Robustness Analysis when Estimating Economic Capital for Credit Risk

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1 Abstract

Credit risk modeling is an important part of the financial protection used by banks during times of turbulence in the economy. More precisely, the modelling is about estimating how much economic capital a bank needs to hold in order to survive during an extreme loss. This thesis is about improving the robustness for the estimation of the economic capital, when it is updated as time passes. A decrease of the variations in the estimate of the economic capital would allow the bank to decrease the frequency of which the value of the economic capital is updated. This is the main aim for the thesis, as banks have a hard time explaining these variations based on any logical ground. The model used for estimating the economic capital is based on a multi-factor Merton model. The study will look at how the correlation matrix of the factors in the model can be updated in order to obtain as little variation as possible for the estimate of the economic capital. Four major approaches will be conducted to try and minimize this variation. The approaches use techniques such as weighting of consecutive correlation matrices, bootstrapping and standardization of the data using a multivariate CCC GARCH model.

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2 Abbreviations

A summary of all abbreviations used throughout the thesis.

CDF	*Cumulative Distribution Function*
EAD	*Exposure At Default*
EC	*Economic Capital*
EL	*Expected Loss*
ES	*Expected Shortfall*
GARCH	*Gereralized AutoRegressive Conditional Heteroskedasticity*
I	*Idicator Function*
IID	*Identically Independently Distributed*
IIND	*Identically Independently Normally Distributed*
L	*Total loss*
LGD	*Loss Given Default*
PD	*Probability of Default*
PDF	*Probability Density Function*
UL	*Unexpected Loss*
VaR	*Value at Risk*

3 Acknowledgements

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4 Introduction

All banks face a variety of financial risks. One of them is credit risk which is the risk that a borrower or counterparty might not be able to pay back money owed to the bank in case of financial distress. When a counterparty cannot pay back what it owes it is called default. To deal with potential financial risks statistical models are used to estimate the economic capital the bank should keep in order to survive during a crisis. It is obviously important that the method used for estimating economic capital has a somewhat accurate result and reflects reality to some extent. This in order to assure the bank will not underestimate the risk and ending up in financial trouble, or potentially overestimate the risk and miss the opportunity to invest some of its capital. At last, it is also desirable that the economic capital needed does not vary too much between the updates as moving assets around takes resources. Moreover, bigger adjustments without substantial support are hard to motivate.

This thesis is about a statistical method for estimation of credit risk. More specifically a study of how the process of updating the correlation matrix between the factors in a multi-factor Merton model is done. The aim is to optimize the updating process to achieve smaller variations of the estimate of the economic capital. Four major approaches will be used to see if the variation in the economic capital estimate can be decreased. These approaches will use and combine techniques with weighting of correlation matrices, bootstrapping to look at correlation distributions as well as using a GARCH model to try and estimate these correlations in a more robust way.

First the theory used throughout the thesis will be stated. Then a consistent model for estimating economic capital is constructed. When a satisfactory model is obtained it will be used to evaluate the different approaches ability to decrease variations of the economic capital between updates.

4.1 Objective

The main objective of this thesis is to evaluate how the economic capital changes over time. That is how the economic capital changes when new data becomes available and used to update the model. To evaluate this, the first step is to select a method for estimation of the economic capital in a consistent manner.

Later, we will investigate how to increase the robustness of the updates of the economic capital as new market data becomes available. More precisely, we will analyze if and how changes can be made in the way the model variables are updated in order to obtain a more robust value for the economic capital. We anticipate to update the model variables with new data biannually.

5 Theory

In this section the risk terms, risk measures and statistical concepts used throughout the thesis are explained. Also statistical methods and models used will be explained in detail.

5.1 General Risk Concepts

The term credit risk refers to the risk associated with credits and loans granted by financial institutions to other parties. The financial institution is exposed to a risk of loosing money as the borrower might fail to make the required repayments.

5.1.1 Single Loan

When managing the risks associated with a single loan to borrower k, the following attributes are usually considered.

- Probability of default, PD_k, is the probability that the borrower k will default and not be able to repay the loan.

- Exposure at default, EAD_k, is the amount of money at exposure, should borrower k default.

- Loss given default, LGD_k, is the percentage of EAD_k that is lost, given that borrower k defaults. Even if a default occurs, this does not necessarily mean all of EAD_k will be lost. Loans are often somewhat insured and parts of EAD_k can then be retrieved.

The values of these attributes are either set by the banks own standards or by a third party, such as a financial firm like Standard & Poor's.

The total loss for each borrower k, will be the stochastic variable L_k, see Equation (1).

$$L_k = EAD_k \cdot LGD_k \cdot D_k \qquad (1)$$

where D_k is the logistic stochastic variable described in Equation (2).

$$D_k = \begin{cases} 1, & \text{if k defaults} \quad \text{probability} = PD_k \\ 0, & \text{if k does not default} \quad \text{probability} = 1\text{-}PD_k \end{cases} \tag{2}$$

The Expected Loss EL_k, is the expected value of the loss L_k. It is calculated in Equation (3).

$$\begin{aligned} EL_k &= E[EAD_k \cdot LGD_k \cdot D_k] = (EAD_k \cdot LGD_k) \cdot E[D_k] \\ &= (EAD_k \cdot LGD_k) \cdot (1 \cdot PD_k + 0 \cdot (1 - PD_k)) \\ &= EAD_k \cdot LGD_k \cdot PD_k \end{aligned} \tag{3}$$

5.1.2 Portfolio of loans

In general, a financial institution does not only grant one loan to one single borrower. Most often institutions have a credit portfolio consisting of multiple loans, where each loan has the properties described in 5.1.1. The amount of money the bank anticipates to loose over a certain time period is no longer due to one single exposure. The anticipated total loss will in this case be the combined outcome of the credit portfolio. We refer to this as the total loss L. If $K = \{1,2,3,...,n\}$, where n is the total number of assets in the portfolio, we can write L as in Equation (4).

$$L = \sum_{k \in K} L_k = \sum_{k \in K} EAD_k \cdot LGD_k \cdot D_k \tag{4}$$

As the outcome, D_k, of each borrower in the credit portfolio is uncertain, L will be a stochastic variable. One possible distribution of L for a general credit portfolio can be seen in Figure 1.

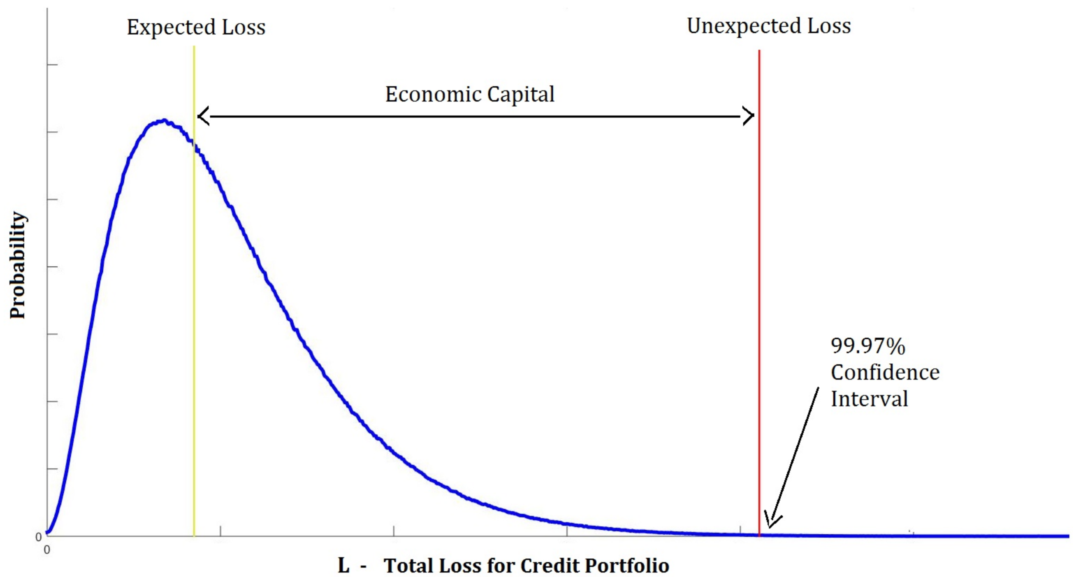

Figure 1: The blue curve shows the probability density function for the total loss L of a typical credit portfolio. The expected loss, unexpected loss and economic capital are important measures in credit risk.

EL is the expected loss for the credit Portfolio, see Figure 1. It is the loss that the bank anticipates to loose in average on a yearly basis. As the name suggests, EL is the expected value of L. See Equation (5).

$$EL = E\,[\sum_{k \in K} EAD_k \cdot LGD_k \cdot D_k\,] = \sum_{k \in K} EAD_k \cdot LGD_k \cdot E[D_k] \tag{5}$$

However, it is not straight forward how to estimate all $E[D_k]$. This since, in any realistic setting, the outcome of each one of the variables D_k is correlated to those of the remaining D_k:s. In the theoretic case where all borrowers outcomes are independent EL can be calculated as in Equation (6).

$$EL = \sum_{k \in K} EAD_k \cdot LGD_k \cdot E[D_k] = \sum_{k \in K} EAD_k \cdot LGD_k \cdot PD_k \tag{6}$$

Unexpected loss, UL, is the total loss exceeding the expected loss, see Figure 1. The unexpected loss can stretch all the way from the expected loss up to the outer parts of the right hand side tail of the total loss distribution L. UL is often defined as VaR_α, for a certain α. In Figure (1), UL is defined as $VaR_{0.9997}$.

Economic capital, EC, is the amount of money that the bank needs to hold in case of an extreme event to stay solvent, see Figure 1. EC can be quantified in many different manners. An often used definition is to set EC as [UL - EL], as is done in Figure (1) and this thesis.

Further descriptions of above mentioned credit risk concepts can be found in Antwi, Constance Mensah, Owusu Amoamah and Joseph[2].

5.1.3 Extreme Measures

VaR, or Value-at-Risk, is an often used measure when describing extreme events in credit risk contexts. Given a confidence level $\alpha \in]0,1[$, the VaR_α of a portfolio L is defined as the smallest value l such that the loss L exceeds l with a probability of at most $(1-\alpha)$. In other words $VaR_\alpha(L)$ is the α-quantile value for a the loss distribution L, to a certain portfolio P. More on VaR can be found in Artzner, Delbaen, Eber and Heath[3]. The mathematical formulation taken from McNeil, Frey and Embrechts[11] is given below:

$$VaR_\alpha(L) = inf\{l \in \mathbb{R} : P(L > l) \leq 1 - \alpha\} \tag{7}$$

Expected Shortfall, ES, sometimes also called tail VaR, is an alternative measure of extreme events for stochastic variables. $ES_\alpha(L)$ is defined as the expected loss of the portfolio given that the loss L exceeds a certain VaR_α. The mathematical formulation is:

$$ES_\alpha(L) = E(L \mid L > VaR_\alpha) \tag{8}$$

where the quantile $\alpha \in]0,1[$. More on ES in Acerbi and Tasche 2001[1].

5.2 Multi-factor Merton Model

In 1974, Robert Merton[12] introduced a model for quantitative modeling of credit risk where a company's equity was defined as a call option on the company's assets. Every company was assumed to have a specific amount of zero coupon debt that would mature at a future time T. If at time T, the value of the company's total assets would be less than the obligated debt repayment, then the company would default. Merton's model has become very popular amongst banks and other financial institutions for estimation of credit risk. Except for estimating the probability of default the model can also be used for estimating risk neutral probabilities or debt credit spread. For more see Hull, Nelken and White[10].

A multi-factor Merton model is simply a Merton model where the modeling of the company asset value X_i is done with a model of two or several parameters. An example is given by Pykhtin[14]:

$$X_i = \sqrt{R_i^2}\; Y_i + \sqrt{1 - R_i^2}\; e_i \tag{9}$$

where e_i is the standardized normally distributed idiosyncratic shock and R_i is a measure for how sensitive the borrower is to systematic risk. Y_i is the systematic factors given by:

$$Y_i = \sum_{k=1}^{N} \alpha_{ik}\; Z_k \tag{10}$$

where α_{ik} are weighting factors and Z_k are independent standard normal systematic factors.

5.3 GARCH Models

General Autoregressive Conditional Heteroskedasticity models, are a popular way of modeling volatility for financial time series. The reason is the model's ability to model heteroskedastic behaviors in the volatility.

5.3.1 Univariate GARCH

The univariate GARCH(p,q) model first stated by Bollerslev[4] has the following appearance:

$$r_t = \sigma_t\, z_t \tag{11}$$

$$\sigma_t^2 = w + \sum_{i=1}^{q} \alpha_i\, \varepsilon_{t-1}^2 + \sum_{i=1}^{p} \beta_i\, \sigma_{t-1}^2 \tag{12}$$

where r_t is the expected return, σ_t the volatility, w, α_i and β_i are parameters, and z_t and ε_t are $N(0,1)$ distributed white noise.

