# A mean-variance portfolio optimizing trading algorithm using regime-switching economic parameters

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**Abstract**

In this master's thesis a model of algorithmic trading is constructed. The model aims to create an optimal investment portfolio consisting of a risk-free asset and a risky asset. The risky asset is in the form of a stock generated using regime-switching parameters with a Markov chain explaining the state of the economy. The optimization of the portfolio is carried out under certain assumptions and reasonable constraints on risk, transaction costs and amount traded. The constraint on financial risk is implemented through the recognized mean-variance criterion, balancing the expected value of the portfolio against the variance of the portfolio after every time period. The algorithm is implemented using quadratic programming techniques in Matlab. By varying parameters of the model a sensitivity analysis is performed. Simulated scenarios and the behaviour of the algorithm is presented in graphs. The algorithm is found to be rational and outperforms a static portfolio in every scenario.

***Keywords***: *financial engineering, algorithmic trading, portfolio optimization, mean-variance criterion, regime-switching, quadratic programming, hidden markov model*

# Contents

# Chapter 1

# Introduction

In modern financial markets mathematical models are used to arrive at, and execute, investment decisions. Such automated technologies are referred to as algorithmic trading. During recent years there has been increased interest in algorithmic trading and large institutional investors tend to use algorithmic trading technologies to e.g. sell large blocks of shares in small tranches.

In this thesis a trading algorithm model have been constructed. The algorithm aims to construct an optimal investment portfolio, i.e. the mathematically optimal allocation among assets. An optimal investment portfolio is a loose term. It is simply a choice of allocations combining the risk-free asset (bank account, $B$) and the risky asset (stock, $S$), given certain assumptions and necessary conditions. [Dombrovskii and Obyedko, 2015] state that the portfolio management problem, with stochastic parameters under constraints, is the key problem of financial engineering.

## 1.1 Purpose

The purpose of this thesis is to investigate whether the trading algorithm's resulting portfolio outperforms a static portfolio, i.e. a portfolio with predetermined weights allocated in the risky asset as well as the risk-free asset. The static portfolio is often used by a pension fund, as mentioned in

[McNeil et al., 2005], where the time horizon is long and the end time often known. This thesis' results are particularly applicable at a pension fund since it has a large responsibility to it's investors.

The constructed algorithm is meant to be managed and updated regularly when new information is gathered. For some preferences it might need harder constraints in order to achieve a desirable result. By continuously carrying out the optimization after every time period all information can be incorporated and the planned trading behaviour updated.

## 1.2 Methodology

### 1.2.1 Trading algorithm

The constructed algorithm builds on a portfolio optimization model that uses predicted values of the stock price, a certain number ($k$) of time-periods ahead, as it's input. It aims to maximize the value of a portfolio at the end of a given time horizon. The algorithm is best used if updated after every time-period, leading to a new planned trading strategy after every time period. The maximization of the portfolio is carried out under certain constraints on risk, transaction costs and amount allowed to be traded every day.

The constraints on risk are implemented through the mean-variance criterion (will be referred to as MVC occasionally) and determined by the risk-aversion of the investor, i.e. how the investor values risk in terms of expected future wealth. Recently several studies have been conducted in the subject of mean-variance optimization problems, see e.g. [Li and Ng, 2000]. In [Costa and Araujo, 2008] the authors express that the main advantage of the mean-variance criterion is that it has a simple and clear interpretation in terms of individual portfolio choice and utility optimization. The model of this thesis uses an intermediate mean-variance criterion as well, i.e. the expected value of the portfolio is balanced against the variance after every time period, which was introduced by [Costa and Nabholz, 2007].

### 1.2.2 Stock values

The predicted values of the stock price will be generated using a regime-switching strategy, i.e. the stock evolves differently depending on the regime, or the state, of an economy. The regime-switching model should be interpreted as a momentum process where it is more likely to continue in the same state than to transition into another. The Hidden Markov Model (HMM), introduced in [Nystrup et al., 2015], incorporates new information as it is available and evaluates what state the economy is in. By analyzing the behaviour of the risky asset the algorithm aims to take advantage of favourable economic regimes, withstand adverse economic regimes and reduce potential drawdowns. This type of strategy is suitable for non-stationary environments, which is usually the case for financial time series.

## 1.3 Challenges and performance

Because of the advanced nature of financial markets there are, of course, limitations to the developed model. When creating models describing financial data one has to make assumptions, often very strong assumptions at that. As a result, the final algorithm is not without faults. Depending on constraints, together with transaction costs, the optimal allocation is found to vary significantly. If the constraints are not strong enough one might assume that the optimal portfolio is allocated 100% in the risky asset. This is due to the assumption that over time the stock market converges to a growing trend. However, due to risk aversion, transaction costs and beforehand determined amount allowed to be traded this should not be the result. An issue to keep in mind is that there is no “general” risky asset as used in our model. There will always be company specific risk tied to a specific stock.

Due to constraints and risk measures the algorithm shows to be rational and it outperforms the static portfolio in every scenario. By applying the results of this thesis to real-time data even more reliable stock values could be generated. If the algorithm would still outperform the static portfolio is left for further investigation.

## 1.4 Contribution

The main contribution of this thesis is the constructed trading algorithm. The behaviour and performance of the algorithm when applied to simulated stock price values are the product of this thesis. The main focus of the works of this thesis is on the mathematical optimization structure rather than the statistical time-series model. To the knowledge of the author this type of portfolio optimization using these constraints have not been studied before.

The representation of the portfolio value is uniquely derived by the author using basic financial knowledge and mathematical simplifications. The generation of stock price values are not unique in itself, yet uniquely implemented in the portfolio optimization carried out in this thesis.

In [Li and Ng, 2000] a multiperiod mean-variance portfolio optimization is solved. The authors use several available risky assets and they constrain the variance by implementing an upper bound on variance of the portfolio wealth. In contrast, this thesis only have one asset available and minimize variance of the portfolio rather than constraining it. In [Zhou and Yin, 2003] a continuous mean-variance portfolio is determined. They do however not restrict non-negative wealth nor include transaction costs. In [Li et al., 2014] and [Celikyurt and Özekici, 2007] the MV efficient frontier is presented using dynamic programming, recursively updating the mean and variance after every time period. Neither [Li et al., 2014] or [Celikyurt and Özekici, 2007] include transaction costs. In this thesis static quadratic programming is used to optimize the portfolio leading to a simpler notation of portfolio evolution. None of the above mentioned papers include the intermediate mean-variance criterion making this thesis' contribution clear in that regard. However, the authors of [Costa and Nabholz, 2007] do consider the intermediete mean and variance, but do not include any other constraints or transaction costs.

In this thesis a trading algorithm, where portfolio optimization is carried out using the mean-variance criterion, is constructed. The author of this thesis implement an intermediate mean-variance criterion through minimizing the variance part of the goal function. The remaining part of the goal function is to maximize the end value of a portfolio, as well as intermediate excess returns. The optimization is carried out using quadratic programming principles, under constraints on amount allowed to be traded and subject to transaction costs. Stock price values are used as input to the portfolio value and are generated using a regime switching strategy.

## 1.5 Outline

This thesis is organized as follows. In Chapter 2, Section 2.1, the theoretical formulations of the algorithm, stock generations as well as constraints are derived. Then in Section 2.2 the implementation in Matlab is explained, the optimization problems are derived and choices are motivated. In Section 2.3 a sensitivity analysis of the optimal solutions to the problems is explained. The solutions are created by varying important parameters of the model. Moving on to Chapter 3 the results of the sensitivity analysis of the optimal portfolios are presented and explained. Resulting graphs showing the behaviour of the algortithm in different scenarios are also presented. In Chapter 4 the results of the sensitivity analysis are discussed and conclusions are provided. Future studies and improvements of this thesis are also proposed and discussed in this chapter.

# Chapter 2

# Method

This chapter provides the theoretical basis of which this thesis is built on. It also explains how the theory is implemented and how the sensitivity analysis is performed. In the context of the purpose of this thesis this chapter aims to provide the building blocks of the constructed trading algorithm.

The following chapters presents the results of the sensitivity analysis as well as discuss the outcomes and choices of this thesis.

## 2.1 Theory

In this section the theoretical basis of which this thesis is based on is explained and formulations as well as equations are derived. In order to understand how the trading algorithm is constructed and under which constraints this section needs to be understood.

### 2.1.1 Trading algorithm

In order to optimize a portfolio the representation of portfolio value need to be derived. In this section the evolution of the portfolio is derived and presented. As mentioned in the introduction, the value of the portfolio is comprised of the value of assets in the bank account and the value of assets in the stock. The portfolio is constructed to be self-financing. This means there is no exogenous infusion or withdrawal of money, i.e a purchase of a stock has to be financed by the current amount available in the bank account, and the sale of a stock results in more wealth in the bank account accordingly. This in turn lead to short selling of a stock not being allowed. The evolution of the portfolio value is presented in Equation (2.1) and Figure 2.1 below.

$$\begin{aligned}
V_t &= \alpha_t B_t + \beta_t S_t \\
&\downarrow \textbf{Evolve} \\
V_{t+1}^- &= \alpha_t B_{t+1} + \beta_t S_{t+1} \\
&\downarrow \textbf{Rebalance} \\
V_{t+1}^+ &= \alpha_{t+1} B_{t+1} + \beta_{t+1} S_{t+1} \\
&= \alpha_t B_{t+1} + \beta_t S_{t+1} - \lambda S_{t+1} |\beta_{t+1} - \beta_t| \\
&\downarrow \textbf{Evolve} \\
V_{t+2}^- &= \alpha_{t+1} B_{t+2} + \beta_{t+1} S_{t+2} \\
&\text{And so on...}
\end{aligned} \tag{2.1}$$


Figure 2.1: The evolution of the portfolio value presented in Equation (2.1) above.

Here $V$ is the total value of the portfolio, $\alpha$ the amount invested in the bank account, $\beta$ the number of stocks, $B_t = 1$, $B_{t+k} = B_t e^{kr_f}$, $S_t$ is the value of the stock at time $t$ and $\lambda$ is the cost factor representing the transaction cost of a trade. The reader should notice that the portfolio weights are described in absolute terms as opposed to relative. This is due to the fact that the practical interpretation is easier to understand and that the relative portfolio weights could result in singularities in the algorithm. The amount of wealth reallocated to, or from, the stock is represented by $S_{t+1}|\beta_{t+1} - \beta_t|$. The process can be described in words as follows:

*An initial portfolio is determined – the value of the stock (S) is evolved over one period – depending on the expected evolution of the stock, the portfolio is rebalanced. (The value of the portfolio is here reduced by the amount of the transaction cost) – the value of the stock is evolved over yet another period changing the value of the portfolio. The process is then repeated over several periods.*

#### Mathematical representation

Now that the evolution of the portfolio has been presented, a mathematical representation of the process is derived. As mentioned in the introduction, the goal function of this thesis is to maximize the expected value of the portfolio at the end time. To represent the value of the portfolio an expression for future $\alpha$:s is derived in Equation (2.2) below. From Equation (2.1) above we have that:

$$\begin{aligned}
\alpha_{t+1}B_{t+1} &= -\beta_{t+1}S_{t+1} + \alpha_t B_{t+1} + \beta_t S_{t+1} - \lambda S_{t+1}|\beta_{t+1} - \beta_t| \\
&\textbf{And} \\
\alpha_{t+2}B_{t+2} &= \alpha_{t+1}B_{t+2} - S_{t+2}(\beta_{t+2} - \beta_{t+1}) - \lambda S_{t+2}|\beta_{t+2} - \beta_{t+1}| \\
&\textbf{Where} \\
B_{t+2} &= B_{t+1}e^{r_f} \\
&\textbf{Inserting the first row:} \\
\alpha_{t+2}B_{t+2} &= e^{r_f}[\alpha_t B_{t+1} - S_{t+1}(\beta_{t+1} - \beta_t) -_{t+1} |\beta_{t+1} - \beta_t|] \\
&\quad - S_{t+2}(\beta_{t+2} - \beta_{t+1}) - \lambda S_{t+2}|\beta_{t+2} - \beta_{t+1}| \\
&\textbf{Leading to k steps ahead:} \\
\alpha_{t+k}B_{t+k} &= \alpha_t B_t e^{kr_f} \\
&\quad - \sum_{i=1}^{k} S_{t+i}e^{(k-i)r_f}[(\beta_{t+i} - \beta_{t+i-1}) + \lambda|\beta_{t+i} - \beta_{t+i-1}|]
\end{aligned} \tag{2.2}$$

This would in turn lead to an expression of the value of the portfolio at $k+1$ periods ahead ($k+1$, not $k$, periods ahead since the last step is, of course, not a rebalancing):

$$\begin{aligned}
V^-_{t+k+1} &= \alpha_{t+k}B_{t+k+1} + \beta_{t+k}S_{t+k+1} \\
&= \alpha_t B_t e^{r_f(k+1)} \\
&- \sum_{i=1}^{k} S_{t+i}e^{(k-i+1)r_f}[(\beta_{t+i} - \beta_{t+i-1}) + \lambda|\beta_{t+i} - \beta_{t+i-1}|] \\
&+ \beta_{t+k}S_{t+k+1}
\end{aligned} \tag{2.3}$$

where the goal is to maximize the expected value of Equation (2.3). To be able to do this in Matlab, we will need to represent the above expression in matrices. This is quite complicated to do when there is an absolute value within the expression. A more detailed description of the implementation is described in Section 2.2 below.

