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DEFAULT CORRELATIONS WITHIN CREDIT VALUATION

Sofie Svensson

Department of Mathematical Statistics, Lund University

Master thesis supervised by Magnus Wiktorsson



LUNDS UNIVERSITET
Lunds Tekniska Högskola

Abstract

This paper focuses on pricing of basket Credit Default Swaps. The credit market instruments such as CDSs and CDOs are introduced. The concepts of trading these derivatives in basket CDSs divided into tranches is also of big importance. Different pricing models are presented and compared. Those are Semi-Analytic Valuation model with a Copula approach used together with the methods compound and base correlation. Also two contagion models are presented, one pure contagion and one contagion Copula mixture model. They are compared to each other to see how the prices differ for different tranches and premium payments. It is impossible to say which model prices right, but the main conclusion is that the prices differ. The models need to be further investigated in order to decide which one is the best for which purpose.

Keywords: Basket CDS, Copula, Contagion model, Credit market, Compound correlation, Base Correlation, Semi-Analytic Valuation

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Chapter 1

Introduction

1.1 Background

Credit risk is the probability of borrowers not being able to pay back their loans. The lender faces the risk of not getting his money back. This risk can be traded in several different ways, in *credit derivatives*. There are many traded credit derivatives on the market and some of them are fairly complex. The risk in the derivatives is not easy to predict since it is the result of many macro economic factors, links and dependencies between companies. Due to their complexity they have been one of many parts contributing to crises such as the financial crises in 2008 [Wang, 2014].

Common credit derivatives are for instance Credit Default Swaps, CDS and collateralized Debt Obligations, CDO. When the CDSs and CDOs were introduced in the late 80's it was a way for the banks to securitize their loans. With many bonds in the same basket it would be a risk diversification if one argues that it is unlikely that many default simultaneously. This is only true for uncorrelated bonds though. This is where the problem started because the reality is definitely not uncorrelated. The impact from this correlation were underestimated. Investors thought, according to the banks rating, that this was a safe investment and insurance companies, pension funds and big banks themselves believed in this safe investment with high return. The truth was the opposite. With many correlated bonds the investment was very risky. On the other hand, those few investors believing CDOs weren't safe bought a lot of CDS, this specific insurance explained later, on those and made a lot of money [Wang, 2014].

Important to emphasize is that this was not an unknown problem. While some, for instance Salmon in [Salmon F, 2009], claim that the crises were all due to the wrong model assumptions, others like [Brigo, 2011] argue that one can not blame the models only for the financial crises. The effect were rather underestimated and the models obviously needs to be improved but are not entirely responsible. People may have different opinions about this, but the Copula model presented in this paper, and ex-

tensions, is commonly used on the market. There are improvements to do but they can still be useful, if considered carefully.

There are mainly three approaches to model the risk and price these credit derivatives. Those are conditional independence, Copula and contagion models. This paper will have its focus on the Copula, Contagion and Contagion Copula mixture model.

1.2 Objectives

For the uninitiated reader it may be suggested to read through chapter 2 introducing the concepts and the credit market before coming back to this section.

One objective of this thesis is to explain the credit market and relevant valuation processes used today. As mentioned in the introduction, these derivatives have been in trouble and modeling the correlation between defaults is still something to improve. How previous defaults changes the probability of future defaults is an important part of the valuation process. It will be investigated what valuation methods are used today and if there are other models possible to use for this problem. The focus will be on *default correlations* and how they should be considered in the most common valuation method with a Copula approach. One of the main objectives is to get a model that prices the basket CDS for a given premium. If an entity wants to insure their loans and pay X every quarter of a year, what is the price of that?

Another approach investigated is to look at the default intensities as for epidemic or contagion models. For epidemics, the more that are infected by the disease the higher is the risk of even more infections. In this case with loans, the more names that have already defaulted the higher is the risk of another default. In short, this thesis is about how one should describe the impact on future defaults from previous when looking at a diversified basket of loans.

Given traded market prices, is it possible to calibrate the model and use it for pricing of other tranches and/or underlying portfolios?

1.3 Outline

Chapter 2 initially explains the credit market, concepts and different derivatives on the market, among them CDSs and CDOs which are the main derivatives of this paper. Definitions, concepts and notations are also introduced in this chapter. In chapter 3 the models are presented and explained. In chapter 4 the implementation is explained on detail for the reader to understand the models deeper. There is also the data presented. Chapter 5 presents the results from the models. Chapter 6 draw significant conclusions

and compare the models. Finally in chapter 7 it is also discussed how the models can be improved and how to continue this research.

Chapter 2

Theory

2.1 Credit Default Swap - CDS

For anyone lending money there is always a risk for the borrower, or the so called *issuer*, not to be able to pay back. To eliminate this risk the lender can buy an insurance from a third party. The lender pays a premium for this insurance, normally one, two or four times a year and then if the issuer defaults meaning that he cannot pay back, the insurer pays for the loss. The company lending the money continues to pay the insurance premium until a default event occurs or the CDS contract ends. The CDS is a derivative instrument that transfers the risk of default from the lender to the insurer. The insurer is obligated to pay the lender if a credit default occurs [Hull, 2009]. In Figure 2.1 the payments are shown from issuer to insurer and the other way around. The figure is from Credit Valuation [”Reference, Credit Valuation”, 2014].

The CDS is the most traded credit derivative. When a default occurs the insurer is obligated to either pay cash or physically deliver the bonds corresponding to the size of the default. The premium is usually paid in arrears and if a default occur the insured company will have to pay for the premium up to the day of the default. For the CDS a *notional value* is defined. That is the total value of the underlying assets. A basket



Figure 2.1: *Payments from credit default swap, issuer(solid lines) and insurer in case of default (dotted line)*

CDS may contain hundreds of underlying loans and the combined value of those are the notional value. The *CDS spread* is the premium as percent of the notional value. The premium should cover the risk free rate and the risk [Hull, 2009].

The trading with CDSs has varied a lot over time. The two years with the most traded CDSs were clearly 2007 – 2008. According to Bank for International Settlements [”OTC, Credit default swaps”, 2016] the notional amount outstanding, the value of the underlying basket went from 24000 billion USD in 2004 to 100000 in 2007 – 2008 and since then the trading has decreased for every year. In 2014 the trading was back on 37000 billions USD. This is based on statistics from central banks and other authorities from 13 different countries including Sweden, United States, Australia, Japan, Canada and some additional western Europe countries [”OTC, Credit default swaps”, 2016].

2.1.1 Basket Credit Default Swap

If there is more than one issuer, companies with loans, insured simultaneously it is called a *basket CDS*. Then the entire basket of loans is considered as the portfolio to pay insurance for. It is possible to create different derivatives from this basket CDS. For instance the *add-up* or *linear* basket CDS which provides payoff for the first and all other defaults. The *first-to-default CDS* provides payoff only for the first default. The *second-to-default CDS* provides payoff for the second default and the *kth-to-default CDS* provides payoff for the *kth* default [Hull, 2009]. There are also *loss basket CDSs* that provide payoffs for sizes of losses within a certain range which will be further explained in a later section.

2.2 Collateralize Debt Obligation - CDO

There are several similarities between CDSs and Collateralize debt obligation, *CDOs*. The CDO is mainly a fancy bond, an asset backed security where the assets are bonds. The underlying portfolio contains numerous, usually hundreds of assets such as loans, mortgages or other CDOs. CDOs can be constructed based on anything with regular cash flows, paybacks of loans, memberships, or credit card receivables. This could even include the premium payments that flow from the CDS [Kyle G, Russel A, 2013].

To distinguish the CDS from the CDO it can be concluded that a CDS is just an insurance against a risk while a CDO is a pool of bonds which is also possible to insure in the same way as in a CDS. Mathematically basket CDS and CDO are essentially the same. For a *cash-flow CDO* the underlying assets are real. It could for instance be possible to insure the underlying bonds and pay the insurance premiums entirely by the income of the bonds. These derivatives can obviously be fairly complex.

2.2.1 Synthetic CDO

A *synthetic CDO* is called that because it does not necessarily have any underlying assets. To fully explain this we look at the general structure of a CDO. Investors buy the CDO to get exposure to the underlying assets, which are often a big number of diversified assets. The importance for the investor is not whether the CDO actually owns the underlying assets but if it pays out the same cash flow. The synthetic CDO is instead made up of different CDS with exposure to the same assets as the corresponding cash-flow CDO. Instead of owning the asset the special purpose entity sells CDS protection for the same asset and sells the cash flow to the investors. In this way the investors obtain the same exposure to the asset as they would have done through a cash-flow CDO. The reason for doing this is because it can be practically hard to collect the specific pool of assets [Kyle G, Russel A, 2013].

While the cash flow CDOs are the easiest to understand those are not so heavily traded compared to the synthetic CDOs due to practical advantages [Altrock F et al., 2006].

2.3 Loss Basket CDS - Tranches

Loss basket CDS means that each investor take care of a certain part of the loss. The bonds in the CDO or basket CDS are usually collected by a bank and sold to a special purpose entity. This entity provides an investment vehicle where investors buy a share in the portfolio with preferred risk and return. The return is provided if credit defaults are avoided and cash flow is available. The CDO and basket CDS are divided into tranches which means that investors themselves chose the risk they want to take in the portfolio. We divide them into the tranches *Senior*, *Mezzanine* and *Equity*. The Senior tranche has the lowest risk and thus also the lowest return. The equity has the highest risk which means that they take care of the default first and receive income after Senior and Mezzanine. This makes it possible for investors to choose which level of risk they are willing to take. Loss sensitive investors should prefer the senior tranche [Kyle G, Russel A, 2013].

The credit risk is moved entirely to the investors in different tranches. In the equity tranche the investors finance say the first 5% of the defaults and will have a return of 30%. The senior investors finance a lot more, say 75% and have a return of 6%. What happens then is that the senior investors get their return first and only if there are resources left the next tranche will get their return. The equity tranche get their return if and only if all other investors already got theirs. This means that if we experience credit defaults, the equity tranche will receive less than 30% as return. This example is further explained in chapter 23 in [Hull, 2009] and figure 2.3 from [Hull, 2009] illustrates it.

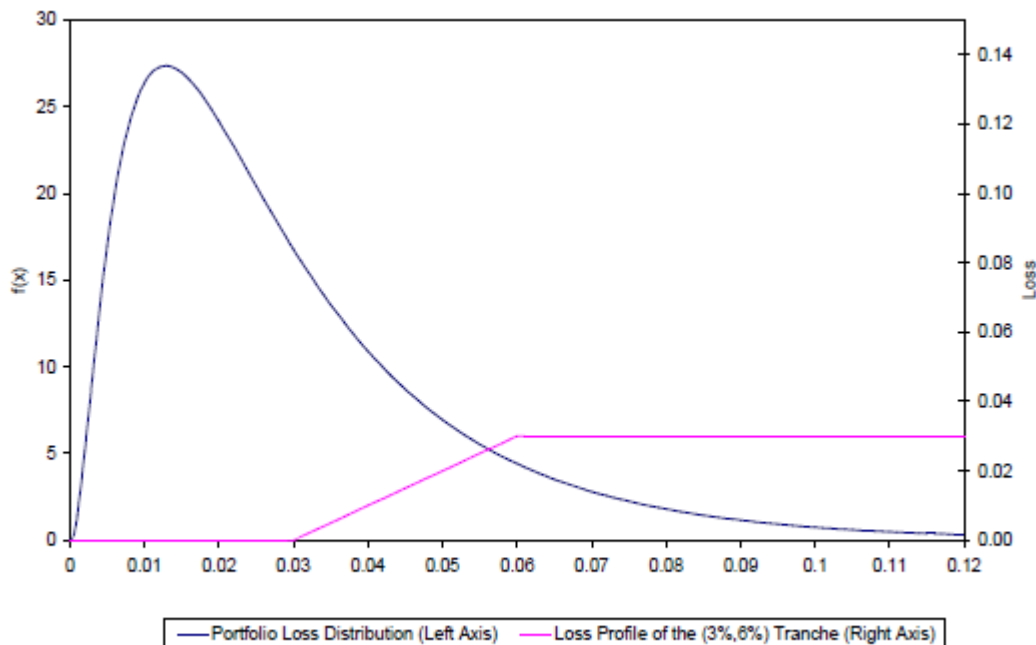


Figure 2.2: *Loss (pink line) for the mezzanine tranche 3 – 6% together with the probability distribution (blue line) for a basket CDS.*

As an example figure 2.2 from [Altrock F et al., 2006] shows the loss for the mezzanine tranche, 3 – 6%, specifically. Up to the first 3% defaults the mezzanine tranche is not influenced and the investors will get their expected return. Between 3% and 6% the return decreases linearly. When 6% has defaulted the mezzanine investors get no return at all. The blue line in the figure explains a standard loss distribution for underlying portfolio.

