7, required 60 hours of computational time. From the result, it is possible to find the standard deviation which can be used to create confidence intervals of different levels. The result is tested against a 95% and a 99% confidence interval and can be seen in Figure 7.17 and Figure 7.18.

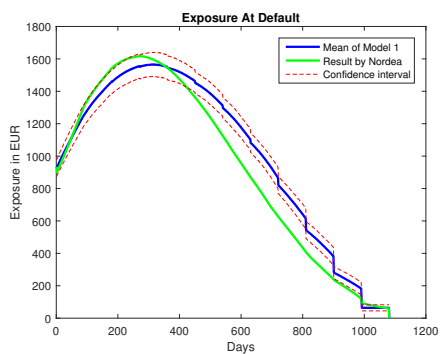


Figure 7.17: The results from Nordea's model together with the mean of 50 simulations with Model 1, where 30 normal and 30 stressed scenarios are used. The confidence level is 95%.

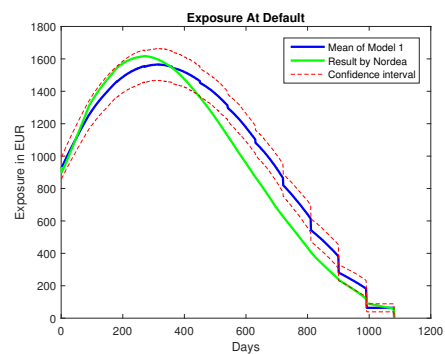


Figure 7.18: The results from Nordea's model together with the mean of 50 simulations with Model 1, where 30 normal and 30 stressed scenarios are used. The confidence level is 99%.

When studying Figure 7.17 and Figure 7.18 the conclusion is drawn that Nordea's proxy

model underestimates the *EAD*. Nordea was provided with this information and recalibrated their model according to the results of this thesis. The result of the recalibration is shown in Figure 7.19 and Figure 7.20.

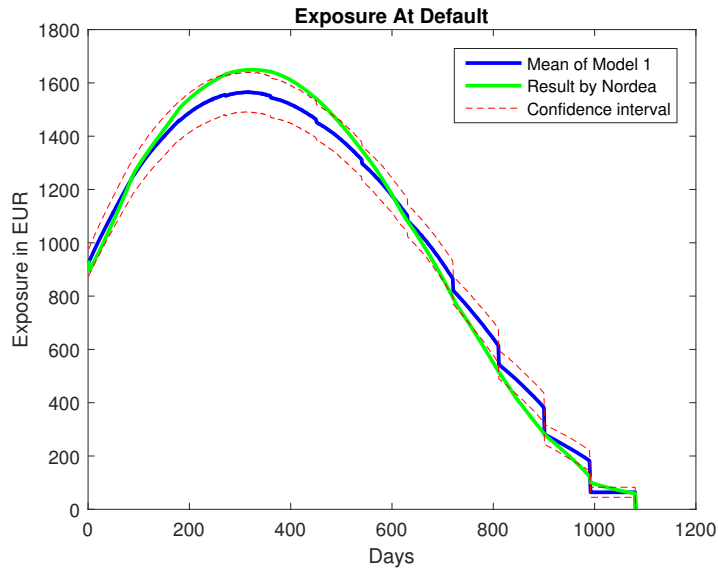


Figure 7.19: The results from Nordea's model together with the mean of 50 simulations with Model 1, where 30 normal and 30 stressed scenarios are used. The confidence level is 95%.

