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Updating bounds on the low-energy constants of Chiral Perturbation Theory from exact bounds on amplitudes

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“ Comment s’étaient-ils rencontrés ? Par hasard, comme tout le monde. Comment s’appelaient-ils ? Que vous importe ? D’où venaient-ils ? Du lieu le plus prochain. Où allaient-ils ? Est-ce que l’on sait où l’on va ? Que disaient-ils ? Le maître ne disait rien; et Jacques disait que son capitaine disait que tout ce qui nous arrive de bien et de mal ici-bas était écrit là-haut.”

Jacques le Fataliste DIDEROT

“Kanske vil der gå både Vinter og Vår,
og næste Sommer med, og det hele År; —
men engang vil du komme, det véd jeg visst;
og jeg skal nok vente, for det lovte jeg sidst.

Gud styrke dig, hvor du i Verden går!
Gud glæde dig, hvis du for hans Fodskammel står!
Her skal jeg vente til du kommer igen;
og venter du histoppe, vi træffes der, min Ven!”

Peer Gynt Henrik Ibsen

Abstract

Chiral perturbation theory is an efficient effective theory to describe meson-meson scattering at low energy. An effective theory naturally introduces low energy constants and they are used, for example, to solve the ultraviolet divergences. The aim of this master thesis is to focus on meson-meson scattering to find some theoretical bounds on the coupling constants, based mainly on arguments from analyticity and unitarity. In this work, new bounds have been derived for the two and three flavours cases, at p^4. Next, properties of the general N_f flavours case have been studied and applied to the four flavours case. The contribution of the p^6 lagrangian has also been considered to estimate the error in our calculations. Finally, upon all the calculated constraints, one might want to keep only the necessary ones. When the number of coupling constants is high, it is a problematic issue. A way to treat this problem has been developed.

Key words: Coupling constants, meson-meson scattering, Chiral perturbation theory, constraints.

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A thought for Johan Relefors, and I wish him good luck for his PhD.

I finally thank everybody in the department for their warm welcome.

Popular science description

Many new physical theories have been developed during the last century. The objects with high velocities are described by Special Relativity, the heavy ones by General Relativity, and the small ones by Quantum Mechanics. All of them show us that physics does not always follow our natural intuitions. Time, space, distance and even shapes appear to be more difficult to define than it sounds. This makes a popular science paper difficult to write. Let us nevertheless try to go down to the world of mesons.

A question of interactions

Physics is governed by four main interactions: Gravitation, which is described by General Relativity; Electromagnetism, the most important one at our scale as most of our devices like computers, phones or lamps are electromagnetic systems; the weak interaction, which can be used, through the natural instability of some materials, to produce energy with nuclear plants and finally, the strong interaction, which makes the atom's nucleus stable. These latter three interactions are described by Quantum Mechanics and Special Relativity, and are unified in the Standard Model.

Standard model

The Standard Model is used to understand low scale physics. The paper you are reading is made of molecules, these molecules, of atoms, an atom of electrons and of a nucleus. The nucleus in turn is made of protons and neutrons, which are both made of 3 quarks. Quarks, of which there are six types, and electrons are the fundamental "bricks" of all the matter. A meson, is a particle made up of two of these quarks.

A question of scale

With a magnet you can attract your keys despite of gravity but not any objects left on the moon. So at our scale, Electromagnetism is the most interesting phenomena, whereas at a bigger range, it becomes negligible with respect to gravity. At each scale, there are preponderant physical phenomena that we can highlight with a particular theory. The same occurs in our study: the Standard Model describes non relevant physics for us, like high energy physics. We use therefore a simplified theory, which introduces "coupling constants" and the more precision we want, the more constants we need. These numbers are really difficult to calculate, and the purpose of this work is to derive some constraints on them.

In the world of light meson particles

Since we want to simplify our problem by getting rid of heavy particles, we first have to look at light mesons. They are, of course, made of the lightest quarks. At the first level of precision we need to introduce two constants, on which constraints have been already found in Vincent Mateu and Aneesh V. Manohar's paper Dispersion Relation Bounds for $\pi\pi$ Scattering. We have derived better ones. Next, the numbers of quarks have been increased to 3 and then 4. Then, a procedure was created to organise our results and select the relevant information. Finally, the error has been evaluated inserting experimental values for the coupling constants in the second level of precision.

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1 Introduction

The Standard Model of particle physics successfully describes electromagnetic, weak and strong interactions. It introduces boson fields, like photons or Higgs bosons, and fermion fields which are devided in two categories: the leptons and the quarks. The quarks are submitted to the strong interaction, which is described by gluon fields. However, these two particles cannot be isolated and they are contained in bigger structures, without any colour charges, called hadrons. This phenomena is called confinement. In particular, when a quark is associated with its anti-quark, the hadron is called meson.

Nevertheless, for light particles physics, the entire description of nature is not required and one might want to highlight the relevant processes through an effective theory. This is the purpose of Chiral perturbation theory. As presented in [1] and [2], if the mass of the quarks were equal to zero, a new symmetry would appear: $SU(N_f)_L \times SU(N_f)_R$, with N_f the number of flavours. It is spountaneaously broken to $SU(N_f)_V$ which creates Goldstone bosons. They represents low mass mesons, like pion states in the two flavours case. Moreover, an explicit symmetry breaking, due to the mass of the quarks, is added. This method provide a lagrangian which is able to describe meson-meson scattering. The terms are ordered through power counting, which means that the terms with high power in momenta are negligible compared to the ones with low power. However, as it is an effective theory, each order introduces coupling constants, or low energy constants, which are hard to calculate without experiments and it is therefore interesting to derive some bounds on them from theoretical arguments.

For the two flavours and three flavours cases at p^4, bounds have been derived for some specific channels in [3] and [4]. Following the same philosophy as in these publications, with more general linear combinations of the scattering amplitudes and using analyticity and unitarity properties, we try to update the bounds.

The more flavours we have, the more coupling constants we need, the constraints are then hard to represent and organise. Finding the useful information in the raw results is then a real problem so we make rigorous the notion of " better set of constraints" and prove a criteria which can be used to select the useful information.

Moreover, the general N_f flavours case has not been considered yet so we treat it analytically. These results are applied to the four flavours case.

In this report, we will first present quickly the main idea of Chiral perturbation theory, the way lagrangians are built and the coupling constants introduced. Then, the two flavours case at p^4 is treated. Next, focusing on the general case with m low energy constants, we derive tools to extract the relevant hyperplanes, which allows us to consider the three flavours case and, theoretically, the general p^4 lagrangian, whatever the number of flavours. Finally, the four flavours case is treated.

2 Chiral perturbation theory

2.1 Introduction

In this section we will discuss how a new symmetry appears in the Quantum Chromodynamic (QCD) lagrangian if we assume that the mass of the quarks are negligible. First we will see, following [1], that a spontaneous symmetry breaking gives us, through the Goldstone's theorem, a first description of light mesons, and then that an explicit symmetry breaking, due to quark masses, completes the model. Finally, we will go on to discuss that, since this is an effective field theory, each order bring in new coupling constants. These constants are also needed for the renormalisation procedure as divergences for ultraviolet energies are observed.

2.2 Chiral symmetry and spontaneous symmetry breaking

If the quark masses were equal to zero, the QCD lagrangian would have a new symmetry described by the group $G = SU(N_f)_L \times SU(N_f)_R$ of dimension n_G. N_f refers to the number of flavours. Considering $H = SU(N_f)_V$, of dimension n_H, the relevant symmetry breaking is $G \to H$. It has been proven [1] that a non vanishing scalar quark condensate would be a sufficient condition to this phenomena. The Goldstone's theorem gives then $n_G - n_H$ bosons, that we can label by ϕ^a. Introducing X^a the Lie algebra generators of H one can define:

$$\phi = \sum_a \phi^a X^a = \phi^a X^a$$

Note that to choose the generators is equivalent to choosing a basis of the Lie algebra. For example, for the two flavours case, a possible choice is the Pauli matrices:

$$\phi = \sum \tau_i \phi_i = \begin{pmatrix} \phi_3 & \phi_1 - i\phi_2 \\ \phi_1 + i\phi_2 & -\phi_3 \end{pmatrix} = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix} \tag{2.1}$$

Since the $n_G - n_H$ Goldstone bosons live on the coset G/H, it is important to understand the action of G on G/H, as we want to build a invariant lagrangian upon G. First, a parametrization of G/H can be expressed, for example, by:

$$U = \exp \frac{i\phi}{F}$$

where F is the pion decay constant in the chiral perturbation limit, in our case. Now, let us consider the simple case of the left coset, following the discussion in [1]. Let $g = (g_L, g_R) \in G$. The left coset can be described as:

$$gH = \{(g_L h_V, g_R h_V), (h_V, h_V) \in H\}$$

But:

$$(g_L h_V, g_R h_V) = (g_L h_V, g_R {g_L}^\dagger g_L h_V)$$

Since $(g_L h_V, g_L h_V) \in H$ then:

$$gH = (1, g_R {g_L}^\dagger)H$$

Recalling that the matrix U can be described by an equivalence class: gH, and taking $g_2 \in G$, the action of G on G/H is:

$$\begin{aligned}
g_2 gH &= (g_{2L}, g_{2R})(1, g_R {g_L}^\dagger)H \\
g_2 gH &= (g_{2L}, g_{2R} g_R {g_L}^\dagger)H \\
g_2 gH &= (1, g_{2R} g_R {g_L}^\dagger {g_{2L}}^\dagger)(g_{2L}, g_{2L})H \\
g_2 gH &= (1, g_{2R} g_R {g_L}^\dagger {g_{2L}}^\dagger)H
\end{aligned}$$

This shows that the equivalence class of $(1, g_R {g_L}^\dagger)$ transform into the one of $(1, g_{2R} g_R {g_L}^\dagger {g_{2L}}^\dagger)$ under G, which suggests that:

$$U \to g_{2R} U {g_{2L}}^\dagger$$

In the general case [2], for $g_2 \in G$, the transformation takes the form of

$$U \to g_{2R} U h^\dagger$$

Where h depends on g_2 and ϕ. It is given by the chosen representation. Note that it can be also written in the following fashion:

$$U \to hU {g_{2L}}^\dagger$$

To study scattering processes, we need to consider the following expression:

$$\partial_\mu U$$

And the transformation is given by

$$\partial_\mu U \to g_R \partial_\mu U h^\dagger$$

And:

$$\partial_\mu U^\dagger \to \partial_\mu (g_R U h^\dagger)^\dagger = h \partial_\mu U^\dagger {g_R}^\dagger$$

Finally, the easiest way to build an invariant scalar under G is to take the traces of product of matrices such as:

$$tr(\partial_\mu U \partial^\mu U^\dagger)$$

Which is really easy to verify:

$$tr(\partial_\mu U \partial^\mu U^\dagger) \to tr(g_R \partial_\mu U h^\dagger h \partial^\mu U^\dagger {g_R}^\dagger) = tr(\partial_\mu U \partial^\mu U^\dagger)$$

Only the spontaneous symmetry breaking has been considered here. The lagrangian has to be invariant upon G, which leads to build elements like: $tr(\partial_\mu U \partial^\mu U^\dagger)$. Note that we can theoretically use the U matrix as many time as we want in the traces and that the previous one was only an example. In the next section, the explicit symmetry breaking will be considered.

2.3 Explicit symmetry breaking

The Goldstone bosons are massless. But as suggested in Eq.(2.1), they have to describe real massive particles. The masses of the quarks are not zero so the Chiral symmetry has to be explicitly broken. Through this process, mass term will appear in the lagrangian. For example, in the two flavours cases, one way to break the symmetry is to introduce a matrix M [1], [2]:

$$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$$

where m_u and m_d are the masses of the up and down quarks. As shown in [1], the quark mass terms in the QCD lagrangian can be introduced with the following explicit symmetry breaking:

$$\mathcal{L}_M = -\bar{q}_R M q_L - \bar{q}_L M^\dagger q_R$$

We would like to translate this phenomena from the QCD case into our effective theory. $\mathcal{L}_M$ is not invariant upon G, but it would be if M transformed like U upon G. This suggest to introduce the following type of structure:

$$tr(MU^\dagger + UM^\dagger)$$

Again, the purpose of this discussion is to give an idea of how the Chiral lagrangian is built. For more details, we refer to [1].

We have seen that the spontaneous and explicit symmetry breaking are the main tools to build lagrangian. However, many different possible combinations of the matrices ∂U, U and M can be introduced and it is important to know which ones are the most important.

2.4 Power counting scheme

The lagrangians, for meson-meson scattering, can be written only with even power combinations of momentum [1],[8]:

$$\mathcal{L} = \mathcal{L}_{p^2} + \mathcal{L}_{p^4} + \mathcal{L}_{p^6} + \dots$$

This can be thought as a Taylor expansion around $p^2 = 0$, where the amplitudes vanish. If p is small enough, the lower the exponent of p is, the more important the term is. The preponderant term here is $\mathcal{L}_{p^2}$ followed by $\mathcal{L}_{p^4}$ and so on.

Each lagrangian ($\mathcal{L}_{p^2}$, $\mathcal{L}_{p^4}, \dots$) are made of traces elements, which have been studied before. Each new trace brings a new low energy constant in front. The number of coupling constants then depends on the level of precision. Moreover, it depends on the number of flavours, as illustrate in [2]. For example, with two flavours, the first term introduces one coupling constant F identified as the pion decay constant, the next one introduces two more, the third one six, ... Our study focuses on p^4 and its coupling constants.

Now that we have seen how the effective lagrangians are build and how the coupling constants are introduced, we will quickly present the methodology used to go from the Chiral lagrangian to the scattering amplitude.

2.5 Scattering amplitude derivation

The following section is strongly inspired by an exercise proposed in [1]. We now know how to build the lagrangian. Let us consider the $SU(2)$ case. A really simple one for the spontaneous symmetry breaking would be, for example:

$$\mathcal{L}_{impl} = c_1 tr(\partial_\mu U \partial^\mu U^\dagger)$$

Where c_1 is a constant. And for the explicit symmetry breaking part, one can choose:

$$\mathcal{L}_{expl} = c_2 tr(MU^\dagger + UM^\dagger)$$

Where c_2 is a constant, and:

$$M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

The approximation $m_u \approx m_d \approx m$ has been made. The full lagrangian becomes:

$$\mathcal{L}_{example} = c_1 tr(\partial_\mu U \partial^\mu U^\dagger) + c_2 tr(MU^\dagger + UM^\dagger)$$

But:

$$U = \exp\frac{i\phi}{F} = 1 + i\frac{\phi}{F} - \frac{1}{2}\frac{\phi^2}{(F)^2} - \frac{i}{6}\frac{\phi^3}{(F)^3} + \frac{1}{24}\frac{\phi^4}{(F)^4}\dots$$

Considering the expansion up to the fourth power of ϕ. Then:

$$\partial_\mu U = i\frac{\partial_\mu\phi}{F} - \frac{1}{2}\frac{\partial_\mu\phi^2}{(F)^2} - \frac{i}{6}\frac{\partial_\mu\phi^3}{(F)^3} + \frac{1}{24}\frac{\partial_\mu\phi^4}{(F)^4} + o(\phi^4)$$

So:

$$\partial^\mu U^\dagger = -i\frac{\partial^\mu\phi}{F} - \frac{1}{2}\frac{\partial^\mu\phi^2}{(F)^2} + \frac{i}{6}\frac{\partial^\mu\phi^3}{(F)^3} + \frac{1}{24}\frac{\phi^4}{(F)^4} + o(\phi^4)$$

We have used the fact that $\phi = \phi^\dagger$. Next:

$$\partial_\mu U \partial^\mu U^\dagger = \frac{\partial_\mu\phi\partial^\mu\phi}{(F)^2} - \frac{i}{2}\frac{\partial_\mu\phi\partial^\mu\phi^2}{(F)^3} - \frac{1}{6}\frac{\partial_\mu\phi\partial^\mu\phi^3}{(F)^4} + \frac{i}{2}\frac{\partial_\mu\phi^2\partial^\mu\phi}{(F)^3} + \frac{1}{4}\frac{\partial_\mu\phi^2\partial^\mu\phi^2}{(F)^4} - \frac{1}{6}\frac{\partial_\mu\phi^3\partial^\mu\phi}{(F)^4} + o(\phi^4)$$

As we see, the odd powers of F cancel ($-\frac{i}{2}\frac{\partial_\mu\phi\partial^\mu\phi^2}{(F)^3} + \frac{i}{2}\frac{\partial_\mu\phi^2\partial^\mu\phi}{(F)^3} = 0$). The term $\frac{\partial_\mu\phi\partial^\mu\phi}{(F)^2}$ is a kinetic term and not an interaction one. The trace of the other part gives us:

$$\mathcal{L}^{\phi^4}_{example} = c_1\frac{1}{6F^4}tr([\phi, \partial_\mu\phi]\phi\partial^\mu\phi) + c_2\frac{1}{12F^4}tr(M\phi^4)$$

Here, we see clearly that we have only even powers of ϕ. Next, the usual procedure, deriving Feynman rules allowed us to calculate the scattering amplitude, T. For the $SU(2)$ case, it can be written in the following fashion [2, 3, 7]:

$$T(p_a, p_b, p_c, p_d) = \delta^{ab}\delta^{cd}A(s,t,u) + \delta^{ac}\delta^{bd}A(t,s,u) + \delta^{ad}\delta^{cb}A(u,t,s)$$

Here, a and b stands for the incoming particles whereas c and d stands for the outgoing ones. Moreover, s, t and u are the usual Mandelstam variables:

$$\begin{aligned} s &= (p_a + p_b)^2 \\ t &= (p_a - p_c)^2 \\ u &= (p_a - p_d)^2 \\ \text{And: } s + t + u &= \sum_{i=1}^{4} m_i^2 \end{aligned}$$

As we will see in the next section, the expression A includes coupling constants.

2.6 Some formulas of the two flavours case to the first loop

From now on, M will stand for a mass constant which is here the pion mass. As shown in [7] the expression of A to the first loop is:

$$A(s,t,u) = \frac{(s - M^2)}{F^2} + B(s,t,u) + C(s,t,u)$$

Where $\frac{(s-M^2)}{F^2}$ is the tree-level contribution and the one loop contributions is given by:

$$\begin{aligned} B(s,t,u) &= \left(\frac{1}{6F^4}\right)\Big[3\left(s^2 - M^4\right)J(s) \\ &+ \left(t\,(t-u) - 2M^2t + 4M^2u - 2M^2\right)J(t) \\ &+ \left(u\,(u-t) - 2M^2u + 4M^2t - 2M^4\right)J(u)\Big] \end{aligned}$$

Moreover, the polynomial part is given by:

$$\begin{aligned} C(s,t,u) &= \left(96\pi^2F^4\right)^{-1}\Big[2\left(\bar{l}_1 - \frac{4}{3}\right)\left(s - 2M^2\right)^2 \\ &+ \left(\bar{l}_2 - \frac{5}{6}\right)\left(s^2 + (t-u)^2\right) - 12M^2s + 15M^4\Big] \end{aligned}$$

With:

$$J(q) = \frac{1}{16\pi^2}\sqrt{1 - \frac{4M^2}{q}}\left(\ln\frac{\sqrt{1 - \frac{4M^2}{q}} - 1}{\sqrt{1 - \frac{4M^2}{q}} + 1} + 2\right)$$

J is well define for $q < 0$ and an analytic extension is used otherwise. $\bar{l}_1$ and $\bar{l}_2$ are the two coupling constants of the first loop.