2.4 Linear Basket CDS

The term *linear basket CDS* means that all investors are exposed to all defaults [”Linear Credit Default Swap”, 2016]. It is similar to a loss basket CDS but the range for each investor is 0-100% so there is only one tranche. All investors bear the same risk and expected return. Compared to the mezzanine tranche the line describing the loss in figure 2.2 would be a straight line between 0% and 100% on the loss scale.

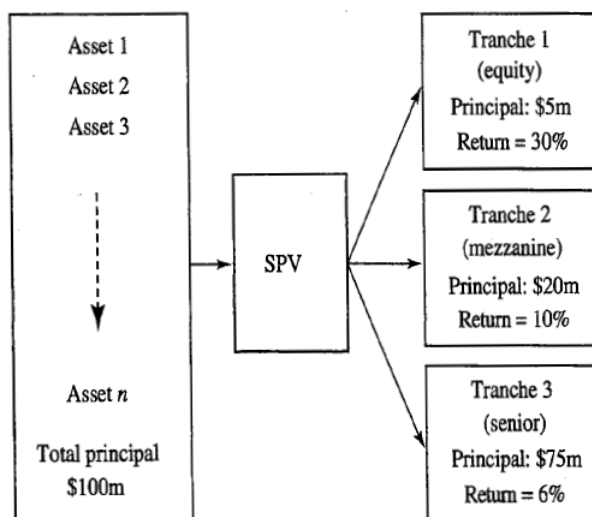


Figure 2.3: *The special purpose vehicle owns the entire basket of assets and divide it into tranches with different risks and returns.*

2.5 iTraxx

The iTraxx indexes are reference points for CDS, tradable credit default swap prices. The iTraxx index is one of two families of indexes. The other is the CDX indexes that contain North America and Emerging Market companies while the iTraxx indexes contain companies from the rest of the world. Which companies that should be included in the index portfolio are determined every six months. The index is available for different times to maturity and sometimes also for different tranches. This is done so that the index portfolio is "up to date" and contains what is actually traded on the market. The iTraxx indexes are managed by the International Index Company and are available in currencies USD, EUR and JPY ["Product Descriptions", 2013]. It was invented to bring greater liquidity, transparency and acceptance to the CDS market when it grew bigger ["Definition of iTraxx", 2016]. Thus should the names, companies and organizations, selected for the iTraxx indexes represent the most liquid, traded part of the market and are chosen by International Index Company ["iTraxx", 2016]. The underlying companies are set to have equal probability distribution over time.

There are many indexes with different standard tranches. They could for instance be for iTraxx 0 – 3%, 3 – 6%, 6 – 9%, 9 – 12% and 12 – 22% while for CDX they are 0 – 3%, 3 – 7%, 7 – 10%, 10 – 15% and 15 – 30% [Fabozzi F, et al., 2006]. Used later in this paper is 0 – 3%, 3 – 6% and 6 – 12% and 0 – 10%, 10 – 20% and 20 – 35% which are combined standard tranches which is also possible to obtain market prices for [Amato J, Gyntelberg J, 2005].

The index is presented in the so called *spread* which represent the credit quality. The spread unit is basis points where $100\text{bps} = 1\%$ which explains how much it costs to insure the basket. With the cost of 500bps it costs 5 to insure 100 for one year. This is a curve that have a value for each time point a coupon is payed out. This value is possible to transform into the default probability, which will be explained in section 2.6.

2.6 Credit Curves

Before the models are presented some things will be said about *credit curves*. The credit curves describe the credit quality such as default probability, spread and default intensity.

2.6.1 Hazard rate

There is a useful relationship between survival probability and hazard rate according to the article "On Bootstrapping Hazard Rates from CDS Spreads" [Castellacci G, 2012]. Hazard rate is another word for default intensity. This is needed for the contagion models described in later sections. The expression is presented by Castellacci as

$$S(t) = \exp \left(- \int_0^t h(u) du \right)$$

where $S(t)$ is the survival probability ($= 1 - p_t$) and $h(t)$ is the hazard rate, or default intensity, as function of time. By taking the inverse the expression for the default intensity becomes

$$h(t) = -\frac{d}{dt} \ln S(t)$$

where $S(t)$ is survival probability before time t . This is possible to solve for bootstrapping methods further described in [Castellacci G, 2012]. A more practical approach is to do the approximation

$$h(t) = -\frac{dS(t)}{S(t)} \frac{1}{dt} \approx -\frac{S(t + \Delta t) - S(t)}{S(t)} \frac{1}{\Delta t}$$

where the hazard rate is approximated to being constant over the interval $t + \Delta t$.

2.6.2 Default probability

An important feature of the spread is the relationship to default probability. The formula is

$$p_t = 1 - e^{-st}$$

where p_t is the cumulative default probability before time t and s_t is the corresponding spread [Manning, 2004]. The probability refer to the probability for a default before every time point where coupons are paid. The spread is the index reference point as previous described in section 2.5.

2.6.3 Hazard rate JPMorgan Model

This is as well according to Castellacci [Castellacci G, 2012]. There is more than one way to calculate the default probability. One of them, a commonly used model is the JP Morgan model. The rate will be calculated from the CDS spreads. Let $PV_D(T)$ denote the present value of the default leg, the payment if a default occurs. Let also $PV_{FIX}(T)$ denote the total value of the insurance payments up to time T . The fair deal is whenever those two are equal. Let $s = s(T)$ be the fair spread for the CDS, then we can introduce the variable $PV_{FIX,S}$ related to PV_{FIX} as $PV_{FIX} = sPV_{FIX,S}$. That means that

$$PV_{FIX,S} = \frac{\text{value of protection, default leg}}{\text{spread}}$$

is value of protection per spread unit. Denote τ the time of default and assume that the payments occur at the end of the periods. Then the present value of the default leg can be written as

$$\begin{aligned} PV_D(T) &= (1 - R) \sum_{i=1}^n df_{T_i} P(T_{i-1} < \tau \leq T_i) = \\ &= (1 - R) \sum_{i=1}^n df_{T_i} (S(T_{i-1}) - S(T_i)) \end{aligned}$$

where R is recovery rate, df_{T_i} is the discount factor, and $S(T_i)$ is probability to survive time T_i . This could be thought on as discounted values for experienced defaults in time periods $\{T_{i-1}, T_i\}$. The recovery rate converts the loss to experienced loss. Now, look at the fixed payments instead. Here is assumed that if the issuer defaults he does halfway through each time period. The following expression explains the present value of the fixed legs

$$\begin{aligned} PV_{FIX}(T) &= s \sum_{i=1}^n \Delta_i df_{T_i} P(T_i < \tau) + \frac{s}{2} \sum_{i=1}^n \Delta_i df_{T_i} P(T_i < \tau \leq T_i) = \\ &= s \sum_{i=1}^n \Delta_i df_{T_i} S(T_i) + \frac{s}{2} \sum_{i=1}^n \Delta_i df_{T_i} (S(T_{i-1}) - S(T_i)) = \\ &= s \sum_{i=1}^n \Delta_i df_{T_i} \frac{S(T_{i-1}) + S(T_i)}{2} \end{aligned}$$

with the same notations as before and Δ_i is the fraction of a year corresponding to the length of the interval $\{T_{i-1}, T_i\}$. The intuitive explanation of this is that the fixed legs are payed out if we survive the time period T_i and half of the following time period.

So given the spread curve it is possible to find all default probabilities. When $i = 1$, the probability of surviving previous time step $S(T_0)$ is 1. Then it is possible to find $S(T_1)$. Given $S(T_1)$ it is for the next spread value possible to find $S(T_2)$ as well. This can be continued for all i and the default probabilities are uniquely determined. This is the method used in this paper to obtain the default probabilities [”Reference, Credit Valuation”, 2014].

Chapter 3

Methods and Models

3.1 Semi-Analytic Valuation

The credit quality is modeled with a factor Copula. This model is presented in [Fabozzi F, et al., 2006]. From this it is possible to obtain the probability distribution of surviving time t conditioned on a variable describing the general status of the market which is equal for all issuers. This is referred to as *The Factor Model*. The conditional joint loss distribution for all issuers is then obtained by the so called *Probability bucketing approach*. This approach is simply a way to put different defaults in the same bucket with corresponding probability. The joint loss unconditioned distribution is then the integral over the conditioned variable. From the loss distribution the present value of the derivatives can be calculated. The only unknown parameter in the model is the correlation in the Copula. This will be discussed in section *Correlation Input*. A summary of the steps are

- Find marginal distribution for each issuer of surviving time t conditioned on the common factor
- Find the joint loss distribution for the loss at a given time point, conditioned on the same common factor.
- Integrate the probability distribution over the common factor to obtain the unconditioned probability distribution of losses at all time points
- Calculate the present value of the CDS contract given the unconditioned probability distribution.
- Find the correlation that gives correct present value given the index spread.

3.1.1 The One Factor Model

The distribution for the default loss is computational time consuming and what is usually used today to model the default correlations is a factor copula [Hull J, White A, 2004].

The joint probability for losses at all times is then created from the marginal distributions and the correlation. There are many different Copula methods that would be possible to use for the valuation. We will consider the standard normal copula model which is the most commonly used model since it is a practical and straightforward method [”Reference, Credit Valuation”, 2014]. The definition of the Copula function is

$$C(u_1, u_2, \dots, u_m, \rho) = P(U_1 \leq u_1, U_2 \leq u_2, \dots, U_m \leq u_m)$$

The Copula function represents the joint distribution including their pairwise correlation. The standard normal Copula method thus means that we have standard normal variables and a correlation to combine within a Copula function.

Set X_i as the credit quality of issuer i . We want to model credit quality of each issuer as a function of a common factor Y_0 and an idiosyncratic factor Y_i . What is meant with idiosyncratic is that it describes the structural behavior of credit quality for issuer i . The common factor on the other hand describes the general status of the market and is equal for all issuers. These two together explains the credit quality for the issuer and can be written as

$$X_i = a_i Y_0 + \sqrt{1 - a_i^2} Y_i$$

where Y_0 and Y_1 both are standard normal random variables [”Reference, Credit Valuation”, 2014]. Thus is X_i also a standard normal random variable but correlated with Y_0 and Y_i , in other words is this a one factor standard normal Copula function.

The correlation between X_i and X_j are $a_i a_j$ but if conditioned on the common factor all issuer’s credit quality, X_i , are independent. It would be possible to use different a_k but standard is to use the same for all k .

What we will obtain here is the probability for the i th issuer to survive a given time T conditioned on the common factor Y_0 . Because of the normal distribution assumption the expected value is

$$E[X_i | Y_0] = E[a_i Y_0 + \sqrt{1 - a_i^2} Y_i | Y_0] = a_i Y_0$$

and the standard deviation

$$D[X_i | Y_0] = D[a_i Y_0 + \sqrt{1 - a_i^2} Y_i | Y_0] = \sqrt{1 - a_i^2}$$

Thus is

$$P(X_i \leq x | Y_0) = \Phi \left(\frac{x - a_i Y_0}{\sqrt{1 - a_i^2}} \right) \tag{3.1}$$

where $\Phi(x)$ is the cumulative distribution function for a standard normal random variable. Think of X_i as a variable with distribution function $F_i(x)$. Under the Copula model the percentile $F_i(x)$ can be mapped to a percentile for the default time t_i , lets call this $Q_i(t) = P(t_i \leq t)$, so that $F_i(x) = Q_i(t)$ or equally $P(x_i \leq x) = P(t_i \leq t)$. Then equation (3.1) can be rewritten as

$$P(t_i \leq t|Y_0) = \Phi\left(\frac{x - a_i Y_0}{\sqrt{1 - a_i^2}}\right)$$

Since $F_i(x) = Q_i(t)$, also $x = F_i^{-1}(Q_i(t))$ and thus

$$P(t_i \leq t|Y_0) = \Phi\left(\frac{F_i^{-1}(Q_i(t)) - a_i Y_0}{\sqrt{1 - a_i^2}}\right)$$

It is also known from before that X_i is a standard normal random variable and thus is the conditional distribution for issuer i to survive time T , $S_i(T|Y_0)$ given as

$$S_i(T|Y_0) = P(t_i \geq T|Y_0) = 1 - P(t_i \leq T|Y_0) = 1 - \Phi\left(\frac{\Phi_i^{-1}(Q_i(t)) - a_i Y_0}{\sqrt{1 - a_i^2}}\right)$$

where $Q_i(t)$ is the probability for issuer i to survive time t obtained from the credit curve.