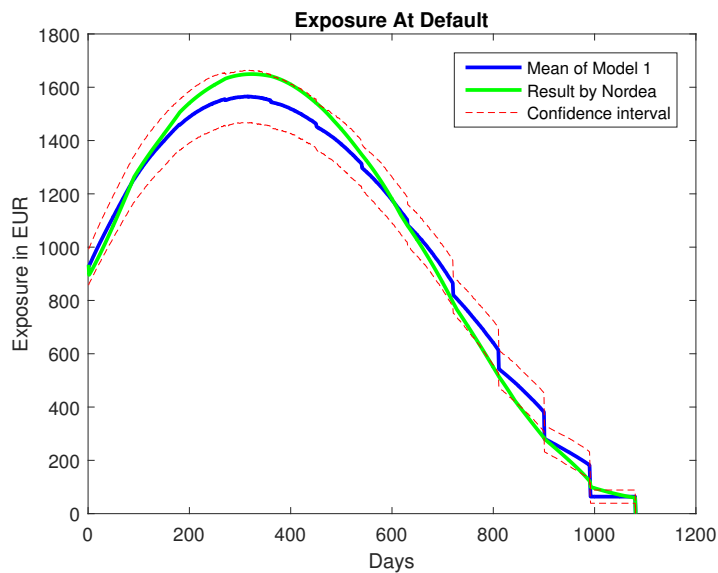


Figure 7.20: The results from Nordea's model together with the mean of 50 simulations with Model 1, where 30 normal and 30 stressed scenarios are used. The confidence level is 99%.

Chapter 8

Discussion

“Absence of evidence is not evidence of absence.”

– Carl Sagan

The results shown in Chapter 7 and the approaches explained in Chapter 6 are discussed in this chapter. Furthermore the choices made in this study are reflected and suggestions for future research provided.

8.1 The Parameters

As mentioned in Section 7.1, the parameters estimated by *MLE* are unrealistic. Studying the parameters in Table 7.1 it can be seen that θ in the normal case is estimated to be 0.2533. Since θ is the long term mean it should be similar to the mean of the previous rates. In this case, the long term mean is estimated to be 100 times larger than the mean of the historical data, which indicates that there is a problem with the parameter estimation. Since the calibration of the model was not supposed to be a central part of this thesis, nor important for Nordea, a simple manual estimation of the parameters was made, described in Section 6.2. Even though the parameters in the stressed calibration can be assumed to be realistic they were manually estimated as well, to be consistent between the two calibrations.

This thesis has used the same parameters when valuing in the different measures. However to be precise, the parameters should be modified when changing measure. The Kalman filter, explained in Section 3.4, consists of two parts, one prediction step and one updating step. The prediction equation can be assumed to be in measure \mathbb{P} since the historical data is in this measure. The measurement equation on the other hand is in measure \mathbb{Q} , see Section 6.2. Because of the fact that there is a strong diffusion term at the same time as there exists a lot of data that enable the measurement residual to be small, the Kalman filter generates short rates more affected by the measurement equation than by the prediction equation. Thus, the optimized parameters are a combination of the parameters from the two measures but are probably closer to the parameters in measure \mathbb{Q} . However, the assumption that the combined parameters can be used in both measure should not affect the result extensively.

8.2 Comparison Between the Developed Models

8.2.1 Justification of Model 1 and Model 2

In Model 2, which is developed in Section 6.5, there is no randomness added. This means that the result is exactly the same in each simulation of the *EAD* curve, as long as the parameters are kept the same. Accordingly, this method calculates the exact value of the *EAD* and no Monte Carlo error is affecting the result. There are however some assumptions made, for example the unknown distribution of the value of the swap is approximated. Of course these assumptions induce shortcomings in the result which can be seen as systematic errors. In Model 1 on the other hand, there exist Monte Carlo errors. When comparing Figure 7.4 with Figure 7.5 it is obvious that a higher number of stressed and normal scenarios are needed to receive an accurate result which is consistent with the theory in Section 3.5. In Figure 7.7 it is possible to see that as the number of simulations is increased, the result of Model 1 converges towards the result in Model 2. Due to that, the results from Model 2 are concluded to be accurate, which means that the systematic error is small. Furthermore, it is the fastest model and therefore Model 2 is considered to be better than Model 1. However, the Monte Carlo simulations from Model 1 were used in the development of Model 2 and therefore Model 1 is essential as well. Moreover, Model 2 requires a short rate model that generates normally distributed short rates while any short rate model is applicable for Model 1.