2.7 About coupling constants

We saw that Chiral perturbation theory is an effective theory which highlights the low energy phenomena. However, in the ultra violet regime, some parts diverge. A regularisation procedure is then introduced to treat those divergences. The poles, which produce this behaviour, appear when one integrates over four dimensional space-time. The coupling constants, are then used to absorb them. They are difficult to calculate directly so they are determined by fitting the theory with experiments. The present work will derive some bounds on them. More information about regularisation can be found in [5, 2, 6].

For the two flavours case, the coupling constants are called either l_i^r or $\bar{l}_i$, and are related by the following type of relation [7]:

$$l_i^r = \frac{\Gamma_i}{32\pi^2}\left(\bar{l}_i + \ln\left(\frac{M_{phys}^2}{\mu^2}\right)\right)$$

Where M_{phys} is a cut off term, and μ is a regulatrisation factor. The numbers Γ are $\Gamma_1 = \frac{1}{3}$ and $\Gamma_2 = \frac{2}{3}$. We will use also the following notations for the N_f flavours case, where $N_f > 2$:

$$l_i^r = \bar{l}_i + \frac{1}{16\pi^2}\Gamma_i \ln\left(\frac{M_{phys}^2}{\mu^2}\right) \tag{2.2}$$

These two types of notation can be disturbing, but they are conventional. When constraints on the coupling constants are derived, the Γ_i can be chosen in order to cancel, in the bounds on the low energy constants, all the terms which include M_{phys} and μ. This makes the study easier. For example, in the $SU(3)$ at p^4 three coupling constants are needed and we find $\Gamma_1 = \frac{3}{64}$, $\Gamma_2 = \frac{3}{32}$ and $\Gamma_3 = 0$, in agreement with [4]. Note that in many cases, it is preferred to use the l_i^r. Nevertheless, historically the two flavours case was expressed first in terms of $\bar{l}_i$.

3 The one loop lagrangian with two flavours

3.1 Introduction

Let us recall the T matrix form, in this case:

$$T(p_a, p_b, p_c, p_d) = \delta^{ab}\delta^{cd}A(s,t,u) + \delta^{ac}\delta^{bd}A(t,s,u) + \delta^{ad}\delta^{cb}A(u,t,s)$$

Some dispersion relation bounds have been established in [3], by calculating the dispersion relation for fixed t. We will use the same strategy. From the cartesian basis, $(|\pi^1\rangle, |\pi^2\rangle, |\pi^3\rangle)$, we can go to the isospin one, $(|\pi^+\rangle, |\pi^-\rangle, |\pi^0\rangle)$. For the isospin basis the generators of $SU(2)_V$ are the Pauli matrices and using Eq.(2.1) we can show that:

$$\begin{aligned} \left|\pi^0\right\rangle &= \left|\pi^3\right\rangle \\ \left|\pi^+\right\rangle &= \left(\frac{1}{\sqrt{2}}\right)\left(\left|\pi^1\right\rangle - i\left|\pi^2\right\rangle\right) \\ \left|\pi^-\right\rangle &= \left(\frac{1}{\sqrt{2}}\right)\left(\left|\pi^1\right\rangle + i\left|\pi^2\right\rangle\right) \end{aligned}$$

There are three relevant isospin channels: $I = 0$ with the amplitude T^0, $I = 1$ with T^1 and $I = 2$ with T^2. Using the previous relations lead to:

$$T^0(s,t,u) = 3A(s,t,u) + A(t,s,u) + A(u,t,s) \tag{3.3}$$

$$T^1(s,t,u) = A(t,s,u) - A(u,t,s) \tag{3.4}$$

$$T^2(s,t,u) = A(t,s,u) + A(u,t,s) \tag{3.5}$$

But since u, s and t are related by $u + s + t = 4M^2$, we may write:

$$\begin{aligned} T^0(s,t) &= 3A(s,t) + A(t,s) + A(u,s) \\ T^1(s,t) &= A(t,s) - A(u,s) \\ T^2(s,t) &= A(t,s) + A(u,s) \end{aligned}$$

We need now to introduce the so-called crossing symmetry relation in order to continue. First, we interchange one of the incoming and outgoing particles:

$$p_a p_b \to p_c p_d \Rightarrow p_a \bar{p}_d \to p_c \bar{p}_b$$

Next, using symmetry properties of the function A upon this exchange of particles and performing some linear combinations in between the amplitudes, expressed in the isospin basis, lead to:

$$\begin{aligned} T^I(s,t) &= C_u^{II'} T^{I'}(u,t) \\ \text{Where: } C_u &= \frac{1}{6}\begin{bmatrix} 2 & -6 & 10 \\ -2 & 3 & 5 \\ 2 & 3 & 1 \end{bmatrix} \\ C_u^{II'} C_u^{I'J} &= \delta_{IJ} \end{aligned}$$

A method to derive new bounds on the low energy constants is to study the sign of the isospin amplitudes, independently of the values of the coupling constants. To do this, we need a real and analytic T. These two conditions give a bounded domain in the Mandelstam plane, illustrated in figure 1.

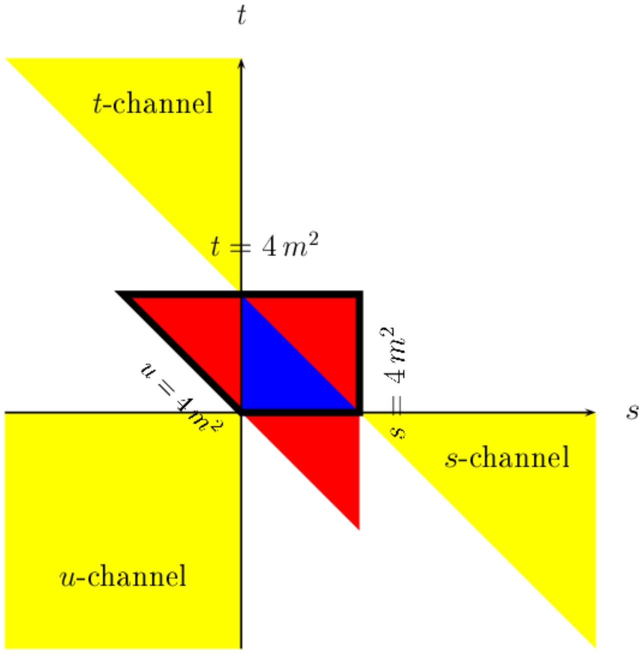


Figure 1: Mandelstam (s,t) plane from [3]. The blue triangle is the Mandelstam triangle. In the red and blue triangles the amplitudes are real and analytic. The region delimited by the black boundary fulfill all positivity conditions. The yellow regions represent the physical regions for the u, s and t channels.

In the red and blue triangles, that is if $s \leq 4M^2, t \leq 4M^2, u \leq 4M^2$, T is analytic and real. Note that the condition on u, gives $4M^2 - s - t \leq 4M^2$ so $s \geq -t$. This means that if we consider the complex s plane with fixed $t \leq 4M^2$, T has to be real and analytic at any point of the segment on the real axis between $[-t; 4M^2]$, as illustrated on Figure 2.

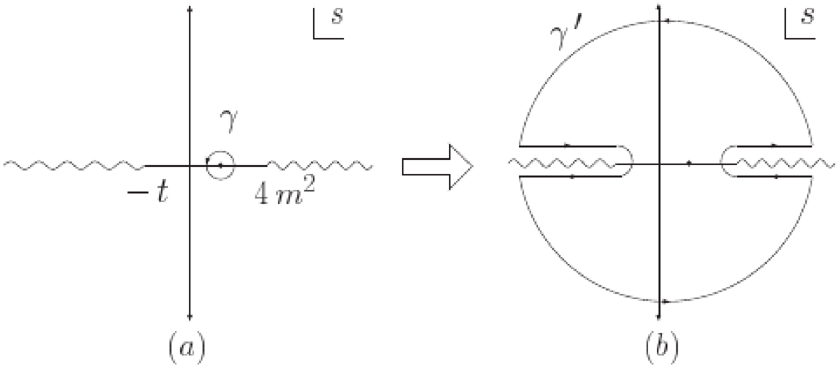


Figure 2: Contour integrals [3] which illustrates of how the contour integral γ can be deformed to $\gamma^{'}$ contour

Using Cauchy's theorem on one of these points and with the contour γ, leads to the following formula:

$$T^I(s,t,u) = \frac{1}{2\pi i}\oint_\gamma dx \frac{T^I(x,t,u)}{x-s}$$

The contour γ contains complex numbers which makes difficult to study the sign of T^I. So we need now to perform the deformation of γ into γ' (cf Figure 2) and focus on the real axis. To do this, it is necessary that the contour of integration vanishes at infinity. By performing successive derivatives, one can hope that the expression $\frac{d^n}{ds^n}T^I$ will decrease fast enough at the spherical part of the contour, when it is sent to infinity. It has been proven [9] that for $\pi\pi$ scattering such good behaviour occurs for $n \geq 2$. Moreover, using the crossing symmetry relation, one can show that[4]:

$$\frac{d^n}{ds^n}T^I(s,t) = \frac{n!}{2\pi i}\int_{4M^2}^{\infty} dx \left[\frac{\delta^{II'}}{(x-s)^{n+1}} + (-1)^n \frac{C_u^{II'}}{(x-u)^{n+1}}\right] \mathrm{Im}T^{I'}(x+i\epsilon,t) \tag{3.6}$$

The partial wave expansion of $\mathrm{Im}T^I$ gives:

$$\mathrm{Im}T^I(x,t) = \sum_{l=0}^{\infty}(2l+1)\mathrm{Im}{f_l}^I(x)P_l\left(\cos(\theta)\right)$$

Where θ is the scattering angle. An equivalent form is:

$$\mathrm{Im}T^I(x,t) = \sum_{l=0}^{\infty}(2l+1)\mathrm{Im}{f_l}^I(x)P_l\left(1+\frac{2t}{x-4M^2}\right)$$

Moreover, the optical theorem relates the imaginary part of partial wave amplitude $\mathrm{Im}{f_l}^I$ with the cross section ${\sigma_l}^I$:

$$\mathrm{Im}{f_l}^I(x) = x\sqrt{1-\frac{4M^2}{x}}{\sigma_l}^I(x)$$

It should be noted that $x \geq 4M^2$ imposes $\sqrt{1-\frac{4M^2}{x}} \geq 0$. Moreover, $\sigma_l{}^I(x)$ is a partial-wave cross-section which has to be positive. So $\mathrm{Im} f_l{}^I(x) \geq 0$. Finally, as explained in [4], if $t \geq 0$ then $1 + \frac{2t}{x^2-4M^2} \geq 1$ and $P_l\left(1 + \frac{2t}{x^2-4M^2}\right) \geq 0$. Restraining ourself to the region highlighted with a dark boundary (see Figure 1) in the Mandelstam plane insures that $t \geq 0$ and consequently that $\mathrm{Im}T^I(s,t) \geq 0$. However, the expression:

$$\left[\frac{\delta^{II'}}{(x-s)^{n+1}} + (-1)^n \frac{C_u^{II'}}{(x-u)^{n+1}}\right]$$

is clearly not positive for every x. Nevertheless, by performing some linear combinations like $\sum a_I T^I$, one can have:

$$a_I \left[\frac{\delta^{II'}}{(x-s)^{n+1}} + (-1)^n \frac{C_u^{II'}}{(x-u)^{n+1}}\right] \geq 0$$

For $n = 2$ it becomes:

$$\frac{a_{I'}}{(x-s)^3} + \frac{C_u^{II'} a_I}{(x-u)^3} \geq 0$$

In [3], the following three physical processes have been studied which satisfy $a_I \geq 0$ and $C_u^{II'} a_I \geq 0$:

$$\begin{aligned}
0 &\leq \frac{d^2}{ds^2} T(\pi^0\pi^0 \to \pi^0\pi^0) \text{ with } a_0 = \frac{1}{3},\ a_1 = 0,\ a_2 = \frac{2}{3} \\
0 &\leq \frac{d^2}{ds^2} T(\pi^+\pi^0 \to \pi^+\pi^0) \text{ with } a_0 = 0,\ a_1 = \frac{1}{2},\ a_2 = \frac{1}{2} \\
0 &\leq \frac{d^2}{ds^2} T(\pi^+\pi^+ \to \pi^+\pi^+) \text{ with } a_0 = 0,\ a_1 = 0,\ a_2 = 1
\end{aligned}$$

We now use the expressions of section 2.6 and look for the values of s and t that gives the best bounds. From [3] we have:

$$\begin{aligned}
\frac{157}{80} - \frac{1}{2}\bar{l}_1 &\leq \bar{l}_2 \\
\frac{27}{20} &\leq \bar{l}_2 \\
1.868 - \frac{1}{3}\bar{l}_1 &\leq \bar{l}_2
\end{aligned}$$

In this chapter we will try to find new bounds considering arbitrary numbers a_0, a_1 and a_2 which fulfill the following conditions:

$$\boxed{\forall I \in \{0,1,2\}, \forall x \in [4M^2; \infty[,\ \frac{a_I}{(x-s)^3} + \frac{C_u^{II'} a_I}{(x-u)^3} \geq 0} \tag{3.7}$$

3.2 Methods

Let a_0, a_1 and a_2, be real numbers. We consider an arbitrary linear combination:

$$T = a_0 T^0 + a_1 T^1 + a_2 T^2 \tag{3.8}$$

On the one hand, as long as (s, t, u) is in the allowed region (c.f Figure 1) and $\forall x > 4M^2$:

$$\frac{a_0}{(x-s)^3} + \frac{a_0 - a_1 + a_2}{3(x-u)^3} \geq 0 \tag{3.9}$$

$$\frac{a_1}{(x-s)^3} + \frac{-a_0 + \frac{1}{2}(a_1 + a_2)}{(x-u)^3} \geq 0 \tag{3.10}$$

$$\frac{a_2}{(x-s)^3} + \frac{\frac{10}{6}a_0 + \frac{5}{6}a_1 + \frac{1}{6}a_2}{(x-u)^3} \geq 0 \tag{3.11}$$

the sign of the second derivative of T is known, independently of the values of the coupling constants:

$$\frac{d^2}{ds^2} T(s,t,u) \geq 0 \tag{3.12}$$

On the other hand, T can be expressed in terms of the coupling constants using the formula of [7], presented in the section 2.6:

$$\frac{d^2}{ds^2} T(s,t) = \left(\frac{6a_0 + a_1 + 3a_2}{24\pi^2 F^4} \right) \bar{l}_2 + \left(\frac{4a_0 - a_1 + a_2}{24\pi^2 F^4} \right) \bar{l}_1 + g(s,t,\dots) \tag{3.13}$$

where g is a function which does not depend on the coupling constants. For given (s, t, u, a_0, a_1, a_2) fulfilling all the constraints , Eq.(3.13) and Eq.(3.12) give bounds on the coupling constants. Dividing Eq.(3.13) by the coefficient in front of $\bar{l}_2$, our constraints then become of the form:

$$\bar{l}_2 \geq d\,\bar{l}_1 - \frac{g(s,t) 24\pi^2 F^4}{6a_0 + a_1 + 3a_2} \tag{3.14}$$

It is thus natural to see them as line, like in [3], with $\bar{l}_1$ as x-axis and $\bar{l}_2$ as y-axis. The slope of the line is:

$$d = -\frac{4a_0 - a_1 + a_2}{6a_0 + a_1 + 3a_2} \tag{3.15}$$

By varying a_0, a_1, a_2, s and t, hundreds of lines will be calculated. We then need to find a criteria to organise them and find the best constraints. A good one would be the following: for each set of lines with the same slope, we keep only the line which has the highest points of intersection with the vertical axis. Now, we will first study the slope, and prove that the only possible values are contained in the interval $[-\frac{1}{2}, 0]$. Next, we will present the algorithm that we have used.

3.2.1 Study of the slope

The following proof was originally an attempt to connect the regions of existence of d with the values of the a_I but instead gives us that $d \in [-\frac{1}{2}, 0]$ which was unexpected. First of all, the constraints can be rewritten as:

$$\forall x \geq 4M^2, \frac{(x-u)^3}{(x-s)^3} a_0 + \frac{1}{3}(a_0 - a_1 + a_2) \geq 0 \tag{3.16}$$

$$\forall x \geq 4M^2, \frac{(x-u)^3}{(x-s)^3} a_1 + (-a_0 + \frac{1}{2}(a_1 + a_2)) \geq 0 \tag{3.17}$$

$$\forall x \geq 4M^2, \frac{(x-u)^3}{(x-s)^3} a_2 + (\frac{10}{6} a_0 + \frac{5}{6} a_1 + \frac{1}{6} a_2) \geq 0 \tag{3.18}$$

If $x \to \infty$ then $\frac{(x-u)^3}{(x-s)^3} \to 1$ and the conditions become:

$$a_0 + \frac{1}{3}(a_0 - a_1 + a_2) \geq 0 \tag{3.19}$$

$$a_1 + (-a_0 + \frac{1}{2}(a_1 + a_2)) \geq 0 \tag{3.20}$$

$$a_2 + (\frac{10}{6} a_0 + \frac{5}{6} a_1 + \frac{1}{6} a_2) \geq 0 \tag{3.21}$$

Now, using Eq.(3.15), a_0 can be written as a function of d:

$$a_0 = -\frac{1}{2} \frac{da_1 + 3da_2 - a_1 + a_2}{3d + 2} \tag{3.22}$$

Using this expression, we can write Eq.(3.19), Eq.(3.20) and Eq.(3.21) as:

$$f_1(d) = -\frac{2}{3} \frac{da_1 + 3da_2 - a_1 + a_2}{3d + 2} - \frac{1}{3} a_1 + \frac{1}{3} a_2 \geq 0$$

$$f_2(d) = \frac{3}{2} a_1 + \frac{1}{2} \frac{da_1 + 3da_2 - a_1 + a_2}{3d + 2} + \frac{1}{2} a_2 \geq 0$$

$$f_3(d) = \frac{7}{6} a_2 - \frac{5}{6} \frac{da_1 + 3da_2 - a_1 + a_2}{3d + 2} + \frac{5}{6} a_1 \geq 0$$

We have to study each function, which will give us constraints on the slope. For example, by taking the derivative of f_1, we have:

$$\frac{\partial f_1}{\partial d} = -\frac{2}{3} \frac{5a_1 + 3a_2}{(3d + 2)^2}$$

So the sign of the derivative depends on the sign of $5a_1 + 3a_2$. For example, if it is negative the function f_1 will grow with d and $-\frac{5}{9} a_1 - \frac{1}{3} a_2$ will be positive. Table 1, Table 2 and Figure 3 present the behavior of the function for both $5a_1 + 3a_2 \geq 0$ and $5a_1 + 3a_2 \leq 0$.