3.1.2 Probability Distribution

From the conditional probability distribution the next step is to find the joint probability distribution for total loss [Hull J, White A, 2004]. There are different ways of doing this and here the so called *probability bucketing approach* will be presented. The goal is to find the probability distribution of the loss for time T. The first thing to do is to put losses into different ranges, “buckets”.

The *recovery rate* is assumed to be known and constant for all issuers. Recovery rate is how much an issuer recovers within a default and (1- recovery rate) times the loss is the experienced loss. If the recovery rate is 40% it means that when an issuer defaults at a size of 100 they immediately recover 40% and the experienced loss is $(1 - 0.4) * 100 = 60$. A commonly used value is 40% according to chapter 23 in Options, Futures and other Derivatives [Hull, 2009].

Let the buckets be $\{0, b_0\}$, $\{b_0, b_1\}$... $\{b_{K-1}, \infty\}$. Thus the first bucket corresponds to no loss and the others have equal widths. If only one tranche is valued it would make sense to have narrow buckets within the tranche and wide outside to optimize computation time. The goal is then to obtain the conditional probability for the loss to be in the k th bucket, and this will be denoted $p_k = P_T(k|Y_0)$.

The mean loss in bucket k is defined as A_k . This means that

$$A_k = \frac{1}{2}(b_{k-1} + b_k)$$

The method is iterative and calculates p_k and A_k by adding one bond at the time. Each bucket is represented with its mean A_k and this has been shown to be a quite accurate model according to [Hull J, White A, 2004].

When there are no bonds in the basket there is also no loss for sure, so the probability of being in the first bucket is $p_0 = 1$, and in all other buckets probability zero, $p_k = 0$ for all $k > 0$. Worth noting is that all these probabilities are still conditioned on Y_0 . Now assume that $i - 1$ bonds are already added and next step is to add the i th. That means that the probability of being in each bucket and corresponding average loss is calculated for each bucket respectively, for the first $i - 1$ bonds. The loss of the default from this bond would be L_{AVG} since it is assumed that all defaults are equal. Denote the probability of this default conditioned on Y_0 as α_i . Adding another default to the mean value A_k imply that the total loss might end up in another bucket than that corresponding to A_k . Denote this new bucket $u(k)$. The impact of this bond is thus that some probability is moved from bucket k to $u(k)$. The updating formulas are

$$\begin{aligned} p_k^n &= p_k^p - p_k^p \alpha_i \\ p_{u(k)}^n &= p_{u(k)}^p + p_k^p \alpha_i \\ A_k^n &= A_k^p \\ A_{u(k)}^n &= \frac{p_{u(k)}^p A_{u(k)}^p + p_k^p \alpha_i (A_k^p + L_{AVG})}{p_{u(k)}^p + p_k^p \alpha_i} \end{aligned}$$

Here $\alpha_i = 1 - S_i(T|Y_0)$ is the probability of default before time T for issuer i conditioned on Y_0 . That probability is the one calculated in the factor model. The probability p_k^n is the new probability of being in bucket k after adding bond i . The probability p_k^p is the previous value of the same probability. There is a minus sign decreasing the probability which is moved to another bucket, $p_{u(k)}$ which is increased with the same size. For each issuer added the updating formulas are used, so first some probability is moved from bucket 0 to 1, then from 1 to 2 and so on. The new $A_{u(k)}$ is calculated to be the mean of the bucket containing loss ($A_k + L_{AVG}$). In the case where a loss is added but the total loss remains in the same bucket for this additional loss L_{AVG} , the updating formulas are simplified to

$$\begin{aligned} p_k^n &= p_k^p \\ A_k^n &= A_k^p + \alpha_i L_{AVG} \end{aligned}$$

When all bonds have been added to the basket the conditional distribution of total loss $P(k|Y_0)$ is determined [Hull J, White A, 2004]. Since Y_0 is a continuous random variable the unconditional probability is obtained by an integral

$$P(k) = \int_{-6}^6 P(k|Y_0) \frac{d\Phi(y)}{dy} dy$$

This has to be solved numerically and is done by Legendre abscissas and weights which will not be further explained here [”Reference, Credit Valuation”, 2014], but is mainly a discrete integration. Recall the law of total probability

$$P(k) = \sum_{Y_0} P(k|Y_0)P(Y_0)$$

By choosing 50 points between -6 and 6 with these particular weights and consider it as a discrete probability distribution the result seems to be a fair approximation.

3.1.3 Many Factor Model

The *many factor model* is found in more detail in [Hull J, White A, 2004]. The one factor model can be extended to many factors. Then the probability of surviving a given time can be derived conditional on all common factors.

$$X_i = a_0Y_{0_0} + a_1Y_{0_1} + \dots + a_mY_{0_m} + Z_i\sqrt{1 - a_0^2 - a_1^2 - \dots - a_m^2}$$

where all Y_i and Z_i are standard normal variables. The corresponding conditional probability for issuer i to survive time T becomes

$$P(t_i \geq T|Y_{0_0}, Y_{0_1} \dots Y_{0_m}) = 1 - \Phi\left(\frac{\Phi_i^{-1}(Q_i(t)) - a_0Y_{0_0} - a_1Y_{0_1} - \dots - a_mY_{0_m}}{\sqrt{1 - a_0^2 - a_1^2 - \dots - a_m^2}}\right)$$

where a_n is the pairwise correlation between issuer i and n and the other notations as before. The probability distribution is obtained as for the one factor model and the result is the probability distribution for k defaults given all conditioned variables $P(k|Y_{0_0}, Y_{0_1} \dots Y_{0_m})$. The unconditioned probability distribution can be derived in a similar way as will be presented in next section for the one factor model. This model immediately becomes difficult since here one correlation is needed for each factor. Already for two factors there are infinitely many solutions when trying to find the correlation for each tranche. Also with 50 integration points as in the one factor model the sum goes over 50^m terms which is computational hard for many factors.

3.1.4 Simplified Theory and Examples Semi-Analytic Valuation

The theory presented above could be simplified in a couple of steps. The loss in the semi-analytic valuation model is assumed to be equal for all defaults and thus is only

the number of defaults interesting. Each bucket represents a number of defaults where bucket b_k corresponds to exactly k defaults. The loss within a default is assumed to be the mean of the nominal amount times (1- recovery rate) for each issuer. Thus the average loss is calculated as

$$L_{AVG} = \frac{1}{n} \sum_{j=1}^n (1 - r_i) M_i$$

where r_i is the recovery rate for issuer i , M_i is the nominal amount of issuer i and n is the number of issuers [”Reference, Credit Valuation”, 2014].

For the updating formulas in the probability distribution the average loss A_k is entirely left out since each bucket b_k corresponds to exactly k defaults, there is no mean to calculate. Being in bucket k is equal to a loss of $k * L_{AVG}$. The two different cases, where a default implies that the total loss ”changes bucket” or not is irrelevant since another default by definition means another bucket.

To easier understand this an example is presented. Assume three issuers are already added to the basket and next step is to add the fourth. Assume for the three first issuers that the probabilities for 0 – 4 defaults are 0.4, 0.3, 0.2, 0.1, 0 respectively. The last probability corresponds to four defaults which is not possible yet since only three issuers are added so far. Assume for current time step that the probability of a default is 0.1. The updating formulas are

$$p_0 = p_0 - p_0 * \alpha = 0.4 - 0.4 * 0.1 = 0.4 - 0.04 = 0.36$$

where the probability of which p_0 is reduced is added to p_1

$$p_1 = p_1 + p_0 * \alpha = 0.3 + 0.04 = 0.34$$

In the next step some of the probability is moved from this new p_1 to p_2 . The size of the moved probability is determined by the probability of being in bucket 1 in the previous step. This is intuitive since if it was very likely to be in bucket 1 after added one issuer it should be more likely to be in bucket 2 after adding two issuers than if we most likely were in bucket 0 for one issuer

$$p_1 = p_1^{new} - p_1^{old} * \alpha = 0.34 - 0.3 * 0.1 = 0.34 - 0.03 = 0.31$$

and p_2 is increased with the same probability

$$p_2 = p_2 + p_1^{old} * \alpha = 0.2 + 0.03 = 0.23$$

In the same way

$$p_2 = 0.23 - 0.2 * 0.1 = 0.21$$

$$p_3 = 0.1 + 0.2 * 0.1 = 0.12$$

$$p_3 = 0.12 - 0.1 * 0.1 = 0.11$$

and the new bucket gets the probability

$$p_4 = 0 + 0.1 * 0.1 = 0.01$$

So the new probabilities after introducing the fourth issuer are

$$p_0 = 0.36 \quad p_1 = 0.31 \quad p_2 = 0.21 \quad p_3 = 0.11 \quad p_4 = 0.01$$

where p_k is the probability of k defaults. The sum of all probabilities is still 1 but some probability is moved from the already existing buckets to the new that appear first when the fourth issuer is introduced. This is then done for all time steps so the probability of k defaults is calculated for all k up to number of issuers in the basket.

3.1.5 Pricing a Loss Basket CDS

Now the probability for k defaults at time t is known for all issuers. The total loss at time t is

$$L(t) = L_{AVG} \sum_{j=1}^n I_j(t)$$

where $I_j(t)$ is the indicator function taking values 1 or 0 depending on if it is a default (1) or not (0). For the loss basket contract payments are done for losses between attachment point C and detachment point D . We denote the total received payments from insurer

$$D(L(t)) = \begin{cases} 0, & L(t) \leq C \\ L(t) - C, & C < L(t) \leq D \\ D - C, & D < L(t) \end{cases}$$

and the nominal amount of owning the tranche, the value paid to the insurer, is

$$N(L(t)) = \begin{cases} D - C, & L(t) \leq C \\ D - L(t), & C < L(t) \leq D \\ 0, & D < L(t) \end{cases} \quad (3.2)$$

The interesting thing here is to compare the present value of the default leg and present value of premiums paid. The present value of the default leg is the expected value of the default leg $E(L(T))$ discounted. Since the probabilities of default now are known the expected default leg is

$$E(D(L(t))) = \sum_{k=0}^K p_t(L_k) D(L_k)$$

and the discounted default leg is

$$PV_{DL} = \sum_{t=1}^T df_t \sum_{k=0}^K (p_t(L_k) - p_{t-1}(L_k)) D(L_k) \quad (3.3)$$

where df_t is the discount factor for time t . The first sum sums over all different time periods and discount the amount $D(L(t))$ that is paid out during time $\{t - 1, t\}$ with probability $p_t(L_k) - p_{t-1}(L_k)$. In the similar way the present value of the premium leg is calculated as the discounted coupons $cN(L(t))$.

$$PV_{PL} = c \sum_{t=1}^T \Delta_t df_t E(N(L(t))) = c \sum_{t=1}^T \Delta_t df_t \sum_{k=0}^K p_t(L_k) N(L_k) \quad (3.4)$$

with the same notations as before, cN are the coupons and c is the coupon in percent of the nominal amount, and Δ_t is the cash flow period in years. This is needed to compensate for the fact that the premiums can be payed out at different time points. If they are payed out quarterly Δ_t should be equal to $1/4$ for all t .

The price of the CDS is now possible to determine given the input data and the unknown correlation.

