NONRADIAL SOLUTIONS OF A NONLINEAR ELLIPTIC EQUATION

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Abstract

We consider the partial differential equation $\Delta u + f(u) = 0$ in two dimensions, where f is a nonlinear function of u. Under certain assumptions on f we show that the equation has non-radial C^2 solutions with $\lim_{|x|\to\infty} u(x) = 0$ with an arbitrary number of zeros on each axis. The equation has applications in Bose-Einstein condensation and water waves among other fields. We also look at numerical solutions in the special case $f(u) = |u|^2 u - u$.

Sammanfattning

I detta arbete studerar vi en partiell differentialekvation som bland annat används som en modell för vattenvågor och Bose-Einstein-kondensat. Vi undersöker lösningar som är icke-radiella och går mot noll i oändligheten. Genom att göra en speciell ansats reducerar vi ekvationen till en ordinär differentialekvation. Vi visar att denna ekvation har lösningar med godtyckligt antal nollställen. Vi undersöker även numeriska lösningar.

Acknowledgements

The Python code used in the numerical part is based on a previous code by Teo Nilsson for radial solutions. I'm grateful to Teo for letting me use his code.

1 Introduction

The semilinear elliptic equation

$$\Delta u + f(u) = 0, \quad x \in \mathbb{R}^n, \tag{1}$$

with $u\colon \mathbb{R}^n \to \mathbb{R}$ or $\mathbb{C}$ appears in many different fields in physics such as quantum mechanics, optics and fluid mechanics [17]. A large amount of research has been dedicated to find radial solutions of (1) with $\lim_{|x|\to\infty} u(x) = 0$. Under certain assumptions on f it can be shown that there is a unique positive radial solution of (1) with $\lim_{|x|\to\infty} u(x) = 0$, [1, 3, 4, 10, 12, 13, 15]. Under additional assumptions one can show that that there is a sequence of real solutions $u_n(r)$, $n = 0, 1, 2, ...$ such that u_n has precisely n positive zeros [2, 8, 14] and there is even a recent uniqueness result for sign-changing solutions [5].

Much less effort has been devoted to find non-radial solutions. In this thesis we will look at the case $n = 2$. P. L. Lions [11] introduced a method for finding non-radial solutions when $f\colon \mathbb{C} \to \mathbb{C}$ satisfies $f(se^{i\theta}) = f(s)e^{i\theta}$, $s, \theta \in \mathbb{R}$. His idea was to look for solutions of the form $u(r,\theta) = e^{im\theta}w(r)$, where m is a positive integer, which reduces (1) to

$$w''(r) + \frac{w'(r)}{r} - \frac{m^2}{r^2}w(r) + f(w) = 0 \tag{2}$$

for $r > 0$. Lions proved that there exists a positive solutions if $f(u) = g(|u|^2)u$, where $g\colon [0,\infty) \to \mathbb{R}$ is continuous with $g(0) = 0$, $F(s) := \int_0^s f(t)\,\mathrm{d}t$ has a positive zero $\gamma > 0$, $F(s) < 0$, on $(0, \alpha)$ for some $\alpha > 0$ and $\lim_{s\to\infty} F(s)\exp(-s^\beta) \leqslant 0$ for some $\beta < 2$. Lions also proved that there is infinitely many distinct decaying solutions if we instead of the last condition require that $\lim_{s\to\infty} |g(s)|\exp(-s^\beta) = 0$ for some $\beta < 2$.

In this theses we instead follow [7] and suppose that the restriction of f to real arguments is an odd locally Lipschitz continuous function satisfying

$$-\infty < -\sigma^2 = \lim_{s\to 0}\frac{f(s)}{s} \leqslant 0, \quad f(s) < 0 \text{ for small } s > 0, \tag{3}$$

$$F(s) := \int_0^s f(t)\,\mathrm{d}t \quad \text{has exactly one positive zero } \gamma \text{ and } f(s) > 0 \text{ for } s \geq \gamma, \tag{4}$$

$$f(s) = k|s|^{p-1}s + g(s) \quad \text{where} \quad k > 0, \quad p > 1, \quad \lim_{s\to\infty} s^{-p}g(s) = 0. \tag{5}$$

We set

$$w(r) = r^m v(r) \tag{6}$$

to obtain

$$v'' + \frac{2m+1}{r} v' + \frac{1}{r^m} f(r^m v) = 0. \tag{7}$$

and look for solutions that satisfy

$$v(0) = d, \quad \text{and} \quad v'(0) = 0. \tag{8}$$

If v is a C^2 solution of (7), (8) it follows that w is a C^2 solution to (2) with

$$\lim_{r \to 0^+} \frac{1}{r^m} w(r) = d \quad \text{and} \quad \lim_{r \to 0^+} \frac{1}{r^{m-1}} w'(r) = md. \tag{9}$$

α is defined as the smallest value of t where $f(t) = 0$ and β is defined as the largest value of t where $f(t) = 0$. Thus we have

$$0 < \alpha \leqslant \beta < \gamma. \tag{10}$$

By multiplying (2) with $r^2 w'$ we obtain

$$\left(\frac{r^2}{2} (w')^2 \right)' - \left(\frac{m^2}{2} w^2 \right)' + r^2 (F(w))' = 0. \tag{11}$$

We integrate over the interval (r_1, r_2) to obtain Pohozaev's identity

$$r^2 \left[\frac{1}{2} (w')^2 + F(w) \right]_{r_1}^{r_2} = \left[\frac{m^2}{2} w^2 \right]_{r_1}^{r_2} + 2 \int_{r_1}^{r_2} s F(w(s)) \, \mathrm{d}s. \tag{12}$$

In [7, Section 2] it is shown that (2), (9) and (7), (8) are equivalent. Moreover the following existence and uniqueness result is shown.

Theorem 1.1. *For each $d \in \mathbb{R}$ there exists a unique solution of the problem* (2), (9)*. The solution is defined for all $r > 0$.*

Note that this is not a standard result since the equation is singular at the origin. The local existence and uniqueness of solutions is proved by considering the problem for v and rewriting it as a fixed point problem for a contraction mapping. The global existence is proved by showing that the quantity

$$\frac{\frac{1}{2} w'^2(r) + F(w(r)) - F_0}{r^{2m-2}} + \frac{m^2}{2} \frac{w^2(r)}{r^{2m}}$$

is non-increasing, where $F_0 := \min F < 0$.

The main result in [7] is the following:

Theorem 1.2. *Let the nonlinearity f have the properties* (3)–(5). *Then, for each nonnegative integer n there is a positive number d and a C^2 solution w to* (2), (9) *such that $w(r) \to 0$ as $r \to \infty$ and w has exactly n positive zeros.*

Unlike the result from [11] we are able to show that a solution can have arbitrary number of zeros in $[0, \infty)$ which requires that f satisfies different conditions, one difference is that we only allow $F(s)$ to have one zero.

In this thesis we will go through the main steps of the proof of Theorem 1.2. The proof is split into three parts. In Section 2 we show that for sufficiently small d, the solution of (2), (9) is positive. Here we mainly follow the arguments in [7, Section 3]. Since this part is not that technical, we have included all the details. In Section 3 we show that as $d \to \infty$, the number of zeros of the solution tends to infinity. We do this by noting that a rescaled version of the solution converges locally uniformly to a solution of a limiting differential equation and showing that this function has infinitely many zeros. We use a new proof to show the latter, which is based on an idea in [6]. We have omitted the convergence proof since it is rather technical, but not that surprising. We then put all the pieces of the proof together in Section 4. Here we again mainly follow [7, Section 5], but we have restructured the proof slightly in order to make it easier to read. The idea is to show that if d_k is defined as the supremum of the set

$$\{d > 0 \mid w(r, d) \text{ has at most } k \text{ positive zeros}\},$$

then $w(r, d_k)$ has precisely k positive zeros and $\lim_{r\to\infty} w(r, d_k) = 0$. Heuristically, an extra zero enters the positive axis from $+\infty$ as d crosses d_k from below. Finally, in Section 5 we include some numerical computations which illustrate the main result. Overall, we have tried to make the presentation easier to follow compared to [7] by including more details and adding some figures.

2 Existence of positive solutions

Lemma 2.1. *Let w be a nontrivial solution of* (2) *and suppose $w(r_0) = 0$. Then there exists a smallest $b > r_0$ such that $|w(b)| = \alpha$. Furthermore, $w \neq 0$ and $w' \neq 0$ on $(r_0, b]$.*

Proof. A nontrivial solution cannot vanish on any non-empty open interval, by uniqueness of solutions to the initial value problems. Hence there is an interval $(r_0, r_0 + \epsilon)$ on which either $w > 0$ and $w' > 0$ or $w < 0$ and $w' < 0$. We assume without loss of generality that $w' > 0$ and $w > 0$ on $(r_0, r_0 + \epsilon)$. There are two possibilities:

(P1) w has a local maximum at some smallest value r_1 of r,

(P2) $$w' \geqslant 0 \quad \text{for all} \quad r \geqslant r_0.$$

If (P1) holds, then at r_1 we have $w'(r_1) = 0$ and $w''(r_1) \leqslant 0$. Substituting into (2) gives

$$-\frac{m^2 w(r_1)}{r_1^2} + f(w(r_1)) \geqslant 0.$$

Thus

$$f(w(r_1)) \geqslant \frac{m^2 w(r_1)}{r_1^2} > 0,$$

which implies

$$w(r_1) \geqslant \alpha.$$

Therefore there is some smallest $b > b_0$ with $b < r_1$ such that $w(b) = \alpha$.

If (P2) holds there are two possibilities:

(P2a) there exists a smallest $b > r_0$ such that $w(b) = \alpha$,

(P2b) $$w'(r) \geqslant 0 \quad \text{and} \quad 0 < w(r) < \alpha \quad \text{for all } r > r_0.$$

If (P2b) is true we get $f(w) < 0$ for all $r > r_0$ which results in

$$w'' + \frac{w'}{r} - \frac{m^2 w}{r^2} = -f(w) > 0$$

for all $r > r_0$, which is equivalent to

$$(r^{2m+1} v')' \geqslant 0.$$

Integrating on (r_0, r) gives

$$v'(r) \geqslant \frac{r_0^{2m+1}}{r^{2m+1}} v'(r_0).$$

Integrating again on (r_0, r) gives

$$v(r) \geqslant v(r_0) + \frac{r_0}{2m} v'(r_0) \left[1 - \left(\frac{r_0}{r} \right)^{2m} \right]. \tag{13}$$

If $r_0 = 0$ then $v(r_0) = d > 0$ and $v'(r_0) = 0$ which gives

$$w(r) \geqslant d r^m.$$

If $r_0 > 0$ then $v(r_0) = 0$ and $v'(r_0) > 0$ which gives

$$w(r) \geqslant \frac{r_0}{2m} v'(r_0) \left[r^m - \frac{r_0^{2m}}{r^m} \right].$$

In either of these cases w grows without bound which contradicts (P2b).