Table 1: Variation table, case $5a_1 + 3a_2 \leq 0$

d	$-\infty$		$-\frac{2}{3}$		0		$+\infty$
$f_1'(d)$	‖	$+$	‖	$+$	⋮	$+$	‖
$f_1(d)$	‖	$-\frac{5}{9}a_1 - \frac{1}{3}a_2 \geq 0$ ↗ $+\infty$	‖	$-\infty$ ↗	0	↗ $-\frac{5}{9}a_1 - \frac{1}{3}a_2 \geq 0$	‖

Table 2: Variation table, case $5a_1 + 3a_2 \geq 0$

d	$-\infty$		$-\frac{2}{3}$		0		$+\infty$
$f_1'(d)$	‖	$-$	‖	$-$	⋮	$-$	‖
$f_1(d)$	‖	$-\frac{5}{9}a_1 - \frac{1}{3}a_2 \leq 0$ ↘ $-\infty$	‖	$+\infty$ ↘	0	↘ $-\frac{5}{9}a_1 - \frac{1}{3}a_2 \leq 0$	‖

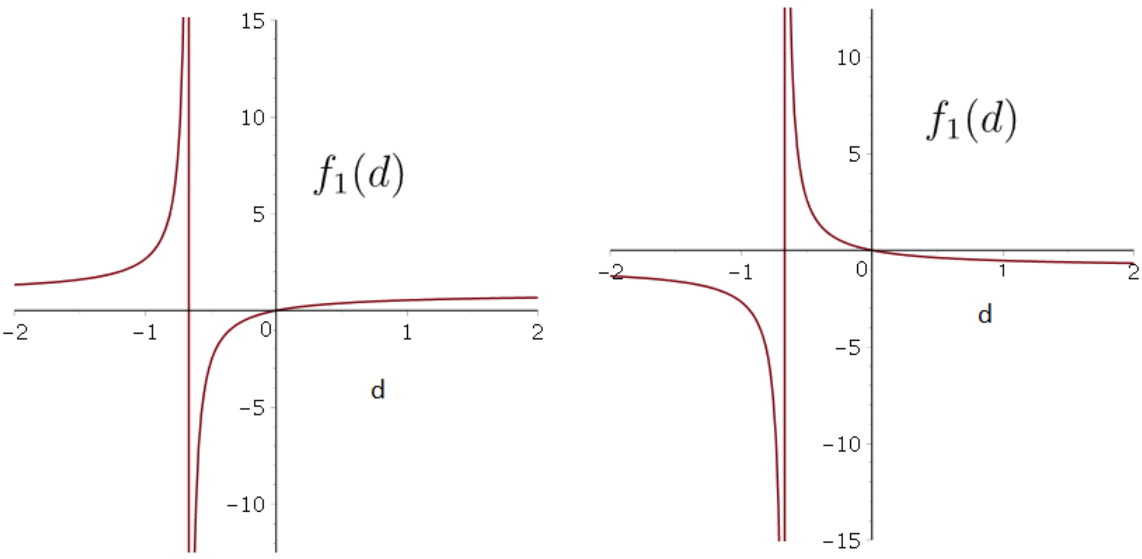


Figure 3: The function f_1 in two cases: $5a_1 + 3a_2 \leq 0$ (left) and $5a_1 + 3a_2 \geq 0$ (right)

As we see, if $5a_1 + 3a_2 \leq 0$, $f_1(d)$ is positive if and only if $d \in]-\infty; -\frac{2}{3}[\ \cup \ [0; \infty[$. If $5a_1 + 3a_2 \geq 0$, $f_1(d)$ is then positive if and only if $d \in]-\frac{2}{3}; 0]$. We repeat the exact same study for all the constraints and we find the conditions listed in Table 3.

Table 3: Positivity constraints for the three conditions

	$5a_1 + 3a_2 \geq 0$	$5a_1 + 3a_2 \leq 0$
$f_1(d) \geq 0$	$d \in]-\frac{2}{3};0]$	$d \in]-\infty;-\frac{2}{3}[\cup[0;\infty[$
$f_2(d) \geq 0$	$d \in]-\infty;-\frac{2}{3}[\cup[-\frac{1}{2},\infty[$	$d \in]-\frac{2}{3};-\frac{1}{2}[$
$f_3(d) \geq 0$	$d \in]-\infty;-\frac{3}{2}[\cup]-\frac{2}{3},\infty[$	$d \in]-\frac{3}{2};-\frac{2}{3}[$
conditions are verified if	$d \in [-\frac{1}{2};0]$	*never*

We just proved that $d \in [-\frac{1}{2};0]$ is a necessary and sufficient condition to Eq.(3.19…3.21). Let us now show that it is a necessary condition to Eq.(3.16…3.18). These conditions are of the form:

$$f(x)A + B \geq 0$$

Suppose $b = A + B < 0$, since $f(x)$ goes to one when x goes to infinity we can write:

$$\forall\epsilon > 0, \exists M > 0, \forall x \geq M, |f(x) - 1| \leq \epsilon$$

If $A > 0$, we take $\epsilon = -\frac{b}{2A}$, and we have $f(x) \leq 1 + \epsilon$, else, $\epsilon = \frac{b}{2A}$ and $f(x) \geq 1 - \epsilon$. In both cases we have:

$$f(x)A + B \leq (A - \frac{b}{2}) + B = A + B - \frac{b}{2} = \frac{b}{2} < 0$$

So the condition $A + B \geq 0$ is necessary. This basically means that Eq.(3.19…3.21) are necessary for Eq.(3.16…3.18) to be valid. We have just proved that the condition $d \in [-\frac{1}{2}, 0]$ is a necessary condition of Eq.(3.16…3.18). Moreover, if $d \in [-\frac{1}{2}, 0]$, then there exist a_0, a_1 and a_2 which fulfill Eq.(3.19…3.21). If $u = s$, then Eq.(3.16…3.18) become Eq.(3.19…3.21). This shows that if $d \in [-\frac{1}{2}, 0]$, there exists s, t, u, a_0, a_1, a_2 such that all the constraints are checked, so it is also a sufficient condition.

Proving that $d \in [-\frac{1}{2}, 0]$ is a necessary condition can also be done in a simpler way proposed by Johan Bijnens. The slope can be written as:

$$d = -\frac{(3.19)}{2((3.19) + (3.20))}$$

So we see directly that $d \leq 0$. Moreover:

$$d = -\frac{1}{2\left(1 + \frac{(3.20)}{(3.19)}\right)}$$

Which implies: $d \geq -\frac{1}{2}$

3.2.2 Algorithm

Let us recall the set of constraints define by Eq.(3.9), Eq.(3.10) and Eq.(3.11) and try to simplify our problem.

$$\frac{a_0}{(x-s)^3} + \frac{a_0 - a_1 + a_2}{3(x-u)^3} \geq 0$$
$$\frac{a_1}{(x-s)^3} + \frac{-a_0 + \frac{1}{2}(a_1 + a_2)}{(x-u)^3} \geq 0$$
$$\frac{a_2}{(x-s)^3} + \frac{\frac{10}{6}a_0 + \frac{5}{6}a_1 + \frac{1}{6}a_2}{(x-u)^3} \geq 0$$

If a_2 is positive, we can divide all the inequalities by it and still have the same problem. So we can assume $a_2 = 1$. If a_2 is negative, we can divide by $|a_2|$, which is equivalent to suppose that $a_2 = -1$. So we can say that the only relevant values of a_2 are: $-1; 1$ and 0 whereas a_1 and a_0 can take any values. Now recalling Eq.(3.14) and using Eq.(3.22) to replace a_0 with d and a_1, our problem becomes:

$$\bar{l_2} \geq d\ \bar{l_1} + C(s, t, d, a_1)$$

Finding the best constraint, defined by the highest intersection between a line and the vertical axis, then becomes equivalent to maximise C for fixed d and a_2.

3.3 Results and comments

For each possible values of a_2, 100 slopes d have been chosen and the other parameters are randomly taken in the allowed region to calculate C. The maximum value is kept and presented in Figure 4. All the maximum lines are obtained at $t = 4M^2$. This phenomena was just observed and not proven.

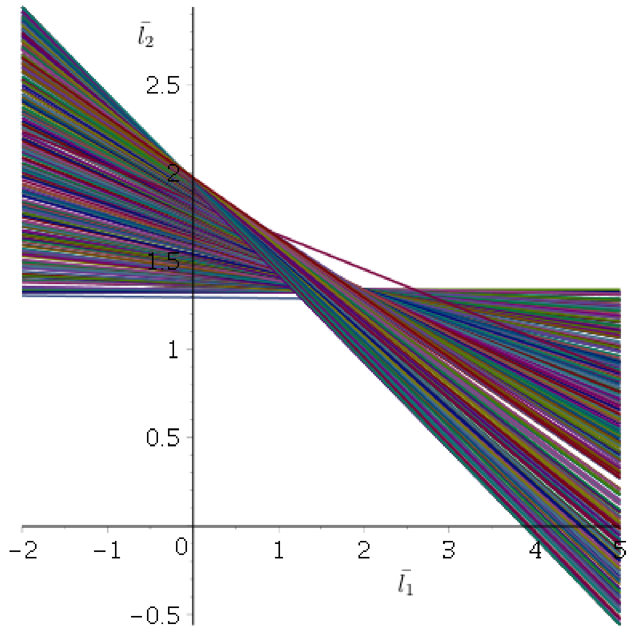


Figure 4: Constraints, presented as line, with $\bar{l}_1$ in the x-axis and $\bar{l}_2$ in the y-axis, for a specified d optimised by varying t, s and a_1.

We then focus around $t = 4M^2$ and for each values of a_2 test approximately 300 slopes. The results are printed in Figure 5.

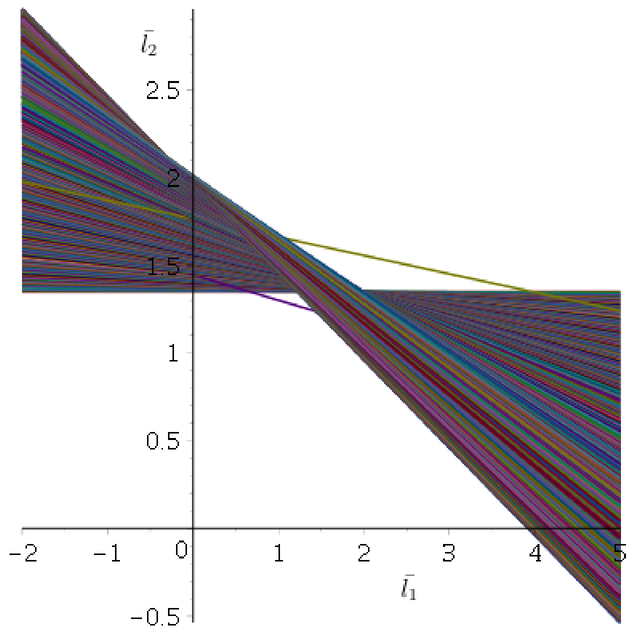


Figure 5: Constraints, presented as line, with $\bar{l}_1$ in the x-axis and $\bar{l}_2$ in the y-axis, for fixed $t = 4M^2$, for a specified d optimised by varying s and a_1.

In both Figure 4 and Figure 5, numerical errors happen, and lead to fake constraints that we have to detect and reject. Finally, Figure 6 presents in green our results and in blue the ones from [3].

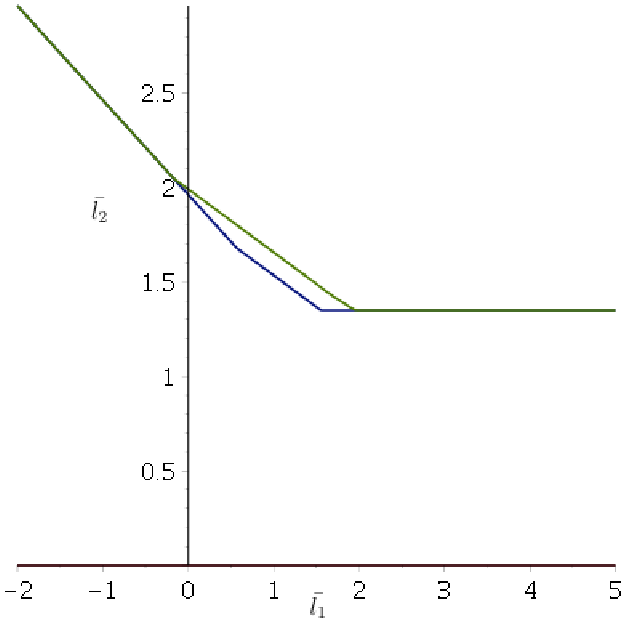


Figure 6: Comparison between our results and the ones from [3].

As can be seen from Figure 6, our constraints are the same as the ones found in [3], except in a bounded region, when $\bar{l}_1 \in [0,2]$, where our bounds are more restrictive.

For completeness we also give the numerical values for the constraints that have been found:

$$\bar{l}_2 \geq \frac{157}{80} - \frac{1}{2}\bar{l}_1 \quad (3.23)$$

$$\bar{l}_2 \geq \frac{27}{20} \quad (3.24)$$

$$\bar{l}_2 \geq 1.971 - \frac{62}{200}\bar{l}_1 \quad (3.25)$$

$$\bar{l}_2 \geq 2.009 - \frac{199}{600}\bar{l}_1 \quad (3.26)$$

$$\bar{l}_2 \geq 2.011 - \frac{1}{3}\bar{l}_1 \quad (3.27)$$

$$\bar{l}_2 \geq 2.012 - \frac{211}{600}\bar{l}_1 \quad (3.28)$$

$$\bar{l}_2 \geq 2.013 - \frac{207}{600}\bar{l}_1 \quad (3.29)$$

First, they are perfectly consistent with the values from [7]:

$$\begin{aligned} \bar{l}_1 &= -2.3 \pm 3.7 \\ \bar{l}_2 &= 6.0 \pm 1.3 \end{aligned}$$

However, the number of lines seems large and only lines (3.23), (3.24), which agree with [4], and (3.27) would be expected to be needed by looking at Figure 6. Nevertheless, it has been verified that to describe fully our result we need all the 7 constraints. Figure 7 presents both the constraints define by Eq.(3.23), Eq.(3.24) and Eq.(3.27) and the difference between these three constraints and all our results.

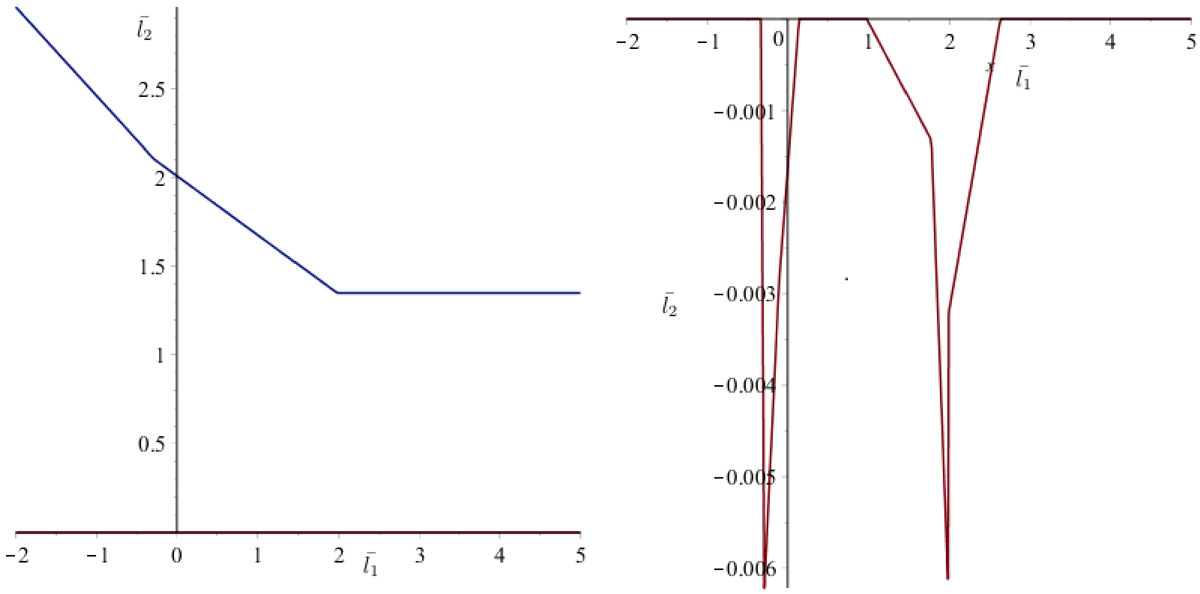


Figure 7: On the left hand side , constraints (3.23), (3.24) and (3.27) are printed. On the right hand side, we present the difference between the three most important constraints and the full set of them.

In particular, they contribute to make the transition between the lines (3.23), (3.24) and (3.27) smoother and Figure 7 shows that they are important around the intersection points of the three most important constraints.

These results were obtained considering the p^4 lagrangian. It is thus interesting to evaluate the contribution of the next order. Using experimental data with the p^6 lagrangian will give us an estimation of the error.

3.4 p^6 contribution

In order to get an estimate of the theoretical errors for the constraints on $\bar{l}_1$ and $\bar{l}_2$ we consider the p^6 expression to $\frac{\partial^2 T}{\partial s^2}$, from [10]. It is evaluated at the coordinates

$(s,\ t,\ u,\ a_0,\ a_1,\ a_2)$ for each of the found constraints and with the pion mass. The coupling constants of p^6 are replaced by experimental values from [11]. The results are presented in Table 4.

Table 4: The constraints and their p^6 contributions

Constraints	p^6 contribution
$\bar{l}_2 \geq \frac{157}{80} - \frac{1}{2}\bar{l}_1$	+ 0.0209
$\bar{l}_2 \geq \frac{27}{20}$	+ 0.0428
$\bar{l}_2 \geq 1.971 - \frac{62}{200}\bar{l}_1$	+ 0.104
$\bar{l}_2 \geq 2.009 - \frac{199}{600}\bar{l}_1$	+ 0.126
$\bar{l}_2 \geq 2.011 - \frac{1}{3}\bar{l}_1$	+ 0.198
$\bar{l}_2 \geq 2.012 - \frac{211}{600}\bar{l}_1$	- 0.00859
$\bar{l}_2 \geq 2.013 - \frac{207}{600}\bar{l}_1$	+ 0.0799

As can be seen in Table 4, the p^6 contribution is quite small compared to the order of magnitude of the constraints, which are thus reliable. Nevertheless, large differences can be observed in the order of magnitude of the error. This phenomena is not well understood yet.

4 General consideration about m coupling constants

4.1 Introduction

Finding constraints is not a problem, but we have to find the best ones. This was really obvious in the one loop case, because the bounds were in two dimensions so it was possible to draw them. Identifying the necessary ones was easy and followed our visual intuition. However, when considering more flavours or the next order lagrangian, the number of coupling constants increase, and we can no longer trust our visual intuition. In this chapter, the main problem will be to explain rigorously what it means to be “the best constraints” when the number of coupling constants is an arbitrary m . Some explanations about the notations used in this chapter can be found in appendix A.

4.2 Constraint ordering

What we mean by “the best constraints” is a set with the minimal number of bounds, which is still sufficient to describe the allowed region of the coupling constants. Constraints can always be written in the form:

$$\langle f|b\rangle \geq c \tag{4.30}$$

Where $\langle .|.\rangle$ is the usual scalar product on $\mathbb{R}^m$, $b \in \mathbb{R}^m$ describes the coupling constants, f is a vector and $c \in \mathbb{R}$. f and c parametrise the constraints. It can be assumed that c is either 1, 0 or -1. Note that later on f and c will be given by a maximisation procedure. From now on, constraints will be denoted by: (f, c). The notation f_i will refer to an element of a set of vectors, and c_i to an element of a set of real numbers.

Definition 4.2.1 (Weaker conditions)**.** *Let g be an element of $\mathbb{R}^m$, and $\{f_i\}_{i\in I}$, with I countable, a set of vectors of $\mathbb{R}^m$. We will say that (g, c_g) is weaker than the $\{(f_i, c_i)\}_{i\in I}$, and note it $(g, c_g) < \{(f_i, c_i)\}_{i\in I}$, if and only if the following statement is true:*

$$\text{For } b \in \mathbb{R}^m, [\forall i \in I\ \langle f_i|b\rangle \geq c_i] \Rightarrow \langle g|b\rangle \geq c_g$$

For example, let us consider the following constraints: $y \geq -x$, $y \geq 1$, $y \geq -1$, illustrated in Figure 8.

