An introduction to Krein strings

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Abstract

Krein strings appear in the study of the motion of a vibrating string where an irregular density is allowed. This thesis presents the theory from the perspective of integral equations and operator theory. It will be shown that each Krein string gives rise to a unique Stieltjes function, by utilizing the compactness of the resolvent operators for short strings and then approximating any long string with a sequence of short strings. The converse is also true: each Stieltjes function gives rise to a unique Krein string and this bijection is called Krein's correspondence. The existence part is proved by constructing Krein strings for a special class of Stieltjes functions. Then, an arbitrary Stieltjes function can be approximated by this class and the limiting procedure yields a string corresponding to this Stieltjes function. The uniqueness part is not treated in this thesis. Instead, some properties and simple examples of Krein's correspondence will be presented.

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Populärvetenskaplig sammanfattning

I den klassiska modellen för en endimensionell vibrerande sträng antas massan vara likformigt fördelad, vilket leder till den ordinära differentialekvationen $f'' = \lambda \varrho f$, där densiteten ϱ är konstant. Kreins strängteori handlar om samma ekvation, men massfördelningen tillåts variera. Denna teori används även för att lösa problemet att förutsäga framtiden med hjälp av information från en ändlig del $-2T \leqslant t \leqslant 0$ av dåtiden för endimensionella stokastiska normalprocesser med väntevärde 0. Detta arbete ger en behandling av Kreins strängteori, med fokus på Kreins korrespondens – problemet där spektraldata är givna i form av en så kallad Stieltjesfunktion och vi vill veta så mycket som möjligt om strängen som funktionen kommer från.

Contents

1 Introduction

A general model for the motion of a vibrating elastic string in one dimension is the linear partial differential equation

$$\varrho(x)u_{tt} = \big(T(x,t)u_x\big)_x + F(x,t), \qquad x \in [0,l),\ t > 0,$$

where the string starts at the coordinate $x = 0$, ends at the coordinate $x = l \leqslant \infty$, and

(1) $u(x,t) : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ denotes the vertical displacement of the string at time t,

(2) $\varrho(x)$ denotes the density of the string at position x,

(3) $T(x,t)$ denotes the tension force on the string at position x and time t, and

(4) $F(x,t)$ is any external force acting on the string at position x and time t.

If the mass of the string is uniformly distributed, the tension force is constant and there are no external forces, we obtain the classical wave equation $u_{tt} = c^2 u_{xx}$, which has been thoroughly studied. In this thesis, we consider the case where the mass might be irregularly distributed, while we set $T(x,t) \equiv 1$ and $F(x,t) \equiv 0$:

$$\varrho(x)u_{tt} = u_{xx}, \qquad x \in [0,l),\ t > 0.$$

The method of separation of variables is used to obtain the eigenequation

$$f''(x) = z\varrho(x)f(x), \qquad z \in \mathbb{C}$$

We denote the mass of the string up to and including the point x as $M(x)$. If $\varrho(x)$ is locally integrable throughout the string, we have

$$M(x) := \int_{[0,x]} \varrho(y)\, dy, \qquad x \in [0,l)$$

and the eigenequation in this case can be rewritten as

$$\frac{d}{dM} f'_+ = zf, \tag{1.1}$$

which motivates the formal definition of the operator

$$\tau = \frac{d}{dM}\frac{d_+}{dx}, \tag{1.2}$$

acting on an appropriate space. In this thesis, ϱ is allowed to be a non-negative Borel measure, which means that we for example allow the density to contain point masses.

Krein string theory, developed in the early 1950's by M. G. Krein, deals with the study of this operator and the corresponding Weyl function $m(z)$, which contains all spectral information of τ. Moreover, it is a Herglotz-Nevanlinna function and $m(-z)$ is a Stieltjes function. Conversely, given a Stieltjes function $h(z)$, a unique string can be found with $h(-z)$ as its Weyl function. This means that the inverse spectral problem for Krein strings is solvable and the solution is unique. The bijection between the class of Stieltjes functions and the class of Krein strings is called Krein's correspondence, which is the most intriguing feature of Krein string theory. If one considers a larger class of strings where ϱ is allowed to be a signed Borel measure, its Weyl function is still a Herglotz-Nevanlinna function, but not all Herglotz-Nevanlinna functions correspond to such a string. A characterization of the Herglotz-Nevanlinna functions corresponding to strings with a sign-changing density is presently unknown. See [15] and [5] for more details.

Krein string theory and its generalizations have attracted a lot of attention also due to its vast applications in many different areas. For instance, it is used to study the prediction of the future from a finite past segment $-2T \leqslant t \leqslant 0$ of a real one-dimensional Gaussian process with mean 0. Since the projection of the family $\{e^{i\gamma(t+T)}\}$ onto the span of $\{e^{i\gamma t}\}_{t \leqslant T}$ is not invariant under the shift $f \mapsto e^{i\gamma T} f$, the theory of Hardy spaces is not applicable, see [4, ix] and [4, pp.146–147]. Another application is within the study of generalized diffusions, see for example Appendix II in [13], or Chapter 15 in [16].

In 2012, a group of mathematicians and physicists published results of a real-life experiment of a special case of Krein's correspondence, in which some weights were attached to a thread with negligible mass at different positions. The eigenvalues were computed and compared to the model for these types of strings made by Krein, using Stieltjes continued fractions. The interested readers are referred to [3].

The focus of this thesis is the following:

(1) to give a treatment of Krein string theory,
(2) to study how the behavior of the mass distribution affects the behavior of its Stieltjes function $m(-z)$.

A common approach in Krein string theory is to use integral equations as in [10], [4] and [13]. In the first part of the thesis, we will combine this approach with an operator theoretical one as in [6], the focus of which is the generalized Sturm-Liouville equation with measure-valued coefficients

$$\frac{d}{d\varrho(x)}\Big(-\frac{d}{d\varsigma(x)} y(x) + \int^x y(t)\, d\chi(t)\Big) = zy(x), \quad -\infty \leqslant a < x < b \leqslant \infty,$$

where the measures ϱ, ς and χ are required to meet some hypotheses as in [6, p.11]. Eigenequation (1.1) is obtained for $\chi \equiv 0$ and ς the restricted Lebesgue measure on $(-\infty, l)$ or $\mathbb{R}$. The first approach provides explicit and insightful constructions. For example, the construction of a string in Section 3.2.2 is very easy to follow and understand. The second approach provides elegant arguments, for example the limit circle and limit point characterization of the operator τ. Alternatively, one could also convert (1.1) into a first-order system of differential equations with measure-valued coefficients, the theory of which is thoroughly investigated by C. Bennewitz in [2].

In the second part, starting from some simple Krein strings, we will try to either compute the Weyl function and the spectrum, or observe as much as possible from the even and odd transforms between the relevant spaces. The even and odd transforms for the classical string $M(x) = x \cdot \mathbb{1}_{x \geqslant 0}$, or equivalently for the operator d^2/dx^2 on an appropriate domain, are in fact

$$\int_0^\infty \cos(\xi x) f(x)\, dx, \quad \text{and} \quad \int_0^\infty \sin(\xi x) f(x)\, dx$$

respectively. This means that we can view the even and odd transforms as a generalization of the cosine and sine transforms, and these prove to be very efficient tools to study Krein's correspondence. We will also mention some famous results about Krein's correspondence. For example, the leading part in the eigenvalue asymptotics for a short string depends solely on the absolutely continuous part $M'(x)$ of the mass distribution

$$\lim_{n\to\infty} \frac{n}{\sqrt{r_n}} = \frac{1}{\pi} \int_0^l \sqrt{M'(x)}\, dx.$$

This is a discovery of M. G. Krein [14]. If $\varrho(x)$ is absolutely continuous, one can obtain sharper asymptotics, as illustrated in Example 3.36, where a point mass is added to a smooth density.

Here is an overview of the structure of this thesis. Chapter 2 is dedicated to the formal settings of the unbounded operator τ, the domains on which τ is a non-positive self-adjoint operator, and the resolvent operator. One has to be careful in choosing a domain for τ, because it is multi-valued on its maximal domain. Also, some choices of domains on which τ is a self-adjoint operator might cause the spectrum to contain positive points. Chapter 3 is about Krein's correspondence. In Section 3.1.1 and 3.1.2, we show how a string yields a Stieltjes function. Then, in Section 3.2.2 and 3.2.3, we first show how a string can be constructed from Stieltjes functions with some additional assumptions and then via limiting process, we show that a string can be found for an arbitrary Stieltjes function. In Section 3.3, we present known results about Krein's correspondence, and demonstrate the correspondence for Stieltjes strings, strings with $M(x) = x^\alpha \cdot \mathbb{1}_{[0,l)}$ with $\alpha > 0$ and $l \leqslant \infty$, and a combination of these two types of strings.

The prerequisites for this thesis are linear functional analysis, integration theory and some knowledge in differential equations. In particular, Section 2.4 and the beginning of Section 3.3.1 will require some acquaintance with spectral theory for unbounded self-adjoint operators. Readers who are unfamiliar with this can skip the beginning of Section 2.4 and go directly to the discussion of boundary conditions on pages 13 and 14.

2 Krein strings

2.1 Notation

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Frequently, we use the following notations:

$$\begin{aligned} f(a-) &= \lim_{x\to a,\, x<a} f(x) = \lim_{x\uparrow a} f(x), \\ f(a+) &= \lim_{x\to a,\, x>a} f(x) = \lim_{x\downarrow a} f(x), \\ \frac{d_- f}{dx}(a) &= f'_-(a) = \lim_{x\uparrow a} \frac{f(x)-f(a)}{x-a}, \\ \frac{d_+ f}{dx}(a) &= f'_+(a) = \lim_{x\downarrow a} \frac{f(x)-f(a)}{x-a}. \end{aligned}$$

Also, we will use these conventions:

$$f'_-(a+) = f'_+(a), \quad \text{and} \quad f'_+(a-) = f'_-(a).$$

Let σ be a locally finite Borel measure. The notations for the integral bounds are to be understood in the following way, if a and b are finite numbers:

$$\int_a^b f(y)\, d\sigma(y) = \begin{cases} \int_{(a,b]} f(y)\, d\sigma(y), & \text{if } a \leqslant b \\ -\int_{(b,a]} f(y)\, d\sigma(y), & \text{if } b < a \end{cases}$$

$$\int_{a-}^b f(y)\, d\sigma(y) = \begin{cases} \int_{[a,b]} f(y)\, d\sigma(y), & \text{if } a \leqslant b \\ -\int_{(b,a)} f(y)\, d\sigma(y), & \text{if } b < a \end{cases}$$

$$\int_a^{b-} f(y)\, d\sigma(y) = \begin{cases} \int_{(a,b)} f(y)\, d\sigma(y), & \text{if } a \leqslant b \\ -\int_{[b,a]} f(y)\, d\sigma(y), & \text{if } b < a. \end{cases}$$

Let $(a,b) \subset \mathbb{R}$ be an interval. A function $g : (a,b) \to \mathbb{R}$ is said to be of class $AC_{loc}((a,b), d\sigma)$ if g is right-continuous and it can be written as

$$g(x) = g(c) + \int_c^x h(y)\, d\sigma(y), \qquad x \in (a,b),$$

where h is required to be of class $\mathbf{L}^1_{loc}((a,b), d\sigma)$. The function h will be referred to as a quasi-derivative of g in $AC_{loc}((a,b), d\sigma)$.

2.2 Definition of a string and the initial-value problem

In this thesis, a string is defined by its two physical properties: its length l and its mass function $M(x)$. Hence, we may refer to a string as the pair (l, M). The left endpoint of a string is always at the coordinate $x = 0$ and the right endpoint is at $x = l$, where $l \in (0, \infty]$. The mass function $M(x) : \mathbb{R} \to [0, \infty]$ gives the mass of the string up to and including the point x. $M(x)$ is required to be non-negative, non-decreasing and right-continuous. Also, $M(x) \equiv 0$ for $x < 0$ and M is bounded on every interval of the form $[0, a)$ for $a < l$. We denote the measure associated to $M(x)$ by ϱ, obeying $\varrho((-\infty, x]) = M(x)$. It is easy to see that ϱ is a non-negative Borel measure, locally finite on $(-\infty, l)$. A point x is a growth point of M if for every a and b, such that $a < x < b$, we have $M(a) < M(b)$. We also require that $x = 0$ is a growth point, and that $x = l$ is the supremum of all growth points. This implies that $M(x)$ is constant after $x = l$. In the case $l + M(l-) < \infty$, this includes the possibility that ϱ has a point mass at l, which we require to be finite.

We set
$$\mathbb{I} = \begin{cases} (-\infty, l), & \text{if } l + M(l-) = \infty \\ \mathbb{R}, & \text{if } l + M(l-) < \infty. \end{cases}$$

If $l + M(l-) = \infty$, the string (l, M) is called a long string. Otherwise, it will be referred to as a short string. The operator of concern has the form
$$\tau = \frac{d}{dM}\frac{d_+}{dx},$$
and τ acts on functions in $AC_{loc}(\mathbb{I}, dx)$, with a quasi-derivative in $AC_{loc}(\mathbb{I}, dM)$. Since functions in $AC_{loc}(\mathbb{I}, dM)$ are right-continuous, the quasi-derivative of $f \in AC_{loc}(\mathbb{I}, dx)$ must be the right-derivative f'_+. We define τf to be the quasi-derivative of $f'_+ \in AC_{loc}(\mathbb{I}, dM)$. Given $h \in \mathbf{L}^1_{loc}(\mathbb{I}, dM)$ with $h = \tau f$, the function f can be recovered from h using the formula
$$f(x) = f(0) + f'_-(0)x + \int_0^x d\xi \int_{0-}^{\xi} h(\eta)\, dM(\eta), \qquad x \in [0, l). \tag{2.1}$$

Note that τf is well-defined ϱ-almost everywhere on $\mathbb{I}$ and not well-defined on massless intervals – the intervals on which $M(x)$ is constant. On massless intervals of $\mathbb{I}$, we have the convention that $f(x)$ is linear. In particular, in the long-string case, $f(x)$ to the left of $x = 0$ is
$$f(x) = f(0) + f'_-(0)x, \qquad x \in (-\infty, 0), \tag{2.2}$$
and in the short-string case, we extend $f(x)$ on both $(-\infty, 0)$ and (l, ∞) as
$$f(x) = \begin{cases} f(0) + f'_-(0)x, & x \in (-\infty, 0) \\ f(l) + f'_+(l)(x - l), & x \in (l, \infty). \end{cases} \tag{2.3}$$

$\mathbf{D}_\tau$ will denote the space of functions f given by (2.1) and (2.2) in the long-string case, or by (2.1) and (2.3) in the short-string case. Computing f'_- and f'_+ from (2.1), we have that:
$$\begin{aligned} \frac{f'_+(x) - f'_-(x)}{\varrho(\{x\})} &= \tau f(x), \quad \text{if } \varrho(\{x\}) > 0 \\ f'_+(x) - f'_-(x) &= 0, \qquad\quad \text{if } \varrho(\{x\}) = 0. \end{aligned} \tag{2.4}$$

Equation (2.4) provides an alternative way to compute τf when $\varrho(\{x\}) > 0$.

Consider the initial-value problem
$$(\tau - z)f = g, \quad \text{with} \quad f(c) = d_1 \text{ and } f'_-(c) = d_2, \tag{2.5}$$
where $c \in \mathbb{I}$, $g \in \mathbf{L}^1_{loc}(\mathbb{I}, dM)$, as well as $z, d_1, d_2 \in \mathbb{C}$ are given. The problem (2.5) can be converted into a first-order system of differential equations with measure-valued coefficients. We set $F(x) = (f(x), f'_+(x))$ and observe that
$$f(x) = d_1 + \int_{c-}^{x} f'_+(y)\, dy, \qquad f'_+(x) = d_2 + \int_{c-}^{x} (zf + g)\, dM, \qquad x \in \mathbb{I}.$$

This yields the system

$$F(x) = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} + \int_{c-}^{x} \begin{pmatrix} 0 \\ g \end{pmatrix} dM + \int_{c-}^{x} CF\, d\omega, \tag{2.6}$$

where

$$d\omega = dx + |z| dM, \tag{2.7}$$

and

$$C = \begin{pmatrix} 0 & \frac{dx}{d\omega} \\ z\frac{dM}{d\omega} & 0 \end{pmatrix}. \tag{2.8}$$

The entries $dx/d\omega$ and $dM/d\omega$ are the Radon-Nikodym derivatives.

Theorem 2.1 *The initial-value problem* (2.5)*, or equivalently the integral equation* (2.6)*, has a solution f of class $\mathbf{D}_\tau$ and the solution is uniquely determined by its initial values. Moreover, if g, z, d_1 and d_2 are real, then the solution is real.*

Remark 2.2. For a proof, see [6, pp.156–157]. In [6], Eckhardt and Teschl consider a more general initial-value problem, namely

$$\begin{pmatrix} f(x) \\ f'_-(x) \end{pmatrix} = \begin{pmatrix} f(c) \\ f'_-(c) \end{pmatrix} + \int_{c-}^{x-} C \begin{pmatrix} f(y) \\ f'_-(y) \end{pmatrix} d\omega(y),$$

where $d\omega = |d\mu_1| + |d\mu_2|$ for μ_1, μ_2 locally finite complex Borel measures, and

$$C = \begin{pmatrix} 0 & m_1 \\ m_2 & 0 \end{pmatrix}$$

with the Radon-Nikodym derivatives $m_1 = d\mu_1/d\omega$ and $m_2 = d\mu_2/d\omega$. Since they use the left derivative instead, the initial values and the integral bounds are different from ours. This system is proved to be solvable uniquely under the condition that $(I + C(x)\omega(\{x\}))$ is invertible for each $x < c$. In our case, we look at $\tilde{F}(x) = F(-x)$, where $F(x)$ is as in (2.6), and use the result for the above system. It follows that the invertibility of $(I - C(x)\omega(\{x\}))$ for each $x \geqslant c$ is required instead and it is trivially met with ω and C as in (2.7) and (2.8)

$$I - C(x)\omega(\{x\}) = \begin{pmatrix} 1 & 0 \\ -z\rho(\{x\}) & 1 \end{pmatrix}.$$

2.3 The fundamental system $\{A(x,z), C(x,z)\}$

A consequence of Theorem 2.1 is that the solution space of the equation $(\tau - z)f = 0$ has dimension two. For a fixed $z \in \mathbb{C}$, a fundamental system of the equation $(\tau - z)f = 0$ is a basis of the solution space. For each fixed $x \in \mathbb{I}$, the solutions are entire functions according to the following result, a proof of which can be found in [6, pp.161–162].

Proposition 2.3 *Let $u(x,z)$ be the solution of the initial-value problem* (2.5) *for each $z \in \mathbb{C}$. Then, $u(x,z)$ and $u'_+(x,z)$ are entire functions in z for every $x \in \mathbb{I}$. In particular, for every $x \in [0,l)$, we have the estimate*

$$|u(x,z)| + |u'_+(x,z)| \leqslant Ce^{B\sqrt{|z|}},$$

for some constants $C > 0$ and $B \in \mathbb{R}$.

We define the Wronski determinant, or the Wronskian, of $f, g \in \mathbf{D}_\tau$ by

$$W(f,g)(x) = f(x)g'_+(x) - f'_+(x)g(x), \qquad x \in \mathbb{I}.$$

The Wronskian of fixed f, g is of class $AC_{loc}(\mathbb{I}, dM)$, with quasi-derivative

$$\frac{d}{dM}\, W(f,g) = f\,(\tau g) - (\tau f)\, g.$$

This result is called the Lagrange identity, see [6, pp.157–158]. Suppose u_1, u_2 are solutions to the equation $(\tau - z)f = 0$. Then, according to the Lagrange identity, we have

$$\frac{d}{dM}\, W(u_1,u_2) = u_1\,(\tau u_2) - (\tau u_1)\, u_2 = u_1(zu_2) - (zu_1)u_2 = 0,$$

which implies that $W(u_1, u_2) \equiv C$ on $\mathbb{I}$. The Wronskian vanishes if and only if u_1 and u_2 are linearly dependent, see [6, p.158].

Next, we introduce a convenient fundamental system for the equation $(\tau - z)f = 0$, for each $z \in \mathbb{C}$. It consists of solutions $A(x,z)$ and $C(x,z)$, with the initial values

$$\begin{pmatrix} A(0,z) \\ A'_-(0,z) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} C(0,z) \\ C'_-(0,z) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

yielding $W(A,C) \equiv 1$. By Theorem 2.1, the solutions $A(x,z)$ and $C(x,z)$ are unique in $\mathbf{D}_\tau$. We can in fact give general formulas for $A(x,z)$ and $C(x,z)$. We will also construct an auxiliary solution $D(x,z)$, which is important later on. The set $\{A, D\}$ is also a fundamental system. The initial values of $D(x,z)$ will be

$$\begin{pmatrix} D(0,z) \\ D'_-(0,z) \end{pmatrix} = \begin{pmatrix} D(0) \\ -1 \end{pmatrix},$$

yielding $W(A,D)(x) \equiv -1$.

2.3.1 The solution $A(x,z)$

In this section, we will derive a power expansion of $A(x,z)$. Let $p_0(x) = 1$ and $\{p_n(x)\}_{n \geqslant 1}$ be defined by

$$p_n(x) = \int_0^x d\xi \int_{0-}^{\xi} p_{n-1}(\eta)\, dM(\eta), \qquad x \in [0,l),$$

in other words, $\tau p_n = p_{n-1} \in \mathbf{D}_\tau$. Define a function $\tilde{A}(x,z)$ by

$$\tilde{A}(x,z) := \sum_{n=0}^{\infty} z^n p_n(x). \tag{2.9}$$

For $\tilde{A}$ to make sense, the series should be absolutely convergent for each x, such that $x + M(x) < \infty$, and for each $z \in \mathbb{C}$. To show this, we prove the estimate

$$p_n(x) \leqslant \frac{1}{n!}\Big(\int_0^x M(y)\, dy\Big)^n, \tag{2.10}$$

for each $n \geqslant 1$, using induction. For $n = 0$, the estimate is immediate. Assuming that it holds for $p_{n-1}(x)$, we estimate

$$\begin{aligned} p_n(x) &= \int_0^x d\xi \int_{0-}^{\xi} p_{n-1}(\eta)\, dM(\eta) \leqslant \int_0^x p_{n-1}(\xi)\, d\xi \int_{0-}^{\xi} dM(\eta) \\ &= \int_0^x p_{n-1}(\xi) M(\xi)\, d\xi \leqslant \int_0^x [(n-1)!]^{-1} \Big[\int_0^{\xi} M(y)\, dy \Big]^{n-1} M(\xi)\, d\xi \\ &= (n!)^{-1} \Big[\int_0^x M(y)\, dy \Big]^n . \end{aligned}$$

Now, the convergence of (2.9) follows from

$$|\tilde{A}(x,z)| \leqslant \sum_{n=0}^{\infty} \frac{|z|^n}{n!} \Big[\int_0^x M(y)\, dy \Big]^n = \exp\Big(|z| \int_0^x M(y)\, dy \Big), \tag{2.11}$$

which is finite for each $z \in \mathbb{C}$, as long as $x + M(x) < \infty$.

The right derivative $\tilde{A}'_+(x,z)$ is

$$\tilde{A}'_+(x,z) = z \sum_{n=0}^{\infty} z^n \int_{0-}^{x} p_{n-1}(\eta)\, dM(\eta),$$

from which it is clear that $\tau \tilde{A} = z \tilde{A}$ and that $\tilde{A}(x,z) \in \mathbf{D}_\tau$. Moreover, $\tilde{A}(0,z) = 1$ and $\tilde{A}'_-(0,z) = 0$, for $z \in \mathbb{C}$. By uniqueness of solutions in $\mathbf{D}_\tau$, we conclude that $A(x,z) = \tilde{A}(x,z)$. The power series (2.9), together with (2.11), shows that $A(x,z)$ is indeed entire in z for each $x \in \mathbb{I}$. Note that Proposition 2.3 provides a sharper estimate for $|A(x,z)|$ than (2.11).

If there is no confusion, we omit r and write $A(x,r)$ as $A(x)$ when $r > 0$. By Theorem 2.1, $A(x)$ is a real-valued function. Some properties of $A(x)$ are:

(1) $A(x) \equiv 1$ when $x \leqslant 0$, A is non-decreasing and $\tau A \geqslant 0$,

(2) $A(l-) < \infty$ if and only if $\int_0^l M(x)\, dx < \infty$,

(3) $A'_+(l-) < \infty$ if and only if $\int_0^l x\, dM(x) < \infty$,

(4) $\int_{0-}^l |A|^2\, dM < \infty$ if and only if $\int_0^l x^2\, dM(x) < \infty$.

