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YANG-MILLS CONDENSATES IN COSMOLOGY

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Abstract

The cosmological constant is one of bigger mysteries in modern fundamental physics and cosmology. In this work we investigate one of the possible interpretations for the cosmological constant. The model a studies spatially-homogeneous mode of the Yang-Mills field known as the Yang-Mills condensate and throughout the work we discuss the dynamics and the time evolution of the homogeneous isotropic Yang-Mills condensate in the expanding Universe. The description of the Yang-Mills field is given for a completely classical field and an effective Lagrangian in the one-loop approximation. The stability of those solutions has been studied in both perturbative and non-perturbative cases. The other approach featuring the functional renormalization group method is discussed in last part of this work and a non-perturbative Yang-Mills Lagrangian is retrieved. This result is compared to the one-loop toy model. The conclusion discusses the cosmological implications of the condensate and its possible connection to the Dark Energy.

Populärvetenskapligt sammanfattning

I den här uppsatsen diskuterar vi flera möjligheter om hur man kan förklara den kosmologiska konstanten. År 1998 genomfördes observationer av typ 1a supernovor som visade att universums expansionshastighet ökar. Det här fenomenet blev kallat mörk energi. Sedan dess har flera olika hypoteser framlagts. Vi ska studera en teori som använder Yang-Millsfält som en mekanism som hjälper att klargöra flera problem med det här fenomenet. Vi finner en ny form för fältens effektiva Lagrangetäthet, som inte använder störningsteori men ger snarlika resultat som de kända approximationerna.

Contents

1 Introduction

The observations of the type 1a supernovae and the cosmic background radiation led to a conclusion that the Universe is expanding with acceleration.[1] The data obtained from the WMAP suggests the effective energy of the substance that causes the acceleration corresponds to 73% of the total energy composition of our Universe, while other 23% being dark matter and the rest being the baryonic matter and radiation.[2] Although at this point phenomenology of this "Dark Energy" is quite satisfactory, a working theoretical model for this component of the Universe still remains undiscovered.

The ground state of Yang-Mills fields plays an important role in particle physics in the form of gluon condensate, while the Yang-Mills field of the SU(3) gauge symmetry is a primary model for the Quantum Chromodynamics (QCD) interactions. Yang-Mills fields also have several applications in cosmology, being an adequate model for explanation of Cosmic Inflation and Dark Energy. This means that further investigation of the ground state dynamics is imperative for understanding the fundamental laws of physics.

Cosmological Constant (CC) scenario with the equation of state $w = p/\epsilon = -1$ is the most accepted observational model and it is supported by a significant amount of observational data. The initial theoretical ideas were to connect the existence of the Dark Energy (DE) to the levels of the vacuum energy predicted by the Quantum Field Theory (QFT), but the results had an extreme divergence with the theoretical observations. For example, the vacuum expectation value for the Higgs field at the scale of electroweak symmetry breaking $\langle 0|H(x)|0\rangle \sim 100\text{GeV}$ yields the Higgs condensate contribution $\Lambda^{EW}_{vac} \sim \langle 0|H(x)|0\rangle^4 \sim 10^8\text{GeV}^4$, while the corresponding value based on observations for the DE is $\Lambda_{\text{cosm}} = 2.5 \times 10^{-47}\text{GeV}^4$. This drastic difference(known as "the vacuum catastrophe") is a serious issue since it implies that the existing models fail to describe the Universe accurately. This means that any theoretical framework for the CC should not only explain the smallness of the CC but also its positivity and should give strong reasons for its existence. Many of the existing theoretical models, for example, those that utilize simple scalar fields cannot provide sufficient explanations for this behavior. Although this problem has been investigated for almost two decades a consensus in the scientific community regarding the model for CC has not yet been reached.[7]

The ground state of the Universe receives contributions from various existing quantum fields, such as corrections from quantum gravity and QCD.[7] This correction appears due to the graviton-exchange interactions between virtual elementary particles and it provides a contribution term $\Lambda_{\text{cosm}} \sim Gm^6$ to the energy density, where m is a characteristic mass of light particles. Later this relation was more accurately specified and written in terms of the fundamental constants, through the minimal and the maximal mass scales for hadrons[7]

$$\Lambda_{\text{cosm}} = \frac{m_\pi^6}{(2\pi)^4 M_{Pl}^2} \simeq 3.0 \times 10^{-47}\text{GeV}^4 \tag{1}$$

This value is extremely close to the value of the CC that is currently observed, which is a remarkable coincidence. The way the QCD contributes to the ground state is quite unique, since in this case it emerges from the perturbative quantum-topological fluctuations of the quark and gluon fields predicted in the instanton theory[7]

$$\Lambda_{\rm inst}^{\rm QCD} = -(5 \pm 1) \times 10^{-3} {\rm GeV}^4.$$

The theory that could explain the evolution of such contribution has not been developed yet. Thus CC from the aforementioned effects is considered to be

$$\overline{\Lambda} = \Lambda_{\rm cosm} + \Lambda_{\rm inst}^{\rm QCD} \tag{2}$$

There are several possible ways to avoid discrepancies between the predictions for the DE. The first one interprets the observed CC as a non-vanishing effect in the expanding Friedmann-Lemaitre-Robertson-Walker (FLRW) Universe. In this way, the de-Sitter phase can be dynamically initiated in expanding Universe by a topological (auxiliary) non-propagating field in the non-perturbative QCD vacuum which does not possess a canonical kinetic term in analogy to the topologically ordered phases in condensed matter systems.[7]

Another possible pathway to resolve the CC problem relies on a mechanism of dynamical compensation for short-distance vacuum fluctuations, in particular, during the electroweak and QCD phase transition epochs, such that they no longer affect the macroscopic evolution of the Universe.

In this work one of the possible models for the DE will be explored, which links the dynamics of the CC to a Yang-Mills condensate (YMC). This model states that the effect perceived as the Cosmological Constant is composed of several different components and the YMC corresponds to the dynamical component of the physical vacuum that contributes to its ground state. This allows for the compensation mechanism to occur without a specific fine-tuning. This means that we can require from the general quantities such as energy density to take specific values at the current times, but we do not need to specify most of the parameters.

The perturbation theory for the Yang-Mills fields was studied for one-, two- and three-loop approximations. [4, 5] While giving satisfactory qualitative results, these approximations are not sufficient for a complete understanding of dynamics of these fields. The results given by the non-perturbative approach retain most of the properties of the fixed-order perturbative models, although these models also take some approximations, such as an imposed self-duality of the field.[3, 8] In the final section of this paper we will attempt to expand this work, and compare it to the one-loop approximation.

One of the key features that is evident from the one-loop approximation is that the equation of state for the condensate develops from $w = \frac{1}{3}$ to $w = -1$ over time.[8, 18] This means that it is close to $w = \frac{1}{3}$ for high values of the redshift (early Universe) then at some point, which is determined by the initial conditions, it goes through a transition and eventually becomes close to the current observed value for the CC, $w = -1$. This makes the YMC one of the best models for the Dark Energy. The value $w = \frac{1}{3}$ corresponds to the classical Yang-Mills field, thus we will start the analysis with its description.

2 Classical Yang-Mills fields.

As we will see, the classical YM field does not behave as the observed CC, but it is still an instructive example that shows differences between quantum and classical systems. We start with Yang-Mills field strength, which is similar to the one used in QCD, although it is important to stress that this field is a separate entity from the one that we know from the Standard Model:

$$\mathcal{F}^a_{\mu\nu} = \partial_\mu A^a_m - \partial_\nu A^a_\mu + g_{\rm YM} e^{abc} A^b_\mu A^c_\nu \tag{3}$$

where $a, b, c = 1 \ldots N^2 - 1$ are isotopic (adjoint representation) indices and Lorenz $\mu, \nu = 1 = 0, 1, 2, 3$ are Lorentz indices. $g_{\rm YM}$ is the gauge coupling constant.

The classical Lagrangian has a standard form

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a \tag{4}$$

One typically uses temporal (Hamilton) gauge that makes the asymptotic states of the S-matrix to contain the physical transverse modes only. The corresponding condition is

$$A^a_0 = 0. \tag{5}$$

Due to the local isomorphism of the isotopic SU(2) gauge group to the SO(3) group of spacial rotations in 3-dimensional space, the unique (up to a rescaling) SU(2) YM configuration can be parameterized in terms of a scalar time-dependent spatially homogeneous field. [14, 15, 16]

With a mixed space-isotopic orthonormal basis e^a_i, $a, i = 1, 2, 3$ in the temporal gauge (5), such that the field A^a_μ transforms into a tensor field A_{ik} as follows

$$e^a_i A^a_k \equiv A_{ik}, \quad e^a_i e^a_k = \delta_{ik}, \quad e^a_i e^b_i = \delta_{ab}. \tag{6}$$

Then the tensor A_{ik} can be split into two parts

$$A_{ik}(t, \overrightarrow{x}) = \delta_{ik} V(t) + \tilde{A}_{ik}(t, \overrightarrow{x}), \quad \langle \tilde{A}_{ik}(t, \overrightarrow{x}) \rangle = \int \mathrm{d}^4 \tilde{A}_{ik}(t, \overrightarrow{x}) = 0. \tag{7}$$