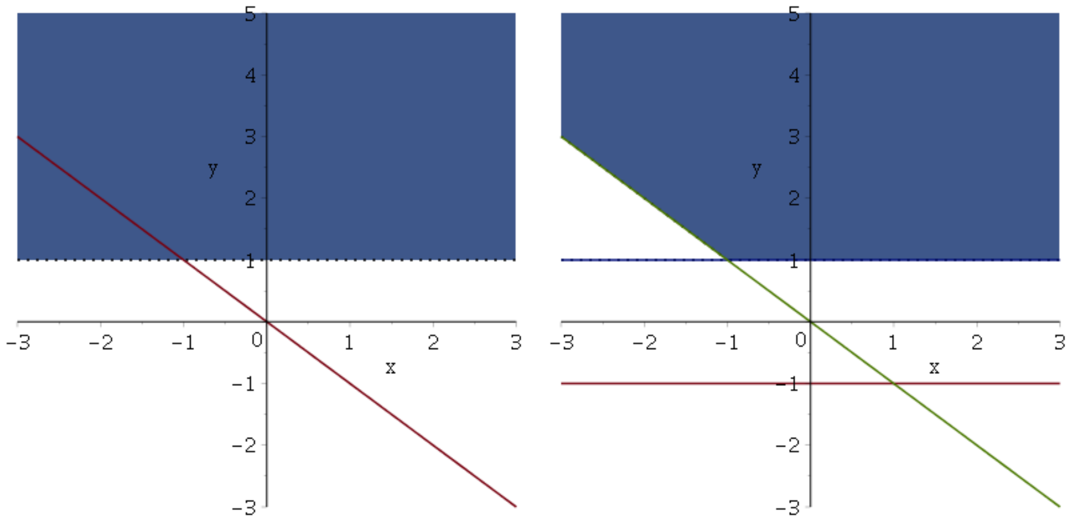


Figure 8: In the left hand side, we see that $y \geq -x$ is not weaker in the right hand side , $y \geq 1$ whereas $y \geq -1$ is weaker than $\{y \geq -x, y \geq 1\}$.

In the left figure, we see that all the points in the blue part (where $y \geq 1$) do not satisfy the condition $y \geq -x$. It is thus needed and has to be kept. In the right picture however, all the points in the blue part fulfill $y \geq -1$, which means that this condition is superfluous.

This definition encodes the idea of having a non necessary condition. Nevertheless, it is not so easy to use it in higher dimensions because we cannot represent them so easily. That is why we need the following property:

Proposition 4.2.1. *For c=1. Let g be an element of $\mathbb{R}^m$, and $\{f_i\}_{i\in[1;n]}$ a set of vectors of $\mathbb{R}^m$. Then $(g,1) < \{(f_i,1)\}_{i\in[1;n]}$ if and only if there exists $\{\alpha_i\}_{i\in[1;n]}$ such that:*

$$\forall i \in [1,..,n]\ \alpha_i \geq 0 \tag{4.31}$$

$$\sum_{i=1}^{n} \alpha_i \geq 1 \tag{4.32}$$

$$g = \sum_{i=1}^{n} \alpha_i f_i \tag{4.33}$$

Moreover, we can express g as a linear combination of at most m functions from $\{f_i\}_{i\in[1;n]}$.

Before proving this property, we need to recall, quickly, a definition and a property.

Definition 4.2.2. *Let f be a linear functional. The kernel of f is the vector space defined as:*

$$N_f = \{v \in \mathbb{R}^m, f(v) = 0\}$$

Lemma 4.2.1. *Two functionals which have the same kernel are proportional.*

Proof of the Lemma. Let us assume that the functionals f and g have the same non trivial kernel ($N \neq \emptyset$). The kernel of f is of dimension $m-1$. Let us then take an element x which is not in N_f, then:

$$\mathbb{R}^m = N_f \bigoplus x\mathbb{R}$$

So:

$$\forall v \in \mathbb{R}^m, \exists n_v \in N_f, \lambda_v \in \mathbb{R}, v = n_v + \lambda_v x$$

And:

$$f(v) = \lambda_v f(x)$$

Moreover:

$$g(v) = g(n_v + \lambda_v x) = g(n_v) + \lambda_v g(x)$$

But $n_v \in N_f = N_g$, so $g(n_v) = 0$. Finally:

$$\forall v \in \mathbb{R}^m, g(v) = \lambda_v g(x) = \lambda_v f(x)\frac{g(x)}{f(x)} = \frac{g(x)}{f(x)} f(v)$$

□

Proof of the Proposition 4.2.1. In all this proof we assume that the vectors are distinct and not piecewise colinear. If $n < m$, let us assume that:

$$(g, 1) < \{(f_i, 1)\}_{i \in [1..n]} \tag{4.34}$$

We define:

$$I_j = [1, .., j-1, j+1, .., n]$$

We will first prove that

$$\exists j \neq k \textit{ such as } N_g \cap \bigcap_{i \in I_j} N_{f_i} = N_g \cap \bigcap_{i \in I_k} N_{f_i}$$

Assume that it is not true. For any $1 \geq k \geq n$, $N_g \cap \bigcap_{i \in I_k} N_{f_i}$ is a vector space and $\dim(N_g \cap \bigcap_{i \in I_k} N_{f_i}) = m - n > 0$, so it is non-empty. If:

$$\forall (j,k)\ N_g \cap \bigcap_{i \in I_j} N_{f_i} \neq N_g \cap \bigcap_{i \in I_k} N_{f_i}$$

Then we choose $\{x_i\}_{i \in [0.n]}$ such that:

$$\forall x_i \in [2..n], x_i \in N_g \cap \bigcap_{k \in I_i} N_{f_k} \textit{ but } x_i \notin N_g \cap \bigcap_{k \in I_1} N_{f_k}$$

And

$$x_1 \in N_g \cap \bigcap_{i \in I_1} N_{f_i} \textit{ but } x_1 \notin N_g \cap \bigcap_{i \in I_2} N_{f_i}$$

So by construction we have that:

$$\forall x_i, \forall j \neq i, x_i \in N_{f_j} \; and \; x_i \notin N_{f_i}$$

We then define the following number:

$$v = \sum_{i=1}^{n} \frac{x_i}{\langle f_i | x_i \rangle} \tag{4.35}$$

Clearly,

$$\forall i \in [1..n], \langle f_i | v \rangle = 1 \; but \; \langle g | v \rangle = 0$$

Which contradicts Eq.(4.34). So

$$\exists (k,j), k, j \in [1..n], k \neq j \; such \; that \; N_g \cap \bigcap_{i \in I_j} N_{f_i} = N_g \cap \bigcap_{i \in I_k} N_{f_i}$$

We can relabel our vectors in order to have $k = 1$ and $j = 2$ so:

$$N_g \cap \bigcap_{i \neq 1} N_{f_i} = N_g \cap \bigcap_{i \neq 2} N_{f_i}$$

So the functionals $(f_1)_{|N_g \cap \bigcap_{i \neq 1,2} N_{f_i}}$ and $(f_2)_{|N_g \cap \bigcap_{i \neq 1,2} N_{f_i}}$ have the same kernel on the vector space $N_g \cap \bigcap_{i \neq 1,2} N_{f_i}$. Thus according to the lemma, there exists a constant c_1 such that:

$$(f_1)_{|N_g \cap \bigcap_{i \neq 1,2} N_{f_i}} = -c_1 (f_2)_{|N_g \cap \bigcap_{i \neq 1,2} N_{f_i}}$$

Then:

$$(f_1 + c_1 f_2)_{|N_g \cap \bigcap_{i \neq 1,2} N_{f_i}} = 0$$

and $(f_1 + c_1 f_2)_{|N_g \cap \bigcap_{i \neq 1,2,3} N_{f_i}}$ vanishes on $N_g \cap \bigcap_{i \neq 1,2} N_{f_i}$ and so has the same kernel as $(f_3)_{|N_g \cap \bigcap_{i \neq 1,2,3} N_{f_i}}$ regarded as linear functionals of the vector space $N_g \cap \bigcap_{i \neq 1,2,3} N_{f_i}$. So there exists c_2 such as:

$$(f_1 + c_1 f_2)_{|N_g \cap \bigcap_{i \neq 1,2,3} N_{f_i}} = -c_2 (f_3)_{|N_g \cap \bigcap_{i \neq 1,2,3} N_{f_i}}$$

We can continue this process and we find finally that:

$$g = c_n \left(f_1 + \sum_{i=2}^{n} c_{i-1} f_i \right)$$

Redefining $\alpha_i = c_{i-1} c_n$, if $i \neq 1$, and else, c_n, we can write:

$$g = \left(\sum_{i=1}^{n} \alpha_i f_i \right) \tag{4.36}$$

We next need to prove that the α_i are all positive. Let us take a vector x_1 which is in the kernels of all the functionals except f_1. Next x_2 in the kernel of all the functionals except

f_2, and so on. These vectors exist because $\forall k, \dim(N_k) > \dim(\bigcap_{i\neq k} N_i)$. We define for all $\mu > 1$

$$v(\mu) = \mu \frac{x_1}{\langle f_1|x_1\rangle} + \sum_{i=2}^{n} \frac{x_i}{\langle f_i|x_i\rangle}$$

Then, we have:

$$\langle f_1|v(\mu)\rangle = \mu \geq 1$$
$$\forall i \neq 1\, \langle f_1|v(\mu)\rangle = 1 \geq 1$$

and using Eq.(4.34) and Eq.(4.36) we get:

$$\langle g|v(\mu)\rangle = \alpha_1\mu + \left(\sum_{i=2}^{n} \alpha_i\right) \geq 1 \tag{4.37}$$

But Eq.(4.37) has to be true for any $\mu > 1$. So $\alpha_1 > 0$. The same method can be used to prove that the other constants are positive. Finally, for $\mu = 1$, we have

$$\langle g|v\rangle = \left(\sum_{i=i}^{n} \alpha_i\right) \geq 1$$

Which proves Eq.(4.32).

Now if $n \geq m$, we have in the worst case, for any set $M \subset [0..n]$ with $m-1$ elements, that $dim(N_g \cap \bigcap_{i\in M} N_{f_i}) = 0$. So it is obvious that:

$$N_g \cap \bigcap_{i=[1,3,..m]} N_{f_i} = N_g \cap \bigcap_{i\in[2,..m]} N_{f_i} \tag{4.38}$$

The exact same proof as before give us that g is a linear combination of at most m vectors of $\{f_i\}_{i\in[1..n]}$.

Conversely, if we have a vector that satisfies the properties (4.31), (4.32) and (4.33), then obviously $g < \{f_i\}_{i\in[1..n]}$, which completes the proof of the proposition. □

This proposition is the main result of this chapter. It will be used to extract the necessary constraints in our results. In order to understand this statement, let us assume that we have an equal sign in Eq.(4.32):

$$\sum_{i=1}^{n} \alpha_i = 1$$

Then, the set of our bounds would have been convex, and the best constraints would have been the convex hull. To understand the difference between the set we are looking for and the convex hull, let us take a simple example. In two dimensions, the constraints can be represented as vectors of $\mathbb{R}^2$, which are here represented by points in Figure 9. Note that the cross on each figures represents the origin.

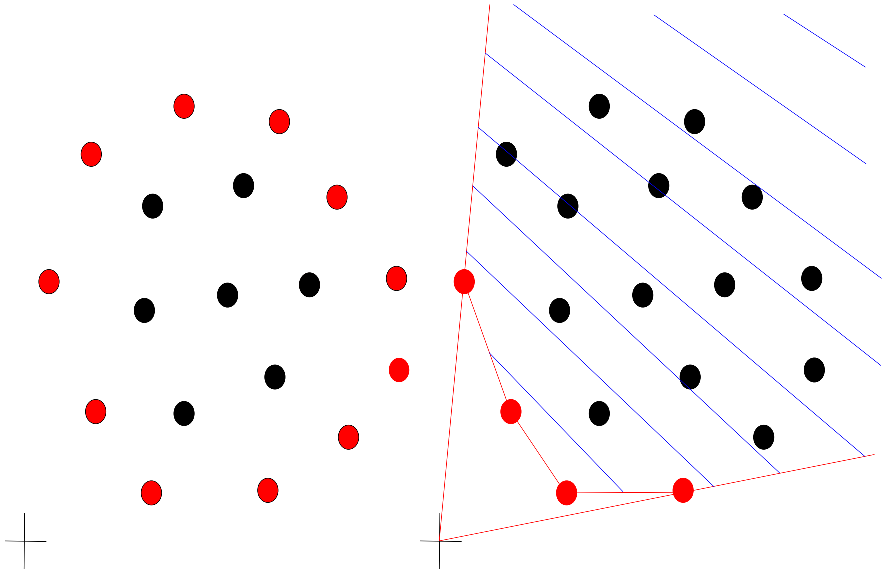


Figure 9: Comparison between a convex hull and the set we are looking for. To the left, constraints, in red and black, and theirs convex Hull in red are printed. To the right, we see constraints, in red and black, and the set of the necessary points in red. The set of non necessary constraints are highlighted in blue.

As we see in Figure 9, only four constraints are necessary and the region limited by the red lines and highlighted in blue is the set of weaker constraints than the four selected ones. We see that we are looking for a special subset of the convex hull. One possible strategy is then to focus on algorithms which search convex hull and adapt them to our problem. We did not have enough time to do this work and the algorithm we used will be described later.

We can prove, with the same type of arguments the following propositions:

Proposition 4.2.2. *For c=-1. Let g be an element of $\mathbb{R}^m$, and $\{f_i\}_{i\in[1;n]}$ a set of vectors of $\mathbb{R}^m$. $(g,-1) < \{(f_i,-1)\}_{i\in[1;n]}$ if and only if there exist $\{\alpha_i\}_{i\in[1;n]}$ such that:*

$$\forall i \in [1;n]\ \alpha_i \geq 0 \quad (4.39)$$

$$\sum_{i=1}^{n} \alpha_i \leq 1 \quad (4.40)$$

$$g = \sum_{i=1}^{n} \alpha_i f_i \quad (4.41)$$

Moreover, we can express g as a linear combination of at most m functions from $\{f_i\}_{i\in[1;n]}$.

Proposition 4.2.3. *For c=0. Let g be an element of $\mathbb{R}^m$, and $\{f_i\}_{i\in[1;n]}$ a set of vectors* *of* $\mathbb{R}^m$*.* $(g,0) < \{(f_i,0)\}_{i\in[1;n]}$ *if and only if there exist* $\{\alpha_i\}_{i\in[1;n]}$ *such that:*

$$\forall i \in [1;n]\ \alpha_i \geq 0 \tag{4.42}$$

$$g = \sum_{i=1}^{n} \alpha_i f_i \tag{4.43}$$

Moreover, we can express g *as a linear combination of at most* m *functions from* $\{f_i\}_{i\in[1;n]}$*.*

In this report, only the first proposition will be used but the two other ones could be important in more detailed studies. They can be put together in the following proposition:

Proposition 4.2.4 (representation of weaker constraint)**.** *Let* g *be an element of* $\mathbb{R}^m$*, and* $\{f_i\}_{i\in[1;n]}$ *a set of vectors of* $\mathbb{R}^m$*. Assume that we are interested in the following constraints:* $\{(f_i,1)\}_{i\in[1;n_1]}$*,* $\{(f_i,0)\}_{i\in[n_1+1;n_2]}$ *and* $\{(f_i,-1)\}_{i\in[n_2+1;n]}$*.*
Then, $(g,c) < \left\{\{(f_i,1)\}_{i\in[1;n_1]}, \{(f_i,0)\}_{i\in[n_1+1;n_2]}, \{(f_i,-1)\}_{i\in[n_2+1;n]}\right\}$ *if and only if there exists* $\{\alpha_i\}_{i\in[1;n_1]}$*,*$\{\alpha_i\}_{i\in[n_1+1;n_2]}$ *and* $\{\alpha_i\}_{i\in[n_2+1;n]}$ *such that:*

$$\forall i \in [1;n]\ \alpha_i \geq 0 \tag{4.44}$$

$$\sum_{i=1}^{n_1} \alpha_i \geq c \tag{4.45}$$

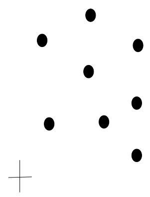
$$\sum_{i=n_2+1}^{n} \alpha_i \leq -c \tag{4.46}$$

$$g = \sum_{i=1}^{n} \alpha_i f_i \tag{4.47}$$

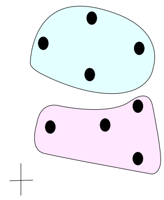
Moreover, we can express g *as a linear combination of at most* m *functions from* $\{f_i\}_{i\in[1;n]}$*.*

4.3 Algorithm

Finding an optimized program appeared to be an difficult task so it turned out to be preferable to use a recursive algorithm instead. We used the so-called divide and conquer method, which increase the speed of programs, as long as the output is not to big. To illustrate how the algorithm works, we consider again the two-dimension case. In the following diagram, the cross is the origin and the points represent constraints.

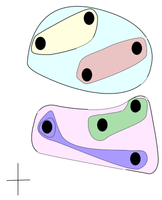


Step 0: Input

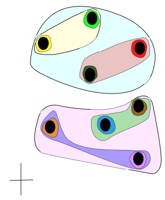


Step 1: Division in two subgroups.

First, the input set of constraints is divided in two subset of equal size.

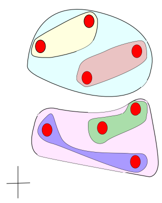


The function is called again. Each subgroup is an input.

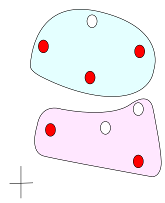


The function stops when there is only one point in each groups.

Each subset is consider as a new input and then, it is divided again. The process stop when all the subsets have one and only one constraints. The proposition is applied for each isolated points. A set of one vector cannot be simplified so all of them are kept.

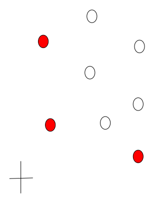


Step 2: The lowest groups are trivially solved and the merging process start.



Step 3: The groups are merged, using the propositions. The white points are rejected.

Each subset is then merge with another one, using the proposition. In Step 2, no simplification occurs whereas in Step 3 white points are rejected, because they are not necessary.



Step 4: The algorithm gives the selected point (in red).

When all the subsets are merged, the program stops and returns the red points, which are the necessary ones.

4.4 Example and comments

To illustrate how different constraints are compared, let us take, for example, the following ones:

$$\begin{aligned} C_1 &= 0.0006294215400 - 0.001861051254x - 0.3321187659y \\ C_2 &= 0.0005668861930 - 0.002504750162x - 0.005701327123y \\ C_3 &= 0.0006164694886 - 0.001215026534x - 0.09372952850y \end{aligned}$$

Figure 10 presents both the difference of the maximum of the two first one with the maximum of all of them and the maximum of the two first constraints with the third one.