3.1.6 Pricing a First-to-Default Basket CDS

The first-to-default basket CDS is a special case of the Nth-to-default basket CDS so we will derive the premium leg and default leg for the Nth-to-default basket CDS. As for the loss basket CDS the indicator function will be used to derive the total loss which is here equal to number of losses.

$$I(t) = \sum_{i=1}^n I_i(t)$$

For a Nth-to-default basket CDS is the insurance payed out when $I(t) = N$. The total amount payed out at time t is

$$A(I(t)) = \begin{cases} 0, & I(t) < N \\ L_{AVG}, & I(t) \geq N \end{cases}$$

So when the Nth default occurs the average loss for a default is payed out. Nothing is paid before the Nth default and more than N defaults has no further impact. This is an insurance against the Nth default and only against that. The corresponding nominal amount is

$$M(I(t)) = \begin{cases} M, & I(t) < N \\ 0, & I(t) \geq N \end{cases}$$

The expected value for the total amount payed out is

$$E(A(I(t))) = \sum_{k=0}^K p_t(k)A(k)$$

As for the loss basket CDS the present value of the default leg is obtained as

$$PV_{DL} = \sum_{t=1}^T df_t \sum_{k=0}^K (p_t(k) - p_{t-1}(k))A(k)$$

and the value for the premium leg is in the corresponding way

$$PV_{PL} = c \sum_{t=1}^T \Delta_t df_t E(M(I(t))) = c \sum_{t=1}^T \Delta_t df_t \sum_{k=0}^K p_t(k)M(k) \quad (3.5)$$

3.2 Correlation input

Now the model is presented and the valuation procedure is unambiguous given the input correlation. The interpretation of default correlations is how likely it is that another issuer defaults when one already has. If the correlation is -1 they would never default at the same time. If the correlation is 1 one default would immediately cause another default. If the correlation is positive but less than 1 they do not necessary default at the same time but if one name defaults the risk increases for other defaults. With negative correlation the risk would not be so high, since if one defaults it would be less likely that another issuer also does and crises when many companies default at the same time would be extremely rare.

The model is based on the assumption of all companies being equally likely to default [Livesey M, O'Kayne D, 2004]. Within a tranche the correlation is set to be pairwise equal but the pairwise correlation differs between tranches. If there is a very high correlation it means that if one defaults then it is very likely that others also will. It means that the senior tranches are closer to the equity tranche in terms of riskiness and would increase the value of the equity tranches and decrease the senior tranches. If the correlation is very low, the senior tranches are "safe" because the likelihood that many issuers default at the same time is quite unlikely and thus their value is increased and the opposite for the equity tranche.

For the following theory it is needed to understand the term *present value* of the CDS tranche. The present value of the CDS tranche is equal to $PV_{tr} = Price_{tr} + PV_{PL} - PV_{DL}$ where $Price_{tr}$ is the upfront payment or the price of the insurance. This is a one time payment compared to the regular coupons.

3.2.1 Tranche Correlation / Compound Correlation

One approach to obtain the input correlation is the so called *compound correlation*. Compound correlation is used for both N th to default and loss protection. The compound correlation is obtained by assuming that $PV_{tr} = 0$ or equally $Price_{tr} = PV_{DL} - PV_{PL}$. This is similar to section 2.6.3 where the two legs were set to be equal to back out the default probability. Here is the goal instead to back out the correlation.

$$Price_{tr} = \sum_{t=0}^T df_t \sum_{k=0}^K (p_t(L_k) - p_{t-1}(L_k)) A(L_k) - c \sum_{t=1}^T \Delta_t df_t \sum_{k=0}^K p_t(L_k) N(L_k) \quad (3.6)$$

where c are the coupons paid regularly during the insurance period, the spread which is known. The price is also observed on the market and thus equation (3.6) is known except for the correlation of the tranche ρ_{tr} . For example the coupons paid every year could be 1% and the price for that derivative is 1000, then by putting the coupon to 1% there is a correlation corresponding to the price 1000. The calculations are done by numerical optimization but does not necessarily have a solution or in some cases even more than one. This can be done for all standard tranches using their index spreads and traded prices.

A drawback of compound correlation is that it is not good to use in other intervals than standard tranches. This will be discussed later.

3.2.2 Base Correlation

This chapter explaining *Base correlation* is a detailed explanation of the method presented in Base Correlations [Galiani S, et al., 2004]. Base correlations are used only for loss protection contracts. They are created from base tranches, all starting from zero representing all tranches as a subtraction between detachment and attachment point. As in figure 3.1 the tranche $K_1 - K_2$ will be calculated as a subtraction between $0 - K_2$ and $0 - K_1$. Take for instance the CDX index tranches $0 - 3\%$, $3 - 7\%$ and $7 - 10\%$. The corresponding base tranches are then $0 - 3\%$, $0 - 7\%$ and $0 - 10\%$. The question is how to use the standard tranche spreads to find base correlations. After finding correlations for all standard tranches the base correlations can be interpolated to find correlations and spreads even for non standard tranches.

The first base tranche correlation $0 - 3\%$ is implied in the same way as the compound correlation for the equity tranche. Then the tranche $0 - 7\%$ can be seen as the sum of $0 - 3\%$ and $3 - 7\%$. Looking at the equality

$$Price_{tr}(0 - 7\%) = Price_{tr}(0 - 3\%) + Price_{tr}(3 - 7\%) \quad (3.7)$$

where the relationship between price and present values are the same as explained in the introduction to this section. For each tranche it is known which premium is used

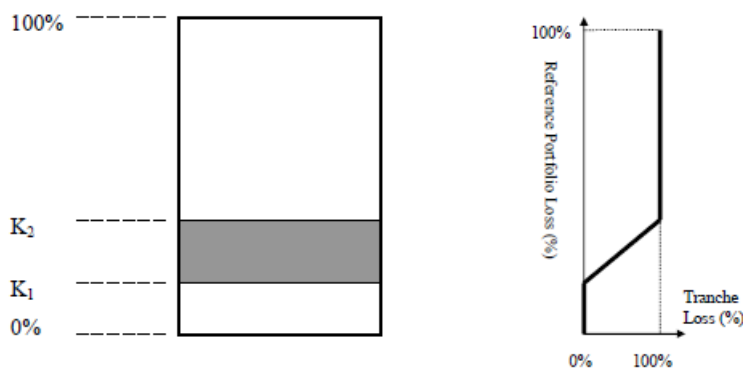


Figure 3.1: A mezzanine tranche with attachment K_1 and detachment K_2 with linear loss within the certain tranche.

and to what price the derivative is traded. This premium is called the *market spread* and is usually equal for the tranches. The method to find the correlation for 0 – 7% is first to find the price that tranche would have been traded for. This is found as the sum of the prices for the tranches 0 – 3% and 3 – 7%. To compensate for the different widths of intervals the nominal should be scaled in accordance to the width in terms of the 0 – 7% tranche. That implies the equation

$$\frac{7}{7}Price_{tr}(c^{market}, \rho_{0-7}) = \frac{3}{7}Price_{tr}(0-3) + \frac{4}{7}Price_{tr}(3-7) \quad (3.8)$$

The right hand side is then possible to calculate since the prices for both 0 – 3% and 3 – 7% tranches are known. Then it is possible by linear optimization to obtain the correlation ρ_{0-7} .

In the exactly same way it is possible to obtain the base correlations for all other base tranches. For instance the 0 – 10% tranche base correlation is obtained by the steps

- Look at the price of the 0 – 10% tranche as the sum of 0 – 7% and the standard tranche 7 – 10%
- Take the price obtained in previous calculation for 0 – 7% tranche and use price for 7 – 10% tranche
- Scale the prices in accordance to tranche widths and calculate the price of the 0 – 10% tranche
- Use linear optimization to find the tranche correlation for tranche 0 – 10%

3.2.3 Pricing Standard Tranches

When all base correlations are estimated the task left is to price the tranches with help of those. It is possible to price the standard tranches in accordance with equation 3.7, but not to forget the scaling due to different nominal values. That equation becomes

$$Price_{tr}(c_{price}, \rho_{3-7}) = Price_{tr}(c_{price}, \rho_{0-7}) - Price_{tr}(c_{price}, \rho_{0-3})$$

where c_{price} is the chosen premium that is desired to price for. The scaling should be such that the price calculated on the right hand side is obtained "in terms of the 3–7% tranche". The equation now looks like

$$Price_{tr}(c_{price}, \rho_{3-7}) = \frac{7}{4}Price_{tr}(c_{price}, \rho_{0-7}) - \frac{3}{4}Price_{tr}(c_{price}, \rho_{0-3})$$

and that is the final pricing formula for standard tranches from base correlations.

3.2.4 Finding Base Correlations for Non Standard Tranches

One thing that is possible to do with base correlation but not with compound correlation is to derive prices and correlations for non standard tranches. Base correlation has a unique solution which is an advantage compared to compound correlation. Base correlation is a more intuitive way of interpolation of tranche correlations than compound correlation. Base correlation is an increasing function while the compound correlation is normally higher for equity and senior tranche and lower for the mezzanine tranche. This is referred to as *correlation smile*. There are many explanations of this smile. One could be the uncertainty in the model and since the equity and senior tranches are more sensitive to changes in correlations, they need a "model risk premium" included in the derivative. This and other reasons to the smile are further discussed in [Amato J, Gyntelberg J, 2005].

This could be compared to implied volatility surfaces for European options. According to the Black and Scholes theory for pricing of European options should the volatility be independent of strike price and expiration date. But the volatility is in reality a function of time to maturity and strike price. Some points on the surface are known because they correspond to traded options. The rest of the surface is interpolated between those points. Then the entire surface is used for pricing of European options with non observable prices. The implied volatility is commonly a U-shaped "smile" as a function of strike price [Daglish T, et al., 2006].

Prior to the stock market crash in 1987 where the volatility surface fairly flat but this changed afterwards in order to obtain better models for the pricing [Derman E, 2003].

In this paper is instead the volatility, or correlation, a function of defaults. This have been further developed since the crash in 2008 and the similarities between this smile and the implied volatility smile for European options can be recognized.

3.2.5 Pricing Non Standard Tranches with Base Correlation

Say we want to price 5 – 8% tranche. Looking at this 5 – 8% tranche it can be represented by owning a 0 – 8% tranche (long position) and selling a 0 – 5% tranche (short position). The steps to find the price are as follows

- The first step is to interpolate between the two closets standard tranche detachment points, 0 – 7% and 0 – 10% to obtain the 0 – 8% base correlation and between 0 – 3% and 0 – 7% to obtain 0 – 5% base correlation.
- The next step is to price both tranches as $PV_{tr} = PV_{PL} - PV_{DL}$ in accordance with the factor model with those correlations and spreads.
- Then the price of the tranche 5 – 8% can be calculated as the difference between the price for the base tranches weighted by the corresponding width ratios. $Price_{tr}(5 - 8\%) = \frac{8}{3}Price_{tr}(0 - 8\%) - \frac{5}{3}Price_{tr}(0 - 5\%)$

It is shown that the difference between compound and base correlation is most significant in the mezzanine tranche [Galiani S, et al., 2004].

3.3 Contagion Model

Another approach to handle the correlation between defaults is to draw the parallel to contagion or epidemic proliferation. Instead of looking at correlation between issuers, varying intensity for defaults can be considered. After one default the intensity for another default is increased by this contagion ratio. There are several studies bringing this theory up and also the combination of copula and contagion, a so called *copula contagion mixture model*. What are mostly brought up there is the analysis of the effect of different correlations (copula part) and contagion ratio (contagion part) on the spread for the basket CDS.

3.3.1 Simple Contagion Model

The approach is to represent the default intensity by a mathematical model. The simplest form is chosen to have only one parameter, a contagion ratio for each tranche. The equation to solve for every tranche is as before

$$Price_{tr} = PV_{DL} - PV_{PL}$$

where expressions for the expected value of present value for default and premium fees are needed to match the market price. With those it should be possible to find a contagion rate for each tranche which would easily be possible to use for pricing of non-standard tranches and other basket CDSs. Non standard tranches could for instance be priced by an interpolated contagion ratio. This model is from *À la Carte*

Correlation Models: Which One to Choose [Zheng H, 2012]. The simplest form of the model contains the intensity on the form

$$\lambda_i(t) = a \left(1 + \sum_{j=1, j \neq i}^n c \mathbf{1}_{\{\tau_j \leq t\}} \right), i = 1, \dots, n \quad (3.9)$$

where $\lambda_i(t)$ is the intensity for the i :th default after $i-1$ defaults have already occurred. Parameter a is the unconditional default intensity for issuer i and c is the *contagion ratio* that is a constant but different between tranches. Parameter n is the number of names in the basket CDS. The intensity increases with number that have defaulted before time t , $\tau_j \leq t$. The default time τ_k is the default time for the k :th default and the formal definition is

$$\tau_i = \inf \left\{ t > 0 : \int_0^t \lambda_i(s) ds \geq E_i \right\} \quad (3.10)$$

where $E_i = -\ln(1 - U_i)$, $i = 1 \dots n$. In other words is the default time i the smallest time such that the integral over the intensity exceeds E_i .