To fulfill the purpose of this thesis, i.e. finding an optimal portfolio, the expected value of the portfolio need to be calculated for several scenarios. To be able to calculate the expected value of the portfolio, the expected value of the stock is needed.

### 2.1.2 Stock values

To calculate the expected value of the stock a time series model, representing the financial data of the stock is needed. The technique used in this thesis is a regime-switching time series model based on a Hidden Markov Model (HMM), introduced in the works of [Nystrup et al., 2015] and [Nystrup et al., 2016]. There has recently been increased interest in the study of financial models where key parameters vary and are modulated by a Markov chain, see e.g. [Bauerle and Rieder, 2004], [Çakmak and Özekici, 2006], [Yin and Zhou, 2004], [Zhang, 2001] and [Zhou and Yin, 2003].

In a Hidden Markov Model (HMM) the probability distribution that generates an observation depends on the state of an unobserved Markov chain. A sequence of discrete random variables $\{X_t : t \in N\}$ is said to be a first-order Markov chain if it, for all $t \in N$ satisifies the Markov property $\Pr(X_{t+1}|X_t, ..., X_1) = \Pr(X_{t+1}|X_t)$. The conditional probabilities $\Pr(X_{t+1} = j|X_t = i) = q_{ij}(t)$ are called the transition probabilities and are represented by the Markov-Switching parameters in this thesis. In this thesis we consider the two-state Markov model with Gaussian conditional distributions, where $X_t$ represents the state of the economy(1 is good and 2 is bad):

$$Y_t \sim N(\mu_{X_t}, \sigma^2_{X_t})$$

The geometric distribution is here without memory, implying that the time until the next transition from the current state is independent of the time spent in the state. The initial state of the economy can be represented by $p_0 = [p_{good}\, p_{bad}]$. $p_{good}$ is a parameter which is chosen as the probability that we are in a good economic state, $X_t = 1$. $p_{bad}$ is, respectively, $1 - p_{good}$ and represent the probability that we are in a bad economic state, $X_t = 2$. These parameters represent our view of the current economic state. The Markov switching parameters, $q_{ij}$ represent the tendency to go to from state $i$ to state $j$. Together they create the matrix Q below.

$$Q = \begin{bmatrix} q_{11} & (1-q_{11}) \\ (1-q_{22}) & q_{22} \end{bmatrix}.$$

$\mu_1$, $\mu_2$, $\sigma_1$ and $\sigma_2$ represent the expected return and volatility respectively, when the economy is in the good state or the bad state. These parameters can be interpreted as to represent the behaviour of the economy. For further interest in the Hidden Markov Model, the reader is referred to [Nystrup et al., 2016] and [Lindström et al., 2015].

In [Nystrup et al., 2016] the authors construct a Hidden Markov Model (HMM) to model the daily returns and to infer the hidden states of the financial market. The states represent the dynamics of the market value of assets. They refer to several studies showing the profitability of dynamic asset allocation based on these classes of models, see e.g. [Bulla et al., 2011] and [Nystrup et al., 2015]. One reason is due to the persistence of the volatility, also known as *Volatility Clustering*. That is why it is important to explore the model's ability to reproduce the long memory and forecast future returns. Volatility clustering is explained further in [McNeil et al., 2005]. [Nystrup et al., 2016] present an adaptive estimation model which allow the parameters of the estimated model to be time-varying, as opposed to fixed-length forgetting factors. They found that the parameters vary quite significantly over time and they argue that failure to account for the time-varying behaviour is likely part of the reason why a simple random walk model usually outperforms regime-switching models when used out-of-sample, as discussed in [Dacco and Satchell, 1999]. However, the regime-based approach has the flexibility to adapt to changing economic conditions within a benchmark-based investment policy, leading to a better result in their case. The goal is not to predict regime shifts or future market movements, but to identify when a regime shift has occurred and then benefit from the persistence of returns and volatilities.

As proposed in [Costa and Araujo, 2008] such models based on HMMs can better reflect the market environment, since the overall assets usually move according to a major trend given by the state of the underlying economy, or by the general mood of the investor. The benefits of models with regime-switching parameters are illustrated in the results of [Costa and Araujo, 2008].

In [Yiu et al., 2010] the authors also stress the fact that there can be substantial fluctuations in economic variables, which affect the dynamics of the market values of the assets, over a long period of time. The authors state it is of practical importance and relevance to incorporate the switching behavior of the economic states in modeling the dynamics of the market values.

### 2.1.3 Constraints

The trading algorithm is implemented subject to certain constraints given by the preferences of the investor. The constraints are incorporated as to regulate the algorithm not to exceed risks or overuse resources. The following constraints are implemented in the algorithm.

#### Mean-variance criterion

Maximizing the value of the portfolio is, of course, an important issue. However, an equally important one is to manage the risk the investor is taking. Various methods of risk management have been proposed in literature, e.g. in [McNeil et al., 2005]. In this thesis the financial risk is managed through implementation of the mean-variance criterion. It is defined as the trade-off between high expected value of the portfolio and low variance of the portfolio. The following formula is used in the implementation of this problem:

$$\text{maximize } E[V_i] - \gamma Var[V_i] \tag{2.4}$$

where $\gamma \geq 0$ is called the *risk-aversion parameter* and represents the risk-aversion of the investor, see e.g. [Boyd and Vandenberghe, 2004]. Risk aversion represents the willingness to trade higher expected wealth for lower variance. A large $\gamma$ means that a larger emphasis is placed on the second part of the maximized function, a very risk-averse investor. In this thesis the notation $E[V_i]$ is replaced with $V_{k+1}^{-}$ from Equation (2.3) above.

In [Li and Ng, 2000], where an analytical optimal solution to the mean-variance formulation in multiperiod portfolio selection is investigated, this way of formulating the mean-variance portfolio selection problem is preferred especially when the investor is able to specify it's trade-off between expected wealth and variance. According to [Jorion, 1992] an advantage of the MVC is the fact that it can uniquely incorporate portfolio objectives with policy constraints and efficient use of information. It's ability to implement investors' constraints makes the criterion a flexible tool.

The multiperiod mean-variance selection problem was solved analytically in [Li and Ng, 2000] with no Markovian jumps, and with Markovian jumps in [Çakmak and Özekici, 2006] and [Zhou and Yin, 2003]. The mean-variance criterion was used to develop a solution of a problem of investment portfolio optimization with serially correlated returns using Model Predictive Control (MPC), or receeding horizon, in [Dombrovskii and Obyedko, 2015]. In [Zhou and Yin, 2003] a continuous mean-variance portfolio selection problem with regime-switching parameters is solved, however without non-negative constraints on terminal wealth and without transaction costs. In [Li et al., 2014] the multiperiod investment model with self-financing constraints were studied and the time-consistent optimal investment policy and the resulting MV efficient frontier was explicitly derived. In [Celikyurt and Özekici, 2007] several multiperiod portfolio optimization models are considered. The stochastic evolution is represented by a Markov-chain with perfectly observable states and the objective functions depend only on the mean and the variance of the final wealth. In both [Li et al., 2014] and [Celikyurt and Özekici, 2007] the efficient frontier is generated using dynamic programming to identify optimal portfolio management policies.

### Intermediate mean-variance criterion

In the above formulation of the mean-variance criterion, the variance at the end time, $k$ steps ahead, is considered. This in turn results in the initial variance not playing a large enough role in regulating the allocation decisions. To regulate the intermediate variances, meaning the variances up to every time step, a new intermediate mean-variance criterion is constructed. The main advantage of implementing this intermediate mean-variance criterion is the possibility to control the intermediate behaviour of a portfolio's expected value and variance, as proposed by [Costa and Nabholz, 2007]. This will in turn allow the investors to weight greater wealth at a price of lower variance after every time period, by varying the risk-aversion parameter $\gamma$. The new formulation will contain several covariance matrices, one for every sequential time step.

This new formulation should result in more careful initial trades, since the variance up to every period is considered in the mean-variance criterion. This means that large initial trades should be regulated even further.

**Allocation constraint**

To make sure that the portfolio is self-fincancing a constraint on the allocation is needed. This is an upper bound, lower bound constraint. The allocation constraint is constructed as to never trade beyond its limits, i.e. the upper bound is constructed as if the algorithm were to sell every day it would never run out of funds, and the lower to never sell more than currently owned. The resulting constraint will make sure that the investor only use the resources available, and that only a certain amount per day can be traded.

## 2.2 Implementation

The implementation of the problem at hand is done by creating an algorithm in Matlab. The algorithm is formulated as to maximize the value of the portfolio at a given number of time periods ahead. This is done using mathematical optimization theory of Linear Programming and Quadratic Programming, see [Boyd and Vandenberghe, 2004] for further explanation of optimization theory. The optimization is carried out using the Matlab programs `linprog.m` for the Linear Programming problem, and `quadprog.m` for the Quadratic Programming problem.

### 2.2.1 Stock values

As explained above the expected values of the stock is determined through the regime-based prediction strategy. The HMM will create several trajectories of possible stock values over a $k$-period time series. The evolution of the stock is generated as the mean value of the stock at every time step. This is calculated as the expected value of the simulated sample trajectories generated by the HMMs. For more details regarding the HMMs the reader is again referred to [Nystrup et al., 2016] and [Lindström et al., 2015]. The generated stock price values are represented in the vector $\boldsymbol{S}$ below.

In Figure 2.2 below an illustration of 100 simulations of the stock value, as well as the mean value of the stock, is presented.

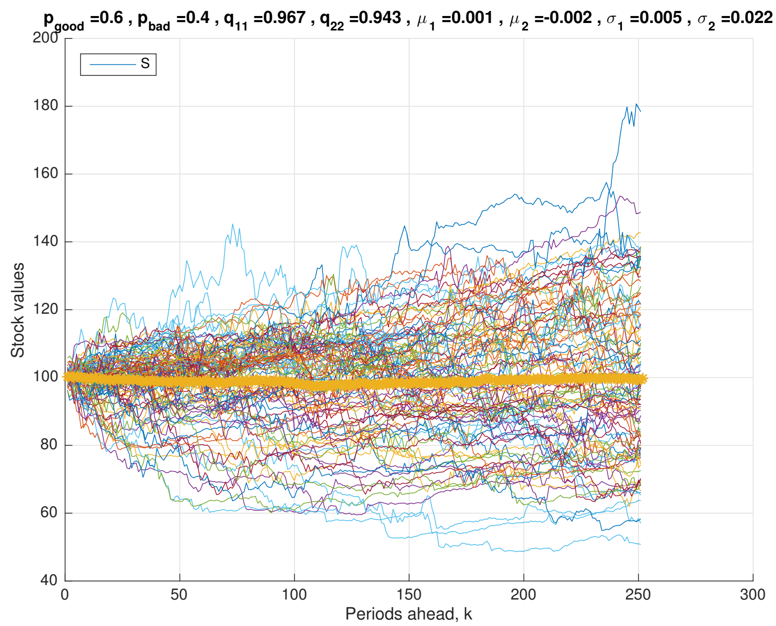


Figure 2.2: Seen in the graph are 100 generated simulations of the evolution of the stock. The fat *-line in the middle is the expected value of the stock used as input to the portfolio optimizing algorithm.

### 2.2.2 Optimization

#### Linear Programming

When the mean-variance criterion is not included the problem is a Linear Programming problem (LP-problem). The LP-problem is of the form:

$$\begin{aligned} \text{minimize } & \boldsymbol{f^T x} \\ \text{Subject to: } & \boldsymbol{Ax} \leq \boldsymbol{b} \\ & \boldsymbol{Aeq} \cdot \boldsymbol{x} = \boldsymbol{beq} \\ & \boldsymbol{lb} \leq \boldsymbol{x} \leq \boldsymbol{ub} \end{aligned} \tag{2.5}$$

where **Aeq** and **beq** are non-existent in the case of this thesis.

### Quadratic Programming

Once the mean-variance criterion is included in the algorithm the problem is transformed to a quadratic programming problem (QP-problem):

$$\begin{aligned} \text{minimize} \;& \boldsymbol{f^T x} + \frac{1}{2}\boldsymbol{x^T H x} \\ \text{Subject to:} \;& \boldsymbol{Ax} \leq \boldsymbol{b} \\ & \boldsymbol{Aeq} \cdot \boldsymbol{x} = \boldsymbol{beq} \\ & \boldsymbol{lb} \leq \boldsymbol{x} \leq \boldsymbol{ub} \end{aligned} \tag{2.6}$$

The constraints of the problem are all unchanged. The goal function, however, is updated to a quadratic function. Equation (2.4) above is changed to:

$$\text{minimize } \boldsymbol{f^T x} + \gamma Var(\boldsymbol{f^T x})$$

where the problem is now to minimize the goal function as well as the variance simultaneously, where the prioritization between the two parts is represented by $\gamma$. $\gamma$ can, as mentioned above, be qualitatively interpreted as the level of risk-aversion of the investor, see e.g. [Boyd and Vandenberghe, 2004]. The reason why we are using the variance of the goal function, and not the variance of the portfolio value is because they are the same.