Property (1) is obvious from (2.9). For a proof of (2)–(4), see [4, pp.162–163]. In fact, item (4) is true for $A(x,z)$, for all $z \in \mathbb{C}$, see [4, pp.171–172].

2.3.2 The solutions $C(x,z)$ and $D(x,z)$

We prove the identity

$$C(x,z) = A(x,z) \int_0^x \big[A(y,z)\big]^{-2}\, dy, \qquad x \in \mathbb{I}, \tag{2.12}$$

by showing that the right-hand side is a solution of $(\tau - z) f = 0$ with the correct initial values.

Let $\tilde{C}(x,z)$ denote the right-hand side of (2.12). For each $r > 0$, the function $A(x)$ is bounded from below by

$$A(y) \geqslant 1 + r \int_0^y M(\xi)\, d\xi \geqslant 1 + rM(\epsilon)(y - \epsilon),$$

for y not too close to 0, and $A(y) \geqslant 1$ for y close to 0. This guarantees the convergence of $\int_0^x A(y)^{-2}\, dy$ for each $x \in \mathbb{I}$. When $z \notin (0,\infty)$, the convergence of $\int_0^x A(y,z)^{-2}\, dy$ is a delicate issue, which is discussed in [4, pp.172–176].

It is easily seen that $\tilde{C}(x,z)$ has the required initial values for each $z \in \mathbb{C}$. It is a solution because

$$\begin{aligned} d\,\tilde{C}'_+(x,z) &= d\,(A'_+ \int_0^x A^{-2}\, dy + A^{-1}) \\ &= d\,A'_+ \int_0^x A^{-2}\, dy + A'_+\, A^{-2} dx - A^{-2}\, A'_+ dx \\ &= z\tilde{C}\, dM. \end{aligned}$$

Identity (2.12) is now established.

From $A(x,z)$ and $C(x,z)$, we define a new solution $D(x,z)$ by making the following linear combination of A and C:

$$D(x,z) = \Big(\int_0^{l+k} \big[A(y,z)\big]^{-2}\, dy\Big) A(x,z) - C(x,z), \qquad x \in (-\infty, l+k], \tag{2.13}$$

or equivalently

$$D(x,z) = A(x,z) \int_x^{l+k} \big[A(y,z)\big]^{-2}\, dy, \qquad x \in (-\infty, l+k].$$

The constant $0 \leqslant k \leqslant \infty$ depends on the self-adjoint domains of τ, which will be introduced in the next section.

When there is no confusion, we write $D(x,r) = D(x)$, for $r > 0$. The solution $D(x)$ has interesting properties – especially item (6):

(5) $D(x)$ is non-negative, non-increasing and convex, i.e. $\tau D \geqslant 0$;

(6) $\int_{0-}^{l+k} D^2\, dM < \infty$.

Item (5) is immediate. For a proof of item (6), we refer to [4, p.164–166].

2.4 Domains on which τ is a self-adjoint operator

To apply the theory for unbounded self-adjoint operators to τ, we need to restrict $\mathbf{D}_\tau$ to an appropriate Hilbert space. This is chosen to be the complex Hilbert space $\mathbb{M} = \mathbf{L}^2(\mathbb{I}, dM)$, equipped with the scalar product

$$(f,g)_\varrho = \int_{0-}^{l} f(x) g(x)^*\, dM(x),$$

where g^* is the complex conjugate of g. The new domain is

$$\mathbf{D}_{max} = \{f \in \mathbf{D}_\tau \,:\, \|f\|_\varrho + \|\tau f\|_\varrho < \infty\}.$$

A problem arises: we might have $\tau f \neq \tau g$ on a set of positive ϱ-measure, even if $f = g$ ϱ-almost everywhere. Equivalently, there are functions f which are 0 ϱ-almost everywhere but satisfy $\tau f \neq 0$ on a set of positive ϱ-measure. An example of such a function $f \in \mathbf{D}_{max}$ is

$$f(x) = \begin{cases} C_0 u_0(x), & x \in (-\infty, 0] \\ 0, & x \in (0, l] \\ C_l u_l(x), & x \in (l, \infty) \end{cases}$$

for the case when $l + M(l-) < \infty$. We choose the fundamental system $\{u_0, u_l\}$ to the equation $\tau f = 0$, with the initial values $u_0(0) = u_l(l) = 0$ and $u'_{-,0}(0) = u'_{+,l}(l) = 1$. The function f vanishes ϱ-almost everywhere, and yet by (2.4), we obtain

$$\tau f = C_0 \mathbb{1}_{\{0\}} + C_l \mathbb{1}_{\{l\}}.$$

This means, if $\varrho(\{0\}) > 0$ or $\varrho(\{l\}) > 0$, $\tau f \neq 0$ on a set of positive measure and τ will be multi-valued. See [6, pp.164–165] for a complete proof of this fact. Hence, we are forced to work with τ on $\mathbf{D}_{max}$ as a so-called relation, which is a generalization of the notion of an operator. There are many similarities between the theory for relations and the theory for operators. View [6, pp.213–214] for further details. We will denote an operator by $(\tau, \mathbf{D})$, and a relation by $\{\tau, \mathbf{D}\}$.

As in the classical theory for unbounded self-adjoint operators, we specify the minimal domain. Consider

$$\mathbf{D}_0 = \{f \in \mathbf{D}_{max} \,:\, \operatorname{supp}(f) \text{ is compact in } \mathbb{I}\}.$$

The adjoint of $(\tau, \mathbf{D}_0)$ is indeed $\{\tau, \mathbf{D}_{max}\}$. The minimal domain $\mathbf{D}_{\min}$ is defined as the closure of $\mathbf{D}_0$ in $\mathbb{M}$, and τ on $\mathbf{D}_{min}$ is always an operator. See [6, pp.168,170].

Any self-adjoint extension of $(\tau, \mathbf{D}_{min})$ lies between $\mathbf{D}_{min}$ and $\mathbf{D}_{max}$. The fact that $(\tau, \mathbf{D}_{min})$ is an operator is necessary for our search. If it is a relation, then any extension of it will be a relation. The multi-valuedness of τ will be carried further into all larger domains.

We use the limit-point/limit-circle classification of endpoints, as in the classical Sturm-Liouville theory. Let $x = a$ be the left endpoint. The operator τ is limit-circle at a if for all $z \in \mathbb{C}$, there exists a fundamental system $\{u, v\}$ to $(\tau - z)f = 0$, such that u and v are of the class $\mathbf{L}^2([a, a + \epsilon), dM)$ for some $\epsilon > 0$, or shortly, u and v are $\mathbf{L}^2$ near $x = a$. Let $x = b$ be the right endpoint. We have a similar definition for the limit-circle case at b, but instead with the interval $(b - \epsilon, b]$ for some $\epsilon > 0$ if b is finite, and $[N, \infty)$ for some N if b is infinite. If τ is not limit-circle, then it is limit-point at that endpoint. In fact, τ is limit-circle at an endpoint if and only if for *some* $z_0 \in \mathbb{C}$, there is a fundamental system which is $\mathbf{L}^2$ near that endpoint, see Lemma 5 in [6, p.172].

In this thesis, τ is always limit-circle at $x = 0$. The classification of the endpoint $x = l$ breaks into three cases.

Case 1: when $l + M(l-) = \infty$ and $\int_{0-}^{l} x^2 \, dM = \infty$, the solution $A(x, z)$ fails to be $\mathbf{L}^2$ near $x = l$ for all $z \in \mathbb{C}$, according to item (4) of Section 2.3.1 . Hence, τ is limit-point at $x = l$.

Case 2: when $l + M(l-) = \infty$ and $\int_{0-}^{l} x^2 \, dM < \infty$, both $A(x, r)$ and $D(x, r)$ are $\mathbf{L}^2$ near $x = l$ whenever $r > 0$, according to item (4) of Section 2.3.1 and item (6) of Section 2.3.2. Hence, τ is limit-circle at $x = l$.

Case 3: when $l + M(l-) < \infty$, the condition $\int_{0-}^{l} x^2 \, dM < \infty$ is trivially satisfied. Hence, the solutions $A(x, r)$ and $D(x, r)$ are $\mathbf{L}^2$ near $x = l$ whenever $r > 0$ by item (4) and (6) in 2.3.1 and 2.3.2 respectively. Hence, τ is limit-circle at $x = l$.

A complete characterization of self-adjoint extensions for each of the above cases, and when these are operators, can be found in [6]. A lot can be simplified, using the fundamental system $\{A, C\}$, and the facts that ϱ places a mass far from $x = -\infty$ and that $\varrho(\{l\}) = 0$ in case 1 and 2. The characterization depends on the endpoints' types. If τ is limit-point at an endpoint, there are no requirements on how the functions should behave at that endpoint, except that they should be $\mathbf{L}^2$ near it. If τ is limit-circle, we always ask for more. For example, the boundary conditions at $x = 0$ are:

$$\mathbf{D}_-(\tau) = \{f \in \mathbf{D}_\tau \,:\, f \text{ is } \mathbf{L}^2 \text{ near } x = 0 \text{ and } f'_-(0) = 0\}.$$

A function $f \in \mathbf{D}_\tau$ is said to satisfy the boundary conditions at $x = 0$ if $f \in \mathbf{D}_-(\tau)$. The boundary conditions at the endpoint $x = l$ are given case-wise.

Case 1: Since τ is limit-point at $x = l$, we only require

$$\mathbf{D}_+(\tau) = \{f \in \mathbf{D}_\tau \,:\, f \text{ is } \mathbf{L}^2 \text{ near } x = l\}.$$

Case 2: The only choice possible for the spectrum to be non-positive is the Neumann boundary condition

$$\mathbf{D}_+(\tau) = \{f \in \mathbf{D}_\tau \,:\, f \text{ is } \mathbf{L}^2 \text{ near } x = l \text{ and } f'_-(l) = 0\}.$$

See [11, pp.74–75].

Case 3: We can "tie down" the strings in many different ways thanks to the finiteness of l and $M(l)$ and choose among the following boundary conditions:

$$\mathbf{D}_{+,k}(\tau) = \{f \in \mathbf{D}_\tau \,:\, f \text{ is } \mathbf{L}^2 \text{ near } x = l \text{ and } f(l) + kf'_+(l) = 0\} \text{ or}$$
$$\mathbf{D}_{+,\infty}(\tau) = \{f \in \mathbf{D}_\tau \,:\, f \text{ is } \mathbf{L}^2 \text{ near } x = l \text{ and } f'_+(l) = 0\}$$

for $0 \leqslant k < \infty$. Only when $\varrho(\{l\}) = 0$ can the parameter k be chosen to be 0 because when $k = 0$ and $\varrho(\{l\}) > 0$, we cannot erase the multi-valuedness of τ at $x = l$, see Corollary 7.8–9 in [6, pp.184–185]. The requirement $f(l) + kf'_+(l) = 0$ is the same as $f(l + k) = 0$.

A function $f \in \mathbf{D}_\tau$ is said to satisfy the boundary condition at $x = l$ if $f \in \mathbf{D}_+(\tau)$. Because of the tying constant k, we sometimes denote the short string as (l, M, k).

Finally, the domains on which τ is self-adjoint are given by the recipe

$$\mathbf{D}(\tau) = \mathbf{D}_-(\tau) \cap \mathbf{D}_+(\tau) \cap \mathbf{D}_{max}.$$

The boundary condition at $x = 0$ is of Neumann type, which means that we let the string slides freely at $x = 0$. We may choose many other boundary conditions, for example the Dirichlet boundary condition with $f(0) = 0$, or in general $\cos(\theta)f(0) + \sin(\theta)f'_-(0) = 0$ for some $\theta \in [0, 2\pi)$. When τ is limit-circle at both endpoints, there are more complicated boundary conditions, for example the coupled boundary conditions, see [6]. However, in this thesis, we only consider the separated boundary conditions, which means that the boundary condition at $x = 0$ is not dependent on the boundary condition at $x = l$.

Theorem 2.4 *The operator $(\tau, \mathbf{D}(\tau))$ is non-positive and self-adjoint.*

For the proof of the non-positivity of $(\tau, \mathbf{D}(\tau))$, we refer to [4, pp.153–156].

Proposition 2.5 *Each eigenvalue of τ is simple.*

Proof. A consequence of Theorem 2.4 is that the eigenvalues are real and non-positive. Let $\gamma \in (-\infty, 0]$ be an eigenvalue, so that $(\tau - \gamma)f = 0$ has two non-trivial solutions u_1 and u_2. Then, u_1 and u_2 must be of class $\mathbf{D}_-(\tau)$, hence $u'_{1,-}(0) = u'_{2,-}(0) = 0$. Computing the left limit of the Wronskian at the point $x = 0$ gives

$$\begin{aligned} W(u_1, u_2)(0-) &= u_1(0-)u'_{2,+}(0-) - u'_{1,+}(0-)u_2(0-) \\ &= u_1(0)u'_{2,-}(0) - u'_{1,-}(0)u_2(0) \\ &= 0, \end{aligned}$$

which implies that u_1 and u_2 are linearly dependent. □

Now, we can clarify the constant k in the definition of the solution $D(x, z)$. In the short-string case with (l, M, k), we have

$$D(x, z) = A(x, z) \int_x^{l+k} [A(y, z)]^{-2} \, dy,$$

and in the long-string case, we have $k = 0$.

Lemma 2.6 *The fundamental system $\{A(x, z), D(x, z)\}$ has the properties*

$$A(x, z) \in \mathbf{D}_-(\tau), \quad \text{and} \quad D(x, z) \in \mathbf{D}_+(\tau)$$

for all $z \in \mathbb{C}$, and

$$A(x, z) \notin \mathbf{D}_+(\tau), \quad \text{and} \quad D(x, z) \notin \mathbf{D}_-(\tau)$$

for $z \in \mathbb{C}\backslash(-\infty, 0]$. Furthermore, $A(x, z)$ and $D(x, z)$ are unique, up to constant multiples, in $\mathbf{D}_-(\tau)$ and $\mathbf{D}_+(\tau)$ respectively, for all $z \in \mathbb{C}$.

Remark 2.7. For a proof of $D(x, z) \in \mathbf{D}_+(\tau)$, see [4, pp.164–166, 175]. The rest is easy to prove. By construction, $A'_-(0, z) = 0$ for all $z \in \mathbb{C}$. Since τ is limit-circle at $x = 0$, all solutions are $\mathbf{L}^2$ near $x = 0$ for all $z \in \mathbb{C}$. Hence, the solution $A(x, z)$ is of class $\mathbf{D}_-(\tau)$. Let $z_0 \in \mathbb{C}\backslash(-\infty, 0]$ and suppose that $A(x, z_0) \in \mathbf{D}_+(\tau) \cap \mathbf{D}_-(\tau)$. Then, z_0 would be an eigenvalue, as $A(x, z_0)$ becomes an eigenfunction. This contradicts the fact that $(\tau, \mathbf{D}(\tau))$ is non-positive and self-adjoint. Hence, $A(x, z) \notin \mathbf{D}_+(\tau)$ for $z \in \mathbb{C}\backslash(-\infty, 0]$. Let $\tilde{A}(x, z)$ be another solution of class $\mathbf{D}_-(\tau)$. Then, $W(A, \tilde{A}) \equiv 0$, which shows that the solution space as a subspace of $\mathbf{D}_-(\tau)$ has dimension one. Similar arguments apply for $D(x, z)$.

2.5 The resolvent operator

Since $(\tau, \mathbf{D}(\tau))$ is self-adjoint and non-positive, the spectrum $\Sigma(\tau)$ is contained in the non-positive real axis $(-\infty, 0]$. The resolvent set

$$\rho(\tau) = \{z \in \mathbb{C} : \mathfrak{G}_z = (z - \tau)^{-1} \text{ is a bounded and bijective operator on } \mathbb{M}\}$$

includes $\mathbb{C}\backslash(-\infty, 0]$. The map $z \mapsto \mathfrak{G}_z$ is an analytic map from $\rho(\tau)$ to the space of bounded linear operators on $\mathbb{M}$. The resolvent operator $\mathfrak{G}$ is called the Green operator. Since we only treat separated boundary conditions, we have the following simplified representation.

Theorem 2.8 *The Green operator $\mathfrak{G}_z : \mathbb{M} \to \mathbb{M}$ admits the integral representation*

$$\begin{aligned}\mathfrak{G}_z f &= \int_{0-}^{l} G_z(x,y) f(y)\, dM(y)\\ &= D(x,z)\int_{0-}^{x} A(y,z) f(y)\, dM(y) + A(x,z)\int_{x}^{l} D(y,z) f(y)\, dM(y),\end{aligned}$$

where the kernel $G_z(x,y)$ is given by

$$G_z(x,y) = \begin{cases} A(x,z)D(y,z), & \text{if } x \leqslant y\\ A(y,z)D(x,z), & \text{if } y \leqslant x.\end{cases}$$

For a proof, see [4, pp.166–170]. A more operator-theoretical approach can be found in [6, pp.188–189].

Remark 2.9. The function $y' \mapsto G_r(x,y')$ on an interval $I = [y-\delta, y+\delta] \subset [0,l)$ is Lipschitz continuous. This is because the derivatives of $A(y')$ or $D(y')$ are uniformly bounded on I whenever they exist, and they exist λ-almost everywhere on I.

Next, we want to know when $\mathfrak{G}_z$ is compact.

Corollary 2.10 *If τ is limit-circle at $x = l$, or if (l, M) satisfies one of the following conditions*

$$\int_{0-}^{l} x\, dM(x) < \infty, \ \text{or} \ \int_{0}^{l} M(x)\, dx < \infty, \tag{2.14}$$

then $\mathfrak{G}_z$ is a Hilbert-Schmidt operator, meaning that

$$\int_{0-}^{l}\int_{0-}^{l} |G_z(x,y)|^2\, dM(x)\, dM(y) < \infty.$$

Proof. If τ is limit-circle, then $A(x,z)$ and $D(x,z)$ are both of class $\mathbf{L}^2(\mathbb{I}, dM)$ and the statement follows.

Suppose that (l, M) satisfies one of the conditions in (2.14). It is sufficient to prove that $\mathfrak{G}_r$ is a Hilbert-Schmidt operator for $r > 0$. Indeed, assume $(r-\tau)^{-1}$ is a Hilbert-Schmidt operator for $r > 0$. Then, by the resolvent formula,

$$(z-\tau)^{-1} - (r-\tau)^{-1} = -(z-r)(z-\tau)^{-1}(r-\tau)^{-1}, \qquad z, r \in \rho(\tau).$$

The right-hand side is a Hilbert-Schmidt operator because $(z-\tau)^{-1}$ is bounded. Moving $(r-\tau)^{-1}$ from the left-hand side to the right-hand side, $(z-\tau)^{-1}$ is a Hilbert-Schmidt operator because it is the sum of two Hilbert-Schmidt operators.

For the real Green kernel $G_r(x,y)$, we have the inequality

$$G_r(x,y) \leqslant \min\{G_r(x,x), G_r(y,y)\},$$

implying that $G_r(x,y)^2 \leqslant G_r(x,x)G_r(y,y)$ and

$$\int_{0-}^{l}\int_{0-}^{l} G_r(x,y)^2\, dM(x)\, dM(y) \leqslant \Big(\int_{0-}^{l} G_r(x,x)\, dM(x)\Big)\Big(\int_{0-}^{l} G_r(y,y)\, dM(y)\Big).$$

If $\int_{0-}^{l} M(x)\, dx < \infty$, we have $A(l-) < \infty$ by item (2) in Section 2.3.1. Using the monotonicity of the real solutions $A(x)$ and $D(x)$, we estimate:

$$\int_{0-}^{l} G_r(x,x)\, dM(x) = \int_{0-}^{l} A(x)D(x)\, dM(x) \leqslant D(0)A(l-) < \infty.$$

If $\int_{0-}^{l} x\,dM(x) < \infty$, we have $A'_+(l-) < \infty$ by property (5) of $A(x)$. This gives

$$\int_{0-}^{l} G_r(x,x)\,dM(x) \leqslant D(0)\int_{0-}^{l} A(x)\,dM(x) = r^{-1}\,D(0)\,A'_+(l-) < \infty.$$

The last step in the above is obtained by differentiating the formula

$$A(x) = 1 + r\int_0^x d\xi \int_{0-}^{\xi} A(\eta)\,dM(\eta),$$

yielding

$$A'_+(x) = r\int_0^x A(\eta)\,dM(\eta).$$

The corollary is now established. □

If τ is limit-circle at $x = l$, then $\Sigma(\tau)$ is discrete. By Proposition 2.5, the eigenvalues must be simple.

The following estimate will come in handy.

Lemma 2.11 $\int_{0-}^{l} G_r(x,y)\,dM(y) \leqslant r^{-1}$.

Proof.

$$\begin{aligned}
&r\int_{0-}^{l} G(x,y)\,dM(y)\\
&= rD(x)\int_{0-}^{x} A(y)\,dM(y) + rA(x)\int_x^l D(y)\,dM(y)\\
&= D(x)\int_{0-}^{x} \big(\tau A\big)(y)\,dM(y) + A(x)\int_x^l \big(\tau D\big)(y)\,dM(y)\\
&= D(x)\int_{0-}^{x} \frac{dA'_+}{dM}(y)\,dM(y) + A(x)\int_x^l \frac{dD'_+}{dM}(y)\,dM(y)\\
&= D(x)A'_+(x) + A(x)\big[D'_-(l) - D'_+(x) + \big(\tau D\big)(l)\varrho(\{l\})\big].
\end{aligned}$$

If $\varrho(\{l\}) = 0$, the above sum reduces to:

$$\begin{aligned}
r\mathfrak{G}_r 1 &= D(x)A'_+(x) + A(x)\big[D'_-(l) - D'_+(x)\big]\\
&\leqslant D(x)A'_+(x) - A(x)D'_+(x)\\
&= W(D,A)\\
&= 1.
\end{aligned}$$

In the above inequality, we use that D is non-increasing, hence $D'_-(l) \leqslant 0$, and that $A(x) \geqslant 0$ for $x \in [0,l)$.

If $\varrho(\{l\}) > 0$, then by (2.4),

$$\frac{D'_+(l) - D'_-(l)}{\varrho(\{l\})} = (\tau D)(l),$$

which gives

$$\begin{aligned}
&r\,\mathfrak{G}_{ib}1\\
&= D(x)A'_+(x) + A(x)\big[D'_+(l) - D'_+(x)\big]\\
&\leqslant D(x)A'_+(x) - A(x)D'_+(x)\\
&= 1.
\end{aligned}$$

The proof is now complete. □

3 Krein's correspondence

3.1 The Weyl function and the spectral measure σ

Definition 3.1 Let $\{\theta_z, \phi_z\}$ be a fundamental system of the equation $(\tau+z)f = 0$, such that $\phi_z \in \mathbf{D}_-(\tau)$ and $W(\theta_z, \phi_z) = 1$. A singular Weyl-Titchmarsh-Kodaira function, or just a Weyl function, is a function $m(z)$ defined on $-\rho(\tau) = \{-z \,|\, z \in \rho(\tau)\}$, such that the linear combination

$$\psi_z = \theta_z + m(z)\phi_z$$

is of class $\mathbf{D}_+(\tau)$. The solution ψ_z is called a Weyl solution associated to $m(z)$.

From Section 2.3.2, we are provided with the formula

$$D(x, -z) = -C(x, -z) + D(0, -z)\, A(x, -z), \qquad z \in \mathbb{C}\backslash(-\infty, 0].$$

The functions $C(x, -z), A(x, -z)$ and $D(x, -z)$ solve the equation $(\tau + z)f = 0$ and meet the requirements of Definition 3.1. Hence,

$$m(z) := D(0, -z) = \int_0^{l+k} \big[A(y, -z)\big]^{-2}\, dy, \qquad z \in \mathbb{C}\backslash[0, \infty)$$

is a Weyl function, where k is the tying constant in the short-string case, and $k = 0$ in the long-string case.

The Weyl function has always been of interest in the study of Sturm-Liouville operators because it contains all spectral information, which will be discussed more in details in Section 3.3.1. As in the classical theory, $m(z)$ is a Herglotz-Nevanlinna function, i.e. $m(z)$ is analytic in the upper half-plane $\mathbb{H}^+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and it maps $\mathbb{H}^+$ into $\mathbb{H}^+$. Also, $m(z^*) = m(z)^*$, see Corollary 9.5 and Theorem 9.2 in [6].

More can be said: the function $m(-z)$ has the limit $\lim_{r\uparrow\infty} m(-r) = 0$, and we will prove that $m(-z)$ belongs to the following class.

Definition 3.2 A complex function f is said to be of the class $\mathbb{S}$ (or the class of Stieltjes functions) if it satisfies

(1) $f \in \mathrm{Hol}(\mathbb{C}\backslash(-\infty, 0])$;

(2) $\mathrm{Im} f(z) \leqslant 0$ if $\mathrm{Im}\, z > 0$;

(3) $f(x) \geqslant 0$ for $x \in (0, \infty)$.