8.2.2 Assumptions made in Model 3

The calculations in Model 3 are more time consuming than in the other models, which is shown in Table 7.4. This is because of the fact that all rates that will be generated in the future must be stored unlike in Model 2 where only the rates generated in the current time step are stored, see Section 6.4 and Section 6.6. Moreover, as indicated in Section 3.1 the calculations become more complex in the real world probability measure than in the risk neutral probability measure. Furthermore, Model 3 require that some assumptions are made which might affect the accuracy of the result. For example the 1 day *EURIBOR* spot rate is approximated. Another approximation in this model is that the 3 month *EURIBOR* spot rate, which is supposed to be used in the calculations, equals the generated short rate. As seen in Figure 7.1 and Figure 7.2 the short rates are in reality lower than the *EURIBOR* spot rates¹. Due to this, the cash flows that the floating rates originate are lower than they should be when calculating with the short rates instead of 3 month *EURIBOR* spot rates. An indication of how this assumption has affected the calculations can be seen in Table 7.3 where it is shown that the fixed rate in Model 3 is lower than in the other models. Since the fixed rate depends on the future cash flows as explained in Section 4.2, the fixed rate would be underestimated with this argumentation. A further analysis of the accuracy of Model 3 is discussed in Section 8.2.5 below.

8.2.3 The Starting Point of the Expected Exposure Curve

Due to the fact that the valuation of the fixed rate is dependant on the future cash flows in measure \mathbb{P} , see Section 6.6, contracts with the same fundamental conditions could give

¹An exception is during the Eurozone crisis where the short rate exceeded *EURIBOR* spot rates

different fixed rates. The reason to this is that when estimating the fixed rate, future floating rates are generated and the value of these decide what the value of the fixed rate should be. Consequently two different investors would be offered two different fixed rates despite that the contract otherwise would be exactly the same. This is not solid with the arbitrage theory discussed in Section 2. Due to this problem a fixed rate is derived from the expected curve of future rates as described in Section 6.6. This means that every investor will be offered equal and fair contracts where individual futures are not considered. Now the requirements of an arbitrage free market should be fulfilled but there is one disadvantage. Since the fixed rate is set equally for all different simulations of future floating rates and not individually for each contract, the contracts are not all worth zero at the starting point. Due to that, nor the *EE* is valued to be zero at the starting point and this differs from the results in Model 1 and Model 2, see Section 6.4, Figure 7.15 and Figure 7.16. A natural intuition is that when the contract is At The Money at the contracting time, the *EE* should be zero since the contract should be equally worth for both parts. This is the main reason why the comparison between Nordea's proxy model and Model 3 is omitted in the result.

8.2.4 The Shape of the Expected Exposure Curve

Studying Figure 7.12 and Figure 7.13 it can be seen that the *EE* curve has a pattern of a sawtooth with jumps at the payment dates. The observant reader will discover that the first jumps are upwards while the last ones are downwards. The intuition behind this is that when passing a payment date where the concerned part has to pay more than what is received, the value of the swap for this part is increased. This depends on that the debt to the counterpart has decreased. Of course, if the concerned part instead receives a positive coupon the value is decreased. In this case the concerned part has received a positive cash flow and has less to look forward to during the remaining lifetime of the swap. Accordingly, for the payer of the swap, positive jumps should occur as long as the fixed rate is higher than the floating rate and negative jumps otherwise. When comparing Figure 7.10 with Figure 7.12 and Figure 7.13 it seems as if the change in the directions of the jumps arrives at a later time point than when the floating rate crosses the fixed rate. This depends on the cash flow at the first payment date that is handled separately as a fixed payment, as described in Section 4.1. Therefore, Figure 7.11 which shows the 3 month *EURIBOR* spot rate that is used in the fixed payment, should be considered as well. Here, it is possible to see that the *EURIBOR* spot rates exceed the fixed rate about three months before the jumps in the *EE* curves turn downwards and since the 3 month *EURIBOR* spot rates are used to calculate the cash flows three months later, the intuition is proved to be correct.

The *EE* changes during the lifetime of the interest rate swap which can be seen in Figure 7.12 and Figure 7.13. These figures show the *EE* in a payer's perspective and the following analysis is therefore in this perspective. There are two effects fighting towards each other to affect the exposure. The first one depends on the diffusion, namely that there exists a possibility that the floating rate can increase. This would increase the value of the swap and thus push the *EE* curve upwards. The other effect arises from the amortisation since that decreases the future risk of the holder. This effect pushes the *EE* curve downwards. In the beginning of the lifetime of the interest rate swap, the diffusion effect dominates while the amortisation effect dominates when the maturity approaches. This is because in the beginning, no or few amortisations has been made. Furthermore, when there is long time left until maturity the rates are given the opportunity to vary more which increases the exposure

and thus giving more weight to the diffusion effect. When the time to maturity decrease and more amortisations has been made, a new amortisation constitutes a larger percentage of the remaining cash flows, giving more weight to the amortisation effect. Finally, the last payment date is passed and then there is no obligations for any part and that is why the EE becomes zero.