### Goal function

From now on we will assume $B_i$=1 for all $i$, i.e. $r_f$=0, because of the current economic environment. For simplicity, in the rest of this thesis the notation of $t+k$ will be replaced by $k$, where just $t$ will be replaced by 0. From here on $\delta\beta_i = (\beta_i - \beta_{i-1})$ as well. Equation (2.3) above will then be formulated as:

$$\begin{aligned} V^-_{k+1} &= \alpha_k + \beta_k S_{k+1} \\ &= \alpha_0 - \sum_{i=1}^{k} S_i[\delta\beta_i + \lambda|\delta\beta_i|] + \beta_k S_{k+1} \end{aligned} \tag{2.7}$$

where we can see that $\beta$ and $\alpha$ is updated recursively for every trading period from 1 to $k$. Every $\alpha$ and $\beta$ can be represented in the following way:

$$\alpha_j = \alpha_0 - \sum_{i=1}^{j} S_i[\delta\beta_i + \lambda|\delta\beta_i|]$$
$$\beta_j = \beta_0 + \sum_{i=1}^{j} \delta\beta_i$$

for all $j = 0, \ldots, k$. In matrix form this can be represented as:

$$\boldsymbol{\alpha} = \boldsymbol{\alpha_0} - \boldsymbol{X}\boldsymbol{S_m}(\boldsymbol{\delta\beta} + \lambda|\boldsymbol{\delta\beta}|)$$
$$\boldsymbol{\beta} = \boldsymbol{\beta_0} + \boldsymbol{X}\boldsymbol{\delta\beta}$$

where

$$\boldsymbol{\alpha}= \begin{bmatrix}\alpha_1\\ \vdots\\ \vdots\\ \vdots\\ \alpha_k\end{bmatrix}, \boldsymbol{\beta}= \begin{bmatrix}\beta_1\\ \vdots\\ \vdots\\ \vdots\\ \beta_k\end{bmatrix}, \boldsymbol{\delta\beta}= \begin{bmatrix}\delta\beta_1\\ \vdots\\ \vdots\\ \vdots\\ \delta\beta_k\end{bmatrix}, |\boldsymbol{\delta\beta}|= \begin{bmatrix}|\delta\beta_1|\\ \vdots\\ \vdots\\ \vdots\\ |\delta\beta_k|\end{bmatrix}, \boldsymbol{\beta_0}= \beta_0 \begin{bmatrix}1\\ \vdots\\ \vdots\\ \vdots\\ 1\end{bmatrix},$$

$$\boldsymbol{\alpha_0}= \alpha_0 \begin{bmatrix}1\\ \vdots\\ \vdots\\ \vdots\\ 1\end{bmatrix}, \mathbf{X}= \begin{bmatrix}1 & 0 & \ldots & \ldots & 0\\ \vdots & \ddots & & & \vdots\\ \vdots & & 1 & & \vdots\\ \vdots & & & \ddots & 0\\ 1 & \ldots & \ldots & \ldots & 1\end{bmatrix}, \boldsymbol{S}_m= \begin{bmatrix}S_1 & 0 & \ldots & \ldots & 0\\ 0 & \ddots & & & \vdots\\ \vdots & & \ddots & & \vdots\\ \vdots & & & \ddots & 0\\ 0 & \ldots & \ldots & 0 & S_k\end{bmatrix}$$

The function to be maximized is the one of Equation (2.7) above. This can be reformulated as:

$$\begin{aligned} V_{k+1}^{-} &= \alpha_k + \beta_k S_{k+1} \\ &= \alpha_0 - (\boldsymbol{\delta\beta}^T + \lambda|\boldsymbol{\delta\beta}|^T)\boldsymbol{S} + \beta_k S_{k+1} \\ &= \alpha_0 - (\boldsymbol{\delta\beta}^T + \lambda|\boldsymbol{\delta\beta}|^T)\boldsymbol{S} + \boldsymbol{\delta\beta}^T S_{k+1}\mathbf{1} + \beta_0 S_{k+1} \end{aligned} \tag{2.8}$$

where

$$\boldsymbol{S}= \begin{bmatrix} S_1 \\ \vdots \\ \vdots \\ \vdots \\ S_k \end{bmatrix}, \, \mathbf{1}= \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \end{bmatrix}$$

$\boldsymbol{S}$ and $S_{k+1}$ represent the stock prices, or expected stock prices, and the only variables are $\delta\beta_i$ and $|\delta\beta_i|$.

In Equation (2.5) above $f^T x$ represents the goal function. In the optimization the goal function is represented by the non-constant part of Equation (2.8) above. Since $\alpha_0$ and $\beta_0 S_{k+1}$ are constants, the only variable part is represented by:

$$-(\boldsymbol{\delta\beta}^T + \lambda|\boldsymbol{\delta\beta}^T|)\boldsymbol{S} + \boldsymbol{\delta\beta}^T S_{k+1}\mathbf{1}$$

In the goal function there is a problem. $|\delta\beta|$ can not be expressed as $\delta\beta$ in a simple way. Therefore a change of variables is performed by replacing $\lambda|\delta\beta|_i S_i$ with $t_i$ and implementing it in the following manner:

$$\begin{aligned} \lambda|\delta\beta|_i S_i &\leq t_i \\ &\Longleftrightarrow \\ \lambda\delta\beta_i S_i &\leq t_i \\ -\lambda\delta\beta_i S_i &\leq t_i \\ &\Longleftrightarrow \\ \lambda\delta\beta_i S_i - t_i &\leq 0 \\ -\lambda\delta\beta_i S_i - t_i &\leq 0 \end{aligned}$$

leading to the goal function:

$$\begin{aligned} &\text{maximize } -(\boldsymbol{\delta\beta}^T\boldsymbol{S} + t_1 + ... + t_k) + \boldsymbol{\delta\beta}^T S_{k+1}\mathbf{1} \\ &\Longleftrightarrow \\ &\text{minimize } \boldsymbol{\delta\beta}^T\boldsymbol{S} + t_1 + ... + t_k - \boldsymbol{\delta\beta}^T S_{k+1}\mathbf{1} \\ &\Longleftrightarrow \\ &\text{minimize } [(\boldsymbol{S} - S_{k+1}\mathbf{1})^T|1...1][\delta\beta_1...\delta\beta_k|t_1...t_k]^T \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{f} &= [(\boldsymbol{S} - S_{k+1}\mathbf{1})^T|1...1]^T \\ \boldsymbol{x} &= [\delta\beta_1...\delta\beta_k|t_1...t_k]^T \end{aligned}$$

which is now of the form as in Equation (2.5) above.

The constraint of the new variable is then represented in Matlab as to fit the LP- and QP-problem in Equation (2.5), and Equation (2.6) above:

$$\lambda\delta\beta_i S_i - t_i \leq 0$$
$$-\lambda\delta\beta_i S_i - t_i \leq 0$$

which now creates a new constraint on $\mathbf{x}$ of the form $\boldsymbol{Ax} \leq \boldsymbol{b}$ where:

$$\boldsymbol{A} = \begin{bmatrix} \lambda\boldsymbol{S}_m & -\boldsymbol{I} \\ -\lambda\boldsymbol{S}_m & -\boldsymbol{I} \end{bmatrix}, \boldsymbol{b} = 0$$

$$\text{and } \boldsymbol{I} = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix} \boldsymbol{S}_m = \begin{bmatrix} S_1 & 0 & \dots & \dots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \dots & \dots & 0 & S_k \end{bmatrix}$$

### Mean-variance criterion

In order to implement the mean-variance criterion the variance of the goal function need to be derived. This is done using the covariance matrix, $\Sigma$. The goal function can be written as:

$$\begin{aligned} \boldsymbol{f^T x} &= \boldsymbol{\delta\beta}^T \boldsymbol{S} + t_1 + ... + t_k - \boldsymbol{\delta\beta}^T S_{k+1}\mathbf{1} \\ &= \boldsymbol{\delta\beta}^T \boldsymbol{S} - \boldsymbol{\delta\beta}^T \mathbf{1} S_{k+1} + [t_1 ... t_k]\mathbf{1} \end{aligned}$$

where the variance can be found as:

$$Var(\boldsymbol{f^T x}) = Var(\boldsymbol{\delta\beta}^T \boldsymbol{S}) = \boldsymbol{\delta\beta}^T \Sigma_S^k \boldsymbol{\delta\beta} = \boldsymbol{x}^T \Sigma_k \boldsymbol{x} \tag{2.9}$$

where

$$\Sigma_S^k = \begin{bmatrix} \sigma_1^2 & Cov_{12} & \dots & \dots & Cov_{1k} \\ Cov_{21} & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ Cov_{k1} & \dots & \dots & \dots & \sigma_k^2 \end{bmatrix}$$

$$Cov_{i,j} = Cov(S_i, S_j),\ \sigma_i^2 = Var(S_i) \text{ and } \Sigma_k = \begin{bmatrix} \Sigma_S^k & \mathbf{0}_{k\times k} \\ \mathbf{0}_{k\times k} & \mathbf{0}_{k\times k} \end{bmatrix}$$

which can be inserted into the form of the QP-problem in Equation (2.6) above. In order to fit this into the form of the goal function in Equation (2.6) above, the covariance matrix of the goal function, $\Sigma_k$, need to be implemented as $H = 2\gamma\Sigma_k$.

The new goal function is then represented by:

$$\text{minimize } \boldsymbol{f^T x} + \gamma \boldsymbol{x}^T \Sigma_k \boldsymbol{x} \tag{2.10}$$

which is solved using quadprog.m in Matlab. This is in line with the mean-variance optimization problem proposed in [Boyd and Vandenberghe, 2004].

### Intermediate mean-variance criterion

The implementation of the intermediate mean-variance criterion is inspired by the motivation of [Costa and Nabholz, 2007]. In this thesis, however, we will consider the mean-variance regulation as in the formula in Equation (2.4) above, and not as constraints proposed in [Costa and Nabholz, 2007].

The final covariance matrix will be formulated as follows:

$$\Sigma = \Sigma_1 + ... + \Sigma_k$$

where $\Sigma_i$ is constructed as $\Sigma_k$ from Equation (2.9) above. The resulting covariance matrix $\Sigma$ is constructed using a simple for-loop in Matlab.

### Hyperopic limit

To make sure the investor is not too hyperopic, meaning long-sighted, in its investments maximization of intermediate excess returns is added to the goal function. This is implemented as follows:

$$
\begin{aligned}
&(\boldsymbol{S}_{t+1} - \boldsymbol{S}_t)^T \boldsymbol{\beta}_t \\
=&(\boldsymbol{S}_{t+1} - \boldsymbol{S}_t)^T [\boldsymbol{X}\boldsymbol{\delta\beta}_t] \\
=&[(\boldsymbol{S}_{t+1} - \boldsymbol{S}_t)^T \boldsymbol{X}]\boldsymbol{\delta\beta}_t \\
=&\boldsymbol{S}_{diff}\boldsymbol{\delta\beta}_t
\end{aligned}
$$

where $\boldsymbol{S}_{diff}$ is added to $\boldsymbol{f}$ in the goal function above. This additional term in the goal function should lead to additional trading to maximize every excessive return. This will in turn lead to a higher variance, however a very small addition, but is implemented because of the increased rationality of maximizing excess returns.

#### Allocation Constraint

To make sure the algorithm will not buy more than the wealth allows the upper bound is constructed as the inital wealth not invested in stock divided by the intial stock price and the number of time-periods. The lower bound is constructed as the initial amount of stocks divided by the number of time-periods.

The upper- and lower bounds are constructed only to affect the first half of the vector $\mathbf{x}$, $(\delta\beta_i)$, since that is the only part where it is applicable. The second part of $\mathbf{x}$ is, however, not constrained.

### 2.2.3 Choice of parameters

**Mean-variance criterion:**
Since there is not a unique value of $\gamma$ representing a typical investor, the parameter is varied over several values. The variation in end value of the portfolio and variance, as well as the trading behaviour, are discussed below for different values of $\gamma$. If $\gamma$ were to be zero, the variance would not be considered and it would simply be a LP-problem as in Equation (2.5) above.

**Transaction cost, $\lambda$:**
$\lambda$, representing the transaction costs of trading, is another variable that impacts the resulting trading pattern. It is given several values in order to compare the resulting trading pattern.

**Initial values:**
The initial allocation of the portfolio will be chosen the same as the static one. The reason is that they should be evaluated according to the same starting point. If the algorithm were to start at a "less favourable" allocation it will be at a disadvantage due to the limit set on the amount traded.

**Number of stocks, $\beta$:**
$\beta$ is not an integer. This is not a problem since the amount of money initially invested is reasonable. Large enough initial investment makes the continous aproximation of an integer number of stocks.

**Risk-free rate, $r_f$:**
Because of the nature of the current economy, the decision was made not to include the risk-free rate in the bank account.

## 2.3 Sensitivity analysis

To test the performance of the trading algorithm a sensitivity analysis is conducted. Parameters are varied and several scenarios are created. First the more obvious tests are performed to ensure the reliability and rationality of the algorithm, then interesting scenarios are evaluated. During the sensitivity analysis three portfolios will be compared, the *static portfolio*, the mean-variance constrained *MVC portfolio* and the *non-MVC portfolio*. The static portfolio has a set allocation ratio between the stock and the bank account, meaning there is no optimization carried out. The MVC portfolio is constrained by all means possible and it is represented by a QP-problem as described above. The non-MVC portfolio is the same as the MVC portfolio but with the risk aversion paremeter, $\gamma = 0$, and is thereby not constrained by the mean-variance criterion. The sensitivity analysis is conducted to see how the MVC-portfolio behaves and performs compared to the other portfolios, when parameters are changed.