The default times can be obtained from Monte Carlo simulations. Based on this model the default times for all issuers can be determined in the following way

- Generate standard uniform variables U_i , $i = 1, \dots, n$
- Set $E_i = -\ln(U_i)$, $i = 1, \dots, n$ and sort E_i in increasing order such that $E_1^* < E_2^* < \dots < E_n^*$.
- Then the default times can be found as

$$\tau_1 = \frac{E_1^*}{a}, \tau_k = \tau_{k-1} + \frac{E_k^* - E_{k-1}^*}{a(1 + (k-1)c)} \quad k = 2, \dots, n$$

Where a is the unconditional default intensity. From the default times it is possible to price the CDS by calculating present value of default and premium legs. This will be done in the section after the following where a more complex contagion model first is presented.

3.3.2 Copula Contagion Mixture Model

This model is as the name indicate a mixture of the pure Contagion and Copula models. The first thing here is the difference in the expression for the intensity.

$$\lambda_i(t) = a \left(1 + \sum_{j=1, j \neq i}^n ce^{-d(t-\tau_j)} \mathbf{1}_{\{\tau_j \leq t\}} \right), i = 1, \dots, n \quad (3.11)$$

where another constant d is introduced. The Copula part is represented with, as before, a model that tries to represent the credit quality as

$$X_i = \rho Z + \sqrt{1 - \rho^2} Z_i$$

So the defaults are again correlated by this ρ . One difference compared to the pure contagion model is that the random samples from where the default times are generated are correlated. Except that, step 1 and step 2 in the algorithm for finding the default times will be the same for the simple contagion model and this mixture model.

The correlated uniform variables are generated by first drawing correlated normal distributed random samples. This is done by generating independent normal random samples X_1 and X_2 and from those construct $X_3 = \rho X_1 + \sqrt{1 - \rho^2} X_2$ where X_1 and X_3 are correlated.

From the correlated normal distributed observations the cumulative normal distribution function is evaluated in those sample points. By taking the inverse of the uniform cumulative distribution function the correlated random uniform samples are obtained. The more tricky part in this model is to find the default time. Equation 3.11 can be rewritten as

$$\int_0^t \lambda_k(s) ds = at + \frac{ac}{d} \sum_{i=1}^{k-1} (1 - e^{-d(t-\tau_i)})$$

and from the relationship 3.10

$$\int_0^{\tau_k} \lambda_k(s) ds = E_k^*$$

Define

$$F_k(t) = \int_0^t \lambda(s) ds$$

and thus is

$$F_k(t) = \int_0^t \lambda_k(s) ds = at + \frac{ac}{d} \sum_{i=1}^{k-1} (1 - e^{-d(t-\tau_i)})$$

and

$$F_k(\tau_k) = \int_0^{\tau_k} \lambda_k(s) ds = E_k^*$$

Thus is the default time found by solving the equation

$$F_k(\tau_k) - E_k^* = 0$$

By using the starting point τ_{k-1} the Newton algorithm converges quadratically to the root τ_k which is a good property and can be further read about in [Zheng H, 2012].

3.3.3 Pricing of contagion models

Given all the ordered default times it is possible to price the loss basket CDS. At first the loss needs to be defined as

$$L(t) = \sum_{k=1}^n \frac{k}{n} \mathbb{1}_{\{\tau_k \leq t < \tau_{k+1}\}}$$

where n is the number of names in the portfolio and τ_k is default time for issuer k . This expression could possibly only be a sum of one non-zero term since a given t can only be in one interval, between default k and $k + 1$. If there are 50 names in the basket and t is between default 10 and 11 the loss becomes $10/50 = 20\%$. The loss is thus 20% of the nominal value. The loss at time t for an entire tranche l can be obtained as

$$L_l(t) = (L(t) - k_{l-1}) \mathbb{1}_{\{k_{l-1} \leq L(t) \leq k_l\}} + \Delta k_l \mathbb{1}_{\{L(t) > k_l\}}$$

where k_l is the attachment point for tranche l and $\Delta k_l = k_{l-1} - k_l$. This means that the tranche loss is the loss minus the attachment point given that the loss is between attachment and detachment point plus the difference between detachment and attachment point given that the loss is greater than detachment. With this notations it is possible to express the present value of the default leg as

$$PV_{DL} = E\left(\sum_{i=1}^n df_{t_i}(L_l(t_i) - L_l(t_{i-1}))\right)$$

so that the loss is calculated and discounted for all time points. The present value of the premium leg is

$$PV_{PL} = s_l E\left(\sum_{i=1}^n (t_i - t_{i-1}) df_{t_i}(\Delta k_l - L_l(t_i))\right)$$

where s_l is the spread for tranche l and $\Delta k_l - L_l(t_i)$ is the nominal value minus the loss. This is discounted with df_{t_i} and multiplied with the size of the time step between premium payments.

The model can be used to find a contagion ratio c for a given spread s_l . Then there will be one ratio for each index tranche. There is no previous work found on how this could be used for pricing CDSs but the simplest way would be to interpolate the ratio to use for other tranches and non-index portfolios as well. Say that c_0, c_1 for 0 – 3% and 3 – 7% tranche are calculated respectively. The corresponding c for 2 – 6% tranche would be $\frac{1}{4}c_0 + \frac{3}{4}c_1$. The c is independent of size of the tranche and thus it is only interesting how much of the new c that is taken from each of the other tranches 2 – 3% and 3 – 6% which is here $\frac{1}{4}$ from the first tranche 0 – 3% and $\frac{3}{4}$ from tranche 3 – 7%.

Chapter 4

Data and Implementation

4.1 Calibration Data

The data is provided from Handelsbanken Capital Markets system which includes credit curves such as spread and discount factor for given time periods and indexes. The default probabilities are the individual probability of default for one company in the underlying portfolio. The data is from different series and maturities. Here follows a presentation of the CDSs used

- iTraxx Main Europe S 24, maturity 3 years, **itrxEurMain/3Y**
 - Underlying index is iTraxx Europe
 - 125 underlying names
 - Calibration tranches 0 – 3%, 3 – 6% and 6 – 12%.
 - Nominal 1000000 for each tranche
 - Market premium 1%
- iTraxx Main Europe S 24, maturity 5 years, **itrxEurMain/5Y**
 - Underlying index is iTraxx Europe
 - 125 underlying names
 - Calibration tranches 0 – 3%, 3 – 6% and 6 – 12%.
 - Nominal 1000000 for each tranche
 - Market premium 1%
- iTraxx Xover Europe S 24, maturity 5 years, **itrxEurXover/5Y**
 - Underlying index is iTraxx Europe Crossover
 - 75 underlying names
 - Calibration tranches 0 – 10%, 10 – 20% and 20 – 35%.
 - Nominal 1000000 for each tranche
 - Market premium 5%

Where market premium refers to the premium that the price is calculated for. The yearly premium paid to the insurer. All data for each CDS is taken from Handelsbankens system at the same time point to be able to compare the tranches within the

CDS. The Xover index is a subset of the main index and represents the 75 most liquid obligors. The S above represent *Series* referring to the series with roll date September 21, 2015. Roll date is meaning that the indexes are traded from that date. It may be known a few weeks earlier though [”Markit iTraxx”, 2015].

4.2 Implementation

For implementation of the presented theory, Python 2.6.6 has been used together with Eclipse. The libraries NumPy and SciPy are used a lot because of their mathematical and statistical purposes. Also Handelsbankens system Prime, provided by Sungard Front Arena is the foundation for calibration of the models. Given the price Prime presents, which are the market prices for the standard tranches the models have been calibrated. The term market price refers to the price from the index. The assignment has then been to compare the prices the models obtain to each other.

For the implementation, another approach is taken to implement the recovery rate which has been done to be able to compare with the valuation models in Prime. This is as far as the author knows not a commonly used method. The difference here is that the tranches do not contain what would have been expected from previous theory. The computing programs have been easy to modify in order to price with the same tranches. It has been necessary to do this adaption since Primes models give prices that would not be possible with the definitions in this paper. As defined before, the recovery rate means that the tranche 0 – 10% with nominal 1000000 contains the 10% first defaults. Say there are 100 names in the underlying index and thus 10 defaults in the tranche. With recovery rate 40% this would imply an experienced loss of

$$\frac{1000000(1 - 0.4)}{10} = 60000$$

for each defaulted name. The maximum experienced loss would be $\text{Nominal} * (1 - \text{recoveryRate}) = 1000000 * (1 - 0.4) = 600000$. This turns out not to be the case here.

When the underlying names have recovery rate 40% does it not mean that the tranche 0 – 10% contains 10% of the underlying names but the number of names needed to lose up to the nominal value with that recovery rate. Assume that the nominal value is 100, recovery rate is still 40% with 100 underlying names. The tranche 0 – 10% would correspond to 10 defaults out of 100 names with no recovery. Each default correspond to a loss of 10. In the case of recovery is the loss only $10 * (1 - \text{recovery rate}) = 6$ and thus is, since $16 * 6.25 = 100$, 17 names included in the tranche. There are 16 entire defaults and then 25% of the 17th default included in the tranche 0 – 10% instead of 10. This does not necessary have an impact on the rest of the theory but is useful to be aware of in order to understand all results. The main interest is to compare the models and see how the pricing of the derivatives differ. The most obvious appearance

of this approach is that for some tranches the present value is higher than would be possible with the previous definition, for instance a price above 600000 for a tranche with nominal 1000000.

Both contagion models are calibrated with 10000 simulations. The results of the contagion models are presented together with standard deviation. For the pricing 1000 simulations are made with 100 repetitions, the price is the mean of all prices and the standard deviation can easily be found by looking at all prices.

The models are calibrated towards already existing CDSs in Prime. These are not the standard tranches but combined standard tranches which should not matter to compare the models.

Chapter 5

Result

Each of the indexes have been valuated with all four pricing approaches. Afterwards all the models are used to price other CDSs and tranches. Those are compared to each other to see how the models differ in pricing.

5.1 Calibration

All models are calibrated against data from Handelsbanken. There are market prices for the CDS for which the models are tried to be calibrated. The point is that after calibration be able to price other tranches and premiums. All models presented earlier in the paper are considered. To remind the reader those are Copula model with both compound and base correlation, and two different contagion models, one simple with only one unknown parameter and one with three parameters. The compound correlation is only used for pricing standard tranches, since as mentioned in chapter 3 it is not good for non standard tranches due to the correlation smile. Base correlation is used for both standard and non standard tranches. Since the contagion ratio during the calibration phase has turned out to be increasing over tranches (approximately), with the same argument as for compound and base correlation it may be enough for pricing even non standard tranches with just interpolation of the ratio. This is investigated to find out if it is a good approach.

For the contagion models the default times are generated. The obtained parameters are gained from 10000 simulations of default times.

5.1.1 iTraxx Europe Main S24 3Y Calibration

In Table 5.1 are the parameters for compound correlation and contagion model (simple) models presented. The parameters are calibrated towards the prices in the first column in Table 5.1 which is the market price. The underlying index is the main index Series 24 with coupon 1% payed every quarter of a year with maturity 3 years. In Table 5.2

is the parameters for the two other models, base correlation and complex contagion model shown. For the complex model is all tranches needed to obtain the parameters while for the other models there are one parameter found for each tranche.

Table 5.1: iTraxx Europe Main S24 3Y, Parameter values Calibration, coupon 1%

Tranche \ Model	Market Val	Comp Corr	Cont Mod
0 – 3%	213200	0.6711	0.8464
3 – 6%	8060	0.3004	1.452
6 – 12%	-16660	0.9713	1.871

Table 5.2: iTraxx Europe Main S24 3Y, Parameter values Calibration, coupon 1%

Tranche \ Model	Base Corr	Parameter \ Model	Cont Mod 3Par
0 – 3%	0.6711	Cont ratio, c	17.67
0 – 6%	0.7548	Decay ratio, d	49.90
0 – 12%	0.8466	Correlation, ρ	0.2151

5.1.2 iTraxx Europe Main S24 5Y Calibration

In Table 5.3 are the parameters for compound correlation and contagion model (simple) models presented. The parameters are calibrated towards the prices in the first column in Table 5.3 which is the market price. The underlying index is the main index Series 24 with coupon 1% payed every quarter of a year with maturity 5 years. In Table 5.4 is the parameters for the two other models, base correlation and complex contagion model shown. For the complex model is all tranches needed to obtain the parameters while for the other models there are one parameter found for each tranche.