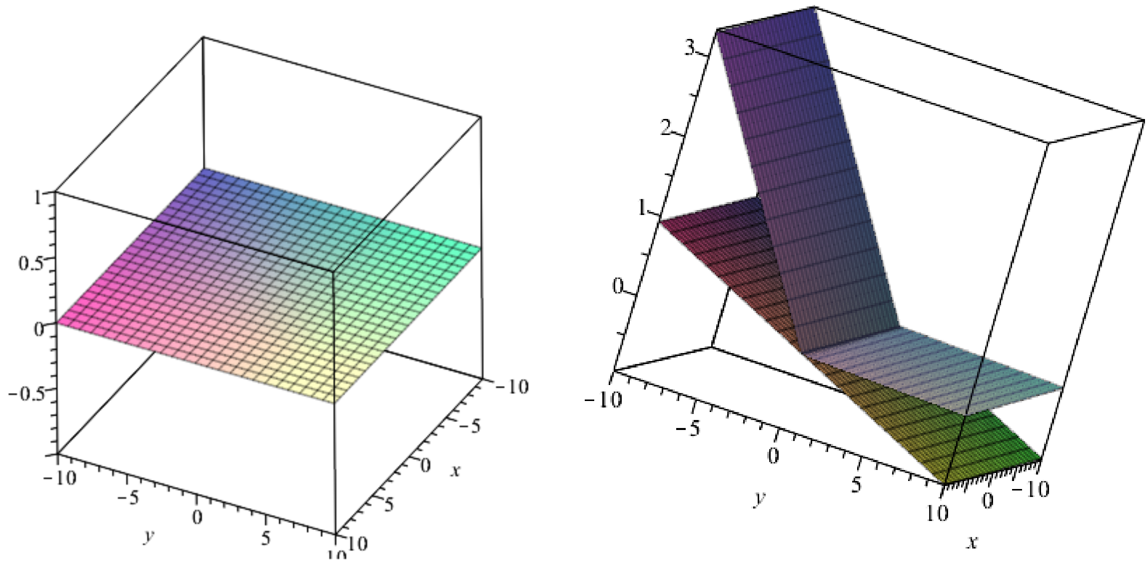


Figure 10: In the left hand side, we present $\max(\{C_1, C_2\})$ minus $\max(\{C_1, C_2, C_3\})$. In the right hand side $\max(\{C_1, C_2\})$ and C_3 are plotted.

As we see, the third plane seems useless. Nevertheless, a program based on the previous propositions will keep it, because it exists a region where it is more important than the two first planes, as shown in Figure 11.

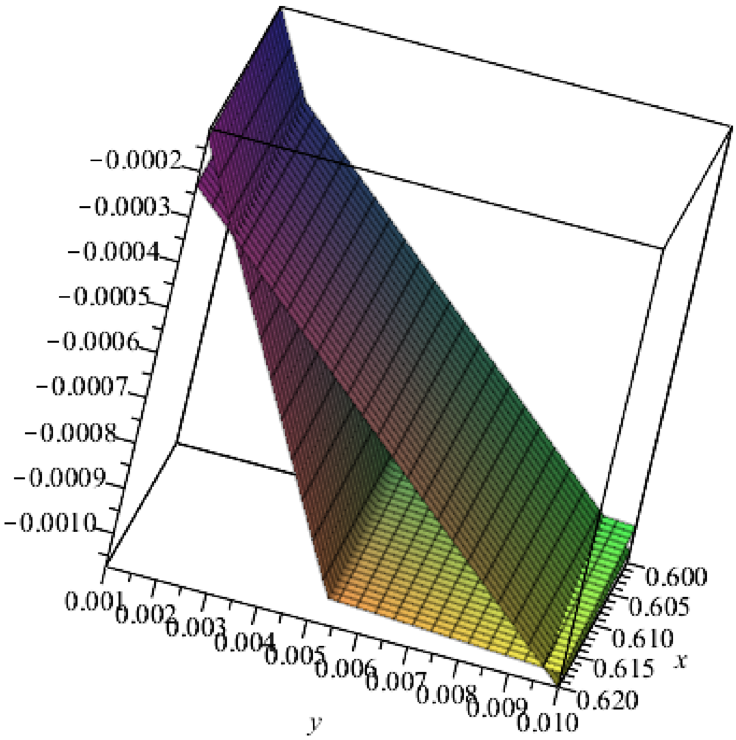


Figure 11: Same figure as 15 but zoomed on the relevant region

Hence, the previous description is really precise and seems to overcome a naive visual treatment of the data. Indeed, this region is not visible in Figure 10, and it seems hard to find it.

5 N flavours, general considerations

The two flavours case has been studied in detail in the previous sections. We will generalised our study to the general N_f case. Our discussion will be valid for N_f greater or equal than four. The three flavour case will be studied later. In order to simplify our expressions, n will be used instead of N_f. Moreover, it should be highlighted that from now one, we assume that all the mesons have the same mass, labeled M.

5.1 The isospin amplitudes

Starting from [2] some interesting properties of the general N_f flavours case will be presented. First, the amplitudes in Cartesian coordinates for the $SU(n)$ case is given by:

$$\begin{aligned} T(s,t,u) = & \left[tr\left(X^a X^b X^c X^d\right) + tr\left(X^a X^d X^c X^b\right)\right] B(s,t,u) \\ & \left[tr\left(X^a X^c X^d X^b\right) + tr\left(X^a X^b X^d X^c\right)\right] B(t,u,s) \\ & \left[tr\left(X^a X^d X^b X^c\right) + tr\left(X^a X^c X^b X^d\right)\right] B(u,s,t) \\ & + \delta^{ab}\delta^{cd} C(s,t,u) + \delta^{ac}\delta^{bd} C(t,u,s) + \delta^{ad}\delta^{bc} C(u,s,t) \end{aligned}$$

Where the X^a are generators of the $SU(n)$ group. Again, it is much more convenient to consider independent isospin amplitudes. In $SU(2)$ there are three amplitudes, presented in the third chapter Eq.(3.3), Eq.(3.4) and Eq.(3.5), for $SU(3)$, five, and for more than three flavours, there are seven channels although only six are independent:

$$\begin{aligned} T_I &= 2\left(n-\frac{1}{n}\right)\left[B(s,t,u)+B(t,u,s)\right] - \frac{2}{n}B(u,s,t) + (n^2-1)C(s,t,u) \\ & \quad +C(t,u,s) + C(u,s,t) \\ T_S &= 2\left(n-\frac{4}{n}\right)\left[B(s,t,u)+B(t,u,s)\right] - \frac{4}{n}B(u,s,t) + C(t,u,s) + C(u,s,t) \\ T_A &= n\left[-B(s,t,u)+B(t,u,s)\right] + C(t,u,s) - C(u,s,t) \\ T_{AS} = T_{SA} &= C(t,u,s) - C(u,s,t) \\ T_{SS} &= 2B(u,s,t) + C(t,u,s) + C(u,s,t) \\ T_{AA} &= -2B(u,s,t) + C(t,u,s) + C(u,s,t) \end{aligned}$$

The labels come from the decomposition of the adjoint-adjoint representation for the $SU(n)$ case [2]:

$$Adj \otimes Adj = R_I \bigoplus R_S \bigoplus R_A \bigoplus R_S^A \bigoplus R_A^S \bigoplus R_A^A \bigoplus R_S^S$$

$R_I, R_S, R_A, R_S^A, R_A^S, R_A^A$ and R_S^S stands for some representations, for example R_I is a singlet representation. More information can be found in [2]. The seven previous amplitudes are projections of the scattering amplitude following this decomposition. Note that $T_{AS} = T_{SA}$ which shows that only six channels are independent and relevant to study.

From these formulae, it is easy to derive the crossing symmetry relations:

$$T_J(s,t,u) = {C_u}^{JJ'} T_{J'}(u,t,s)$$

with $J \in \{I, S, A, SA, SS, AA\}$ and we have, in agreement with [14]:

$$
{C_u}^{JJ'} = \begin{bmatrix}
\frac{1}{n^2-1} & 1 & -1 & -\frac{1}{2}n^2+2 & \frac{n^2(n+3)}{4n+4} & \frac{n^2(n-3)}{4n-4} \\
\frac{1}{n^2-1} & \frac{n^2-12}{2n^2-8} & -\frac{1}{2} & 1 & \frac{1}{4}\frac{n^2(n+3)}{(n+1)(n+2)} & -\frac{1}{4}\frac{n^2(n-3)}{(n-1)(n-2)} \\
-\frac{1}{n^2-1} & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{n^2(n+3)}{4n+4} & -\frac{n^2(n-3)}{4n-4} \\
-\frac{1}{n^2-1} & \frac{2}{n^2-1} & 0 & \frac{1}{2} & \frac{1}{4}\frac{n(n+3)}{(n+1)(n+2)} & \frac{1}{4}\frac{n(n-3)}{(n-1)(n-2)} \\
\frac{1}{n^2-1} & \frac{1}{n+2} & \frac{1}{n} & -\frac{1}{2}\frac{2-n}{n} & \frac{1}{4}\frac{n^2+n+2}{(n+1)(n+2)} & \frac{(n-3)}{4n-4} \\
\frac{1}{n^2-1} & -\frac{1}{n-2} & -\frac{1}{n} & \frac{1}{2}\frac{n+2}{n} & \frac{(n+3)}{4n+4} & \frac{1}{4}\frac{n^2-n+2}{(n-1)(n-2)}
\end{bmatrix} \tag{5.48}
$$

5.2 The constraints

Following the exact same strategy as in the second chapter, the constraints on a_J from demanding $\frac{\partial^2 a_J T^J}{\partial s^2} \geq 0$, are:

$$\forall x \in [4M^2; \infty[, \ \frac{a_{J'}}{(x-s)^3} + \frac{C_u^{JJ'} a_J}{(x-u)^3} \geq 0$$

Which gives us:

$$
\begin{gathered}
\left(\frac{1}{n^2-1} + \frac{(x-u)^3}{(x-s)^3}\right) a_I - \frac{a_A}{(n^2-1)} + \frac{a_{AA}}{(n^2-1)} + \frac{a_S}{(n^2-1)} - \frac{a_{SA}}{(n^2-1)} + \frac{a_{SS}}{(n^2-1)} \geq 0 \\
a_I - \frac{1}{2}a_A - \frac{a_{AA}}{n-2} + \left(\frac{n^2-12}{2n^2-8} + \frac{(x-u)^3}{(x-s)^3}\right) a_S + \frac{2a_{SA}}{n^2-4} + \frac{a_{SS}}{n+2} \geq 0 \\
-a_I + \left(\frac{1}{2} + \frac{(x-u)^3}{(x-s)^3}\right) a_A - \frac{a_{AA}}{n} - \frac{1}{2}a_S + \frac{a_{SS}}{n} \geq 0 \\
a_I\left(-\frac{1}{2}n^2+2\right) + \frac{1}{2}\frac{a_{AA}(n+2)}{n} + a_S + \left(\frac{1}{2} + \frac{(x-u)^3}{(x-s)^3}\right) a_{SA} - \frac{1}{2}\frac{a_{SS}(2-n)}{n} \geq 0 \\
\frac{n^2(n+3)a_I}{4n+4} + \frac{a_A n(n+3)}{(4n+4)} + \frac{a_{AA}(n+3)}{4n+4} + \frac{1}{4}\frac{a_S n^2(n+3)}{(n+1)(n+2)} + \frac{1}{4}\frac{a_{SA} n(n+3)}{(n+1)(n+2)} + \left(\frac{1}{4}\frac{n^2+n+2}{(n+1)(n+2)} + \frac{(x-u)^3}{(x-s)^3}\right) a_{SS} \geq 0 \\
\frac{n^2(n-3)a_I}{4n-4} - \frac{a_A n(n-3)}{(4n-4)} + a_{AA}\left(\frac{(x-u)^3}{(x-s)^3} + \frac{n^2-n+2}{4(n-1)(n-2)}\right) - \frac{1}{4}\frac{a_S n^2(n-3)}{(n-1)(n-2)} + \frac{1}{4}\frac{a_{SA} n(n-3)}{(n-1)(n-2)} + \frac{a_{SS}(n-3)}{4n-4} \geq 0
\end{gathered}
$$

The relation also have to be true when x goes to infinity, which gives:

$$
\begin{gathered}
R_I = \left(\frac{1}{n^2-1} + 1\right) a_I - \frac{a_A}{(n^2-1)} + \frac{a_{AA}}{(n^2-1)} + \frac{a_S}{(n^2-1)} - \frac{a_{SA}}{(n^2-1)} + \frac{a_{SS}}{(n^2-1)} \geq 0 \\
R_S = a_I - \frac{1}{2}a_A - \frac{a_{AA}}{n-2} + \left(\frac{n^2-12}{2n^2-8} + 1\right) a_S + \frac{2a_{SA}}{n^2-4} + \frac{a_{SS}}{n+2} \geq 0 \\
R_A = -a_I + \frac{3}{2}a_A - \frac{a_{AA}}{n} - \frac{1}{2}a_S + \frac{a_{SS}}{n} \geq 0 \\
R_{SA} = a_I\left(-\frac{1}{2}n^2+2\right) + \frac{1}{2}\frac{a_{AA}(n+2)}{n} + a_S + \frac{3}{2}a_{SA} - \frac{1}{2}\frac{a_{SS}(2-n)}{n} \geq 0 \\
R_{SS} = \frac{n^2(n+3)a_I}{4n+4} + \frac{a_A n(n+3)}{(4n+4)} + \frac{a_{AA}(n+3)}{4n+4} + \frac{1}{4}\frac{a_S n^2(n+3)}{(n+1)(n+2)} + \frac{1}{4}\frac{a_{SA} n(n+3)}{(n+1)(n+2)} + \left(\frac{1}{4}\frac{n^2+n+2}{(n+1)(n+2)} + 1\right) a_{SS} \geq 0 \\
R_{AA} = \frac{n^2(n-3)a_I}{4n-4} - \frac{a_A n(n-3)}{(4n-4)} + a_{AA}\left(1 + \frac{n^2-n+2}{4(n-1)(n-2)}\right) - \frac{1}{4}\frac{a_S n^2(n-3)}{(n-1)(n-2)} + \frac{1}{4}\frac{a_{SA} n(n-3)}{(n-1)(n-2)} + \frac{a_{SS}(n-3)}{4n-4} \geq 0
\end{gathered} \tag{5.49}
$$

Note that:

$$R_{SS} = \left(-1 + \frac{1}{2}n + \frac{1}{2}n^2\right) R_I + \left(\frac{1}{4}n - \frac{1}{2}\right) R_S + (\frac{1}{4}n + \frac{1}{2})R_A + \frac{1}{2}R_{SA}$$

so when $n \geq 2$ this constraints is always verified as long as the other ones are. We then do not need it.

5.3 Generalised slopes

Using the amplitudes in terms of the low energy constants from [2], the second derivative of this expression is:

$$\begin{aligned}\frac{d^2}{ds^2}T(s,t) =& \frac{4M^4}{nF^4}\left(6n^2a_I - n^2a_A + 3n^2a_S - 2na_{AA} + 2na_{SS} - 8a_I - 16a_S\right)\bar{l}_3 \\ &+ \frac{8M^4}{F^4}\left(n^2a_I + 2a_I + a_A + 3a_{AA} + 3a_S + a_{SA} + 3a_{SS}\right)\bar{l}_2 \\ &+ \frac{16M^4}{F^4}\left(n^2a_I - a_A + a_{AA} + a_S - a_{SA} + a_{SS}\right)\bar{l}_1 \\ &+ \frac{8M^4}{nF^4}\left(2n^2a_I + n^2a_A + n^2a_S - 2na_{AA} + 2na_{SS} - 4a_I - 8a_S\right)\bar{l}_0 \\ &+ G(s,t,a_I,a_S,a_{SA},a_{SS},a_{AA})\end{aligned} \tag{5.50}$$

where G does not contain any coupling constants. Note that this formula and the set of constraints derived before are also true for the three flavours case if we put $a_{AA} = 0$. Using Eq.(5.50) and Eq.(5.49), it is easy to show that:

$$\begin{aligned}\frac{4}{n}\left(6n^2a_I - n^2a_A + 3n^2a_S - 2na_{AA} + 2na_{SS} - 8a_I - 16a_S\right)\bar{l}_3 &= \frac{4\bar{l}_3}{n}\left((16n^2-16)R_I \right. \\ &\quad \left. +(8n^2-32)R_S\right) \geq 0 \\ 8\left(n^2a_I + 2a_I + a_A + 3a_{AA} + 3a_S + a_{SA} + 3a_{SS}\right)\bar{l}_2 &= 8\bar{l}_2\left((16n^2-16)R_I + 16R_A + \right. \\ &\quad 16R_{SA}) \geq 0 \\ 16\left(n^2a_I - a_A + a_{AA} + a_S - a_{SA} + a_{SS}\right)\bar{l}_1 &= 16\bar{l}_1\left((16n^2-16)R_I\right) \geq 0 \\ \frac{8}{n}\left(2n^2a_I + n^2a_A + n^2a_S - 2na_{AA} + 2na_{SS} - 4a_I - 8a_S\right)\bar{l}_0 &= \frac{8\bar{l}_0}{n}\left((16n^2-16)R_I + (8n^2-32)R_S \right. \\ &\quad +8nR_A) \geq 0\end{aligned}$$

In the two flavour case only two coupling constants were needed and we had:

$$\frac{d^2}{ds^2}T(s,t) = \gamma_2\bar{l}_2 + \gamma_1\bar{l}_1 + \ldots$$

where:

$$\gamma_2 = \left(\frac{6a_0 + a_1 + 3a_2}{24\pi^2F^4}\right) = 2((3.19) + (3.20))$$

$$\gamma_1 = \left(\frac{4a_0 - a_1 + a_2}{24\pi^2F^4}\right) = (3.19)$$

It was then easy to determine a bounded region of existence for the slope, which simplified our maximisation procedure. Then, it is natural, in the N_f flavour case to introduce the following positive numbers:

$$\begin{aligned} \alpha_3 &= \frac{4}{n}\Big((16n^2-16)R_I + (8n^2-32)R_S\Big) \geq 0 \\ \alpha_2 &= 8\Big((16n^2-16)R_I + 16R_A + 16R_{SA}\Big) \geq 0 \\ \alpha_1 &= 16\Big((16n^2-16)R_I\Big) \geq 0 \\ \alpha_0 &= \frac{8}{n}\Big((16n^2-16)R_I + (8n^2-32)R_S + 8nR_A\Big) \geq 0 \end{aligned} \tag{5.51}$$

As we see, we cannot say that one coefficient is always greater than the other ones. It is then hard to convert our unbounded slopes problem into a bounded one. We therefore introduce a linear combination of $\bar{l}_0$ and $\bar{l}_2$ in order to build a coefficient that is always larger than the other ones.

$$\begin{aligned} \bar{l}_0 &= \beta_0 L_0 + \beta_1 L_2 \\ \bar{l}_2 &= \beta_2 L_0 + \beta_3 L_2 \end{aligned}$$

Where the coefficients $\beta_0, \beta_1, \beta_2, \beta_1$ are assumed to be positive. We then impose that this system has to be invertible, which means that $\beta_0\beta_3 - \beta_1\beta_2 \neq 0$, so that L_0 and L_2 are well defined:

$$\begin{aligned} L_0 &= -\frac{\beta_1\bar{l}_2 - \beta_3\bar{l}_0}{\beta_0\beta_3 - \beta_1\beta_2} \\ L_2 &= \frac{\beta_0\bar{l}_2 - \beta_2\bar{l}_0}{\beta_0\beta_3 - \beta_1\beta_2} \end{aligned}$$

We then get:

$$\frac{d^2}{ds^2}T(s,t) = \frac{M^4}{F^4}\Big(\Lambda_0 L_0 + \alpha_1\bar{l}_1 + \Lambda_2 L_2 + \alpha_3\bar{l}_3\Big) + G(s,t,\dots) \tag{5.52}$$

with:

$$\begin{aligned} \Lambda_0 &= \beta_0\alpha_0 + \beta_2\alpha_2 \\ \Lambda_2 &= \beta_1\alpha_0 + \beta_3\alpha_2 \end{aligned}$$

Imposing $\beta_1 \geq \beta_0$ and $\beta_3 \geq \beta_2$ leads to $\Lambda_2 \geq \Lambda_0$. Moreover, if $\Lambda_2 = 0$, then from Eq.(5.51) all the $R_J = 0$ so all the other coefficients have to vanish which leads to a trivial case. So, it can be assumed that Λ_2 is strictly positive which allows us to divide the entire expression by this term, without any sign problems. The problem is now described by bounded slopes: $x = \frac{\alpha_1}{\Lambda_2}, y = \frac{\alpha_3}{\Lambda_2}, z = \frac{\Lambda_0}{\Lambda_2}$.