### 2.3.1 Varying parameters

By varying different paramaters which have a constraining effect they will regulate the algorithm in different ways. For example, if a low transaction cost is set, but a high value for the risk aversion parameter, the transaction costs might have little effect on the MVC portfolio - because it is already constrained. By varying the parameters simultaneously infliction points might be reached where one constraint takes over from another.

By taking the parameters to extremes other infliction points should be noticeable. The portfolio values of the three portfolios should converge to each other once the constraining parameters are increased enough.

**Risk aversion parameter, $\gamma$:**
The risk aversion parameter, $\gamma$, defines the difference between the MVC portfolio and the non-MVC portfolio. By increasing $\gamma$ the portfolio will be constrained until it reaches the infliction point of not trading at all - it will converge to the static portfolio. By decreasing $\gamma$ towards 0 the MVC portfolio will converge towards the non-MVC portfolio.

**Transaction cost, $\lambda$:**
An obvious test is varying the transaction cost. This is done as to see that the investor would trade less, or less often, when there is an increase in transaction cost.

There should be an infliction point where the transaction cost is set high enough to impose the algorithm not to trade at all. The reason is that this parameter would constrain the algorithm not to gain enough by buying or selling. This will result in the evolutions of the three portfolios to be the same, i.e. static.

**Number of periods ahead, $k$:**
The intermediate mean-variance criterion explained above should be less effective when $k$ is small. When $k$ is large, however, the trading behaviour should be more stable over time.

By increasing the horizon, $k$, the determination of economic parameters will be even more important. This means that the strategy for generating stock values will be increasingly important.

**Amount allowed to be traded:**
As mentioned above the amount allowed to be traded is set not to use more resources than available. Regardless, it would be interesting to see how the trading strategies change when the amount allowed to be traded is increased or decreased.

### 2.3.2 Scenarios

By changing the parameters that represent the state and behaviour of the economy, i.e. the investor's view of the economy, scenarios are created. The parameters representing the economy are the ones used to generate the expected values of the stock, i.e. the parameters in the regime-switching Markov chain explained above. These parameters will simulate interesting scenarios when changed together.

**The state of the economy, $p_{good}$ & $p_{bad}$:**
By varying the state parameters the initial evolution of the stock should change.

**The Markov switching parameters, $q_{ii}$:**
By varying the Markov switching parameters the tendency to switch between the states of the economy changes. Because the economy converges to the stationary time-series of a growing economy a reasonable scenario would be that the economy has the highest probability of being in a bull market. It would, however, be interesting to see the resulting trading behaviour when we are in a bear market, or a crash, as well.

**The behaviour of the economy, $\mu$ & $\sigma$:**
By varying the behaviour parameters the volatility and the expected return in the different states are changed. Changed behaviour parameters should result in faster or slower change of the values of the stock.

# Chapter 3

# Results

This chapter provides numerical results for the sensitivity analysis of the optimal portfolio. Portfolio performance as well as trading behaviour of the algorithm are presented in this chapter. The evolution of the portfolio values, $\beta$-values as well as resulting variances are presented in plots below.

The results will be presented in a plot divided into four graphs. The top left graph shows the evolution of the three portfolios' values, the top right the evolution of mean value of the stock and the bottom two shows the evolution of the two $\beta$s, i.e. the number of owned stocks in the two non-static portfolios. Here ROE (top left corner) refers to *return on equity* defined as

$$\text{ROE} = \frac{\text{End value of portfolio - Starting value}}{\text{Starting value}}$$

and represent a measure of performance of the portfolio. The variance above the plot in the left bottom corner refers to the variance of the portfolio value.

The reader should keep in mind that the trading strategy seen in the graphs below are set day one for the following $k$ periods. Day two another prediction and optimization should be carried out leading to a slightly different trading strategy for the $k$ periods following day two.

## 3.1 Varying parameters

In this section parameters are varied as to show how the trading behaviour changes when regulated differently.

### 3.1.1 Risk aversion parameter, $\gamma$

Below are three figures, Figure 3.1, Figure 3.2 and Figure 3.3, showing the resulting graphs for three values of $\gamma$. Seen in the figures below are clear changes in the MVC portfolio and the difference between it and the other two portfolios. As expected when $\gamma$ is very low the MVC portfolio value is quite similar to the non-MVC one, and when increased it converges to the static portfolio value. The amount of stocks ($\beta$) at the maximum peak of the MVC portfolio is, however, still less than of the non-MVC portfolio. When $\gamma$ is increased the amount being traded in the MVC portfolio is significantly more constrained. This, in turn, results in a lower end value of the portfolio.

Another clear change when $\gamma$ is varied is the change in portfolio variance - the purpose of the mean-variance criterion. Notice that even though the end variance is constrained, the trading frequency is quite high.

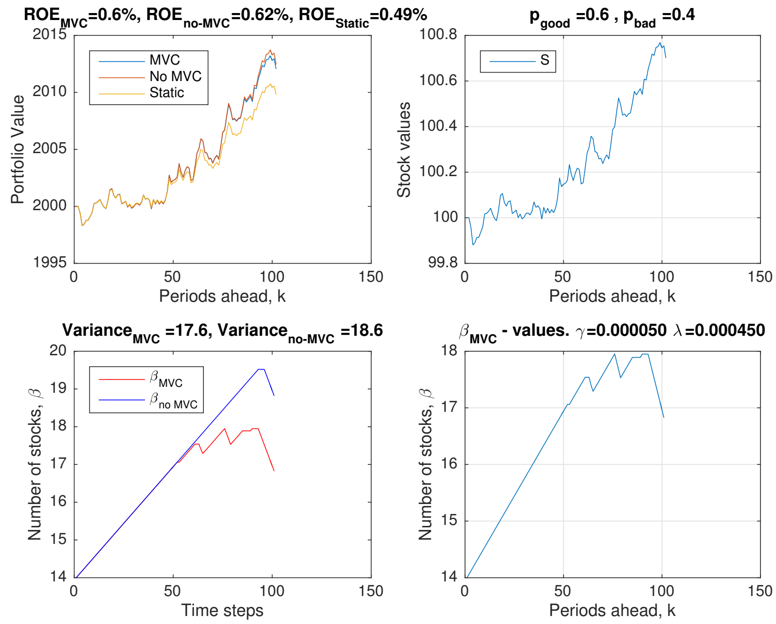


Figure 3.1: The resulting graphs with the risk aversion parameter, $\gamma$, in focus. Here the risk aversion is very low: $\gamma = 0.00005$.

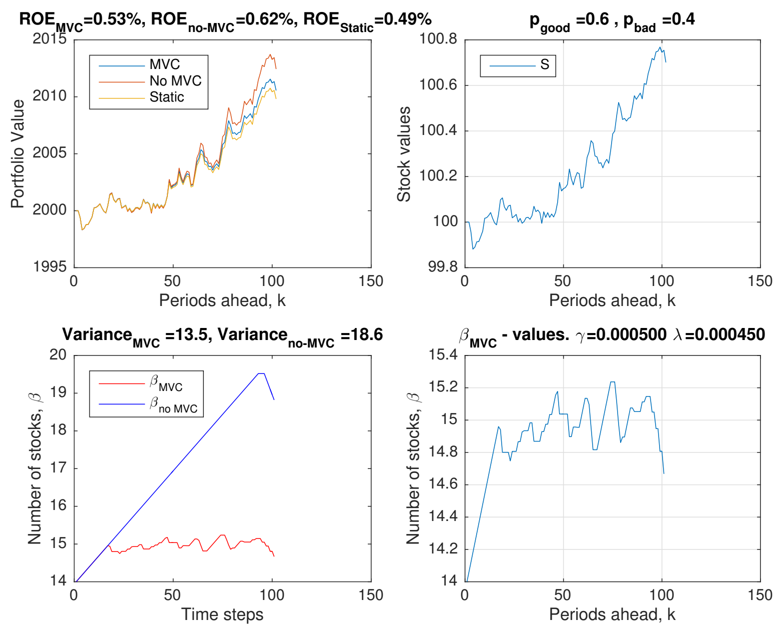


Figure 3.2: The resulting graphs with the risk aversion parameter, $\gamma$, in focus. Here $\gamma = 0.0005$.

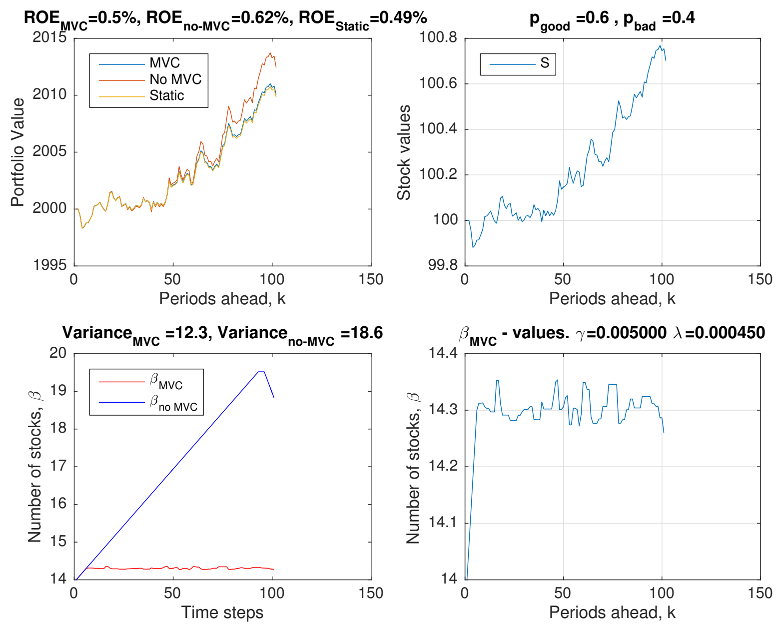


Figure 3.3: The resulting graphs with the risk aversion parameter, $\gamma$, in focus. Here $\gamma = 0.005$.

### 3.1.2 Transaction cost, $\lambda$

Below are three figures, Figure 3.4, Figure 3.5 and Figure 3.6, showing the resulting graphs for three values of $\lambda$. Seen below are the changes in trading behaviour of the MVC portfolio and the non-MVC portfolio, due to changes in the transaction cost. As expected, when the transaction cost is low, significantly more and larger trades are carried out, especially noticeable for the MVC portfolio.

An interesting result is the trading behaviour of the MVC-portfolio in Figure 3.6 below. Notice that the trading frequency is significantly constrained when the transaction cost is high enough, here $\lambda$ is 0.45%.

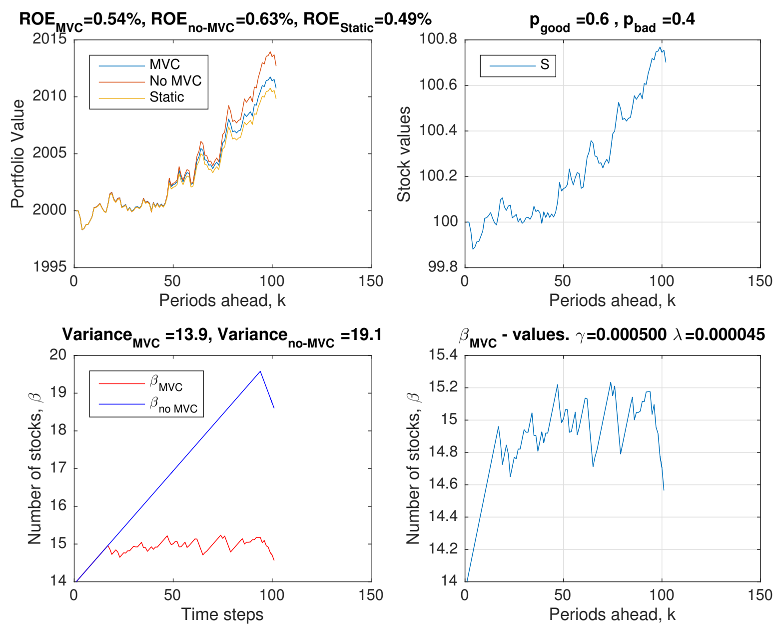


Figure 3.4: The resulting graphs with the transaction cost, $\lambda$, in focus. Here $\lambda = 0.000045$.

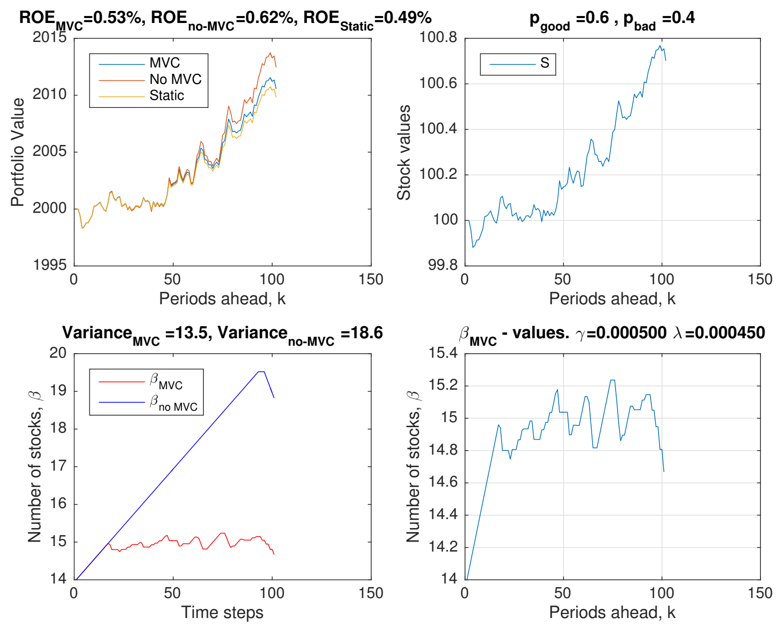


Figure 3.5: The resulting graphs with the transaction cost, $\lambda$, in focus. Here $\lambda = 0.00045$.