5.3.2 Multivariate GARCH

When investigating the co-movements of several financial time series a multivariate GARCH model can be used. One of the simpler and more intuitive multivariate models described in more detail by Silvennoinen and Teräsvirta[16] is the Constant Conditional Correlation GARCH, CCC GARCH. This model is based on the assumption that the conditional covariance matrix can be decomposed into conditional standard deviations and correlations:

$$H_t = D_t \, CC \, D_t \tag{13}$$

where H_t is the conditional covariance matrix, D_t is a diagonal matrix with the conditional standard deviations on the diagonal, and CC is a constant correlation matrix.

5.4 Monte Carlo Method

The Monte Carlo method is an algorithm where a statistical model is repeated a large number of times giving numerical results. According to the Law of Large Numbers the average of the Monte Carlo method will after a large number of repeated samples be close to the expected value of the statistical model. The Monte Carlo method is often used for complicated calculations where analytical computation would be too difficult or impossible. For more on Monte Carlo methods see Eckhardt[6]. Below is the mathematical formulation of the Monte Carlo technique stating that the average of the method will be equal to the expectation for a large N:

$$E(X) = \bar{X} = \frac{1}{N}\sum_{i=1}^{N} X_i \tag{14}$$

Where X_i is the output values from the statistical method.

5.5 Cholesky Decomposition

The main idea of a Cholesky decomposition is finding a root matrix, L, to a hermitian positive semidefinite matrix A. A is decomposed by L as in Equation (15).

$$A = LL^* \tag{15}$$

where L is a lower triangular matrix and L^* is the conjugate transpose of L.

Having found L it can be showed that $L\varepsilon$, where ε is a random vector with identically independent normally distributed elements ($IIND$ elements), will be random variables with zero mean and unit variance. The internal correlations of the random variables will be equal to the original matrix A. The proof can be seen in Equation (16).

$$\begin{aligned} V[L\varepsilon] =& E[(L\varepsilon)(L\varepsilon)^*] - E[L\varepsilon]^2 = \\ & E[(L\varepsilon)(L\varepsilon)^*] = E[L\varepsilon\,\varepsilon^* L^*] = \\ & LE[\varepsilon\varepsilon^*]L^* = LL^* = A \end{aligned} \tag{16}$$

For more theory on Cholesky decomposition we refer to Glyn Holton[9].

5.6 Bootstrapping

Bootstrapping is a technique that creates a distribution of sample estimates by using random sampling with replacement described more thoroughly by Efron and Tibshirani[7]. Bootstrapping can for example be used on the correlations between factors in a multi-factor model. First the sample data is resampled with replacement and then the correlations are estimated. This procedure is then repeated a large number of times to create a distribution of the sample estimates.

5.7 Antithetic Variates

Antithetic Variates is a method used in computer simulations to reduce variance. Suppose the objective is to estimate the expectation $E(f(X))$, where $f(X)$ is a function of X, antithetic variates can be implemented by drawing a sequence of antithetic sample pairs (X_1, X_1'), (X_2, X_2'), (X_3, X_3'),..., (X_n, X_n') where X and X' has the same distribution and the pairs are IID Ocnasu, Besanger, Rognon and Carer[13]. The antithetic variates estimator of $E(f(X))$ is then:

$$\hat{X} = \frac{1}{2n}\Big(\sum_{i=1}^{n} f(X_i) + \sum_{i=1}^{n} f(X_i')\Big) = \frac{1}{n}\sum_{i=1}^{n}\Big(\frac{f(X_i) + f(X_i')}{2}\Big) \tag{17}$$

If we look at the variance of the average of the estimator we get:

$$
\begin{aligned}
\sigma^2 &= Var\Big(\frac{f(X_i)+f(X_i')}{2}\Big) \\
&= \frac{1}{4}\Big(Var(f(X_i)) + Var(f(X_i')) + 2Cov(f(X_i), f(X_i'))\Big) \\
&= \frac{1}{4}\Big(2Var(f(X_i)) + 2Cov(f(X_i), f(X_i'))\Big) \\
&= \frac{1}{2}\Big(Var(f(X_i)) + Cov(f(X_i), f(X_i'))\Big)
\end{aligned}
\tag{18}
$$

We can now see that $Cov(f(X_i), f(X_i')) < 0$ must be true for the antithetic variables X_i and X_i' to be able to reduce variance.

In a simulation with independent standard normal random variables the antithetic variates can be implemented by creating two paired sequences of IID $N(0,1)$ random variables $Z_1, Z_2, Z_3, ..., Z_n$ and $-Z_1, -Z_2, -Z_3, ..., -Z_n$ which do have negative covariances. For more on this see Calzolari[5].

6 Model and Simulation Technique for EC

There are several ways to go around measuring and handling worst case scenarios for a credit portfolio. As stated in the introduction we have chosen an approach where we estimate the EC the bank needs to hold to be protected from bad outcomes for the credit portfolio. The technique investigated in this thesis is to first select a manner in which to estimate the random variable L. Then select a suitable measure for EC that depends on the distribution of L. At last we simulate the distribution of L in order to extract the values needed for the EC estimate.

We wish to estimate EC in a consistent yet relevant manner. By a consistent estimate of EC we mean an estimate that will grant a similar EC in two consecutive estimations - all model parameters fixed. The model parameters will depend on the underlying market and will thus change when the market conditions change. A consistent estimation of EC would yield accurate thresholds when comparing EC from different market situations. Should the estimate of EC change too much between two consecutive runs with identical model parameters, this would cause a problem. More precise, the problem would occur when comparing EC, after having updated the model to describe contemporary market conditions. In such a situation, it would be hard to distinguish whether a change in EC would have arisen from simulation inaccuracy or from an actual change in the market.

We do not, in any way, endeavour to find the "*best*" estimate of EC. Thus our focus is not on assessing the accuracy of EC itself. In fact, it would be very hard to determine when a "*good enough*" estimate of EC is obtained. A very consistent simulation of EC would not, in itself, say anything about how true to reality the EC estimate actually is. However, a consistent manner in which to simulate EC will set a good foundation for viewing changes in EC between different market situations. So, how much should EC be allowed to change between two consecutive simulations then? We do obviously not crave a perfectly deterministic model, but the change in EC should not be too high. The change in EC when using separate model parameters has to be much larger than the potential variation in the estimate of EC when using the same market parameters.

When having decided upon some desired maximum change of EC there are several ways to assure it is obtained. One way could be to make the simulation of the distribution of L more precise. Another alternative is selecting a more *smoothing* estimate for EC. A last improvement, before comparing EC values between different market situations, is to use the same random variables in all evaluations. That is, simulating the random variables in the model once and for all, then keeping them fixed throughout all different evaluations. This would obviously yield a deterministic model, with a zero-variance in the simulation of EC.

6.1 Creating Credit Portfolio

In order to start evaluating the total credit risk, we need a proper definition of a credit portfolio, P. The portfolio used in this thesis is created to inhibit suitable properties in order to mimic a real credit portfolio used by a financial institution. The portfolio defined, P, will be simulated and created once, then used throughout the whole paper.

We assume a portfolio P with N units. Each unit p_i, for i=1,2,3,...,N, represents one borrower or one $exposure\ to\ default$. Each exposure, p_i, will have the following attributes specified: EAD_i, LGD_i and PD_i. These are positive and deterministic for all i. This is logical since PD is a probability which is always positive, EAD is the amount of money lent form the bank to the third party and LGD is a share in percentage.

In credit portfolios used by financial institutions today, one exposure p_i does not always represent a single person's or company's loan. In fact it is common to accumulate a group of single loans then assign them as one borrower in the credit portfolio. In these cases the single loans accumulated have roughly the same properties. Such as; the probability of default, the industry they are exposed to and the country they are affected by. This kind of accumulation has several benefits. First of all; assigning each and everyone of the single loans as one borrower would immensely increase the time it takes to simulate EC. This since the credit portfolio, in this case, would consist of millions of units. At the same time it would not, substantially, grant a more accurate estimate of EC. Altogether it is much more manageable and efficient to densify the portfolio as long as it is done mindfully.

In this thesis, the number of assets, N, is chosen to be 10 000. The number used for N does not, however, reflect any specific bank. It is selected to operate in the same order of magnitude as any standard bank. The PD and EAD values connected to each borrower, are created to mimic how PD and EAD often are distributed in banks' credit portfolios see Figure 2 and Figure 3. The distribution of all PD in a credit portfolio often looks roughly exponential. This is because most borrowers have low risks of defaulting since only a few loans are granted to borrowers with a high PD. EAD is naturally very correlated with the PD as only very stable borrowers with a small PD get very high loans.

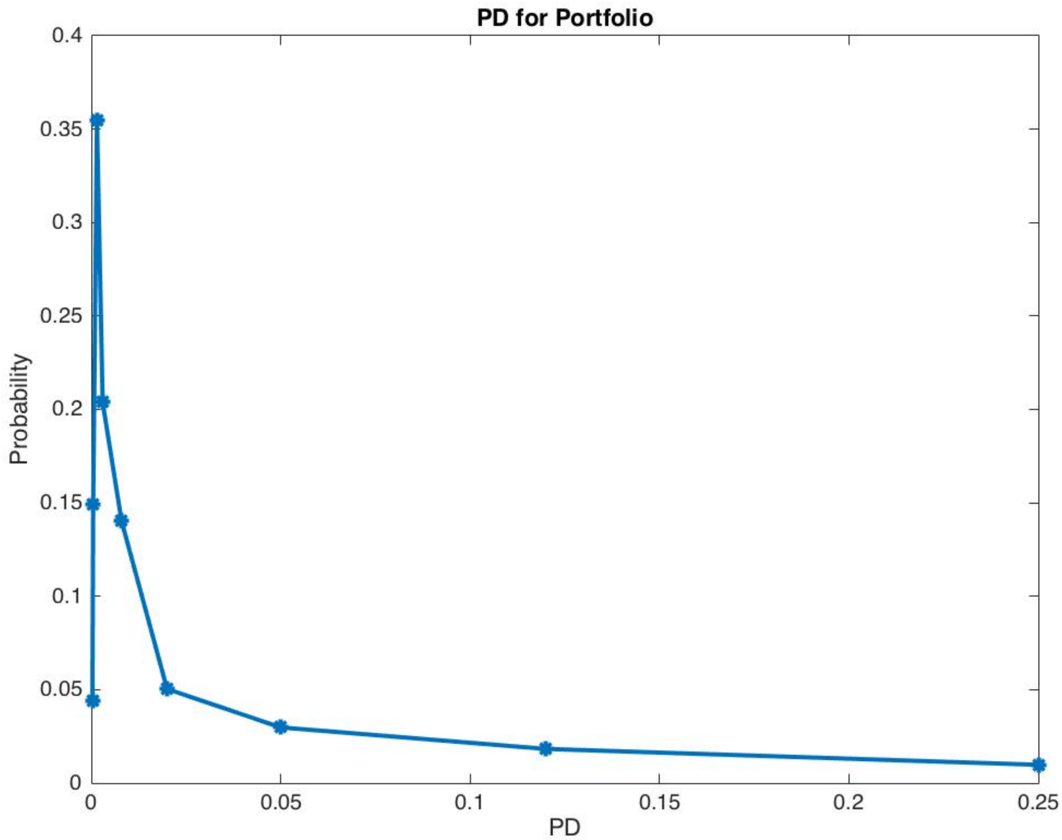

Figure 2: The figure shows a plot of the distribution of PD for the portfolio P. The distribution of PD is created to mimic the properties of a bank's portfolio. Most borrowers have a low PD. PD reaches from 0,0003 to 0,25.