Now we only have to show that $w' \neq 0$ to complete the proof of the lemma. If $w'(r_2) = 0$ for some $r_2 \in (r_0, b]$, (2) gives

$$w''(r_2) = \frac{m^2}{r_2^2} w(r_2) - f(w(r_2)) > 0. \tag{14}$$

But $w(r_2)$ being a local minimum contradicts $w(r) > 0$ and $w'(r) \geqslant 0$ for all $r \in (r_0, b)$, which was proven above. □

Lemma 2.2. *Let w be the solution of (2), (9) for $d > 0$. Let b_d be the smallest positive value of r for which $w(r) = \alpha$. Then as $d \to 0$, $b_d \to \infty$.*

Proof.

$$\frac{f(w)}{w} \geqslant -M \quad \text{for all } w. \tag{15}$$

Therefore

$$w'' + \frac{w'}{r} - \frac{m^2 w}{r^2} = -f(w) \leqslant Mw \tag{16}$$

or, equivalently,

$$v'' + \frac{2m+1}{r} v' \leqslant Mv \quad \text{for} \quad 0 \leqslant r \leqslant b_d. \tag{17}$$

Since $v > 0$ on $[0, b_d]$, dividing by v gives

$$\frac{v''}{v} + \frac{2m+1}{r} \frac{v'}{v} \leqslant M. \tag{18}$$

We set

$$y = \log v \tag{19}$$

and obtain

$$y'' + \frac{2m+1}{r} y' \leqslant y'' + (y')^2 + \frac{2m+1}{r} y' \leqslant M.$$

Thus

$$(r^{2m+1} y')' \leqslant M r^{2m+1}.$$

We note that $r^{2m+1} y' \to 0$ as $r \to 0$ and integrate on $(0, r)$:

$$r^{2m+1} y' \leqslant \frac{M}{2(m+1)} r^{2m+2}.$$

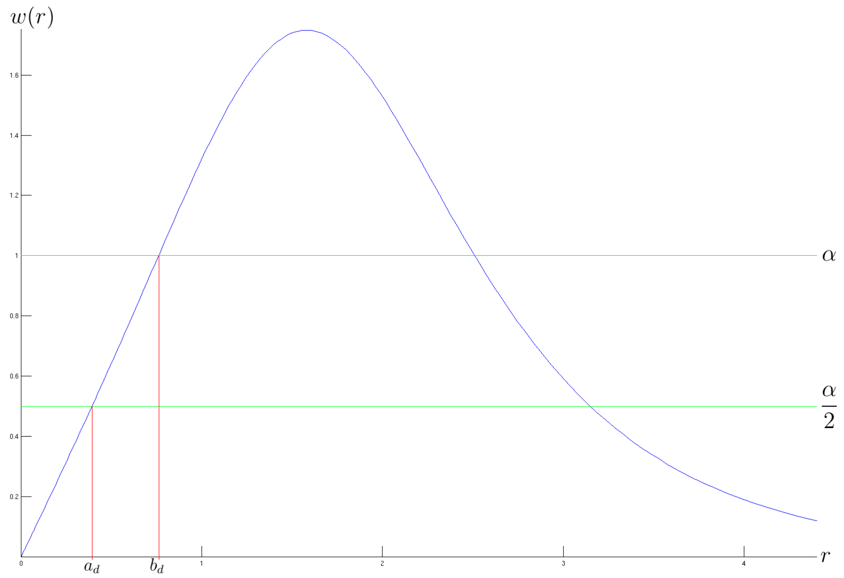


Figure 1: This figure illustrates the numbers α, a_d and b_d for $f(s) = |s|^2 s - s$

Integrating again on $(0, r)$ gives

$$\log \frac{v}{d} \leqslant \frac{Mr^2}{4(m+1)}.$$

Thus

$$w(r) \leqslant dr^m e^{Mr^2/4(m+1)} \quad \text{for } 0 \leqslant r \leqslant b_d. \tag{20}$$

Evaluation at $r = b_d$ gives

$$\alpha \leqslant db_d^m e^{b_d^2/4(m+1)}.$$

Thus $b_d \to \infty$ as $d \to 0$. □

Lemma 2.3. *Let w and b_d be as in Lemma 2.2 and let a_d be the smallest positive number such that $w(a_d) = \alpha/2$. Then $b_d - a_d \geqslant K > 0$, where K is a constant independent of d, for small d.*

Proof. It follows from (20) evaluated at $r = a_d$ that $a_d \to \infty$ as $d \to 0$. Pohozaev's identity (12) for $(0, r)$ is

$$\frac{1}{2} r^2 (w')^2 + r^2 F(w) = \frac{m^2}{2} w^2 + 2 \int_0^r sF(w(s))\, \mathrm{d}s. \tag{21}$$

We see (figure 1) that for $a_d \leqslant r \leqslant b_d$ we have $\alpha/2 \leqslant w \leqslant \alpha$ and $F(w) < 0$. Thus

$$\frac{1}{2}(w')^2 + F(w) < \frac{m^2}{2}\frac{\alpha^2}{a_d^2} \quad \text{for } a_d \leqslant r \leqslant b_d. \tag{22}$$

We define

$$C(d) \equiv \frac{m^2}{2}\frac{\alpha^2}{a_d^2}$$

and note that $\lim_{d\to 0} C(d) = 0$ and

$$\frac{(w')^2}{C(d) - F(w)} \leqslant 2 \quad \text{for } a_d \leqslant r \leqslant b_d. \tag{23}$$

Lemma 2.1 gives $w' > 0$ for $0 \leqslant r \leqslant b_d$. Taking the square root and integrating (23) on (a_d, b_d) gives

$$\int_{\alpha/2}^{\alpha} \frac{\mathrm{d}s}{\sqrt{C(d) - F(s)}} = \int_{a_d}^{b_d} \frac{w'}{\sqrt{C(d) - F(w)}}\mathrm{d}r < \sqrt{2}(b_d - a_d).$$

Since $C(d) \to 0$ as $d \to 0$ we get $b_d - a_d \geqslant K > 0$ for small enough d. □

Lemma 2.4. *Let w be the solution of (2), (9) for $d > 0$. For d chosen small enough we have $0 < w(r) < \gamma$ for all $r > 0$.*

Proof. Recall that $w(r) > 0$ and $w'(r) > 0$ for r small. We first claim that if $w(r) < \gamma$ for $r \in (0, c)$ then $w(r) \neq 0$ for $r \in (0, c)$. We suppose $w(r) = 0$ for some $r \in (0, c)$ and let $z_d \in (0, c)$ be the smallest such value of r. Pohozaev's identity (12) for $(0, z_d)$ is

$$\frac{1}{2}z_d^2(w'(z_d))^2 = 2\int_0^{z_d} rF(w(r))\,\mathrm{d}r.$$

Since $0 < w(r) < \gamma$ for $r \in (0, z_d)$ and $F(w(r)) < 0$ for $r \in (0, z_d)$ we get

$$\frac{1}{2}z_d^2(w'(z_d))^2 \geqslant 0 \quad \text{and} \quad 2\int_0^{z_d} rF(w(r))\,\mathrm{d}r < 0,$$

which is a contradiction. Thus there is no zero of $w(r)$ in the interval $(0, c)$ if $w(r) < \gamma$ on $(0, c)$.

Next we claim that for sufficiently small d, $w(r) < \gamma$ for all $r > 0$. Suppose there is a smallest value c_d of r such that $w(c_d) = \gamma$, $0 < w < \gamma$ on $(0, x_d)$. Pohozaev's identity on $(0, c_d)$ is

$$0 \leqslant \frac{1}{2}c_d^2(w'(c_d))^2 = \frac{m^2}{2}\gamma^2 + 2\int_0^{c_d} rF(w(r))\,\mathrm{d}r. \tag{24}$$

We will now show that the right hand side in (24) is negative for small d resulting in a contradiction to the supposition $w(c_d) = \gamma$. We have the following inequalities

$$0 < a_a < b_d < c_d \quad \text{and} \quad F(w) \leqslant 0 \text{ on } (0, c_d).$$

For $a_d \leqslant r \leqslant b_d$ we have $\alpha/2 \leqslant w \leqslant \alpha$ and thus $F(w) \leqslant F(\alpha/2) < 0$ since F is decreasing on $[\alpha/2, \alpha]$. Thus

$$\int_0^{c_d} rF(w(r))\,\mathrm{d}r \leqslant \int_{a_d}^{b_d} rF(w(r))\,\mathrm{d}r \leqslant \frac{1}{2}F(\alpha/2)(b_d^2 - a_d^2). \tag{25}$$

From Lemmas 2.2 and 2.3 we get

$$b_d^2 - a_d^2 = (b_d - a_d)(b_d + a_d) \geqslant Kb_d \to \infty \quad \text{as } d \to 0.$$

Since $F(\alpha/2) < 0$ we have

$$\int_0^{c_d} rF(w(r))\,\mathrm{d}r \to -\infty \quad \text{as } d \to 0. \tag{26}$$

Hence the right hand side of (24) is negative for small enough d which gives a contradiction. □

3 Solutions with arbitrarily many zeros

Given $\lambda > 0$ let $z_\lambda(r)$ be the solution of (2), (9) with $d \equiv \lambda^{2/(p-1)+m}$ and define

$$u_\lambda(r) = \lambda^{-2/(p-1)} r^{-m} z_\lambda\left(\frac{r}{\lambda}\right). \tag{27}$$

The function u_λ satisfies

$$u_\lambda'' + \frac{2m+1}{r}u_\lambda' + \lambda^{-2p/(p-1)} r^{-m} f(\lambda^{2/(p-1)} r^m u_\lambda) = 0$$

and

$$u_\lambda(0) = 1, \quad u_\lambda'(0) = 0.$$

Due to (5) we have that

$$\lambda^{-2p/(p-1)} r^{-m} f(\lambda^{2/(p-1)} r^m u) \to kr^{(p-1)m}|u|^{p-1}u$$

as $\lambda \to \infty$ for fixed u and r, and this suggests that u_λ should converge to a solution of the equation $u'' + \frac{2m+1}{r}u' + kr^{m(p-1)}|u|^{p-1}u = 0$ as $\lambda \to \infty$. A rigorous proof of this result based on the Arzela-Ascoli theorem can be found in [7, Lemma 4.1]; here we just record it as a lemma.

