To Measure Concentration Risk - A comparative study

Alma Broström and Hanna Scheibenpflug

Department of Mathematical Statistics
Faculty of Engineering at Lund University

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Abstract

Credit risk is one of the largest risks facing a bank and following the Basel regulations, banks are expected to hold capital to protect themselves against credit risk. This thesis aims to evaluate models to calculate the capital requirement for credit concentration risk and compare them to the models suggested by Finansinspektionen.

Credit concentration risk can be split into name and sector concentration and two models are evaluated for each type of concentration risk. For both name and sector concentration a Full Monte Carlo method is implemented but as this is a time consuming method, alternative methods are suggested. For name concentration risk the alternative method splits the portfolio into two sub-portfolios and treats only one of the portfolios as if it contains any name concentration risk. The proposed method for sector concentration builds on the multi-factor Merton model and gives an analytical solution. Each pair of models is tested on separate sets of simulated portfolios containing varying degrees of name respective sector concentration. Both methods assessing name concentration perform well but as the alternative method is faster, this is to be preferred. None of the methods are in perfect agreement with the results of the methods of Finansinspektionen and although this does not necessarily indicate that the models are faulty one should investigate the reasons behind the differing results before continuing with any of the methods. When testing the sector concentration the alternative method appears to be the preferable one but as both methods differ greatly from the results of Finansinspektionen none of the methods should be used before considering the reasons for the large deviations in results.

Keywords: Credit concentration risk, name concentration, sector concentration, Monte Carlo, capital requirement, Partial Portfolio Approach, Pykhtin , Multi factor adjustment

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Alma Broström
Hanna Scheibenpflug

Abbreviations

A summary of all abbreviations used throughout the thesis.

AHI	Adjusted Herfindahl Index
ASRF	Asymptotic Single Risk Factor
BCBS	Basel Committee on Banking Supervision
CMC	Crude Monte Carlo
EAD	Exposure At Default
EL	Expected Loss
FI	Finansinspektionen
FMC	Full Monte Carlo
HI	Herfindahl Index
IRB	Internal Ratings Based
IS	Importance Sampling
LGD	Loss Given Default
MC	Monte Carlo
PD	Probability of Default
PPA	Partial Portfolio Approach
UL	Unexpected Loss
VaR	Value at Risk

Contents

Chapter 1

Introduction

1.1 Background

As one of the core businesses of banks today is to supply the market with various forms of credit, it follows that proper management of the risks associated with the practice is of importance. Credit risk, the risk that a borrower defaults on its obligation towards the bank, is one of the greatest risk facing a bank and banks are therefore required to hold capital as protection against this type of risk (Hull, 2012). The capital requirements are regulated in the Basel Capital Accords. The Capital Accords are only broad supervisory standards but incorporated into European Union law and hence the capital requirements apply to all Swedish banks as well as all other banks within the European Union.

When calculating capital requirements for a credit portfolio, the credit concentration risk needs to be considered. Concentration risk in a bank's credit portfolio arises mainly from two types of imperfect diversification, name and sector concentration (BCBS, 2006). Name concentration occurs when there are large exposures to individual borrowers so that the idiosyncratic risk is not perfectly diversified. Sector concentration arises when there are several sectors correlated with each other, meaning that the financial well being of two obligors can be correlated depending on which sector these obligors belong to (Grippa and Gornicka, 2016). The Basel Committee on Banking Supervision found that nine out of the thirteen analyzed banking crises were affected by risk concentration (BCBS, 2004a) and states that risk concentration is arguably the single most important cause of major problems in banks (BCBS, 2004b). Hence the subject of how to correctly assess the concentration risk is of great importance.