Remark 3.3. Definition 3.2 is practical for our constructions. In the classical work [10], written by Krein and Kac, the $\mathbb{S}$-class is defined differently from here. Their Stieltjes functions are analytic on $\mathbb{C}\backslash[0, \infty)$. The definition in [10] is fulfilled for $f(-z)$ if and only if $f \in \mathbb{S}$ as in Definition 3.2.

Theorem 3.4 *$f \in \mathbb{S}$ if and only if f has the integral representation*

$$f(z) = C + \int_{0-}^{\infty} \frac{d\sigma(\gamma)}{\gamma + z}, \qquad z \in \mathbb{C}\backslash(-\infty, 0],$$

for some constant $C \geqslant 0$ and a unique non-negative Borel measure σ, such that

$$C + \int_{0-}^{\infty} (1 + \gamma)^{-1}\, d\sigma(\gamma) < \infty.$$

Remark 3.5. For a proof, see Chapter 2 in [16]. In Theorem 3.4, it is sufficient to have the integral representation of $f \in \mathbb{S}$ for $z \in (0, \infty)$. There exists a unique analytic extension of f to the cut complex plan $\mathbb{C} \backslash (-\infty, 0]$ and the extension satisfies Definition 3.2.

The measure σ corresponding to $m(-z)$ as in Theorem 3.4 will be referred to as the spectral measure associated to the string (l, M), or $m(-z)$. Note that σ in this context is the spectral measure associated to the operator $(-\tau, \mathbf{D}(\tau))$. Instead of proving its existence directly, we will prove the eigendifferential expansion of the real Green kernel, which is extremely insightful, and from which we only need to set $x = y = 0$ and use that $G_r(0, 0) = D(0, r) = m(-r)$ to identify a spectral measure associated to $m(-r)$, for $r > 0$. The proof will be divided into several sections.

Theorem 3.6 *There exists a non-negative Borel measure σ with support in $[0, \infty)$, such that the real Green kernel can be expressed as*

$$G_r(x, y) = \int_{0-}^{\infty} \frac{A(x, -\gamma) A(y, -\gamma)}{r + \gamma} \, d\sigma(\gamma), \tag{3.1}$$

for $r > 0$, with $0 \leqslant x, y \leqslant l$ in the short-string case, and $x, y < l$ in the long-string case, as long as x and y don't belong to the same massless interval.

3.1.1 Construction of σ for short strings

Proof of Theorem 3.6. Due to self-adjointness of $(\tau, \mathbf{D}(\tau))$ and Corollary 2.10, the resolvent operator $\mathfrak{G}_r = (r - \tau)^{-1}$ is self-adjoint and compact for each $r > 0$. There exists an orthonormal basis $\{f_n\}_{n \geqslant 1}$ consisting of eigenfunctions f_n of $\mathfrak{G}_r$ for the space $\mathbb{M}$. The functions f_n are also eigenfunctions of τ. Because of the non-positivity of $(\tau, \mathbf{D}(\tau))$, we have $(\tau + \gamma_n) f_n = 0$, for $\gamma_n \in [0, \infty)$. According to Corollary 2.5, each eigenvalue is simple. Hence, it must hold that

$$f_n(x) = \alpha_n \, A(x, -\gamma_n),$$

for some complex scalars α_n. In fact, α_n can be chosen to be real, in view of Theorem 2.1, and then $\alpha_n = \|A(x, -\gamma_n)\|_{\varrho}^{-1}$ because f_n has unit length.

The resolvent $\mathfrak{G}_r$ has the eigenvalues $(r + \gamma_n)^{-1}$, which are all positive. Suppose the following equality holds when $x \in [0, l]$ is held fixed:

$$\begin{aligned} G_r(x, y) &= \sum_{n=1}^{\infty} (r + \gamma_n)^{-1} f_n(x) f_n(y) \\ &= \sum_{n=1}^{\infty} (r + \gamma_n)^{-1} \alpha_n^2 A(x, -\gamma_n) \, A(y, -\gamma_n), \end{aligned} \tag{3.2}$$

for $0 \leqslant y \leqslant l$ and x, y not in the same massless interval, and where the equality is to be understood as uniform convergence. We can then design a discrete non-negative Borel measure σ with

$$\sigma(B) := \sum_{\gamma_n \in B} \|A(x, -\gamma_n)\|^{-2}, \qquad B \in \mathfrak{B}(\mathbb{R}). \tag{3.3}$$

Note that $\operatorname{supp}(\sigma) \subset [0, \infty)$ but σ is defined on the whole $\mathbb{R}$. We switch the sum in (3.2) to the integral sign:

$$G_r(x, y) = \int_{0-}^{\infty} \frac{A(x, -\gamma) \, A(y, -\gamma)}{r + \gamma} \, d\sigma(\gamma),$$

which is what we want to prove for $0 \leqslant x, y \leqslant l$ and x, y not in the same massless interval.

Now, we prove (3.2). We claim that the function

$$F_N(x,y) = G_r(x,y) - \sum_{n=1}^{N} (r+\gamma_n)^{-1} f_n(x) f_n(y)$$

is a kernel for a non-negative integral operator for each finite $N \geqslant 1$, that is,

$$\int_{0-}^{l} \Big(\int_{0-}^{l} F_N(x,y) g(y)\, dM(y) \Big) g(x)^*\, dM(x) \geqslant 0,$$

for all $g \in \mathbb{M}$. Indeed, since the functions f_n are real, we have

$$\begin{aligned}
&\int_{0-}^{l} \Big(\int_{0-}^{l} F_N(x,y) g(y)\, dM(y) \Big) g(x)^*\, dM(x) \\
&= (\mathfrak{G}_r\, g, g) - \int_{0-}^{l} \Big(\int_{0-}^{l} \sum_{n=1}^{N} (r+\gamma_n)^{-1} f_n(x) f_n(y) g(y)\, dM(y) \Big) g(x)^*\, dM(x) \\
&= (\mathfrak{G}_r\, g, g) - \sum_{n=1}^{N} (r+\gamma_n)^{-1} (f_n, g^*) \int_{0-}^{l} f_n(x) g(x)^*\, dM(x) \\
&= (\mathfrak{G}_r\, g, g) - \sum_{n=1}^{N} (r+\gamma_n)^{-1} (g, f_n)(f_n, g) \\
&= (\mathfrak{G}_r\, g, g) - \sum_{n=1}^{N} (r+\gamma_n)^{-1} |(g, f_n)|^2.
\end{aligned}$$

Expanding $g = \sum_{n \geqslant 1} (g, f_n) f_n$, we obtain now

$$(\mathfrak{G}_r\, g, g) = \sum_{n \geqslant 1} (g, f_n)(\mathfrak{G}_r\, f_n, g) = \sum_{n \geqslant 1} (r+\gamma_n)^{-1} |(g, f_n)|^2.$$

The claim is now apparent. An implication of this is

$$F_N(x,x) = G_r(x,x) - \sum_{n=1}^{N} (r+\gamma_n)^{-1} f_n(x)^2 \geqslant 0,$$

at every growth point $x \leqslant l$. If not, then $F_N(x,x) < 0$ for some growth point $x \leqslant l$. Since $F_N(x,x)$ is continuous, there is an interval $(a,b) \subset [0,l]$ containing x, such that $F_N(w,w) < 0$ for all $a < w < b$ and that $M(b) - M(a) > 0$. Then

$$\int_{0-}^{l} \Big(\int_{0-}^{l} F_N(x,y) \chi_{(a,b)}(y)\, dM(y) \Big) \chi_{(a,b)}(x)^*\, dM(x) < 0,$$

which is a contradiction. So $F_N(x,x) \geqslant 0$ at all growth points $x \in [0,l]$. Differentiating $F_N(x,x)$ twice on massless intervals, we use that $A'(x) \geqslant 0$ and that $D'(x) \leqslant 0$ to realize the concavity of $F_N(x,x)$ for x on massless intervals:

$$F_N''(x,x) = 2\Big(A'(x) D'(x) - \sum_{n=1}^{N} (r+\gamma_n)^{-1} [f_n'(x)]^2 \Big) \leqslant 0.$$

Hence, we can extend the inequality $F_N(x,x) \geqslant 0$ across massless intervals. This implies that

$$\sum_{n=1}^{\infty} (r+\gamma_n)^{-1} f_n^2(x) \leqslant G_r(x,x) < \infty. \tag{3.4}$$

for every $0 \leqslant x \leqslant l$. This fact, together with the Cauchy-Schwarz inequality can be used to estimate the tail of the sum

$$\begin{aligned}
&\Big| \sum_{n=M}^{N} (r+\gamma_n)^{-1} f_n(x) f_n(y) \Big|^2 \\
&\leqslant \Big(\sum_{n=M}^{N} (r+\gamma_n)^{-1} f_n^2(x) \Big) \Big(\sum_{n=1}^{\infty} (r+\gamma_n)^{-1} f_n^2(y) \Big) \\
&\leqslant \Big(\sum_{n=M}^{N} (r+\gamma_n)^{-1} f_n^2(x) \Big) G_r(y,y),
\end{aligned}$$

which shows that the sum $\sum_{n=1}^{\infty}(r+\gamma_n)^{-1} f_n(x) f_n(y)$ converges uniformly to a continuous function in the variable y as $N \to \infty$ and as x is held fixed. Finally, the difference $G_r(x,y) - \sum_{n=1}^{\infty}(r+\gamma_n)^{-1} f_n(x) f_n(y)$ is orthogonal to every f_k for each fixed x and $k = 1, 2, \dots$ because

$$\begin{aligned}
&\int_{0-}^{l} \Big[G_r(x,y) - \sum_{n=1}^{\infty} (r+\gamma_n)^{-1} f_n(x) f_n(y) \Big] f_k(y)\, dM(y) \\
&= \Big(\mathfrak{G}_r f_k \Big)(x) - (r+\gamma_k)^{-1} f_k(x) = 0,
\end{aligned}$$

which by Bessel's inequality implies that

$$\Big\| G_r(x,y) - \sum_{n=1}^{N} (r+\gamma_n)^{-1} f_n(x) f_n(y) \Big\| \to 0, \ N \to \infty.$$

So, for each fixed $x \in [0,l]$, the series in (3.2) has a uniform limit, which is a continuous function in y. Moreover, its $\mathbf{L}^2$-limit is $G_r(x,\,.\,)$ in $\mathbb{M}$. Since $G_r(x,y)$ is continuous in y, (3.2) must hold pointwise for ϱ–almost every $y \in [0,l]$. If x and y belong to different massless intervals, say $y \in (a,b)$, then both sides of (3.2) are linear functions in y having the same initial values at $y = a$. Hence, the equality in (3.2) also holds for this case. If x and y belong to the same massless interval, say $x, y \in (a,b)$, then $G_r(x,y)$ attains its maximum at $y = x$. However, on the right-hand side of (3.1), because $A(y, -\gamma)$ is linear on massless intervals, the series cannot attain a maximum on (a,b). By symmetry, the same holds when y is held fixed and everything is regarded as continuous functions in x. The claim of Theorem 3.6 is now proved.

□

3.1.2 Existence of σ for long strings

The idea is to approximate the long string (l, M) with a sequence of short strings (l_n, M_n, k_n) and define σ as a weak limit of σ_n associated to (l_n, M_n, k_n).

Let $\{L_n\}_{n=1}^{\infty}$ be a sequence of points with the properties

$$\begin{aligned}
&\varrho(\{L_n\}) = 0, \\
&0 < L_{n-1} < L_n < l \text{ for all } n, \text{ and} \\
&\lim_{n \uparrow \infty} L_n = l.
\end{aligned}$$

The short strings have lengths l_n, which are

$$l_n = \sup\{x < L_n \,|\, x \text{ is a point of growth of } M\},$$

with the mass functions

$$M_n(x) = \begin{cases} M(x), & x \leqslant l_n \\ M(l_n), & x > l_n. \end{cases}$$

The tying constants are

$$k_n = L_n - l_n.$$

Let $(\tau, \mathbf{D}(\tau))$ be the operator from (l, M), and $(\tau_n, \mathbf{D}(\tau_n))$ from (l_n, M_n, k_n). If $A(x)$ is a solution in $\mathbf{D}_-(\tau)$, then $A(x)$ for $x < L_n$ is a solution of class $\mathbf{D}_-(\tau_n)$. Hence, the solutions $D_n(x)$ are given by

$$D_n(x) = A(x) \int_x^{L_n} A(y)^{-2}\, dy,$$

which increase to the solution $D(x) \in \mathbf{D}_+(\tau)$ as $L_n \to l$. Using Theorem 3.6 for short strings, we have a spectral measure σ_n for each (l_n, M_n, k_n), such that

$$G_r^{(n)}(0,0) = D_n(0,r) = \int_{0-}^{\infty} \frac{1}{r+\gamma}\, d\sigma_n(\gamma), \qquad r > 0. \tag{3.5}$$

Since

$$D_n(0,r) \leqslant D(0,r) \quad \text{and} \quad \lim_{n \uparrow \infty} D_n(0,r) = D(0,r), \tag{3.6}$$

the total variations of the measures $d\mathcal{F}_n(\gamma) = (1+\gamma)^{-1}\, d\sigma_n(\gamma)$ are uniformly bounded by $D(0,1)$. Let $\mathcal{F}_n$ denote the distribution functions associated to $d\mathcal{F}_n$. By Helly's Selection Theorem, there is a subsequence $\{\mathcal{F}_{n_k}\}$, and a distribution function $\mathcal{F}(\gamma)$, such that

$$\lim_{k \uparrow \infty} \mathcal{F}_{n_k}(\gamma) = \mathcal{F}(\gamma),$$

for all points of continuity γ of $\mathcal{F}$. This suggests a candidate for the spectral function σ:

$$d\sigma := (1+\gamma)\, d\mathcal{F},$$

where $d\mathcal{F}$ is the bounded positive Borel measure associated to $\mathcal{F}$. The measure σ is simply

$$\begin{aligned} d\sigma &= (1+\gamma) \lim_{k\uparrow\infty} d\mathcal{F}_{n_k} \\ &= (1+\gamma) \lim_{k\uparrow\infty} (1+\gamma)^{-1}\, d\sigma_{n_k} \\ &= \lim_{k\uparrow\infty} d\sigma_{n_k}. \end{aligned}$$

The convergence here is at least in the weak sense, that is, $\int f\, d\mathcal{F}_{n_k} \to \int f\, d\mathcal{F}$, for all bounded continuous functions f on $[0, l)$.

Proof of Theorem 3.6. For simplicity, we drop the index k and write $d\sigma = \lim_n d\sigma_n$. Pick x, y not in the same massless interval and let $r > 0$. We examine the difference

$$\begin{aligned}
&G_r(x,y) - G_1(x,y)\\
&= \lim_{n\uparrow\infty} \left[G_r^{(n)}(x,y) - G_1^{(n)}(x,y)\right]\\
&= \lim_{n\uparrow\infty} \Big[\int_{0-}^{\infty} \frac{A(x,-\gamma)A(y,-\gamma)}{r+\gamma}\, d\sigma_n(\gamma) - \int_{0-}^{\infty} \frac{A(x,-\gamma)A(y,-\gamma)}{1+\gamma}\, d\sigma_n(\gamma)\Big] \qquad (3.7)\\
&= \lim_{n\uparrow\infty} (1-r) \int_{0-}^{\infty} \frac{A(x,-\gamma)A(y,-\gamma)}{(1+\gamma)(r+\gamma)}\, d\sigma_n(\gamma).
\end{aligned}$$

Suppose the following holds:

$$\lim_{n\uparrow\infty} \int_{0-}^{\infty} \frac{A(x,-\gamma)A(y,-\gamma)}{(1+\gamma)(r+\gamma)}\, d\sigma_n(\gamma) = \int_{0-}^{\infty} \frac{A(x,-\gamma)A(y,-\gamma)}{(1+\gamma)(r+\gamma)}\, d\sigma(\gamma). \qquad (3.8)$$

Then we can decompose the integrand in (3.7) and obtain

$$G_r(x,y) - G_1(x,y) = \int_{0-}^{\infty} \frac{A(x,-\gamma)A(y,-\gamma)}{r+\gamma}\, d\sigma(\gamma) - \int_{0-}^{\infty} \frac{A(x,-\gamma)A(y,-\gamma)}{1+\gamma}\, d\sigma(\gamma).$$

This means that

$$G_r(x,y) = \int_{0-}^{\infty} \frac{A(x,-\gamma)A(y,-\gamma)}{r+\gamma}\, d\sigma(\gamma) + C,$$

in which the constant C is independent of the spectral parameter r. To show that $C = 0$, we fix y and use that $y' \mapsto G_r(x,y')$ on a small interval $[y-\delta, y+\delta]$ is Lipschitz-continuous, meaning

$$G_r(x,y) \leqslant G_r(x,y') + K|y-y'|,$$

for all $y' \in I = [y-\delta, y+\delta] \subset [0,l)$ and for some constant $K > 0$, see Remark 2.9. Applying Lemma 2.11, we have

$$\begin{aligned}
G_r(x,y) &\leqslant [\varrho(I)]^{-1} \int_{y-\delta}^{y+\delta} G_r(x,y')\, dM(y') + [\varrho(I)]^{-1} \int_{y-\delta}^{y+\delta} K\,|y-y'| dM(y')\\
&\leqslant [\varrho(I)]^{-1} r^{-1} + [\varrho(I)]^{-1} \delta\, K\, \varrho(I)\\
&= \varrho(I)^{-1} r^{-1} + \delta\, K,
\end{aligned}$$

for each $r > 0$. Letting $r \to \infty$ and $\delta \to 0$, we get that the constant term C must be 0.

It remains to prove (3.8). We split the integrals into two pieces:

$$\begin{aligned}
&\int_{0-}^{\infty} \frac{A(x,-\gamma)A(y,-\gamma)}{(1+\gamma)(r+\gamma)}\, d\sigma_n(\gamma)\\
&= \int_{0-}^{N} \frac{(A(x,-\gamma)A(y,-\gamma)}{(1+\gamma)(r+\gamma)}\, d\sigma_n(\gamma) + \int_{N}^{\infty} \frac{A(x,-\gamma)A(y,-\gamma)}{(1+\gamma)(r+\gamma)}\, d\sigma_n(\gamma).
\end{aligned}$$

The first piece converges to the integral

$$\lim_{n\uparrow\infty} \int_{0-}^{N} \frac{A(x,-\gamma)A(y,-\gamma)}{(r+\gamma)} \frac{d\sigma_n(\gamma)}{1+\gamma} = \int_{0-}^{N} \frac{A(x,-\gamma)A(y,-\gamma)}{(r+\gamma)} \frac{d\sigma(\gamma)}{1+\gamma},$$

because the integrand $\frac{A(x,-\gamma)A(y,-\gamma)}{r+\gamma}$ is continuous and bounded on $[0,N)$. The second piece goes to zero independently of n as $N\to\infty$. Indeed, for each n, we have from (3.5), (3.6) and then from (3.4) the inequalities

$$\int_{0-}^{\infty}\frac{\big[A(x,-\gamma)\big]^2}{r+\gamma}\,d\sigma_n(\gamma)\leqslant\int_{0-}^{\infty}\frac{\big[A(x,-\gamma)\big]^2}{r+\gamma}\,d\sigma(\gamma)\leqslant G_r(x,x),$$

for every $x\leqslant l$. Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&\int_{N}^{\infty}\frac{A(x,-\gamma)A(y,-\gamma)}{(1+\gamma)(r+\gamma)}\,d\sigma_n(\gamma)\\ \leqslant&\Big(\sup_{\{\gamma>N\}}(1+\gamma)^{-1}\Big)\int_{N}^{\infty}\frac{A(x,-\gamma)A(y,-\gamma)}{r+\gamma}\,d\sigma_n(\gamma)\\ \leqslant&(N+1)^{-1}\Big[\int_{N}^{\infty}\frac{\big[A(x,-\gamma)\big]^2}{r+\gamma}\,d\sigma_n(\gamma)\Big]^{1/2}\Big[\int_{N}^{\infty}\frac{\big[A(y,-\gamma)\big]^2}{r+\gamma}\,d\sigma_n(\gamma)\Big]^{1/2}\\ \leqslant&(N+1)^{-1}\,G_r(x,x)^{1/2}G_r(y,y)^{1/2},\end{aligned}$$

which goes to 0 as $N\to\infty$ when x and y are held fixed. The proof is now complete. □

3.2 The inverse spectral problem

In the prior sections, we associate each string (l,M) with a spectral measure σ, which is automatically unique in virtue of Theorem 3.4. The question now is: given *any* non-negative Borel measure σ with $\operatorname{supp}(\sigma)\subset[0,\infty)$ and $\int_{0-}^{\infty}(1+\gamma)^{-1}\,d\sigma(\gamma)<\infty$, is there a pair (l,M) with σ as its spectral measure?

Krein's correspondence has confirmed that such a string can be found. Moreover, such a string is unique. The existence part of Krein's correspondence will be dealt with in this section. Due to the complexity of the proof for uniqueness, which involves for example de Branges spaces, it will not be included in this thesis. For the interested readers, we refer to [4] and [10].

3.2.1 Definition and structure of the space $\mathbf{L}^2(\mathbb{R},d\Delta)$

Let (l,M) be given and let σ be the associated spectral measure. We define the measure Δ on $(\mathbb{R},\mathfrak{B}(\mathbb{R}))$ by introducing the variable $\gamma=\xi^2$ in the following way: for all $r>0$, we have

$$\int_{0}^{\infty}\frac{d\sigma(\gamma)}{r+\gamma}=\frac{1}{2}\int_{0}^{\infty}\frac{d\Delta(\xi)}{r+\xi^2}=\frac{1}{2}\int_{-\infty}^{0-}\frac{d\Delta(\xi)}{r+\xi^2}$$

and $\sigma(\{0\})=\Delta(\{0\})$, so that for all Borel sets $B\subset(0,\infty)$, we have

$$\sigma(B)=\int_B d\sigma(\gamma)=\int_{-\sqrt{B}\cup\sqrt{B}}d\Delta(\xi)=\frac{1}{2}\Delta(\sqrt{B})+\frac{1}{2}\Delta(-\sqrt{B}),\tag{3.9}$$

where $\sqrt{B}$ and $-\sqrt{B}$ denote the sets $\{\sqrt{b}\,|\,b\in B\}$ and $\{-\sqrt{b}\,|\,b\in B\}$ respectively. Naturally, if $0\in B\subset[0,\infty)$, then

$$\sigma(B)=\sigma(\{0\})+\int_{B\backslash\{0\}}d\sigma(\gamma)=\sigma(\{0\})+\frac{1}{2}\Delta\big(\sqrt{B\backslash\{0\}}\big)+\frac{1}{2}\Delta\big(-\sqrt{B\backslash\{0\}}\big).\tag{3.10}$$

We have redistributed the mass of the spectral measure σ over the whole real line symmetrically around the origin and the new measure Δ has a double jump at the origin. By construction, we have

$$\int_{0-}^{\infty} \frac{d\sigma(\gamma)}{1+\gamma} = \int_{-\infty}^{\infty} \frac{d\Delta(\xi)}{1+\xi^2} < \infty.$$

Conversely, whenever we are provided with a measure Δ with the properties

(1) Δ is a non-negative Borel measure,

(2) Δ distributes its mass symmetrically around the origin,

(3) $\int_{\mathbb{R}} (1+\xi^2)^{-1}\, d\Delta(\xi) < \infty$,

a spectral measure σ can be extracted.

Definition 3.7 A measure Δ with property (1)–(3) is called a principal spectral measure. Given a string (l, M), we say Δ is the principal spectral measure associated to (l, M) if

$$D(0,r) = \int_{-\infty}^{\infty} \frac{d\Delta(\xi)}{r+\xi^2}$$

holds for all $r > 0$.

Instead of finding a string for a given measure σ, we find a string for a given Δ with property (1)–(3).

We denote $\mathbf{L}^2(\mathbb{R}, d\Delta)$ by $\mathbf{Z}(\Delta)$. Let $\mathbf{Z}_e(\Delta)$ be the subspace of even functions in $\mathbf{Z}(\Delta)$, and $\mathbf{Z}_o(\Delta)$ the subspace of odd functions in $\mathbf{Z}(\Delta)$. We introduce the even transform, which maps $\mathbb{M}$ isomorphically onto $\mathbf{Z}_e(\Delta)$, and the odd transform, which maps the subspace $\mathbf{S}$ of Lebesgue square-integrable functions, which are constant on massless intervals, onto $\mathbf{Z}_o(\Delta)$. For the string with $l = \infty$ and $M(x) = x \cdot \mathbb{1}_{[0,\infty)}$, the even and odd transforms correspond to the cosine and sine transforms

$$f \mapsto \int_0^{\infty} \cos(\xi x) f(x)\, dx, \qquad f \mapsto \int_0^{\infty} \sin(\xi x) f(x)\, dx$$

respectively.