8.2.5 Differences Between the Measures

Studying Figure 7.3 it is obvious that the EAD is underestimated when calculated in measure \mathbb{P} , see Section 6.6, compared to when it is calculated in measure \mathbb{Q} , see Section 6.4 and Section 6.5. As explained in Section 3.1 the risk must be considered when pricing derivatives since the buyers have risk preferences. When pricing in measure \mathbb{Q} the risk premium is included automatically by the risk neutral valuation formula (3.1.2), but in measure \mathbb{P} it has to be added to the value after the discounting of the future cash flows. To calculate the risk premium is complex since all investors risk preferences must be collected and valued and this thesis has omitted this part of the price in Model 3. Due to that a difference between the EAD calculated in the different measures was expected. Most investors are risk averse and are thus willing to pay less for a risky derivative. Due to that the risk premium should be a negative contribution to the price of the interest rate swap, valued in measure \mathbb{P} . This is a contradiction of the underestimation compared to the results from measure \mathbb{Q} . However the results in this thesis reflect a payer's perspective and thus a negative risk premium would instead match the price correctly from a receiver's perspective. In summary, it might be difficult to decide the sign of the risk premium without actually calculating it and studying other articles within the field, among others [17] and [18] both signs are usual.

When pricing under measure \mathbb{Q} there are more aspects to consider than in measure \mathbb{P} , for example the risk preferences. Therefore this measure has a tendency of faster adapting changes in the surroundings while measure \mathbb{P} is more stable when the risk premium is not considered. For example if the market gets stressed, the risk increases and the exposure in measure \mathbb{Q} , which includes the risk premium, is affected more than the exposure in measure \mathbb{P} . This can be a reason to that the normally calibrated EE curve in measure \mathbb{P} lies above the normally calibrated EE curve in measure \mathbb{Q} in Figure 7.15 while it is the other way around when the stressed calibrated EE curves are considered in Figure 7.16. This conclusion demonstrates that the EAD curve derived in measure \mathbb{P} would lie above the EAD curve derived in measure \mathbb{Q} during normal conditions and below during stressed conditions. Thus the risk premium should change sign depending on the market conditions which seems realistic since risk preferences are highly correlated to the expectations on the market.

In summary the difference between the results obtained in Model 3 and the other two models is assumed to depend on the absence of the risk premium in Model 3. Following the discussion above, the risk premium has not been calculated but it is realistic that is formed as the difference between the EAD curves obtained in the different measures. Thus even the result from Model 3 is assumed to be accurate, except for the absence of the risk premium, even though some assumptions has been made as discussed in Section 8.2.

8.3 Comparison with Nordea's Proxy Model

In Section 7.5, confidence intervals were generated to validate Nordea's proxy model. Usually when producing confidence intervals, a large number of simulations are made. Since Model 1, which was used to produce the confidence intervals, is very time consuming, 50 simulations of the *EAD* curve were made which required 60 hours of computational time. As can be seen in Figure 7.7 the mean of these simulations are very close to the result of Model 2. Therefore the assumption that the confidence intervals produced by the 50 simulations are accurate was made.

Studying Figure 7.17 and Figure 7.18, it can be seen that the *EAD* curve generated by Nordea's proxy model falls outside both the 95% and the 99% confidence interval. This means that there is a difference in the *EAD* approximated with Nordea's proxy model and the real value calculated by the models in this thesis. Since Nordea's *EAD* curve is outside the 95% interval, this difference is that large that this thesis must reject the hypothesis that the proxy model generates accurate results. The recommendation to Nordea is therefore that they should review their proxy model.

After this recommendation, a recalibration of the proxy model was performed. Thereby, the result is most often within the 95% confidence interval and almost always within the 99% confidence interval, see Figure 7.19 and Figure 7.20. It is a question of interpretation whether the new results can be assumed to be accurate or not. In a discussion with Nordea the conclusion that their updated model is trustworthy is drawn.