The capital requirements of concentration risk are not covered under Pillar 1 of the Basel Framework. Instead banks are expected to compensate for the concentration risk by setting aside capital buffers based on their own estimates, which are then assessed by financial supervisors under Pillar 2 (BCBS, 2004b). For Swedish banks the supervisory authority is Finansinspektionen (FI). FI has proposed methods for estimation of both name and sector concentration risk. However banks with permission from FI can use their own models to estimate the risk as long as these models are approved by FI (Finansinspektionen, 2014).

The ability to correctly estimate the level of risk is highly important to banks. If the bank holds too little capital it will not be protected in the case of a large default. On the other hand, if too much capital is held the bank can lose investment opportunities. The

models used to calculate the risk can therefore have a large impact as they will determine how much capital needs to be held. The models proposed by FI are straight forward to use but as the models are the same regardless of bank it can be to the bank's advantage to use its own models and estimation of variables. This allows a bank to better estimate the bank specific risk level and specify the capital requirement based on the bank's own characteristics.

1.2 Problem formulation

The aim of this thesis is to use different kind of techniques for assessing credit concentration risk and compare these for different portfolios. Models for both name and sector concentration will be studied. The models need to be in agreement with the requirements decided by the supervisory authority. For Swedish banks this is FI. Furthermore the thesis aims to give an insight to the regulatory works that governs how credit risk is currently handled as well as the theory concerning credit concentration risk.

1.3 Chapter Outline

The chapter outline of the thesis is presented below.

Chapter 2: The chapter introduces the main variables considered when modeling credit risk as well as the theory related to concentration risk that is needed for the thesis. The subject of concentration risk is presented further. The chapter also includes an introduction to the regulatory works of the Basel Committee on Banking Supervision.

Chapter 3: The theory behind the technical solutions used in this thesis is presented in this chapter. The following chapters rely heavily on the theory presented here.

Chapter 4: As a benchmark, the models considered by Finansinspektionen are used and these models are presented here. The models considered are the ones used by FI to calculate capital requirements for name and sector concentration.

Chapter 5: The approaches considered in the thesis when calculating capital requirements for name concentration risk are presented in this chapter. Both the theory behind the methods as well as the implementation are described.

Chapter 6: The 6th chapter has much the same contents as the 5th but this chapter considers the models for sector concentration.

Chapter 7: Chapter 7 presents the data used when testing the models presented in chapter 5 and 6. The data is simulated and the main concepts and assumptions behind the simulations are presented.

Chapter 8: The methods presented in chapter 5 and 6 are evaluated on the portfolios presented in chapter 7 and the results are given in this chapter.

Chapter 9: Chapter 9 discusses the results of chapter 8 and gives suggestions for further research.

Chapter 10: The final chapter gives a short summary of the thesis and conclusions drawn by the authors.

Chapter 2

Credit risk - theory and regulations

Credit risk describes the risk that a counterparty defaults, meaning that the they are not able to make their payment in time. Banks are required to hold capital for this type of risk, which is specified in the Basel accords' Pillar 1 (Hull, 2012). The method considered for the credit risk capital requirement in Pillar 1 does not take into account the risk that arises from concentration of exposures in the portfolio (Grippa and Gornicka, 2016). Basel II states that "A risk concentration is any single exposure or group of exposures with the potential to produce losses large enough (relative to a bank's capital, total assets, or overall risk level) to threaten a bank's health or ability to maintain its core operations" (BCBS, 2004b). Basel II further states that credit concentration risk is an important factor behind major bank problems and that it needs to be considered when calculating the capital requirements required for credit risk (BCBS, 2004b). In this chapter two subgroups of credit concentration risk, name and sector risk concentration, are presented. Before this is possible the Basel regulations need to be presented closer as well as the mathematical models that the credit risk capital requirement is based on. The chapter also includes an introduction to the risk measures required to measure credit risk.