$V(t)$ is identified with isotopic and homogeneous classical YM condensate, and $\tilde{A}_{ik}(t, \overrightarrow{x})$ are spatially-inhomogeneous space YM-waves.. In QFT formulation, the Yang-Mills the inhomogeneous waves are interpreted as YM quanta (gluons) while $V(t)$ contributes to the vacuum ground state of the theory. In the following we will not be considering the non-homogeneous part; we assume that the contribution from this part is either very small or it dissipates entirely. We introduce the following quantities

$$\mathcal{A}^a_\mu \equiv g_{\rm YM} A^a_\mu, \quad \mathcal{F}^a_{\mu\nu} \equiv g_{\rm YM} F^a_{\mu\nu} = \partial_\mu \mathcal{A}^a_\mu + f^{abc} \mathcal{A}^b_\mu \mathcal{A}^c_\nu. \tag{8}$$

$U(t)$ is the quantity that will be used from now on to refer the YM condensate instead of $V(t)$ because it is more convenient:

$$U(t) \equiv g_{\rm YM} V(t). \tag{9}$$

The energy-momentum tensor for this field takes form:

$$T_{\mu\nu} = -\mathcal{L} g_{\mu\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \mathcal{A}^a_\nu)} \partial_\nu \mathcal{A}_\mu = \frac{1}{4} \mathcal{F}^a_{bc} \mathcal{F}^{bc}_a g_{\mu\nu} - \frac{1}{4} \frac{\partial(\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta})}{\partial(\partial_\mu \mathcal{A}^a_\nu)} \partial_\nu \mathcal{A}_\mu \tag{10}$$

$$= \frac{1}{4} \mathcal{F}^a_{bc} \mathcal{F}^{bc}_a g_{\mu\nu} - \frac{1}{2} \mathcal{F}_{\alpha\beta} \frac{\partial \mathcal{F}_{\alpha\beta}}{\partial(\partial_\mu \mathcal{A}^a_\nu)} \partial_\nu \mathcal{A}_\mu = \frac{1}{4} \mathcal{F}^a_{bc} \mathcal{F}^{bc}_a g_{\mu\nu} - \frac{1}{2} \mathcal{F}_{\alpha\beta} (\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha) \partial_\nu \mathcal{A}_\mu \tag{11}$$

$$= \frac{1}{4} \mathcal{F}^a_{bc} \mathcal{F}^{bc}_a g_{\mu\nu} - \mathcal{F}^a_{\mu\lambda} \mathcal{F}^{\lambda b}_a g_{b\nu} \tag{12}$$

And the equation of motion is

$$\partial^\mu \ F^a_{\mu\nu} + g e^{abc} A^{bm} \mathcal{F}^c_{\mu\nu} = 0 \tag{13}$$

In the following we are using the spatially-flat Friedman-Lemaítre-Robertson-Walker conformal metric

$$g_{\mu\nu} = a^2(\eta) \mathrm{diag}(1, -1, -1, -1), \quad \sqrt{-g} = a^4(\eta), \quad t = \int a(\eta) \mathrm{d}\eta,$$

where $a(\eta)$ is the scale factor. The equation of motion and the Einstein equation for YMC read [7]

$$\left(\frac{\delta^{ab}}{\sqrt{-g}} \partial_\nu \sqrt{-g} - f^{abc} \mathcal{A}^c_\nu \right) \frac{\mathcal{F}^{\mu\nu}_b}{\sqrt{-g}} = 0 \tag{14}$$

$$\frac{1}{\varkappa} \left(R^\nu_\mu - \frac{1}{2} \delta^\nu_\mu R \right) = - \frac{\mathcal{F}^a_{\mu\lambda} \mathcal{F}^{\nu\lambda}_a}{g^2_{\mathrm{YM}} \sqrt{-g}} + \delta^\nu_\mu \frac{\mathcal{F}^\sigma_{\alpha\beta} \mathcal{F}^{\alpha\beta}_\sigma}{4 g^2_{\mathrm{YM}}}, \tag{15}$$

where $\varkappa$ is the gravitational constant and R^ν_μ and R are the Ricci tensor and the Ricci scalar respectively. If we ignore the inhomogeneous fluctuations, these equations of motion are reduced to

$$\frac{3}{\varkappa} \frac{(a')^2}{a^4} = \frac{3}{2 g^2_{\mathrm{YM}} a^4} \left((U')^2 + U^4 \right), \quad U'' + 2U^3 = 0, \tag{16}$$

The second equation in (16) can be exactly integrated leading to the following solutions

$$(U')^2 + U^4 = C, \quad \int_{U_0}^{U} \frac{\mathrm{d}x}{\sqrt{C - x^4}} = \eta, \quad U_0, C = const, \tag{17}$$

$$U'(0) = 0, \quad U_0 = C \quad \to \quad U(\eta) \simeq U_0 \cos(\frac{6}{5} U_0 \eta) \tag{18}$$

So the classical YM condensate behaves as an ultra-relativistic medium with energy density $\varepsilon \sim 1/a^4$ and the equation of state $p_{\mathrm{YM}} = \varepsilon_{\mathrm{YM}}/3$ (radiation) [9].

3 Yang-Mills effective Lagrangian.

As it was shown in the previous section, the classical Yang-Mills condensate behaves in a way similar to radiation and thus is vastly different from the observed dark energy. It is now necessary to incorporate the quantum corrections into the classical action and, as it will be shown in the following sections, these additions drastically change the picture. The way these additions are usually handled is through the effective Lagrangian.

We will name the field contraction as J:

$$J = \mathcal{F}^{\sigma}_{\alpha\beta}\mathcal{F}^{\alpha\beta}_{\sigma} = \mathcal{F}^2. \tag{19}$$

The effective Lagrangian for Yang-Mills field has a form

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}\frac{\mathcal{F}^2}{g_{\text{YM}}(J)^2} \tag{20}$$

The main difference here is that the coupling strength is no longer considered to be a constant and depends on the field strength. So, if we follow a similar derivation, the energy-momentum tensor will take form

$$\begin{aligned} T^{\nu}_{\mu} &= \frac{1}{g^2_{\text{YM}}}\left(-\mathcal{F}^a_{\mu\sigma}\mathcal{F}^{\nu\sigma}_a + \frac{1}{4}\delta^{\nu}_{\mu}\mathcal{F}^a_{\rho\sigma}\mathcal{F}^{\rho\sigma}_a\right) + \frac{\mathrm{d}}{\mathrm{d}J}\left(\frac{1}{g^2_{\text{YM}}(J)}\right)\mathcal{F}^a_{\mu\sigma}\mathcal{F}^{\nu\sigma}_a \\ &= \frac{1}{g^2_{\text{YM}}}\left(-\mathcal{F}^a_{\mu\sigma}\mathcal{F}^{\nu\sigma}_a + \frac{1}{4}\delta^{\nu}_{\mu}\mathcal{F}^a_{\rho\sigma}\mathcal{F}^{\rho\sigma}_a + \frac{\beta(g^2_{\text{YM}})}{2}\mathcal{F}^a_{\mu\sigma}\mathcal{F}^{\nu\sigma}_a\right) \end{aligned} \tag{21}$$

$$\tag{22}$$

The trace of the energy-momentum tensor for the classical field vanishes, while for this one it is non-zero and equals

$$T^{\mu}_{\mu} = \frac{\beta(g^2_{\text{YM}})}{2g^2_{\text{YM}}}J \tag{23}$$

4 Dynamics of Yang-Mills condensate.

If we do not introduce any constraints on $\beta(g^2_{\text{YM}})$, the energy-momentum tensor for the ground state of the field can be written as

$$T^{\nu}_{\mu} = -\left[1 - \frac{1}{2}\beta(g^2_{\text{YM}})\right]\frac{\mathcal{F}^a_{\mu\lambda}\mathcal{F}^{\nu\lambda}_a}{g^2_{\text{YM}}\sqrt{-g}} + \delta^{\nu}_{\mu}\frac{\mathcal{F}^{\sigma}_{\alpha\beta}\mathcal{F}^{\alpha\beta}_{\sigma}}{4g^2_{\text{YM}}} \tag{24}$$

The equation of motion for the condensate is

$$\left(\frac{\delta^{ab}}{\sqrt{-g}}\partial_{\nu}\sqrt{-g} - f^{abc}\mathcal{A}^c_{\nu}\right)\left[1 - \frac{1}{2}\beta(g^2_{\text{YM}})\right]\frac{\mathcal{F}^{\mu\nu}_b}{g^2_{\text{YM}}\sqrt{-g}} = 0 \tag{25}$$

In the FLRW metric

$$\frac{1}{\varkappa}\left(R^{\nu}_{\mu}-\frac{1}{2}\delta^{\nu}_{\mu}R\right)=-\left[1-\frac{1}{2}\beta(g^2_{\rm YM})\right]\frac{\mathcal{F}^{a}_{\mu\lambda}\mathcal{F}^{\nu\lambda}_{a}}{g^2_{\rm YM}\sqrt{-g}}+\delta^{\nu}_{\mu}\frac{\mathcal{F}^{\sigma}_{\alpha\beta}\mathcal{F}^{\alpha\beta}_{\sigma}}{4g^2_{\rm YM}} \tag{26}$$

If we put $\beta(g^2_{\rm YM})=0$ then we will come back to

$$\left(\frac{\delta^{ab}}{\sqrt{-g}}\partial_{\nu}\sqrt{-g}-f^{abc}\mathcal{A}^{c}_{\nu}\right)\frac{\mathcal{F}^{\mu\nu}_{b}}{g^2_{\rm YM}\sqrt{-g}}=0 \tag{27}$$

$$\frac{1}{\varkappa}\left(R^{\nu}_{\mu}-\frac{1}{2}\delta^{\nu}_{\mu}R\right)=-\frac{\mathcal{F}^{a}_{\mu\lambda}\mathcal{F}^{\nu\lambda}_{a}}{g^2_{\rm YM}\sqrt{-g}}+\delta^{\nu}_{\mu}\frac{\mathcal{F}^{\sigma}_{\alpha\beta}\mathcal{F}^{\alpha\beta}_{\sigma}}{4g^2_{\rm YM}} \tag{28}$$

If we, on the the other hand, put $\beta(g^2_{\rm YM})=2$ we will get the exact partial solution of this system.