We present only a summary of the relevant results here.

Theorem 3.8 *The mapping, known as the even transform,*

$$f \mapsto \widehat{f}_e(\xi) = \int_{0-}^{l} A(x, -\xi^2) f(x)\, dM(x)$$

is an isomorphism from $\mathbb{M}$ to $\mathbf{Z}_e(\sigma)$. It has the inverse

$$\widehat{f}_e \mapsto \big(\widehat{f}_e(\xi)\big)^{\vee} = f(x) = \int_{-\infty}^{\infty} A(x, -\xi^2) \widehat{f}_e(\xi)\, d\Delta(\xi).$$

The even transform satisfies the Plancherel identity

$$\|f\|_{\varrho}^2 = \int_{0-}^{l} |f(x)|^2\, dM(x) = \int_{-\infty}^{\infty} |\widehat{f}_e(\xi)|^2\, d\Delta(\xi) = \|\widehat{f}_e\|_{\Delta}^2.$$

Remark 3.9. When $l + M(l-) = \infty$, the integrals in Theorem 3.8 might not exist in the traditional sense. In such case, the even transform and its inverse are to be understood as improper integrals

$$\lim_{x\uparrow\infty} \left\| \widehat{f}_e - \int_{-x}^{x} Af\, dM \right\|_\Delta = 0, \qquad \lim_{x\uparrow l} \left\| f - \int_{0-}^{x} A\widehat{f}_e\, d\Delta \right\|_\varrho = 0.$$

The proof idea is to use the eigendifferential expansion (3.1) of $G_r(x, y)$ to prove the Plancherel identity for functions $f \in \mathbb{M}$ which vanish near $x = l$. Then, the even transform can be extended to $\mathbf{Z}_e(\Delta)$. The challenging part of the proof is to show its surjectivity. For more details, see [4, pp.186–188]. An alternative route with a spectral theoretical approach can be found in [12], in which (3.1) is not assumed.

The appropriate domain for the odd transform is the space $\mathbf{S} \subset \mathbf{L}^2([0, l + k], dx)$, which consists of functions constant on massless intervals. The kernel for the odd transform is

$$B(x, \xi) = \xi^{-1} A'_+(x, -\xi^2).$$

The motivation behind $B(x, \xi)$ is that the equation

$$\tau f = -\xi^2 f$$

can be rewritten as the system

$$\begin{pmatrix} df \\ dg \end{pmatrix} = \begin{pmatrix} 0 & \xi \\ -\xi\varrho & 0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}.$$

We present the counterpart of Theorem 3.8 for the odd transform.

Theorem 3.10 *The mapping, known as the odd transform,*

$$f \mapsto \widehat{f}_o(\xi) = \int_0^{l+k} B(x, \xi) f(x)\, dx,$$

is an isomorphism from $\mathbf{S}$ *to* $\mathbf{Z}_o(\Delta)$ *with the inverse*

$$\widehat{f}_o \mapsto \big(\widehat{f}_o(\xi)\big)^{\vee} = f(x) = \int_{-\infty}^{\infty} B(x, \xi) \widehat{f}_o(\xi)\, d\Delta(\xi).$$

The odd transform satisfies the Plancherel identity

$$\|f\|_x^2 = \int_0^{l+k} |f|^2\, dx = \int_{-\infty}^{\infty} |\widehat{f}_o|^2\, d\Delta = \|\widehat{f}_o\|_\Delta^2.$$

Remark 3.11. Similar to Remark 3.9, $\widehat{f}_o \in \mathbf{Z}_o(\Delta)$ and $\big(\widehat{f}_o\big)^{\vee} \in \mathbf{S}$ are to be understood as the $\mathbf{L}^2$-limits in the respective spaces when the integrals do not exist in the traditional sense.

Theorem 3.8 and 3.10 have the following useful consequences.

Corollary 3.12 *We have for* $0 \leqslant x < l$ *and* $x = l$ *only when* $l + M(l-) < \infty$

(1)

$$\|A(x, -\xi^2)\|_\Delta^2 = \varrho(\{x\})^{-1}, \tag{3.11}$$

whenever x *is a growth point of* M*, with* $x = l$ *excluded in the case* $(l, M, k = 0)$.

(2)

$$\|B(x,\xi)\|_{\Delta}^2 = (x^* - x_*)^{-1}, \tag{3.12}$$

with

$$x_* := \sup\{y < x \,|\, y \textit{ is a growth point of } M\}$$
$$x^* := \sup\{z > x \,|\, z \textit{ is a growth point of } M\},$$

and $x^* = l + k$ *when* $x = l$ *in the short-string case.*

(3)

$$\int_{-\infty}^{\infty} \frac{d\Delta(\xi)}{\xi^2} = \begin{cases} l+k, & \textit{when } l + M(l-) < \infty \\ l, & \textit{when } l + M(l-) = \infty. \end{cases} \tag{3.13}$$

Remark 3.13. For a proof, see [4, pp.185–194]. One should compare (3.11) to how σ (and hence Δ via (3.9)) is constructed in (3.3), as it reveals a duality between ϱ and σ (or Δ).

Lastly, we discuss the dimension of $\mathbf{Z}(\Delta)$. If the number of growth points of Δ is finite, that is,

$$\Delta = \sum_{k=1}^{n} a_k \delta_{-\xi_k^2} + \sum_{k=1}^{n} a_k \delta_{\xi_k^2}, \quad 0 < a_k < \infty, \text{ and } \xi_k^2 < \xi_{k+1}^2, \text{ for all } k = 1, \ldots, n,$$

we observe that the dimension cannot be infinite. It must in fact equal the number of jumps of Δ. Indeed, the functions

$$\mathbb{1}_{(-\infty,-\xi_n^2)}, \ldots, \mathbb{1}_{(-\xi_2^2,-\xi_1^2)}, \mathbb{1}_{(\xi_1^2,\xi_2^2)}, \ldots, \mathbb{1}_{(\xi_n^2,\infty)}$$

belong to the equivalence class $[\,0\,]$ in $\mathbf{Z}(\Delta)$. Hence, the activity of any function on those intervals will not contribute to its $\mathbf{Z}$-norm. An orthogonal basis of $\mathbf{Z}(\Delta)$ consists of $\mathbb{1}_{\{-\xi_k^2\}}$ and $\mathbb{1}_{\{\xi_k^2\}}$, for $k = 1, \ldots, n$. This discussion prepares the way for the construction of (l, M) when Δ is discrete.

3.2.2 Construction of (l, M) for discrete Δ

Let $d \leqslant \infty$ denote the number of jumps of Δ. The key here is to define the functions

$$A(x, -\xi^2), \quad \text{and} \quad B(x,\xi).$$

We assume that the powers $\{\xi^n\}_{n=0}^d$ all have finite $\mathbf{Z}$-norm, and that they form a basis for $\mathbf{Z}(\Delta)$. Then, there is a perpendicular basis of alternatively even and odd real polynomials in the variable ξ:

$$A_0, B_1, A_2, \ldots, A_{2n}, B_{2n+1}, \ldots,$$

where $\deg(A_{2n}) = 2n < d$ and $\deg(B_{2n+1}) = 2n+1 < d$. Note that A_{2n} and B_{2n+1} are not assumed to have unit length. The space spanned by $\{A_{2k}\}_{k\geqslant 0}$ is $\mathbf{Z}_e(\Delta)$ and the space spanned by $\{B_{2k+1}\}_{k\geqslant 0}$ is $\mathbf{Z}_o(\Delta)$.

Since $\xi^{-1}\left(A_{2n}(\xi) - A_{2n}(0)\right)$ is an odd polynomial of degree $2n-1$, it will be perpendicular to $B_{2n+1}(\xi)$, meaning

$$\int_{-\infty}^{\infty} \frac{A_{2n}(\xi) - A_{2n}(0)}{\xi}\, B_{2n+1}(\xi)\, d\Delta(\xi) = 0$$

and hence,

$$A_{2n}(0)\int_{-\infty}^{\infty}\frac{B_{2n+1}(\xi)}{\xi}\,d\Delta(\xi)=\int_{-\infty}^{\infty}A_{2n}(\xi)\frac{B_{2n+1}(\xi)}{\xi}\,d\Delta(\xi).$$

In the above, $\xi^{-1}B_{2n+1}(\xi)$ is an even polynomial of degree $2n$. Hence, the right-hand side cannot vanish for $2n+1<d$. By the same reasoning, the quantity

$$A_{2n+2}(0)\int_{-\infty}^{\infty}\frac{B_{2n+1}(\xi)}{\xi}d\Delta(\xi)=-\int_{-\infty}^{\infty}\frac{A_{2n+2}(\xi)-A_{2n+2}(0)}{\xi}\,B_{2n+1}(\xi)\,d\Delta(\xi)$$

is non-zero, for $2n+2<d$. This allows us to rescale the even and odd polynomials, so that

$$A_{2n}(0)=1,\quad\text{and}\quad\int_{-\infty}^{\infty}\frac{B_{2n+1}(\xi)}{\xi}\,d\Delta(\xi)=-1,\tag{3.14}$$

for $2n<d$ and $2n+1<d$. The following lemma relates A_{2n-2} and A_{2n} to B_{2n-1} and B_{2n+2}, and vice versa.

Lemma 3.14 *We have:*

(1) for $2\leqslant 2n+2<d$,

$$\frac{A_{2n+2}(\xi)-A_{2n}(\xi)}{\xi\,B_{2n+1}(\xi)}=\|B_{2n+1}\|_{\Delta}^{-2},$$

(2) for $1\leqslant 2n+1<d$,

$$\frac{B_{2n+1}(\xi)-B_{2n-1}(\xi)}{-\xi\,A_{2n}(\xi)}=\|A_{2n}\|_{\Delta}^{-2},$$

where we define $B_{-1}:=0$.

Proof. To prove item (1) and (2) requires the same technique. For (1), let $2\leqslant 2n+2<d$. We consider the polynomial

$$P(\xi):=\frac{A_{2n+2}(\xi)-A_{2n}(\xi)}{\xi}.$$

By (3.14), the constant term of $A_{2n+2}-A_{2n}$ vanishes. Hence, $P(\xi)$ is an odd polynomial of degree $2n+1$. Furthermore, $P(\xi)$ is perpendicular to the powers $\xi,\xi^3,\dots,\xi^{2n-1}$ because

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{A_{2n+2}(\xi)-A_{2n}(\xi)}{\xi}\cdot\xi^{2k+1}\,d\Delta(\xi)\\=&\int_{-\infty}^{\infty}A_{2n+2}(\xi)\,\xi^{2k}\,d\Delta(\xi)-\int_{-\infty}^{\infty}A_{2n}(\xi)\,\xi^{2k}\,d\Delta(\xi)\\=&0,\end{aligned}$$

where we have used that A_{2n+2} and A_{2n} belong to the orthogonal complement of

$$\operatorname{span}(A_0,\dots,A_{2n-2})=\operatorname{span}(1,\xi^2,\dots,\xi^{2n-2}).$$

Hence, we must have

$$\frac{A_{2n+2}(\xi)-A_{2n}(\xi)}{\xi}=c\cdot B_{2n+1}(\xi),$$

which leads us to the desired equation

$$
\begin{aligned}
c\|B_{2n+1}\|_\Delta^2 &= \int_{-\infty}^{\infty} \frac{A_{2n+2}(\xi) - A_{2n}(\xi)}{\xi}\, B_{2n+1}(\xi)\, d\Delta(\xi) \\
&= -\int_{-\infty}^{\infty} A_{2n}(\xi)\, \frac{B_{2n+1}(\xi)}{\xi}\, d\Delta(\xi) \\
&= -A_{2n}(0) \int_{-\infty}^{\infty} \frac{B_{2n+1}(\xi)}{\xi}\, d\Delta(\xi) \\
&= 1.
\end{aligned}
$$

In the second line, we use that A_{2n+2} is perpendicular to $\xi^{-1}B_{2n+1}$, as $\deg(\xi^{-1}B_{2n+1}) = 2n$. In the third line, we use that $\xi^{-1}\big[A_{2n}(\xi) - A_{2n}(0)\big]$ is perpendicular to B_{2n+1} because it is an odd polynomial of degree exactly $2n-1$. The proof for (1) is now done.

The proof for (2) is similar. For $n = 0$, we have $A_0 = 1$ and $B_1 = c\xi$. Hence, by using (3.14), we obtain

$$
-1 = \int_{-\infty}^{\infty} \frac{B_1(\xi)}{\xi}\, d\Delta(\xi) = c \int_{-\infty}^{\infty} d\Delta(\xi) = c\,\|A_0\|^2.
$$

Since $B_{-1} = 0$, the case when $n = 0$ is proved. For $1 < 2n+1 < d$, we observe that

$$
\frac{B_{2n+1}(\xi) - B_{2n-1}(\xi)}{\xi}
$$

is an even polynomial of degree $2n$ and it is perpendicular to the powers $1, \xi^2, \ldots, \xi^{2n-2}$. Thus, it must be parallel to A_{2n}, and consequently,

$$
\begin{aligned}
c\|A_{2n}\|_\Delta^2 &= \int_{-\infty}^{\infty} \frac{B_{2n+1}(\xi) - B_{2n-1}(\xi)}{\xi}\, A_{2n}(\xi)\, d\Delta(\xi) \\
&= \int_{-\infty}^{\infty} \frac{B_{2n+1}(\xi)}{\xi}\, A_{2n}(\xi)\, d\Delta(\xi) \\
&= A_{2n}(0) \int_{-\infty}^{\infty} \frac{B_{2n+1}(\xi)}{\xi}\, d\Delta(\xi) \\
&= -1.
\end{aligned}
$$

The proof for (2) is now complete. $\square$

We define the masses ϱ_n and the spacings $x_{n+1} - x_n$ by

$$
\varrho_n := \|A_{2n}\|_\Delta^{-2}, \qquad \text{and} \qquad x_{n+1} - x_n := \|B_{2n+1}\|_\Delta^{-2},
$$

with $x_0 := 0$, for $2n < d$ and $2n+1 < d$.

Theorem 3.15 *Suppose $d < \infty$. Let N be such that $d = 2N+1$ or $d = 2N+2$. Let (l, M, k) be the string*

$$
\begin{aligned}
l &= x_N \\
M(x) &= \sum_{x_n \leqslant x} \varrho_n, \qquad\qquad n \in \{0, 1, \ldots, N\}, \\
k &= \begin{cases} x_{N+1} - x_N, & \text{if } d = 2N+2 \\ \infty, & \text{if } d = 2N+1 \end{cases}.
\end{aligned}
$$

Then, Δ is the principal spectral measure associated to (l, M, k).

Remark 3.16. If $d = 2N+2$, the positions of the masses are

$$\begin{aligned}
&\varrho_0 \text{ is placed at } x_0 = 0,\\
&\varrho_1 \text{ is placed at } x_1,\\
&\dots\\
&\varrho_N \text{ is placed at } x_N.
\end{aligned}$$

While x_{N+1} is defined, the mass ϱ_{N+1} is not. So there is no mass placed at the point x_{N+1}, hence $k = x_{N+1} - x_N$ in this case. If $d = 2N+1$, the above placements of masses are the same up to x_N. However, the point x_{N+1} is not defined, hence $k = \infty$ in this case.

Proof. We define the function $A(x, -\xi^2)$ by

$$A(x, -\xi^2) := A_{2n}(\xi) + (x - x_n)\xi B_{2n+1}(\xi), \qquad x \in [x_n, x_{n+1}) \tag{3.15}$$

for $0 \leqslant n \leqslant N$, and $A(x, -\xi^2) \equiv 1$ for $x < 0$. On each interval $[x_n, x_{n+1})$, $A(x, -\xi^2)$ is linear with the slope $\xi B_{2n+1}(\xi)$, meaning

$$A'_+(x, -\xi^2) = \xi\, B_{2n+1}(\xi), \qquad x \in [x_n, x_{n+1}),$$

and hence

$$B(x, \xi) = B_{2n+1}(\xi), \qquad x \in [x_n, x_{n+1}). \tag{3.16}$$

At each point $x_n \in [0, l]$, we have the left limit

$$\begin{aligned}
&\lim_{x \uparrow x_n} A(x, -\xi^2)\\
&= \lim_{x \uparrow x_n} A_{2n-2}(\xi) + (x - x_{n-1})\xi B_{2n-1}(\xi)\\
&= A_{2n-2}(\xi) + (x_n - x_{n-1})\xi B_{2n-1}(\xi)\\
&= A_{2n-2}(\xi) - \big[A_{2n-2}(\xi) - A_{2n}(\xi)\big]\\
&= A_{2n}(\xi),
\end{aligned}$$

using item (1) of Lemma 3.14. The right limit is also $A_{2n}(\xi)$. Hence, the function $A(x, -\xi^2)$ is continuous at all points $x \in [0, l]$.

We want to check that $A(x, -\xi^2)$ is a solution to $(\tau + \xi^2)f = 0$ with $A(0, -\xi^2) = 1$ and $A'_-(0, -\xi^2) = 0$. According to (2.4), this is the same as checking that

$$\frac{A'_+(x_n, -\xi^2) - A'_-(x_n, -\xi^2)}{\varrho_n} = -\xi^2 A(x_n, -\xi^2),$$

for $0 \leqslant x_n \leqslant l$, which is easily done using item (2) of Lemma 3.14

$$\begin{aligned}
\frac{A'_+(x_n, -\xi^2) - A'_-(x_n, -\xi^2)}{\varrho(\{x_n\})} &= \frac{\xi\big[B_{2n+1}(\xi) - B_{2n-1}(\xi)\big]}{\varrho_n}\\
&= -\xi^2 A_{2n}(\xi)\\
&= -\xi^2 A(x_n, -\xi^2).
\end{aligned}$$

Let Δ^* be the principal spectral measure of the string (l, M, k). Denote $\mathbf{L}^2(\mathbb{R}, d\Delta^*)$ by $\mathbf{Z}(\Delta^*)$. Let $\mathbf{Z}_e(\Delta^*)$ be the subspace of even functions in $\mathbf{Z}(\Delta^*)$ and $\mathbf{Z}_o(\Delta^*)$ the subspace of odd functions in $\mathbf{Z}(\Delta^*)$. The even transform from $\mathbb{M}$ to $\mathbf{Z}_e(\Delta^*)$ has the discrete form

$$\begin{aligned}\widehat{f}_e(\xi) &= \int_{0-}^{l} A(x,-\xi^2) f(x)\, dM(x) \\ &= \sum_{j=0}^{N} f(x_j) A(x_j,-\xi^2)\varrho(\{x_j\}) \\ &= \sum_{j=0}^{N} f(x_j) A_{2j}(\xi)\varrho_j.\end{aligned}$$

The space $\mathbb{M}$ is spanned by the orthogonal basis $\{\mathbb{1}_{\{x_j\}}\}_{j=0}^{N}$. Via the even transform, we obtain an orthogonal basis for $\mathbf{Z}_e(\Delta^*)$, which is

$$\widehat{\mathbb{1}_{\{x_j\}}}_e(\xi) = A_{2j}(\xi)\varrho_j,$$

because the even transform is a vector space isomorphism. In addition, according to (3.11), we have

$$\|A_{2j}(\xi)\|_{\Delta^*}^2 = \varrho_j^{-1} = \|A_{2j}(\xi)\|_{\Delta}^2.$$

We have shown that the spaces $\mathbf{Z}_e(\Delta^*)$ and $\mathbf{Z}_e(\Delta)$ share the basis $\{A_{2j}\}_{j=0}^{N}$ and the basis elements have the same length in both spaces. It is also clear that $(f,g)_{\Delta^*} = (f,g)_{\Delta}$. So, $\mathbf{Z}_e(\Delta^*) = \mathbf{Z}_e(\Delta)$.

Recall that $\mathbf{S}$ is a subspace of $\mathbf{L}^2([0,l+k],dx)$ and it consists of functions constant on the massless intervals of $M(x)$. By (3.16), the odd transform from $\mathbf{S}$ to $\mathbf{Z}_o(\Delta^*)$ is given by

$$\begin{aligned}\widehat{f}_o(\xi) &= \int_{0}^{l+k} B(x,\xi) f(x)\, dx \\ &= \sum_{j=0}^{N-1} f(x_j) B(x_j,\xi)(x_{j+1}-x_j) \\ &= \sum_{j=0}^{N-1} f(x_j) B_{2j+1}(\xi)(x_{j+1}-x_j),\end{aligned}$$

when $d = 2N+1$, and

$$\widehat{f}_o(\xi) = \sum_{j=0}^{N} f(x_j) B_{2j+1}(\xi)(x_{j+1}-x_j),$$

when $d = 2N+2$, see Remark 3.16. We take the orthogonal basis $\{\mathbb{1}_{\{[x_j,x_{j+1})\}}\}$ for j from 0 up to $N-1$ or N, depending on the dimension d, in $\mathbf{S}$ to produce the orthogonal basis $\{B_{2j+1}\}$ in $\mathbf{Z}_o(\Delta^*)$. Then, we argue as in the above to establish that $\mathbf{Z}_o(\Delta^*) = \mathbf{Z}_o(\Delta)$.

Finally, we must check that $\Delta^*(B) = \Delta(B)$ for all Borel sets B, that is,

$$\int_{-\infty}^{\infty} \mathbb{1}_B(\xi)\, d\Delta^*(\xi) = \int_{-\infty}^{\infty} \mathbb{1}_B(\xi)\, d\Delta(\xi).$$

Since

$$(\mathbb{1}_B, A_{2j})_{\Delta^*} = (\mathbb{1}_B, A_{2j})_{\Delta},$$

which we denote by α_j, and

$$(\mathbb{1}_B, B_{2j+1})_{\Delta^*} = (\mathbb{1}_B, B_{2j+1})_{\Delta},$$

which we denote by β_j, we get

$$\begin{aligned}\int_{-\infty}^{\infty} \mathbb{1}_B(\xi)\, d\Delta^*(\xi) &= \|\mathbb{1}_B\|^2_{\Delta^*} \\ &= \sum_{j=0}^{N} |\alpha_j|^2 \|A_{2j}\|^2_{\Delta^*} + \sum_{j=0}^{N-1} |\beta_j|^2 \|B_{2j+1}\|^2_{\Delta^*} \\ &= \sum_{j=0}^{N} |\alpha_j|^2 \|A_{2j}\|^2_{\Delta} + \sum_{j=0}^{N-1} |\beta_j|^2 \|B_{2j+1}\|^2_{\Delta} \\ &= \|\mathbb{1}_B\|^2_{\Delta} \\ &= \int_{-\infty}^{\infty} \mathbb{1}_B(\xi)\, d\Delta(\xi),\end{aligned}$$

when $d = 2N + 1$. A similar argument works for $d = 2N + 2$. The proof is now complete. □

For $d = \infty$, we instead construct the string as follows.

Theorem 3.17 *Suppose $d = \infty$. Let (l, M, k) be the string with*

$$\begin{aligned} l &= \lim_{n\uparrow\infty} x_n \\ M(x) &= \sum_{x_k \leqslant x} \varrho_k \\ k &= \begin{cases} 0, & \text{if } \| \xi^{-1} \|_{\Delta} < \infty \\ \infty, & \text{if } \| \xi^{-1} \|_{\Delta} = \infty. \end{cases}\end{aligned}$$

Then, Δ is the principal spectral measure of (l, M, k).

Remark 3.18. The tying constant k is only relevant in the case $l + M(l-) < \infty$.

Let Δ^* be the principal spectral measure of the string in Theorem 3.17. As in the finite dimensional case, we have matching moments, meaning

$$\int_{-\infty}^{\infty} \xi^{2n}\, d\Delta^*(\xi) = \int_{-\infty}^{\infty} \xi^{2n}\, d\Delta(\xi), \qquad n = 0, 1, \dots. \tag{3.17}$$

Since the dimension is infinite, it is difficult to conclude that $\mathbf{Z}(\Delta^*) = \mathbf{Z}(\Delta)$. The following lemma provides an essential tool.