2.1 BCBS and the Basel Accords

The Basel Committee on Banking Supervision (henceforth called BCBS) was founded in 1974 by the central bank Governors of the Group of Ten countries (BCBS, 1999). Following in the serious disturbances in the international currency and banking markets the BCBS was established to enhance financial stability by improving the quality of the banking supervision worldwide, and to serve as a forum for regular cooperation between its member countries on banking supervisory matters. Since its inception the BCBS has expanded its members from G10 to 45 institutions from 28 jurisdictions (BCBS, 1999). Furthermore, they have devised three major global publications of the accords on capital adequacy which are commonly known as Basel I, Basel II and Basel III.

Basel Capital Accord, Basel I, was published in 1988 and covered only credit risk (BCBS, 1999). The goal was to set minimum capital requirements for commercial banks as a buffer against financial losses. It called for minimum ratio of capital to risk-weighted assets of 8%. The primary objective was to promote the safety and soundness of the global financial system. Uniform minimum standards also created a level playing field for internationally

active banks, which was the secondary objective. The Amendment to Capital Accord to incorporate market risk was issued in 1996 to incorporate a capital requirement for market risks.

The New Capital Framework, Basel II, was released in 2004 and consisted of three pillars (BCBS, 2004b). The First Pillar specifies minimum capital requirements, which aim to cover credit, market and operational risk. The Second Pillar is a supervisory review of an institutions capital adequacy and internal assessment process. The Third Pillar covers the disclosure requirement used as a lever to strengthen market discipline and encourage sound banking practices.

Basel III builds on Basel II and contains, in addition to stronger capital requirements, two new capital buffers to strengthen the banks' abilities to resist losses and to lower the probability for new financial crises. The implementation of Basel III started in 2013 and will gradually continue until 2019.

The Basel Committee formulates broad supervisory standards but its conclusions do not have legal force (BCBS, 2016). However the European Union has incorporated the Basel rules into EU law through regulations and directives making them into national law in all member States, including Sweden (European Banking Authority, 2010). Finansinspektionen is the regulatory body in Sweden that monitors the companies on the Swedish financial market (Finansinspektionen, 2017). It is FI's responsibility to supervise that the Basel regulations are complied with.

2.2 Risk measures

Probability of Default

In BCBS (2004b) the probability of default (PD) is described as the probability that a counterparty defaults within one year. The PD will take values between 0-100%, where 0% indicates no probability of default while 100% indicates default. However the Basel framework states that the lowest possible PD to be assigned for bank or corporate exposures is 0.03%. As a default event can be defined in many ways the definition of BCBS (2004b) is presented. The Basel framework defines a default event to have happened if one or both of the two events listed below has transpired:

1. It is considered unlikely, by the bank, that the counterparty will be able to repay the credit obligations the counterparty has to the bank.

2. It has been more than 90 days since the loan expired and the counterparty has still to make the payment.

Exposure at Default

The total exposure a bank has towards an obligor is defined as the Exposure at Default (EAD). The EAD is given as the sum of the obligors debt to the bank and is therefore given in the corresponding currency (Bluhm, Overbeck, and Wagner, 2016).

Loss Given Default

The Loss Given Default (LGD) can be described as the part of the exposure that is not recoverable in the case that the obligor defaults. In the case of default it can still be

possible for the bank to recover some of the losses and the LGD is therefore given as a fraction of the total exposure towards each obligor. As LGD is given as a fraction it can take values between 0%, meaning that the bank can recover all losses, and 100%, meaning that the LGD is the full exposure towards the obligor. However in some cases the LGD can take a value larger than 100%. This can happen due to costs that arise when the bank tries to recover part of the loss. If the recovery attempt fails, meaning that if no part of the loss can be recovered, the LGD will be larger than the EAD (Hibbeln, 2010).

In this thesis LGD will be seen, in most cases, as a known quantity or, in some cases, as a random variable. To distinguish between these two cases LGD will be used for the case when the loss given default is seen as known while the random variable will be called $\tilde{LGD}$. We also define $E[\tilde{LGD}] = \mu_{LGD}$ and $V[\tilde{LGD}] = \sigma^2_{LGD}$.