5.4 The study of $x,\ y$ and z

Like in the two flavours case, we want to find the region of existence of the slopes $x,\ y$ and z. Let us introduce four numbers $(\theta_0;\theta_1;\theta_2;\theta_3)$. We will first find necessary conditions on the slopes, of the form:

$$\begin{aligned}\theta_0\Lambda_0+\theta_1\alpha_1+\theta_2\Lambda_2+\theta_3\alpha_3 = \Bigg(&\left(\frac{8(16n^2-16)\beta_0}{n}+8(16n^2-16)\beta_2\right)\theta_0\\ &+\left(\frac{8(16n^2-16)\beta_1}{n}+8(16n^2-16)\beta_3\right)\theta_2\\ &+16(16n^2-16)\theta_1+\frac{16(4n^2-4)\theta_3}{n}\Bigg)R_I\\ &+\left(\frac{8(8n^2-32)\beta_0\theta_0}{n}+\frac{8(8n^2-32)\beta_1\theta_2}{n}\frac{4(8n^2-32)\theta_3}{n}\right)R_S\\ &+\left(\left(64\beta_0+128\beta_2\right)\theta_0+\left(64\beta_1+128\beta_3\right)\theta_2\right)R_A\\ &+\left(128\beta_2\theta_0+128\beta_3\theta_2\right)R_{SA}\geq 0\end{aligned} \tag{5.53}$$

which can be written as:

$$\theta_0\Lambda_0+\theta_1\alpha_1+\theta_2\Lambda_2+\theta_3\alpha_3=\Theta_I R_I+\Theta_S R_S+\Theta_A R_A+\Theta_{SA}R_{SA}\geq 0 \tag{5.54}$$

In order to delimit the region of existence of the three numbers x, y, z, we have to control the sign of this expression for all possible values of the slopes. We choose to keep it positive. Moreover, in Eq.(5.49), the constraints R_I, R_A, R_S, R_{SA} are positive and linearly independent, if $N_f \geq 4$. One can then choose a_J to cancel as many constraints R_I, R_A, R_S, R_{SA} as one want and, for example, keep only one non vanishing constraint. This means that, in order to fulfill Eq.(5.54) for all the possible slopes which verify R_I, R_A, R_S, R_{SA}, all the Θ_J have to be positive or equal to zero. Finally, if one of the constraints R_I, R_A, R_S, R_{SA} were always equal to zero, we would have more freedom to choose the slopes $x,\ y$ and z. So we will try to cancel as many Θ_J as we can to determine the less stringent constraints. As we will show now, there are a finite numbers of bounds on x, y, z. It is important to remark that, in this procedure, we did not impose $R_{AA} \geq 0$ and this condition can be violated. However, this is not yet a problem as we are looking for necessary conditions.

Let us take a simple example with $\theta_0=\theta_1=0$ and $\theta_2\neq 0$, $\theta_3\neq 0$. Then:

$$\begin{aligned}\theta_2\Lambda_2+\theta_3\alpha_3 = \Bigg(&\left(\frac{8(16n^2-16)\beta_1}{n}+8(16n^2-16)\beta_3\right)\theta_2+\frac{16(4n^2-4)\theta_3}{n}\Bigg)R_I\\ &+\left(\frac{8(8n^2-32)\beta_1\theta_2}{n}+\frac{4(8n^2-32)\theta_3}{n}\right)R_S\\ &+\left(\left(64\beta_1+128\beta_3\right)\theta_2\right)R_A+\left(128\beta_3\theta_2\right)R_{SA}\geq 0\end{aligned} \tag{5.55}$$

We want to use as few constraints as possible. If we try to cancel 2 or more factors in front of the constraints R_J, the solution would be $\theta_2 = 0$ and $\theta_3 = 0$ which we do not want in this case. So we can only cancel one factor. This leads to four different cases. First we can try to get rid of the constraint R_I by imposing that:

$$\left(\frac{8(16n^2-16)\beta_1}{n} + 8(16n^2-16)\beta_3\right)\theta_2 + \frac{16(4n^2-4)\theta_3}{n} = 0$$

Which gives $\theta_3 = -(2(n\beta_3 + \beta_1))\theta_2$. With Eq.(5.55) we then get:

$$\theta_2\Lambda_2 - (2(n\beta_3 + \beta_1))\theta_2\alpha_3 \geq 0$$

or equivalently:

$$y \leq \frac{1}{(2(n\beta_3 + \beta_1))}$$

Next we can try to cancel the factor of R_S:

$$\frac{8(8n^2-32)\beta_1\theta_2}{n} + \frac{4(8n^2-32)\theta_3}{n} = 0$$

Which gives $\theta_3 = -2\beta_1\theta_2$ and with Eq.(5.55):

$$\theta_2\Lambda_2 - 2\beta_1\theta_2\alpha_3 \geq 0$$

or equivalently :

$$y \leq \frac{1}{2\beta_1}$$

If we try to cancel the factor in front of the constraints R_A or R_{SA}, θ_2 would be equal to zero, which is not wanted in this case. Moreover, the following conditions have to be verified in any case: $\theta_2 \geq 0$, $\theta_3 \geq -2\beta_1\theta_2$ and $\theta_3 \geq -(2(n\beta_3 + \beta_1))\theta_2$ in order to have positive Θ_J. As a conclusion, keeping $\theta_3 = -2\beta_1\theta_2$, we can say that

$$\boxed{y \leq \tfrac{1}{2\beta_1}} \tag{5.56}$$

Let us now consider all the possible cases for θ.

One non vanishing θ: This case shows that x, y and z have to be positive.
The other cases are summarised in Tables 4, 5 and 6.

Table 5: **Two non vanishing θ**

$\theta_0 = \theta_1 = 0$	$0 \leq y \leq \frac{1}{2\beta_1}$
$\theta_0 = \theta_2 = 0$	$0 \leq -\frac{1}{4n}x + y$
$\theta_0 = \theta_3 = 0$	$0 \leq x \leq \frac{1}{\frac{1}{2}\left(\frac{\beta_1}{n}+\beta_3\right)}$
$\theta_1 = \theta_2 = 0$	$z - 2\beta_0 y \geq 0$
$\theta_1 = \theta_3 = 0$	$min(\frac{\beta_0}{\beta_1}, \frac{\beta_2}{\beta_3}) \leq z \leq max(\frac{\beta_0}{\beta_1}, \frac{\beta_2}{\beta_3})$
$\theta_2 = \theta_3 = 0$	$0 \leq z - \frac{n\beta_2+\beta_0}{2n}x$

Table 6: **All the θ are non-vanishing**

$\beta_0\beta_3 > \beta_1\beta_2$	$-\frac{\beta_1+2\beta_3}{\beta_0+2\beta_2}z - \frac{1}{2}\frac{\beta_0\beta_3-\beta_1\beta_2}{\beta_0+2\beta_2}x + \frac{4(\beta_0\beta_3-\beta_1\beta_2)}{\beta_0+2\beta_2}y + 1 \geq 0$
$\beta_0\beta_3 < \beta_1\beta_2$	$-\frac{\beta_1+2\beta_3}{\beta_0+2\beta_2}z - \frac{1}{2}\frac{\beta_0\beta_3-\beta_1\beta_2}{\beta_0+2\beta_2}x + \frac{4(\beta_0\beta_3-\beta_1\beta_2)}{\beta_0+2\beta_2}y + 1 \leq 0$

Table 7: **Three non vanishing θ**

$\theta_0 = 0$		$-\frac{1}{2}\beta_3 x - 2\beta_1 y + 1 \geq 0$
$\theta_1 = 0$	$\beta_0\beta_3 > \beta_1\beta_2$	$-\frac{\beta_3}{\beta_2} z + \frac{2(\beta_0\beta_3 - \beta_1\beta_2)}{\beta_2} y + 1 \leq 0$
	$\beta_0\beta_3 < \beta_1\beta_2$	$-\frac{\beta_3}{\beta_2} z + \frac{2(\beta_0\beta_3 - \beta_1\beta_2)}{\beta_2} y + 1 \geq 0$
$\theta_2 = 0$		$-\frac{1}{2}\beta_2 x - 2\beta_0 y + z \geq 0$
$\theta_3 = 0$	$\beta_0\beta_3 > \beta_1\beta_2$	$-\frac{\beta_1}{\beta_0} z - \frac{1}{2}\frac{\beta_0\beta_3 - \beta_1\beta_2}{\beta_0} x + 1 \geq 0$
		$-\frac{\beta_3}{\beta_2} z + \frac{1}{2}\frac{\beta_0\beta_3 - \beta_1\beta_2}{\beta_2 n} x + 1 \leq 0$
	$\beta_0\beta_3 < \beta_1\beta_2$	$-\frac{\beta_1}{\beta_0} z - \frac{1}{2}\frac{\beta_0\beta_3 - \beta_1\beta_2}{\beta_0} x + 1 \leq 0$
		$-\frac{\beta_3}{\beta_2} z + \frac{1}{2}\frac{\beta_0\beta_3 - \beta_1\beta_2}{\beta_2 n} x + 1 \geq 0$

We just proved that these conditions are necessary. We did not have enough time to look for necessary and sufficient conditions. However, these results will be good enough for the three and four flavours case.

In the next section we will focus on the 3 flavours and 4 flavours case and used these conditions in the same way as before: choosing randomly the generalised slopes of the hyperplanes and solve a maximization problem.

6 The 3 flavours p^4 lagrangian

In a previous study [4], bounds on the coupling constants have been derived. We try here to update them using the same strategy as in the $SU(2)$ case.

6.1 The specific case of $SU(3)$

Most of the considerations for n greater than 4 remains true, but the $SU(3)$ case has some specific properties. The only relevant isospin channels are T_I, T_S, T_A, T_{SA} and T_{SS}, and the crossing symmetry becomes, in agreement with [4]:

$$C_u = \begin{bmatrix} \frac{1}{8} & 1 & -1 & -\frac{5}{2} & \frac{27}{8} \\ \frac{1}{8} & -\frac{3}{10} & -\frac{1}{2} & 1 & \frac{27}{40} \\ -\frac{1}{8} & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{9}{8} \\ -\frac{1}{8} & \frac{2}{5} & 0 & \frac{1}{2} & \frac{9}{40} \\ \frac{1}{8} & \frac{1}{5} & \frac{1}{3} & \frac{1}{6} & \frac{7}{40} \end{bmatrix}$$

Which leads to rewrite the positivity conditions as:

$$\begin{aligned} \left(\frac{1}{8} + \frac{(x-u)^3}{(x-s)^3}\right) a_I - \frac{1}{8}a_A + \frac{1}{8}a_S - \frac{1}{8}a_{SA} + \frac{1}{8}a_{SS} &\geq 0 \\ a_I - \frac{1}{2}a_A + \left(-\frac{3}{10} + \frac{(x-u)^3}{(x-s)^3}\right) a_S + \frac{2}{5}a_{SA} + \frac{1}{5}a_{SS} &\geq 0 \\ -a_I + \left(\frac{1}{2} + \frac{(x-u)^3}{(x-s)^3}\right) a_A - \frac{1}{2}a_S + \frac{1}{3}a_{SS} &\geq 0 \\ -\frac{5}{2}a_I + a_S + \left(\frac{1}{2} + \frac{(x-u)^3}{(x-s)^3}\right) a_{SA} + \frac{1}{6}a_{SS} &\geq 0 \\ \frac{27}{8}a_I + \frac{9}{8}a_A + a_S\frac{27}{40} + \frac{9}{40}a_{SA} + \left(\frac{7}{40} + \frac{(x-u)^3}{(x-s)^3}\right) a_{SS} &\geq 0 \end{aligned}$$

And for $x \to \infty$:

$$\begin{aligned} R_I = \left(\frac{9}{8}\right) a_I - \frac{1}{8}a_A + \frac{1}{8}a_S - \frac{1}{8}a_{SA} + \frac{1}{8}a_{SS} &\geq 0 \\ R_S = a_I - \frac{1}{2}a_A + \left(\frac{7}{10}\right) a_S + \frac{2}{5}a_{SA} + \frac{1}{5}a_{SS} &\geq 0 \\ R_A = -a_I + \left(\frac{3}{2}\right) a_A - \frac{1}{2}a_S + \frac{1}{3}a_{SS} &\geq 0 \\ R_{SA} = -\frac{5}{2}a_I + a_S + \left(\frac{3}{2}\right) a_{SA} + \frac{1}{6}a_{SS} &\geq 0 \\ R_{SS} = \frac{27}{8}a_I + \frac{9}{8}a_A + a_S\frac{27}{40} + \frac{9}{40}a_{SA} + \left(\frac{47}{40}\right) a_{SS} &\geq 0 \end{aligned} \tag{6.57}$$

Moreover, as derived in the previous chapter we have:

$$\frac{d^2}{ds^2}T(s,t) = \frac{M^4}{F^4}\left(\alpha_3\bar{l}_3 + \alpha_2\bar{l}_2 + \alpha_1\bar{l}_1 + \alpha_0\bar{l}_0\right) + G(s,t,\dots) \geq 0 \tag{6.58}$$

Nevertheless, from [12] we have:

$$tr\left(\Delta^\mu U^\dagger \Delta^\nu U \Delta_\mu U^\dagger \Delta_\nu U\right) = -2tr\left(\Delta^\mu U^\dagger \Delta_\mu U \Delta^\nu U^\dagger \Delta_\nu U\right) + \frac{1}{2}tr\left(\Delta^\mu U^\dagger \Delta_\mu U\right)^2 + tr\left(\Delta^\mu U^\dagger \Delta^\nu U\right) tr\left(\Delta_\mu U^\dagger \Delta_\nu U\right)$$

Which means that the operator related to $\bar{l}_0$ is linearly dependent on the other ones:

$$\mathcal{O}_0 = -2\mathcal{O}_3 + \frac{1}{2}\mathcal{O}_1 + \mathcal{O}_2$$

So we have only three independent operators and:

$$(\bar{l}_3 - 2\bar{l}_0)\mathcal{O}_3 + (\bar{l}_1 + \frac{1}{2}\bar{l}_0)\mathcal{O}_1 + (\bar{l}_2 + \bar{l}_0)\mathcal{O}_2$$

Which means that $\mathcal{O}_0$ is not required in the lagrangian and only three coupling constants are left: $\bar{l}_3, \bar{l}_2$ and $\bar{l}_1$:

$$\frac{d^2}{ds^2}T(s,t) = \frac{M^4}{F^4}\left(\alpha_3\bar{l}_3 + \alpha_2\bar{l}_2 + \alpha_1\bar{l}_1\right) + G(s,t,\dots) \geq 0$$

With α_i given by [2], like in the previous chapter:

$$\begin{aligned}
\alpha_3 &= \frac{128}{3}R_I + \frac{40}{3}R_S \geq 0 \\
\alpha_2 &= 128R_I + 16R_S + 16R_{SA} \geq 0 \\
\alpha_1 &= 128R_I \geq 0
\end{aligned}$$

Moreover from Eq.(6.57) one can derive:

$$R_{SA} = -4R_I + \frac{5}{2}R_S + \frac{1}{2}R_A$$

which means that we can also write α_2 as:

$$\alpha_2 = 64R_I + 56R_S + 8R_A$$

The two expressions for α_2 show that it is the largest coefficient, and it vanishes if and only if the other two coefficients vanish. We then define the slopes:

$$\begin{aligned}
x &= \frac{\alpha_1}{\alpha_2} \\
y &= \frac{\alpha_3}{\alpha_2}
\end{aligned}$$

Using the same method as in the last chapter, we get the allowed region:

$$0 \leq x \leq 1 \tag{6.59}$$

$$0 \leq y \leq \frac{1}{2} \tag{6.60}$$

$$y \geq -\frac{1}{6} + \frac{2}{3}x \tag{6.61}$$

$$y \geq \frac{1}{3}x \tag{6.62}$$

$$y \leq \frac{1}{3} + \frac{1}{6}x \tag{6.63}$$

which is illustrated in Figure 12.

6.2 Method

The numbers x and y have been chosen randomly in the allowed region and two of the a_J can be written as a function of the slopes and of the remaining parameters. Moreover, we have fixed α_2 equal to one. So the total numbers of free parameters is reduced to four. For each x and y, the function $-G$ is then maximized, using Maple Optimization tools. The maximum seems to always be reached at $t = 4M^2$. So the maximization process has been done again with this assumption in order to increase the speed of the program.

6.3 Results

1026 slopes have been studied and the raw results are presented in Figure 12.

Moreover, using the properties derived in chapter four, we can extract the most important planes as illustrated in Figure 13.

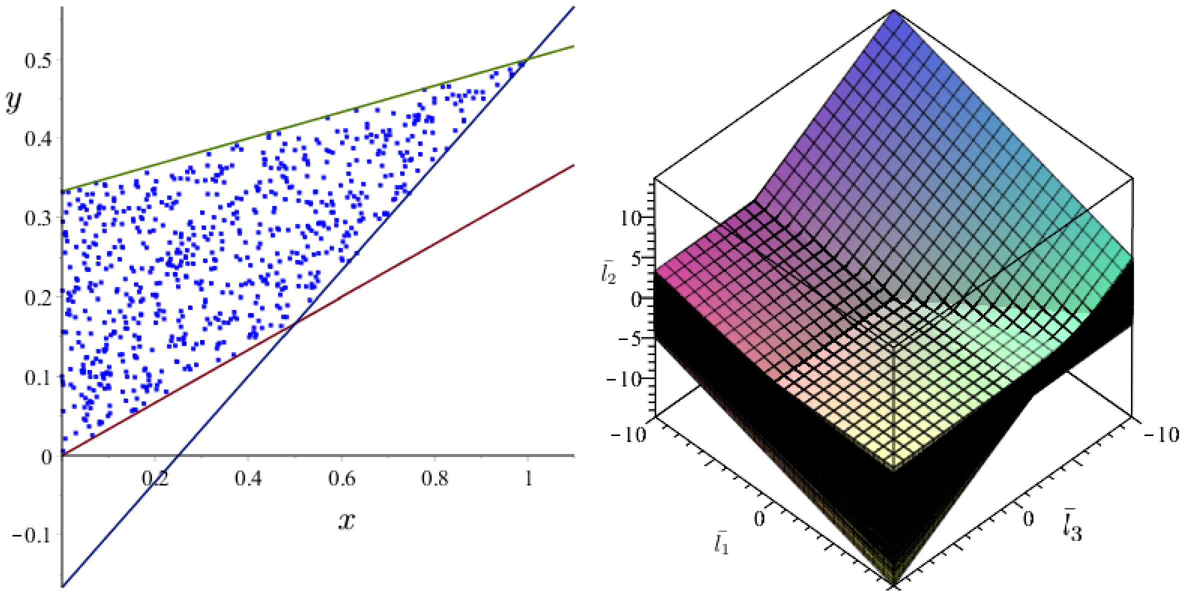


Figure 12: In the left hand side, the set of all tested slopes, x and y, consistent with (6.59…6.63) are plotted. The constraints with $\bar{l}_2$ in the z-axis, $\bar{l}_1$ in the x-axis and $\bar{l}_3$ in the y-axis, obtained with the Maple maximisation procedure, for these specific slopes and fixed $t = 4M^2$ are presented in the right hand side.