Lemma 3.1. *As $\lambda \to \infty$, $u_\lambda(r) \to u(r)$ uniformly on compact subset of* $[0, \infty)$, *where u is the solution of*

$$u'' + \frac{2m+1}{r} u' + k r^{m(p-1)} |u|^{p-1} u = 0 \tag{28}$$

with

$$u(0) = 1, \quad u'(0) = 0. \tag{29}$$

We next use this information to show that we can obtain solutions to (2), (9) with arbitrarily many zeros by choosing d sufficiently large. We do this by showing that u has infinitely many zeros. A proof of this fact is contained in [7, Section 4]. Here we present an original proof, which is based on ideas in [9, 19] and [6, Chapter 19.6].

Lemma 3.2. *Let u be the solution of* (28), (29). *Then u has infinitely many zeros.*

Proof. We make the change of variables

$$u(r) = r^{-\frac{2}{p-1}-m} x(\log r) \tag{30}$$

which gives

$$u'(r) = r^{-\frac{2}{p-1}-m-1} \left[x' + \left(\frac{-2}{p-1} - m \right) x \right] (\log r).$$

Setting $R \equiv r^{-\frac{2}{p-1}-m-2}$, we obtain

$$\begin{aligned} u'' &= \left[\frac{-2}{p-1} - m - 1 \right] \left[x' + \left(\frac{-2}{p-1} - m \right) x \right] R + \left[x'' + \left(\frac{-2}{p-1} - m \right) x' \right] R \\ &= Rx'' + \left[\frac{-4}{p-1} - 2m - 1 \right] Rx' + \left[\frac{4}{(p-1)^2} + \frac{4m}{p-1} + m^2 + \frac{2}{p-1} + m \right] Rx, \end{aligned}$$

where x, x' and x'' are evaluated at $t = \log r$. From (28) we get the equation

$$x'' - \frac{4}{p-1} x' + \left(\frac{4}{(p-1)^2} - m^2 \right) x + kx|x|^{p-1} = 0$$

We now have an autonomous equation of the form

$$x'' + Ax' + Bx + kx|x|^{p-1} = 0, \tag{31}$$

where

$$A = \frac{-4}{p-1}, \quad B = \frac{4}{(p-1)^2} - m^2, \quad p > 1, \quad k > 0. \tag{32}$$

The second order differential equation (31) can be rewritten as a system of first order differential equations:

$$\begin{cases} x' = y, \\ y' = -Ay - Bx - k|x|^{p-1}x. \end{cases} \tag{33}$$

We define

$$E(x,y) = \frac{k|x|^{p+1}}{p+1} + \frac{Bx^2}{2} + \frac{y^2}{2} \tag{34}$$

and get

$$E' = k|x|^{p-1}xy + Bxy + y(-Ay - Bx - k|x|^{p-1}x) = -Ay^2, \tag{35}$$

where E' denotes the derivative of $E(x,y)$ along a solution to (33). Equation (32) tells us that $A < 0$ and hence $E' \geqslant 0$. Note also that

$$\frac{y^2}{2} = E(x,y) - \frac{k|x|^{p+1}}{p+1} - \frac{Bx^2}{2} \leqslant E(x,y) - E_{\min},$$

where $E_{\min} = \min_{(x,y)} E(x,y) = \min_x \left(\frac{k|x|^{p+1}}{p+1} + \frac{Bx^2}{2}\right) > -\infty$ since $p > 1$. Since $E' \leqslant 2|A|(E - E_{\min})$ we find that

$$E_{\min} \leqslant E(x(t),y(t)) \leqslant \begin{cases} E_{\min} + (E_0 - E_{\min})e^{2|A|t}, & t \geqslant 0, \\ E_0, & t \leqslant 0, \end{cases}$$

where $E_0 = E(x(0),y(0))$. Also,

$$\lim_{|(x,y)|\to\infty} E(x,y) = \infty. \tag{36}$$

Together, this implies that any solution to (33) must exist for all $t \in \mathbb{R}$. Note also for future use that

$$E(x,y) \leqslant C_1(x^2 + y^2 + C_2)^{(p-1)/2}. \tag{37}$$

We now divide our analysis into two cases, depending on the sign of B.

Case 1 ($B \geqslant 0$): We note that the only fixed point is $x = y = 0$. The linearization has the eigenvalues

$$\lambda = \frac{-A}{2} \pm \sqrt{\frac{A^2}{4} - B}. \tag{38}$$

From (32) we notice that $\frac{1}{4}A^2 - B = m^2 \geqslant 0$. This tells us that the eigenvalues are real and that $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 0$ since we have restricted B to non-negative values. If $B > 0$ both eigenvalues are in fact positive and therefore no solution can approach

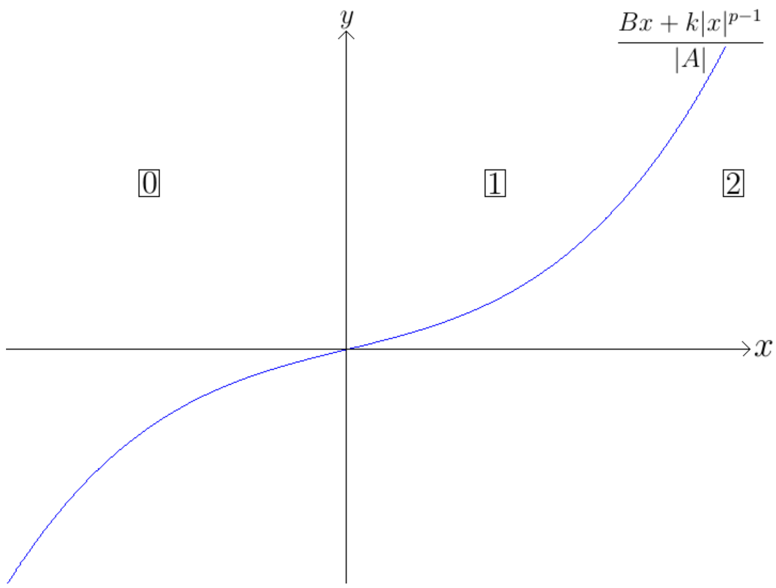


Figure 2: Phase portrait for $B \geqslant 0$; a non-trivial solution will spiral out clockwise

the origin as $t \to \infty$. In the case $B = 0$, one of the eigenvalues is 0. However, since E attains its minimum value 0 at the origin and is strictly increasing along any non-trivial solution (note that the x-axis contains no orbits), we again find that no solution can approach the origin as $t \to \infty$. By monotonicity it follows that $E_\infty = \lim_{t\to\infty} E(x(t), y(t))$ exists for any solution. We claim that $E_\infty = \infty$ unless $(x(t), y(t)) \equiv (0, 0)$. Indeed, otherwise the solution $(x(t), y(t))$ is uniformly bounded for $t \in \mathbb{R}$ by (36) and hence has a non-empty, invariant ω-limit set Ω (see e.g. [18, Chapter 6.3]). Moreover, E is constant on Ω. The latter is only possible if Ω is contained in the x-axis. By (33) this means that $\Omega = \{(0, 0)\}$, which is only possible if $(x(t), y(t)) \equiv (0, 0)$ due to the discussion above. Now that we have concluded that $E_\infty = \infty$, from (37) we get

$$(x(t), y(t)) \to \infty \quad \text{when } t \to \infty$$

Consider now the different (open) subsets of the phase plane indicated in Figure 2. In zone 0, $y' > 0$ and $x' > 0$. Moreover, a solution starting at (x_0, y_0) satisfies $x' \geq y_0 > 0$ as long as the solution remains in the closure of zone 0. Thus the solution must end up in zone 1. Note also that a solution starting on the negative x-axis will wander into zone 0 since $y' > 0$. Therefore it will also end up in zone 1. In zone 1 (and the positive y-axis), $y' > 0$, $x' > 0$ and therefore y can be written as a function of x along an orbit. We have

$$y'(x) = \frac{y'}{x'} = -A - \frac{Bx}{y} - \frac{k|x|^{p-1}x}{y} \leqslant |A|$$

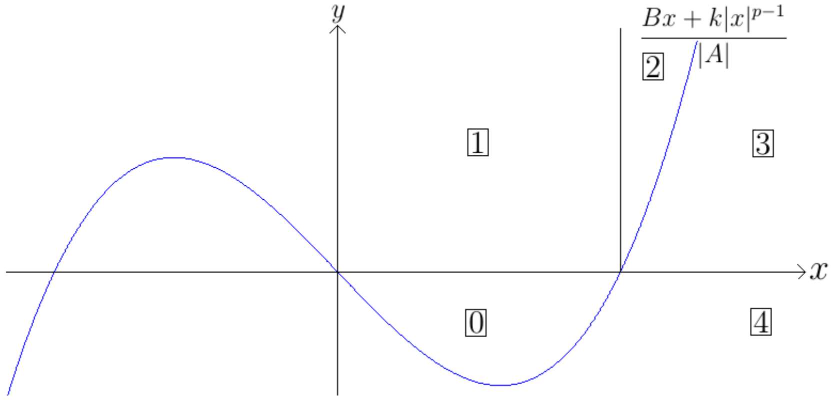


Figure 3: Phase portrait for $B < 0$, a solution will spiral out clockwise for big E

Thus $y(x) \leq y_0 + |A|x$ while the function $(Bx + k|x|^{p-1}x)/|A|$ grows with a power of $p > 1$. Hence in zone 1 (and the positive y-axis) a solution must end up in zone 2. In zone 2 we get $y' < 0$ and $x' > 0$. Moreover, $y' \leqslant |A|y_0 - Bx_0 - k|x_0|^{p-1}x_0 < 0$ for a solution not starting at (x_0, y_0) on the boundary between zones 1 and 2. If the solution starts on this boundary, it will leave it immediately since $x' > 0$ there. Hence, a solution starting in zone 2 (or the boundary to zone 1) must cross the x axis. By symmetry reasons (note that (33) is invariant under the reflection $(x, y) \mapsto (-x, -y)$) it will eventually end up in zone 0. Thus, all solutions (except for $(x(t), y(t)) \equiv (0, 0)$) spiral infinitely many times around the origin on their way to infinity.