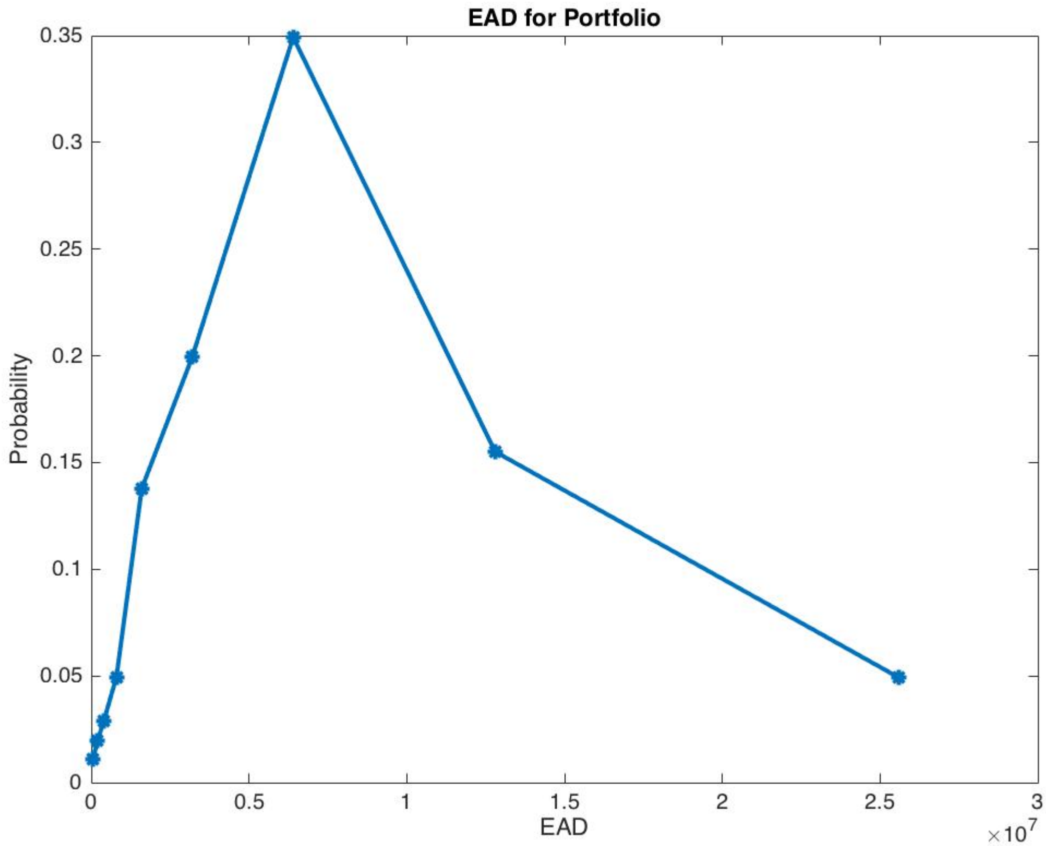

Figure 3: The figure shows a plot of the distribution of EAD for the portfolio P. The distribution of EAD is created to be very correlated with PD as this is often the case in a bank's portfolio. A low value of PD corresponds to a high value of EAD. EAD reaches from $5 * 10^4$ to $2,56 * 10^7$.

6.2 Selecting Model for Total Loss

Regardless of what exact measurement is used to evaluate EC, we need to find a manner in which to estimate the stochastic variable L. The formula for total loss is specified in Equation (4). Using this formula we obtain the total loss for the portfolio P to be as in Equation (19).

$$L = \sum_{i=1}^{N} EAD_i \cdot LGD_i \cdot D_i \tag{19}$$

It is clear form Equation (19) that in order to estimate L we need to define a method for determining D_i for each borrower p_i. That is, determining whether or not a certain borrower will default.

6.2.1 Assessing Default for Borrower

We want to find a way to determine the stochastic variable D_i, see Equation (2). We want D_i to take value 1 if borrower p_i delfault, and 0 otherwie. An important criteria for each D_i is for its expected value to be PD_i; $E[D_i] = PD_i$.

The way we will determine D_i in this thesis is to first create a borrower specific variable X_i. This variable should reflect a measure of the financial well-being of the corresponding borrower. Then, in an adequate manner, X_i is compared to PD_i, the probability of default for that same borrower. In this thesis X_i is created to have a distribution that is standard normal, thus we can easily construct a function of PD_i as a threshold for default. Each borrower, p_i, is said to have defaulted if $X_i < \phi^{-1}(PD_i)$, where ϕ^{-1} is the inverse CDF of the normal distribution. See Equation (20).

$$D_i = \begin{cases} 1, & \text{if} \quad X_i < \phi^{-1}(PD_i) \\ 0, & \text{if} \quad X_i \geq \phi^{-1}(PD_i) \end{cases} \tag{20}$$

p_i will default if $D_i = 1$ and not default if $D_i = 0$. We need to confirm that each D_i will also fulfill the criteria mentioned above, $E[D_i] = PD_i$. See Equation (21) for this estimation.

$$\begin{aligned} E[D_i] &= 1 \cdot P[X_i < \phi^{-1}(PD_i)] \ + \ 0 \cdot P[X_i \geq \phi^{-1}(PD_i)] \\ &= 1 \cdot PD_i \ + \ 0(1 - PD_i) \\ &= PD_i \end{aligned} \tag{21}$$

The formula for total loss, as seen in Equation (19), can now be updated to Equation (22).

$$L = \sum_{i=1}^{N} EAD_i \cdot LGD_i \cdot I(X_i < \phi^{-1}(PD_i)) \tag{22}$$

Where I($\cdot$) is the idicator function. I($\cdot$) takes the value 1 if the statement inside the brackets occurs and 0 elsewise.

As mentioned above, we have chosen to create all X_i to be stochastic standardized normal variables. Yet the only restraint on the distribution of X_i is, in fact, for it to be known. Both Equation (20) and (21) could have been constructed in a similar manner would X_i have had an other, known, distribution. The only modification of the equations would have been a substitution of ϕ to the other, known, distribution. We selected a standard normal distribution since it seemed like a natural choice and it was straight forward to construct. We didn't see any particular reason to why some other more complex distribution for X_i would improve the estimate of D_i.

6.2.2 Constructing X_i

From the previous section we learnt that in order to find all the values D_i we need to construct and estimate X_i for each borrower. We wish for the value of X_i to reflect the financial well-being of the corresponding borrower. Hence, it seems reasonable to create X_i to somehow be affected by the market conditions observable at the moment. Doubtlessly, there are many ways to accomplish this.

In this thesis, we have settled on an approach using country and industry indices connected to each borrower. Correlations between market indices tend to increase during a crisis. A consequence of a crisis is also a higher systematic risk for the borrowers affected by it. Hence, we want the model to be built in such a way that a high correlation between indices connected to a borrower will increase the systematic risk for that borrower. More precise we want the probability of $X_i < \phi^{-1}(PD_i)$ to increase the higher the correlation between the country and the industry connected to p_i is. We have access to data over performance indices for 13 countries and 23 industries. This makes a total of 36 indices. However, each borrower will be assigned two of these 36 factors. One country index and one industry index. The indices chosen for every exposure p_i are the ones that are believed to influence the specific borrower the most.

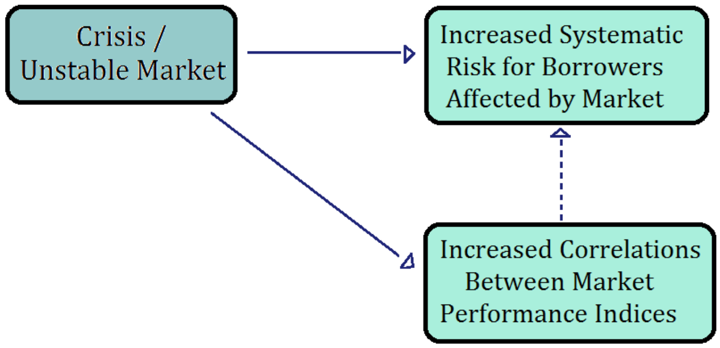

Figure 4: The current market situation is believed to affect both the correlations between performance indices on the market as well as the systematic risk of the borrowers affected by the market. We use this fact to model the systematic risk as if it was directly affected by the correlations between indices. See the dashed line.

As a first step towards creating X_i, we create a variable r_i for each borrower. The outcome of each r_i should be affected by the correlation between the country and industry index for the corresponding borrower p_i. When constructing r_i, a multi-factor Merton model is used. Since we have 36 factors, the model is a 36-factor-model. However, only two factors for each r_i will have non-zero corresponding parameters. See α_{ik} in Equation (10). This results in a 2-factor-model for each measure r_i. The model for each r_i is stated in Equation (23).

$$r_i = \sqrt{R^2}\ (w_I I_i + w_C C_i)\ +\ \sqrt{1-R^2}\ e_i \tag{23}$$

The residual e_i corresponds to the idiosyncratic risk which is an asset specific zero mean unit variance residual independent from C_i and I_i. C_i and I_i are simulated to be normal variables with a correlation corresponding to that between the indices for borrower p_i. We will explain more exactly how the indices are simulated in Section 6.2.4.

The correlation between the indices is now obviously incorporated in the model. How do we know that a high correlation between indices connected to a borrower will in fact, grant a higher systematic risk for said borrower? When viewing the model it is rather obvious that this will be the case. A high correlation between the indices C and I will increase the variance of r_i. The higher variance will in turn increase the probability that X_i is small enough to cause a default for borrower p_i.

To read more about general traits of a factor model, we refer you to the theory chapter. The specific factor model used in this thesis is, a part form a few changes, similar to the one stated by Pykhtin[14]. Pykhtin used an n-factor Merton model with n independent factors. In this thesis we will have a total of 36 factors which will all be correlated. Since our factors are correlated, our model is in some sence more similar to the one used by García Céspedes and García Martín[8]. In our case a model with correlated indices was believed to be the most natural approach. This as industries and countries are highly correlated economically which will probably affect the outcome of borrowers affected by them.

Using r_i straight off, would grant a measure of the financial well-being for each borrower. It is not, however, normalized and thus not comparable to the inverse CDF of PD_i in any logical manner. To convert all the r_i to normalized variables X_i, each r_i is divided by its standard deviation as in Equation (24).

$$X_i = \frac{r_i}{\sqrt{2R^2 w_I w_C \rho_i + (1 - R^2) + R^2(w_I^2 + w_C^2)}} \tag{24}$$

The returns now have a unit variance and we assign this result to the stochastic variable X_i. Knowing X_i enables us to compare it to the threshold $\phi^{-1}(PD_i)$, as in Equation (20). This in turn, will enable us to obtain all D_i needed in order to estimate the total loss as in Equation (19).

6.2.3 C and I - Data

As mentioned in the previous section, the correlation between C_i and I_i are incorporated in the model to capture the market conditions observable at the moment.

C and I are mutually correlated which is intuitive as industries and countries have a natural impact on each other in the economy. The correlation corresponds to the correlation between the different countries and industries according to public macro economic Morgan Stanley indices. The data is displayed in Figure 14.

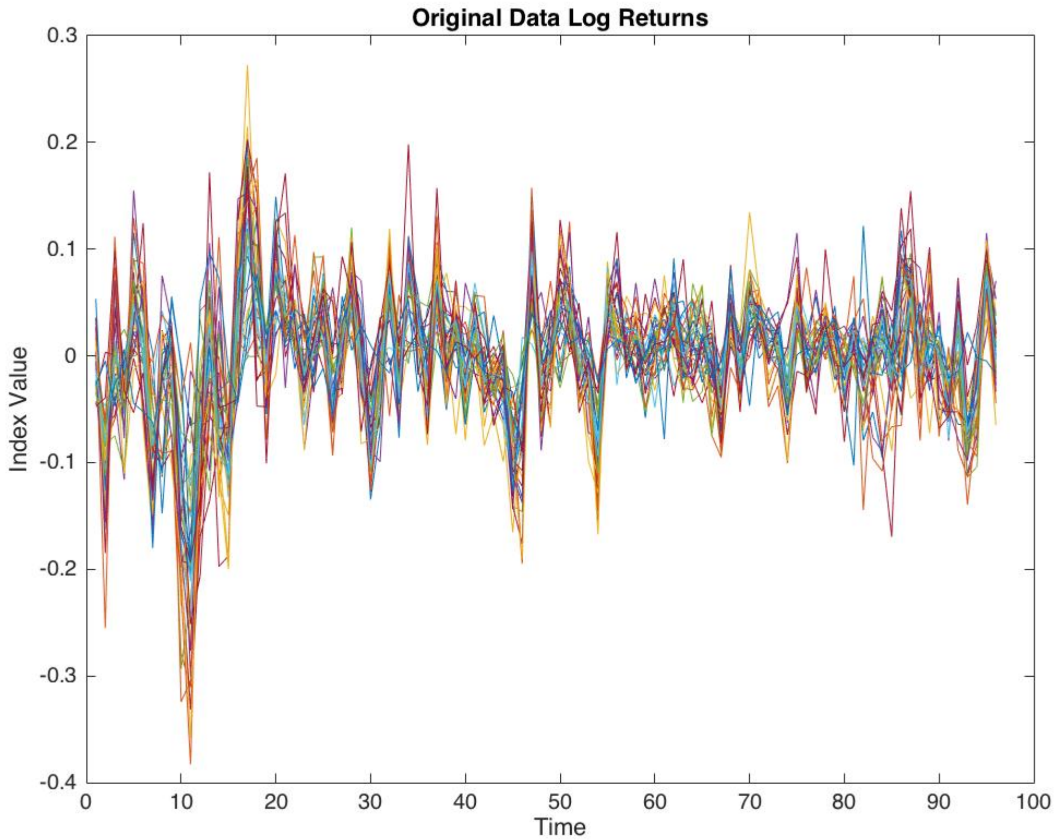

Figure 5: Here we can see all the 36 Morgan Stanley indices with 96 observations each from 2007/12/31 - 2015/11/31, plotted in one figure. The economic crisis in 2008 is clearly visible around index 10 on the time axis.