Table 5.3: iTraxx Europe Main S24 5Y, Parameter values Calibration, coupon 1%

Tranche \ Model	Market Val	Comp Corr	Cont Mod
0 – 3%	426600	0.6701	0.9730
3 – 6%	85590	0.9403	1.131
6 – 12%	5826	0.3006	1.388

Table 5.4: iTraxx Europe Main S24 5Y, Parameter values Calibration, coupon 1%

Tranche \ Model	Base Corr	Parameter \ Model	Cont Mod 3Par
0 – 3%	0.6701	Cont ratio, c	1.979
0 – 6%	0.7544	Decay ratio, d,	3.224
0 – 12%	0.8500	Corr, ρ	0.1244

5.1.3 iTraxx Europe Xover S24 5Y Calibration

In Table 5.5 are the parameters for compound correlation and contagion model (simple) models presented. The parameters are calibrated towards the prices in the first column in Table 5.3 which is the market price. The underlying index is the Xover index Series 24 with coupon 5% payed every quarter of a year with maturity 5 years. In Table 5.6 is the parameters for the two other models, base correlation and complex contagion model shown. For the complex model is all tranches needed to obtain the parameters while for the other models there are one parameter found for each tranche.

Table 5.5: iTraxx Europe Xover S24 5Y, Parameter values Calibration, coupon 5%

Tranche \ Model	Market Val	Comp Corr	Cont Mod
0 – 10%	654200	0.5534	0.6060
10 – 20%	113500	0.9573	0.5894
20 – 35%	-101000	0.4934	0.7064

Table 5.6: iTraxx Europe Xover S24 5Y, Parameter values Calibration, coupon 5%

Tranche \ Model	Base Corr	Parameter \ Model	Cont Mod 3Par
0 – 10%	0.5534	Cont ratio, c	1.822
0 – 20%	0.6531	Decay ratio, d	7.128
0 – 35%	0.7735	Corr, ρ	-0.0906

5.1.4 Compound Correlation Illustrations

In figure 5.1 - 5.3 is the relationship between price and correlation shown for the equity tranche for the three indexes. The market price is also plotted as a black line and their intersection is the correlation corresponding to market price. In figure 5.4 - 5.6 are the corresponding relationship for the mezzanine tranche and in figure 5.7 - 5.9 the senior tranche.

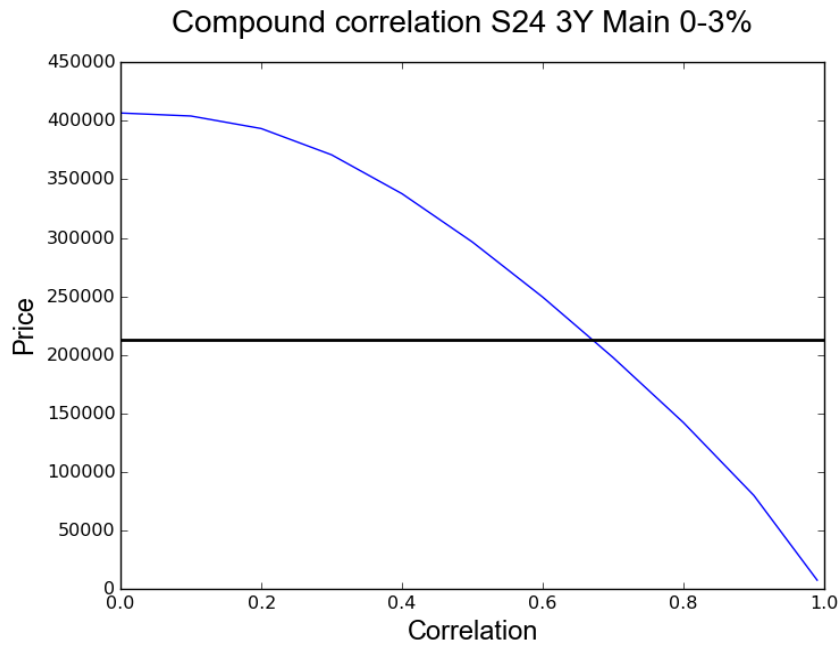


Figure 5.1: *Correlation for the 0–3% iTraxx Main 3Y tranche (blue). Price to calibrate against (black). Matching correlation is 0.6711*

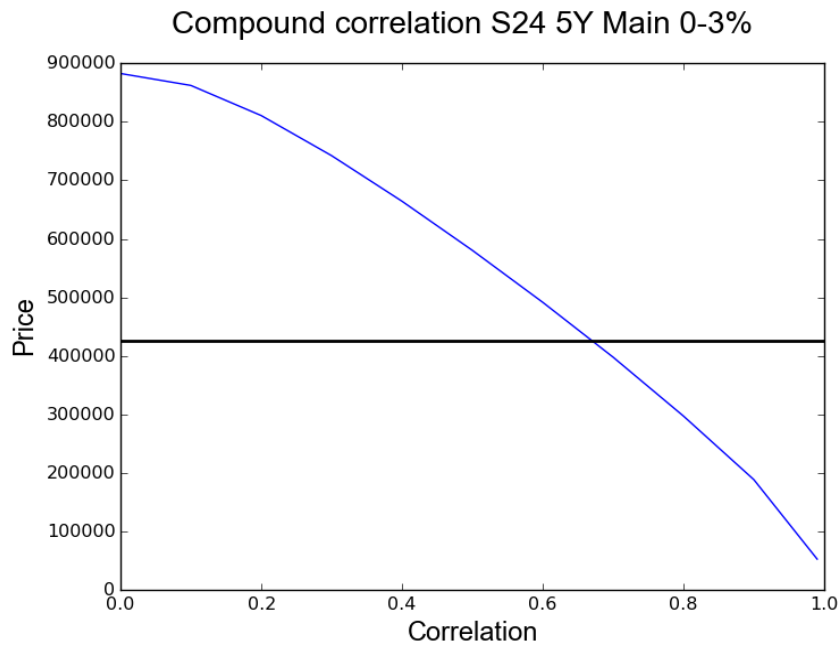


Figure 5.2: *Correlation for the 0–3% iTraxx Main 5Y tranche (blue). Price to calibrate against (black). Matching correlation is 0.6701.*

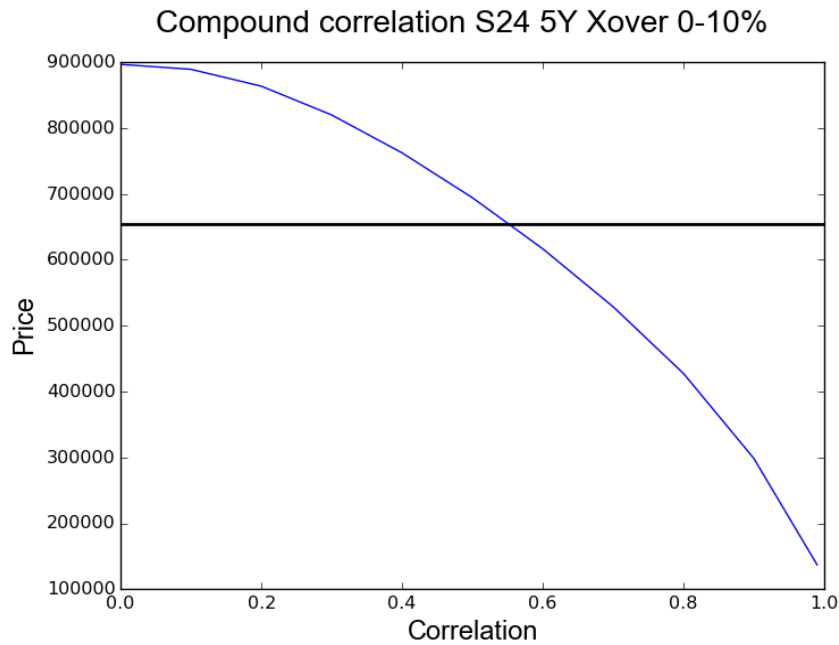


Figure 5.3: *Correlation for the 0 – 10% iTraxx Xover 5Y tranche (blue). Price to calibrate against (black). Matching correlation is 0.5534.*

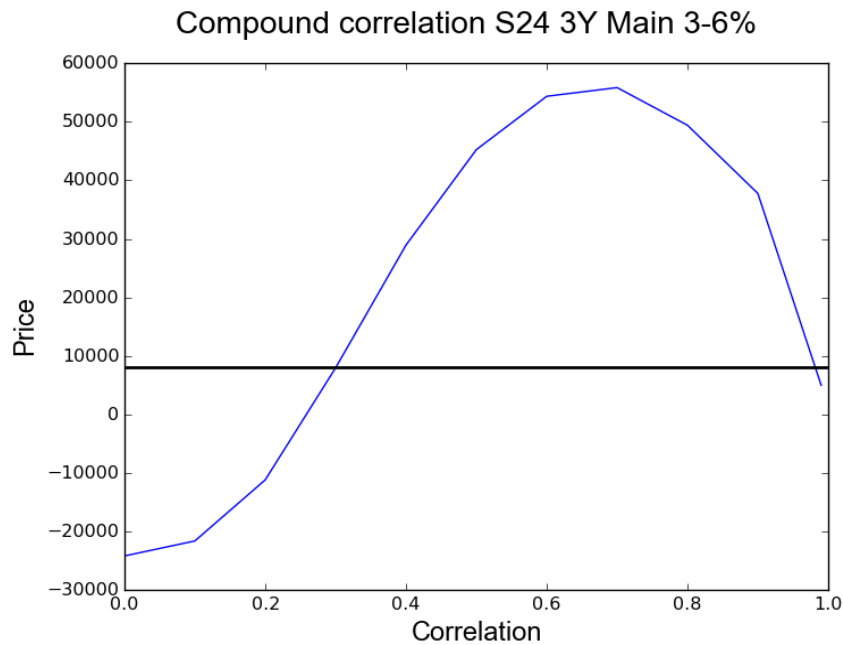


Figure 5.4: *Correlation for the 3–6% iTraxx Main 3Y tranche (blue). Price to calibrate against (black). Matching correlations are 0.3004 and 0.9713*

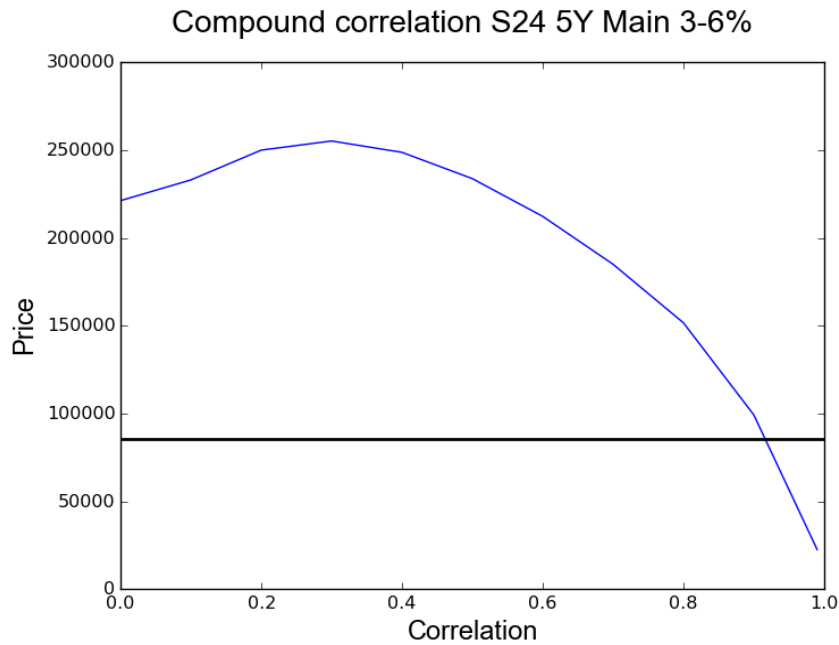


Figure 5.5: *Correlation for the 3–6% iTraxx Main 5Y tranche (blue). Price to calibrate against (black). Matching correlation is 0.9403.*