The field contraction can be written in terms of the condensate and the scale factor as

$$\mathcal{F}^{\sigma}_{\alpha\beta}\mathcal{F}^{\alpha\beta}_{\sigma}=-\frac{6}{a^4}\left(a^2\dot{U}^2-\frac{1}{4}U^4\right) \tag{29}$$

The energy density T^0_0

$$T^0_0=\frac{3}{2g^2_{\rm YM}a^4}\left[\frac{2-\beta(g^2_{\rm YM})}{2}\left(a^2\dot{U}^2+\frac{1}{4}U^4\right)-\frac{\beta(g^2_{\rm YM})}{2}\left(a^2\dot{U}^2-\frac{1}{4}U^4\right)\right] \tag{30}$$

The trace of the energy momentum tensor is

$$T^{\mu}_{\mu}=-\frac{3\beta(g^2_{\rm YM})}{g^2_{\rm YM}a^4}\left(a^2\dot{U}^2-\frac{1}{4}U^4\right) \tag{31}$$

This exact partial solution presents an important object for further analysis.

4.1 One loop approximation for $\beta(g^2_{\rm YM})$.

In order to understand the dynamics of the YMC with the effective Lagrangian it is necessary to employ the simplest approximation. While not being incredibly accurate, this approximation, however, shows some the of the required properties.

We identify

$$J=\frac{\mathcal{F}^{\sigma}_{\alpha\beta}\mathcal{F}^{\alpha\beta}_{\sigma}}{\sqrt{-g}}$$

The dependence of the coupling constant is determined by the RG equation

$$2J\frac{{\rm d}g^2_{\rm YM}}{{\rm d}J}=g^2_{\rm YM}\beta(g^2_{\rm YM}) \tag{32}$$

For the one-loop approximation the form of β-function is determined by

$$\beta(g^2_{\rm YM})=-\frac{bg^2_{\rm YM}}{16\pi^2},\ g^2_{\rm YM}=\frac{32\pi^2}{b\ln(|J|/\lambda^4)} \tag{33}$$

where λ is the scale parameter and b is the one-loop β-function coefficient. The Lagrangian will take form

$$\mathcal{L}_{\rm eff}^{1-\rm loop} = -\frac{bJ}{128\pi^2}\ln\left(\frac{|J|}{(\xi\lambda)^4}\right) \tag{34}$$

Using standard variational techniques we obtain the equations of motion and after substitution into the general relativity equations the result is

$$\begin{aligned}\frac{1}{\varkappa}\left(R_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R\right) = T_\mu^{\nu,\rm matter} + \overline{\Lambda}\delta_\mu^\nu + \frac{b}{32\pi^2}\frac{1}{\sqrt{-g}}\Big[\Big(-\mathcal{F}_{\mu\lambda}^a\mathcal{F}_a^{\nu\lambda} \\ + \frac{1}{4}\delta_\mu^\nu\mathcal{F}_{\sigma\lambda}^a\mathcal{F}_a^{\sigma\lambda}\Big)\ln\frac{e|\mathcal{F}_{\alpha\beta}^a\mathcal{F}_a^{\alpha\beta}|}{\sqrt{-g}} - \frac{1}{4}\delta_\mu^\nu\mathcal{F}_{\sigma\lambda}^a\mathcal{F}_a^{\sigma\lambda}\Big]\end{aligned} \tag{35}$$

$$\left(\frac{\delta^{ab}}{\sqrt{-g}}\partial_\nu\sqrt{-g} - f^{abc}\mathcal{A}_\nu^c\right)\left(\frac{\mathcal{F}_b^{\mu\nu}}{\sqrt{-g}}\ln\frac{e|\mathcal{F}_{\alpha\beta}^a\mathcal{F}_a^{\alpha\beta}|}{\sqrt{-g}(\xi\lambda)^4}\right) = 0 \tag{36}$$

The Friedman equation for condensate and the expansion law are

$$\frac{6}{\varkappa}\frac{\dot{a}^2}{a^3} = \overline{\Lambda} + \frac{3b}{16\pi^2 a^4}\left[(U')^2 - \frac{1}{4}U^4\right] \tag{37}$$

$$\frac{\partial}{\partial\eta}\left(U'\ln\frac{6e|(U')^2 - \frac{1}{4}U^4|}{a^4(\xi\lambda)^4}\right) + \frac{1}{2}U^3\ln\frac{6e|(U')^2| - \frac{1}{4}U^4}{a^4(\xi\lambda)^4} = 0 \tag{38}$$

The first integral of this system is

$$\frac{3}{\varkappa}\frac{\dot{a}^2}{a^4} = \overline{\Lambda} + T_0^{0,\rm YM} \tag{39}$$

Where

$$T_0^{0,\rm YM} = \frac{3b}{64\pi^2 a^4}\left(\left[((U')^2 + \frac{1}{4}U^4)\right]\ln\frac{6e|((U')^2 - \frac{1}{4}U^4|}{a^4(\xi\lambda)^4} + (U')^2 - \frac{1}{4}U^4\right) \tag{40}$$

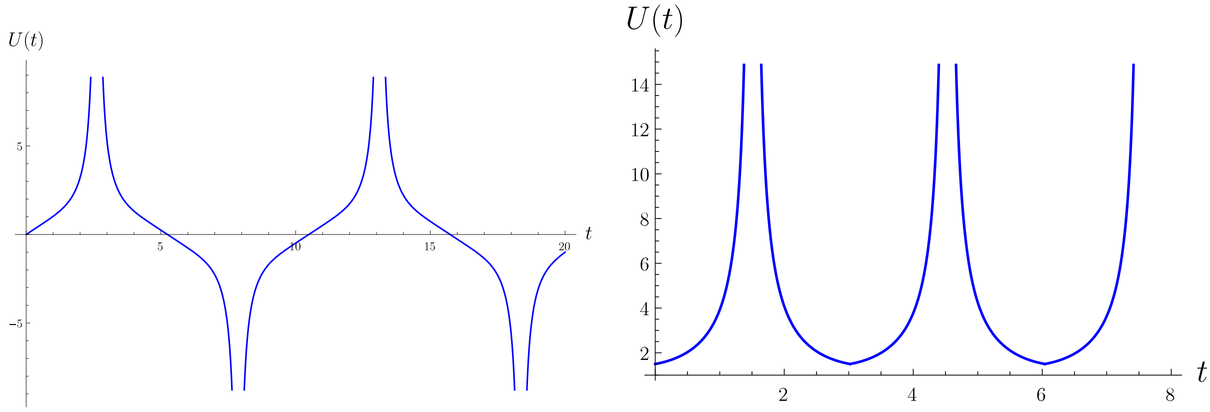
4.2 Solutions.

The logarithm in the equation (35,38,39) can vanish if its argument is equal to 1 and this leads to two exact partial solutions. We can represent this as a transcendent equation:

$$|Q| = 1, \quad Q \equiv \frac{32\pi^2 e}{11(\xi\Lambda_{QCD})^4} \qquad T_\mu^{\mu,\rm YM} = \frac{6\left[(U')^2 - \frac{1}{4}U^4\right]}{a^4(\xi\Lambda_{QCD})}, \tag{41}$$

which provides two possible distinct cases $Q = \pm 1$. Here the Λ_{QCD} is taken as the scale λ. One of these $Q = 1$ yields a positive constant energy density of the gluon condensate with minimum corresponding to the chromoelectric condensate

$$(U')^2 - \frac{1}{4}U^4 > 0, \qquad T_0^{0,\rm YM*} \equiv \frac{3b}{64\pi^2}\frac{(\xi\Lambda_{QCD})^4}{6e} > 0. \tag{42}$$



(a) Plot of the U(t) for the chromoelectric case.

(b) Plot of the U(t) for the chromomagnetic case. The minimums are supposed to be smooth, but the separate branches were retrieved numerically and this is the limitation of the algorithm; the exact solutions should be smooth near the minimums.

Figure 1: The graphical solutions for the equations (42),(43).