Lemma 3.19 *Assume* (3.17) *holds for Δ and Δ^*. Then, we have the estimate*

$$\frac{C(x,r)}{A(x,r)} \leqslant \int_{-\infty}^{\infty} \frac{d\Delta(\xi)}{r + \xi^2} \leqslant \frac{C'_+(x,r)}{A'_+(x,r)},$$

for $x < l$ and $r > 0$, where the function $A(x,z)$ is the broken line as in (3.15)*, and the function $C(x,z)$ is given by*

$$C(x,z) = A(x,z) \int_0^x \big[A(y,z)\big]^{-2}\, dy.$$

Proof. Let (l, M, k) be the string in Theorem 3.17. Recall that $C(x,r)$ is the unique solution to the equation $(\tau - r)f = 0$ with the initial values $C(0,r) = 0$ and $C'_-(0,r) = 1$. Define

$$\tilde{C}(x,r) := \int_{-\infty}^{\infty} \frac{A(x,r) - A(x,-\xi^2)}{r+\xi^2} d\Delta(\xi), \tag{3.18}$$

when $0 < x < l$, and $\tilde{C}(x,r) = x$ when $x \leqslant 0$. We want to show that $C(x,r) = \tilde{C}(x,r)$ using uniqueness of a solution with given intial values. We already have that $\tilde{C}'_-(0,r) = C'_-(0,r) = 1$. By the definition of $\tilde{C}$, the left limit is $\lim_{x\uparrow 0} \tilde{C}(x,r) = 0$. For the right limit, we use that $A(0,-\xi^2) = A(0,r) = 1$ in the following

$$\tilde{C}(0,r) = \int_{-\infty}^{\infty} \frac{A(0,r) - A(0,-\xi^2)}{r+\xi^2}\, d\Delta(\xi) = 0.$$

According to (2.4), it is sufficient to check that

$$\tilde{C}'_+(x_n,r) - \tilde{C}'_-(x_n,r) = \varrho_n\, r\, \tilde{C}(x_n,r),$$

for all $0 < x_n < l$ to show that $\tilde{C}$ is a solution. On the intervals $x_n \leqslant x < x_{n+1}$, we can interchange integration and differentiation to compute $\tilde{C}'_-$ and $\tilde{C}'_+$. Indeed, we have

$$\frac{d_+}{dx}(A(x,r) - A(x,-\xi^2)) = i\sqrt{r}B_{2n+1}(i\sqrt{r}) - \xi B_{2n+1}(\xi)$$

is a polynomial in ξ^2 of degree $n+1$ with a root $\xi^2 = -r$, which implies that the integrand

$$\frac{A'_+(x,r) - A'_-(x,-\xi^2)}{r+\xi^2}$$

is a polynomial in ξ^2 of degree n. The hypothesis placed on Δ is that all even polynomials are integrable, which implies that all polynomials are integrable and hence the integrand is integrable. So differentiation under the integral sign is justified in this case. The left and right derivatives of $\tilde{C}(x,r)$ at $0 < x = x_n < l$ are

$$\tilde{C}'_+(x_n,r) = \int_{-\infty}^{\infty} \frac{i\sqrt{r}B_{2n+1}(i\sqrt{r}) - \xi B_{2n+1}(\xi)}{r+\xi^2}\, d\Delta(\xi),$$
$$\tilde{C}'_-(x_n,r) = \int_{-\infty}^{\infty} \frac{i\sqrt{r}B_{2n-1}(i\sqrt{r}) - \xi B_{2n-1}(\xi)}{r+\xi^2}\, d\Delta(\xi).$$

We use item (2) of Lemma 3.14, then add and subtract $\varrho_n A(x_n,-\xi^2)$ to obtain

$$\begin{aligned}\tilde{C}'_+(x_n,r) - \tilde{C}'_-(x_n,r) &= \varrho_n\, r\, \tilde{C}(x_n,r) + \varrho_n \int_{-\infty}^{\infty} A_{2n}(\xi)\, d\Delta(\xi) \\ &= \varrho_n\, r\, \tilde{C}(x_n,r) + \varrho_n \big(A_{2n}, A_0\big)_\Delta \\ &= \varrho_n\, r\, \tilde{C}(x_n,r),\end{aligned}$$

because A_{2n} is assumed to be orthogonal to A_0 in $\mathbf{Z}(\Delta)$ for $n \geqslant 1$. We have now confirmed that

$$\tilde{C}(x,r) = A(x,r) \int_0^x \big[A(y,r)\big]^{-2} dy = C(x,r).$$

Define

$$\tilde{D}(x,r) := \int_{-\infty}^{\infty} \frac{A(x,-\xi^2)}{r+\xi^2}\, d\Delta(\xi),$$

for $x < l$ and $r > 0$. Note that

$$\tilde{C}(x,r) = \int_{-\infty}^{\infty} \frac{d\Delta(\xi)}{r+\xi^2}\, A(x,r) - \tilde{D}(x,r)$$

and

$$\tilde{C}'_+(x,r) = \int_{-\infty}^{\infty} \frac{d\Delta(\xi)}{r+\xi^2}\, A'_+(x,r) - \tilde{D}'_+(x,r),$$

where $\tilde{D}'_+(x,r) = \int_{-\infty}^{\infty} A'_+(x,-\xi^2)(r+\xi^2)^{-1}\, d\Delta$. Here, differentiation under the integral sign can be justified as before. Replacing $\tilde{C}$ by C, the proof will be complete once $\tilde{D} \geqslant 0$ and $\tilde{D}'_+ \leqslant 0$ is proved.

For $y \geqslant x$, the claim is that the integrand in (3.18) is orthogonal to $A(y,-\xi^2)$, that is

$$\int_{-\infty}^{\infty} \frac{A(x,r) - A(x,-\xi^2)}{r+\xi^2}\, A(y,-\xi^2)\, d\Delta(\xi) = 0.$$

Indeed, the numerator of the integrand in (3.18) is a polynomial in ξ^2 with a root $\xi^2 = -r$. Hence, the integrand in (3.18) has a lower degree than $A(y,-\xi^2)$ and these are orthogonal to each other by construction. Letting $y = x$ and rearranging terms, we find that

$$\tilde{D}(x,r) = \big[A(x,r)\big]^{-1} \int_{-\infty}^{\infty} \frac{\big[A(x,-\xi^2)\big]^2}{r+\xi^2}\, d\Delta(\xi).$$

Since $A(x,r) > 0$ for all $x < l$ and $r > 0$, we have $\tilde{D}(x,r) \geqslant 0$.

The right derivative of $\tilde{D}$ is

$$\tilde{D}'_+(x,r) = \int_{-\infty}^{\infty} \frac{A'_+(x,-\xi^2)}{r+\xi^2}\, d\Delta(\xi) = \int_{-\infty}^{\infty} \frac{\xi B(x,\xi)}{r+\xi^2}\, d\Delta(\xi),$$

where $B(x,\xi) = B_{2n+1}(\xi)$, for $x_n \leqslant x < x_{n+1}$. We repeat the argument as before to realize that the polynomial

$$\frac{\xi B(x, i\sqrt{r}) - i\sqrt{r} B(x,\xi)}{r+\xi^2}$$

is of lower degree than that of $B(y,\xi)$ for $y \geqslant x$, and thus orthogonal to $B(y,\xi)$. Letting $y = x$ and rearranging terms again, we find that

$$\tilde{D}'_+(x,r) = \frac{i\sqrt{r}}{B(x,i\sqrt{r})} \int_{-\infty}^{\infty} \frac{\big[B(x,\xi)\big]^2}{r+\xi^2}\, d\Delta(\xi).$$

Replacing $B(x,i\sqrt{r})$ by $(i\sqrt{r})^{-1} A'_+(x,r)$, then using that $A'_+(x,r) \geqslant 0$ and $r > 0$, the right derivative $\tilde{D}'_+$ is non-positive as we can see:

$$\tilde{D}'_+(x,r) = -\frac{r}{A'_+(x,r)} \int_{-\infty}^{\infty} \frac{\big[B(x,\xi)\big]^2}{r+\xi^2}\, d\Delta(\xi) \leqslant 0.$$

We now have

$$\tilde{D}(x,r) = \int_{-\infty}^{\infty} \frac{d\Delta(\xi)}{r+\xi^2}\, A(x,r) - C(x,r) \geqslant 0$$
$$\tilde{D}'_+(x,r) = \int_{-\infty}^{\infty} \frac{d\Delta(\xi)}{r+\xi^2}\, A'_+(x,r) - C'_+(x,r) \leqslant 0,$$

for $x < l$ and $r > 0$, which gives the desired estimate. □

Lemma 3.19 is sufficient to prove Theorem 3.17 for the case $l + M(l-) = \infty$.

Proof of Theorem 3.17 when $l + M(l-) = \infty$. The goal is to establish

$$\int_{-\infty}^{\infty} \frac{d\Delta(\xi)}{r+\xi^2} = \int_{-\infty}^{\infty} \frac{d\Delta^*(\xi)}{r+\xi^2},$$

for all $r > 0$, because the functions $\{(r+\xi^2)^{-1}\}_{r>0}$ are dense in the class of even continuous functions of ξ vanishing at ∞, see [4, p.179].

According to Lemma 3.19, we already have

$$\frac{C(x,r)}{A(x,r)} \leqslant \int_{-\infty}^{\infty} \frac{d\Delta(\xi)}{r+\xi^2} \leqslant \frac{C'_+(x,r)}{A'_+(x,r)},$$

for $x < l$ and $r > 0$. When $l + M(l-) = \infty$, the claim is that the solutions A and C obey

(1)

$$\lim_{x\uparrow l} \frac{C(x,r)}{A(x,r)} - \frac{C'_+(x,r)}{A'_+(x,r)} = 0,$$

(2)

$$\lim_{x\uparrow l} \frac{C'_+(x,r)}{A'_+(x,r)} = D(0,r).$$

Item (1) is a consequence of the Wronskian

$$A(x,r)C'_+(x,r) - A'_+(x,r)C(x,r) = 1,$$

for all $x < l$. Rearranging terms, we find that

$$\frac{C'_+(x,r)}{A'_+(x,r)} - \frac{C(x,r)}{A(x,r)} = \frac{1}{A(x,r)}\,\frac{1}{A'_+(x,r)}.$$

Since $l + M(l-) = \infty$, $l = \infty$ or $M(l-) = \infty$. Consequently, $\int_0^l M\,dx$ or $\int_{0-}^l x\,dM$ fails to be finite. By item (2) or (3) in Section 2.3.1, we must have

$$\lim_{x\uparrow l} A(x,r) = \infty, \quad \text{or} \quad \lim_{x\uparrow l} A'_+(x,r) = \infty.$$

Item (1) is now established. Item (2) is easy:

$$\frac{C'_+(x,r)}{A'_+(x,r)} = \int_0^x \frac{1}{A(x,r)^2}\,dy + \frac{1}{A(x,r)}\,\frac{1}{A'_+(x,r)},$$

which has the limit

$$\lim_{x\uparrow l} \frac{C'_+(x,r)}{A'_+(x,r)} = \int_0^l \frac{1}{A(x,r)^2}\,dy = D(0,r).$$

Representing $D(0,r) = G_r(0,0)$ as in (3.1), we obtain the desired conclusion:

$$\int_{-\infty}^{\infty} \frac{d\Delta(\xi)}{r+\xi^2} = \lim_{x\uparrow l} \frac{C'_+(x,r)}{A'_+(x,r)} = D(0,r) = \int_{-\infty}^{\infty} \frac{d\Delta^*(\xi)}{r+\xi^2}, \quad \text{for all } r > 0.$$

□

For the case $l + M(l-) < \infty$, the strategy is to modify (l, M, k) to a long string and apply the above result. The following estimate prepares the way.

Lemma 3.20 *Let p be a polynomial with $deg(p) \leqslant 2n$ and $\omega \in \mathbb{C}$. We have*

$$|p(\omega)|^2 \leqslant \|p\|_\Delta^2 \Big[\int_{0-}^{x_n} |A(x, -\omega^2)|^2 \, dM(x) + \int_0^{x_n} |B(x, \omega)|^2 \, dx \Big],$$

where $A(x, -\omega^2)$ is as in (3.15) *and $M(x)$ as in Theorem 3.17.*

Proof. We expand $p(\xi)$ for $\xi \in \mathbb{R}$ with respect to the basis $\{A_0, B_1, A_2, \ldots\}$:

$$p(\xi) = \sum_{k=0}^{n} a_k A_{2k}(\xi) + \sum_{k=1}^{n} b_k B_{2k-1}(\xi).$$

The extension of the above polynomial into $\mathbb{C}$ must equal $p(\omega)$. We have:

$$\begin{aligned} \|p\|_\Delta^2 &= \sum_{k=0}^{n} |a_k|^2 \|A_{2k}\|_\Delta^2 + \sum_{k=1}^{n} |b_k|^2 \|B_{2k-1}\|_\Delta^2 \\ &= \sum_{k=0}^{n} |a_k|^2 \varrho_k^{-1} + \sum_{k=1}^{n} |b_k|^2 (x_k - x_{k-1})^{-1}. \end{aligned}$$

The integrals are simply

$$\begin{aligned} \int_{0-}^{x_n} |A(x, -\omega^2)|^2 \, dM(x) &= \sum_{k=0}^{n} |A(x_k, -\omega^2)|^2 \, \varrho_k \\ \int_0^{x_n} |B(x, \omega)|^2 \, dx &= \sum_{k=1}^{n} |B(x_{k-1}, \omega)|^2 (x_k - x_{k-1}). \end{aligned}$$

Putting the pieces together and applying the Cauchy-Schwarz inequality, we find that

$$\begin{aligned} |p(\omega)|^2 &\leqslant \Big(\sum_{k=0}^{n} |a_k| \, |A_{2k}(\omega)| + \sum_{k=1}^{n} |b_k| \, |B_{2k-1}(\omega)| \Big)^2 \\ &\leqslant \Big[\sum_{k=0}^{n} |a_k|^2 \varrho_k^{-1} + \sum_{k=1}^{n} |b_k|^2 (x_k - x_{k-1})^{-1} \Big] \\ &\qquad \cdot \Big[\sum_{k=0}^{n} |A_{2k}(\omega)|^2 \varrho_k + \sum_{k=1}^{n} |B_{2k-1}(\omega)|^2 (x_k - x_{k-1}) \Big] \\ &\leqslant \|p\|_\Delta^2 \Big[\int_{0-}^{x_n} |A(x, -\omega^2)|^2 \, dM(x) + \int_0^{x_n} |B(x, \omega)|^2 \, dx \Big]. \end{aligned}$$

□

A method to modify a short string is suggested in the following lemma.

Lemma 3.21 *Let $r > 0$ be fixed and set*

$$\tilde{\Delta}([0, \xi]) = \int_{0-}^{\xi} \frac{1}{r + (\xi')^2} \, d\Delta(\xi').$$

Let $\tilde{l}$ and $\tilde{M}$ be the length and the mass function which are defined via the polynomials $\tilde{A}_0, \tilde{B}_1, \tilde{A}_2, \ldots$ in $\mathbf{Z}(\tilde{\Delta})$. Then, $\tilde{l} + \tilde{M}(\tilde{l}-) = \infty$.

Proof. We claim that we can always find a polynomial $p \in \mathbf{Z}(\tilde{\Delta})$ such that $\|p\|^2_{\tilde{\Delta}}$ is arbitrarily small and $|p(i\sqrt{r})| = 1$. For $\xi' \in \mathbb{R}$, we have

$$|\xi' - i\sqrt{r}|^2 = (\xi')^2 + r.$$

The length of p in $\mathbf{Z}(\tilde{\Delta})$ is

$$\begin{aligned}\|p\|^2_{\tilde{\Delta}} &= \int_{-\infty}^{\infty} |p(\xi)|^2 \, d\tilde{\Delta}(\xi) = \int_{-\infty}^{\infty} \frac{|p(\xi')|^2}{r + (\xi')^2} \, d\Delta(\xi') \\ &= \int_{-\infty}^{\infty} \left| \frac{1}{\xi' - i\sqrt{r}} + \frac{p(\xi') - 1}{\xi' - i\sqrt{r}} \right|^2 d\Delta(\xi').\end{aligned}$$

Set $p(\xi') = (\xi' - i\sqrt{r})q(\xi') + 1$, where q is a polynomial in $\mathbf{Z}(\Delta)$. It is obvious that $(\xi' - i\sqrt{r})^{-1} \in \mathbf{Z}(\Delta)$ because $\int_{-\infty}^{\infty} (r + (\xi')^2)^{-1} \, d\Delta(\xi') < \infty$. Thus, we can approximate $-(\xi' - i\sqrt{r})^{-1}$ by a polynomial $q(\xi')$. The claim is established.

Lemma 3.20 gives us

$$1 = |p(i\sqrt{r})|^2 \leqslant \|p\|^2_{\tilde{\Delta}} \Big[\int_{0-}^{\tilde{l}} |\tilde{A}(x, r)|^2 \, d\tilde{M}(x) + \int_{0}^{\tilde{l}} |\tilde{B}(x, i\sqrt{r})|^2 \, dx \Big].$$

Since $\|p\|^2_{\tilde{\Delta}}$ can be made arbitrarily small, the estimate cannot hold unless we have $\tilde{l} + \tilde{M}(\tilde{l}-) = \infty$. □

Proof of Theorem 3.17 for $l + M(l-) < \infty$. Fix $r = 1$. Let Δ_k^* denote the principal spectral measure of the short string (l, M, k), as given in Theorem 3.17. For short strings in general, it is easily seen that

$$D_k(0, 1) = \frac{kC'_+(l, 1) + C(l, 1)}{kA'_+(l, 1) + A(l, 1)},$$

using that all functions in $\mathbf{D}_\tau$ have linear extensions to the right of $x = l$ in the short-string case, and the definition of $D(x, r)$. Note that $C(x, 1)$ is of class $\mathbf{D}_\tau$ and it neither belongs to $\mathbf{D}_-(\tau)$ or $\mathbf{D}_+(\tau)$. Hence, $kC'_+(l, 1) + C(l, 1)$ is non-vanishing for all k. Neither can $kA'_+(l, 1) + A(l, 1)$ because $A \notin \mathbf{D}_+(\tau)$ for $r > 0$ according to Lemma 2.6. By Lemma 3.19, we have

$$\lim_{x \uparrow l} \frac{C(x, 1)}{A(x, 1)} \leqslant \int_{-\infty}^{\infty} \frac{d\Delta(\xi)}{1 + \xi^2} \leqslant \lim_{x \uparrow l} \frac{C'_+(x, 1)}{A'_+(x, 1)}.$$

For fixed $r > 0$, $D_k(0, r)$ is a real continuous function in the parameter k. For $k = 0$ and $k = \infty$, we have

$$D_0(0, 1) = \lim_{x \uparrow l} \frac{C(x, 1)}{A(x, 1)}, \quad \text{and} \quad D_\infty(0, 1) = \lim_{x \uparrow l} \frac{C'_+(x, 1)}{A'_+(x, 1)}.$$

Hence, we can choose k such that

$$\int_{-\infty}^{\infty} \frac{d\Delta_k^*(\xi)}{1 + \xi^2} = D_k(0, 1) = \int_{-\infty}^{\infty} \frac{d\Delta(\xi)}{1 + \xi^2}.$$

Set

$$d\tilde{\Delta}_k^*(\xi) := (1 + \xi^2)^{-1} d\Delta_k^*(\xi), \quad \text{and} \quad d\tilde{\Delta}(\xi) := (1 + \xi^2)^{-1} d\Delta(\xi).$$

Then, $\tilde{\Delta}_k^*$ and $\tilde{\Delta}$ have matching moments

$$\int_{-\infty}^{\infty} \xi^{2n}\, d\tilde{\Delta}_k^*(\xi) = \int_{-\infty}^{\infty} \xi^{2n}\, d\tilde{\Delta}(\xi).$$

Indeed, we can write

$$\begin{aligned}\int_{-\infty}^{\infty} \xi^{2n}\, d\tilde{\Delta}_k^*(\xi) &= \int_{-\infty}^{\infty} \frac{\xi^{2n}}{1+\xi^2}\, d\Delta_k^*(\xi) \\ &= \int_{-\infty}^{\infty} \frac{\xi^{2n}-(-1)^n}{1+\xi^2}\, d\Delta_k^*(\xi) + \int_{-\infty}^{\infty} \frac{(-1)^n}{1+\xi^2}\, d\Delta_k^*(\xi).\end{aligned}$$

Observe that $\xi^{2n} - (-1)^n$ has a zero $\xi^2 = -1$. Hence, we have the factorization

$$\xi^{2n} - (-1)^n = (1+\xi^2)\, Q(\xi^2),$$

where $Q(\xi^2)$ is a polynomial in ξ^2. Now, the integral becomes

$$\begin{aligned}\int_{-\infty}^{\infty} \xi^{2n}\, d\tilde{\Delta}_k^*(\xi) &= \int_{-\infty}^{\infty} Q(\xi^2)\, d\Delta_k^*(\xi) + \int_{-\infty}^{\infty} \frac{(-1)^n}{1+\xi^2}\, d\Delta_k^*(\xi) \\ &= \int_{-\infty}^{\infty} Q(\xi^2)\, d\Delta(\xi) + \int_{-\infty}^{\infty} \frac{(-1)^n}{1+\xi^2}\, d\Delta(\xi) \\ &= \int_{-\infty}^{\infty} \frac{\xi^{2n}}{1+\xi^2}\, d\Delta(\xi) \\ &= \int_{-\infty}^{\infty} \xi^{2n}\, d\tilde{\Delta}(\xi).\end{aligned}$$

In the second line, we use that Δ and Δ_k^* have matching moments. In the third line, we reason backwards to return to $\xi^{2n}/(1+\xi^2)$. The claim is now established.

All powers have finite norm with respect to the measure $\tilde{\Delta}$ and the span of these is dense in the space $\mathbf{Z}(\tilde{\Delta})$. By Lemma 3.21, the string $(\tilde{l}, \tilde{M})$ defined by $\tilde{\Delta}$ satisfies $\tilde{l} + \tilde{M}(\tilde{l}-) = \infty$. So, Theorem 3.17 for the long-string case applies and $\tilde{\Delta} = \tilde{\Delta}_k^*$. Thus, $\Delta = \Delta_k^*$.

Lastly, we prove that k can only be either 0 or ∞. Assume the contrary and consider the odd transform of $\mathbb{1}_{[l,l+k]}(x) \in \mathbf{S} \subset \mathbf{L}^2([0, l+k], dx)$ for $k \in (0, \infty)$:

$$\int_l^{l+k} B(x,\xi)\, dx = B(l,\xi)\cdot k \quad \in \quad \mathbf{Z}_o(\Delta_k^*)$$

The function $\mathbb{1}_{[l,l+k]}(x)$ is non-zero whenever $k > 0$. Since the odd transform is an isomorphism, $\mathbb{1}_{[l,l+k]}(x)$ cannot be mapped to $0 \in \mathbf{Z}_o(\Delta_k^*)$. Hence, $B(l,\xi) \not\equiv 0$. However, by construction, $B(l,\xi)$ lies in the orthogonal complement of $\operatorname{span}\big(\xi^n\big)_{n\in\mathbb{N}} = \mathbf{Z}(\Delta)$, which only happens when $B(l,\xi) \equiv 0$. Hence, k can only take value 0 or ∞, depending on whether $\int_{-\infty}^{\infty} \xi^{-2}\, d\Delta(\xi)$ is finite or not, according to (3.13)

$$\int_{-\infty}^{\infty} \xi^{-2}\, d\Delta(\xi) = l + k.$$

$\square$

3.2.3 Existence of (l, M, k) for general Δ

Theorem 3.22 *Let Δ be any principal spectral measure. Then, there exists a string (l, M, k), for which Δ is the associated principal spectral measure.*

Remark 3.23. There is a shorter proof of the fact that $M_j(x) \to M(x)$ as $j \uparrow \infty$ on the set of continuity points of M implies that $A_j(x,r) \to A(x,r)$ pointwise for each $r > 0$. The idea is to show that A_j are equicontinuous and uniformly bounded in j, using (2.11). Then, the Arzelà-Ascoli Theorem gives a subsequence A_{j_k} which converges to A^* uniformly. Using uniform convergence, A^* is also a solution and using uniqueness of a solution, A^* must equal A. This proof is found in the proof of Theorem 1, in [8].

Proof. The measure Δ is the strong limit of the sequence $\{\Delta_n\}_{n=1}^{\infty}$ given by

$$\Delta_n(B) = \Delta(B \cap (-n, n)),$$

and Δ_n is 0 on $\mathbb{R}\backslash(-n,n)$. For each n, the space $\mathbf{Z}(\Delta_n) = \mathbf{L}^2(\mathbb{R}, d\Delta_n)$ is spanned by the powers $\{\xi^j\}_{j=0}^{\infty}$. By Theorem 3.17, Δ_n defines a string (l_n, M_n, k_n). Motivated by Corollary 3.12, we must have

$$l + k = \int_{-\infty}^{\infty} \xi^{-2}\, d\Delta(\xi), \tag{3.19}$$

which exists, and

$$l + k = \lim_{n\uparrow\infty} \int_{-\infty}^{\infty} \xi^{-2}\, d\Delta_n(\xi) = \lim_{n\uparrow\infty} \big(l_n + k_n\big).$$

Next, we claim that there exists a subsequence $\{n_j\}_{j=0}^{\infty}$, for which

$$\lim_{j\uparrow\infty} M_{n_j}$$

is convergent pointwise to a mass function on the set of continuity points at least. This limit will be our candidate for the mass function $M(x)$.