6.2.4 C and I - estimation

As stated above, C and I are created to be zero mean and unit variance random variables. We also need to assure all C and I are correlated according to the right correlation matrix. A Cholesky decomposition is used for creating all C_i and I_i as well as assuring they fulfill the above stated requirements.

In comparison with the theory chapter describing Cholesky decomposition, in this thesis the original matrix A is the $n \times n$ correlation matrix for all the n indices, C and I. We perform a Cholesky decomposition of A to find the root matrix L. Having found L, the estimates of C_i and I_i are constructed as $L\varepsilon$ where ε is a $n \times 1$ vector with identically independent normally distributed, $IIND$, elements. Why this construction will grant the estimates the desirable properties is described in theory Section 5.5, see Equation (16).

An important observation at this stage is that all market variables are incorporated in I and C. Thus updating the model when new market data becomes available, will solely mean an update of all I and C. Since I and C are generated from the correlation matrix A, this will mean an update of the correlation matrix A.

6.3 Selecting EC measure

We need the value of EC to be high enough so that the risk of loosing more than EC is minimal. Well what is minimal? This varies from what standard the financial institution chooses to use. Basel II, the international capital requirement law supervised by Finansinspektionen, suggests a VaR-level in the 99,96% - 99,98% interval. In this thesis we have decided to create EC in such a way that the VaR is at least at a 99,97% level. The next step is selecting how we should measure EC.

$$EC = VaR_\alpha[L] - E[L] \tag{25}$$

$$EC = ES_\alpha[L] - E[L] \tag{26}$$

Should the VaR_α or the ES_α quantile be used to reassure that the risk of loosing more than EC is minimal? See Equations (25) and (26). In this thesis the VaR_α quantile will be used as this is believed to give more robust results when the model is updated. Outliers might induce a higher variance for the ES_α quantile than for the VaR_α quantile. This as the ES_α quantile is a mean of the tail losses of the distribution. We select the EC measure in Equation (25) since this will most certainly cause a smaller change of the EC estimate between runs using fixed market parameters.

6.4 Simulating Total Loss Distribution

Since L, see Equation (19), is a stochastic variable, it will not land on exactly the same value every time it is estimated. In this section we select a technique to simulate the distribution of L. Regardless of what measure of EC we pick, the distribution of L will always be needed. This since EC will always contain some probability operation on the distribution of L, such as $E[L]$, $VaR_\alpha[L]$ or $ES_\alpha[L]$. Hence a proper simulation of the distribution of L is a fundamental part in estimating EC.

As discussed in the introduction to this chapter we need the estimates of EC to be somewhat consistent. L itself and the simulation of its distribution is the only stochastic contribution towards the value for EC. Thus it is important to obtain minimal variance in the simulation of the distribution of L. The distribution should not change too much, should we evaluate it two times using the same market data.

The approach that will be used when simulating the distribution of L and later extracting EC, is a Monte Carlo simulation. The exact amount of Monte Carlo simulations needed will be evaluated in 6.4.2. Obviously a larger amount of simulations will grant a more precise estimate of the true distribution for L. We will also investigate the use of antithetic variables, to minimize the simulation variance further.

It is hard to evaluate if the simulation of L is accurate enough just by viewing the a histogram of the results. We can, however, easily evaluate the value of the EC estimate. Thus we chose to observe EC in the following evaluations and in this manner decide when the simulation of L is "good enough". This is in fact a very logical approach, since the value of EC is the outcome that the bank is fundamentally interested in.

6.4.1 Accuracy Demands for Simulation of L

In order to decide what accuracy we should demand for the simulation of EC, we need to get an idea of what EC value magnitude we are dealing with. For this sake a smaller trial estimation is made. This using correlation matrices from 5 different time intervals. Knowing approximately how big we can expect EC to be using the different market variables, will give us a hint of how consistent the simulation technique has to be.

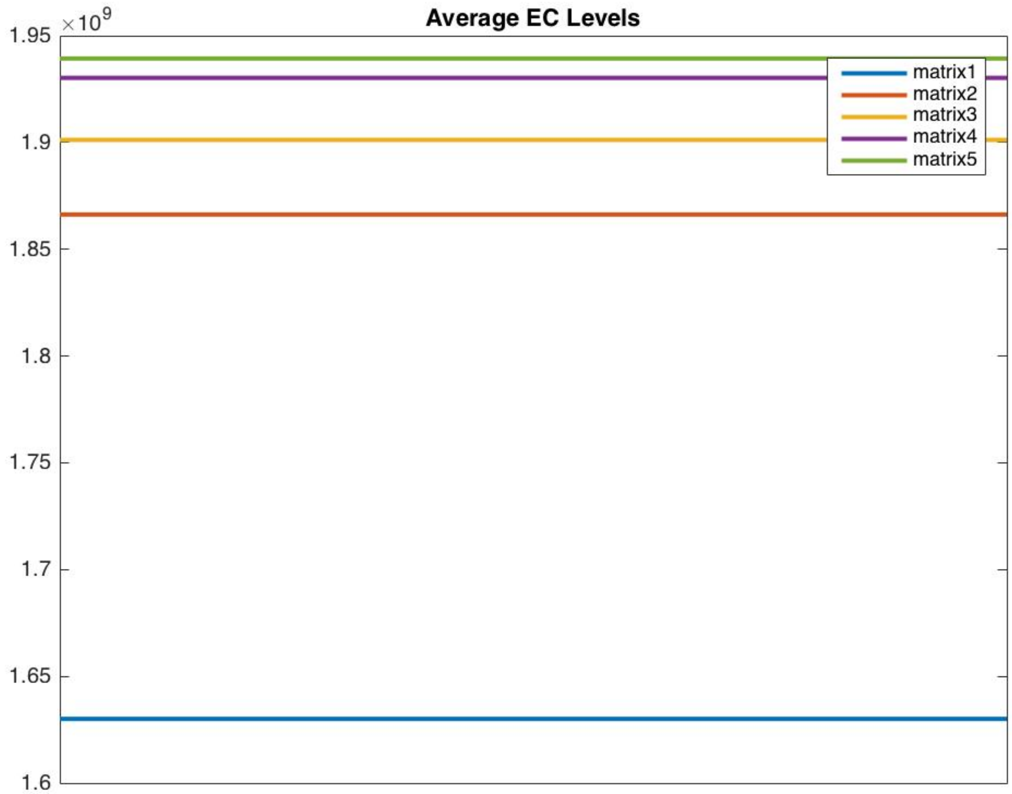

Figure 6: Each line shows the average EC level for the corresponding time interval. Each level is calculated from 10 EC simulations using a very basic Monte Carlo simulation of 1 million samples.

We decide a demand, for the accuracy of the EC simulation, based on the smallest EC estimate. As seen in Figure 6 this is obviously the one around the value $1{,}63{\cdot}10^9$. For each evaluated simulation technique, EC will be simulated a large number of times. By the law of large numbers, this will roughly yield a normal distribution among the EC outcomes. As an accuracy demand we crave that 2 standard deviations of the EC outcomes for a certain simulation technique, should be a maximum of 1% of $1{,}63{\cdot}10^9$. Se Equation (27)

$$2 \cdot \sigma \leq 16,3 \cdot 10^6 \tag{27}$$

We will evaluate the performance of three different simulation techniques in the following chapter.

6.4.2 Empirical Testing of Simulation Techniques

We want to evaluate the accuracy of the simulation technique for L and EC. The simulation technique we select should fulfill the accuracy demands specified in 6.4.1, Equation (27). Three different versions of a Monte Carlo simulation will be evaluated. These evaluations will be done in an empirical manner, where trial simulations for each technique is done 40 times. When doing these trial simulations, we will use all the data available. The Simulation techniques investigated are specified below;

- Simulation Technique 1: 2 millions Monte Carlo runs
- Simulation Technique 2: 2 millions Monte Carlo runs - using antithetic variables
- Simulation Technique 3: 10 millions Monte Carlo runs - using antithetic variables

For each round k in the Monte Carlo Simulations, the statistic variable sampled is the total loss L. See Equations (28) and (29).

$$L_k = \sum_{i=1}^{N} EAD_i \cdot LGD_i \cdot I(X_{ik} < \phi^{-1}(PD_i)) \tag{28}$$

Where X_i is the only stochastic part that will change for each Monte Carlo Sample. See Equation (29).

$$X_{ik} = \frac{\sqrt{R^2}\ (w_I I_{ik} + w_C C_{ik})\ +\ \sqrt{1-R^2}\ e_{ik}}{\sqrt{2R^2 w_I w_C \rho_i + (1-R^2) + R^2(w_I^2 + w_C^2)}} \tag{29}$$

After this the corresponding estimate of EC is easily extracted as in Equation (25). In Table 1 the standard deviation, 2 standard deviations and the mean are summarized for each simulation technique.

	1 Standard Deviation	2 Standard Deviations	Mean
Simulation Technique 1: 2 millions MC	$9{,}15 \cdot 10^6$	$18{,}3 \cdot 10^6$	$17844 \cdot 10^5$
Simulation Technique 2: 2 millions MC antithetic	$8{,}105 \cdot 10^6$	$16{,}21 \cdot 10^6$	$17813 \cdot 10^5$
Simulation Technique 3: 10 millions MC antithetic	$3{,}534 \cdot 10^6$	$7{,}068 \cdot 10^6$	$17833 \cdot 10^5$

Table 1: The table displays trial simulation measurements for the three different simulation techniques; 2 million Monte Carlo runs, 2 million Monte Carlo runs with antithetic variables and 10 million Monte Carlo runs with antithetic variables.

Our demand from Section 6.4.1 was that 2 standard deviations, of the simulation technique, should be less than 1% of $1{,}63 \cdot 10^9$. That is less than $16{,}3 \cdot 10^6$. The empirical testing has resulted in two simulation techniques that are good enough to meet this sufficiency demand. Simulation technique 2 and simulation technique 3.

Not surprisingly the simulation technique chosen is Technique 3. The one using 10 million Monte Carlo samples selected in an antithetic manner. Nevertheless, technique 2, the one using 2 million Monte Carlo samples selected in an antithetic manner performed over our expectations. This technique did also fulfill the demands, although with a much smaller marginal than technique 3. Since technique 3 performed overall better than technique 2, and yet is fast enough to run, it is the obvious choice as our final simulation technique.

A last step, towards perfecting the consistency of the EC simulation, is to fix the random variables used in the Monte Carlso simulations. More precise, we fix the manner in which the random numbers are drawn. This will assure a zero variation of the EC estimate, when using a fix correlation matrix to simulate the model variables. this will make it easy to quantify exactly how much a change of the underlying model parameters - the values of the correlation matrix A - have affected the EC estimate.

We have now selected a final simulation technique that will be used for estimating EC throughout this thesis. We will use 10 million Monte Carlo simulations, with antithetic variates and fixed random variables.

6.5 Summary for EC Estimation

We will now summarize the technique we have chosen to estimate EC. The total loss L, for the credit portfolio, is described by Equation (30).

$$L = \sum_{i=1}^{N} EAD_i \cdot LGD_i \cdot I(X_{ik} < \phi^{-1}(PD_i)) \tag{30}$$

Where X_i is described in Equation (31).

$$X_i = \frac{\sqrt{R^2}\ (w_I I_{ik} + w_C C_{ik})\ +\ \sqrt{1-R^2}\ e_i}{\sqrt{2R^2 w_I w_C \rho_i + (1-R^2) + R^2(w_I^2 + w_C^2)}} \tag{31}$$

This gives us a total function for L, as in Equation (32).

$$L = \sum_{i=1}^{N} EAD_i \cdot LGD_i \cdot I(\frac{\sqrt{R^2}\ (w_I I_{ik} + w_C C_{ik})\ +\ \sqrt{1-R^2}\ e_i}{\sqrt{2R^2 w_I w_C \rho_i + (1-R^2) + R^2(w_I^2 + w_C^2)}} < \phi^{-1}(PD_i)) \tag{32}$$

At the next stage the stochastic variable L is simulated, in order to obtain its distribution. This is done using a Monte Carlo technique using 10 million samples and antithetic variates. Exactly what parts of L that are stochastic, and thus will be updated between the Monte Carlo runs, is shown in Equations (28) and (29). At last the estimate of EC is easily extracted from the obtained distribution of L, as in Equation (25).

An example of an execution of the total procedure is shown in Figure 7.