The second solution with $Q = -1$ corresponds to the chromomagnetic part of the condensate. This contribution is negative and has the same magnitude as the chromoelectric counterpart

$$(U')^2 - \frac{1}{4}U^4 < 0, \qquad T_0^{0,\mathrm{YM}*} \equiv -\frac{3b}{64\pi^2}\frac{(\xi\Lambda_{QCD})^4}{6e} < 0, \tag{43}$$

In order to identify the normalization constant ξ we employ the following compensation condition as it was done in [6]

$$\frac{33}{64\pi^2}\frac{(\xi\Lambda_{QCD})^4}{6e} = |\Lambda_{\mathrm{inst}}^{\mathrm{QCD}}|, \quad \xi \simeq 4 \tag{44}$$

In the present Universe with $a = a_0 = 1$ with the boundary condition $\tilde{U}_0 = 0$ implicit partial solutions for the homogeneous condensate reads

$$Q = \pm 1, \qquad \int_{\tilde{U}_0}^{\tilde{U}} \frac{\mathrm{d}u}{\sqrt{\frac{1}{4}u^4 \pm 1}} = \tilde{\eta}, \qquad \tilde{U} = U\frac{6e^{1/4}}{\eta\Lambda_{QCD}}, \qquad \tilde{\eta} = \eta\frac{\xi\Lambda_{QCD}}{6e^{1/4}}. \tag{45}$$

corresponding to the chromoelectric (42) and chromomagnetic (43) solutions. These solutions exhibit the following properties:

- Symmetry: $\tilde{U}(-\tilde{\eta}) = -\tilde{U}(\tilde{\eta})$
- Periodicity: $\tilde{U}(\tilde{\eta} \pm T) = \tilde{U}(\tilde{\eta})$
- Continuous intervals and singularities: $\tilde{U}(\tilde{\eta} \to \pm T/4) \to \pm\infty$

where T is the period of oscillations.

Fig. 1a and Fig. 1b show the simplified version of the solutions, where all constants are taken to be one. This simplification still retains the key features, such as the quasiperiodic singularities.

So it can be concluded that the time evolution of the gluon condensate can be seen as a regular sequence of quantum tunneling transitions through the regular singularities in the quantum vacuum solution of the effective model (34). That implies the homogeneous gluon condensate is analogous to the topological condensate in the instanton theory of the QCD vacuum interpreted in terms of spatially-inhomogeneous gluon field fluctuations induced by quantum tunneling of the field through topological (spatial) barriers between different classical vacuums. [6]

4.3 Asymptotic analysis of the YM gluon condensate.

Now we will construct a general analytic approximation for the solution of this system in the case Q is not far from unity. One would expect that the general solution would exhibit a similar behavior to the partial one. If it is assumed that the initial energy density is not extremely large, one can show for the $Q = +1$ solution that the period of oscillations in conformal time is estimated in the following way:

$$T_\eta^+ \simeq \frac{4k(6e)^{1/4}}{a\xi\Lambda_{QCD}}, \quad k \equiv \int_0^\infty \frac{du}{1+\frac{1}{4}u^4} = \frac{\Gamma(1/4)^2}{2\sqrt{2\pi}} \approx 2.622. \tag{46}$$

In physical time this reads

$$T_t^+ \simeq \frac{4k(6e)^{1/4}}{\xi\Lambda_{QCD}} \simeq \frac{5.3}{\Lambda_{QCD}} \tag{47}$$

Following the same calculation one gets the corresponding period for the negative solution:

$$T_t^- \simeq \frac{\sqrt{2}k(6e)^{1/4}}{\xi\Lambda_{QCD}} \simeq \frac{1.86}{\Lambda_{QCD}} \tag{48}$$

The conformal time derivative of the gluon condensate energy density for $Q_0 > 1$ given by

$$\frac{\partial T_0^{0,\mathrm{YM}}}{\partial\eta} = -\frac{33\varkappa}{16\pi^2}\frac{a'}{a^5}\left[(U')^2 + \frac{1}{4}U^4 \ln\frac{6e|(U')^2 - \frac{1}{4}U^4|}{a^4(\xi\Lambda_{QCD})^4}\right]. \tag{49}$$

It is negative at $t = t_0$; its absolute value decreases and the value of $T_0^{0,\mathrm{YM}}(t)$ approaches $T_0^{0,\mathrm{YM}*}$, which corresponds to the exact de-Sitter solution given by (42). From the dimensions of the given quantities a rough approximate for the relaxation time of the YM energy can be retrieved:

$$t_r \simeq \frac{1}{\sqrt{\varkappa\varepsilon}}, \quad \varepsilon \equiv T_0^{0,\mathrm{YM}}(t = t_0) \tag{50}$$

It was shown that although the unobservable function $U(t)$ and its combination $(U')^2 + \frac{1}{4}U^4$ have periodic singularities the actual physical quantities $T_0^{0,\mathrm{YM}}(t = t_0)$,$T_\nu^{\mu,\mathrm{YM}}(t)$ are continuous. This continuity condition can only be met if $Q(t)$ oscillates with the same period as $U(t)$ and reached unity, so that the logarithm vanishes and thus the singularities that are coming from the term $(U')^2+\frac{1}{4}U^4$ are compensated. The $Q(t)$ cannot cross through $Q = 1$, since that would lead to a change of sign in the first term in 30, when $Q = 1$. From these arguments one can conclude that the function $Q(t)$ satisfies the constraints $0 < Q(t) \leq 1$ for $0 < Q_0 < 1$ and $Q(t) \geq 1$ for $Q_0 > 1$. Thus the relaxation time for $Q(t)$ is t_r and is the same as for $T_0^{0,\mathrm{YM}}(t = t_0)$. The period of oscillations can be approximated as

$$T \equiv \frac{T_t}{2} \simeq \frac{2k(6e)^{1/4}}{\xi \Lambda_{QCD}} \tag{51}$$

It can be concluded that all of the important features of the general solution can be derived from qualitative arguments only, without choosing any particular parameters and without any reference to numerical approximations.

4.4 Analysis of the general solution.

Omitting the matter component in (35), the equations of the YMC in physical time read

$$\frac{6}{\varkappa}\Big[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\Big] = \overline{\Lambda} + T_\mu^{\mu,\mathrm{YM}}, \quad \frac{3}{\xi}\frac{\dot{a}^2}{a^2} = \overline{\Lambda} + T_0^{0,\mathrm{YM}} \tag{52}$$

In order to find the general solutions of these equations we introduce an auxiliary continuous function g=g(t)

$$T^{\mu,\mathrm{YM}} = (g(t)+1)\Big[T_0^{0,\mathrm{YM}} - \frac{C}{4}\Big], \quad C \equiv -4\Lambda_{\mathrm{inst}}^{\mathrm{QCD}} = \frac{33}{16\pi^2}\frac{(\xi\Lambda_{\mathrm{QCD}})^4}{6e} \tag{53}$$

Thus the (52) can be rewritten as

$$\frac{6}{\varkappa}\Big[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\Big] = 4\Lambda_{cosm} + (g(t)+1)\Big[T_0^{0,\mathrm{YM}} - \frac{C}{4}\Big] \tag{54}$$

$$\frac{3}{\varkappa}\frac{\dot{a}^2}{a^2} = \Lambda_{\mathrm{cosm}} - \frac{C}{4} + T_0^{0,\mathrm{YM}}, \quad T_0^{0,\mathrm{YM}} = T_0^{0,\mathrm{YM}}(U, \dot{U}, a), \tag{55}$$

The equation for the scale factor is retrieved by excluding $T_0^{0,\mathrm{YM}}$

$$6\frac{\ddot{a}}{a} + 3(1-g(t))\frac{\dot{a}^2}{a^2} + \varkappa\Lambda_{\mathrm{cosm}}(g(t)-3) = 0 \tag{56}$$

The general solution for this equation is

$$a(t) = a^* \exp\Big[\sqrt{\frac{\varkappa\Lambda_{\mathrm{cosm}}}{3}}\times \int_{t_0}^{t} \frac{1+\sqrt{\frac{\Lambda_{\mathrm{cosm}}}{\varepsilon_0}} + \Big(1-\sqrt{\frac{\Lambda_{\mathrm{cosm}}}{\varepsilon_0}}\Big)\exp\Big(\sqrt{\frac{\varkappa\Lambda_{\mathrm{cosm}}}{3}}\Big)\Big[-3(t'-t_0)+\int_{t_0}^{t'} g(\tau)\mathrm{d}\tau\Big]}{1+\sqrt{\frac{\Lambda_{\mathrm{cosm}}}{\varepsilon_0}} - \Big(1-\sqrt{\frac{\Lambda_{\mathrm{cosm}}}{\varepsilon_0}}\Big)\exp\Big(\sqrt{\frac{\varkappa\Lambda_{\mathrm{cosm}}}{3}}\Big)\Big[-3(t'-t_0)+\int_{t_0}^{t'} g(\tau)\mathrm{d}\tau\Big]}\mathrm{d}t'\Big] \tag{57}$$

in terms of total energy density ε_0 and the initial value of the scale factor $a^* \equiv a(t = t_0)$ respectively.