Case 2 ($B < 0$): There are now the additional fixed points $(\pm(-B/k)^{1/(p-1)}, 0)$. The origin is a saddle point since $B < 0$ (see (38)). At the fixed point $((-B/k)^{1/(p-1)}, 0)$ we get the eigenvalues

$$\lambda = \frac{-A}{2} \pm \sqrt{\frac{A^2}{4} + B(p-1)}.$$

We get two positive real eigenvalues or two eigenvalues where $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = -A/2 > 0$. Hence the fixed point $((-B/k)^{1/(p-1)}, 0)$ is a source and for symmetry reasons, so is the fixed point $(-(-B/k)^{1/(p-1)}, 0)$. Hence a solution forward in time can only converge to the point $(0, 0)$. If it doesn't a similar argument as in case 1 shows that

$$(x(t), y(t)) \to \infty \quad \text{as } t \to \infty.$$

We now consider the different regions in Figure 3. In zone 1 $y' > 0$, $x' > 0$ and we get

$$y'(x) = \frac{y'}{x'} = -A - \frac{Bx}{y} - \frac{k|x|^{p-1}x}{y} \leqslant |A| + \frac{|B|x}{y_0}$$

and thus

$$y(x) \leqslant y_0 + |A|x + \frac{|B|x^2}{2y_0}.$$

Hence a solution in zone 1 must end up in zone 2. In zone 2 we have

$$|B| \leqslant k|x|^{p-1} \quad \text{and therefore} \quad y'(x) = \frac{y'}{x'} \leqslant |A|.$$

Thus $y(x) \leq y_0 + |A|x$ while the function $(Bx + k|x|^{p-1}x)/|A|$ grows with a power of $p > 1$ for big x. Hence in zone 2 the solution must end up in zone 3. In zone 3, $x' > 0$ and $y' < 0$, and a similar argument as in case 1 (zone 2) shows that the solution must end up in zone 4. In zone 4 the solution will cross the y-axis if it stays out of zone 0 since $y' < 0$ and $x' < 0$. Again, it will eventually end up in zone 1 by symmetry reasons. In zone 0, $x' < 0$, $y' > 0$ and the solution must end up in zone 4 or zone 1 eventually. If E is big enough the solution cannot end up in zone 0 since $E' > 0$ for $y \neq 0$ and from (37) we get that $B + k|x|^{p-1} > 0$ or $y < Bx + k|x|^{p-1}$ for $x > 0$, $y < 0$.

Note in particular that a non-trivial solution with $(x(t), y(t)) \to 0$ as $t \to -\infty$ must satisfy $E(x(t), y(t)) > 0$ for $t > 0$. Hence, it cannot converge to a fixed point as $t \to \infty$ and therefore, as in case 1, it will cross the y-axis infinitely many times as it spirals out towards infinity.

Coming back to the original problem (28), (29), we note that x defined by (30) satisfies

$$x(t) = e^{\left(\frac{2}{p-1}+m\right)t}u(e^t) \to 0$$

and

$$y(t) = x'(t) = e^{\left(\frac{2}{p-1}+m+1\right)t}u'(e^t) + \left(\frac{2}{p-1} + m\right) e^{\left(\frac{2}{p-1}+m\right)t}u(e^t) \to 0$$

as $t \to -\infty$. It follows that x has infinitely many zeros and hence so does u. $\square$

We end this section with a technical lemma which is needed in the next section. Heuristically, it says that at most one zero can appear from $+\infty$ as d crosses d_i from below. For a proof of the lemma we refer to Lemma 4.4 and the note on p. 557 in [7].

Lemma 3.3. *Denote by $w(r, d)$ the solution to* (2), (9). *Let d_i be a value for which $w(r, d_i)$ has exactly i zeros and $w(r, d_i) \to 0$ as $r \to \infty$. If $|d - d_i|$ is sufficiently small $w(r, d)$ has at most $i + 1$ zeros on $[0, \infty)$.*

4 Proof of main theorem

We are now going to prove Theorem 1.2. We define

$$A_0 = \{d > 0 \mid w(r,d) \text{ has no positive zeros}\}. \tag{39}$$

Lemma 2.4 shows that the set A_0 is nonempty while Lemmas 3.1 and 3.2 imply that the set A_0 is bounded above. Let $d_0 = \sup A_0$. We will show that

$$w(r,d_0) > 0 \quad \text{for} \ \ r > 0 \quad \text{and} \quad w(r,d_0) \to 0 \ \text{ as } \ r \to \infty. \tag{40}$$

If $w(r,d_0)$ has a zero at some finite r the continuity of $w(r,d)$ implies that $w(r,d)$ has a zero for d slightly smaller than d_0, This contradicts the definition of d_0 and we get

$$w(r,d_0) > 0 \quad \text{for all} \ \ r > 0. \tag{41}$$

We now have two possibilities

(P1) $\qquad w$ is monotone for large r,

(P2) $\qquad w(r,d_0)$ has local minima at arbitrarily large values of r.

Lemma 4.1. *(P2) implies that* $\limsup_{j\to\infty} w(M_j) \leqslant \beta$*, where* M_j *is the local minimum number* j *of* $w(r,d_0)$*,* $M_j \to \infty$*.*

Proof. At a minimum point M_j of w we have $w'(M_j) = 0$ and $w''(M_j) \geqslant 0$. Substituting into (2) gives

$$-\frac{m^2}{M_j^2} w(M_j) + f(w(M_j)) \leqslant 0. \tag{42}$$

Thus since $w(M_j) > 0$ we obtain

$$\frac{f(w(M_j))}{w(M_j)} \leqslant \frac{m^2}{M_j^2}$$

and since $\lim_{j\to\infty} M_j = \infty$ we have

$$\limsup_{j\to\infty} \frac{f(w(M_j))}{w(M_j)} \leqslant 0. \tag{43}$$

From (5) we see that

$$\frac{f(s)}{s} > \frac{k}{2} s^{p-1} \quad \text{for big enough } s.$$

Since $p > 1$, $w(M_j)$ is bounded from above. But this implies that

$$\limsup_{j\to\infty} w(M_j) \leqslant \beta$$

since $\limsup_{j\to\infty} w(M_j) \in (\beta,\infty)$ would contradict (43). $\square$

Lemma 4.2. *$w(r, d_0)$ is monotone for large values of r.*

Proof. We will show that (P2) is impossible. For sufficiently large j we have

$$w(M_j, d_0) \leqslant \frac{3}{4}\beta + \frac{1}{4}\gamma.$$

We chose d slightly larger than d_0 so that $w(r, d)$ has a zero at z with $z \to \infty$ as $d \to d_0$. By virtue of continuous dependence on initial values, for d close enough to d_0, $w(r, d)$ will also have local minima at N_J for $J = 1, ..j$ where N_j is largest value of r less than z which w has a minimum. Evaluation of Pohozaev's identity (12) between 0 and z gives

$$0 \leqslant \frac{z^2}{2}(w'(z))^2 = 2\int_0^z rF(w)\,\mathrm{d}r \tag{44}$$

Evaluating between 0 and N_j gives

$$N_j^2 F(w(N_j)) = \frac{m^2}{2}(w(N_j))^2 + 2\int_0^{N_j} rF(w)\,\mathrm{d}r. \tag{45}$$

We may furthermore choose d sufficiently close to d_0 so that

$$w(N_j, d) \leqslant \frac{1}{2}\beta + \frac{1}{2}\gamma < \gamma. \tag{46}$$

Thus

$$F(w(N_j)) < 0. \tag{47}$$

From (45) we see that

$$2\int_0^{N_j} rF(w)\,\mathrm{d}r \leqslant 0. \tag{48}$$

Combining (44), (48) gives

$$2\int_{N_j}^{z} rF(w)\,\mathrm{d}r \geqslant 0.$$

In view of (47) and the last inequality, there exist

$$N_j < c < z \quad \text{such that} \;\; w(c) = \gamma. \tag{49}$$

Further, c may be chosen so that $w' \geqslant 0$ on $[N_j, c]$ because N_j is the largest value of r at which w has a minimum before the zero, z, of w. Pohozaev's identity evaluated between N_j and c gives

$$0 \leqslant \frac{1}{2}c^2(w'(c))^2 = N_j^2 F(w(N_j)) + \frac{m^2}{2}(\gamma^2 - w(N_j)^2) + 2\int_{N_j}^{c} rF(w)\,\mathrm{d}r$$

$$\leqslant \frac{m^2}{2}(\gamma^2 - w(N_j)^2) + 2\int_{N_j}^{c} rF(w)\,\mathrm{d}r. \tag{50}$$

Let a, b be such that $N_j < a < b < c$ and $w(a) = \frac{1}{4}\beta + \frac{3}{4}\gamma$. Then since $w' \geqslant 0$, $F(w) \leqslant 0$ on $[a, c]$ and $f(w) = F'(w) \geqslant 0$ on $[(\beta+\gamma)/2, (\beta+3\gamma)/4]$ we obtain

$$2\int_{N_j}^{c} rF(w)\,\mathrm{d}r \leqslant 2\int_{a}^{b} rF(w)\,\mathrm{d}r \leqslant F\left(\frac{\beta+3\gamma}{4}\right)(b^2 - a^2).$$

Thus from (50) we see that

$$0 \leqslant \frac{m^2}{2}(\gamma^2 - w(N_j)^2) + F\left(\frac{\beta+3\gamma}{4}\right)(b^2 - a^2). \tag{51}$$

We will next show that $b - a \geqslant \epsilon > 0$ where ϵ is independent of d. We note that for $N_j \leqslant r \leqslant c$ we have

$$\int_0^r 2sF(w)\,\mathrm{d}s = \int_0^{N_j} 2sF(w)\,\mathrm{d}s + \int_{N_j}^{r} 2sF(w)\,\mathrm{d}s$$

From (48) we see that the first integral on the right is negative. The second integral to the right is non-positive because $0 < w \leqslant \gamma$ on $[N_j, c]$ and thus $F(w) \leqslant 0$. Pohozaev's identity between 0 and r gives

$$\frac{1}{2}(w')^2 + F(w) \leqslant \frac{m^2}{2}\frac{w^2}{r^2} \quad \text{for} \;\; N_j \leqslant r \leqslant c.$$

Choosing d close enough to d_0 we can always ensure that $N_j \geqslant 1$ and we get

$$\frac{1}{2}(w')^2 + F(w) \leqslant C \equiv \frac{m^2}{2}\gamma^2 \quad \text{for} \;\; N_j \leqslant r \leqslant c.$$

Since $w' \geqslant 0$ on $[N_j, c]$ we obtain

$$\frac{w'}{\sqrt{C - F(w)}} \leqslant \sqrt{2} \quad \text{for} \;\; N_j \leqslant r \leqslant c.$$

Now since $[a, b] \subset [N_j, c]$ we can integrate on $[a, b]$ and get

$$\int_{(\beta+\gamma)/2}^{(\beta+3\gamma)/4} \frac{\mathrm{d}s}{\sqrt{C - F(s)}} = \int_a^b \frac{w'}{\sqrt{C - F(w)}}\,\mathrm{d}r \leqslant \int_a^b \sqrt{2}\,\mathrm{d}r = \sqrt{2}(b - a). \tag{52}$$

Thus $b - a \geqslant \epsilon$ as claimed.