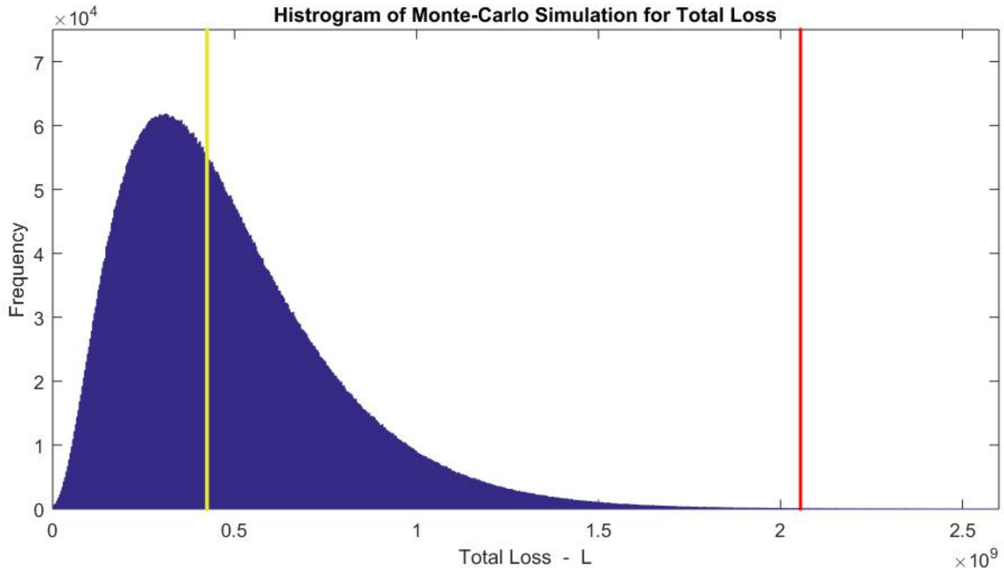

Figure 7: Monte Carlo simulation of the total loss for the credit portfolio constructed in 6.1. The simulation is performed using 10 million Monte Carlo samples with antithetic variates. The simulations are presented as a histogram in blue. The yellow line represents the EL and the red line represents the 99,97% quantile of the sampled losses. The EC is calculated as the 99.97% quantile - the EL.

In the next chapter we will go on evaluating how EC might change as the market conditions change and an update of EC is necessary.

7 Updating EC with new Model Variables

In this chapter we will evaluate how EC changes as market conditions change. The main goal is to investigate and possibly increase the robustness of EC. A big change in EC is in general not preferable for financial institutions, as it might cause a variety of practical problems. At the same time it is necessary for financial institutions to update EC as new market data becomes available. The controversy between the preference of keeping EC stable and the necessity to have an up-to-date value of EC, is why we want to attempt to increase the robustness of EC.

The method used for estimating EC is the one derived and discussed in chapter 6. At first, the multi-factor model is used to obtain the stochastic variables r_i describing the financial well-being of each borrower. Each r_i is then converted into a normal standardized variable X_i which is compared to the corresponding threshold $\phi^{-1}(PD_i)$. Borrower p_i is said to have defaulted if $X_i < \phi^{-1}(PD_i)$. If a certain borrower p_i defaults, the corresponding loss for that borrower will be $LGD_i \cdot EAD_i$. To obtain the total loss value for the whole credit portfolio, the possible losses for each exposure of the portfolio are added up.

The total loss for the credit portfolio will be a stochastic variable and in order to simulate its distribution a Monte Carlo technique is used. The Monte Carlo simulation is run 10 million times including the antithetic variates that are used to reduce variance in the simulation results. Having all the possible outcomes of the total loss will enable us to calculate statistical properties that are needed in order to estimate the EC. Also, the outcomes plotted as a histogram will give us a clear picture over the distribution of the total loss. It is concluded that the distribution is similar to what one could expect of a loss distribution for credit portfolios to most banks. An example of an execution of this procedure is shown in Figure 7.

7.1 Setup for Updating EC

When the model is updated after a certain time interval this is done by updating the correlation matrix with the new data estimating EC as described above. This will make the $VaR_{99,97}$, the red line in Figure 7, change value. We will now take a look at how much the EC will change between evaluations using different correlation matrices. The standard test procedure throughout this thesis will be the simulation technique described above with data over a time period of 16 years with monthly observations.

7.1.1 Selecting Time Intervals

The time window for computations of each correlation matrix will be 8 years. The interval was chosen as it is believed to give a good representation of the market behavior including different business cycles in the market and at the same time not be long enough to include old and irrelevant data. The time frequency for updating the correlation matrix will be 2 years. A higher frequency would have given less changes in the updates and as we want to visualize and study these changes 2 years seemed like an appropriate frequency. Updating the correlation matrix even more seldom would have given a less realistic result as that is a very low frequency for updating a credit risk model for any bank. This setting gives us 5 different correlation matrices and a plot of the changes in EC between the updates can be seen in Figure 8.

7.2 Updating EC without Modification

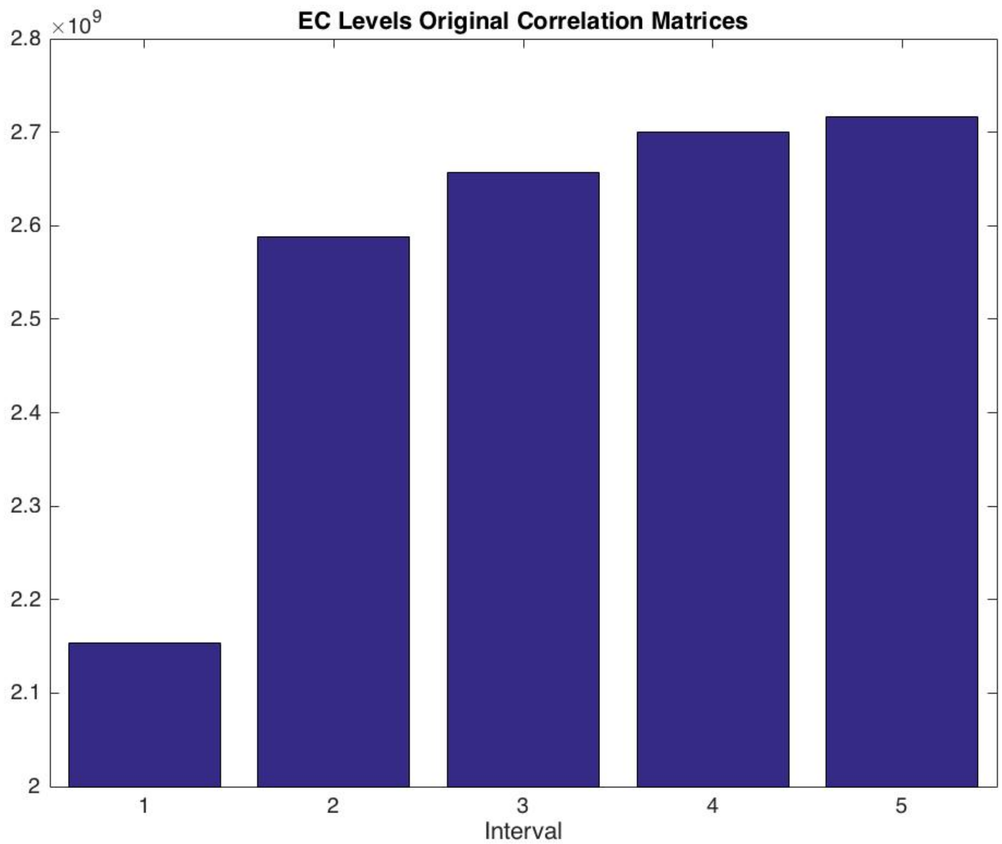

Figure 8: The plot displays the EC levels for the 5 different correlation matrices.

7.2.1 Possible Improvements

For a bank it can be hard to motivate variations of EC of this magnitude and therefore we will now try to slightly alter the way we update the value of the EC. Four major approaches will be used to try to decrease these fluctuations of the EC:

- Weighting of the correlation matrices
- Bootstrapping the correlations
- Estimating the correlation matrix with a multivariate GARCH process
- Standardizing the data with the volatilities from the GARCH process and then bootstrapping

7.2.2 Performance Tests

To evaluate and compare the performance of each possible updating method we need to establish some performance tests. The tests we select should somehow be correlated with the robustness of EC between updates. This since the robustness is essentially what a bank wishes to improve when it comes to EC updates. The tests will be calculated for all the approaches.

- Spread
  By this performance test we wish to capture the overall range covered by EC from all five time ranges. The Spread for a certain update technique is defines as; [Highest EC Level - Lowest EC Level]/reference level. The reference level is set to be the lowest EC level for the original update method.

- Percental change between updates
  This performance test is used to capture the percental change between two consecutive EC calculations.

- Average percental change
  This is, as the name suggests, an average of the four percental changes when EC is calculated.

- # Updates
  This performance test describes how many times the value of EC is updated during the four updates of the model.

In Table 2 we can see the performance test measurements for the original method.

	Spread	Change 1-2	Change 2-3	Change 3-4	Change 4-5	Average Change	# Updates
Original Correlation Matrices	26,1%	20,2%	2,6%	1,6%	0,6%	6,3%	4

Table 2: The table shows the measurements used for performance testing for the model run with the original correlation matrices.

In Table 2 we can see that with a $VaR_{99,97}$, the spread interval of the EC levels corresponds to 18,9% of the reference level. The biggest jump of the value of the EC is 14,7% and the average change is 4,6%.

7.3 Approach 1: Weighting of the Correlation Matrices

The first approach to reduce the variations of the EC when the model is updated is to use a weighted correlation matrix C_{w_t} that consists of a weighting with a parameter $\lambda \in [0,1]$ of the old correlation matrix C_{t-1} and the new correlation matrix C_t. As the first correlation matrix does not have a previous matrix to create a weighted correlation matrix with, this first matrix will be kept like C_t as it was in the original procedure. When weighting the other 4 matrices only the original matrices C_t will be used and not the previous weighted matrix $C_{w_{t-1}}$ as this would accumulate old data in the weighting procedure.

$$C_{w_t} = \lambda\, C_{t-1} + (1-\lambda)\, C_t \tag{33}$$

This technique will hopefully cause a smoother transition of the correlations and decrease the variations in the model, giving smaller changes of the $VaR_{99,97}$ between the updates. Implementing this technique in the model using weighting parameters $\lambda = 0,3, \lambda = 0,5, \lambda = 0,7$ gives us the following results of the EC levels:

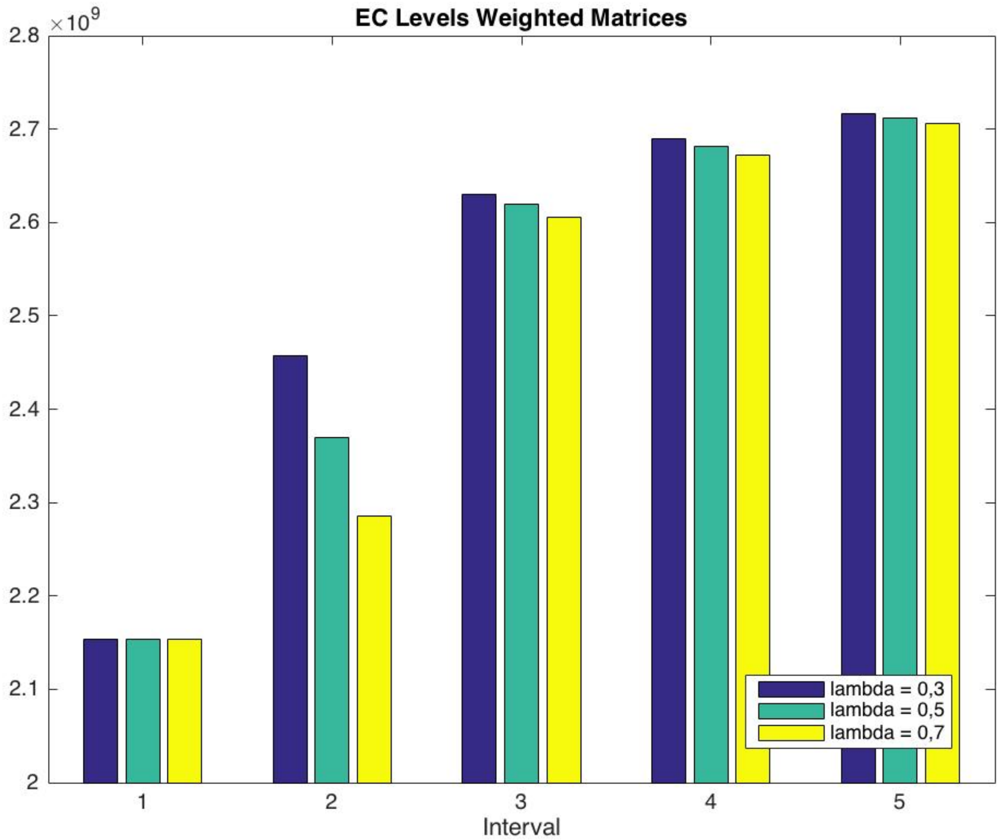

Figure 9: The plot compares the EC levels for the 5 weighted correlation matrices using weighting parameters $\lambda = 0,3$, $\lambda = 0,5$, $\lambda = 0,7$.