Fix x and δ so that $0 < x < l + k$ and $0 < x + \delta < l + k$. For N large enough, we have

$$x + \delta < l_n + k_n = \int_{-\infty}^{\infty} \frac{d\Delta_n(\xi)}{\xi^2}, \qquad \text{for all } n \geqslant N.$$

The above motivates a lower bound for $D_n(0,r)$, when $r > 0$ is small enough and $n \geqslant N$

$$x + \delta < \int_{-\infty}^{\infty} \frac{d\Delta_n(\xi)}{r + \xi^2} = D_n(0,r),$$

from which we obtain an upper bound for $M_n(x)$, when $0 < x < l + k$ is fixed:

$$\begin{aligned} 1 &\geqslant r\mathfrak{G}_r^{(n)} 1 \geqslant r \int_{0-}^{x} G_r^{(n)}(x,y)\, dM_n(y) \\ &\geqslant r D_n(x,r) \int_{0-}^{x} dM_n(y) \\ &\geqslant r\left(D_n(0,r) - x\right) M_n(x) \\ &\geqslant r\,\delta\, M_n(x). \end{aligned}$$

In the first line, we use Lemma 2.11. In the third line, we use the estimate

$$D_n(x,r) = D_n(0,r) - x + r\int_0^x d\xi \int_{0-}^{\xi} D_n(\eta,r)\, dM_n(\eta) \geqslant D_n(0,r) - x,$$

because $D_n \geqslant 0$. The last estimate is from the lower bound for $D_n(0,r)$. We have thus shown that for each $0 < x < l+k$, the sequence $\{M_n(x)\}_{n=N}^{\infty}$ is uniformly bounded in n. According to Helly's Selection Theorem, there is a non-negative, non-decreasing and right-continuous function M, and a subsequence $\{M_{n_j}\}_{j=0}^{\infty}$ such that

$$\lim_{j\uparrow\infty} M_{n_j}(x) = M(x)$$

on the set of continuity points $x < l+k$ of M, see [17, pp.319–320]. For simplicity, the index n will be omitted and the limit of M_j will be referred to as M, with the understanding that equality holds pointwise everywhere except at the jumps of M.

We need to check that $\lim_{j\uparrow\infty} M_j$ really is a mass function, that is, it also satisfies that $x = 0$ is its growth point. Note that there are no negative growth points because $M_j(x) \equiv 0$ for all $x < 0$ and for all j. Assume $x = 0$ is not a growth point of M. Then there is an interval $[0,\epsilon)$, such that $M(x) \equiv 0$ when $0 \leqslant x < \epsilon$. Applying (2.11), we obtain for each j and $\xi \in \mathbb{R}$ the upper bound

$$A_j(x,-\xi^2) \leqslant \exp\Big[\xi^2\, x\, M_j(x)\Big], \qquad \text{for } x + M_j(x) < \infty.$$

Hence, for $0 < x < \epsilon$, we have

$$\big|A_j(x,-\xi^2) - 1\big| \leqslant \xi^2\, x\, M_j(x) \exp\big[\xi^2 x M_j(x)\big] \to \xi^2 x\, M(x) \exp\big[\xi^2 x\, M(x)\big] = 0,$$

as $j \uparrow \infty$, which gives rise to the following contradiction

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{d\Delta(\xi)}{r+\xi^2} &= \lim_{j\uparrow\infty} \int_{-\infty}^{\infty} \frac{d\Delta_j(\xi)}{r+\xi^2} \\
&\leqslant \lim_{j\uparrow\infty} \int_{-\infty}^{\infty} \frac{\big[A_j(x,-\xi^2)\big]^2}{r+\xi^2}\, d\Delta_j(\xi) \\
&= \lim_{j\uparrow\infty} G_r^{(j)}(x,x) \\
&= \lim_{j\uparrow\infty} A_j(x,r) D_j(x,r) \\
&= \lim_{j\uparrow\infty} D_j(x,r) \\
&= \lim_{j\uparrow\infty} A_j(x,r) \int_x^{l_j+k_j} \big[A_j(y,r)\big]^{-2}\, dy \\
&= \lim_{j\uparrow\infty} \int_x^{l_j+k_j} \big[A_j(y,r)\big]^{-2}\, dy \\
&= \lim_{j\uparrow\infty} \Big[D_j(0,r) - \int_0^x \big[A_j(y,r)\big]^{-2}\, dy\Big] \\
&= \lim_{j\uparrow\infty} D_j(0,r) - \int_0^x dy \\
&= \int_{-\infty}^{\infty} \frac{d\Delta(\xi)}{r+\xi^2} - x.
\end{aligned}$$

So, $x = 0$ must be the first point of growth of M. To complete the definition of the string, we define

$$l := \sup\{x \leqslant l + k \,|\, x \text{ is a growth point of } M(x)\}.$$

The string (l, M, k) is now well-defined. It remains to show that Δ is the principal spectral measure of (l, M, k), which means

$$D(0, r) = \int_{-\infty}^{\infty} \frac{d\Delta(\xi^2)}{r + \xi^2},$$

or equivalently,

$$\int_0^{l+k} \big[A(y, r)\big]^{-2}\, dy = \lim_{j\uparrow\infty} \int_0^{l_j+k_j} \big[A_j(y, r)\big]^{-2}\, dy,$$

where $A(x, r)$ is the solution corresponding to (l, M, k), and $A_j(x, r)$ to (l_j, M_j, k_j). We want to show that

$$\lim_{j\uparrow\infty} A_j(x, r) = A(x, r),$$

for each fixed $r > 0$. To do so, we use the power series of $A_j(x, r)$ and $A(x, r)$:

$$\begin{aligned}\lim_{j\uparrow\infty} |A_j(x, r) - A(x, r)| &\leqslant \lim_{j\uparrow\infty} \sum_{n=0}^{\infty} r^n\, |p_n^{(j)}(x) - p_n(x)| \\ &= \sum_{n=0}^{\infty} r^n \Big(\lim_{j\uparrow\infty} |p_n^{(j)}(x) - p_n(x)| \Big),\end{aligned}$$

where we use (2.10) on $p_n^{(j)}$ for each fixed $x \in [0, l + k)$ and n:

$$0 \leqslant p_n^{(j)}(x) \leqslant \frac{1}{n!} \Big[\int_0^x M_j(y)\, dy \Big]^n$$

and the fact that M_j are uniformly bounded in j for each $0 \leqslant x < l + k$ to justify the interchange of the sum and the limit in the second line. This also shows that the functions $p_n^{(j)}$ are uniformly bounded in j for each x and n. Now, we need to prove that

$$\lim_{j\uparrow\infty} |p_n^{(j)}(x) - p_n(x)| = 0, \qquad \text{for } n = 0, 1, \ldots,$$

and for every $x \in [0, l + k)$. For $n = 0$, we have $p_0(x) = p_0^{(j)}(x) \equiv 1$ for all j. Suppose that $\lim_j |p_n^{(j)}(x) - p_n(x)| = 0$ for some n and for all $x \in [0, l + k)$. Let $x \in [0, l + k)$ be fixed. A consequence of the uniform boundedness of $\{p_n^{(j)}(x)\}_j$ in j is the estimate

$$\int_{0-}^{\xi} |p_n^{(j)}(\eta)|\, dM_j(\eta) \leqslant \frac{1}{n!} \xi^n M_j(\xi)^{n+1} \leqslant \frac{1}{n!} \xi^n\, C, \qquad \xi \in [0, x],$$

in which the upper bound is of class $\mathbf{L}^1([0, x], dx)$. By the Dominated Convergence Theorem, we have

$$\lim_{j\uparrow\infty} |p_{n+1}^{(j)}(x) - p_{n+1}(x)| \leqslant \int_0^x d\xi \lim_{j\uparrow\infty} \Big| \int_{0-}^{\xi} p_n^{(j)}(\eta)\, dM_j(\eta) - \int_{0-}^{\xi} p_n(\eta)\, dM(\eta) \Big|,$$

and now

$$
\begin{aligned}
&\lim_{j\uparrow\infty}\Big|\int_{0-}^{\xi} p_n^{(j)}(\eta)\,dM_j(\eta)-\int_{0-}^{\xi} p_n(\eta)\,dM(\eta)\Big|\\
&\leqslant \lim_{j\uparrow\infty}\Big|\int_{0-}^{\xi} p_n(\eta)\,dM_j(\eta)-\int_{0-}^{\xi} p_n(\eta)\,dM(\eta)\Big| + \lim_{j\uparrow\infty}\int_{0-}^{\xi} |p_n^{(j)}(\eta)-p_n(\eta)|\,dM_j(\eta)\\
&\leqslant \lim_{j\uparrow\infty}\Big|\int_{0-}^{\xi} p_n(\eta)\,dM_j(\eta)-\int_{0-}^{\xi} p_n(\eta)\,dM(\eta)\Big| + \lim_{j\uparrow\infty}\Big(\sup_{\eta\in[0,\xi]}|p_n^{(j)}(\eta)-p_n(\eta)|\,M_j(\xi)\Big),
\end{aligned}
$$

for all $0 \leqslant \xi \leqslant x$. The first limit is 0 due to

$$\lim_{j\uparrow\infty} \varrho_j = \varrho$$

in the weak sense, which is a consequence of the convergence of M_j to M in the aforementioned way, see [17, pp.310, 314]. Here, the measures ϱ_j correspond to M_j, and ϱ to M. If $p_n^{(j)}$ converges uniformly to p_n on $[0,x]$ for each fixed $x \in [0,l+k)$, the second limit is 0, and the pointwise convergence of $p_{n+1}^{(j)}$ for each $x \in [0,l+k)$ is clear. In the following, we show that this is indeed the case. More specifically, we claim that the conditions

(i) $\{p_n^{(j)}\}_j$ is a sequence of non-decreasing continuous functions,

(ii) the pointwise limit in j of $p_n^{(j)}$ is continuous on $[0,x]$,

together imply that $p_n^{(j)}$ converges uniformly in j on the interval $[0,x]$.

Since the limit p_n is continuous on $[0,l+k)$, it is uniformly continuous on $[0,x]$. Hence, for each fixed $\epsilon > 0$, there exists a partition

$$0 = t_0 < t_1 < t_2 < \ldots < t_K = x$$

of $[0,x]$, for which

$$\max_{t\in[t_i,t_{i+1}]} p_n(t) - \min_{t\in[t_i,t_{i+1}]} p_n(t) < \epsilon, \tag{3.20}$$

for all $i = 0,1,\ldots,K-1$. By (ii), we have for this ϵ a natural number J, such that

$$p_n(t_i) - \epsilon \leqslant p_n^{(j)}(t_i) \leqslant p_n(t_i) + \epsilon, \qquad \text{for all } j > J, \tag{3.21}$$

for all $i = 0,1,\ldots,K-1$. By (i), we have

$$p_n^{(j)}(t_i) \leqslant p_n^{(j)}(t) \leqslant p_n^{(j)}(t_{i+1}), \qquad t \in [t_i, t_{i+1}].$$

Together with (3.21), we obtain new bounds for $p_n^{(j)}(t)$ on $[t_i,t_{i+1}]$:

$$p_n(t_i) - \epsilon < p_n^{(j)}(t) < p_n(t_{i+1}) + \epsilon, \qquad \text{for all } j > J.$$

Finally, together with (3.20), we arrive at the bounds

$$p_n(t) - 2\epsilon < p_n^{(j)}(t) < p_n(t) + 2\epsilon, \qquad \text{for all } j > J,\ t \in [t_i,t_{i+1}],$$

for all $i = 0,1,\ldots,K-1$. This proves the claim.

Now that the convergence of $A_j(x,r)$ to $A(x,r)$ as $j \to \infty$ for each fixed $x \in [0, l+k)$ is proved, the next observation is that $\big[A_j(x,r)\big]^{-2}$ is dominated by an $\mathbf{L}^1([0, l+k], dx)$ function, which is shown by manipulating the lower bound of $A_j(x,r)$:

$$\begin{aligned}
\big[A_j(x,r)\big]^2 &\geqslant \Big[1 + r\int_0^x M_j(y)\,dy\Big]^2 \\
&\geqslant \big[1 + r\,M_j(\epsilon)(x-\epsilon)\big]^2 \\
&\geqslant C'[1 + (x-\epsilon)]^2,
\end{aligned}$$

where $0 < \epsilon \leqslant x < l+k$, for some fixed ϵ. In the second line, it is used that $M_j(\epsilon)$ is bounded away from 0 uniformly in j. For $0 < x < \epsilon$, we can use that $A_j \geqslant 1$ for j. Applying the Dominated Convergence Theorem, we arrive at the desired conclusion. $\square$

3.3 Examples of Krein's correspondence

In this section, given an $\mathbb{S}$-class function h, the goal is to extract as much information as possible about the string associated to h. First, we list some standard results – some are simply the byproducts of our previous constructions and some are stated without proofs due to their level of difficulty. Next, we restrict our attention to meromorphic $\mathbb{S}$-functions h, which are known to yield for example short strings, strings with compact resolvent and the so-called Stieltjes strings – a finite version of which appears in Theorem 3.15. For the non-meromorphic $\mathbb{S}$-functions, we give two simple examples of strings which can be related to a Sturm-Liouville operator and in the last example we examine the spectrum of a short string, whose mass function has a discontinuity. In Discussion, we discuss how to develop interesting examples further, and which parts of this theory we would like to learn more in order to understand Krein's correspondence better.

3.3.1 General properties of Krein's correspondence

According to Theorem 3.4, any $\mathbb{S}$-function is related uniquely to a non-negative Borel measure σ with $\int_{0-}^{\infty}(1+\gamma)^{-1}\,d\sigma(\gamma) < \infty$. Since the coming results are formulated using σ rather than h, a way to compute σ from h is of some relevance. It is clear that $h(-z)$ is a Herglotz-Nevanlinna function whenever $h(z)$ is a Stieltjes function. Consequently, Stieltjes' inversion formula for Herglotz-Nevanlinna functions gives

$$\sigma((\gamma_1,\gamma_2]) = \lim_{\delta\downarrow 0}\lim_{\epsilon\downarrow 0}\frac{1}{\pi}\int_{\gamma_1+\delta}^{\gamma_2+\delta}\operatorname{Im}(h(-\gamma-i\epsilon))d\gamma, \qquad \gamma_1<\gamma_2. \tag{3.22}$$

Let h belong to the string (l,M), with the differential operator $(\tau,\mathbf{D}(\tau))$ associated to the tying constant k, and the fundamental system $\{A(x,z),C(x,z)\}$. It is known from before that $D(0,z)=h(z)$ is given by

$$h(z) = \begin{cases} \lim_{x\uparrow l}\dfrac{C(x,z)}{A(x,z)}, & \text{if } l+M(l-)=\infty,\\ \dfrac{C(l,z)+k\,C'_+(l,z)}{A(l,z)+k\,A'_+(l,z)}, & \text{if } l+M(l-)<\infty, \end{cases} \tag{3.23}$$

with the convention

$$h(z) = \frac{C'_+(l,z)}{A'_+(l,z)}, \qquad \text{when } k=\infty.$$

From Definition 3.2, $h(z)$ is analytic everywhere in $\mathbb{C}\backslash(-\infty,0]$. In fact, it can be analytically extended to a larger set, namely $\mathbb{C}\backslash -\operatorname{supp}(\sigma)$, where

$$-\operatorname{supp}(\sigma) = \{-r \mid r\in\operatorname{supp}(\sigma)\}.$$

More is true. Let $\mathbf{D}_h$ denotes the set where h can be analytically extended, $\Sigma(\tau)$ the spectrum of the operator $(\tau,\mathbf{D}(\tau))$ and σ the spectral measure of $(\tau,\mathbf{D}(\tau))$. Then, we have

$$\Sigma(\tau) = -\operatorname{supp}(\sigma) = \mathbb{C}\backslash\mathbf{D}_h.$$

The first equality is a consequence of the spectral theorem for unbounded operators, see Lemma 3.12 in [18]. For the second equality, it is not hard to realize that $-\operatorname{supp}(\sigma)\subset\mathbb{C}\backslash\mathbf{D}_h$. Indeed, suppose that the Herglotz–Nevanlinna function $m(z)=h(-z)$ is analytic at $z=z_0\in\mathbb{R}$, and hence in a neighborhood U_{z_0} of the point z_0. Then, there is a small interval $(a,b)\subset U_{z_0}\cap\mathbb{R}$, such that $z_0\in(a,b)$ and $m(x^*)=m(x)^*$ for all $x\in(a,b)$, which implies that $\mathrm{Im}(m(x)) = 0$ on (a,b). It follows from (3.22) that $\sigma((a,b)) = 0$, hence $z_0 \notin \mathrm{supp}(\sigma)$. For a proof of the other inclusion, see Theorem 3.10 in [18]. Hence, to identify the spectrum of τ, we identify the subset of $(-\infty, 0]$, on which h cannot be analytically extended, that is $\mathbb{C}\backslash\mathbf{D}_h$, or $-\mathrm{supp}(\sigma)$.

The following lemma collects some properties of the string belonging to h.

Lemma 3.24 *Let $h(z) \in \mathbb{S}$ be associated with the measure σ. Then,*

(1)

$$\int_{0-}^{\infty} \frac{d\sigma(\gamma)}{\gamma} = \begin{cases} l, & \text{if } h(z) \text{ belongs to a long string} \\ l+k, & \text{if } h(z) \text{ belongs to a short string} \end{cases},$$

(2) h belongs to a string (l, M) with a jump at the origin of the amount

$$\varrho(\{0\}) = \frac{1}{\sigma([0,\infty))},$$

(3) $h(z)$ belongs to a string with the total mass

$$M(l) = \frac{1}{\sigma(\{0\})},$$

if $\sigma(\{0\}) > 0$. Hence, h either belongs to a long string with $M(l) < \infty$, or a short string with the tying constant $k = \infty$.

Proof. Item (1) is item (3) in Corollary 3.12 after changing the measure Δ to σ. Item (2) is due to the fact that $A(0,z) = 1$ for all $z \in \mathbb{C}$. By item (1) in Corollary 3.12, we have

$$\varrho(\{0\})^{-1} = \|A(0,-\xi^2)\|_{\Delta}^2 = \int_{-\infty}^{\infty} 1\, d\Delta(\xi) = \int_{0-}^{\infty} 1\, d\sigma(\gamma) = \sigma([0,\infty)).$$

If $\sigma(\{0\}) > 0$, then $\sigma(\{0\}) = \Delta(\{0\}) > 0$. We apply the inverse even transform on the function $\mathbb{1}_{\{0\}}(\xi) \in \mathbf{Z}_e(\Delta)$ and then use the Plancherel identity:

$$\begin{aligned} \Delta(\{0\}) = \int_{-\infty}^{\infty} \mathbb{1}_{\{0\}}(\xi)^2\, d\Delta(\xi) &= \int_{0-}^{l} \left(\int_{-\infty}^{\infty} A(x,-\xi^2)\mathbb{1}_{\{0\}}(\xi)\, d\Delta(\xi)\right)^2 dM(x) \\ &= \int_{0-}^{l} A(x,0)^2\, \Delta(\{0\})^2\, dM(x). \end{aligned}$$

It is easily seen that $A(x,0) \equiv 1$ from (2.9). After rearrangements, we find that

$$\frac{1}{\sigma(\{0\})} = \frac{1}{\Delta(\{0\})} = \int_{0-}^{l} 1\, dM(x) = M(l).$$

Since $\sigma(\{0\}) > 0$, $M(l)$ is finite. If h belongs to a short string, then by item (1) of this lemma, we have

$$l + k = \int_{0-}^{\infty} \gamma^{-1}\, d\sigma(\gamma) \geqslant \lim_{\gamma\downarrow 0} \frac{\sigma(\{0\})}{\gamma} = \infty.$$

□

The following proposition presents a connection between the asymptotic behavior of the Weyl function $m(z) = h(-z)$ and the behavior of $M(x)$ close to $x = 0$.

Proposition 3.25 *For a given $\alpha \in (0, \infty)$, there is a constant $C > 0$ such that*

$$m(z) = C^{-1}C'(-z)^{\frac{-1}{1+\alpha}}(1 + o(1))^{-1}, \tag{3.24}$$

with

$$C' = \frac{\Gamma\left(\frac{\alpha+2}{\alpha+1}\right)}{\Gamma\left(\frac{\alpha}{\alpha+1}\right)}\left(\frac{(\alpha+1)^2}{\alpha}\right)^{\frac{1}{\alpha+1}}$$

as $z \to \infty$ uniformly in any non-real sector, if and only if

$$M(x) = C^{1+\alpha}x^{\alpha}(1 + o(1)), \quad x \to 0. \tag{3.25}$$

Remark 3.26. For a proof, see Theorem 3 in [8] or Theorem 4.1 in [2]. This proposition will be illustrated in Example 3.35.

3.3.2 Meromorphic $\mathbb{S}$-functions

An important subclass of $\mathbb{S}$-functions is the meromorphic $\mathbb{S}$-functions. These are analytic everywhere in $\mathbb{C}$, except at a countable sequence of isolated points

$$0 \geqslant -r_0 > -r_1 > \ldots > -r_n > \ldots, \quad \text{where } r_n > 0. \tag{3.26}$$

This implies that the associated measure σ is of the form $\sum_{n=0}^{\infty} a_n \delta_{r_n}$. Using Theorem 3.4, we have

$$h(z) = \int_{0-}^{\infty} \frac{d\sigma(\gamma)}{\gamma + z} = \frac{\sigma(\{r_0\})}{r_0 + z} + \frac{\sigma(\{r_1\})}{r_1 + z} + \ldots + \frac{\sigma(\{r_n\})}{r_n + z} + \ldots,$$

which offers a simpler method to compute σ from h than (3.22), namely,

$$\sigma(\{r_n\}) = \lim_{r \to r_n} (r_n - r)\, h(-r), \quad \text{for } r > 0. \tag{3.27}$$

Here are some remarkable results which connect properties of meromorphic $\mathbb{S}$-functions to properties of their strings.

Proposition 3.27

(1) Suppose (l, M) is a long string. Then, the associated $\mathbb{S}$-function h is meromorphic if and only if one of the following holds true

$$\lim_{x \uparrow l} x\big(M(l) - M(x)\big) = 0, \quad \text{if } l = \infty \tag{3.28}$$

$$\lim_{x \uparrow l} M(x)\big(l - x\big) = 0, \quad \text{if } M(l) = \infty. \tag{3.29}$$

(2) h belongs to a string with either

$$\int_{0-}^{l} x\, dM(x) < \infty, \quad \text{or} \quad \int_{0}^{l} M(x)\, dx < \infty \tag{3.30}$$

if and only if h is meromorphic and that it has poles at the sequence (3.26) with the property

$$\sum_{n=1}^{\infty} \frac{1}{r_n} < \infty.$$

(3) Suppose h belongs to a short string. If h is non-rational, then the asymptotic behavior of the sequence $\{r_n\}_{n=0}^{\infty}$ depends solely on the absolutely continuous part $M'(x)$ of $M(x)$ in the following way

$$\lim_{n\to\infty}\frac{n}{\sqrt{r_n}}=\frac{1}{\pi}\int_0^l\sqrt{M'(x)}\,dx.$$

Remark 3.28. The proof of item (1) can be found in [11]. If the string is long, the conditions in (3.30) implies either (3.28) or (3.29), because for example

$$x\bigl(M(l)-M(x)\bigr)\leqslant\int_x^l y\,dM(y)\to 0,\qquad \text{as } x\to l,$$

due to $\int_{0-}^{l} x\,dM(x)<\infty$. The converse is not true. The proof of item (2) can be found in, for example, Chapter 5.6 in [4]. Note that if the string is short, both conditions (3.30) are satisfied. Item (3) is proven in [14] by Krein and this can be extended to the so-called Pontryagin class of strings, as shown in [20].

Next, the classical Stieltjes strings, from which meromorphic $\mathbb{S}$-functions can be obtained, are presented.