8.4 Choices Made in the Study and Further Research

The study was performed in two different measures, the real word measure and the risk neutral probability measure. The reason to this is that the \mathbb{Q} measure was chosen due to the simplifications of the calculations explained in Section 3.1 and discussed in Section 8. It is however more common among banks to derive the *EAD* in the \mathbb{P} measure, according to Nordea. Therefore, Nordea requested a result in this measure. Furthermore the authors were interested in studying the difference between the measures.

Since *EURIBOR* spot rates are needed when valuing the interest rate swap in Model 3, it seems natural to practise a model that directly generates this type of rates, for instance the *LIBOR* market model. Unfortunately, it is very time consuming to generate rates with such models, since a correlation matrix consisting of the correlations between the simulated rates must be stored. Since all future rates already need to be stored when calculating in Model 3, a short rate model was considered instead to improve speed of the code. This resulted in that the short rate had to be approximated to equal the *EURIBOR* spot rate.

The choice of short rate model was not essential to the comparison with Nordea's model since the parameters received from the calibration in this thesis are used in Nordea's proxy model as well. Thus the comparison would not be affected by a bad choice of short rate model and therefore a simple short rate model could be used. The possibility of generating negative interest rates, which is a property of the Vasicek model, has previously been seen as a disadvantage of the model. This is because negative interest rates were considered unrealistic but with the current market conditions this property is actually an advantage. Therefore thanks to this property and the simplicity of the model, the Vasicek model was

used in this study. Worth to mention is that Model 2 is based on the property that the Vasicek model generates normally distributed short rates. This is a necessary property if Model 2 is going to be used and thus a more complex short rate model without this property would require another approximation of the distribution of the value of the swap in Model 2. However, the Vasicek model caused problems during the work since the model was difficult to fit to the complex data and unnecessary time was spent on this. Moreover, as discussed in Section 6.1 the desired stressed period could not be used since the Vasicek model did not manage the high volatility. Therefore, we are convinced that the result would be improved if a two factor model is used instead of the Vasicek model because it would simplify the fitting of the model to the data. Furthermore, it would probably also be more time effective even though a more complex short rate model would be used.

We chose to develop a new method, Model 2, to improve speed of Model 1 and to reduce the Monte Carlo error. Alternatively, the nested Monte Carlo simulation could have been improved by using for instance sequential Monte Carlo methods.

In Section 6.7 a Maclaurin expansion of first order of the price of an interest rate swap in the real world probability measure is made. The expansion depends on the 3 month *EURIBOR* spot rate and was made to find a less complex expression of the price. The result was not satisfying when compared to the Monte Carlo simulations in Model 3 in Section 6.6 but since three operating models had been developed prior to the attempt to develop this fourth model, no further effort was added to this problem. If a less time consuming derivation of the *EAD* in the real world probability measure is needed, a higher order Maclaurin expansion or a Taylor expansion around a general point can be derived. This might generate a more accurate result but is left for future research to prove.

In Section 6.8 a weight is introduced to linearly increase the influences of stress in the market. The weight is chosen to be linearly increasing in this thesis since the models are tested for the worst case scenario of stress, which means that the market should evolve towards this stress level. Otherwise, a hidden Markov chain might be a good and more realistic reflection of the movements in financial markets and therefore an interesting analysis might be to implement a hidden Markov chain as a weight parameter, instead of the linearly increasing parameter.

Finally, this thesis has only considered interest rate swaps and the reason for that is that the interest rate swap is a popular product on financial markets and thus an important product for Nordea. It is also rather simple to value since the value of an interest rate swap only depends on one risk factor, namely the rate, unlike for instance options which are dependent on the rate as well as the underlying asset. A suggestion for further analysis is to consider more advanced derivatives. Moreover, this thesis has not considered netting, collateral or portfolio effects which therefore is another suggestion for further research. Since derivatives most often are traded together, in portfolios, such a study would actually be more realistic.

Chapter 9

Summary and Conclusion

“An approximate answer to the right problem is worth a good deal more than an exact answer to an approximate problem.”