In this study a $\lambda = 0,5$ will be used, weighting half of two consecutive matrices together. The value of λ could of course be changed but a smaller λ would focus more on the new matrix C_t than the old matrix C_{t-1} and make the model less robust. A higher value of λ would focus more on the old matrix, making the model more robust but less realistic as the correlations would not really reflect the true values. How the EC levels are affected by the value of λ can be seen in Figure 9 above.

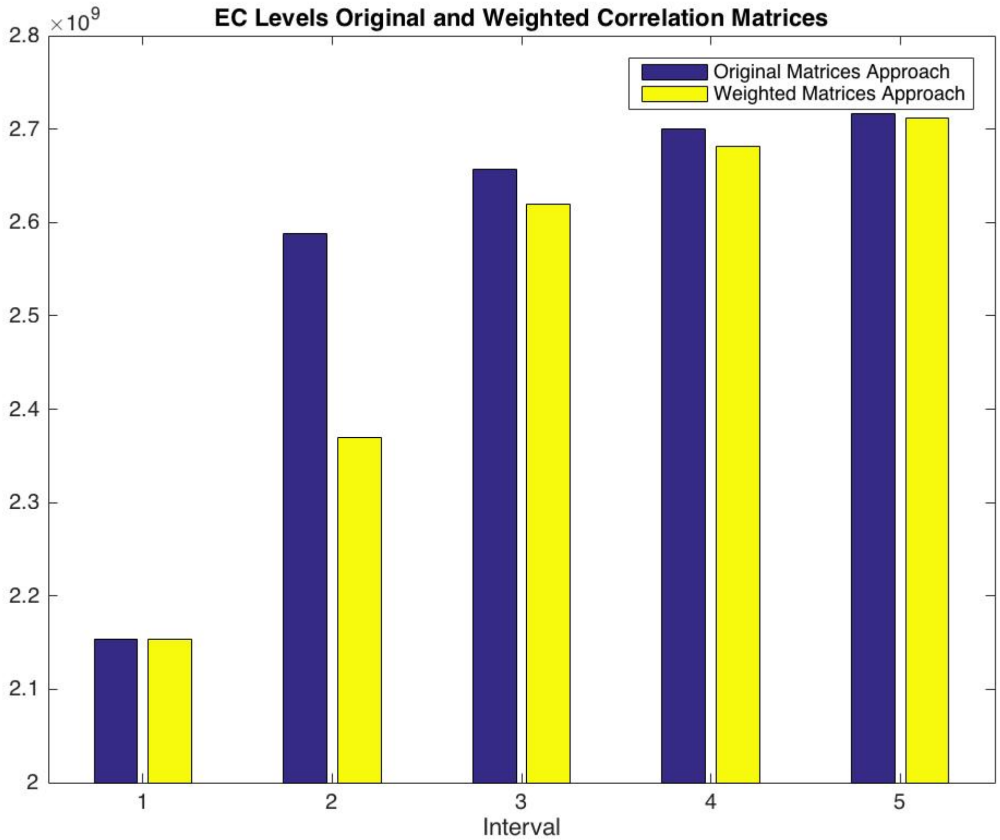

Figure 10: The plot compares the EC levels between the approach of using the original correlation matrices and the approach of using weighted correlation matrices with $\lambda = 0,5$. It is obvious that using weighted correlation matrices gives smaller jumps between the updates of the correlation matrices.

	Spread	Change 1-2	Change 2-3	Change 3-4	Change 4-5	Average Change	# Updates
Approach 1 Weighted Matrices	25,9%	10%	10,5%	2,4%	1,1%	6%	4

Table 3: The table shows the measurements used for performance testing for the model run with the weighted correlation matrices.

In Figure 10 we can clearly see that using weighted correlation matrices while updating the model gives smaller jumps between the updates. The biggest jump corresponds to a change of 7,6% compared to 14,7% with the original correlation matrices. However after the total of 4 updates we still land at approximately the same value for EC for both approaches so they have roughly the same spread, 18,9% for the original model and 18,7% for the weighted approach. This means that the approach of using weighted correlation matrices does smooth out the difference between each update but it does not lower the total magnitude of all the updates very much.

7.4 Approach 2: Bootstrapping the Correlations

A financial institution like a bank would prefer not to move assets around at all if possible. One approach that would address this preference is the one of bootstrapping. The idea is to bootstrap the correlations of the old correlation matrix creating a distribution for every correlation in the matrix. All the correlations in the new correlation matrix is then compared to the old distributions. If the new correlation is outside of a certain two sided confidence interval of the distribution, that specific correlation is updated in the correlation matrix. We will use a 5% and a 95% quantile for this purpose. If the new correlation is within the interval inside the quantiles of the distribution, the new correlation is simply considered to be too similar to the old one and that specific correlation is then not updated.

Creating the correlation distributions has to be done separately for all the index pairs. Since we use 36 unique indices, we will get $(36 \cdot 35)/2$ index pairs to estimate the correlation distribution for. This is done by drawing data equivalent to 8 years of measurements randomly with replacement for every index. When drawing the 8 years of data, the data is drawn in blocks of one year of measurements to erase the possibility of missing periodic correlations. The correlation between the indices is then computed as usual. This is called one bootstrap and the procedure is repeated 10 000 times, creating nice distributions for every index pair correlation.

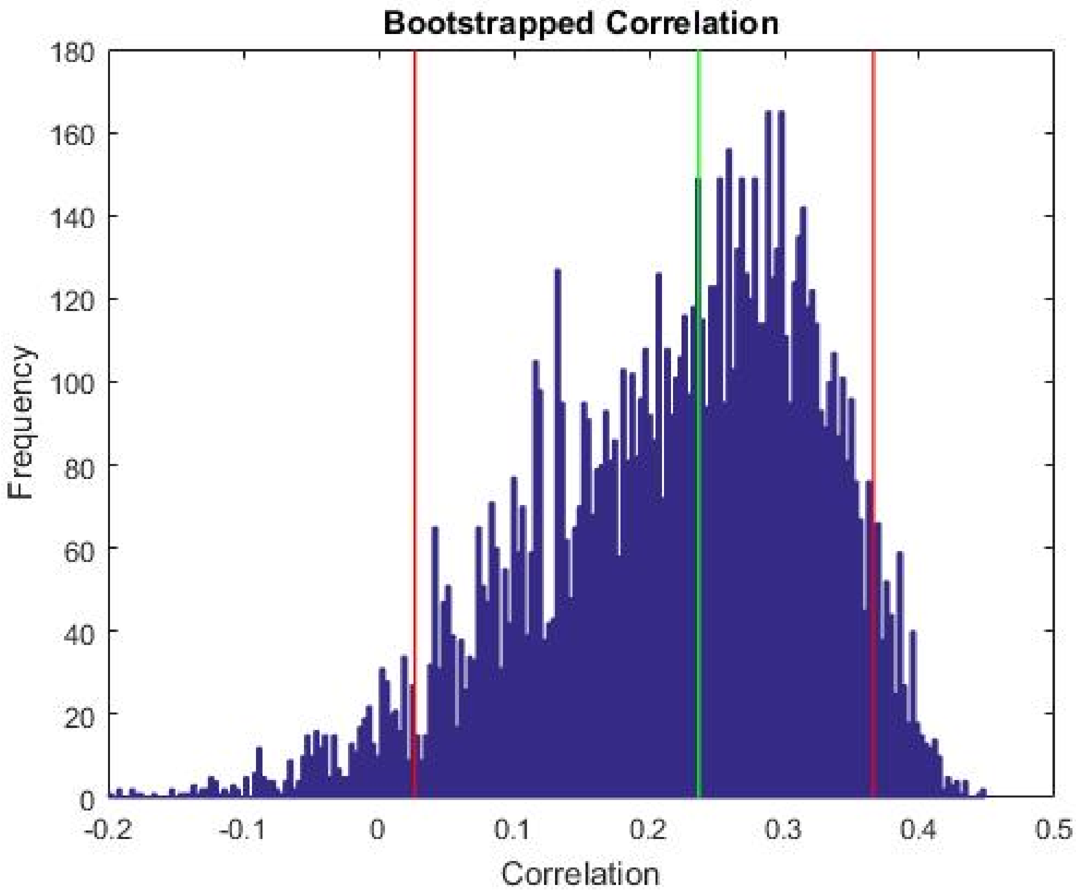

Figure 11: The figure shows a histogram of bootstrapped correlations between two indices. The green line represents the original estimate of the correlation and the two red lines represent the 5% and the 95% quantiles of the confidence interval.

When trying to implement this theory in practice we ran into a problem. When we bootstrap the correlation matrices and swap the correlation elements that are outside of the quantiles for new correlations, we also destroy the delicate properties of the correlation matrix. A symmetric correlation matrix C is by default positive semidefinite as:

$$x^T\, C\, x \geq 0 \tag{34}$$

for all non zero vectors x of real numbers.

When we swap some of the elements in the correlation matrix this is however no longer true. Hence the matrix is no longer by default positive semidefinite and the Cholesky decomposition can no longer be executed, see sections 5.5 and 6.2.4. If the Cholesky decomposition cannot be executed, we cannot create the correlated random variables C and I and we can therefore not run the model. To go around this problem the bootstrapping approach was slightly altered. The new execution method will be to bootstrap the correlations as described before and count the number of correlations in the update that are outside of their quantiles. If the number of correlations that are outside the quantiles is larger than a certain value, the whole correlation matrix will be updated. We will use a two sided confidence interval of 90% with a 5% and a 95% quantile. The null hypothesis in this case states that the correlation will not change with the relaxation that 10% of the correlations are allowed to be outside of the quantiles. If more than 10% is outside, the null hypothesis has to be rejected and the full correlation matrix has to be updated. When the new correlations are compared to the previous bootstrapped correlation distributions, they are always compared to the correlation distributions of the correlation matrix that initiated the last update of the value of EC.

Update	1	2	3	4
Percentage Outside Quantiles	51	7,9	16,8	5,7

Table 4: The table shows the percent of the correlations that are outside of their bootstrap quantiles for the four correlation matrix updates.

In Table 4 we can see that update 1 and 3 have a percentage of correlations outside of the quantiles big enough to reject the null hypothesis. Hence, the result of this approach would be to update the correlation matrix between the first and the second run, and between the third and the fourth run. We can see the results of these updates in Figure 12.

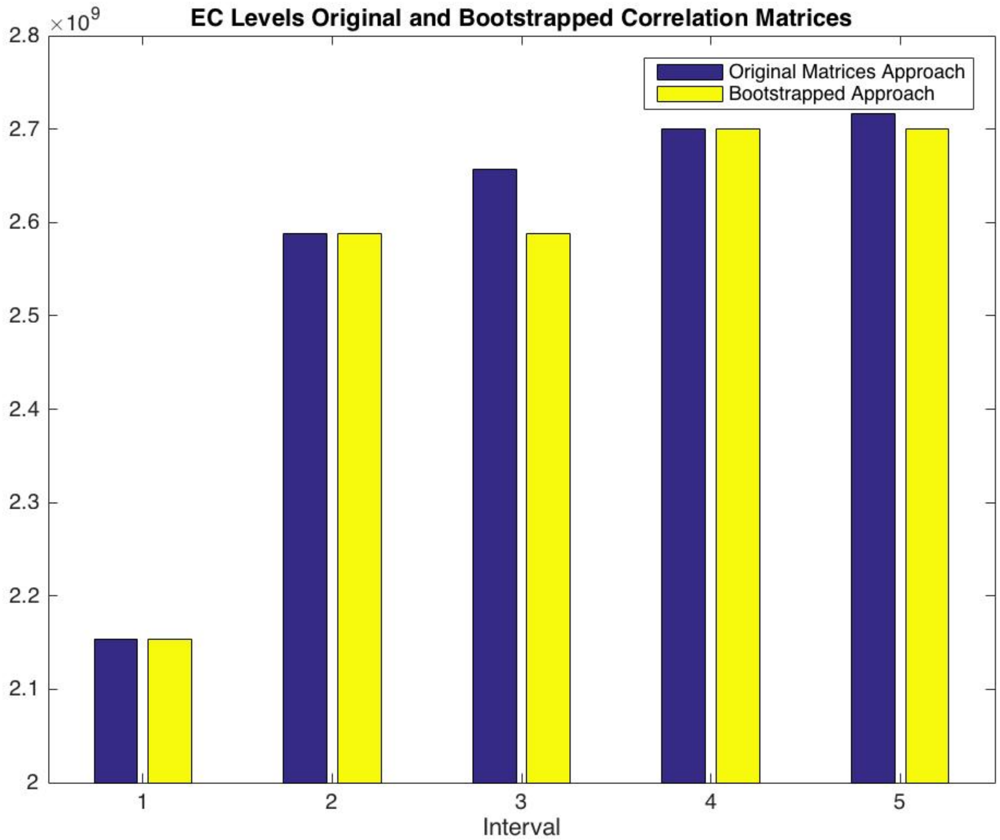

Figure 12: The plot compares the EC levels between the approach of using the original correlation matrices and the approach of using bootstrapped correlation matrices. This gives a more stable model in the sense that the value of EC is updated with a lower frequency.