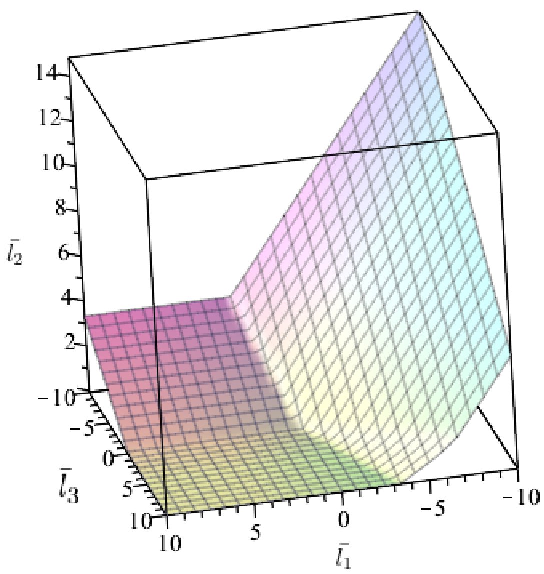


Figure 13: The 73 necessary planes found using the properties of the Chapter four.

Upon the 1026 planes we found, 73 appear to be necessary. However, considering Figure 13, which presents the 73 most important constraints, the visual intuition suggests that only the four following planes are required:

$$\bar{l}_2 \geq 6.294 \cdot 10^{-4} - 1.861 \cdot 10^{-3}\bar{l}_1 - 0.3321\bar{l}_3 \tag{6.64}$$
$$\bar{l}_2 \geq 5.669 \cdot 10^{-4} - 2.505 \cdot 10^{-3}\bar{l}_1 - 5.701 \cdot 10^{-3}\bar{l}_3 \tag{6.65}$$
$$\bar{l}_2 \geq 7.258 \cdot 10^{-4} - 0.5317\bar{l}_1 - 0.1959\bar{l}_3 \tag{6.66}$$
$$\bar{l}_2 \geq 8.518 \cdot 10^{-4} - 0.9885\bar{l}_1 - 0.4932\bar{l}_3 \tag{6.67}$$

Those planes are not enough to describe our results, as illustrated in Figure 14, where the maximum of 73 planes minus the maximum of the four previous ones are represented. As we see, there are some important positive parts, which show that all the 73 planes seem to have a contribution, even negligible, to the solution and they can be found in appendix B.

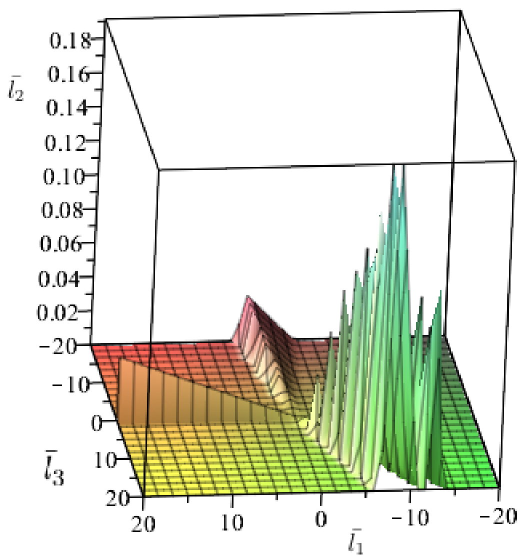


Figure 14: The maximum of all the 73 planes minus the maximum of (6.64. . . 6.67)

Previously, [4] had derived such kind of bounds. We compare our results and theirs in Figure 15 and the equation of the three most relevant ones are available in Table 10.

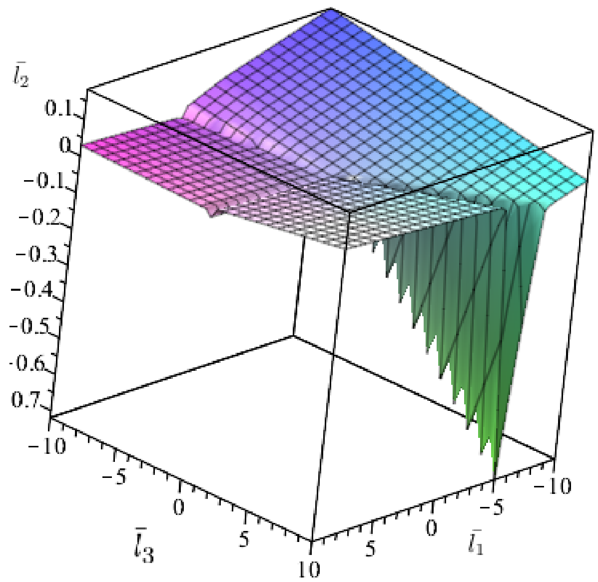


Figure 15: The maximum of the planes found in [4] minus the maximum of all the 73 planes

As we see from Figure 15 the results are complementary. The negative area shows us that we have a good improvement in this region. We actually have found new planes. Nevertheless, it seems that our bounds are often slightly less restrictive than the one in [4]. Note that in [4] the process $a + b \to a + b$ has been studied considering $m_a \neq m_b$ whereas in this report, as stated before, we have assumed $m_a = m_b$.

We have derived the new constraints in terms of the $\bar{l}_i$. In these units, the mass dependence is absorbed in the $\bar{l}_i$ and then does not explicitly appear in the bounds as we see in (6.64. . . 6.67). It is then relevant to go from the $\bar{l}_i$ to the l_i^r, to study the influence of the mass. Moreover, the p^6 corrections depend a lot on the mass term. In the next part, recalling section 2.7, we will use the following transformation:

$$l_i^r = \bar{l}_i + \frac{1}{16\pi^2}\Gamma_i \ln\left(\frac{M_{phys}^2}{\mu^2}\right)$$

with $\Gamma_1 = \frac{3}{64}$, $\Gamma_2 = \frac{3}{32}$, $\Gamma_3 = 0$ and the regularisation factor $\mu = 0.77$. The bounds will be expressed in these units, and the p^6 contribution will be studied.

6.4 p^6 correction

We calculate the p^6 correction using the results from [2] with input from [13], and they are presented in parentheses after our results (see appendix B). We consider only the four most important planes in Table 8.

The larger M_{phys} is, the higher the p^6 contribution is. Moreover, the limits on l_i^r become more restrictive when the mass is increased. We would like now to find a mass range which

Table 8: Three flavours bounds for three different mass channels of the four main constraints.

Linear combination	$M_{phys} = M_\pi = 0.135$	$M_{phys} = M_K = 0.495$	$M_{phys} = M_\eta = 0.548$
$l_2^r + 1.86\ 10^{-3} l_1^r + 0.332 l_3^r \geq$	$-1.44.10^{-3} + (+8.29.10^{-4})$	$+1.03.10^{-4} + (+2.22)$	$+2.25.10^{-4} + (+4.12)$
$l_2^r + 2.505.10^{-3} l_1^r + 5.701.10^{-3} l_3^r \geq$	$-1.50.10^{-3} + (-3.63.10^{-3})$	$+4.04.10^{-5} + (+5.17)$	$+1.62.10^{-4} + (+11.0)$
$l_2^r + 0.5317 l_1^r + 0.1959 l_3^r \geq$	$-1.89.10^{-3} + (-3.83.10^{-3})$	$+6.02.10^{-5} + (+2.41)$	$+2.14.10^{-4} + (+5.79)$
$l_2^r + 0.9885 l_1^r + 0.4932 l_3^r \geq$	$-2.24.10^{-3} + (+1.43.10^{-4})$	$+6.79.10^{-5} + (+0.936)$	$+2.48.10^{-4} + (+1.84)$

would give us the most stringent constraints and, at the same time, have a reasonable error. If we want a correction lower than 10%, we need to fulfill the conditions presented in Table 9.

Table 9: Mass limit for 10% precision

Linear combination	$M_{phys} \leq M_{limit}$	Bounds at M_{limit}
$l_2^r + 1.86\ 10^{-3} l_1^r + 0.332 l_3^r$	$M_{phys} \leq 0.115$	$\geq -1.63.10^{-3}$
$l_2^r + 2.505\ 10^{-3} l_1^r + 5.701\ 10^{-3} l_3^r$	$M_{phys} \leq 0.0742$	$\geq -2.21.10^{-3}$
$l_2^r + 0.5317 l_1^r + 0.1959 l_3^r$	$M_{phys} \leq 0.0830$	$\geq -2.62.10^{-3}$
$l_2^r + 0.9885 l_1^r + 0.4932 l_3^r$	$M_{phys} \leq 0.143$	$\geq -2.14.10^{-3}$

Three of our planes are really close to the one of [4] (see Table 10). For example $l_2^r + 1.86\ 10^{-3} l_1^r + 0.332 l_3^r$, the first one in table 8, looks like $l_2^r + \frac{1}{3} l_3^r$. Then, we focus on the three channels we have in common with [4], and the results are available in Table 10.

Table 10: Three flavours bounds corrected for three different mass channels

Linear combination	Our results at $M_{phys} = M_\pi = 0.135$	Results from [4] at $M_{phys} = M_\pi = 0.135$
$l_2^r + \frac{1}{3} l_3^r \geq$	$-1.44.10^{-3} + (-1.11.10^{-3})$	$+1.21.10^{-3} + (\pm 6.68.10^{-5})$
$l_2^r \geq$	$-1.50.10^{-3} + (+3.49.10^{-3})$	$-1.30.10^{-3} + (\pm 2.00.10^{-4})$
$l_2^r + l_1^r + \frac{1}{2} l_3^r \geq$	$-2.25.10^{-3} + (-8.49.10^{-4})$	$-1.94.10^{-3} + (\pm 1.00.10^{-4})$

Our results are a little bit different from [4]. In particular, the correction is lower than our. These differences could come from the numerical values of the different parameters, like masses or the pion decay constant, which can be different between this work and [4]. Next, we have assumed that all the particles involve in the scattering process have the same mass, whereas in [4], this assumption is not made. Moreover, it has to be highlighted that we did not use the same expression of the p^6 correction than in [4]. This could explain the great differences in the errors.

Finally, from [12] we have:

$$
\begin{aligned}
l_1^r &= (9 \pm 3)\ 10^{-4} \\
l_2^r &= (1.7 \pm 0.7)\ 10^{-3} \\
l_3^r &= (-4.4 \pm 2.5)\ 10^{-3}
\end{aligned}
$$

We replace these values in the constraints presented in Table 8 and it shows that our results are consistent with [12], as we can see in Table 11.

Table 11: Comparison between Table 8 and [12]

$l_2^r + 1.86\ 10^{-3} l_1^r + 0.332 l_3^r \geq -1.63.10^{-3}$	0.00187 (±0.000163)
$l_2^r + 2.505\ 10^{-3} l_1^r + 5.701\ 10^{-3} l_3^r \geq -2.21.10^{-3}$	0.00256 (±0.000221)
$l_2^r + 0.5317 l_1^r + 0.1959 l_3^r \geq -2.62.10^{-3}$	0.00389 (±0.000262)
$l_2^r + 0.9885 l_1^r + 0.4932 l_3^r \geq -2.14.10^{-3}$	0.00394 (±0.000214)

7 The 4 flavours p^4 lagrangian

The following discussion is a numerical application, to the four flavours case, of chapter four and five.

7.1 Method

The general case has been treated in chapter five. Now the coefficients $\beta_0, \beta_1, \beta_2, \beta_3$ have to be chosen. In this section we impose:

$$\begin{aligned} \beta_0 &= 1 \\ \beta_1 &= 1 \\ \beta_2 &= 0 \\ \beta_3 &= 1 \end{aligned}$$

The allowed region is then described by:

$$\begin{aligned} 0 \le y \le \frac{1}{2} \\ 0 \le -\frac{1}{16}x + y \\ 0 \le z - 2y \\ 0 \le x \le \frac{8}{5} \\ 0 \le z \le 1 \\ 0 \le z - \frac{1}{16}x \\ 0 \le -\frac{1}{2}x - 2y + 1 \\ 0 \le -z - \frac{1}{2}x + 1 \\ 0 \le -3z - \frac{1}{2}x + 4y + 1 \end{aligned}$$

The same method as in the two and three flavours cases is repeated here. We randomly choose the values of x, y, z and we maximize $-G$ defined in Eq.(5.52). It is observed, but not proven, that the maximum is reached at $t = 4M^2$. In order to increase the speed of our algorithm, we then fix t to this value. Finally we use again the discussion of chapter four to extract the necessary hyperplanes.

7.2 Results

We have found 1096 hyperplanes but the algorithm did not have enough time to converge. In order to get an idea of what our results look like, we have chosen randomly $\bar{l}_0, \bar{l}_1$ and

$\bar{l}_3$ between -1000 and 1000 and kept only the relevant hyperplanes. Note that the order of magnitude of the $\bar{l}_i$ are much lower than 10^3, but the purpose here is to find the most important planes, independently of the values of the low energy constants .

$$\begin{aligned}
\bar{l}_2 &\geq 755 - 0.141\ \bar{l}_1 - 0.0308\ \bar{l}_3 - 0.0655\ \bar{l}_0 && (7.68)\\
\bar{l}_2 &\geq 1.22\ 10^{-4} - 1.899\ \bar{l}_1 - 0.155\ \bar{l}_3 - 0.322\ \bar{l}_0 && (7.69)\\
\bar{l}_2 &\geq 1.21\ 10^{-4} - 1.83\ \bar{l}_1 - 0.121\ \bar{l}_3 - 0.263\ \bar{l}_0 && (7.70)\\
\bar{l}_2 &\geq 1.23\ 10^{-4} - 1.83\ \bar{l}_1 - 0.121\ \bar{l}_3 - 0.257\ \bar{l}_0 && (7.71)\\
\bar{l}_2 &\geq 9.57\ 10^{-5} - 0.610\ \bar{l}_1 - 5.51\ \bar{l}_3 - 11.2\ \bar{l}_0 && (7.72)\\
\bar{l}_2 &\geq 1.21\ 10^{-4} - 1.77\ \bar{l}_1 - 1.18\ \bar{l}_3 - 2.38\ \bar{l}_0 && (7.73)\\
\bar{l}_2 &\geq 1.22\ 10^{-4} - 1.98\ \bar{l}_1 - 0.395\ \bar{l}_3 - 0.790\ \bar{l}_0 && (7.74)
\end{aligned}$$

The hyperplanes (7.68) seems wrong as the order of magnitude of the first term, which is the maximum of $-G$, is huge with respect to the ones of the other constraints. If we remove this bound and repeat the procedure we get 29 hyperplanes, upon which 7 are enough to constraint 96% of the tested points:

$$\begin{aligned}
\bar{l}_2 &\geq 9.83\ 10^{-5} - 0.0571\ \bar{l}_1 - 6.06\ 10^{-3}\ \bar{l}_3 - 1.54\ 10^{-2}\ \bar{l}_0 && (7.75)\\
\bar{l}_2 &\geq 1.08\ 10^{-4} - 0.864\ \bar{l}_1 - 0.198\ \bar{l}_3 - 0.505\ \bar{l}_0 && (7.76)\\
\bar{l}_2 &\geq 1.11\ 10^{-4} - 0.829\ \bar{l}_1 - 0.0851\ \bar{l}_3 - 0.223\ \bar{l}_0 && (7.77)\\
\bar{l}_2 &\geq 8.84\ 10^{-5} - 0.219\ \bar{l}_1 - 0.0171\ \bar{l}_3 - 0.181\ \bar{l}_0 && (7.78)\\
\bar{l}_2 &\geq 1.10\ 10^{-4} - 0.972\ \bar{l}_1 - 0.377\ \bar{l}_3 - 0.880\ \bar{l}_0 && (7.79)\\
\bar{l}_2 &\geq 9.44\ 10^{-5} - 1.05\ \bar{l}_1 - 0.0940\ \bar{l}_3 - 0.400\ \bar{l}_0 && (7.80)\\
\bar{l}_2 &\geq 1.11\ 10^{-4} - 1.34\ \bar{l}_1 - 0.147\ \bar{l}_3 - 0.406\ \bar{l}_0 && (7.81)
\end{aligned}$$

We would like to highlight again that these results are not representative of our study, and that we have to wait for the program to converge. Moreover, the error analysis is yet to be done.

8 Conclusion and open problems

We managed to find better bounds on the coupling constants than in [3] and complementary one with [4]. Moreover, using [2],[10], [11] and [13], p^6 corrections have been derived. The SU(2) case seems to have 3 important constraints and the SU(3) case, 4. For the 3 flavours case, mass limits, which were derive imposing that the p^6 corrections do not exceed 10% of the p^4 contributions, have been derived for the 4 main results. Nevertheless, a program based on a proposition proven in chapter 4 suggests that many more bounds exist. Finally, a general method have been establish to study the general N_f flavours case at p^4, new necessary constraints on the generalised slopes have been found and it was applied for SU(4). However, some questions were not treated during this thesis and they could be a good starting point to a deeper study.

First, it has been observed that the maximum always occurred for $t = 4M^2$. We believe that this can be proven, using analyticity for example. This could reduce the complexity of the studies and increase the speed of the search for maxima.

As we saw, for example in the SU(3) results in appendix B, that the program that selects the best results is too sensitive as it looks for exact solutions. Moreover, it is not optimized, and converges because the set of points treated were around 1000, which is small. Studying high dimension approximated algorithms for the search of convex hulls seems to be a possible way to overcome those difficulties. This treatment is a problematic issue when a lot of coupling constants are required.

The p^6 case has been treated as a correcting term, to check the validity of results. One might want to repeat the exact same study in this case. Note that, in this case, " the slopes " depend on s and t. So it is harder to produce the same type of strategy as in chapter 5, where we could go from a unbounded problem to a bounded one. A new strategy has then to be found. Moreover, much more coupling constants will be required, which could be a problematic issue.

A Notations of the chapter four

In this section, E is a real vector space of finite dimension.

$\forall$: Mathematical notation which stands for: " for all".

$\exists$: Mathematical notation which stands for: " it exists".

$\in$: Mathematical notation which stands for: " in" or " belong to".

$\emptyset$: Mathematical notation which stands for: "the empty space".

!: Mathematical notation which stands for: "unique".

$\langle . | . \rangle$: This notation stands for the usual scalar product.