Now since $b^2 - a^2 = (b - a)(b + a)$ and $b + a > 2N_j \to \infty$ as $d \to d_0$ we obtain that $b^2 - a^2 \to \infty$ as $d \to d_0$. Therefore the right hand side of (51) approaches $-\infty$ as $d \to d_0$ which is a contradiction and hence (P2) must be impossible. In other words, $w(r, d_0)$ must be monotone for large values of r. $\square$

The monotonicity of $w(r,d_0)$ for large r and implies that $w(r,d_0)$ must be bounded since

$$w'' < \frac{m^2 w}{r^2} - \frac{k}{2} w^p < -c, \quad c > 0, \quad \text{for big } w \text{ and } r$$

which contradicts $w' > 0$ for big w. Thus we must have

$$\lim_{r\to\infty} w(r,d_0) = \zeta, \quad \zeta \in [0,\infty). \tag{53}$$

Taking limits in

$$\frac{1}{2}(w')^2 + F(w) = \frac{m^2}{2}\frac{w^2}{r^2} + \frac{2}{r^2}\int_0^r sF(w)\,\mathrm{d}s,$$

we obtain

$$\lim_{r\to\infty} \frac{1}{2}(w')^2 + F(\zeta) = \lim_{r\to\infty} \frac{2}{r^2}\int_0^r sF(w)\,\mathrm{d}s.$$

Using l'Hôpital's rule we get

$$\lim_{r\to\infty} \frac{2\int_0^r sF(w(s))\,\mathrm{d}s}{r^2} = \lim_{r\to\infty} \frac{2rF(w(r))}{2r} = F(\zeta).$$

Thus

$$\lim_{r\to\infty} w'(r,d_0) = 0. \tag{54}$$

It follows from (2) that

$$\lim_{r\to\infty} w''(r,d_0) = -f(\zeta).$$

Thus we must have

$$f(\zeta) = 0.$$

Lemma 4.3. $\lim_{r\to\infty} w(r,d_0) = 0$

Proof. Suppose $\zeta > 0$. Then since $w(r,d) \to w(r,d_0)$ uniformly on a compact sets. given $\epsilon > 0$ we can choose a large value q and $d > d_0$ sufficiently close to d_0 such that

$$\zeta - \epsilon \leqslant w(q,d) \leqslant \zeta + \epsilon.$$

We know $w(r,d)$ has a zero $z > q$. Evaluating Pohozaev's identity between q and z gives.

$$0 \leqslant \frac{1}{2}z^2(w'(z))^2 = q^2\left[\frac{1}{2}(w'(q))^2 - \frac{m^2}{2}\frac{w(q)^2}{q^2} + F(w(q))\right] + 2\int_q^z rF(w)\,\mathrm{d}r.$$

For $d = d_0$ the expression in the bracket is negative if q is big enough by (54) and the fact that $F(\zeta) < 0$. For large q and $d > d_0$ sufficiently close to d_0 it will be negative by continuous dependence of $w(q,d)$ and $w'(q,d)$ on d. Thus we must have

$$\int_q^z rF(w)\,\mathrm{d}r \geqslant 0.$$

Thus there is an s with $q < s < z$ such that $w(s) = \gamma$ and $w(r) < \gamma$ on (q,s). We note that $F(w) \leqslant 0$ for $r \in [q,s]$. Evaluating Pohozaev's identity between q and s.

$$0 \leqslant \frac{1}{2}s^2(w'(s))^2 \leqslant q^2\left[\frac{1}{2}(w'(q))^2 + \frac{m^2}{2}\frac{\gamma^2}{q^2} - \frac{m^2}{2}\frac{w(q)^2}{q^2} + F(w(q))\right].$$

As above we can make the expression in the brackets negative by choosing q large enough and d sufficiently close to d_0 which is a contradiction.

$$\zeta = \lim_{r\to\infty} w(r,d_0) = 0.$$

Thus there exists a positive solution of (2),(9) with $w(r) \to 0$ as $r \to \infty$. □

Next we define

$$A_1 = \{d > 0 \mid w(r,d) \text{ has at most one positive zero}\}.$$

It follows from the definition of d_0 and Lemma 3.3 that A_1 is nonempty. From Lemma 3.2 it follows that A is bounded above. We define

$$d_1 = \sup A_1.$$

As above we can show that $w(r,d_1)$ has exactly one zero and $w(r,d_1) \to 0$ as $r \to \infty$. Proceeding inductively we can show that there exist a solution that tend to zero at infinity for any prescribed number of zeros which is theorem 1.2 we wanted to prove.

5 Numerical solutions

In this final section we present some numerical computations for the particular case $f(s) = s^3 - s$ for real s which satisfies (3),(4),(5) with $\sigma = -1$, $p = 3$, $k = 1$, $g(s) = s$, $F(s) = \frac{s^4}{4} - \frac{s^2}{2}$ and $\gamma = \sqrt{2}$. The parameter m ranges between 1 and 4.

The simple shooting method [16, Chapter 7.3] was used to find numerical approximations of v. The method was implemented in Python using the `odeint` function from the `SciPy` library. The interval $(0,\infty)$ was replaced by the bounded interval $(0,R)$ for some large $R > 0$ and the condition $\lim_{r\to\infty} v(r) = 0$ was replaced by $v(R) = 0$. R was set to 50 in most cases but had to be increased when the number of roots was large. In order to avoid the singularity at $r = 0$, the initial condition was posed at $D/2$, where D is the step size. The bisection method was used to find values of d such that $v(R) = 0$.

The simple shooting method tends to be unstable when used over long intervals. In order to produce a more accurate approximation for large r, finite-difference discretization and the Newton method were used. The solution from the shooting method was used as an initial guess after modifying it to make it zero for large r. In the finite-difference discretization we set up a grid with N gridpoints between 0 and R:

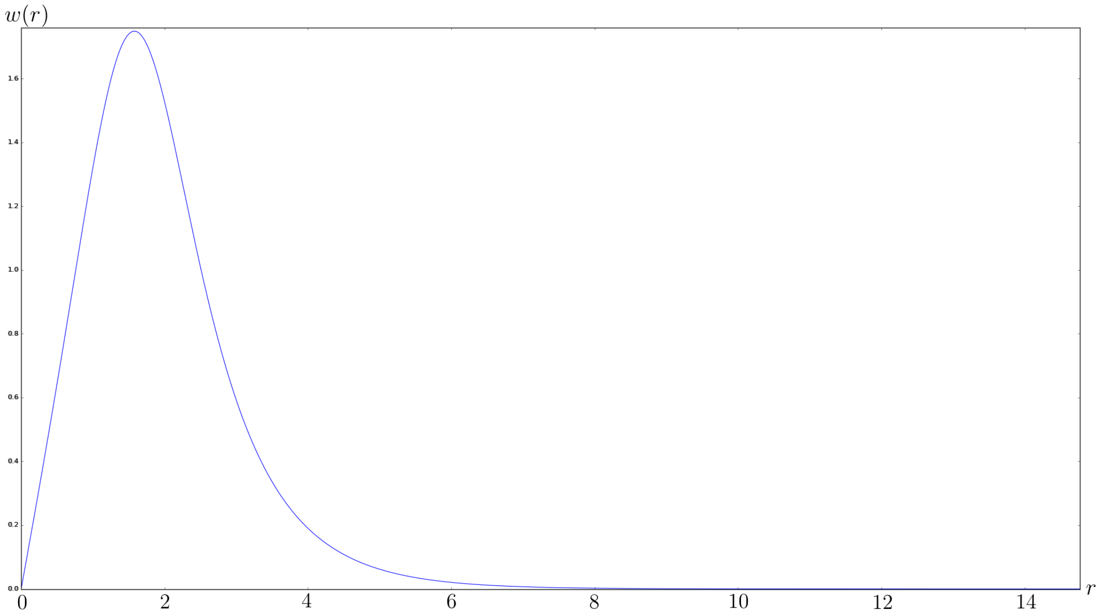
$$r_0 = \frac{-D}{2}, \quad r_1 = \frac{D}{2}, \quad r_2 = \frac{3D}{2}, \quad \ldots, \quad r_N = R - D, \quad r_{N+1} = R,$$

where $D = R/(N + \frac{1}{2})$. Since a solution $v(r)$ must satisfy $v'(0) = 0$ and $v(r) \to 0$ as $r \to \infty$, we imposed the boundary conditions $v(r_1) = v(r_0)$ and $v(r_{N+1}) = 0$. The central difference was taken to get a numerical approximation for the first and second derivatives:

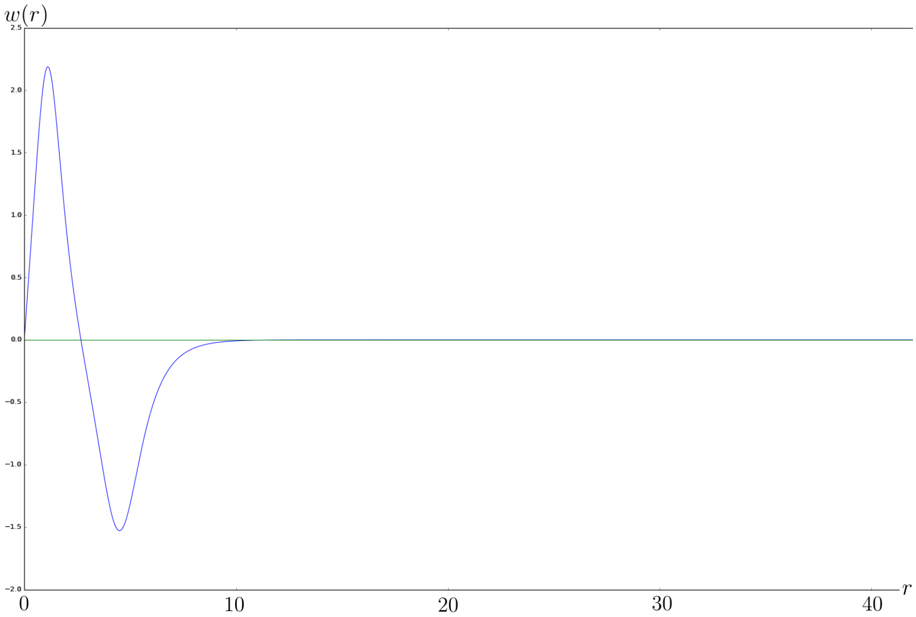
$$v'(r_{n+1}) \approx \frac{v(r_{n+2}) - v(r_n)}{2D}, \quad v''(r_{n+1}) \approx \frac{v(r_{n+2}) - 2v(r_{n+1}) + v(r_n)}{D^2}.$$

After computing v, we plotted $w(r) = r^m v(r)$ (see the figures below). The results became less accurate for larger r or m. For large r the numerical solution w oscillated around ± 1 for $m = 1$ and the solution could even cross the x-axis after oscillating around ± 1 for a long while, resulting in the shooting method not giving the correct value for d. This started to be a common problem for $R > 1000$. We also plotted the d-values against the number of roots. Surprisingly, it seems like the d-values increase linearly with the number of roots for $m = 1$ (see figure 4). This is something which would be interesting to analyze further. For the other values of m, the relationship between d and the roots was not that clear (see figures 5 to 7).