– John Tukey

The purpose of this thesis was to validate if Nordea’s proxy model for *EAD* approximation generates trustworthy results. To succeed with this, this thesis has studied euro interest rate swaps of different maturities. First, calibrations of a short rate model for normal and stressed periods were performed to be able to forecast future short rates. This calibration used data of historical *EURIBOR* spot rates of different maturities and was implemented through *MLE* and Kalman filtering. Then the short rate model could be used to simulate future short rates which in turn gave rise to a valuation of the swap. After that it was possible to calculate the *EAD*.

Three different models were developed for the purpose. First, the risk neutral probability measure was considered since that measure provides simpler calculations than in the real world probability measure. Model 1 was developed with Monte Carlo methods but since the simulations were very time consuming another model, Model 2, was produced as well. This model was built on an analytic solution and the speed of the calculations were greatly improved. Finally, the real world was studied as well and Model 3 was developed and compared to the results from the other models.

The results from Model 1 converge towards the results from Model 2 when the number of simulations increases. With that observation, combined with the fact that the theory in the risk neutral probability measure is applied without many assumptions, these models are concluded to generate a correct result. There is a difference between the *EAD* curves generated by Model 1 and Model 2 and the *EAD* curve generated by Model 3. This difference is assumed to depend on the absence of the risk premium in Model 3 and except for the fact that the risk premium is missing, the model is concluded to generate a correct result.

The *EAD* curve generated from Model 2 was compared to the *EAD* curve generated by Nordea’s proxy model. The proxy model generated an extensive underestimation of the *EAD* curve. Therefore, the recommendation from this thesis was that the proxy model should not be used in calculations of the *EAD* unless an improvement of the model is performed. After the recommendation and based on the results from this thesis, Nordea performed a recalibration of their proxy model. This resulted in a better estimation of the *EAD*.

The final conclusion is that the proxy model did not generate trustworthy results but using the results of this thesis, Nordea was able to improve their proxy model to generate accurate results.

Appendices

Appendix A

Variance in Model 2

“Beauty is the first test: there is no permanent place in the world for ugly mathematics.”

– G. H. Hardy

The approximated variance of the interest rate swap, used in Model 2 is derived to be

$$\begin{aligned}
 V[\Pi] &= V \left[K(\delta(L[T_{i-1}, T_i] - R) + 1)p(t, T_i) - K \sum_{j=i+1}^{n-1} R\delta p(t, T_j) - K(1 + R\delta)p(t, T_n) \right] \\
 &= K^2(\delta(L[T_{i-1}, T_i] - R) + 1)^2 V \left[p(t, T_i) \right] + K^2 R^2 \delta^2 \sum_{j=i+1}^{n-1} V \left[p(t, T_j) \right] + K^2(1 + R\delta)^2 V \left[p(t, T_n) \right] \\
 &\quad + 2K^2(\delta(L[T_{i-1}, T_i] - R) + 1)R\delta \sum_{j=i+1}^{n-1} C \left[p(t, T_i), p(t, T_j) \right] \\
 &\quad + 2K^2(\delta(L[T_{i-1}, T_i] - R) + 1)(1 + R\delta)C \left[p(t, T_i), p(t, T_n) \right] \\
 &\quad + 2K^2 R^2 \delta^2 \sum_{j=i+1}^{n-1} \sum_{k>j}^{n-1} C \left[p(t, T_j), p(t, T_k) \right] + 2K^2 R\delta(1 + R\delta) \sum_{j=i+1}^{n-1} C \left[p(t, T_j), p(t, T_n) \right],
 \end{aligned}$$

where V is the variance of the different terms and C the correlation between the terms in Expression (6.5), calculated to be

$$\begin{aligned}
 V \left[p(t, T) \right] &= V \left[e^{A(t, T) - B(t, T)r(t)} \right] = e^{2(A(t, T) - B(t, T)E[r(t)] + B(t, T)^2\sigma^2\Delta t)} - e^{2(A(t, T) - B(t, T)E[r(t)] + B(t, T)^2\sigma^2\Delta t)}, \\
 C \left[p(t, T_i), p(t, T_j) \right] &= E \left[p(t, T_i)p(t, T_j) \right] - E \left[p(t, T_i) \right] E \left[p(t, T_j) \right] \\
 &= E \left[e^{A(t, T_i) + A(t, T_j) - (B(t, T_i) + B(t, T_j))r(t)} \right] - E \left[e^{A(t, T_i) - B(t, T_i)r(t)} \right] E \left[e^{A(t, T_j) - B(t, T_j)r(t)} \right] \\
 &= e^{A(t, T_i) + A(t, T_j) - (B(t, T_i) + B(t, T_j))E[r(t)] + (B(t, T_i) + B(t, T_j))^2\sigma^2\Delta t/2} \\
 &\quad - e^{A(t, T_i) + A(t, T_j) - (B(t, T_i) + B(t, T_j))E[r(t)] + (B(t, T_i)^2 + B(t, T_j)^2)\sigma^2\Delta t/2}.
 \end{aligned}$$