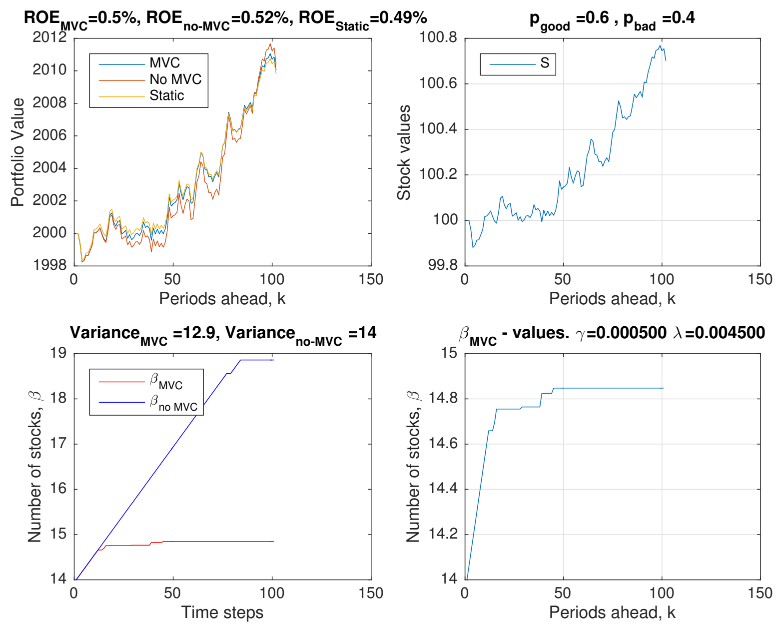


Figure 3.6: The resulting graphs with the transaction cost, $\lambda$, in focus. Here $\lambda = 0.0045$.

### 3.1.3 Amount allowed to be traded

Below are three figures, Figure 3.7, Figure 3.8 and Figure 3.9, showing the resulting graphs for three values of the amount allowed to be traded. As mentioned above the allocation constraint is set not to use more resources than available, i.e. the reliability of this algorithm is violated when it is changed. However, the results are interesting and are therefore presented anyway.

As expected when the amount is increased the gap between the portfolio values are increased. An interesting fact is that the maximum amount of stocks in the MVC portfolio increases when the amount allowed is increased. Notice that in Figure 3.9 the amount of stocks owned by the non-MVC portfolio are above the amount of available stocks.

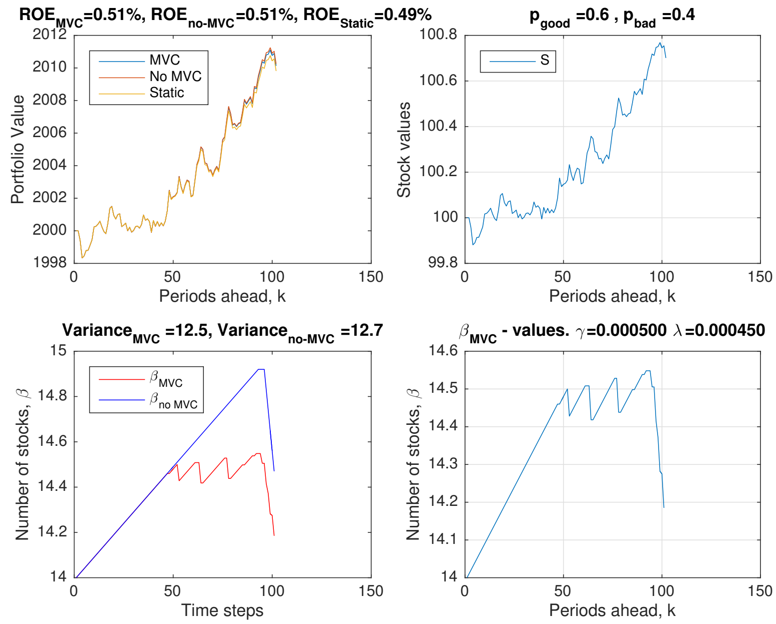


Figure 3.7: The resulting graphs with the amount allowed to be traded in focus. Here the amount available are 5 stocks less than originally.

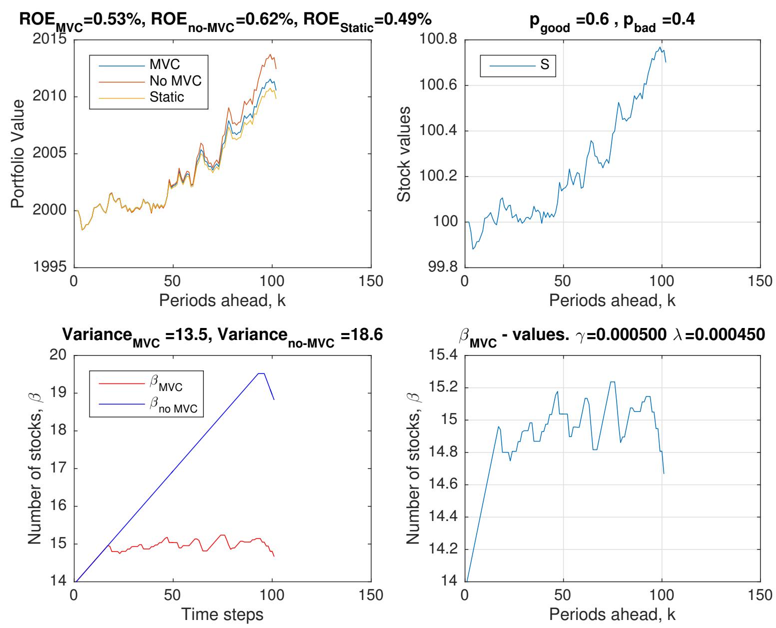


Figure 3.8: The resulting graphs with the amount allowed to be traded in focus. Here the amount available are 20 as originally.

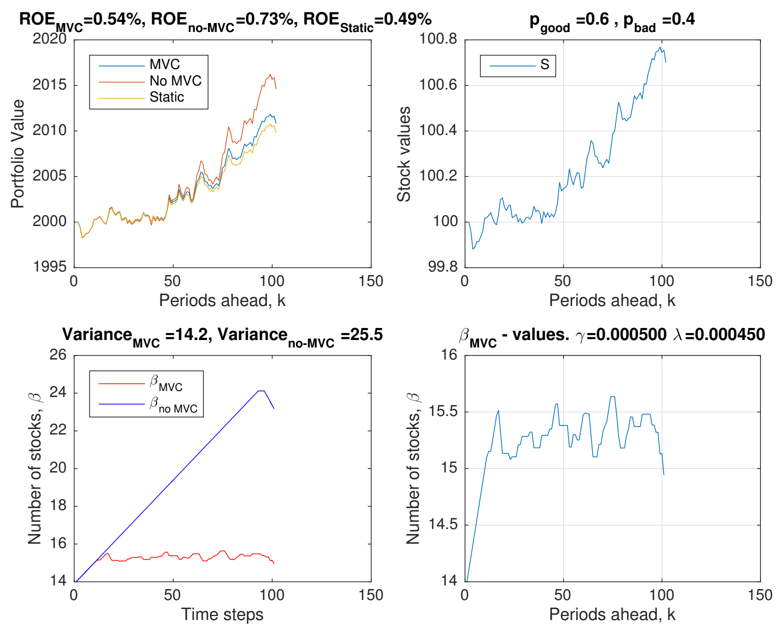


Figure 3.9: The resulting graphs with the amount allowed to be traded in focus. Here the amount available are 5 additional stocks than originally.

## 3.2 Scenarios

In this section several scenarios have been constructed as to show different reactions due to the properties of an economy. Initially a good, a middle and a bad scenario are presented. Graphs representing the evolution of the stock as well as the behaviour of the trading algorithm are presented for every scenario below.

Secondly some interesting scenarios are presented as to show the behaviour of the trading algorithm in different unique economic environments.

### 3.2.1 Good scenario

In Figure 3.10 and Figure 3.11 below the evolution of the stock as well as the resulting behaviour of the trading algorithm are presented. Both the MVC and the non-MVC portfolios behave as expected, increasing its stock exposure as much as possible. The MVC portfolio has lower share allocated in the stock due to the MVC criterion.

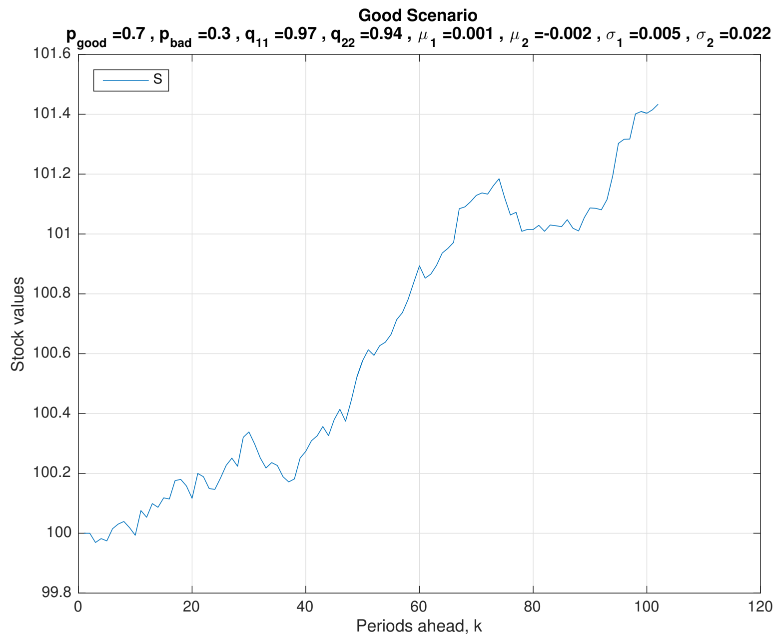


Figure 3.10: The evolution of the stock in a good economic state. A so called *bull market*.

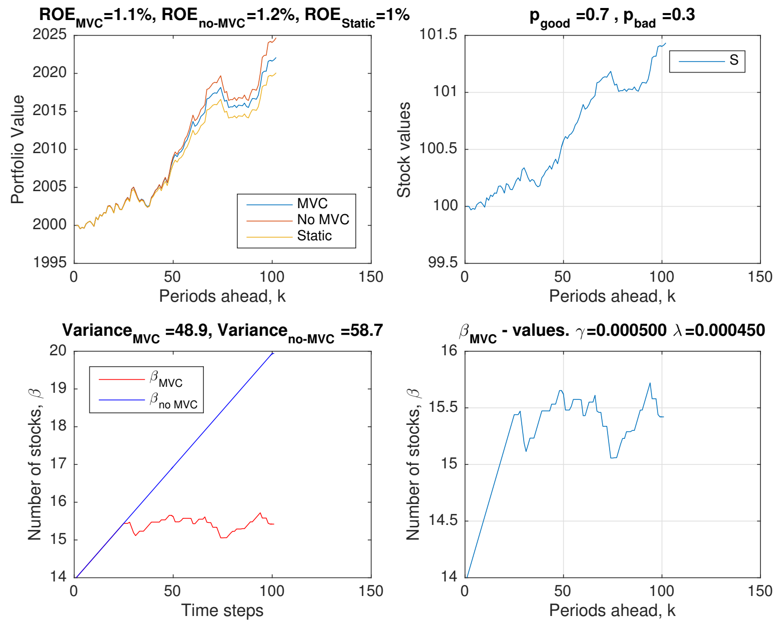


Figure 3.11: The resulting graphs showing the resulting portfolio of a good economic state. A so called *bull market*.

### 3.2.2 Middle scenario

In Figure 3.12 and Figure 3.13 below the evolution of the stock as well as the resulting behaviour of the trading algorithm are presented. Here the MVC and non-MVC portfolios barely outperform the static one. Notice the behaviour of the non-MVC portfolio, it increasies it's position in the stock even though there is no clear trend upwards. This is due to the end value of the stock being higher than the starting value.

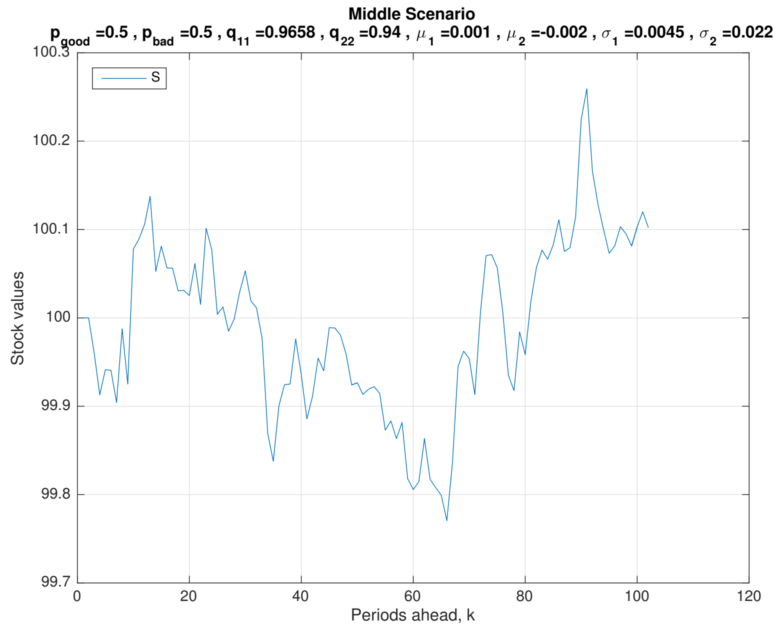


Figure 3.12: The evolution of the stock in a normal economic state.