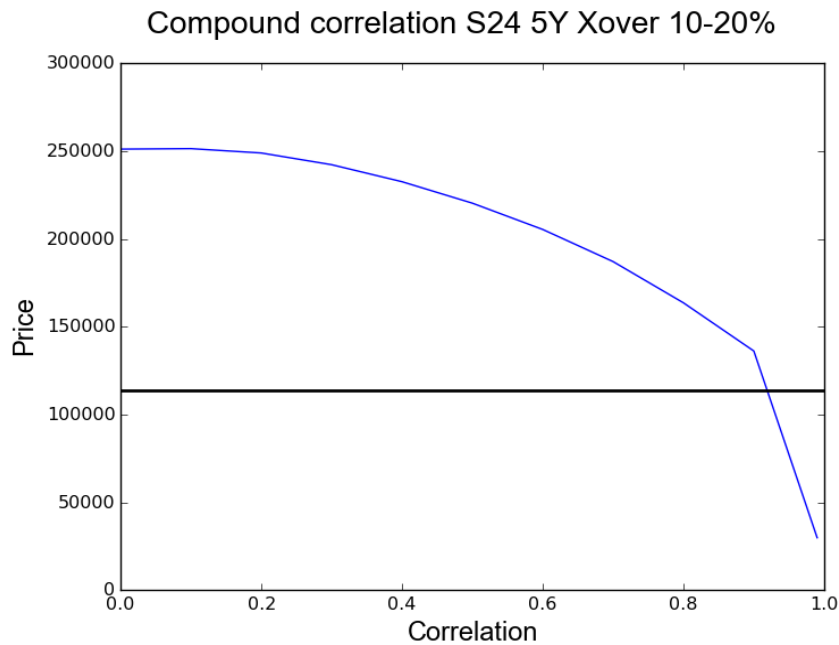


Figure 5.6: *Correlation for the 10 – 20% iTraxx Xover 5Y tranche (blue). Price to calibrate against (black). Matching correlation is 0.9573.*

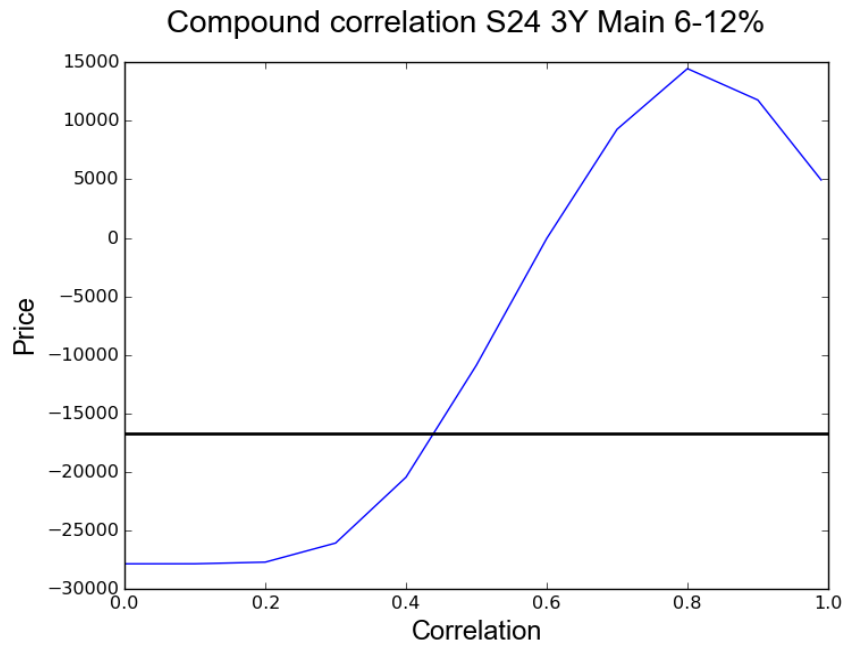


Figure 5.7: *Correlation for the 6 – 12% iTraxx Main 3Y tranche (blue). Price to calibrate against (black). Matching correlation is 0.4433*

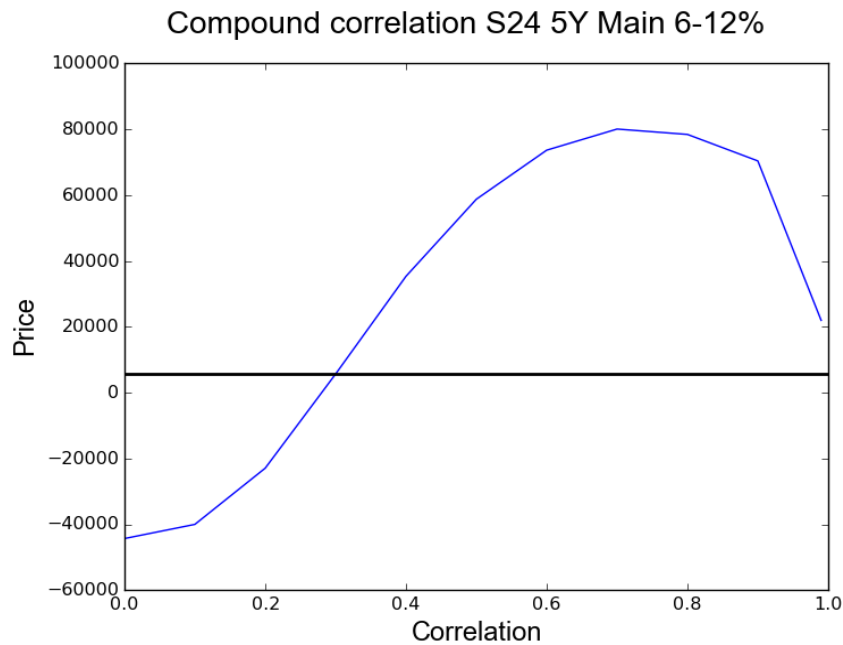


Figure 5.8: *Correlation for the 6 – 12% iTraxx Main 5Y tranche (blue). Price to calibrate against (black). Matching correlation is 0.3006.*

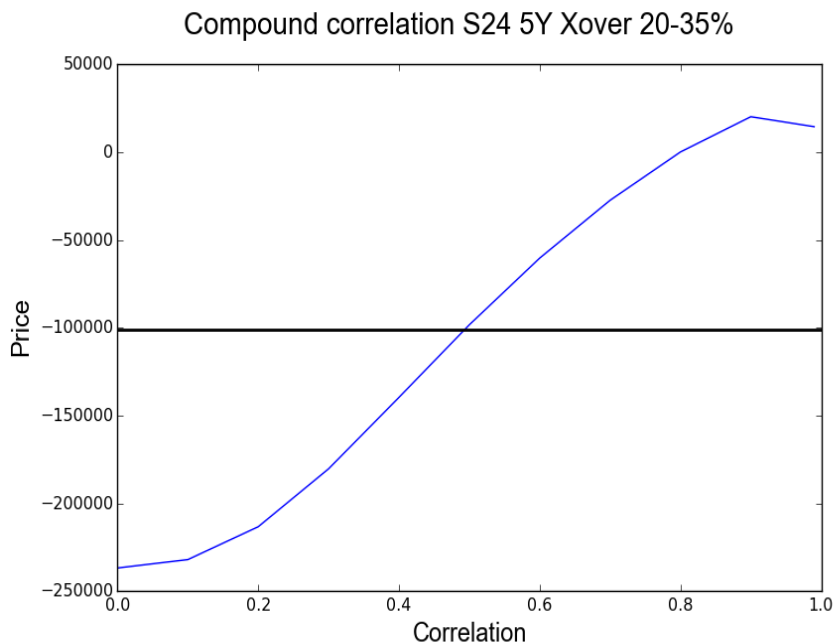


Figure 5.9: *Correlation for the 20 – 35% iTraxx Xover 5Y tranche (blue). Price to calibrate against (black). Matching correlation is 0.4934.*

5.2 Validation

The data in each table is presented so that each row corresponds to a specific tranche and coupon, priced with different models. The models are the columns and the interesting thing becomes to compare how the prices differs over the columns in each row. The column called "Prime" is the value that Handelsbankens system provides and for the three first tranches this is equal to the market price, given by the index.

The non standard tranches are chosen so that they represent diversified tranches in terms of seniority and widths.

5.2.1 iTraxx Europe Main S24 3Y Validation

In Table 5.7 - 5.9 are all models used for pricing of index Main 3Y for different tranches and coupons given the calibration from Table 5.1 and 5.2.

5.2.2 iTraxx Europe Main S24 5Y Validation

In Table 5.10 - 5.12 are all models used for pricing of index Main 5Y for different tranches and coupons given the calibration from Table 5.3 and 5.4.

Table 5.7: iTraxx Europe Main S24 3Y, Prices using Calibrated Models, coupon 1%

Tranche	Model				
	Prime	Comp Cor	Base Cor	Cont M (Std)	Cont M2 (Std)
0 – 3%	213100	213200	213200	208300 (2398)	212500 (3387)
3 – 6%	8060	8062	8063	8542 (1546)	9902 (1544)
6 – 12%	-16660	8065 -16660	16670	-17320 (759.0)	-18170 (848.9)
Non stand tranche					
0 – 6%	110600	-	110600	115000 (1719)	111600 (2408)
2 – 5%	49860	-	50380	22150 (1671)	32250 (1835)
5 – 10%	-11770	-	-11320	-15020 (731.0)	-13370 (1126)

Table 5.8: iTraxx Europe Main S24 3Y, Prices using Calibrated Models, coupon 2%

Tranche	Model				
	Prime	Comp Cor	Base Cor	Cont M (Std)	Cont M2 (Std)
0 – 3%	189000	189100	189100	183300 (2811)	188400 (2978)
3 – 6%	-19510	-19470	-19350	-18840 (1542)	-17150 (1679)
6 – 12%	-44540	-19160 -4440	-44380	-45340 (823.9)	-45960 (865.4)
Non stand tranche					
0 – 6%	84740	-	84890	89110 (1732)	85670 (2075)
2 – 5%	22910	-	23590	-5522 (1829)	5457 (2033)
5 – 10%	-39590	-	-38980	-42980 (761.4)	-41150 (950.7)

5.2.3 iTraxx Europe Xover S24 5Y Validation

In Table 5.13 - 5.15 are all models used for pricing of index Xover 5Y for different tranches and coupons given the calibration from Table 5.5 and 5.6.