Bibliography

- [1] Basel Committee on Banking Supervision. “Basel II: International Convergence of Capital Measurement and Capital Standards”. In: *Banks for International Settlements* (2004).
- [2] Basel Committee on Banking Supervision. “Basel III: A Global Regulatory Framework for more Resilient Banks and Banking Systems”. In: *Banks for International Settlements* (2011), pp. 342–351.
- [3] Basel Committee on Banking Supervision. “The standardised approach for measuring counterparty credit risk exposures”. In: *Banks for International Settlements* (2014).
- [4] T. Björk. *Arbitrage Theory in Continuous Time*. 3rd edn. Oxford University Press Inc, 2009. ISBN: 9780199574742.
- [5] P. de Boer. *Risk Neutral Versus Real World Valuation*. 2009. URL: <http://dare.uva.nl/cgi/arno/show.cgi?fid=164512> (visited on 05/29/2016).
- [6] D. Brigo and F. Mercurio. *Interest Rate Models – Theory and Practice, with Smile, Inflation and Credit*. 2nd edn. Springer, 2007. ISBN: 9783540221494.
- [7] D. Filipovic. *Term Structure Models - A Graduate Course*. 1st edn. Springer Finance, 2009. ISBN: 9783540097266.
- [8] A. Green, C. Kenyon, and C. Dennis. “KVA: Capital valuation adjustment”. In: *Risk, December* (2014).
- [9] A. Jakobsson. *Time Series Analysis and Signal Modeling*. 2014. URL: <http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.187.1616&rep=rep1&type=pdf> (visited on 05/03/2016).
- [10] E. Lindström, H. Madsen, and J. Nygaard Nielsen. *Statistics for Finance*. 1st edn. CRC Press, 2015. ISBN: 9781482228991.
- [11] C. F. Lo. “Research Article – The Sum and Difference of Two Lognormal Random Variables”. In: *Journal of Applied Mathematics* vol. 2012. Article ID 838397 (2012), 13 pages. DOI: 10.1155/2012/838397.
- [12] S. M. Ross. *Introduction to Probability Models*. 9th edn. Academic Press, 2007. ISBN: 9780125980623.
- [13] I. Ruiz. *A Complete XVA Valuation Framework, Why the “Law of One Price” is dead*. 2015. URL: http://iruizconsulting.com/wp-content/uploads/2015/04/iRuiz_XVA.pdf (visited on 04/22/2016).
- [14] O. Vasicek. “An Equilibrium Characterization of the Term Structure”. In: *Journal of Financial Economics* 5 (1977), pp. 177–188.

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- [15] G. Welch and G. Bishop. *An Introduction to the Kalman Filter*. 2001. URL: http://www.cs.unc.edu/~tracker/media/pdf/SIGGRAPH2001_CoursePack_08.pdf (visited on 05/11/2016).
- [16] M. Wiktorsson. *Monte Carlo and Empirical Methods for Stochastic Inference (MASM11/FMS091)*. 2016. URL: <http://www.maths.lth.se/matstat/kurser/fms091mas221/2016/material/L1.pdf> (visited on 05/05/2016).
- [17] S. Yanbin, J.H.M. Anderluh, and J.A.M. Van Der Weide. “Algorithmic Counterparty Credit Exposure for Multi-Asset Bermudan Options”. In: *International Journal of Theoretical and Applied Finance* 18.01 (2015), p. 1550001.
- [18] J. Yoo. *Risk-Neutral and Real-World Considerations from an Investment Perspective*. 2015. URL: <https://www.soa.org/Files/Pd/2015/las/2015-las-session-61.pdf> (visited on 05/26/2016).