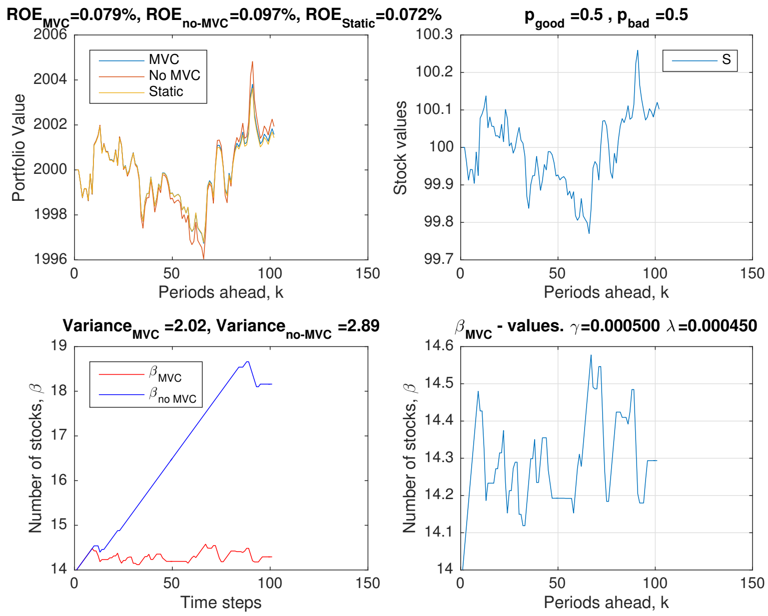


Figure 3.13: The resulting graphs showing the resulting portfolio of a normal economic state.

### 3.2.3 Bad scenario

In Figure 3.14 and Figure 3.15 below the evolution of the stock as well as the resulting behaviour of the trading algorithm are presented. The non-MVC portfolio is clearly outperforming the other two. Notice that the MVC portfolio has a higher final variance than the non-MVC one. This is due to the sharp downward evolution of the stock which the non-MVC portfolio eliminates exposure to, while the MVC portfolio is bound by the intermediate mean-variance criterion.

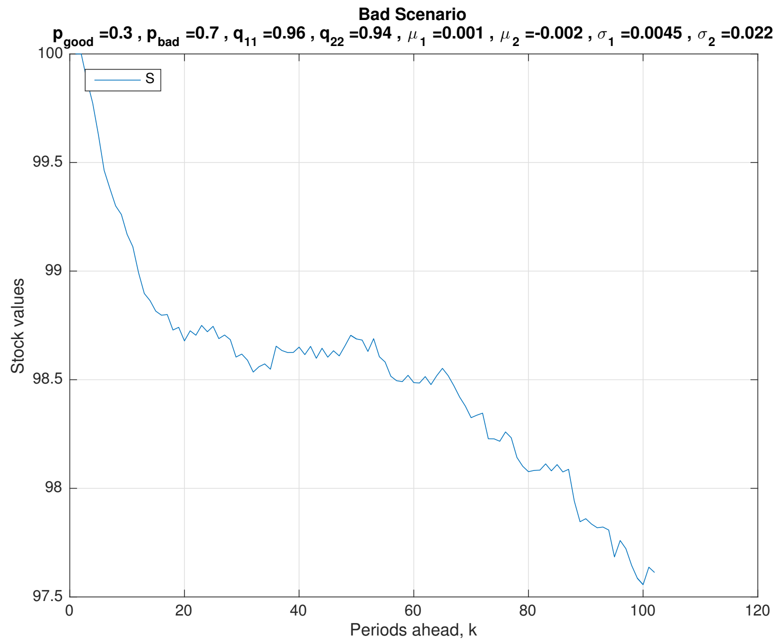


Figure 3.14: The evolution of the stock in a bad economic state. A so called *bear market.*

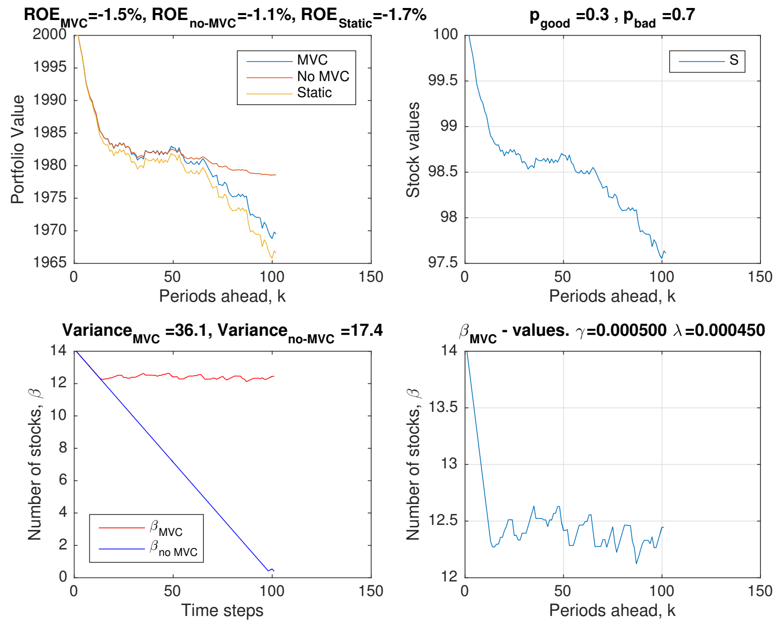


Figure 3.15: The resulting graphs showing the resulting portfolio of a bad economic state. A so called *bear market*.

### 3.2.4 Shifting scenario

In Figure 3.16 and Figure 3.17 below the resulting behaviour of the trading algorithm is presented when the market starts falling as in a bear market then bottoms and turns into a bull market. The resulting behaviours of the algorithms are very interesting. They both react rationally by first selling as much as possible and then buying it back. As usual the non-MVC portfolio's behaviour is more significant due to the mean-variance criterion.

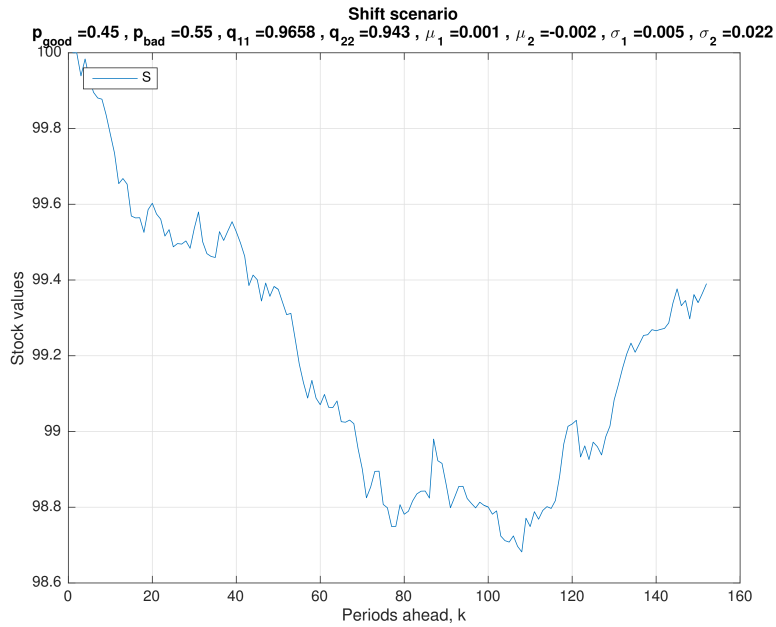


Figure 3.16: The evolution of the stock in a shifting economic state.

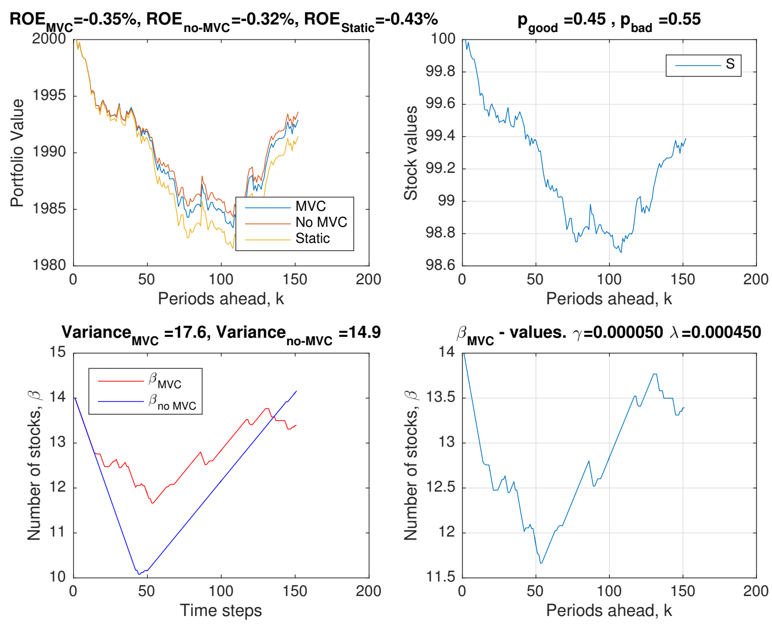


Figure 3.17: The resulting graphs showing the resulting portfolio of a shifting economic state.

### 3.2.5 Technical scenario: Questionable trading behaviour

Due to the limitations of a quadratic goal function and linear constraints in the optimization, the algorithm is not perfect. An example where the trading behaviour is questionable is the following presented in Figure 3.18 and Figure 3.19 below.

Notice that the non-MVC portfolio reacts by buying as much as possible even though the stock is initially falling. The MVC-portfolio, however, seems to follow a more reasonable trading pattern. An important fact to notice is that the non-MVC still outperforms the two other portfolios.

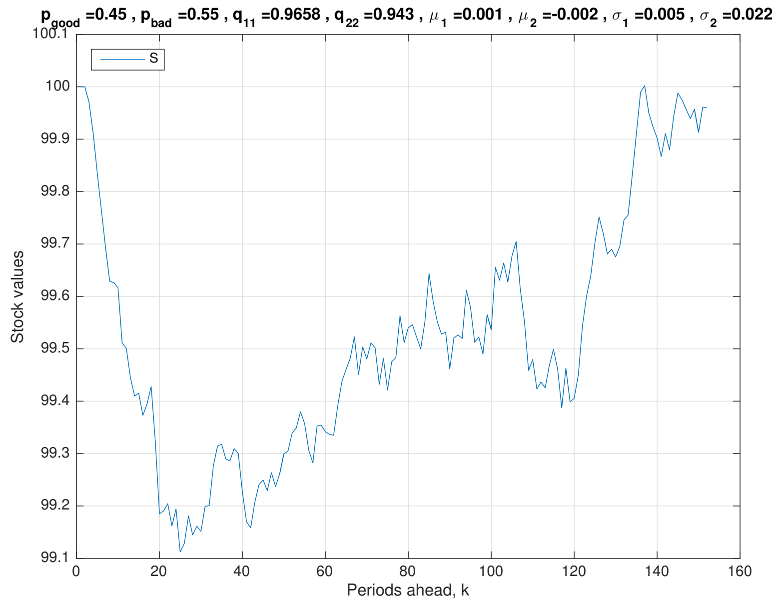


Figure 3.18: The evolution of the stock in a questionable scenario.

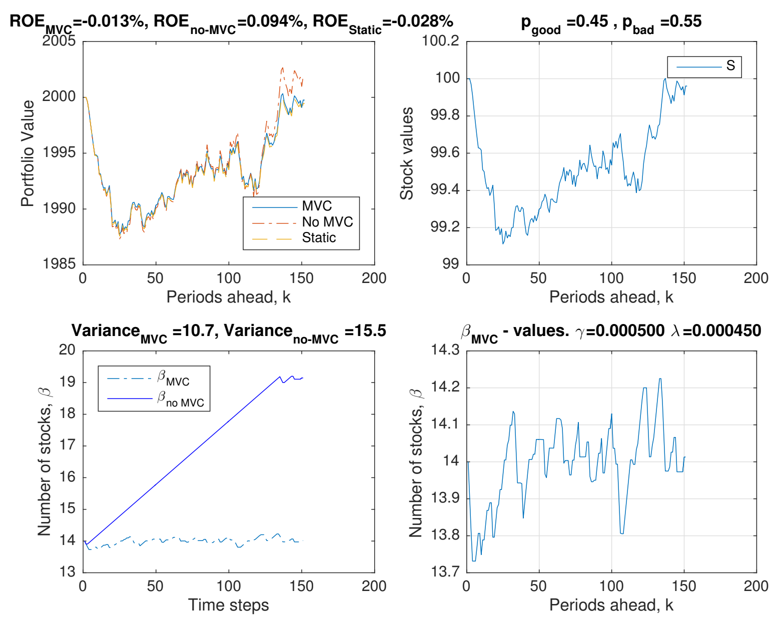


Figure 3.19: The resulting portfolio showing a questionable trading behaviour.