Table 5.9: iTraxx Europe Main S24 3Y, Prices using Calibrated Models, coupon 5%

Tranche \ Model	Prime	Comp Cor	Base Cor	Cont M (Std)	Cont M2 (Std)
0 – 3%	116500	117100	117100	109500 (2974)	116200 (3055)
3 – 6%	-102200	-102100	-101600	-101700 (1465)	-99220 (2009)
6 – 12%	-128200	-127600	-127500	-128500 (776.9)	-129200 (939.0)
Non stand tranche					
0 – 6%	7141	-	7752	10910 (1918)	8633 (2281)
2 – 5%	-57950	-	-56780	-87630 (1817)	-75150 (2435)
5 – 10%	-123000	-	-121900	-126200 (819.6)	-124000 (1042)

Table 5.10: iTraxx Europe Main S24 5Y, Prices using Calibrated Models, coupon 1%

Tranche \ Model	Prime	Comp Cor	Base Cor	Cont M (Std)	Cont M2 (Std)
0 – 3%	426600	426600	426600	422000 (3968)	414600 (3867)
3 – 6%	85590	85590	85620	86540 (2826)	92950 (2848)
6 – 12%	5825	5831	5830	4040 (1736)	-7958 (1764)
Non stand tranche					
0 – 6%	256100	-	256100	252200 (2873)	254000 (3485)
2 – 5%	173600	-	174700	143500 (3331)	152200 (3533)
5 – 10%	22850	-	22090	12770 (1912)	11340 (1922)

Table 5.11: iTraxx Europe Main S24 5Y, Prices using Calibrated Models, coupon 2%

Tranche \ Model	Prime	Comp Cor	Base Cor	Cont M (Std)	Cont M2 (Std)
0 – 3%	389700	390000	390000	384200 (4021)	377800 (3985)
3 – 6%	39460	40280	39760	40170 (2672)	46640 (3049)
6 – 12%	-41740	-41750	-41480	-43550 (1838)	-55500 (1785)
Non stand tranche					
0 – 6%	214600	-	214900	210000 (2790)	212600 (3081)
2 – 5%	129500	-	130900	97540 (3208)	107400 (3733)
5 – 10%	-24420	-	-24920	-34970 (1735)	-34750 (1916)

Table 5.12: iTraxx Europe Main S24 5Y, Prices using Calibrated Models, coupon 5%

Tranche \ Model	Prime	Comp Cor	Base Cor	Cont M (Std)	Cont M2 (Std)
0 – 3%	279000	280100	280100	270400 (4497)	265700(4540)
3 – 6%	-98940	-95640	-97810	-99670 (2978)	-91420 (3398)
6 – 12%	-184400	-184500	-183400	-186500 (1959)	-198200 (1894)
Non stand tranche					
0 – 6%	90030	-	91140	83730 (3470)	87870 (3258)
2 – 5%	-2844	-	-575.4	-39220 (3985)	-27280 (3909)
5 – 10%	-166200	-	-165900	-177200 (2109)	-178000 (2276)

Table 5.13: iTraxx Europe Xover S24 5Y, Prices using Calibrated Models, coupon 5%

Tranche \ Model	Prime	Comp Cor	Base Cor	Cont M (Std)	Cont M2 (Std)
0 – 10%	654200	654200	654200	650700 (3443)	625800 (3425)
10 – 20%	113500	113500	113500	107600 (3802)	158500 (4523)
20 – 35%	-101000	-101000	-101000	-100300 (2413)	-133100 (2725)
Non stand tranche					
0 – 15%	501000	-	502400	507400 (3707)	498700 (3595)
5 – 12%	442300	-	444600	457200 (4964)	424500 (4600)
10 – 30%	15910	-	17200	44120 (2892)	28200 (3593)

Table 5.14: iTraxx Europe Xover S24 5Y, Prices using Calibrated Models, coupon 1%

Tranche \ Model	Prime	Comp Cor	Base Cor	Cont M (Std)	Cont M2 (Std)
0 – 10%	758000	757000	757000	769800 (3376)	736900 (3418)
10 – 20%	283500	277200	282500	286700 (4287)	325500(4426)
20 – 35%	84190	84390	83110	87540 (2643)	54190 (2395)
Non stand tranche					
0 – 15%	624800	-	625100	644200 (3275)	626100 (3343)
5 – 12%	579800	-	580900	610300 (4666)	568300 (4467)
10 – 30%	192900	-	193100	226500 (3506)	205600 (3059)

Table 5.15: iTraxx Europe Xover S24 5Y, Prices using Calibrated Models, coupon 10%

Tranche \ Model	Prime	Comp Cor	Base Cor	Cont M (Std)	Cont M2 (Std)
0 – 10%	524400	525700	525700	501600 (3483)	487800 (4612)
10 – 20%	-98970	-91130	-97730	-116900 (4389)	-53040 (4558)
20 – 35%	-332500	-332700	-331100	-336000 (3142)	-367300 (2843)
Non stand tranche					
0 – 15%	346300	-	349200	335700 (3952)	338100 (4620)
5 – 12%	270500	-	274200	265500 (5095)	246400 (5230)
10 – 30%	-205300	-	-202600	-183300 (3755)	-192600 (3949)

Chapter 6

Conclusions

6.1 Calibration parameters

6.1.1 Compound Correlation

The mezzanine tranche for Main 3Y ρ , in Table 5.1 is the only tranche having more than one solution. It has one solution that is significantly lower than the corresponding correlation for the other indexes and one that is higher, close to 1. This can be seen in figure 5.4. Mezzanine tranche is not as sensitive to compound correlation as the other tranches. That could possibly explain why there are bigger differences in correlations and even more than one that give satisfying models. If the correlations goes up, the senior tranche gets more risky and should be more expensive to insure. The opposite holds for the equity tranche. For the mezzanine tranche, it is not obvious how it affects the price but the figures 5.4 - 5.6 shows that the price goes up for a while before decreasing again. The same pattern can be seen for the senior tranche in figures 5.7 - 5.9 where the price goes down after a while, but for higher correlation than for the mezzanine tranche.

6.1.2 Base correlation

Base correlation has in opposite to compound correlation a unique solution [Galiani S, et al., 2004] and are quite similar for iTraxx Main 3Y and 5Y while a bit lower for the Xover index. It may be reasonable to think that the correlation for 5Y and 3Y should be the same since the underlying names are the same for both indexes. Base correlations has turned out to be increasing with tranches in accordance to earlier theory. The base correlations are very similar for the two main indexes which is reasonable since they have the same underlying names.

6.1.3 Simple Contagion Model

For both the Main indexes are the contagion ratios for the different tranches both strictly increasing. The Main indexes are relatively more equal than the Xover index. The Xover index has significantly lower contagion ratios for all tranches than the main indexes. This may also here be explained by the fact that the Main indexes have the same underlying portfolio and the default should affect the remaining names similarly independent of time to maturity. The contagion ratios are not increasing between the 0 – 10% and 10 – 20% tranche, but they are relatively close. Since the ratios are increasing with tranches for two of the three indexes and the third is close to constant, the contagion ratios have been interpolated in order to try to price non standard tranches. This is with the same argument as for the increasing base correlation.

6.1.4 Complex Contagion Model

For all indexes the parameter optimization gives the decay as the biggest parameter value and the correlation the smallest. The correlation is negative for Xover 5Y index. This may not be expected since the correlation obviously were high in the previous calibrations. It may on the other hand have been compensated by the other parameters so negative correlation should not be excluded as a possibility.

6.2 Model Pricing

In this section will the pricing be compared between the models. The models are calibrated towards the price given by Prime. For the Tables 5.7 5.10 and 5.13 with the coupons used for calibration (1% for the Main indexes and 5% for the Xover index) should the three first tranches be as close to Primes value as possible. For the other tranches and coupons there is no "should be" since Prime is just one of the pricing models here as well. The important thing is that there is no right or wrong here, they are just compared. So what is more interesting is the non standard tranches and the tables with different coupons than those used for calibration.

For the equity tranche compound and base correlation will always generate the same price since the base correlation is equal to the compound correlation. As seen there are the two contagion models less accurate than the semi-analytic valuation and this is of course due to the simulation done for those models.

What is interesting is that their confidence interval will not cover the calibration value for many of the calibrations. To remind the reader is the simulation for both the contagion models done with 1000 simulations and 100 repetitions and the normal approximated 95% confidence interval would thus become

$$price \pm \sigma * 1.96/\sqrt{100} \approx price \pm \sigma/5$$

where σ is the standard deviation presented after the price in the contagion columns. This is the case for all indexes. Since the interval does not cover the calibration price the conclusion is that it is probably not an accurate price and the contagion ratio could have been improved by calibration with more simulations. The complex contagion model is sometimes not so close to the calibration values. It is because of the optimization with three parameters that it not finds an exact solution. The contagion models are calibrated and priced with the same number of simulations. The limitation here is the computational time since the algorithm must include all simulations in every value it goes over within the optimization.

6.2.1 iTraxx Main S24 3y

Consider Table 5.8 and 5.9 the upper part, three first tranches. This index was the only one with more than one solution for one of the tranches. Comp corr lies between Prime and base correlation value for the lower correlation, and slightly above for the high correlation. Since the calibration is more accurate with the lower correlation, this could be seen as the one to use and that imply for the other coupons prices closer to Primes values. This is also the compound correlation used as the first base correlation. Comparing the compound and base correlation for the standard tranches and different coupons, the lower compound correlation is slightly closer to Prime for all tranches and coupons than the base correlation.

In the calibration in Table 5.7 differs the complex contagion model most from the calibration values. Compared to the other indexes is this a quite accurate calibration for the complex contagion model though. The mezzanine tranche is above the calibration value and the senior and equity tranche is below. This can be seen in the prices for standard tranches with other coupons as well, the mezzanine tranche is above and the senior and equity are below. For the non standard tranches is base correlation the model that prices most equal to Prime. Between the contagion models is the complex model closer to Primes values for all non standard tranches.

6.2.2 iTraxx Main S24 5y

For this index, the complex contagion model when calibrated is a bit off for the senior tranche. The calibration errors in the complex contagion model can be seen in standard tranches for other coupons. It prices higher than Prime and base correlation for the mezzanine tranche and below for the other equity and senior tranche. Both contagion models price below Prime and base correlation for all non standard tranches. Simple contagion model has calibration values closer to Prime and keeps generating standard tranche prices closer to Prime compared to the complex contagion model. The interesting thing is what happens with the non standard tranches. There the complex model is closer to the other models than the simple one. Semi-analytic valuation with base correlation as input and Prime are definitely the models that are most similar.

6.2.3 iTraxx Xover S24 5y

The complex contagion model perform worse than for the other two indexes in terms of close to the calibration value. All tranches are quite different from the calibration value. This can be seen in Table 5.13. The simple contagion model is usually closer to Prime and base correlation values for standard tranches while for non standard tranches it is again the opposite. The complex model does actually perform more like the other models for non standard tranches than for the standard tranches in this case. Compound and base correlation are usually closer to Primes values than the other two models, but it differs between them which is closest and has the highest/lowest price. For this index the base correlation and simple contagion model varies which one is bigger but for almost all prices are Primes values in between of those. For the non standard tranches is over all base correlation definitely the approach that performs most like Primes values.

6.2.4 Conclusions

Comparing the two contagion models gives that the simple contagion model is easier to calibrate accurately because of fewer parameters. For all indexes the simple model prices more similar to Prime and base correlation for standard tranches compared to the complex model. The interesting thing is that for the non standard models is it the opposite for all indexes.

One conclusion to draw from this is that the contagion models do not calibrate as accurate as compound and base correlation. The pattern seen within the calibration of those are usually possible to find in standard tranches for other coupons as well. Semi-Analytic valuation with Base correlation as input is definitely the approach that over all prices most similar to Prime.

Chapter 7

Discussion and Future Research

What would have made the contagion models perform better which may be of great impact is a lower standard deviation which could be done by more simulations both for the calibration and the pricing part. Maybe some other optimization algorithms should be considered or restrictions to find the parameters easier. Also some variance reduction could be useful such as antithetic sampling. Since the pattern from the calibration were possible to trace even in pricing for standard tranches with other coupons, it would be of interest to calibrate the contagion models really accurate to see how the pricing differs then.

There are other models that would give additional pricing options which could be interesting to investigate. For instance could the copula model be used with other distributions than normal such as t -distribution or any other known distribution.

This is just a presentation of different pricing models. If one would choose one to use, it could not be done based on this paper. What can be done further, and was the initially purpose was to compare the pricing with these models with market prices, traded between parties. The reason to why this was not done here is that all the data available were traded in SEK. The CDS index is not available in SEK and thus it is not possible to compare the prices without a method to convert the spreads between currencies. This is because of the correlation between defaults and the currency, some underlying names are connected to what happens with a specific currency if they default. This is a future area of work to be able to compare with actual prices on the market.

Without any further measurement it is hard to tell which model is best and the impact from the calibration accuracy needs to be investigated. Even though the simple contagion model seemed to alone have the most different pricing for the non standard tranches the idea of interpolating the contagion ratio for different tranches should not be rejected yet. It could also be further investigated how to find the contagion ratios for non standard tranches, maybe the simple interpolation is not the best. It would be

interesting to see how the contagion models price with more simulations for the calibrations since the calibration affects all later pricing for those tranches. The complex contagion model is probably not so useful when the calibration not goes well. To be able to price the derivatives accurately the calibration needs to be accurate.

As a conclusion there is not an answer which is best, this needs a clear view of how to measure that and data to compare the pricing towards. One thing that can be said is that the differences in prices do not grow with size of the coupons, it stays approximately the same for different coupons. The biggest challenge is to price non standard tranches. There does the prices differ most and can affect the trading the most.

It is also clear that the pricing do differ between models and some models can probably be good for pricing different tranches, indexes and coupons. One question to answer in future research is: what is a good model and how to measure that? This is definitely an area where a lot of research is left and improvements are to come.

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