We also retrieve the total energy density and the trace of the energy-momentum tensor as functions of physical time:

$$\frac{T_0^0(t)}{\Lambda_{\rm cosm}} = \frac{1 + \sqrt{\frac{\Lambda_{\rm cosm}}{\varepsilon_0}} + \left(1 - \sqrt{\frac{\Lambda_{\rm cosm}}{\varepsilon_0}}\right) \exp\left(\sqrt{\frac{\varkappa\Lambda_{\rm cosm}}{3}}\right) \left[-3(t' - t_0) + \int_{t_0}^{t'} g(\tau){\rm d}\tau\right]}{1 + \sqrt{\frac{\Lambda_{\rm cosm}}{\varepsilon_0}} - \left(1 - \sqrt{\frac{\Lambda_{\rm cosm}}{\varepsilon_0}}\right) \exp\left(\sqrt{\frac{\varkappa\Lambda_{\rm cosm}}{3}}\right) \left[-3(t' - t_0) + \int_{t_0}^{t'} g(\tau){\rm d}\tau\right]} \tag{58}$$

$$\frac{T_\mu^\mu(t)}{\Lambda_{\rm cosm}} = 4 + \frac{4(g(t) + 1)\left(1 - \sqrt{\frac{\Lambda_{\rm cosm}}{\varepsilon_0}}\right) \exp\left[\sqrt{\frac{\varkappa\Lambda_{\rm cosm}}{3}} \left(-3(t - t_0) + \int_{t_0}^{t'} g(\tau){\rm d}\tau\right)\right]}{1 + \sqrt{\frac{\Lambda_{\rm cosm}}{\varepsilon_0}} - \left(1 - \sqrt{\frac{\Lambda_{\rm cosm}}{\varepsilon_0}}\right) \exp\left(\sqrt{\frac{\varkappa\Lambda_{\rm cosm}}{3}}\right) \left[-3(t' - t_0) + \int_{t_0}^{t'} g(\tau){\rm d}\tau\right]} \tag{59}$$

where ε_0 is the energy density of the gluon condensate, and $\Lambda_{\rm cosm} \ll \varepsilon_0$ is the observed cosmological constant given by the equation (1). The expressions (58),(59) do not use any additional approximations and yield the general solutions for the system (52) as long as the $g(t)$ is known.

The exact form of the function $g(t)$ can be implicitly found from this differential equation:

$$\dot{g}^4 - \frac{8(\xi\Lambda_{\rm QCD})^4}{3e}(1 - g^2)^3 = 0 \tag{60}$$

A good approximation for the solution is:

$$g(t) \simeq A\cos(\frac{2\pi t}{T_g}) + (1 - A)\cos(\frac{6\pi t}{T_g}), \tag{61}$$

where

$$A = \frac{2}{k}\int_0^1 \frac{g}{(1 - g^2)^{3/4}} \cos\left(\frac{\pi}{2k}\int_0^1 \frac{{\rm d}x}{(1 - x^2)^{3/4}}{\rm d}g\right) \approx 1.14, \tag{62}$$

$$T_g = \frac{2(6e^{1/4})}{\xi\Lambda_{\rm QCD}} \int_0^1 \frac{g}{(1 - g^2)^{3/4}} \tag{63}$$

and k is defined by (46).

Assuming the asymptotic compensation condition

$$T_0^{0,{\rm YM}*}(t) > 0, \quad T_0^{0,{\rm YM}} + \Lambda_{\rm inst}^{\rm QCD} = 0, \quad T_0^0(t \to \infty) \to \Lambda_{\rm cosm} \tag{64}$$

for large times $t \gg T_g$ the general solution (57) takes the form

$$a(t) \simeq a^* \left[\frac{1}{2}\sqrt{\frac{\varepsilon_0}{\Lambda_{\rm cosm}}} \left(1 + \sqrt{\frac{\Lambda_{\rm cosm}}{\varepsilon_0}} - \left(1 - \sqrt{\frac{\Lambda_{\rm cosm}}{\varepsilon_0}}\right) e^{-3t\sqrt{\frac{\varkappa\Lambda_{\rm cosm}}{3}}}\right)\right]^{2/3} e^{t\sqrt{\frac{\varkappa\Lambda_{\rm cosm}}{3}}}, \tag{65}$$

where $a^* \equiv a(t = 0)$.

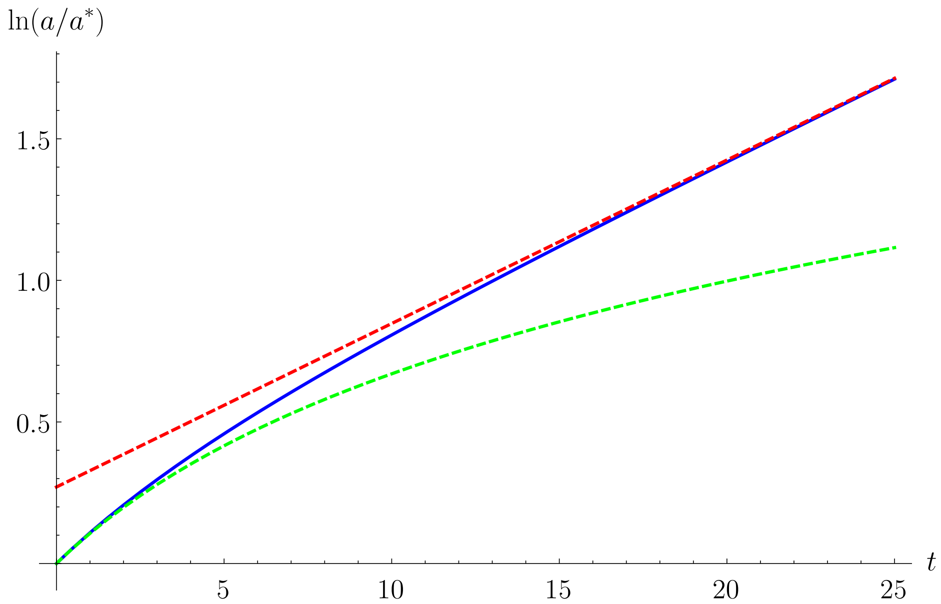


Figure 2: Plot of the time dependence of the logarithm of the scale factor $a(t)$ represented by the primary approximation (65) (blue), power-like law (67) (green and dashed), and the de-Sitter solution (71) (red and dashed). Most constants have been assigned with simple values, thus this plot is largely qualitative and the units of time are arbitrary.

The initial total energy of the Universe is large $\varepsilon \gg \Lambda_{\text{cosm}}$ at time scales

$$T_g \ll t \ll \frac{1}{\varkappa \Lambda_{\text{cosm}}} \tag{66}$$

the corresponding power-like solutions are

$$a(t) \simeq a^* \left(1 + \frac{3}{2}\sqrt{\frac{\varkappa \varepsilon}{3}} t\right)^{2/3}, \quad T^0_0(t) \simeq \frac{\varepsilon_0}{\left(1 + \frac{3}{2}\sqrt{\frac{\varkappa\varepsilon}{3}}t\right)^2}, \quad T^\mu_\nu(t) \simeq \frac{(g(t)+1)\varepsilon_0}{\left(1 + \frac{3}{2}\sqrt{\frac{\varkappa\varepsilon}{3}}t\right)^2}. \tag{67}$$

For the period in the case

$$t_r \sim \frac{1}{\sqrt{\varkappa \varepsilon_0}} \tag{68}$$

the trace and the energy density becomes independent of

$$a(t) = a^*(3\frac{\varkappa\varepsilon_0}{4})^{1/3} t^{2/3}, \quad T^0_0(t) \simeq \frac{4}{3\varkappa t^2}, \quad T^\mu_\mu = \frac{4g(t)+1}{\frac{4}{3\varkappa t^2}}. \tag{69}$$

This corresponds with absence of oscillations in eq. (50). At asymptotically large t

$$t \gg \frac{1}{\varkappa \Lambda_{\text{cosm}}} \tag{70}$$

the solution approaches the de-Sitter solution and the energy density approaches the value of the current cosmological constant:

$$a(t) \simeq a^* \frac{1}{2} \left(\sqrt{\frac{\varepsilon_0}{\Lambda_{\rm cosm}}} + 1 \right)^{2/3} e^{t\sqrt{\frac{\varkappa \Lambda_{\rm cosm}}{3}}}, \quad T^0_0(t) \simeq \Lambda_{\rm cosm}, \quad T^\mu_\mu(t) \simeq 4\Lambda_{\rm cosm}. \tag{71}$$

So we get an important result: under these conditions the one-loop Lagrangian YMC-filled Universe develops from the power-like law expansion, corresponding to the radiation stage into the DE stage, which corresponds to the exponential expanding. It is clear now that the partial solution for $Q = 1$ is an attractor, since $a(t)$ law approaches the exponential evolution at large time scales. Thus we established the stability of the one-loop solution and described its evolution. This behavior matches perfectly with the expectations for the development of the cosmological constant, but the one-loop approximation is just an approximation and so it is important to check whether this behavior will be present in better approximations. The figure (2) demonstrates the results of this section graphically.

5 Functional renormalization group approach for SU(2) Yang-Mills condensate.

Now we shall follow a new approach to the non-perturbative analysis of the Lagrangian (20) and then we will compare the results with the one-loop approximation.

Functional renormalization group is a tool that allows us to study QFT in a non-perturbative fashion in situations where perturbative techniques are not available, for example, when the couplings are not small enough. This approach is based on the path integral formulation of the QFT. The information about the critical exponents and correlation functions of a given system is incorporated into the effective action Γ. This action is derived by integrating out momentum fluctuations momentum shell by momentum shell. This procedure leads to a flow equation for a scale-dependent effective action Γ_k, where k is a momentum scale, above which all quantum fluctuations have been integrated out. The dependence of Γ_k on the momentum scale is determined by the Wetterich equation.