### 3.2.6 Technical scenario: Simulated stockvalues

In a perfect world an infinite amount of sample values would be generated and the mean value would be highly reliable. However, due to limited computer power only up to 100 000 simulations are reasonable to generate. During this sensitivity analysis 1000 samples have been used. An illustration of the changes due to variations in number of samples is presented in Figure 3.20, Figure 3.21 and Figure 3.22 below, representing 1000, 10 000 and 100 000 simulations respectively. Both the evolution of the stock as well as the resulting trading behaviour experience a "smoothing" effect when more samples are generated.

Notice especially the change in trading behaviour of the MVC portfolio. When the number of simulations increase, the stock values are more reliable and the resulting frequency of daily trading is decreased leading to reduced variance of the portfolio value. Notice that the constraining effect of high amount of simulations is similar to that of higher transaction cost.

In Figure 3.23 & Figure 3.24 as well as Figure 3.25 & Figure 3.26 we see the effect of changing the parameters $\gamma$ and $\lambda$ using 100 000 simulations. As expected the changing trading behaviour is the same as when the parameters are varied above, when 1000 simulations are used. When the parameters are increased trades are carried out less often and in smaller sizes. Notice in Figure 3.26 where the trading behaviour of the MVC-portfolio is as close to constant as possible due to increased transaction costs.

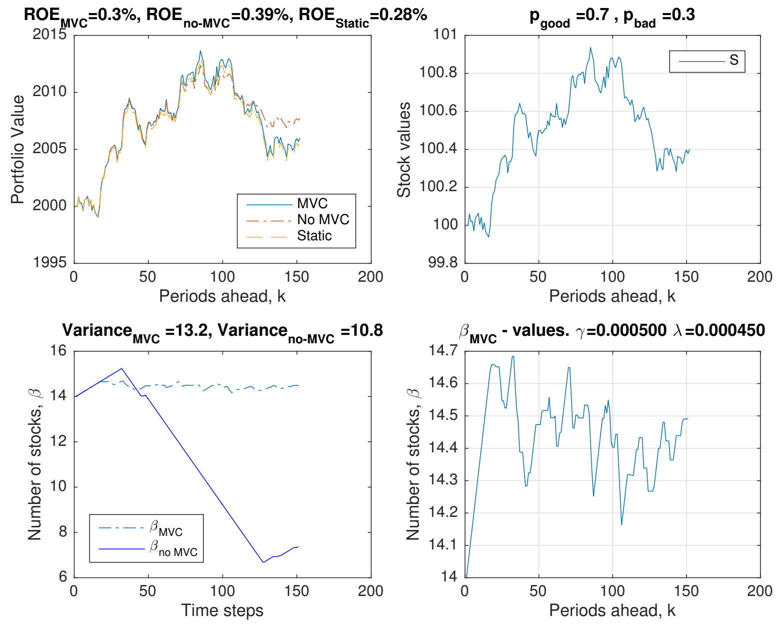


Figure 3.20: The resulting trading when 1000 simulations are used to calculate the expected value of the stock.

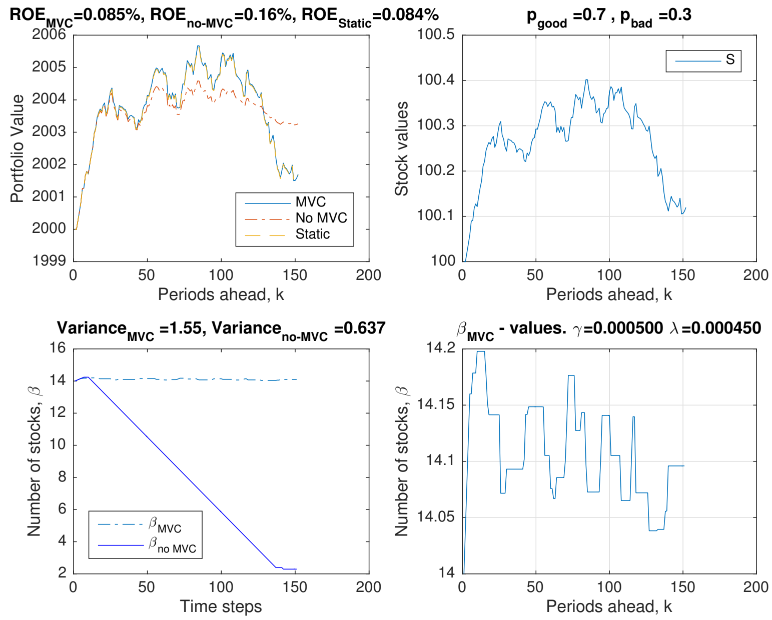


Figure 3.21: The resulting trading when 10 000 simulations are used to calculate the expected value of the stock.

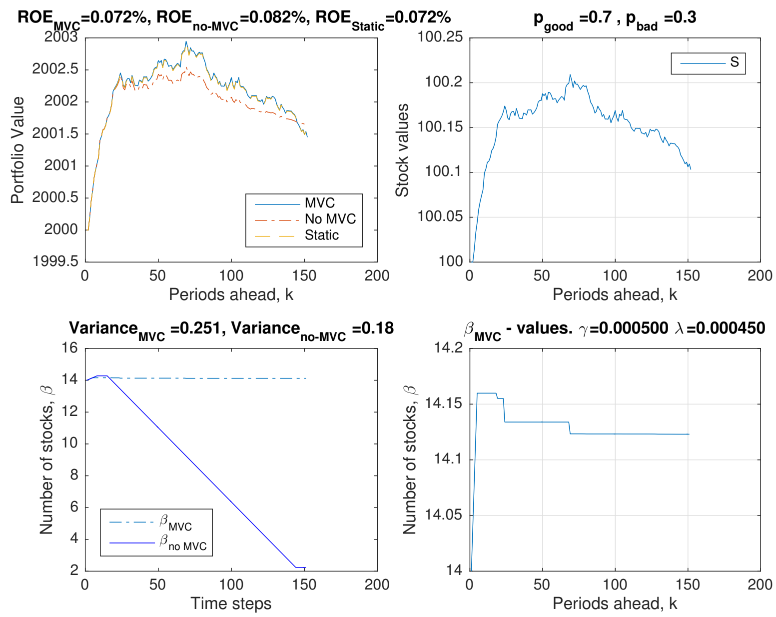


Figure 3.22: The resulting trading when 100 000 simulations are used to calculate the expected value of the stock.

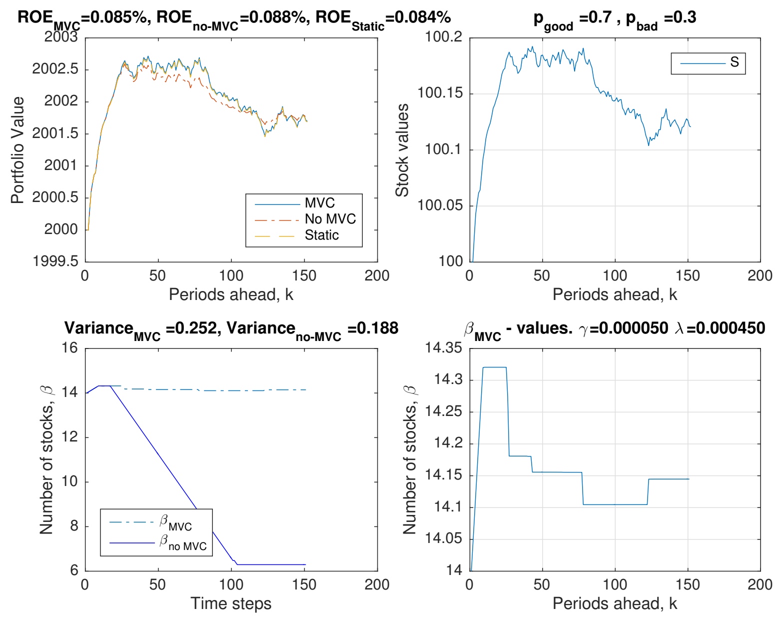


Figure 3.23: The resulting trading when 100 000 simulations are used to calculate the expected value of the stock. Here the risk aversion paremeter is low, $\gamma = 0.00005$.

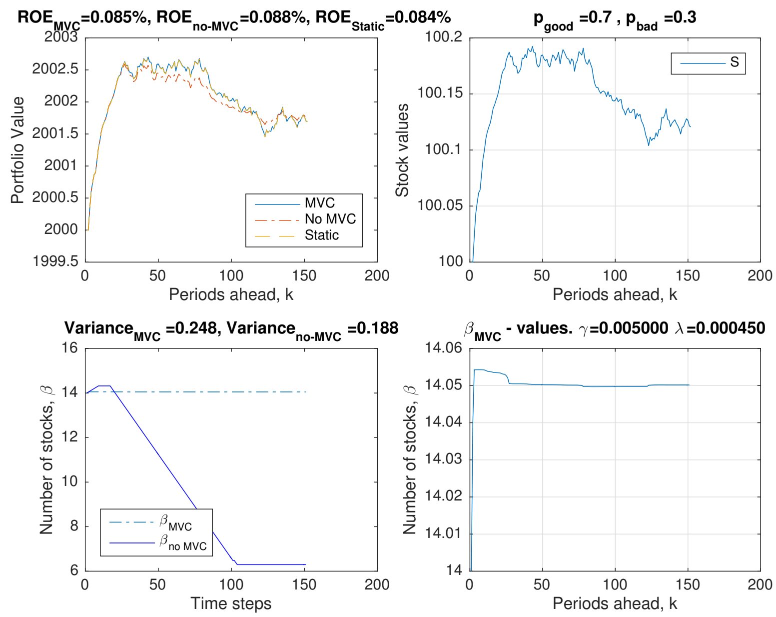


Figure 3.24: The resulting trading when 100 000 simulations are used to calculate the expected value of the stock. Here the risk aversion paremeter is high, $\gamma = 0.005$.

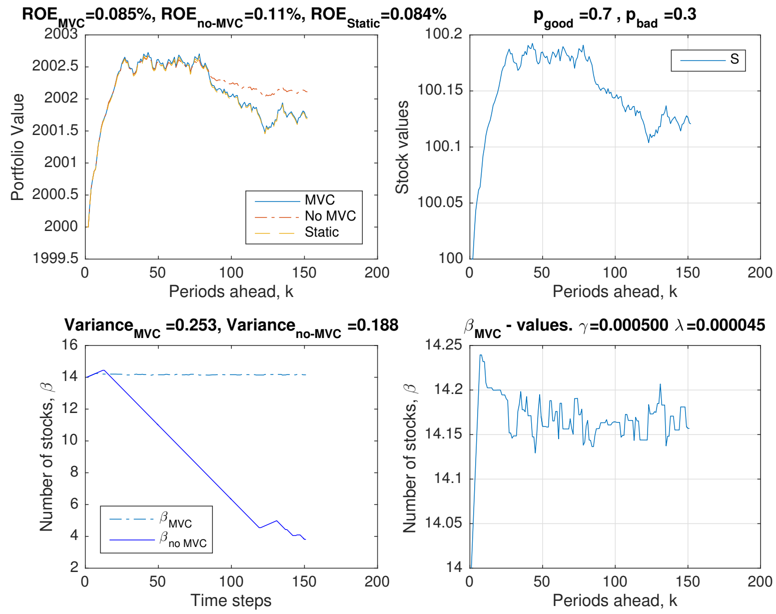


Figure 3.25: The resulting trading when 100 000 simulations are used to calculate the expected value of the stock. Here the transaction costs are low, $\lambda = 0.000045$.

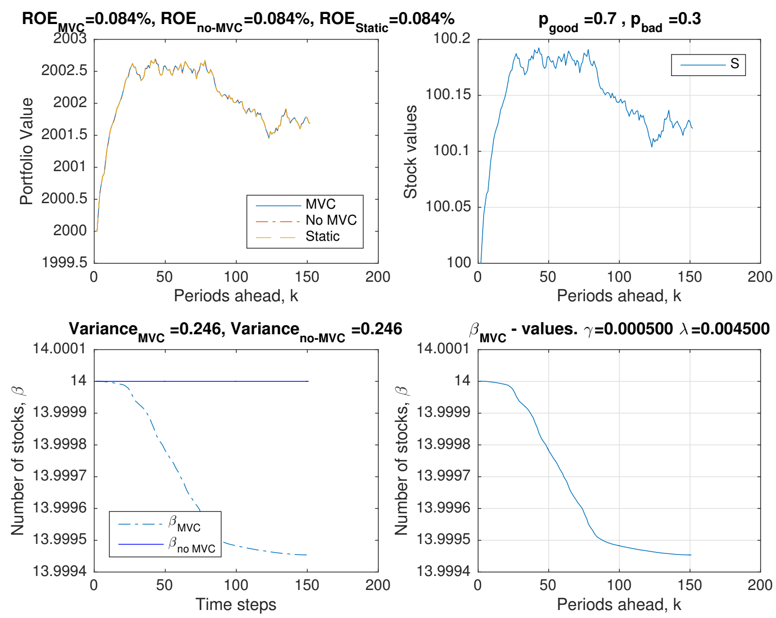


Figure 3.26: The resulting trading when 100 000 simulations are used to calculate the expected value of the stock. Here the transaction costs are high, $\lambda = 0.0045$.

### 3.2.7 1 year scenario

The figure seen below, Figure 3.27, presents the trading behavior of the algorithm 1 year ahead of time. Due to the long time horizon the MVC portfolio trades close to the initial amount of stocks. The reason is simply the fact that the end variance is constrained.