	Spread	Change 1-2	Change 2-3	Change 3-4	Change 4-5	Average Change	# Updates
Approach 2 Bootstrapped Correlations	25,3%	20,2%	0%	4,3%	0%	6,1%	2

Table 5: The table shows the measurements used for performance testing for the model run with the bootstrapped correlation matrices.

We can see in Figure 12 and Table 5 that the bootstrapped correlation matrices approach has about the same spread, biggest change, and average change as the model run with the original correlation matrices. It is however more robust in the sense that it does not update the value of EC as often as the original method.

7.5 Approach 3: Estimating the Correlation Matrix with a Multivariate GARCH Process

As argued before, simply updating the correlation matrix normally does not give a perfectly robust model. Another way of updating the correlation matrix that might be more robust is using a multivariate GARCH model. In this thesis a multivariate constant conditional correlation GARCH model was used to extract the constant correlation matrix that supposedly could replace our old correlation matrix. This GARCH model would standardize the volatilities in the data over time, hopefully giving more stable correlations between the model factors. Kevin Sheppard's MFE Toolbox[15] for matlab was used to compute the conditional correlations using the cccmvgarch function. The output from this function is the conditional covariance matrix H_t described in the theory above, which changes over the time t. The constant correlation matrix CC can then be extracted as:

$$CC = D_t^{-1}\, H_t\, D_t^{-1} \tag{35}$$

where D is a diagonal matrix with the conditional standard deviations on the diagonal.

An important note here is that we are trying to estimate the changes in the correlation matrix by using a method that assumes that the correlation matrix is constant. This is somewhat contradictive but by assuming that the correlation matrix is piecewise constant on the intervals we used to create the original correlation matrices, we still believe that the model could produce interesting results.

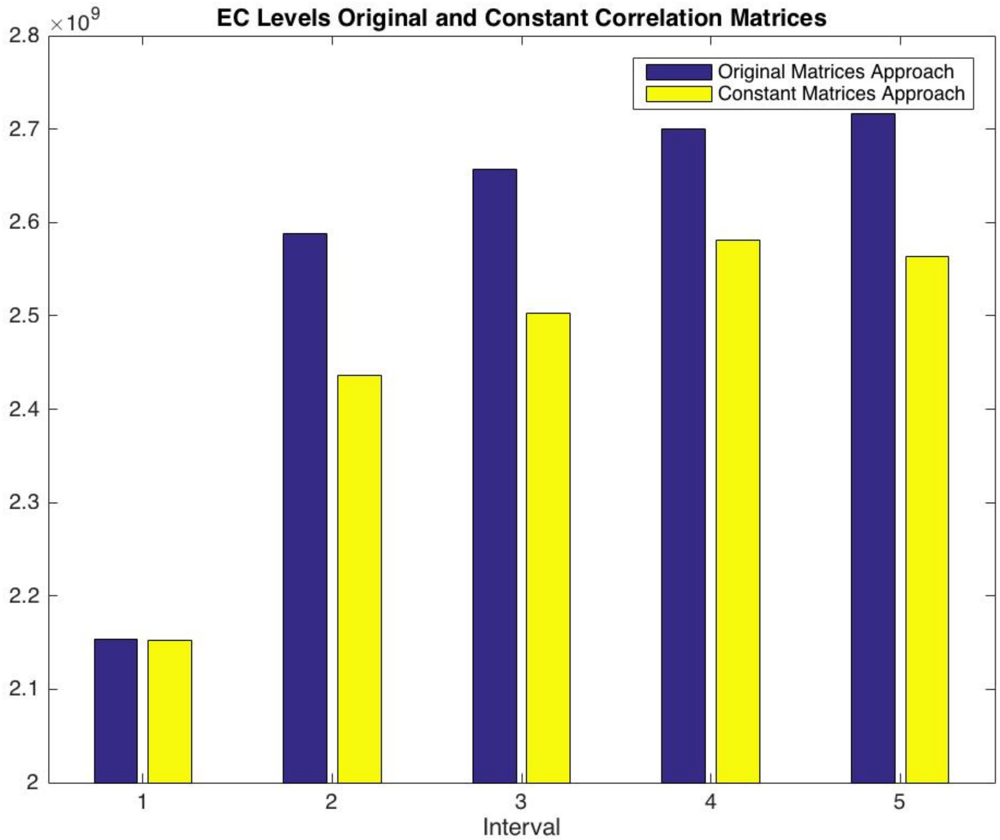

Figure 13: The figure compares the EC levels between the approach of using the original correlation matrices and the approach of using constant correlation matrices. We can see that the EC levels produced with the constant correlation matrices are spread over a smaller interval than the EC levels produced with the original correlation matrices.

	Spread	Change 1-2	Change 2-3	Change 3-4	Change 4-5	Average Change	# Updates
Approach 3 Constant Conditional Correlations	19,9%	13,1%	2,8%	3,1%	0,7%	4,9%	4

Table 6: The table shows the measurements used for performance testing for the model run with the constant correlation matrices from the CCC GARCH.

As we can see in Figure 13 the *EC* levels produced with the constant correlation matrices are spread over a smaller interval than the *EC* levels produced with the original correlation matrices. The spread interval of the constant correlation matrix approach corresponds to 14,4% compared to 18,9% for the original matrix approach. For the constant correlation approach we have a biggest change of 9,6% between consecutive runs compared to 14,7% for the model with the original matrices. Also the average change is much lower with 3,6% compared to 4,6%. Another interesting fact to notice is that the value of *EC* now is higher for interval 4 than for interval 5, while for previous approaches it was the other way around.

7.6 Approach 4: Standardizing the Data with the Volatilities from the GARCH Process and then Bootstrapping

If we take a look at the plotted data it looks like this:

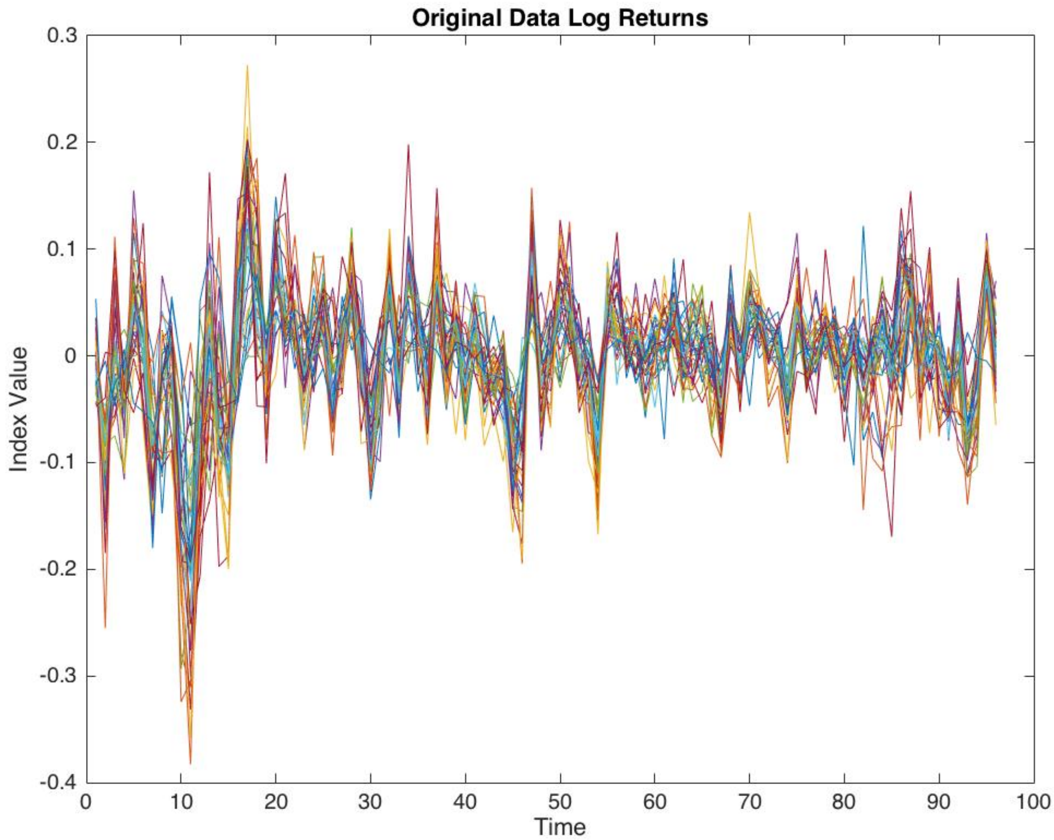

Figure 14: Here we can see all the 36 Morgan Stanley indices with 96 observations from 2007/12/31 - 2015/11/31, plotted in one figure. The economic crisis in 2008 is clearly visible around index 10 on the time axis.

We can see that there is an obvious common trend in the data's volatility. In turbulent times in the economy, like the crisis in 2008 around index 10 on the x-axis in Figure 14, data seems to be perfectly correlated. This is however an effect caused by macroeconomic events. It is also somewhat unclear to take the correlation between two time series when their volatilities change over time. A way to get an intuitively more accurate estimate of the indices internal correlations would be to standardize the indices with their conditional volatilities. These conditional volatilities can be extracted from the same multivariate CCC Garch model that was used in the previous section. After standardizing the data we will hopefully get a more accurate result while bootstrapping for the uncertainty of the correlation matrix and get better estimates for the distributions of the correlations.

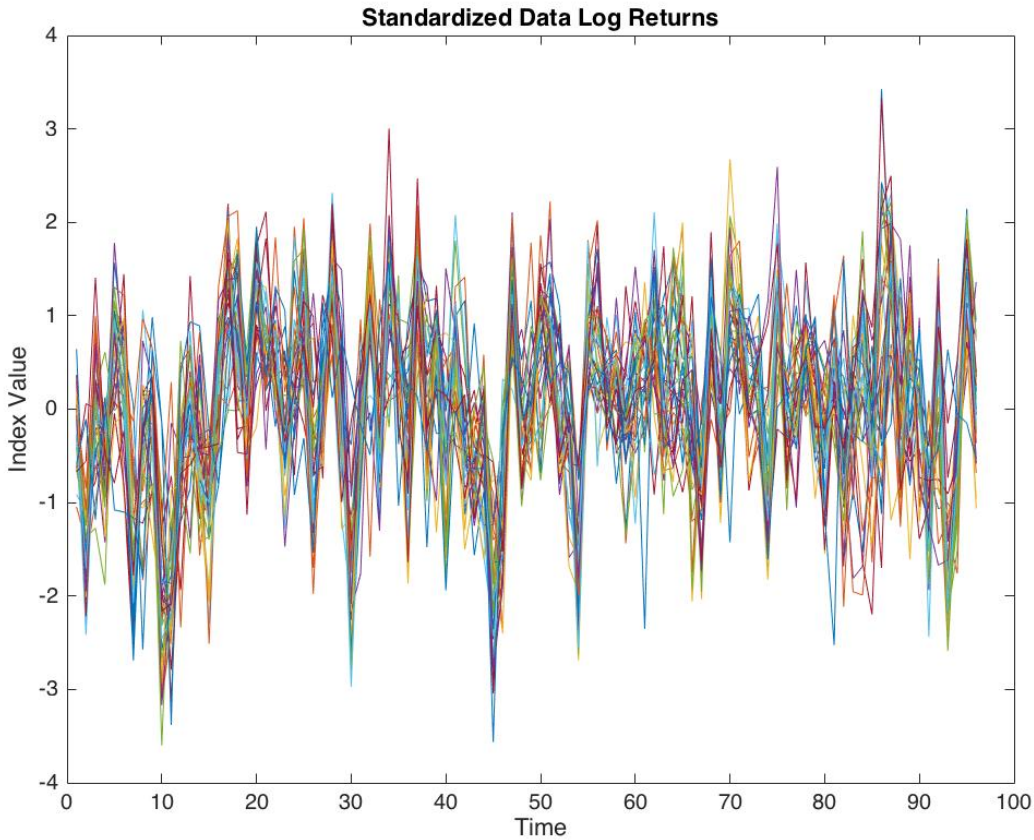

Figure 15: Here we can see all the 36 standardized Morgan Stanley indices with 96 observations from 2007/12/31 - 2015/11/31, plotted in one figure. Now the volatility is more even through time amongst the different indices.