Example 3.29 (Stieltjes strings) A Stieltjes string (l,M) has a discrete measure as its density, that is,

$$M(x)=\sum_{x\geqslant x_j}\varrho_j,$$

where $0=x_0<x_1<x_2<\dots$, and $l=\lim_n x_n$. The differential equation $(\tau-z)f=0$ associated to a Stieltjes string is a difference equation

$$\begin{aligned} f'_+(x_j)-f'_-(x_j)&=\varrho_j\,z\,f(x_j),\\ f'_+(x)-f'_-(x)&=0, &&\text{if } x\notin\{x_0,x_1,\dots\}.\end{aligned}$$

We have earlier adopted the convention that f is linear on massless interval. Hence, with a Stieltjes string, f is piecewise linear on $\mathbb{R}$, thus $f'_+(x_{j-1})=f'_-(x_j)$ and

$$f(x_j)-f(x_{j-1})=f'_-(x_j)\,(x_j-x_{j-1}).$$

Let

$$\begin{aligned} A(x_j,z)&=A_{2j} &&\text{and} && A'_+(x_j,z)=A_{2j+1}\\ C(x_j,z)&=C_{2j} &&\text{and} && C'_+(x_j,z)=C_{2j+1}.\end{aligned}$$

By the above, the sequence $\{A_k\}_{k=0}^{\infty}$ satisfies the Wallis–Euler recurrence relations

$$\begin{aligned} A_{2j}&=(x_j-x_{j-1})\,A_{2j-1}+A_{2j-2},\\ A_{2j+1}&=\varrho_j\,z\,A_{2j}+A_{2j-1},\end{aligned}$$

for $j=1,2,\dots$, with $A_{-1}=0$ and $A_0=1$. Similarly, the sequence $\{C_k\}_{k=0}^{\infty}$ satisfies the above recurrence relation with $C_0=0$ and $C_1=1$. The quotients of the terms in these sequences can be written as continued fractions

$$\begin{aligned}\frac{C(x_j,z)}{A(x_j,z)}&=\frac{C_{2j}}{A_{2j}}=\frac{1\,|}{|\,\varrho_0 z}+\frac{1\,|}{|\,x_1-x_0}+\frac{1\,|}{|\,\varrho_1 z}+\dots+\frac{1\,|}{|\,x_j-x_{j-1}},\\ \frac{C'_+(x_j,z)}{A'_+(x_j,z)}&=\frac{C_{2j+1}}{A_{2j+1}}=\frac{1\,|}{|\,\varrho_0 z}+\frac{1\,|}{|\,x_1-x_0}+\frac{1\,|}{|\,\varrho_1 z}+\dots+\frac{1\,|}{|\,\varrho_j z}.\end{aligned}$$

According to (3.23), the corresponding $\mathbb{S}$-function is

$$h(z) = D(0,z) = \lim_{n\to\infty} \frac{1|}{|\varrho_0 z} + \frac{1|}{|x_1 - x_0} + \ldots + \frac{1|}{|\varrho_n z} + \frac{1|}{|x_{n+1} - x_n}, \tag{3.31}$$

in the long-string case, and

$$h_k(z) = D_k(0,z) = \lim_{n\to\infty} \left(\frac{1|}{|\varrho_0 z} + \frac{1|}{|x_1 - x_0} + \ldots + \frac{1|}{|\varrho_n z} + \frac{1|}{|k} \right), \tag{3.32}$$

in the short-string case with tying constant k, and

$$h_k(z) = \frac{1|}{|\varrho_0 z} + \frac{1|}{|x_1 - x_0} + \ldots + \frac{1|}{|\varrho_n z} + \frac{1|}{|k},$$

if M has finitely many jumps. Let $z = r > 0$. The convergence in (3.31) is guaranteed by the fact that for each fixed r, we have

$$\sum_{j=0}^{\infty} \varrho_j r + (x_{j+1} - x_j) = \infty,$$

as the string is long, and therefore

$$\frac{C_0}{A_0} < \frac{C_2}{A_2} < \ldots < \lim_{j\uparrow\infty} \frac{C_{2j}}{A_{2j}} = \lim_{j\uparrow\infty} \frac{C_{2j+1}}{A_{2j+1}} < \ldots < \frac{C_3}{A_3} < \frac{C_1}{A_1}.$$

The limit of the odd terms and the limit of the even terms coincide, as in the above, if and only if the string is long, see Theorem 30.1 in [19]. If the string is short, the even terms C_{2j}/A_{2j} still form a strictly increasing sequence and the odd terms C_{2j+1}/A_{2j+1} form a strictly decreasing sequence. Since the even terms are all bounded above by C_1/A_1, the limit exists for this sequence and similarly for the odd terms. However, the limits will never coincide. In fact, this illustrates Lemma 3.19

$$\frac{C(x,r)}{A(x,r)} \leqslant \int_{-\infty}^{\infty} \frac{d\Delta(\xi)}{r+\xi^2} \leqslant \frac{C'_+(x,r)}{A'_+(x+r)}, \qquad 0 < x < l.$$

In (3.31), the sequence

$$\frac{1|}{|\varrho_0 z} + \frac{1|}{|x_1 - x_0} + \ldots + \frac{1|}{|\varrho_n z} + \frac{1|}{|k}$$

is strictly decreasing and bounded below. Hence, the limit exists and it is exactly

$$\lim_{j\uparrow\infty} \frac{C_{2j-2} + kC_{2j-1}}{A_{2j-2} + kA_{2j-1}} = \lim_{j\uparrow\infty} \frac{C(x_{j-1},z) + kC'_+(x_{j-1},z)}{A(x_{j-1},z) + kA'_+(x_{j-1},z)} = D(0,z).$$

To learn more about the spectral measure σ, the even transform can be considered. Solving $A(x,-\xi^2)$ accordingly to the above difference equation, it is found that $A(x_j,-\xi^2)$ is a polynomial in ξ of order $2j$. From Corollary 3.12, we have

$$\|A(x_j,-\xi^2)\|_\Delta^2 = \varrho_j^{-1} < \infty.$$

This gives that

$$\int_{-\infty}^{\infty} \xi^{4j}\, d\Delta(\xi) < \infty, \qquad j = 0,1,2,\ldots.$$

A necessary condition for an $\mathbb{S}$-function $h(z)$ to yield a Stieltjes string is that its prinicipal spectral measure has finite even moments. Adding the requirement that the even polynomials are dense in $\mathbf{Z}_e(\Delta)$, we have the following correspondence result.

PROPOSITION 3.30 *Let $h(z)$ be an $\mathbb{S}$-function with the associated principal spectral measure Δ, such that dim $\mathbf{Z}_e(\Delta) = n \leqslant \infty$. Then, $h(z)$ belongs to a Stieltjes string, if and only if the even polynomials are dense in $\mathbf{Z}_e(\Delta)$, and all even moments are finite*

$$\int_{-\infty}^{\infty} \xi^{4j}\, d\Delta(\xi) < \infty,$$

for all $2j \leqslant n$ if n is finite, and $2j < n$ if $n = \infty$.

Remark 3.31. By changing back $\gamma = \xi^2$, it is seen that the space $\mathbf{Z}_e(\Delta)$ is in fact $\mathbf{L}^2(\mathbb{R}, d\sigma)$, and the requirement that even polynomials are dense in $\mathbf{Z}_e(\Delta)$ corresponds to the requirement that polynomials in γ are dense in $\mathbf{L}^2(\mathbb{R}, d\sigma)$. Note that in our previous construction in Section 3.2.2, we also assume that the odd polynomials in $\mathbf{Z}_o(\Delta)$, which leads to the limited choices of tying constants $k = 0$ or $k = \infty$. This proposition is proved in a different way in [11], which does not require denseness of odd polynomials in $\mathbf{Z}_o(\Delta)$. For further details, see Chapter 13 in [11].

If the Stieltjes string is short, it will certainly yield a meromorphic $\mathbb{S}$-function. However, not all Stieltjes strings automatically have a meromorphic $\mathbb{S}$-function.

COROLLARY 3.32 *Suppose (l, M) is a long Stieltjes string, that is,*

$$\sum_{n=0}^{\infty} \varrho_n = \infty \quad \textit{or} \quad \lim_{n\to\infty} x_n = \infty.$$

Then, it has a meromorphic $\mathbb{S}$-function $h(z)$, if and only if either of the following holds true:

$$\textit{if } \lim_{n\to\infty} x_n = \infty, \ \textit{then } \lim_{n\uparrow\infty} \Big(\sum_{j=0}^{n}(x_{j+1}-x_j)\Big)\Big(\sum_{k=n}^{\infty}\varrho_k\Big) = 0,$$
$$\textit{if } \sum_{n=0}^{\infty} \varrho_n = \infty, \ \textit{then } \lim_{n\uparrow\infty} \Big(\sum_{j=n}^{\infty}(x_{j+1}-x_j)\Big)\Big(\sum_{k=0}^{n}\varrho_k\Big) = 0.$$

Proof. This is a direct application of item (1) in Proposition 3.27. □

To close this example, we demonstrate item (3) in Proposition 3.27 by studying the behavior of the short Stieltjes string with

$$\varrho_j = 2^{-j}, \quad \text{and} \quad x_j - x_{j-1} = 2^{-j}, \quad j = 0, 1, \ldots$$

and tying constant $k = \infty$, which is clearly of the case $l + M(l-) < \infty$. Hence, the associated $\mathbb{S}$-function $h(z)$ is meromorphic and has the continued-fraction form

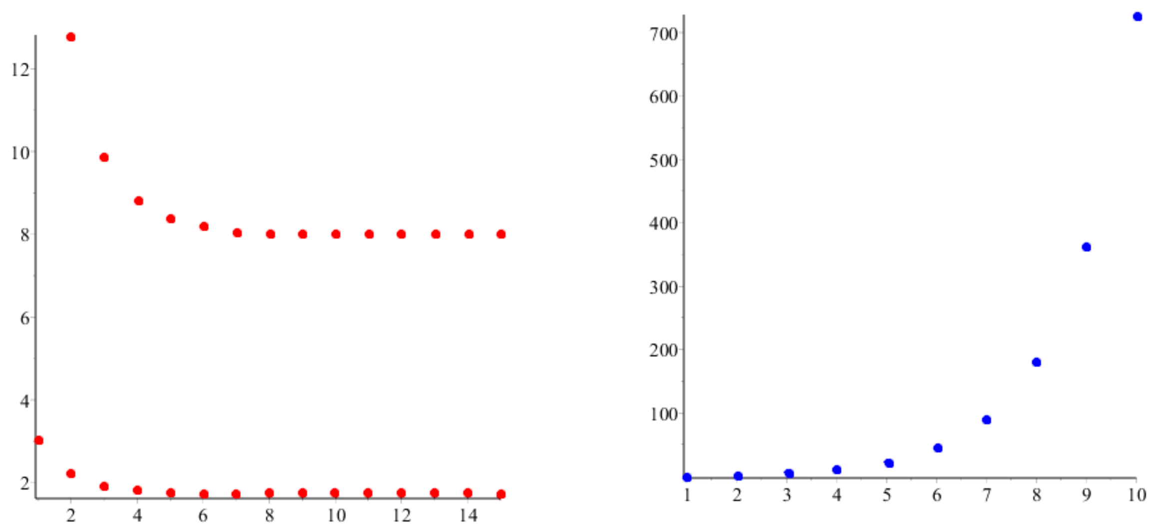
$$h(z) = \lim_{j\uparrow\infty} \left(\frac{1|}{|z} + \frac{1|}{|1} + \frac{1|}{|2^{-1}z} + \frac{1|}{|2^{-1}} + \ldots + \frac{1|}{|2^{-j}z} \right)$$

Since it is very difficult to locate the poles of $h(z)$ from its continued-fraction form, we compute the poles of the approximations

$$h^{(k)}(z) = \frac{1|}{|z} + \frac{1|}{|1} + \ldots + \frac{1|}{|2^{-k}z},$$

	$r_1^{(j)}$	$r_2^{(j)}$	$r_3^{(j)}$	$r_4^{(j)}$	$r_5^{(j)}$	$r_6^{(j)}$	$r_7^{(j)}$	$r_8^{(j)}$	$r_9^{(j)}$	$r_{10}^{(j)}$
$j=1$	3.									
$j=2$	2.19	12.82								
$j=3$	1.90	9.82	51.27							
$j=4$	1.79	8.81	39.31	205.09						
$j=5$	1.73	8.38	35.27	157.25	820.37					
$j=6$	1.71	8.18	33.56	141.10	628.99	3281.48				
$j=7$	1.69	8.08	32.76	134.23	564.38	2515.95	13125.90			
$j=8$	1.69	8.03	32.38	131.04	536.91	2257.54	10063.82	52503.60		
$j=9$	1.68	8.01	32.19	129.50	524.16	2147.63	9030.16	40255.28	$2.10 \cdot 10^5$	
$j=10$	1.68	8.00	32.09	128.75	518.01	2096.63	8590.51	36120.63	$1.61 \cdot 10^5$	$8.40 \cdot 10^5$
$j=11$	1.68	7.99	32.05	128.37	514.99	2072.03	8590.51	34362.02	$1.44 \cdot 10^5$	$6.44 \cdot 10^5$
$j=12$	1.68	7.99	32.02	128.19	513.49	2059.94	8590.51	33546.01	$1.37 \cdot 10^5$	$5.78 \cdot 10^5$
$j=13$	1.68	7.99	32.01	128.09	512.74	2053.95	8590.51	33152.43	$1.34 \cdot 10^5$	$5.50 \cdot 10^5$
$j=14$	1.68	7.99	32.01	128.05	512.37	2050.97	8590.51	32959.09	$1.33 \cdot 10^5$	$5.37 \cdot 10^5$
$j=15$	1.68	7.99	32.00	128.02	512.19	2049.49	8590.51	32863.27	$1.32 \cdot 10^5$	$5.30 \cdot 10^5$

Table 1: approximations of the first ten eigenvalues from $h^{(j)}(z)$ belonging to the Stieltjes string with $\varrho_j = 2^{-j}$ and $x_j - x_{j-1} = 2^{-j}$ for $j = 0, 1, \ldots, 15$.



(a) Convergence of the first two eigenvalues after 15 approximations.

(b) This plot illustrates the growth of the square roots of the first ten eigenvalues of the 15$^{\text{th}}$ approximation.

Figure 1: illustrations of eigenvalue approximations from Table 1.

using the software Maple. For an overview of the approximation of the first ten eigenvalues, see Table 1 and Figure 1b. We also include a graph over the convergence of the first two eigenvalues, see Figure 1a. Although one should be careful drawing conclusions from a finite number of eigenvalues, Figure 1b suggests that the square roots of the eigenvalues grow much faster than linearly, which is in line with item (3) of Proposition 3.27 where the absolutely continuous part is absent.

Example 3.33 In this example, the short string (l, M) with $l = 1$ and

$$M(x) = x \cdot \mathbb{1}_{[0,1]} + 1 \cdot \mathbb{1}_{(1,\infty)},$$

is taken into consideration. On the interval $[0, 1]$, the eigenequation is the differential equation

$$\left(\frac{d^2}{dx^2} - z\right) f = 0,$$

which has the general solution f with the linear extension to the left of $x = 0$ and to the right of $x = 1$

$$f(x) = \begin{cases} f(0) + f'_-(0)x, & x \in (-\infty, 0) \\ f(0)\cosh(\sqrt{z}\,x) + \frac{f'_+(0)}{\sqrt{z}}\sinh(\sqrt{z}\,x), & x \in [0, 1) \\ f(1) + f'_+(1)(x-1), & x \in [1, \infty). \end{cases}$$

Since M has no discontinuity at $x = 0$ or $x = 1$, we have $f'_-(0) = f'_+(0)$ and $f'_-(1) = f'_+(1)$. Plugging in the initial values of the fundamental system $\{A(x, z), C(x, z)\}$, we find that

$$h_k(z) = \frac{C(1, z) + k\, C'_+(1, z)}{A(1, z) + k\, A'_+(1, z)} = \frac{z^{-1/2}\sinh(\sqrt{z}) + k\, \cosh(\sqrt{z})}{\cosh(\sqrt{z}) + k\sqrt{z}\,\sinh(\sqrt{z})},$$

where $0 \leqslant k \leqslant \infty$. We want to compute and compare the spectral measures σ_0 and σ_∞ for $k = 0$ and $k = \infty$ respectively. The function h_0 is given by

$$h_0(-r) = \frac{1}{\sqrt{-r}}\frac{\sinh(\sqrt{-r})}{\cosh(\sqrt{-r})} = \frac{1}{i\sqrt{r}}\frac{i\,\sin(\sqrt{r})}{\cos(\sqrt{r})} = \frac{\tan(\sqrt{r})}{\sqrt{r}},$$

for $r > 0$. Since $\lim_{r\to 0} x^{-1}\tan(x) = 1$, there is no pole at $x = 0$. The poles are located at the sequence $\{-r_j\}_{j=0}^{\infty}$

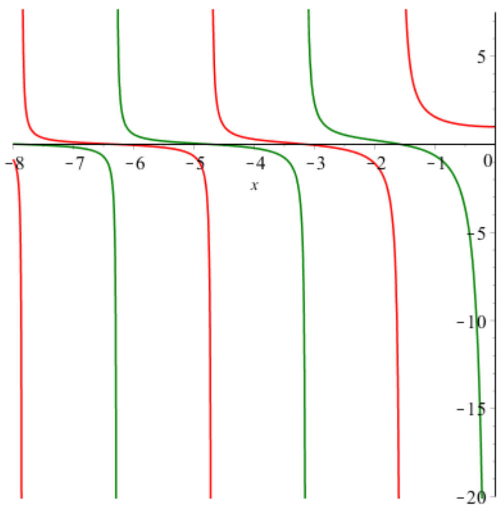
$$r_j = \left(j + \frac{1}{2}\right)^2 \pi^2, \qquad j = 0, 1, \ldots.$$

Employing (3.27), we find that the measure σ_0 jumps with the amount

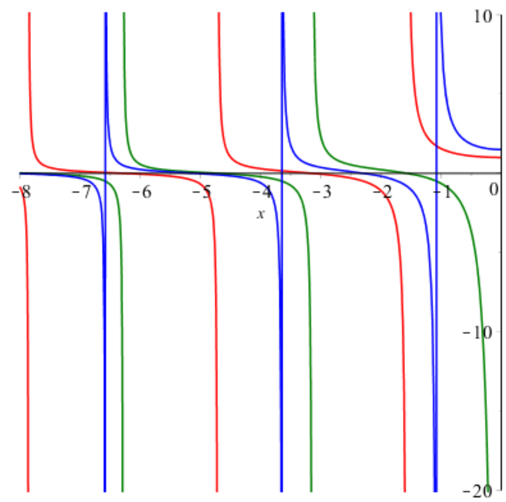
$$\begin{aligned} \sigma_0(\{r_j\}) &= \lim_{r\to r_j} (r_j - r)\, h_0(-r) \\ &= \lim_{r\to r_j} \left(\left(j + \frac{1}{2}\right)^2 \pi^2 - r\right) \frac{\sin(\sqrt{r})}{\sqrt{r}\,\cos(\sqrt{r})} \\ &= \lim_{r\to r_j} \left(\left(j + \frac{1}{2}\right)\pi + \sqrt{r}\right)\left(\left(j + \frac{1}{2}\right)\pi - \sqrt{r}\right) \frac{\sin(\sqrt{r})}{\sqrt{r}(-1)^j \sin\left(\left(j + \frac{1}{2}\right)\pi - \sqrt{r}\right)} \\ &= (2j + 1)\pi \frac{1}{\left(j + \frac{1}{2}\right)\pi} \\ &= 2, \end{aligned}$$

for all $j = 0, 1, \ldots$. Similarly, the $\mathbb{S}$-function h_∞ on $(-\infty, 0]$ is given by

$$h_\infty(-r) = -\frac{1}{\sqrt{r}\,\tan(\sqrt{r})},$$



(a) Plots of $h_0(-r^2)$ (red) and $h_\infty(-r^2)$ (green) for $r \in [-8, 0]$. The zeroes of h_0 are the poles of h_∞.



(b) Plots of $h_0(-r^2)$ (red), $h_{1/2}(-r^2)$ (blue) and $h_\infty(-r^2)$ (green) for $r \in [-8, 0]$. Let $r_j^{(0)}$ and $r_j^{(\infty)}$ denote the j-th pole of h_0 and h_∞ respectively. Then $r_j^{(0)} \leqslant r_k^{(1/2)} \leqslant r_j^{(\infty)}$ for some $k \leqslant j$. Since Maple fails to recognize the discontinuities of $h_{1/2}$, the blue vertical lines appear.

Figure 2: illustrations of poles of h_k of a short string with different tying constants k.

with poles at

$$r_k = j^2\pi^2, \qquad j = 0, 1, \dots,$$

yielding the spectral measure σ_∞ with the jumps

$$\sigma(\{0\}) = 1, \qquad \sigma(\{r_j\}) = 2, \qquad j = 1, 2, \dots.$$

The poles of h_k for $0 < k < \infty$ are included between the poles of h_0 and h_∞, see Figure 2a and 2b for illustrations.

To close this example, we observe that the moments are all highly divergent

$$\sum_{j=0}^{\infty} r_j^n \sigma_0(\{r_j\}) = \sum_{j=0}^{\infty} \left(j + \frac{1}{2}\right)^{2n} \pi^{2n} \cdot 2 = \infty, \qquad n = 0, 1, \dots,$$

that the square roots of the eigenvalues grow at the linear rate

$$\frac{1}{\pi} = \lim_{j\to\infty} \frac{j}{j\pi + \pi/2} = \frac{1}{\pi}\int_0^1 1\, dx,$$

and that

$$\sum_{j=0}^{\infty} \frac{1}{\left(j + \frac{1}{2}\right)^2 \pi^2} < \infty.$$

This illustrates Proposition 3.30 and items (3) and (2) in Proposition 3.27.

3.3.3 Krein strings as Sturm-Liouville operators

A Krein string (l, M) with

$$M(x) = \int_0^x \varrho(s)\, ds, \qquad \varrho(x) > 0,\ x \in [0, l)$$

has the eigenequation

$$\Big(\frac{d^2}{dx^2} - z\,\varrho(x)\Big) f = 0.$$

It is clear that the above is a Sturm-Liouville equation. We have already had an example of a short Krein string with an absolutely continuous mass function in Example 3.33. Here comes two examples of long Krein strings. In the last example, we examine the eigenvalue asymptotics for a short Krein string where a point mass is added to the smooth density $\varrho(x)$.