The momentum shell integrations are usually performed in theories, where gauge fixing is possible, otherwise it is generally impossible to find the effective action. The effective action can be constructed from the gauge-invariant blocks following the background field method. The full gauge field $\mathcal{A}_\mu$ is separated into the the background A_μ and the fluctuations field a_μ in the following way $\mathcal{A}_\mu = A_\mu + a_\mu$. After this the gauge of a_μ is fixed with respect to the background field, but the action remains invariant under an auxiliary gauge transformation of the full gauge field and the background field. The invariance of the standard effective action $\Gamma[A] = \Gamma[a = 0, A]$ is, however, retained. In the FRG the setting $a = 0$ is only allowed once all of the fluctuations have been integrated out, thus the disadvantage of this method is the dependence of the newly constructed action on two gauge fields.[3]

The Wetterich flow equation for the scale-dependent action reads [10]

$$\partial_t \Gamma_k[a, A] = \frac{1}{2}\mathrm{STr}\left(\Gamma_k^{(2)}[a, A] + R_k\right)^{-1} \partial_t R_k \tag{72}$$

where $\Gamma_k^{(n,m)} = \frac{\delta^n}{(\delta a)^n}\frac{\delta^m}{(\delta a)^m}\Gamma_k$ and $\partial_t \equiv k\frac{\mathrm{d}}{\mathrm{d}k}$. The action Γ_k is an interpolation between the microscopic action S_Λ at the UV cutoff Λ and the full quantum effective action Γ, i.e., $\Gamma_{k\to\Lambda} \to S_\Lambda$ and $\Gamma_{k\to 0} \to \Gamma$. The solution of the flow equation provides an RG trajectory of action functionals Γ_k that interconnects the two extremes. The term R_k is a regulator function depending on an infrared cutoff k that suppresses the propagation of momenta smaller than k. The trace (STr) runs over all internal indices, momenta and field components, which means degrees of freedom for gluons and ghosts, including a negative sign for the ghosts.

Since the field fluctuations are not considered in the picture of the CC that we are discussing, we are only interested in the background field action:

$$\Gamma[A] = \Gamma[a = 0, A]. \tag{73}$$

The problem is simplified by the fact that only the propagators of the fluctuation part in a background field are required in order to retrieve the flow of $\Gamma[a = 0, A]$.

It is generally very difficult to solve the (72) equation exactly, thus in order to solve it some approximations must be implemented. Γ_k is replaced with the bare action S^2 and the equation can be integrated:

$$\begin{aligned}\Gamma_k = -\int \mathcal{L}_{\mathrm{eff}} = -\int \mathcal{W}_k(\theta) = \int \mathrm{d}k\frac{1}{2}\mathrm{STr}\left(S^{(2)} + R_k\right)^{-1}\partial_t R_k \\ = \frac{1}{2}\mathrm{STr}\ln\left(S^{(2)} + R_k\right) + const.\end{aligned} \tag{74}$$

The action is selected to be $S = \frac{1}{4}\int \mathrm{d}x F_a^{\mu\nu}F_{\mu\nu}^a$, which corresponds to the UV limit of this effective theory. The integration constant will be fixed in a way that requires the effective action to vanish if the field strength also vanishes. Following our notation from the previous sections, the operators that contain the background field A_μ and not the full field $\mathcal{A}_\mu$ have the plain notation.

The next step in the FRG procedure is to invert the regularized propagator. This is done by fixing the gauge and introducing the associated ghosts. The corresponding actions are

$$S_{\mathrm{gauge}} = \frac{1}{2\alpha}\int \mathrm{d}x D_\mu a_\nu^a D_\nu a_\mu^a, \quad S_{\mathrm{ghost}} = \int \mathrm{d}x D_\mu c_\nu D^\mu c^\nu \tag{75}$$

We can split the original supertrace into the longitudinal, transversal and ghost parts

$$\begin{aligned}\frac{1}{2}\mathrm{STr}\frac{\partial_t R_k}{S^{(2)} + R_k} &= \frac{1}{2}\mathrm{Tr}_T\left(S^{(2)} + R_k\right)^{-1}\partial_t R_k + \frac{1}{2}\mathrm{Tr}_L\left(S^{(2)} + R_k\right)^{-1}\partial_t R_k \\ &- \frac{1}{2}\mathrm{Tr}_{\mathrm{ghost}}\left(S^{(2)} + R_k\right)^{-1}\partial_t R_k \\ &= \frac{1}{2}\mathrm{Tr}_T\frac{\partial_t R_k}{D^{\mu\nu} + R_k} + \frac{1}{2}\mathrm{Tr}_L\frac{\alpha\partial_t R_k}{D^{\mu\nu} + \alpha R_k} - \frac{1}{2}\mathrm{Tr}_{\mathrm{ghost}}\frac{\partial_t R_k}{\Box + R_k}\end{aligned} \tag{76}$$

and after application of Landau-DeWitt gauge condition $\alpha \to 0$, the longitudinal term disappears

$$\frac{1}{2}\mathrm{STr}\frac{\partial_t R_k}{S^{(2)}+R_k} = \frac{1}{2}\mathrm{Tr}_T\frac{\partial_t R_k}{D^{\mu\nu}+R_k} - \frac{1}{2}\mathrm{Tr}_{\mathrm{ghost}}\frac{\partial_t R_k}{\Box + R_k}. \tag{77}$$

the operators are $D_T^{\mu\nu} = \Box\delta_{cb}\delta^{\mu\nu} + gF^{a\mu\nu}f_{abc}$ and $D_{ghost}^{\mu\nu} = \eta^{\mu\nu}\Box$, are made of background fields and g is the coupling of the Yang-Mills field.

After the integration of the effective action we get

$$\frac{1}{2}\mathrm{STr}\ln(S^{(2)}+R_k) = \frac{1}{2}\mathrm{Tr}_T\ln\left(D_T^{\mu\nu}+R_k(D_T^{\mu\nu})\right) - \frac{1}{2}\mathrm{Tr}_{\mathrm{ghost}}\ln\left(\eta^{\mu\nu}\Box + R_k(\eta^{\mu\nu}\Box)\right). \tag{78}$$

Now, following [3] we need to pick the form of the regulator. The simplest one is the mass-like cutoff that we use for both the transversal and the ghost sector:

$$R_k(D_T^{\mu\nu}) = R_k(\eta^{\mu\nu}\Box) = k^2. \tag{79}$$

The specific choice of the background field does not affect the the flow of Γ_k [3], thus any covariantly constant colormagnetic field with $D_\mu F^{\mu\nu} = 0$, would be enough to retrieve the effective potential. However, as discussed in [13], [3], purely magnetic backgrounds lead to a tachyonic mode in the spectrum of fluctuations, thus such systems may be unstable. The other source of constraints arises from the fact that spectrum of differential operators, like $D_T^{\mu\nu}$ has to be known.

The particular choice of the background is based on its stability. The only known stable background is the self-dual field $F_{\mu\nu} = \tilde{F}_{\mu\nu}$. This condition leads to the identification $F_{01} = F_{23} \equiv B = \mathrm{const}$. Thus apart from its diagonal zero elements, the field strength depends only on B. This implies that the field strength has the Abelian form since the commutator of the gauge potentials in the Lagrangian vanishes. It is important to point out that this particular choice of the background does not affect the generality of conclusions. In fact any other choice would lead to negative fluctuation modes, which signify a non-physical solution.[15]

The eigenvalues of the relevant operators are [3, 11, 12]:

$$\begin{aligned}
\mathrm{Spec}\,[\Box] &= 2gB_l(n+m+1), \quad n,m\in\mathbb{N}\\
\mathrm{Spec}\,[D_{\mathrm{T}}] &= \begin{cases} 2gB_l(n+m+2) & \text{, with multiplicity } 2\\ 2gB_l(n+m) & \text{, with multiplicity } 2\end{cases},
\end{aligned}$$

where $B_l = |\nu_l B|$; ν_l denotes the eigenvalues of the adjoint color matrix n^aT^a $\nu_l = \mathrm{Spec}\left[(n^aT^a)^{bc} \mid n^2 = 1\right]$. It is important to emphasize that there is a degeneracy for $n = m = 0$ that occurs because of the symmetry between colormagnetic and colorelectric part. The degeneracy factor is $NB^2/2\pi^2$, where $N = 2$ for the $SU(2)$ group.