The indices are standardized for every time t by dividing with the conditional volatilities, the standard deviations, for every index at time t. The volatilities, D, are extracted from the conditional covariances, H_t, which are an output from the cccmvgarch method described in the previous approach.

$$D_t = \sqrt{diag(H_t)} \tag{36}$$

The correlations are then computed with the standard correlation function like in the first two approaches combined with the bootstrap scheme that we used in approach 2. Once again we use a 90% confidence interval with 5% and 95% quantiles of the bootstrapped correlations to decide whether to update the correlation matrix or not. Meaning that a maximum of 10% of the correlations are allowed to be outside of the quantiles for the null hypothesis not to be rejected. We get the following table of correlations that are outside of the quantiles for the four updates:

Update	1	2	3	4
Percentage Outside Quantiles	37	8,4	20	9,8

Table 7: The table shows the percent of the correlations that are outside of their bootstrap quantiles for the four correlation matrix updates.

In Table 7 we can see that once again only update 1 and 3 have a percentage of correlations outside of the quantiles big enough to reject the null hypothesis. As for approach 2 the result of this approach would be to update the correlation matrix between the first and the second run as well as between the third and the fourth run.

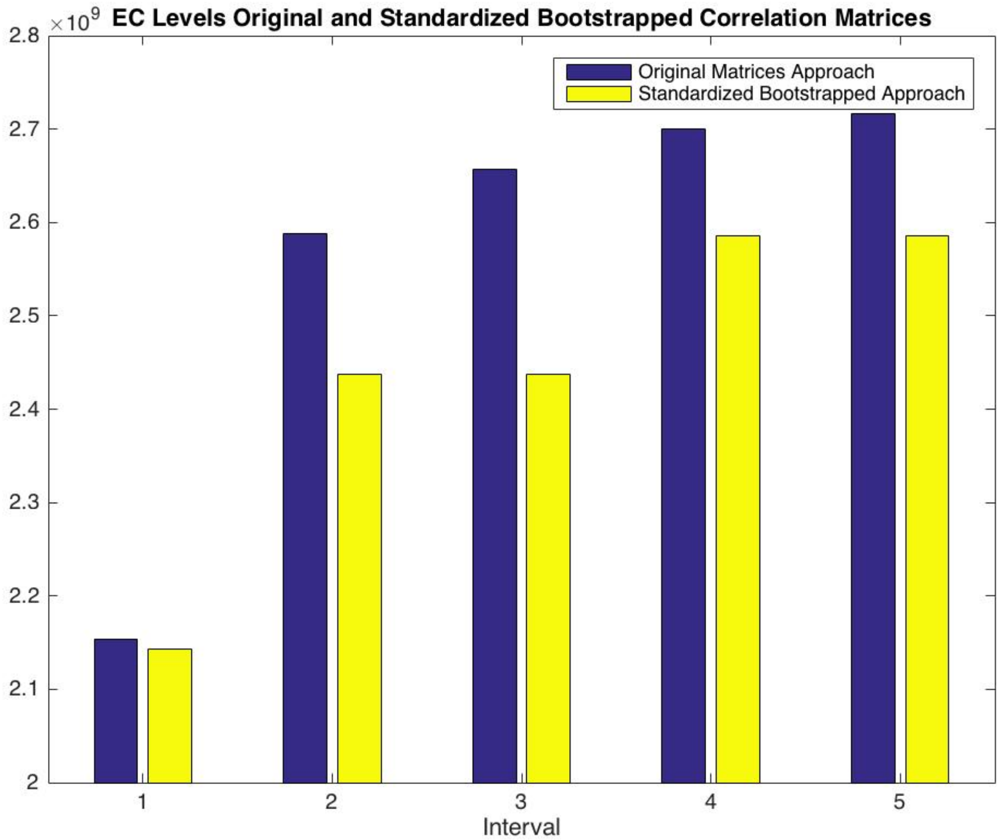

Figure 16: The figure compares the EC levels between the approach of using the original correlation matrices and the approach of using standardized bootstrapped correlation matrices. We can see that the EC levels produced with the standardized bootstrapped correlation matrices have a lower spread, lower biggest change and lower average change than the EC levels produced with the original correlation matrices.

	Spread	Change 1-2	Change 2-3	Change 3-4	Change 4-5	Average Change	# Updates
Approach 4 Standardized Bootstrapped Correlations	20,6%	13,8%	0%	6,1%	0%	5%	2

Table 8: The table shows the measurements used for performance testing for the model run with the standardized bootstrapped correlation matrices.

In Figure 16 and Table 8 we can see that the standardized bootstrapped correlation matrices approach has relatively small values of the spread, the biggest change and the average change with 15%, 10,2% and 3,7% compared to 18,9%, 14,7% and 4,6% for the original method. It is also robust in the sense that it only updates the value of EC twice.

7.7 Summary Results

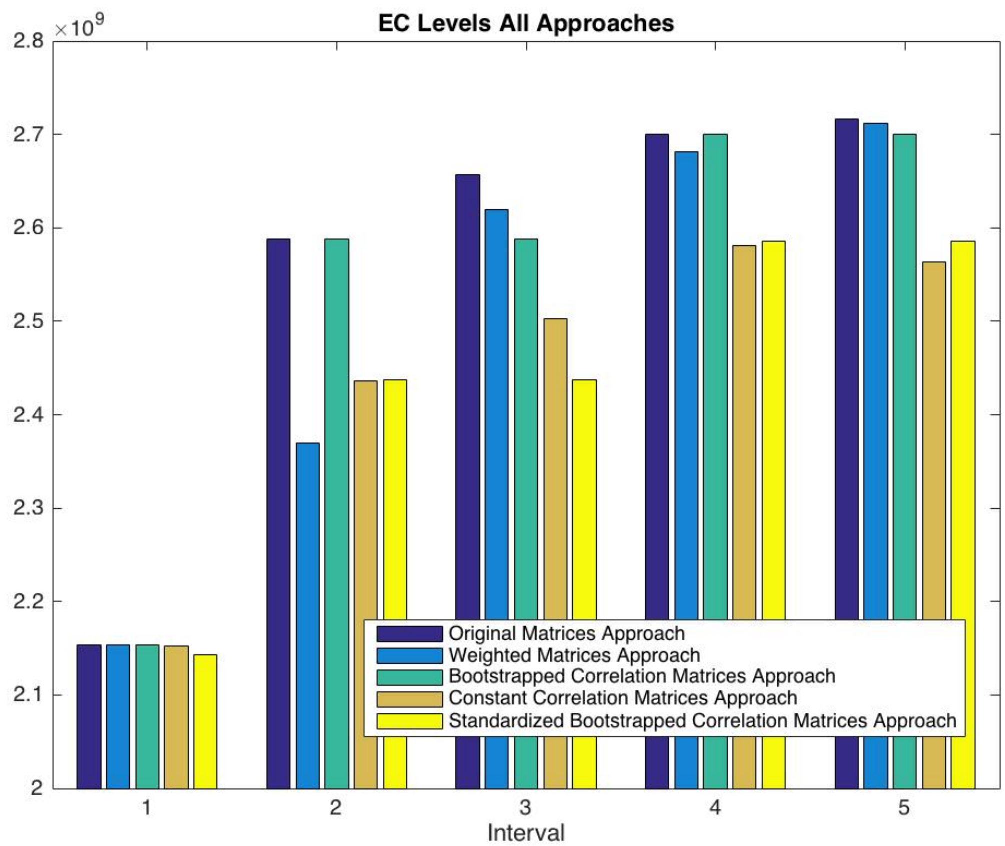

Figure 17: The figure shows a summary of the EC levels of all the four approaches compared to the model with the original correlation matrices.

	Spread	Change 1-2	Change 2-3	Change 3-4	Change 4-5	Average Change	# Updates
Original Correlation Matrices	26,1%	20,2%	2,6%	1,6%	0,6%	6,3%	4
Approach 1 Weighted Matrices	25,9%	10%	10,5%	2,4%	1,1%	6%	4
Approach 2 Bootstrapped Correlations	25,3%	20,2%	0%	4,3%	0%	6,1%	2
Approach 3 Constant Conditional Correlations	19,9%	13,1%	2,8%	3,1%	0,7%	4,9%	4
Approach 4 Standardized Bootstrapped Correlations	20,6%	13,8%	0%	6,1%	0%	5%	2

Table 9: The table shows the measurements used for performance testing for all the four approaches and the original method.

8 Conclusion

The conclusion that can be drawn after testing the four different approaches is that they all outperform the model simply using the original correlation matrices when it comes to robustness. We shall now discuss and compare the results between the different approaches.

Approach 1 with the weighted correlation matrices is the most effective when it comes to smoothing out big jumps between the updates. It is on the other hand not as effective at minimizing the total magnitude of the spread of the EC levels and had about the same average change of the EC levels as the original method. Another positive feature with this approach is that it is very intuitive and easy to implement.

The approach that at first sight seems to be the overall most robust is approach 3, using constant correlation matrices. It has the lowest measurement values for the spread, the biggest change as well as the average change. The approach does have a relatively strong reaction to the economic crises in the first update but is overall robust between the updates due to the standardization of the volatilities of the data over time. And maybe a strong reaction to such a severe crisis is not a bad thing as a crisis of this magnitude does have a major impact on a bank. The problem with this approach is that it always updates the model even if the change of the EC might be small and unnecessary.

The bootstrapping approaches did not turn out as expected as we were not able to update single correlations in the correlation matrices due to problems with the Cholesky decomposition. The alternation of the bootstrap approaches did however turn out to be even better. With these slightly altered approaches we got a threshold method that prevented an update if the change in the correlations between the time intervals was unnecessarily small. From a bank's perspective that is a very desirable feature of a credit risk model as the bank does not want to make small unnecessary changes of the EC. The bootstrap approach with the original data did not except for the low update frequency impress much on the spread, the biggest change or the average change. The bootstrap approach with the standardized data did however like the constant conditional correlation approach provide a very low spread, a low biggest change and a low average change.

Depending on the aim for the user utilizing this credit risk model, different approaches could be preferred. For the purpose of implementing a credit risk model for a bank we would however argue that approach 4 with the combination of standardized data and the bootstrap method is the most useful. The approach combines the low spread, the low biggest change and the low average change with few unnecessary updates, which makes a very effective model for this purpose. The different approaches could also be mixed and used simultaneously for desired results. As an example, using both the weighting approach and the standardized bootstrapped approach would probably give even smoother and more robust results.

9 Final Recommendations for EC calculation

In this section we will give a final recommendation of how to determine and update the economic capital - EC. Our recommendations are based on our findings in this thesis.

- Each EC calculation is made with the technique derived in chapter 6 and summarized in Section 6.5.
- EC is updated every 2 years. When calculating the correlation matrix needed for the model, 8 years of retroactive data with monthly observations is used.
- When updating the correlation matrix, the technique described in approach 4 in Section 7.6 is used. First standardizing the data with the conditional volatilities extracted from a multivariate GARCH process. Then estimating the correlation matrix like in the original method. And at last deciding whether or not to update using a bootstrap technique.

See Figure 16 of how the recommended technique performs compared to the original technique. By the original technique we refer to a technique where the correlation matrix is calculated in a classical manner.

The recommended approach did very well in the experiments in this thesis and outperformed the original method by far when it comes to robustness of the calculation of EC. It has overall smaller changes in EC between updates. The smaller changes are however not due to a failure to capture market conditions. On the contrary we believe that approach 4 actually captures the ongoing market conditions better than the original technique. This is due to the standardization of the data with the GARCH process, which gives a more robust and intuitive measurement of the correlations between the indices. The approach also has the nice feature of reducing noise in the calculations of the EC as the bootstrap technique effectively prohibits small irrelevant changes.

10 Further Discussions

10.1 Related Topics

We will here discuss some additional topics we would have found interesting to investigate if we had the time.

Why is a multi-factor model using only one country index and one industry index the best choice for creating a credit risk model? Of course a variety of different models could have been used. A three factor model with one country index and two industry indices would probably have been more accurate. It is however a trade off as the complexity of the correlation matrix and the computations would have increased greatly.

Though not a part of the aim in this thesis, an interesting topic to investigate is perfecting the actual value of EC. As focus in this thesis has been investigating how EC changes, and how the robustness of EC might be increased, we have not in depth discussed the accuracy for the value of EC itself. Then again an exact value of the EC can never be found as EC is defined by the creator of the model.

Life is like a small magical creature, beyond the horizon, 'neath a rainbow[17]

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