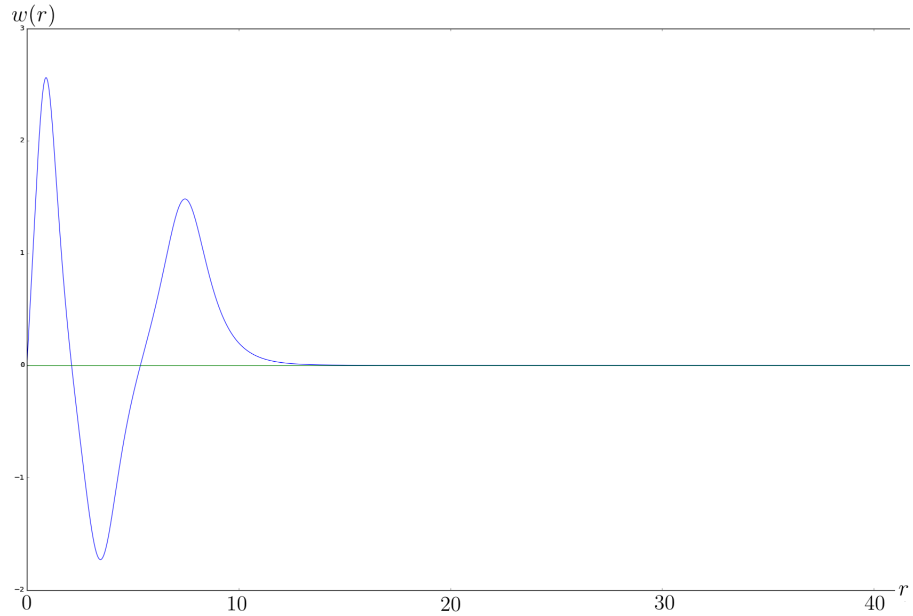
5.1 m=1



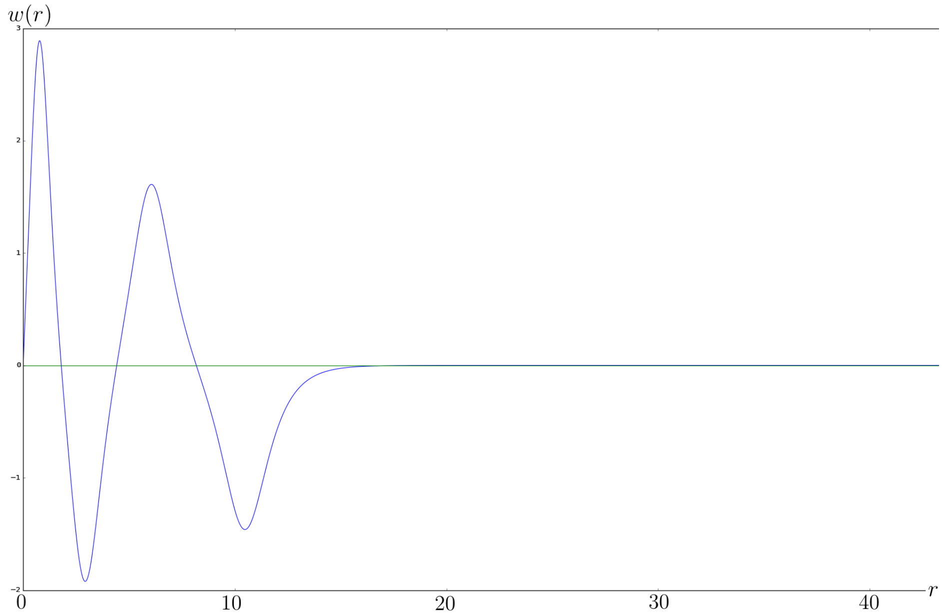
0 roots, $d = 1.25232237967$



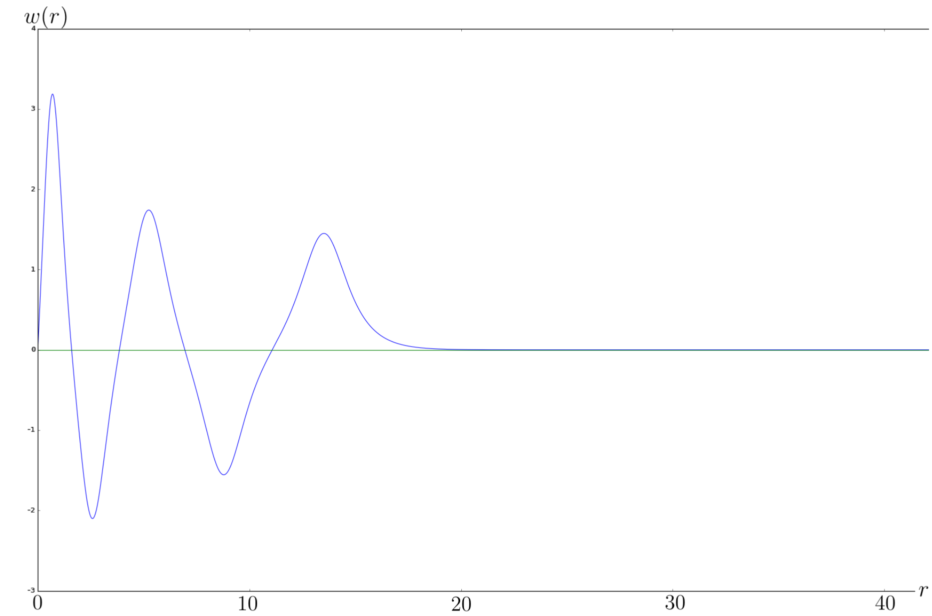
1 root, $d = 2.41551202341$



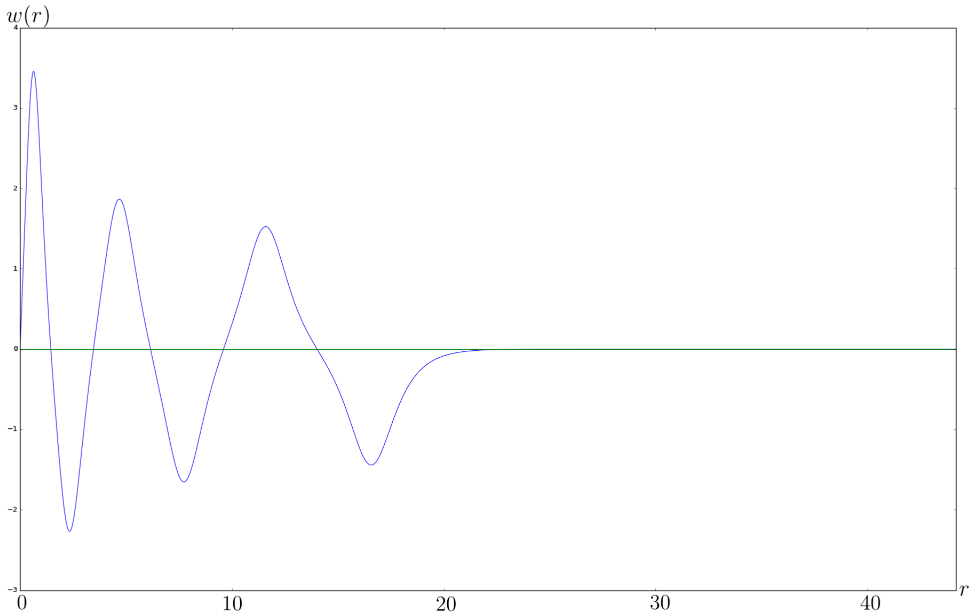
2 roots, $d = 3.57167319666$



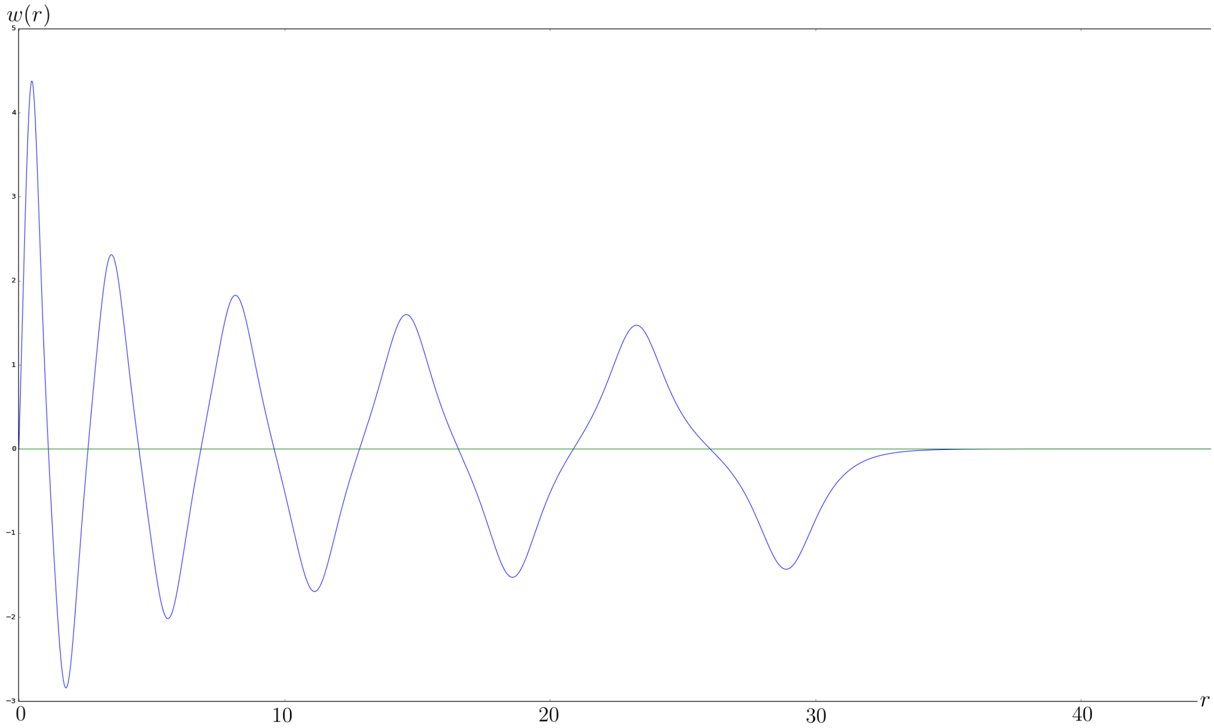
3 roots, $d = 4.7260529563$