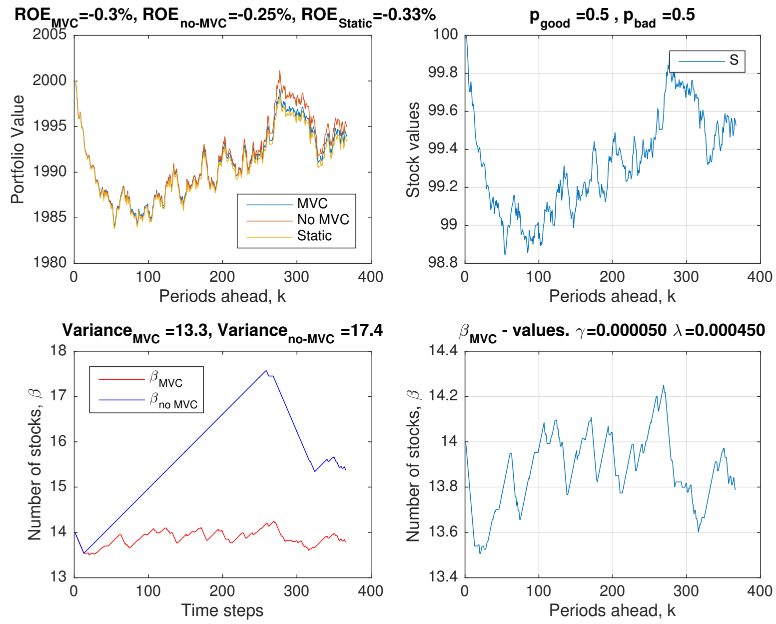


Figure 3.27: The behaviour of the algorithm 1 year (365 days) ahead.

# Chapter 4

# Discussion

A desirable behaviour of the trading algorithm would be one where few reallocations are made during the time horizon, i.e. the trading frequency is low. This is due to the uncertainty involved, which is increased with the trading frequency. The uncertainty mentioned is whether the algorithm trades because of the actual stock movement or because of the limitations of the HMM. In most of the results presented above non-desirable high trading frequencies are present. However, these results lead to different insights and test other functions in the algorithm. After all the main product of this thesis is the behaviour of the algorithm. Contributing results and how to achieve desirable results are discussed below.

In the following section the limitations of this thesis are presented and discussed. Later the results presented above are discussed and future studies are proposed. At last a short summary of the main conclusions is presented.

## 4.1 Limitations

This thesis would be significantly improved if real-time data applications were available. The parameter switching stock value generation could then be implemented with more reliable parameters. By analyzing real-time data the algorithm could be tested in a real environment. An interesting result would be to see whether the algorithm would still outperform the static portfolio.

### 4.1.1 Allocation constraint

A desirable allocation constraint would be to set the share of wealth allocated in stocks between two values, e.g. minimum 10% and maximum 90%. However, this would make the constraint dependant on the stock values and the number of stocks, $\beta$. Due to the fact that our goal function is dependant on $\delta\beta$ rather than $\beta$ it is complicated to construct the constraint as linear, which it needs to be to fit the LP- or QP-problem presented above. If applicable non-linear constraints can be seen as more general than the linear ones. Implementation of non-linear constraints could result in a model that is applicable in a more practical environment.

Ignoring the problems related to making the constraint non-linear, there is another trade off in this case. If we were to switch from $\delta\beta$ to $\beta$ the allocation constraint would be more desirable, but the goal function would change as well. This would mean that implementing the transaction cost would not be possible in the way it is done in this thesis. This creates a trade-off where this thesis emphasize the consideration of transaction costs, rather than a different allocation constraint.

The formulation of the allocation constraint can also be motivated by that it is unusual that the share of wealth allocated in the stock is a hard constraint. A fund are more probable to have a vision of a certain maximum or minimum share invested in the stock. If they were to exceed these limits they would probably try to rebalance within a few days, rather than strictly every day. They would not want a "buy-action" or "sell-action" to launch just as the limit is crossed.

### 4.1.2 Market liquidity

A phenomenon not implicitly incorporated in this thesis, but important to keep in mind, is market liquidity. The market liquidity is the availability of buyers and sellers of a stock. If an investor sells large enough amounts of a stock the seller will need to find several buyers willing to buy at different prices. Another example of an illiquid market is the case when an investor is unable to sell a stock because there is no buyer available. This is called the *cost of liquidation*, meaning that if a large amount of stocks is to be sold at an illiquid market the price will decrease. Further explanation on this subject is presented in [McNeil et al., 2005]. The stock used in this thesis is simply assumed to be of a sufficiently traded stock on a liquid market. By implementing the constraint of maximum amount of stocks allowed to be traded this thesis reduces some effects of the market liquidity. However, a large enough initial amount of wealth should still imply a cost of liquidation. Implementation of a cost of liquidiation is left for further investigation. The results are imagined to be the same as an exponential or quadratic term added to the transaction cost.

## 4.2 Varying parameters

### 4.2.1 Risk aversion parameter, $\gamma$

One could argue that the risktaking investor, represented by the non-MVC portfolio, buys in the early time periods to have maximum exposure in the long term because the overall trend is upwards.

An interesting discussion point is the fact that the MVC-portfolio trades a lot more both up and down. This is due to the MVC-criterion; because it is bound by the end variance not being too large it is profitable to buy and sell during the ups and downs of the stock. It is, however, important to keep in mind that the transaction cost, $\lambda$, is here quite low resulting in a high trading frequency, which is an undesirable result. This can be seen in Figure 3.3 above.

### 4.2.2 Transaction cost, $\lambda$

An interesting result is the fact that the behaviour of the non-MVC portfolio barely changes until the transaction cost is very high. In Figure 3.6 a desirable constraining effect can be seen. Here the transaction costs are 0.45% which can be seen as unreasonable. However, this does not matter since the point is only to constrain the algorithm to achieve a desirable trading behaviour. Without a doubt high transaction cost has the best constraining effect among the parameters of this model. However, in this case the transaction cost is so high that the non-MVC portfolio barely ends up with the highest portfolio value.

### 4.2.3 Amount allowed to be traded

The amount allowed to be traded has a clear constraining effect on the behaviour of the algorithm. This is shown in Figure 3.7, Figure 3.8 and Figure 3.9, as well as in Figure 3.19 above. This is an unfortunate result of the trade-off in Section 4.1.1 above.

The explanation behind that the maximum amount of stocks owned by the MVC portfolio increases is due to the fact that the portfolio has time to sell back shares. This can be done without increasing the end variance above the trade-off limit the mean-variance criterion creates.

## 4.3 Scenarios

### 4.3.1 Bad scenario

This scenario could be seen as to represent when the economy is facing a crash. Imagine a scenario where the investor has been through a bull market allocating 70% in the stock, then a crash hits. As expected the algorithm will sell as much as possible as fast as possible - a desirable result.

A limitation here is the fact that the mean-variance constrained portfolio has higher variance than the non constrained one when the market is in a crash or a downward trend. This could be motivated as a precaution implemented in the MVC-portfolio. The MVC constrains the end variance of $\delta\beta S$ since that is the goal function implemented in Equation (2.10) above, leading to the algorithm being constructed not to make drastic moves. This result is, however, not an undesirable result because of the constraining effects on trading of the MVC-portfolio.

### 4.3.2 Shifting scenario

As seen in Figure 3.17 above the resulting trading behaviour is reasonable. Firstly the non-MVC portfolio sells as much as possible then buys as much possible, while the MVC portfolio is restricted. The MVC portfolio then prioritizes to buy and sell small shares, during the selling and buying trend, to maximize the short-term portfolio values.

### 4.3.3 Technical scenario: Questionable trading behaviour

The trading behaviour shown by the non-MVC portfolio in Figure 3.19 above can be explained by the goal function representation. The most weight is allocated to maximizing the end value of the portfolio. Due to transaction costs and end value focus the non-MVC portfolio will maximize it's amount of stocks in the end.

Another reason is the effect of the maximum trade constraint. Because of it the trading algorithm does not have time to sell all stocks and then buy them back. The algorithm then simply prioritizes having a lot of stocks in the end.

### 4.3.4 Technical scenario: Simulated stock values

Even though the trading behaviour change the strategy and the reactions do not. The algorithm simply reacts differently when the evolution of the stock is less volatile. Important to point out is that the high trading frequency in Figure 3.20 might be a result of the limitations of the Markov chain. The generated stock values are simply not as reliable as in the case of Figure 3.21 and Figure 3.22.

As can be seen in Figure 3.23, Figure 3.24, Figure 3.25 and Figure 3.26 the reactions to varying parameters are as expected and the same as in Section 3.1 above. This shows that the only change when increasing number of simulations is the reliability of the mean value of the stock. However, as mentioned above the point is to regulate the algorithm to react as desirable as possible. By increasing the simulations the trading behaviour converge toward less frequency - which is desirable. A significantly desirable result is seen in Figure 3.26 where the number of simulations are high and the transaction costs are high - leading to an almost constant allocation of the MVC-portfolio.

As mentioned above an optimal amount of generated simulations would be infinity, but that is not possible. In a real application this algorithm would be used with significantly more simulations using a faster computer, resulting in a better model. However, as seen above a similar result can be achieved by increasing the transaction costs - constraining the algorithm to desirable low frequency trading. One might argue that the only way to constrain the algorithm enough to achieve a desirable result is to set unreasonably high transaction costs.

## 4.4 Future Studies

This section proposes future works which can build upon this thesis. This includes e.g. more advanced tests, theories outside the scope of this thesis or simply changes to assumptions made in this thesis.

### 4.4.1 Risk management

An arguably even better risk management application might be to implement a Value-at-Risk (VaR) constraint. It is widely adopted in the finance industries and describes the maximum losses a portfolio could suffer at a certain probability level, under distributional assumptions. A method using VaR as a constraint is presented in [Yiu et al., 2010]. They introduce a Maximum Value-at-Risk (MVaR) constraint and argue that the traditional log-normal variance model can not catch the extreme movements of the stock. They, however, consider a dynamic programming approach and consumption as well as investment as their optimization problem. This makes their approach outside of the scope of this thesis where quadratic- and linear programming problems are considered.

### 4.4.2 Several assets

An interesting development of this thesis would be to implement several risky assets in the mean-variance portfolio optimization problem described above. This is carried out in e.g. [Costa and Araujo, 2008], [Costa and Nabholz, 2007] and [Li and Ng, 2000]. This approach would be a more realistic scenario than in my case and would result in a better portfolio mix. However, it might be less illustrative as to the behaviour of the trading algorithm. It would simply result in more plots of $\beta$s with different evolutions depending on that stock's evolution.

Other complications would arise if several assets were to be implemented. The covariance-matrix would have to represent the covariances between all assets as well as between the asset values themselves, resulting in a $(k+n)\times(k+n)$ dimensional matrix. When predicting the stock price values several periods ahead, multivariate correlations between the inputs (old stock price values) would need to be analysed. The values of $\beta$ would also have to be represented by a matrix instead of a vector.

### 4.4.3 Short selling

By incorporating short selling in the model an even more realistic model could be created. This is carried out in e.g. [Dombrovskii and Obyedko, 2015]. However, this should only give a result as if there were more resources available to the investor. The behaviour of the trading algorithm would thereby be the same as in the model where short selling is not allowed.

### 4.4.4 Risk-free rate

In the case of this thesis the risk-free rate is, as mentioned above, set to be 0. To evaluate this algorithm in a more academic environment a risk-free bank account with a small interest rate could be incorporated. By varying the risk-free rate of the bank account interesting results might have appeared. This should in turn lead to the stock being more or less desirable. The result would be an algorithm allocating more in the bank account instead of in the risky asset.

### 4.4.5 Goal function

The goal function used in this thesis could obviously be changed in several ways. Depending on the investor's preferences different changes could be made. The possible changes explained below is not carried out in this thesis because the focus is elsewhere; this thesis assumes knowledge of the stocks up to time $t + k$ and wants to maximize the end value of the portfolio and the intermediate excess returns.

One interesting test would be to weight the different stock values higher when they are closer to current time, or equivalently adding a punishment factor increasing with time. This means that the investor would rely more on values predicted close to today. This should result in higher frequency in initial trades and lower frequency further ahead. How this can be reasonably implemented is left for further studies on the subject.

Another change could be to maximize all end values after every time period. This would mean that the goal is to maximize the portfolio value every day instead of the end value. Alternatively the goal function could be changed as to maximize only the excess returns every day. These two approaches should have a similar result as they maximize value every day, leading to the appearance of the undesirable result of high trading frequency.

## 4.5 Conclusions

In this thesis an alternative approach to algorithmic trading by portfolio optimization is proposed. The optimization is carried out using quadratic programming principles subject to an intermediate mean-variance criterion, transaction costs and restrictions on trading amounts. The main product is the trading algorithm and the resulting behaviour in different scenarios. In every case the end value of the MVC portfolio is equal or higher than the static portfolio. The trading behaviour of the algorithm shows rationality and reacts as expected when constraining parameters are varied as well as in different scenarios. Even though several results show undesirable high trading frequencies, ways to regulate that are found. When transaction costs and number of simulations are increased the trading behaviour shows a desirable result of less frequency. The model constructed in this thesis could be applied as a supporting tool to recognize different states of the economy and then automatically reallocate resources to maximize portfolio value. With real-time data and unlimited computer power an even more refined model could be constructed. The trading strategy could then be updated after every time period, using e.g. *model predictive control*, leading to updated long term trading strategies.

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