In order to calculate the traces of the logarithms in (78) we employ the Schwinger's formula [17]:

$$\ln(\hat{A}) = -\int_0^\infty \frac{\mathrm{d}s}{s}e^{-\hat{A}}, \tag{80}$$

where $\hat{A}$ is an arbitrary operator. It can be derived from the appendix A in [17]. We get

$$\Gamma(\omega) = \ln(\det \hat{L}_\omega)) = \operatorname{Tr} \ln \hat{L}_\omega = -\operatorname{Tr} \int_0^\infty \frac{\mathrm{d}s}{s} e^{-is\hat{L}_\omega}, \tag{81}$$

thus

$$\ln \hat{L}_\omega = -\int_0^\infty \frac{\mathrm{d}s}{s} e^{-is\hat{L}_\omega}. \tag{82}$$

The obtained integral can very often be divergent, so, as the procedure requires, after all of the changes and reshuffling it should be renormalized. Now we have a way to represent the logarithms in the equation

$$\frac{1}{2}\mathrm{STr}\frac{\partial_t R_k}{S^{(2)} + R_k} = \frac{1}{2}\mathrm{Tr}_T \ln\left(D_T^{\mu\nu} + k^2)\right) - \frac{1}{2}\mathrm{Tr}_{\mathrm{ghost}} \ln\left(\eta^{\mu\nu}\Box + k^2\right). \tag{83}$$

Combining this with the eigenvalues for the operators we get the following representation for the effective Lagrangian:

$$\begin{aligned}
\mathcal{L}_{\mathrm{eff}} &= \frac{2g^2B^2}{2\pi^2}\int_0^\infty \frac{ds}{s}\sum_{m,n=0}^\infty \left(e^{-(2gB(n+m)+k^2)s} + e^{-(2gB(n+m+2)+k^2)s} - e^{-(2gB(n+m+1)+k^2)s}\right) \\
&= \frac{g^2B^2}{\pi^2}\int_0^\infty \frac{ds}{s}\frac{1 - e^{2gBs} + e^{4gBs}}{(-1 + e^{2gBs})^2} = \frac{g^2B^2}{\pi^2}\int_0^\infty \frac{ds}{s} e^{-k^2 s}\left(\frac{1}{4\sinh^2(gBs)} + 1\right).
\end{aligned}$$

The summation of the exponents is easily performed using Mathematica. After a change of the variable $B^2 \to \theta$ we get the following result

$$\begin{aligned}
\mathcal{L}_{\mathrm{eff}} = \mathcal{W}_k(\theta) &= \frac{g^2B^2}{\pi^2}\int_0^\infty \frac{ds}{s} e^{-\frac{k^2}{gB}s}\left(\frac{1}{4\sinh^2(s)} + 1\right) \\
&= \frac{g^2\theta}{\pi^2}\int_0^\infty \frac{ds}{s} e^{-s\left(\frac{k^4}{g^2\theta}\right)^{1/2}}\left(\frac{1}{4\sinh^2(s)} + 1\right).
\end{aligned} \tag{84}$$

As it was expected, this integral has several divergences at $s = 0$. The first one arises from the fact that $1/(4\sinh^2(s)) + 1 = 1/(4s^2) + 11/12 + O(s^2)$ in the vicinity of $s = 0$, thus in order to remove this divergence we need to subtract $1/(4s^2)$ from $1/(4\sinh^2(s)) + 1$. The second divergence is a logarithmic divergence caused by the term s^{-1} in the expansion of the $\exp(-s\left(\frac{k^4}{g^2\theta}\right)^{1/2})/s$. We will subtract $\exp(-s)/s$ from this expression in order to remove the divergence. The final expression takes the following form

$$\mathcal{L}_{\mathrm{eff}}^{\mathrm{reg}} = \mathcal{W}_k(\theta) = \frac{g^2\theta}{\pi^2}\int_0^\infty \frac{\mathrm{d}s}{s}\left(e^{-s\left(\frac{k^4}{g^2\theta}\right)^{1/2}} - e^{-s}\right)\left(\frac{1}{4\sinh^2(s)} + 1 - \frac{1}{4s^2}\right). \tag{85}$$

In order to see the behavior of this Lagrangian we make another change of variable that consumes most of the constants $\frac{k^4}{g^2\theta} \to \frac{1}{\theta'}$. Now the Lagrangian can be represented graphically.

$$\frac{\mathcal{W}_k(\theta')}{k^4} = \frac{\theta'}{\pi^2}\int_0^\infty \frac{\mathrm{d}s}{s}\left(e^{-\frac{s}{\sqrt{\theta'}}} - e^{-s}\right)\left(\frac{1}{4\sinh^2(s)} + 1 - \frac{1}{4s^2}\right). \tag{86}$$

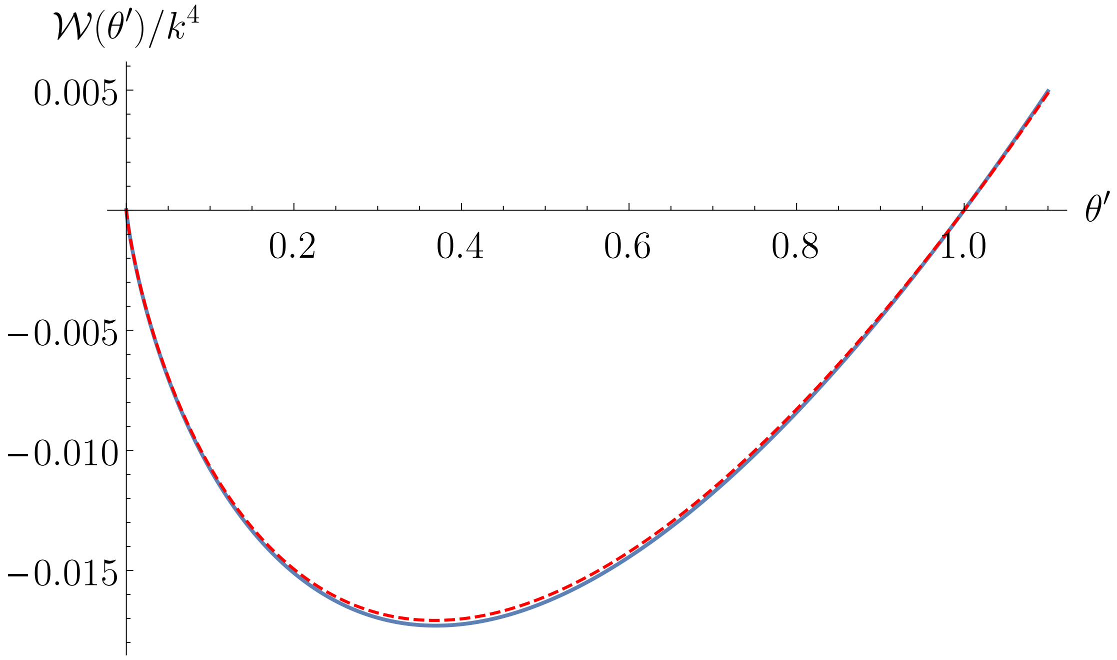


Figure 3: Plot of $\mathcal{L}_{\text{eff}}^{\text{reg}} = \mathcal{W}_k(\theta')$ (blue) and one-loop approximation (90) (red and dashed).

The minimum of the $\mathcal{L}_{\text{eff}}^{\text{reg}}$, $x \approx 0.36927$ is slightly dislocated, when compared to the one-loop approximation minimum $x \approx 0.367879$. This divergence is relatively small and it is not immediately clear whether this will significantly affect the dynamics of the condensate, but, as we will show in the following, this makes almost no difference.

This result differs from a similar result obtained in [8]. The effective Lagrangian [[8], eq. 60]

$$\mathcal{L}_{\text{eff}}^{*} = \mathcal{W}_k(\theta) = \frac{g^2\theta}{\pi^2}\int_0^\infty \frac{\mathrm{d}}{s} e^{-s\left(\frac{k^4}{g^2\theta}\right)^{1/2}}\left(\frac{1}{4\sinh^2(s)} + 1 - \frac{1}{4s^2}\right) \tag{87}$$

presented in this article does not correspond to its alleged plot, since its logarithmic divergence was never addressed and therefore the integral diverges at $s = 0$. There is a possibility that the treatment of this divergence was performed, though it was never included in the final paper, thus it cannot be analyzed. Besides, $\mathcal{L}_{\text{eff}}^{*}$ is clearly positive on the interval $\theta \in [0, 1]$, so the Lagrangian could not have the alleged minimum. The usage of the integral representation for the logarithm function that is employed in this article is also questionable, since this representation may not always be valid for operator expressions and in the paper it requires an additional simplification to retrieve the result; the correct approach should be based on the Schwinger's formula and the subsequent renormalization of all of the divergences. The $SU(2)$ symmetry is also unrepresented in this formula and should have been included in the degeneracy factor.

The one-loop approximation can be deduced from the expression for $\mathcal{L}_{\text{eff}}^{\text{reg}}$. First we Taylor expand $\frac{1}{4\sinh^2(s)}$ around $s = 0$: $\frac{1}{4\sinh^2(s)} = \frac{1}{4s^2} - \frac{1}{12} + O(s^2)$ and after the substitution, $\mathcal{L}_{\text{eff}}^{\text{reg}}$ takes the form:

$$\mathcal{L}_{\rm eff}^{\rm reg} = \mathcal{W}_k(\theta) \approx \frac{g^2\theta}{\pi^2}\int_0^\infty \frac{\mathrm{d}s}{s}\left(e^{-s\left(\frac{k^4}{g^2\theta}\right)^{1/2}} - e^{-s}\right)\left(\frac{1}{4s^2} - \frac{1}{12} + 1 - \frac{1}{4s^2}\right) \tag{88}$$

$$= \frac{11}{12}\frac{g^2\theta}{\pi^2}\int_0^\infty \frac{\mathrm{d}s}{s}\left(e^{-s\left(\frac{k^4}{g^2\theta}\right)^{1/2}} - e^{-s}\right) \tag{89}$$

Using Schwinger's formula backwards we come to a conclusion that the integral equates to $-\ln\left(\frac{k^4}{g^2\theta}\right)^{1/2}$ and thus the expression takes form

$$\mathcal{L}_{\rm eff}^{\rm reg} = \mathcal{W}_k(\theta) \approx \frac{1}{2}\frac{11}{12}\frac{g^2\theta}{\pi^2}\ln\left(\frac{g^2\theta}{k^4}\right) = 2\frac{1}{2}\frac{11}{24}\frac{g^2\theta}{\pi^2}\ln\left(\frac{g^2\theta}{k^4}\right), \tag{90}$$

which is exactly the well-known expression for the one-loop Lagrangian obtained by Savvidy, where the "2" in the front is coming from SU(2) symmetry group.