4 roots, $d = 5.87973020583$



5 roots, $d = 7.03306564512$



9 roots, $d = 11.6449279631$

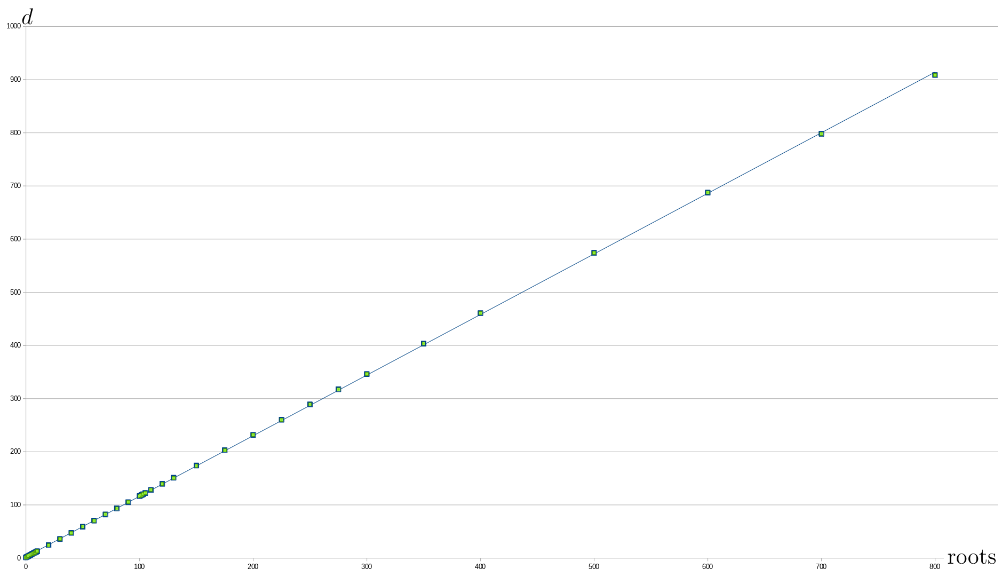
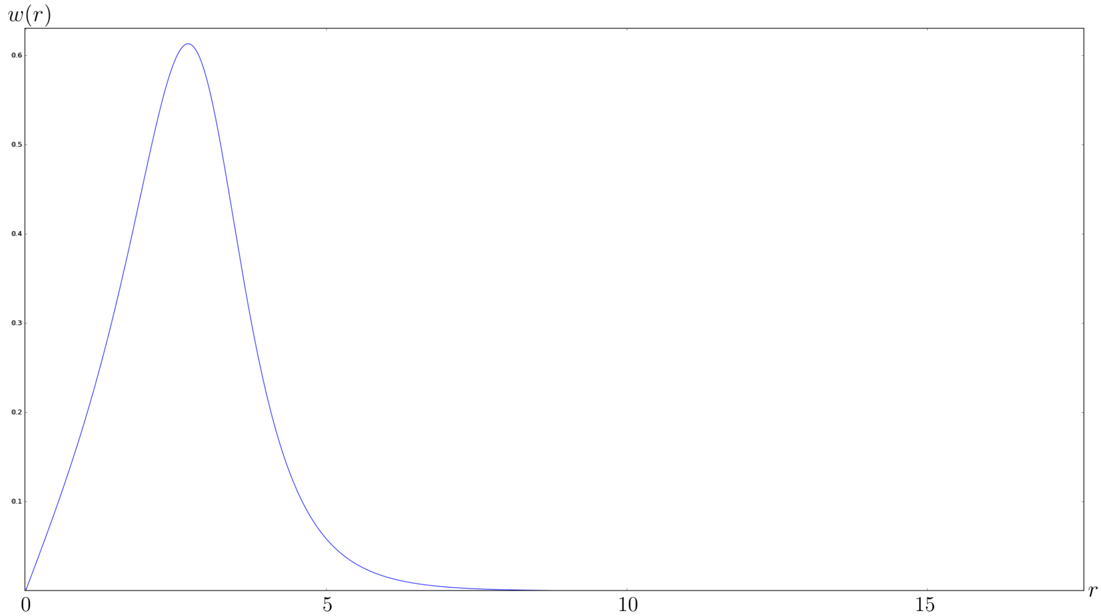
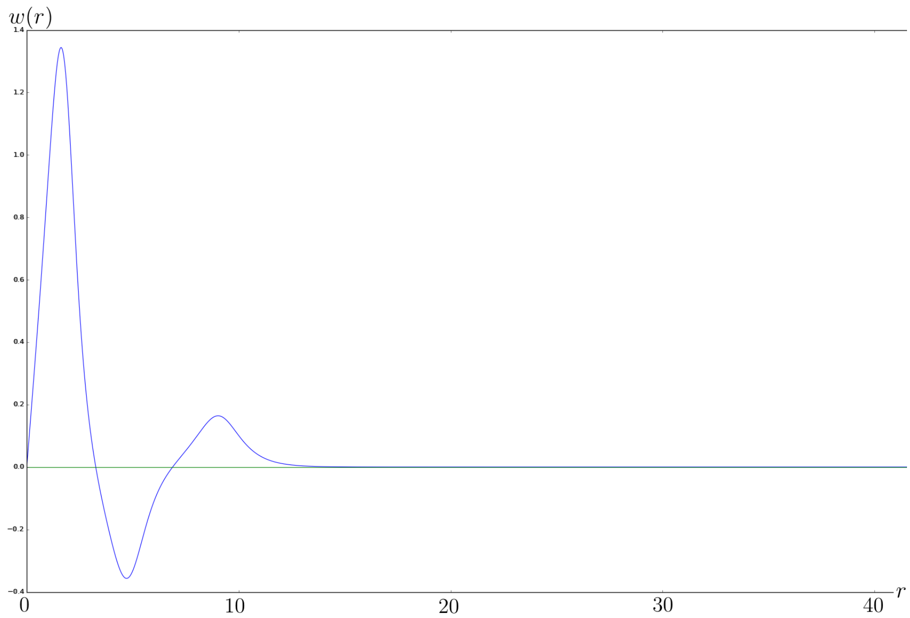


figure 4: d increased linearly with the number of roots in the interval where the program gave reliable results.