$|F$: Let G and F be to set included in E. Then:

$$G_{|F} = \{g \in G \text{ and } g \notin F\}$$

$\bigcap_{i=1}^{n}$: Let us consider n subspaces E_i of E. Then:

$$\bigcap_{i=1}^{n} E_i = E_1 \cap E_2 \cap E_3 \cap \cdots \cap E_{n-1} \cap E_n$$

$\bigoplus$: Direct sum. Let us take E, F and G three real vector spaces. Then:

$$G = E \bigoplus F, \text{ if and only if } \forall z \in G, \exists! x \in E \text{ and } y \in F \text{ such that } z = x + y$$

Linear functional: A linear functional f is a linear map on E such that:

$$\forall x, y \in E \text{ and } \lambda, \mu \in \mathbb{R}, f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$$

Kernel: Let f be a a linear functional on E. We define the kernel of f, N_f, as the subspace of E such as:

$$N_f = \{v \in E, f(v) = 0\}$$

B SU(3) Results

Table 12: All the results for the three flavours case and for three different mass channels

Linear combination	$M_{phys} = M_\pi = 0.135$	$M_{phys} = M_K = 0.495$	$M_{phys} = M_\eta = 0.548$
$l_2^r + 0.100 l_3^r \geq$	$-1.45\ 10^{-3} + (+7.12\ 10^{-3})$	$+9.39\ 10^{-5} + (-0.137)$	$2.16\ 10^{-4} + (-2.47)$
$l_2^r + 9.64\ 10^{-4} l_1^r + 0.208 l_3^r \geq$	$-1.39\ 10^{-3} + (+1.11\ 10^{-2})$	$+1.54\ 10^{-4} + (+5.61)$	$2.28\ 10^{-4} + (+7.29)$
$l_2^r + 1.84\ 10^{-3} l_1^r + 0.295 l_3^r \geq$	$-1.41\ 10^{-3} + (+5.28\ 10^{-3})$	$+1.37\ 10^{-4} + (+2.34)$	$2.26\ 10^{-4} + (+2.78)$
$l_2^r + 1.86\ 10^{-3} l_1^r + 0.332 l_3^r \geq$	$-1.44\ 10^{-3} + (+8.29\ 10^{-4})$	$+1.03\ 10^{-4} + (+2.22)$	$2.25\ 10^{-4} + (+4.12)$
$l_2^r + 2.62\ 10^{-3} l_1^r + 5.61\ 10^{-2} l_3^r \geq$	$-1.47\ 10^{-3} + (+5.50\ 10^{-3})$	$+6.93\ 10^{-5} + (-2.48)$	$1.91\ 10^{-4} + (-6.46)$
$l_2^r + 2.50\ 10^{-3} l_1^r + 5.70 l_3^r \geq$	$-1.50\ 10^{-3} + (+3.63\ 10^{-3})$	$+4.04\ 10^{-5} + (-5.17)$	$1.62\ 10^{-4} + (-11.0)$
$l_2^r + 6.65\ 10^{-3} l_1^r + 0.256 l_3^r \geq$	$-1.37\ 10^{-3} + (+1.20\ 10^{-2})$	$+1.73\ 10^{-4} + (+7.35)$	$2.95\ 10^{-4} + (+10.4)$
$l_2^r + 7.15\ 10^{-3} l_1^r + 2.13\ 10^{-2} l_3^r \geq$	$-1.50\ 10^{-3} + (+4.21\ 10^{-3})$	$+4.98\ 10^{-5} + (-4.35)$	$1.72\ 10^{-4} + (-9.64)$
$l_2^r + 9.19\ 10^{-3} l_1^r + 0.307 l_3^r \geq$	$-1.42\ 10^{-3} + (+3.48\ 10^{-3})$	$+1.27\ 10^{-4} + (+0.994)$	$2.50\ 10^{-4} + (+0.736)$
$l_2^r + 1.73\ 10^{-2} l_1^r + 0.237 l_3^r \geq$	$-1.39\ 10^{-3} + (+1.21\ 10^{-2})$	$+1.70\ 10^{-4} + (+7.07)$	$2.92\ 10^{-4} + (+9.79)$
$l_2^r + 1.63\ 10^{-2} l_1^r + 0.317 l_3^r \geq$	$-1.43\ 10^{-3} + (+8.29\ 10^{-4})$	$+1.20\ 10^{-4} + (-0.127)$	$2.42\ 10^{-4} + (-0.960)$
$l_2^r + 6.53\ 10^{-2} l_1^r + 2.55\ 10^{-2} l_3^r \geq$	$-1.55\ 10^{-3} + (+3.43\ 10^{-3})$	$+3.39\ 10^{-5} + (-5.10)$	$1.65\ 10^{-4} + (-10.8)$
$l_2^r + 8.08\ 10^{-2} l_1^r + 3.98\ 10^{-2} l_3^r \geq$	$-1.55\ 10^{-3} + (+4.16\ 10^{-3})$	$+5.74\ 10^{-5} + (-4.18)$	$1.84\ 10^{-4} + (-9.28)$
$l_2^r + 0.113 l_1^r + 0.263 l_3^r \geq$	$-1.45\ 10^{-3} + (+1.25\ 10^{-2})$	$+1.80\ 10^{-4} + (7.68)$	$3.09\ 10^{-4} + (10.9)$
$l_2^r + 0.103 l_1^r + 5.34\ 10^{-2} l_3^r \geq$	$-1.56\ 10^{-3} + (+4.80\ 10^{-3})$	$+6.41\ 10^{-5} + (-3.38)$	$1.92\ 10^{-4} + (-7.95)$
$l_2^r + 0.124 l_1^r + 0.247 l_3^r \geq$	$-1.47\ 10^{-3} + (+1.12\ 10^{-2})$	$+1.71\ 10^{-4} + (+6.77)$	$3.01\ 10^{-4} + (+9.33)$
$l_2^r + 0.176 l_1^r + 6.57\ 10^{-2} l_3^r \geq$	$-1.61\ 10^{-3} + (+4.03\ 10^{-3})$	$+6.82\ 10^{-5} + (-3.98)$	$2.01\ 10^{-4} + (-8.45)$

$l_2^r + 0.244 l_1^r + 0.281 l_3^r \geq$	$-1.54\ 10^{-3} + (+1.22\ 10^{-2})$	$+1.85\ 10^{-4} + (7.55)$	$3.22\ 10^{-4} + (+10.72)$
$l_2^r + 0.305 l_1^r + 0.287 l_3^r \geq$	$-1.59\ 10^{-3} + (+6.77\ 10^{-3})$	$+1.87\ 10^{-4} + (2.69)$	$3.27\ 10^{-4} + (+3.06)$
$l_2^r + 0.336 l_1^r + 0.121 l_3^r \geq$	$-1.71\ 10^{-3} + (+5.21\ 10^{-3})$	$+9.30\ 10^{-5} + (-2.17)$	$2.35\ 10^{-4} + (-5.71)$
$l_2^r + 0.405 l_1^r + 0.153 l_3^r \geq$	$-1.75\ 10^{-3} + (+5.01\ 10^{-3})$	$+1.07\ 10^{-4} + (-1.84)$	$2.54\ 10^{-4} + (-5.02)$
$l_2^r + 0.403 l_1^r + 0.150 l_3^r \geq$	$-1.75\ 10^{-3} + (+2.89\ 10^{-3})$	$+1.03\ 10^{-4} + (-3.89)$	$2.49\ 10^{-4} + (-8.34)$
$l_2^r + 0.369 l_1^r + 0.127 l_3^r \geq$	$-1.79\ 10^{-3} + (+2.87\ 10^{-3})$	$+3.71\ 10^{-5} + (-4.29)$	$1.81\ 10^{-4} + (-9, 10)$
$l_2^r + 0.415 l_1^r + 0.149 l_3^r \geq$	$-1.54\ 10^{-3} + (+2.90\ 10^{-3})$	$+8.92\ 10^{-5} + (-3.83)$	$2.36\ 10^{-4} + (-8.22)$
$l_2^r + 0.445 l_1^r + 0.401 l_3^r \geq$	$-1.78\ 10^{-3} + (-1.25\ 10^{-3})$	$+1.06\ 10^{-4} + (-2.14)$	$2.54\ 10^{-4} + (-3.75)$
$l_2^r + 0.445 l_1^r + 0.406 l_3^r \geq$	$-1.80\ 10^{-3} + (-8.01\ 10^{-4})$	$+8.50\ 10^{-5} + (-1.84)$	$2.34\ 10^{-4} + (-3.39)$
$l_2^r + 0.558 l_1^r + 0.320 l_3^r \geq$	$-1.78\ 10^{-3} + (+9.04\ 10^{-3})$	$+1.96\ 10^{-4} + (+4.81)$	$3.51\ 10^{-4} + (+6.48)$
$l_2^r + 0.482 l_1^r + 0.175 l_3^r \geq$	$-1.84\ 10^{-3} + (+4.00\ 10^{-3})$	$+7.05\ 10^{-5} + (-2.54)$	$2.21\ 10^{-4} + (-6.08)$
$l_2^r + 0.572 l_1^r + 0.286 l_3^r \geq$	$-1.81\ 10^{-3} + (+8.06\ 10^{-3})$	$+1.76\ 10^{-4} + (+3.23)$	$3.33\ 10^{-4} + (+3.76)$
$l_2^r + 0.533 l_1^r + 0.317 l_3^r \geq$	$-1.76\ 10^{-3} + (+1.15\ 10^{-2})$	$+1.95\ 10^{-4} + (+7.00)$	$3.49\ 10^{-4} + (+9.95)$
$l_2^r + 0.510 l_1^r + 0.226 l_3^r \geq$	$-1.79\ 10^{-3} + (+8.73\ 10^{-3})$	$+1.45\ 10^{-4} + (+2.71)$	$2.98\ 10^{-4} + (+2.59)$
$l_2^r + 0.579 l_1^r + 0.320 l_3^r \geq$	$-1.79\ 10^{-3} + (+1.13\ 10^{-2})$	$+1.95\ 10^{-4} + (6.79)$	$3.52\ 10^{-4} + (+9.60)$
$l_2^r + 0.471 l_1^r + 0.164 l_3^r \geq$	$-1.87\ 10^{-3} + (+3.75\ 10^{-3})$	$+3.73\ 10^{-5} + (-2.98)$	$1.88\ 10^{-4} + (-6.85)$
$l_2^r + 0.612 l_1^r + 0.326 l_3^r \geq$	$-1.82\ 10^{-3} + (+1.12\ 10^{-2})$	$+1.97\ 10^{-4} + (+6.76)$	$3.56\ 10^{-4} + (+9.58)$
$l_2^r + 0.627 l_1^r + 0.327 l_3^r \geq$	$-1.83\ 10^{-3} + (+1.12\ 10^{-2})$	$+1.97\ 10^{-4} + (+6.67)$	$3.57\ 10^{-4} + (+9.44)$
$l_2^r + 0.654 l_1^r + 0.326 l_3^r \geq$	$-1.85\ 10^{-3} + (+1.09\ 10^{-2})$	$+1.95\ 10^{-4} + (+6.35)$	$3.57\ 10^{-4} + (+8.91)$
$l_2^r + 0.532 l_1^r + 0.196 l_3^r \geq$	$-1.89\ 10^{-3} + (+3.83\ 10^{-3})$	$+6.02\ 10^{-5} + (-2.41)$	$2.14\ 10^{-4} + (-5.79)$
$l_2^r + 0.642 l_1^r + 0.305 l_3^r \geq$	$-1.87\ 10^{-3} + (+9.55\ 10^{-3})$	$+1.67\ 10^{-4} + (+4.74)$	$3.28\ 10^{-4} + (+6.22)$
$l_2^r + 0.923 l_1^r + 0.478 l_3^r \geq$	$-2.17\ 10^{-3} + (+1.91\ 10^{-3})$	$+7.89\ 10^{-5} + (+0.811)$	$2.57\ 10^{-4} + (+0.943)$

$l_2^r + 0.611 l_1^r + 0.402 l_3^r \geq$	$-1.88\ 10^{-3} + (-7.50\ 10^{-4})$	$+1.29\ 10^{-4} + (-2.10)$	$2.88\ 10^{-4} + (-3.80)$
$l_2^r + 0.683 l_1^r + 0.341 l_3^r \geq$	$-1.88\ 10^{-3} + (+1.07\ 10^{-2})$	$+1.93\ 10^{-4} + (+6.44)$	$3.56\ 10^{-4} + (+9.13)$
$l_2^r + 0.659 l_1^r + 0.371 l_3^r \geq$	$-1.89\ 10^{-3} + (-6.08\ 10^{-3})$	$+1.64\ 10^{-4} + (-2.10)$	$3.26\ 10^{-4} + (-3.80)$
$l_2^r + 0.665 l_1^r + 0.396 l_3^r \geq$	$-1.91\ 10^{-3} + (-1.40\ 10^{-3})$	$+1.42\ 10^{-4} + (-0.47)$	$3.04\ 10^{-4} + (-1.31)$
$l_2^r + 0.663 l_1^r + 0.405 l_3^r \geq$	$-1.92\ 10^{-3} + (-1.33\ 10^{-4})$	$+1.34\ 10^{-4} + (-1.63)$	$2.96\ 10^{-4} + (-3.07)$
$l_2^r + 0.655 l_1^r + 0.418 l_3^r \geq$	$-1.93\ 10^{-3} + (+1.62\ 10^{-3})$	$+1.21\ 10^{-4} + (+0.263)$	$2.82\ 10^{-4} + (+3.62\ 10^{-2})$
$l_2^r + 0.616 l_1^r + 0.245 l_3^r \geq$	$-1.95\ 10^{-3} + (+4.45\ 10^{-3})$	$+7.13\ 10^{-5} + (-1.07)$	$2.30\ 10^{-4} + (-3.40)$
$l_2^r + 0.731 l_1^r + 0.356 l_3^r \geq$	$-1.93\ 10^{-3} + (+6.83\ 10^{-3})$	$+1.77\ 10^{-4} + (+3.13)$	$3.43\ 10^{-4} + (+3.94)$
$l_2^r + 0.632 l_1^r + 0.255 l_3^r \geq$	$-1.96\ 10^{-3} + (+4.62\ 10^{-4})$	$+7.42\ 10^{-5} + (-0.756$	$2.35\ 10^{-4} + (-2.86)$
$l_2^r + 0.632 l_1^r + 0.255 l_3^r \geq$	$-1.96\ 10^{-3} + (+4.62\ 10^{-3})$	$+7.42\ 10^{-5} + (-0.754)$	$2.34\ 10^{-4} + (-2.86)$
$l_2^r + 0.764 l_1^r + 0.359 l_3^r \geq$	$-1.98\ 10^{-3} + (+8.59\ 10^{-3})$	$+1.49\ 10^{-4} + (+4.64)$	$3.17\ 10^{-4} + (+6.30)$
$l_2^r + 0.801 l_1^r + 0.386 l_3^r \geq$	$-2.00\ 10^{-3} + (+4.22\ 10^{-3})$	$+1.57\ 10^{-4} + (+1.23)$	$3.28\ 10^{-4} + (+1.06)$
$l_2^r + 0.745 l_1^r + 0.331 l_3^r \geq$	$-2.01\ 10^{-3} + (+6.50\ 10^{-3})$	$+1.09\ 10^{-4} + (+2.20)$	$2.76\ 10^{-4} + (+2.24)$
$l_2^r + 0.745 l_1^r + 0.331 l_3^r \geq$	$-2.01\ 10^{-3} + (+6.50\ 10^{-3})$	$+1.09\ 10^{-4} + (+2.20)$	$2.76\ 10^{-4} + (+2.24)$
$l_2^r + 0.795 l_1^r + 0.374 l_3^r \geq$	$-2.01\ 10^{-3} + (+7.89\ 10^{-3})$	$+1.44\ 10^{-4} + (+4.22)$	$3.14\ 10^{-4} + (+5.70)$
$l_2^r + 0.826 l_1^r + 0.387 l_3^r \geq$	$-2.04\ 10^{-3} + (+7.07\ 10^{-3})$	$+1.38\ 10^{-4} + (4.14)$	$3.10\ 10^{-4} + (+5.79)$
$l_2^r + 0.844 l_1^r + 0.410 l_3^r \geq$	$-2.06\ 10^{-3} + (+7.07\ 10^{-3})$	$+1.37\ 10^{-4} + (+4.14)$	$3.10\ 10^{-4} + (+5.79)$
$l_2^r + 0.778 l_1^r + 0.462 l_3^r \geq$	$-2.07\ 10^{-3} + (-7.03\ 10^{-4})$	$+7.17\ 10^{-5} + (-1.51)$	$2.41\ 10^{-4} + (-2.75)$
$l_2^r + 0.839 l_1^r + 0.455 l_3^r \geq$	$-2.09\ 10^{-3} + (-7.01\ 10^{-4})$	$+9.62\ 10^{-5} + (-1.68)$	$2.69\ 10^{-4} + (-3.04)$
$l_2^r + 0.965 l_1^r + 0.479 l_3^r \geq$	$-2.21\ 10^{-3} + (+9,46\ 10^{-4})$	$+7.89\ 10^{-5} + (-0.178)$	$2.59\ 10^{-4} + (-0.676)$
$l_2^r + 0.966 l_1^r + 0.481 l_3^r \geq$	$-2.21\ 10^{-3} + (-6.77\ 10^{-4})$	$+7.73\ 10^{-5} + (-0.376)$	$2.58\ 10^{-4} + (-0.978)$

$l_2^r + 0.962 l_1^r + 0.487 l_3^r \geq$	$-2.21\ 10^{-3} + (-3.54\ 10^{-4})$	$+7.20\ 10^{-5} + (-1.14)$	$2.52\ 10^{-4} + (-2.14)$
$l_2^r + 0.984 l_1^r + 0.494 l_3^r \geq$	$-2.23\ 10^{-3} + (-2.40\ 10^{-4})$	$+6.51\ 10^{-5} + (-0.989)$	$2.47\ 10^{-4} + (-1.89)$
$l_2^r + 0.988 l_1^r + 0.493 l_3^r \geq$	$-2.23\ 10^{-3} + (+1.02\ 10^{-3})$	$+6.79\ 10^{-5} + (+0.0928)$	$2.48\ 10^{-4} + (-0.188)$
$l_2^r + 0.0477 l_1^r + 0.254 l_3^r \geq$	$-1.41\ 10^{-3} + (+1.26\ 10^{-2})$	$+1.78\ 10^{-4} + (+7.72)$	$3.02\ 10^{-4} + (+10.9)$
$l_2^r + 0.183 l_1^r + 0.356 l_3^r \geq$	$-1.57\ 10^{-3} + (+1.48\ 10^{-4})$	$+1.08\ 10^{-4} + (-1.34)$	$2.41\ 10^{-4} + (-2.73)$
$l_2^r + 0.367 l_1^r + 0.387 l_3^r \geq$	$-1.72\ 10^{-3} + (+1.96\ 10^{-4})$	$+1.07\ 10^{-4} + (-1.17)$	$2.51\ 10^{-4} + (-2.40)$
$l_2^r + 0.413 l_1^r + 0.161 l_3^r \geq$	$-1.70\ 10^{-3} + (+6.73\ 10^{-3})$	$+1.13\ 10^{-4} + (-0.156)$	$2.60\ 10^{-4} + (-2.31)$
$l_2^r + 0.643 l_1^r + 0.331 l_3^r \geq$	$-1.84\ 10^{-3} + (+1.11\ 10^{-2})$	$+1.98\ 10^{-4} + (+6.67)$	$3.58\ 10^{-4} + (+9.46)$
$l_2^r + 0.621 l_1^r + 0.289 l_3^r \geq$	$-1.84\ 10^{-3} + (+1.68\ 10^{-2})$	$+1.52\ 10^{-4} + (+10.8)$	$3.11\ 10^{-4} + (+15.74)$
$l_2^r + 0.675 l_1^r + 0.403 l_3^r \geq$	$-1.93\ 10^{-3} + (+6.60\ 10^{-4})$	$+1.37\ 10^{-4} + (-1.01)$	$3.00\ 10^{-4} + (-2.13)$
$l_2^r + 0.677 l_1^r + 0.495 l_3^r \geq$	$-1.96\ 10^{-3} + (-1.49\ 10^{-3})$	$+1.07\ 10^{-4} + (-2.16)$	$3.02\ 10^{-4} + (-3.68)$
$l_2^r + 0.455 l_1^r + 0.307 l_3^r \geq$	$-1.70\ 10^{-3} + (7.13\ 10^{-3})$	$+1.92\ 10^{-4} + (+3.09)$	$3.41\ 10^{-4} + (+3.73)$
$l_2^r + 0.342 l_1^r + 0.279 l_3^r \geq$	$-1.62\ 10^{-3} + (6.90\ 10^{-3})$	$+1.81\ 10^{-4} + (+2.57)$	$3.24\ 10^{-4} + (+2.79)$

As we see, some planes are really close to each other and, sometime, it is difficult to distinguish them. This shows us that our program is too sensitive.

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