If we compare our full final result (85) with the general expression for the effective Lagrangian (20), we can see that $J/g_{\rm YM}^2(J) = \theta$ and thus we identify a non-perturbative expression for the coupling

$$\frac{1}{4g_{\rm YM}^2(J)} = -\frac{1}{\pi^2}\int_0^\infty \frac{\mathrm{d}s}{s}\left(e^{-s\left(\frac{k^4}{J}\right)^{1/2}} - e^{-s}\right)\left(\frac{1}{4\sinh^2(s)} + 1 - \frac{1}{4s^2}\right), \tag{91}$$

In order to calculate the trace anomaly we need the derivative of this expression:

$$\frac{\mathrm{d}}{\mathrm{d}J}\left(\frac{1}{g_{\rm YM}^2(J)}\right) = \frac{2k^2}{J^{3/2}\pi^2}\int_0^\infty \mathrm{d}s\left(e^{-s\left(\frac{k^4}{J}\right)^{1/2}}\right)\left(\frac{1}{4\sinh^2(s)} + 1 - \frac{1}{4s^2}\right) \tag{92}$$

Now we can determine the Callan-Symanzik $\beta-$function, from the RG equation

$$2J\frac{\mathrm{d}g_{\rm YM}^2}{\mathrm{d}J} = g_{\rm YM}^2\beta(g_{\rm YM}^2). \tag{93}$$

We get

$$\frac{\mathrm{d}}{\mathrm{d}J}\left(\frac{1}{g_{\rm YM}^2(J)}\right) = -\left(\frac{1}{g_{\rm YM}^4}\right)\frac{g_{\rm YM}^2\beta(g_{\rm YM}^2)}{2J} = -\frac{\beta(g_{\rm YM}^2)}{2Jg_{\rm YM}^2} \tag{94}$$

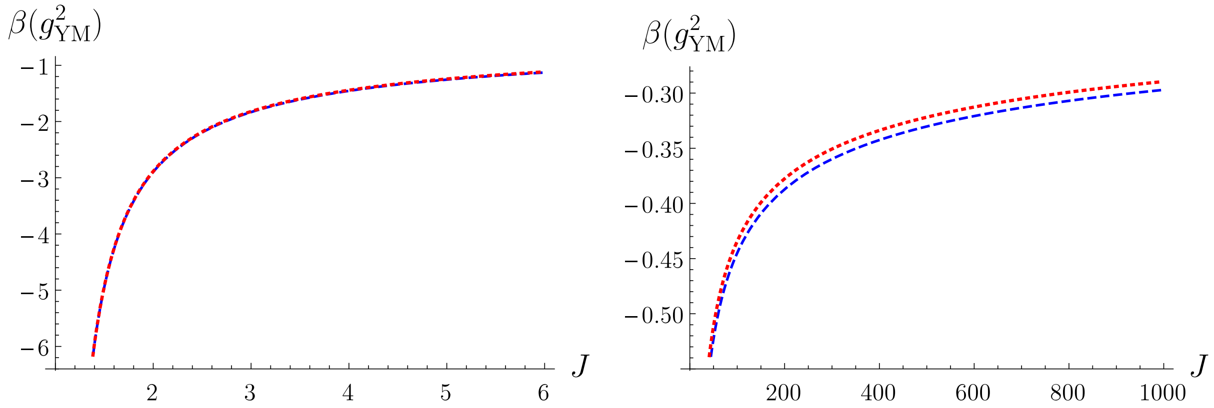
and so we retrieve the expression for the β-function

$$\beta(g_{\rm YM}^2) = -2J\left(\frac{1}{g_{\rm YM}^2}\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}J}\left(\frac{1}{g_{\rm YM}^2(J)}\right). \tag{95}$$

At the point where the Lagrangian (85) reaches its minimum, the β-function has a value:

$$\beta_0(g_{\rm YM}^2(J_{\rm min})) = 2, \tag{96}$$

which is exactly the value we expect, considering the analysis from the previous chapters. The figures 4a and 4b show the form of the β-function $\beta_0(g_{\rm YM}^2(J))$ and we can see that for low energies the one-loop approximation is extremely close to the non-perturbative Lagrangian (85); this means that the one-loop model gives a rather accurate result for our purposes and the analysis that was done in the previous sections holds.



(a) Plot of $\beta(g^2_{\rm YM})$ as a function of J for (85) (blue and dashed) and the one-loop approximation (90) (red and dotted). As one can see, these two functions are overlapping, which means that the one-loop approximation is extremely close to the solution (85) for small values of J. The cutoff is chosen to be $k = 1$GeV.

(b) Plot of $\beta(g^2_{\rm YM})$ as a function of J for (85) (blue and dashed) and the one-loop approximation (90) (red and dotted). In contrast to the low energy sector, the functions do not match exactly for larger values of J. The cutoff is chosen to be $k = 1$GeV.

Figure 4: The comparison of the one-loop and non-perturbative β-functions.

6 Summary and conclusion

In this work we examined the dynamics of the classical YM field and one-loop toy model and gave arguments for stability of the the general and partial solutions. The non-perturbative approach based on the functional renormalization group yielded a Lagrangian that has a very similar shape to the one-loop approximation and can be reduced to it using a series expansion. The analysis did not include the non-homogeneous parts of the condensate (waves), although they were partially taken into account during the FRG derivation. Besides that the main result of this work is that the one-loop approximation is accurate enough and the results obtained in earlier sections still apply. The one-loop YMC model exhibits the characteristics that are required from this component of the CC, such as the equation of state and and its evolution from $w = \frac{1}{3}$ to $w = -1$. Other key features of the one-loop approximation such as behavior of the β-function and the coupling are also valid.

It has been established that there are several components that constitute the physical vacuum. These components are quantum-topological fluctuations, quantum gravity contributions and the ground state of the gluon condensate. The latter has two components that have qualitatively similar, but opposite effects. If both of these solutions co-exist, their attractor nature leads to their compensation and thus their contribution to the vacuum energy diminishes at large times without any fine-tuning. Thus the Yang-Mills ground state provides a compensation mechanism for the cosmological constant and explains its small observed value.

Large early values of the energy density of the condensate can provide a mechanism for the explanation of the inflationary expansion stage. The YM condensate can drive the inflationary epoch in the early Universe and its termination can be potentially explained by means of the same vacuum compensation mechanism as the one which protects the current cosmological constant from being very large.

The Einstein-Yang-Mills equations have been solved for the scale factor, energy density and pressure in the vicinity of the attractor solution with the positive cosmological constant (i.e. de-Sitter solution). Such a solution can be applied for description of the inflationary stage in the Universe evolution as long as the corresponding YM condensates are found in the GUT at large energy scales.

References

[1] A. G. Riess *et al.* [Supernova Search Team Collaboration], Astron. J. **116**, 1009 (1998) [astro-ph/9805201].

[2] D. N. Spergel *et al.* [WMAP Collaboration], Astrophys. J. Suppl. **148**, 175 (2003) [astro-ph/0302209].

[3] A. Eichhorn, H. Gies and J. M. Pawlowski, Phys. Rev. D 83, 045014 (2011) Phys. Rev. D **83**, 069903 (2011)]. [arXiv:1010.2153v2]

[4] T. Y. Xia, Y. Zhang, Phys. Lett. B **656** (2007) 19-24. [arXiv:0710.0077v1]

[5] S. Wang, Y. Zhang and T. Y. Xia, JCAP **10** (2008) 037. [arXiv:0803.2760v3]

[6] R. Pasechnik, V. Beylin and G. Vereshkov, Phys. Rev. D 88, no. 2, 023509 (2013).

[7] Pasechnik R., "Quantum Yang-Mills Dark Energy". Universe. 2016; 2(1):4.

[8] Pietro Don, Antonino Marciano, Yang Zhang, Claudia Antolini, Phys. Rev. D **93**, 043012 (2016). [arXiv:1509.05824]

[9] J. A. R. Cembranos, A. L. Maroto, S. J. Nez Jareo, Phys. Rev. D 87, 043523 (2013).

[10] M. Reuter, C. Wetterich, arXiv:hep-th/9411227 (1994).

[11] M. Reuter and C. Wetterich, arXiv:hep-th/9411227; Phys. Rev. D 56 (1997) 7893.

[12] H. Gies, Phys. Rev. D 66 (2002) 025006 [hep-th/0202207].

[13] N. K. Nielsen and P. Olesen, Nucl. Phys. B 144, 376 (1978).

[14] J. Cervero and L. Jacobs, Phys. Lett. B 78, 427 (1978).

[15] M. Henneaux, J. Math. Phys. 23, 830 (1982).

[16] Y. Hosotani, Phys. Lett. B 147, 44 (1984).

[17] L.C. Albuquerque, C. Farina, S.J. Rabello (1994). arXiv:hep-th/9403002v2

[18] Wen Zhao, Yang Zhang (2006), Class.Quant.Grav.23:3405-3418,2006, arXiv:astro-ph/0510356v3