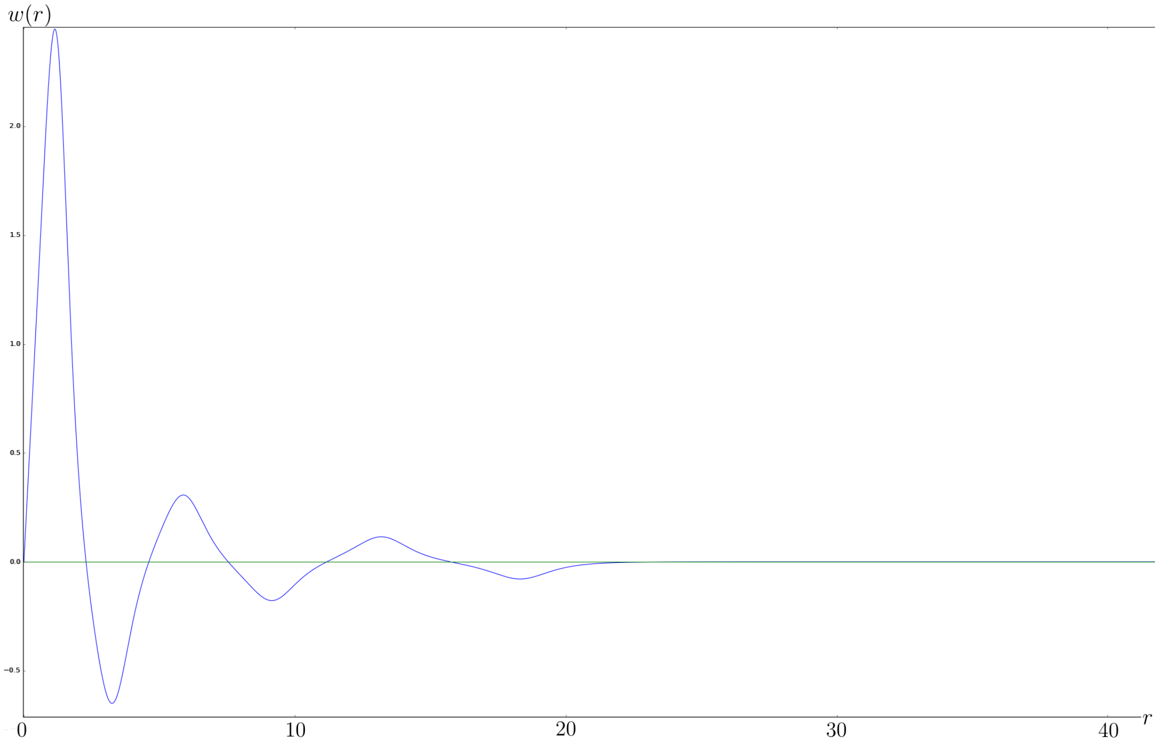
5.2 m=2



0 roots, $d = 0.178781983074$



2 roots, $d = 0.858802996298$



5 roots, $d = 2.38024836036$

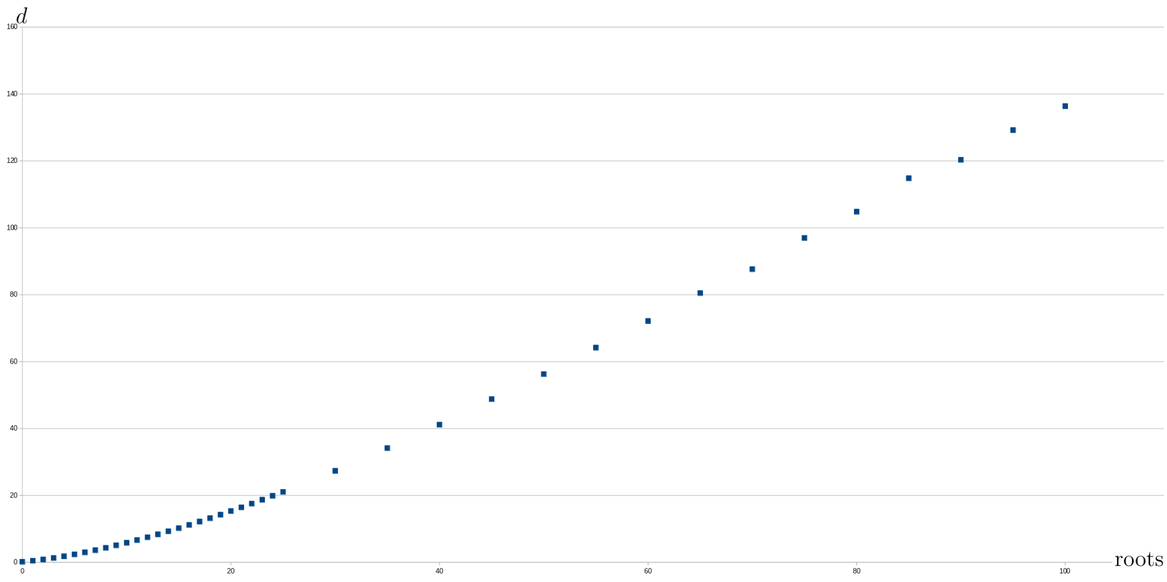
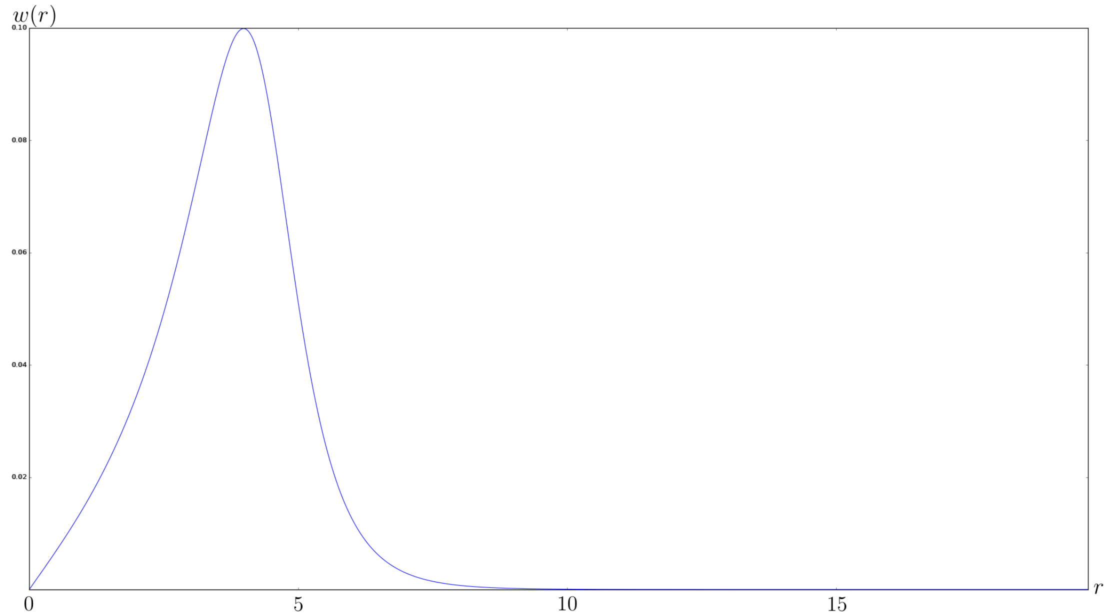
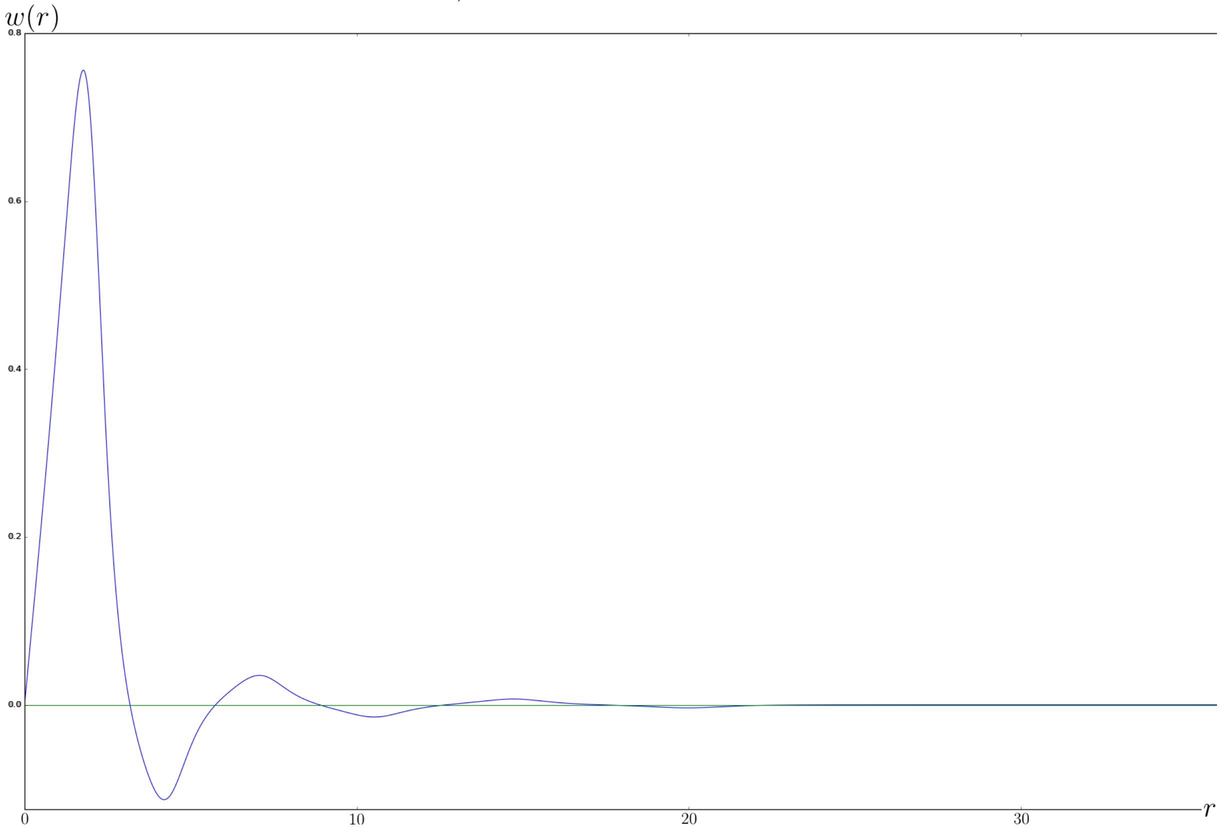


Figure 5

5.3 m=3



0 roots, $d = 0.0135786300215$



5 roots, $d = 0.420248374519$

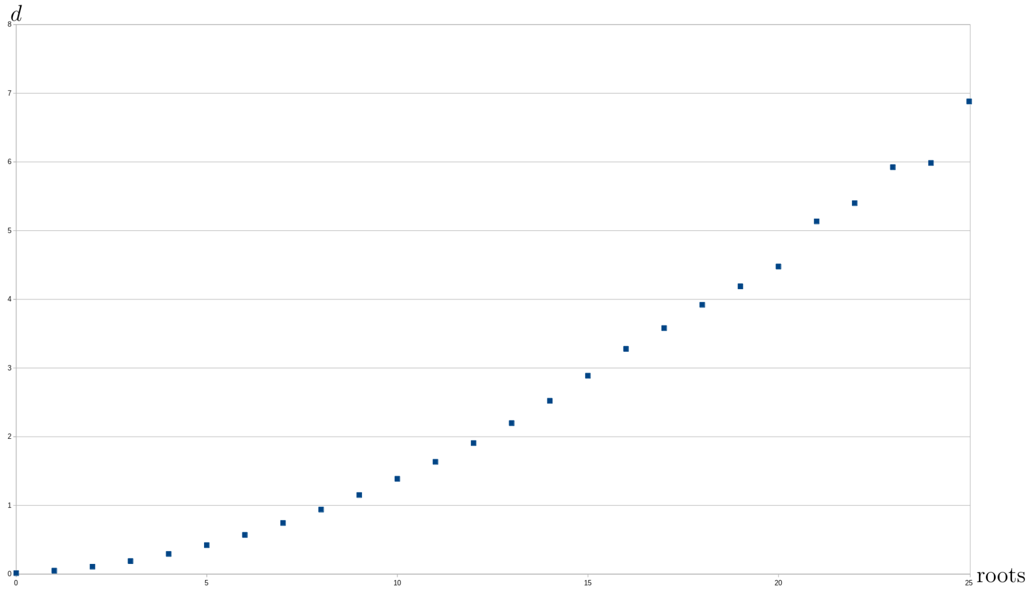


Figure 6

5.4 m=4

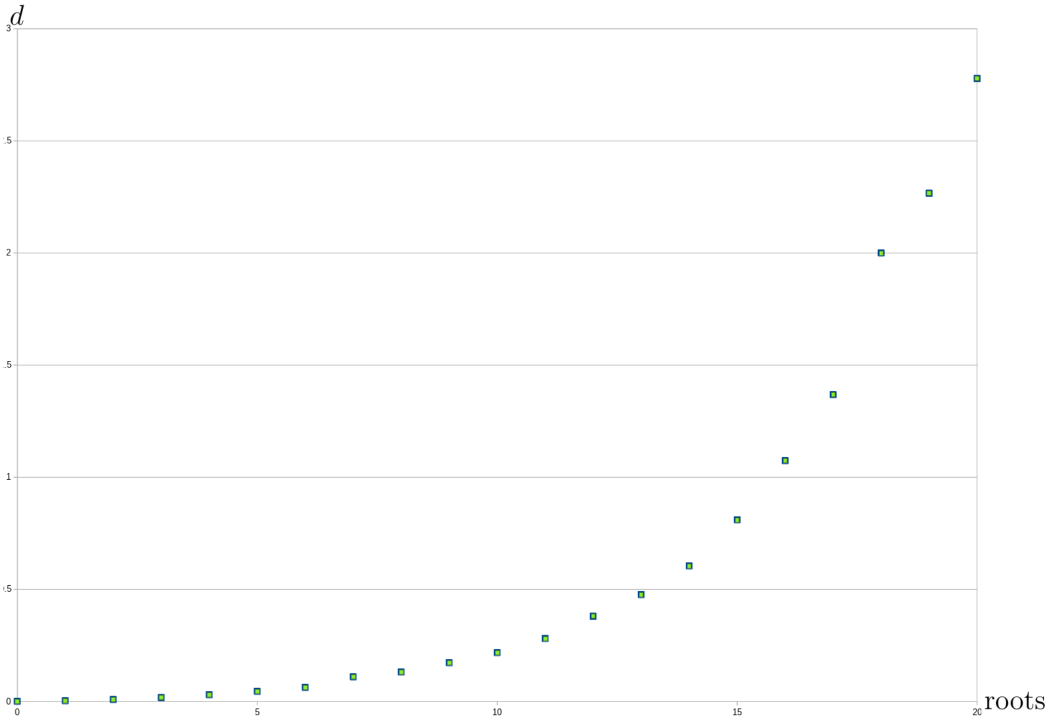


Figure 7

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