Example 3.34 Let (l, M) be the string with

$$l = \infty, \qquad M(x) = x \cdot \mathbb{1}_{[0,\infty)}.$$

The associated eigenequation

$$\Big(\frac{d^2}{dx^2} - z\Big) f = 0, \qquad x \in [0, \infty),$$

has the general extended solution

$$f(x) = \begin{cases} f(0) + f'_-(0)x, & x \in (-\infty, 0) \\ f(0)\cosh(\sqrt{z}\,x) + \frac{f'_+(0)}{\sqrt{z}}\sinh(\sqrt{z}\,x), & x \in [0, \infty). \end{cases}$$

Since $\varrho(\{0\}) = 0$, $f'_-(0) = f'_+(0)$ and hence the fundamental system consists of

$$A(x, z) = \cosh(\sqrt{z}\,x), \qquad \text{and} \qquad C(x, z) = \frac{1}{\sqrt{z}}\sinh(\sqrt{z}\,x),$$

for $z \notin (-\infty, 0]$. By (3.23), the $\mathbb{S}$-function of this string on $(0, \infty)$ is

$$h(r) = \lim_{x\uparrow\infty} \frac{\sinh(\sqrt{r}\,x)}{\sqrt{r}\,\cosh(\sqrt{r}\,x)} = \frac{1}{\sqrt{r}},$$

which has an analytic extension to the cut complex plane $\mathbb{C}\backslash(-\infty, 0)$, and which cannot be extended analytically on the branch $(-\infty, 0)$. Using (3.22), the spectral measure can be computed

$$\sigma((0, r]) = \lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0} \frac{1}{\pi}\int_\delta^{r+\delta} \operatorname{Im}((-y - i\varepsilon)^{-1/2})\, dy.$$

For $y > 0$ and $\varepsilon > 0$, we have $\theta = \arg(y + i\varepsilon) = \arctan\big(\frac{y}{\varepsilon}\big) \in \big(0, \frac{\pi}{2}\big)$ and

$$\frac{1}{\sqrt{-y - i\varepsilon}} = \frac{\sqrt{-y + i\varepsilon}}{\sqrt{y^2 + \varepsilon^2}} = (y^2 + \varepsilon^2)^{-1/4}\Big(\cos\Big(\frac{\pi}{2} - \frac{\theta}{2}\Big) + i\sin\Big(\frac{\pi}{2} - \frac{\theta}{2}\Big)\Big).$$

Using the double angle formula of cosine, we simplify $\sin(\pi/2 - \theta/2)$

$$\sin\Big(\frac{\pi}{2} - \frac{\theta}{2}\Big) = \cos\Big(\frac{\theta}{2}\Big) = \frac{1}{\sqrt{2}}\big(\cos(\theta) + 1\big)^{1/2} = \frac{1}{\sqrt{2}}\Big(\frac{y}{(y^2 + \varepsilon^2)^{1/2}} + 1\Big)^{1/2}.$$

So, the integral of interest is now

$$\begin{aligned}\sigma((0,r]) &= \lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}\frac{1}{\pi}\int_\delta^{r+\delta}\frac{1}{\sqrt{2}}(y^2+\varepsilon^2)^{-1/4}\Big(\frac{y}{(y^2+\varepsilon^2)^{1/2}}+1\Big)^{1/2}dy\\ &= \lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}\frac{\sqrt{2}}{\pi}(y^2+\varepsilon^2)^{1/4}\Big(\frac{y}{(y^2+\varepsilon^2)^{1/2}}+1\Big)^{1/2}\Big|_\delta^{r+\delta}\\ &= \frac{2}{\pi}\sqrt{r}.\end{aligned}$$

To close this example, we observe that $d\sigma/dr = \pi^{-1}h(r)$, and that the estimate in Proposition 3.25 applies for the correspondence

$$M(x) = x \leftrightarrow (-z)^{-1/2} = m(z), \qquad \text{with } \alpha = 1.$$

Example 3.35 Let (l, M) be the string with

$$l = \infty, \qquad M(x) = x^\alpha \cdot \mathbb{1}_{[0,\infty)},$$

for some $\alpha > 0$. The eigenequation is

$$\Big(\frac{d^2}{dx^2} - z\alpha\, x^{\alpha-1}\Big)f = 0, \qquad x \in [0,\infty),$$

with the general extended solution

$$f(x) = \begin{cases} f(0) + f'_-(0)x, & x \in (-\infty, 0)\\ C_1\sqrt{x}\, I_\beta(y) + C_2\sqrt{x}\, I_{-\beta}(y), & x \in [0,\infty),\end{cases}$$

where $I_{\pm\beta}(y)$ are the modified Bessel functions of the first kind, and

$$\beta = \frac{1}{\alpha+1}, \qquad \text{and} \qquad y = 2\sqrt{z\alpha}\,\beta\, x^{\frac{1}{2\beta}}.$$

The fundamental system $\{A(x,r), C(x,r)\}$ for $r > 0$ consists of

$$\begin{aligned}A(x,r) &= \Gamma(1-\beta)(r\alpha)^{\beta/2}\beta^\beta\sqrt{x}\, I_{-\beta}(y)\\ C(x,r) &= \Gamma(1+\beta)(r\alpha)^{-\beta/2}\beta^{-\beta}\sqrt{x}\, I_\beta(y),\end{aligned}$$

for $x \in [0,\infty)$. To compute the $\mathbb{S}$-function, we only need to consider the asymptotic behaviors of $I_{\pm\beta}(y)$, which are

$$I_{\pm\beta}(y) \approx (2\pi y)^{-1/2}\exp(y),$$

since $\mathrm{Arg}(y) = 0$. For more details on the modified Bessel function $I_{\pm\beta}$, see Section 9.6.2, 9.6.10 and 9.6.11 in [1], and for more on the asymptotic behaviors of the Bessel functions, see Section 9.2.1 in [1]. Plugging in $y = \sqrt{4\beta^2 r\alpha}\, x^{\frac{1}{2\beta}}$, then $\beta = (\alpha+1)^{-1}$, and simplifying as much as possible, we obtain the $\mathbb{S}$-function

$$h(r) = \frac{\Gamma\Big(\frac{\alpha+2}{\alpha+1}\Big)}{\Gamma\Big(\frac{\alpha}{\alpha+1}\Big)}\Big(\frac{(\alpha+1)^2}{\alpha}\Big)^{\frac{1}{\alpha+1}} r^{\frac{-1}{\alpha+1}}.$$

This example can be found as Example 1 in [8], where Y. Kasahara also presents the spectral measure σ corresponding to h as above.

In general, if the density $\varrho(x)$ is a sufficiently smooth function, the operator $(\tau, \mathbf{D}(\tau))$ can be rewritten as a Schrödinger operator. Hence, we can use the methods as presented in [7, pp.151–155] to obtain better eigenvalue asymptotics than item (3) in Proposition 3.27. For example, for the Dirichlet boundary condition at $x = l$, that is, when $k = 0$, we get the following asymptotics

$$\sqrt{r_n} = \xi_n = c_0 n + \frac{c_1}{n} + \frac{c_3}{n^3} + \mathcal{O}\Big(\frac{1}{n^5}\Big), \qquad \text{for large } n,$$

where c_0 is already known to be

$$c_0 = \frac{1}{\pi}\int_0^l \sqrt{M'(x)}\,dx$$

and the coefficients c_1 and c_3 depend on (l, M). It is possible that discontinuities of $M(x)$ can be found by observing the eigenvalue asymptotics. We illustrate this with a simple example, in which the eigenvalue asymptotics is given up to c_1.

Example 3.36 Let (l, M) be the string given by

$$M(x) = x \cdot \mathbb{1}_{[0,1]} + \varrho_1 \cdot \mathbb{1}_{[x_1,1]} + (1+\varrho_1)\cdot \mathbb{1}_{(1,\infty)}.$$

The associated eigenequation is

$$\begin{cases} \Big(\frac{d^2}{dx^2} - z\Big) f = 0, & x \in (0, x_1) \cup (x_1, 1] \\ f'_+(x_1) - f'_-(x_1) = \varrho_1\, z\, f(x_1) \end{cases}$$

with the general solution $f(x) \in \mathbf{D}_{max}$

$$f(x) = \begin{cases} f(0) + f'_-(0)x, & x < 0 \\ f(0)\cosh(\sqrt{z}\,x) + \frac{f'_-(0)}{\sqrt{z}}\sinh(\sqrt{z}\,x), & x \in [0, x_1) \\ f(x_1)\cosh(\sqrt{z}(x - x_1)) + \frac{f'_+(x_1)}{\sqrt{z}}\sinh(\sqrt{z}(x - x_1)), & x \in [x_1, 1) \\ f(1) + f'_-(1)(x-1), & x \geqslant 1 \end{cases}$$

from which we obtain the fundamental system

$$\begin{aligned} A(x,z) =& \cosh(\sqrt{z}\,x_1)\cosh(\sqrt{z}(x-x_1)) \\ &+ \big(\varrho_1\sqrt{z}\cosh(\sqrt{z}\,x_1) + \sinh(\sqrt{z}\,x_1)\big)\sinh(\sqrt{z}(x-x_1)), \end{aligned}$$

and

$$\begin{aligned} C(x,z) =& \frac{1}{\sqrt{z}}\sinh(\sqrt{z}\,x_1)\cosh(\sqrt{z}(x-x_1)) \\ &+ \Big(\varrho_1\sinh(\sqrt{z}\,x_1) + \frac{1}{\sqrt{z}}\cosh(\sqrt{z}\,x_1)\Big)\sinh(\sqrt{z}(x-x_1)), \end{aligned}$$

for $x_1 \leqslant x < 1$. The Weyl function $D_0(0, z)$ corresponding to tying constant $k = 0$ is the quotient $C(1, z)/A(1, z)$.

Let $x_1 = \frac{1}{2}$ and set $z = -\xi^2$ for $\xi \in \mathbb{R}$. Since $A(x, z)$ and $C(x, z)$ do not have common zeroes, finding the poles of $D_0(0, -\xi^2)$ is equivalent to finding the zeroes of $A(x, -\xi^2)$. Hence, we consider the equation

$$\cos(\xi) - \frac{\varrho_1\xi}{2}\sin(\xi) = 0,$$

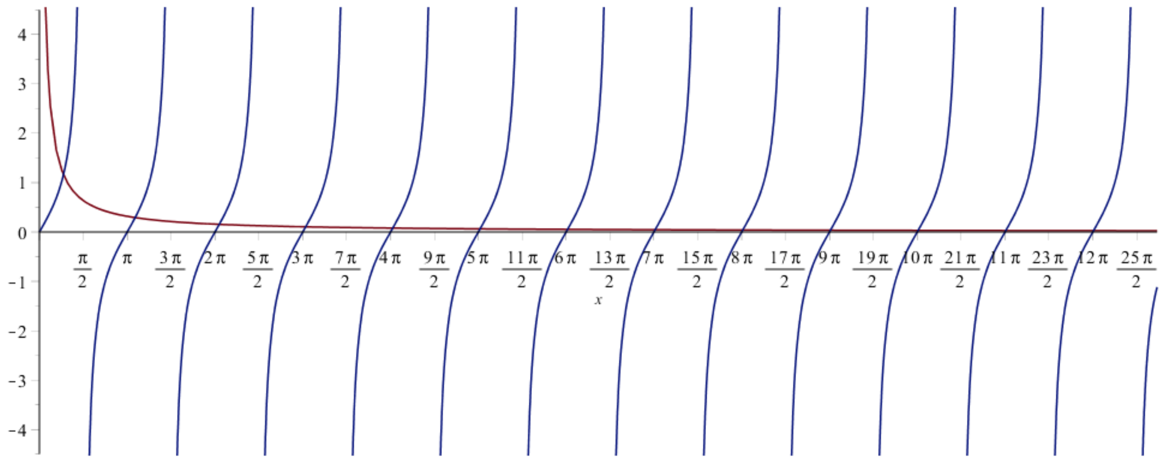


Figure 3: plots of $\tan(\xi)$ and $C\xi^{-1}$. The functions intersect at points near $n\pi$.

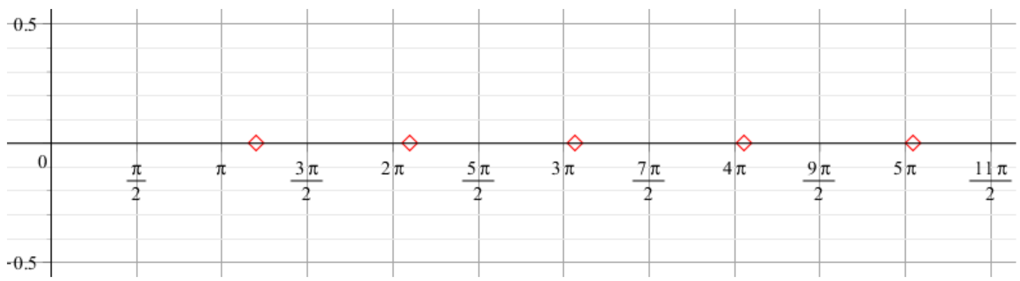


Figure 4: an illustration of $\xi_n = n\pi + \Delta_n$, where $\Delta_n = 2 \cdot (\varrho_1 n\pi)^{-1}$ and $\varrho_1 = 1$. The errors Δ_n are positive, hence ξ_n is always to the right of the points $n\pi$, for $n = 1, 2, \ldots$.

and since $\cos(\xi)$ and $\sin(\xi)$ do not have common zeroes, we rearrange terms to have

$$\tan(\xi) = \frac{2}{\varrho_1 \xi}. \tag{3.33}$$

The values ξ for which (3.33) is satisfied will indeed behave asymptotically like $n\pi$ (see Figure 3), as predicted by item (3) in Proposition 3.27. To obtain a better asymptotic estimate, we set

$$\xi_n = n\pi + \Delta_n, \qquad \text{for} \qquad \Delta_n = \frac{c_1}{n} + \frac{c_2}{n^2} + \mathcal{O}\Big(\frac{1}{n^3}\Big).$$

By Taylor expanding the left-hand side, and the right-hand side of (3.33) near the points $\xi_n = n\pi + \Delta_n$, we get

$$\begin{aligned}\tan(n\pi + \Delta_n) &= \Delta_n + \mathcal{O}(\Delta_n^3)\\ &= \frac{2}{\varrho_1 n\pi} - \frac{2\Delta_n}{\varrho_1 (n\pi)^2} + \frac{2\Delta_n^2}{\varrho_1 (n\pi)^3} + \mathcal{O}(\Delta_n^3)\\ &= \frac{2}{\varrho_1 (n\pi + \Delta_n)},\end{aligned}$$

and then plugging in the expression for Δ_n, we find that

$$\begin{aligned}&\frac{c_1}{n}+\frac{c_2}{n^2}+\mathcal{O}\Big(\frac{1}{n^3}\Big)\\ =&\frac{2}{\varrho_1 n\pi}-\frac{2}{\varrho_1(n\pi)^2}\Big(\frac{c_1}{n}+\frac{c_2}{n^2}+\mathcal{O}\Big(\frac{1}{n^3}\Big)\Big)+\mathcal{O}\Big(\frac{1}{n^5}\Big)\\ =&\frac{2}{\varrho_1 n\pi}+\mathcal{O}\Big(\frac{1}{n^3}\Big),\end{aligned}$$

which leads to the identification

$$c_1=\frac{2}{\varrho_1\,\pi},\qquad c_2=0,$$

and a sharper asymptotic behavior for the eigenvalues is obtained,

$$\xi_n=n\pi+\frac{2}{\varrho_1 n\pi}+\mathcal{O}\Big(\frac{1}{n^3}\Big),$$

which is illustrated in Figure 4.

Similarly, let $x_1=\frac{1}{3}$. Locating the poles of $D_0(0,-\xi^2)$ is equivalent to finding the solutions to the equation

$$\cos(\xi)-\varrho_1\xi\cos\left(\frac{\xi}{3}\right)\sin\left(\frac{2\xi}{3}\right)=0. \tag{3.34}$$

When

$$\xi=3\cdot\frac{\pi(2k+1)}{2}=(3k+1)\pi+\frac{\pi}{2},$$

the functions $\cos(\xi)$, $\cos(\xi/3)$ and $\sin(2\xi/3)$ have value 0 and (3.34) is trivially satisfied. Suppose $\xi\neq n\pi+\pi/2$ for $n\equiv 1\,(\mathrm{mod}\,3)$. Then, we can rearrange terms to arrive at

$$\frac{1}{\varrho_1\xi}=\frac{\cos\left(\frac{\xi}{3}\right)\sin\left(\frac{2\xi}{3}\right)}{\cos(\xi)}. \tag{3.35}$$

Like before, the intersection of the left-hand side and the right-hand side occurs near the points

$$\xi=n\pi,\ \text{for}\,n\equiv 0\,(\mathrm{mod}\,3)\qquad\text{and}\qquad \xi=n\pi-\frac{\pi}{2},\ \text{for}\,n\equiv 2\,(\mathrm{mod}\,3),$$

see Figure 5. Set

$$\begin{aligned}\xi_n&=n\pi+\Delta_n, n\equiv 0\,(\mathrm{mod}\,3),\qquad \Delta_n=\frac{c_1}{n}+\frac{c_2}{n^2}+\mathcal{O}\Big(\frac{1}{n^3}\Big)\\ \xi_n&=n\pi-\frac{\pi}{2}+\Delta_n', n\equiv 2\,(\mathrm{mod}\,3),\qquad \Delta_n'=\frac{c_1'}{n}+\frac{c_2'}{n^2}+\mathcal{O}\Big(\frac{1}{n^3}\Big).\end{aligned}$$

Taylor expanding both sides in (3.35) near $n\pi$ for $n\equiv 0$ (mod 3) and then identifying coefficients gives

$$c_1=\frac{3}{2\varrho_1\pi},\qquad c_2=0.$$

Near the points $(3k+2)\pi-\pi/2$, we get

$$c_1'=\frac{9}{2\varrho_1\pi},\qquad c_2'=0.$$

In summary, the eigenvalue asymptote depends on n in the following way

$$\xi_n = \begin{cases} n\pi + \frac{3}{2\varrho_1\pi}\frac{1}{n} + \mathcal{O}\Big(\frac{1}{n^3}\Big), & n \equiv 0\,(\mathrm{mod}\,3), \\ n\pi + \frac{\pi}{2}, & n \equiv 1\,(\mathrm{mod}\,3), \\ n\pi - \frac{\pi}{2} + \frac{9}{2\varrho_1\pi}\frac{1}{n} + \mathcal{O}\Big(\frac{1}{n^3}\Big), & n \equiv 2\,(\mathrm{mod}\,3). \end{cases} \tag{3.36}$$

This means that near the point $(3k+1)\pi+\pi/2$, there will always be two eigenvalues, one of which is always exactly $(3k+1)\pi+\pi/2$ and the other is on the right of $(3k+1)\pi+\pi/2$. As k goes to infinity, the error Δ'_{3k+2} will go to zero in either case, and $\xi_{3k+2} - \xi_{3k+1}$ goes to 0. See Figure 6 for an illustration.

When the quotient $\frac{x_1}{l}$ is not a rational number, the asymptotic behaviors of ξ_n will be much more complicated.

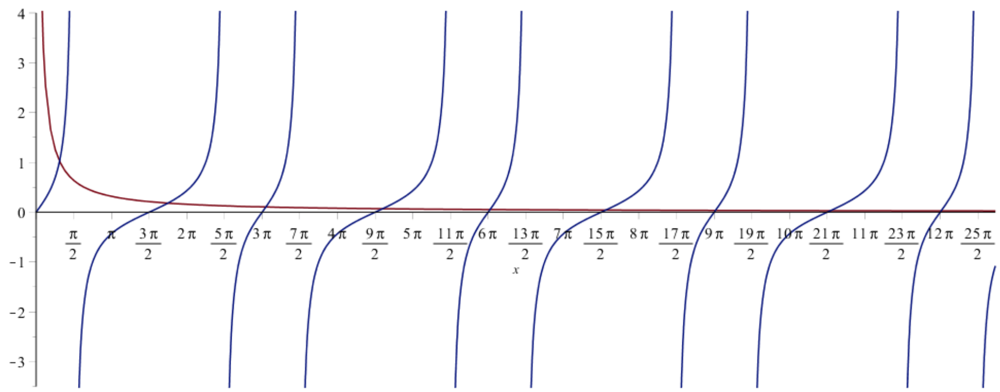


Figure 5: plots of $\sin(2\xi/3)\cos(\xi/3)\cos(\xi)^{-1}$ and $C\xi^{-1}$. The functions intersect at points near $n\pi$ for $n \equiv 0$ (mod 3) and $n\pi - \pi/2$ for $n \equiv 2$ (mod 3).

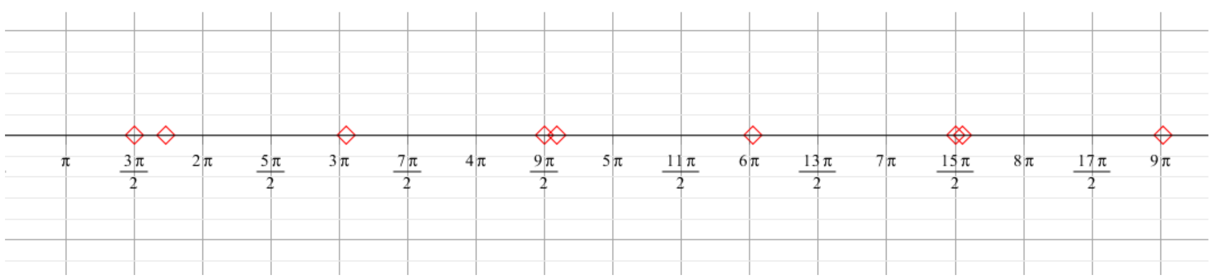


Figure 6: an illustration of the eigenvalue approximations in (3.36). Near $\xi = n\pi + \pi/2$ for $n \equiv 1$ (mod 3), two eigenvalues are found.

3.3.4 Discussion

In general, a characterization of $\mathbb{S}$-functions h corresponding to a string (l, M) is hard to find if (l, M) is not a Stieltjes string. It is as difficult as characterizing all spectral problems corresponding to a certain subclass of Herglotz-Nevanlinna functions, as mentioned in the introduction. The examples in this section, despite their simplicity, have opened up some interesting paths for future investigations on Krein's correspondence.

Example 3.36 can be developed further. For instance, the point mass can be placed at another position on the string, more point masses can be added, or other boundary conditions can be chosen. Everything can be computed explicitly for the simple absolutely continuous part of ϱ as in 3.36. Hopefully, we will be able to observe the number of jumps and where they are positioned from the eigenvalue asymptotics, from which we might be able to generalize for other more complicated absolutely continuous parts.

Another idea is to follow the steps in Example 3.29 for ϱ with the absolutely continuous part 1 and a simple discrete part, that is, for the string with

$$M(x) = x \cdot \mathbb{1}_{[0,l)} + \sum_{0 \leqslant x_j \leqslant x < l} \varrho_j, \qquad 1 \leqslant j \leqslant n.$$

We can solve the differential equation $\tau f = zf$ for this string. From the jumps of M and the solution, we might be able to find a basis for $\mathbf{Z}_e(\Delta)$ and use this to deduce a sufficient condition for h to belong to such a string.

The convergence $\lim_n M_n = M$ in the sense that $\lim_n M_n(x) = M(x)$ for every continuity point x of M induces a topology on the class of mass functions, and the convergence $\lim_n h_n = h$ in the sense that $\lim_n h_n(r) = h(r)$ for every $r > 0$ induces a topology on $\mathbb{S}$. Krein's correspondence $(l, M) \leftrightarrow h$ is continuous with respect to these topologies, as we have seen in Theorem 3.22. In the future, we would like to study other topologies making Krein's correspondence continuous. More specifically, we look for a topology on the mass functions, such that irregular densities are approximated by sufficiently smooth densities. Also, studying the uniqueness part of Krein's correspondence will certainly lead to a better understanding of this problem. In [5], J. Eckhardt and A. Kostenko investigate how to extend the equation $f'' = z\varrho f$ with a signed Borel measure ϱ, so that the corresponding Weyl functions coincide exactly with the class of all Herglotz-Nevanlinna functions. In particular, they do this by allowing ϱ to be a real-valued distribution in H^{-1}_{loc} and adding a term $z^2 \nu f$, where ν is a non-negative Borel measure. Although this problem is about extending and ours is about restricting, the approach made in [5] might be of interest for us.

References

[1] M. Abramowitz & I. A. Stegun, *Handbook of mathematical functions*, National Bureau of Standards Applied Mathematics Series, Washington DC, 1972.

[2] C. Bennewitz, *Spectral asymptotics for Sturm-Liouville equations*, Proc. London Math. Soc. 59 (1989), 294–338.

[3] S. J. Cox, M. Embree & J. M. Hokanson, *One can hear the composition of a string: experiments with an inverse eigenvalue problem*, SIAM Rev. 54 (2012), 157–178.

[4] H. Dym & H. P. McKean, *Gaussian processes, function theory and the inverse spectral problem*, Academic Press, New York – London, 1976.

[5] J. Eckhardt & A. Kostenko, *The inverse spectral problem for indefinite strings*, Invent. Math. 204 (2016), 939–977.

[6] J. Eckhardt & G. Teschl, *Sturm-Liouville operators with measure-valued coefficients*, J. Anal. Math. 120 (2013), 151–224.

[7] H. Hochstadt, *Differential equations – a modern approach*, Dover Publications Inc., New York, 1964.

[8] Y. Kasahara, *Spectral theory of generalized second order differential operators and its applications to Markov processes*, Japan J. Math 1 (1975/1976), 67–84.

[9] I. S. Kats, M. G. Krein, *Criteria for the discreteness of the spectrum of a singular string* (in Russian), Izv. Vyssh. Ucebn. Zaved. Matematika 1958 (1958), 136–153.

[10] S. Kac & M. G. Krein, *R–functions – Analytic functions mapping the upper half-plane into itself*, Amer. Math. Soc. Transl. 103 (1974), 1–18.

[11] S. Kac & M. G. Krein, *On the spectral functions of the string*, Amer. Math. Soc. Transl. 103 (1974), 19–102.

[12] A. Kostenko, A. Sakhnovich & G. Teschl, *Weyl-Titchmarsh theory for Schrödinger operators with strongly singular potentials*, Int. Math. Res. Not. (2012), 1699–1747.

[13] S. Kotani & S. Watanabe, *Krein's spectral theory of strings and generalized diffusion processes*, Lecture Notes in Math., Springer, Berlin – New York, 1982.

[14] M. G. Krein, *On inverse problems for the nonhomogeneous cord* (in Russian), Dokl. Akad. Nauk. SSSR 82 (1951), 669–672.

[15] H. Langer, *Zur Spectraltheorie Verallgemeinerter Gewöhnlicher Differentialoperatoren Zweiter Ordnung mit Einer Nichtmonotonen Gewichtfunktion*, Math. Int. Ber. 14 (1972), 58.

[16] R. L. Schilling, R. Song & Z. Vondracek, *Bernstein functions – theory and its applications*, De Gruyter Studies in Mathematics 37, Berlin, 2012.

[17] A. N. Shiryaev, *Probability*, Graduate Texts in Mathematics, Springer, New York, 1984.

[18] G. Teschl, *Mathematical methods in quantum mechanics with applications to Schrödinger operators*, Graduate Studies in Mathematics, Amer. Math. Soc., Providence, 2014.

[19] H. S. Wall, *Analytic theory of continued fractions*, D. Van Nostrand Company Inc., New York, 1948.

[20] H. Woracek, *Asymptotics of eigenvalues for a class of singular Kreĭn strings*, Collect. Math. 66 (2015), 464–479.

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