Expected Loss

The Expected Loss (EL) is, as indicated by the name, the loss that a bank can expect given the risk measures of the exposures in the bank portfolios. The EL of one portfolio is calculated, using the previous defined risk measures, according to,

$$EL = \sum_{i}^{N} EAD_i \cdot LGD_i \cdot PD_i. \tag{2.1}$$

Here i indicates the index of each obligor in the portfolio and N is the total number of obligors in the portfolio (Hibbeln, 2010). Throughout the thesis we assume that the portfolio is sorted according to EAD so that $EAD_{(1)} \geq EAD_{(2)} \geq \cdots \geq EAD_{(N)}$.

Value at Risk

Given that the loss l has the distribution function $F_{Loss}(l)$, the Value at Risk (VaR) given confidence level α is defined as (Bluhm, Overbeck, and Wagner, 2016)

$$VaR_\alpha = \inf\{l : F_{Loss}(l) \geq \alpha\}. \tag{2.2}$$

This means that the VaR_α is the loss that will not be exceeded in $100 \cdot \alpha\%$ of the cases. For credit risk measures α is often set to 0.999 as the bank is interested in protecting itself from large losses.

Unexpected Loss

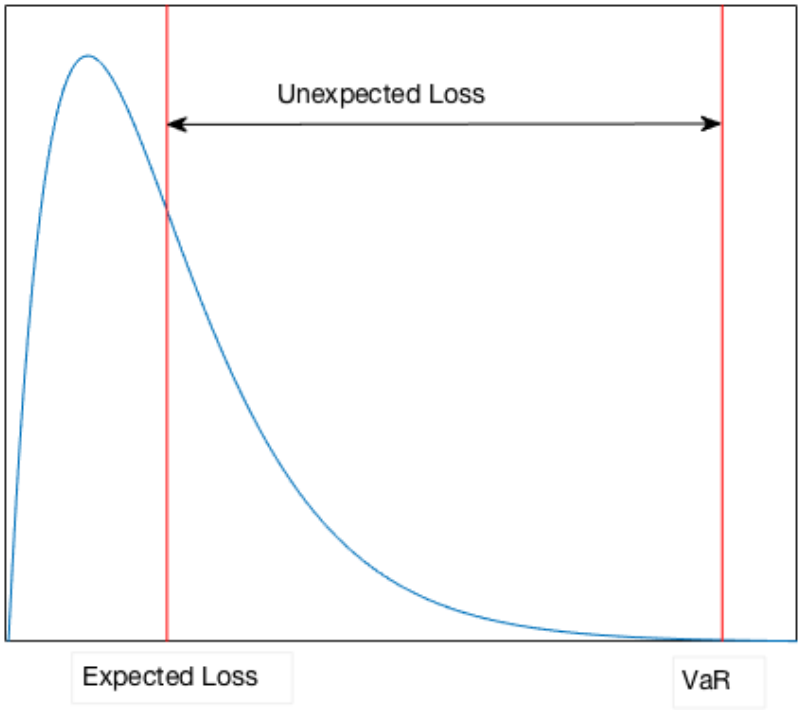

FIGURE 2.1: The Unexpected Loss is calculated as the difference between the VaR and the EL. The blue line represents a simulated loss distribution.

The Unexpected Loss (UL) is calculated as the difference between the VaR and the EL, as illustrated in figure 2.1 and seen in the equation below,

$$UL_\alpha = VaR_\alpha - EL = VaR_\alpha - \sum_i EAD_i \cdot LGD_i \cdot PD_i. \tag{2.3}$$

The expected loss should be expected by the bank and should therefore be covered by for example the pricing of the bank's products. The UL however needs to be covered by capital held by the bank that covers the difference between the VaR of a "worst case scenario" and the EL (Hull, 2012).

2.3 The Asymptotic Single Risk Factor model

The Asymptotic Single Risk Factor (ASRF) is a model based on the work of Vasicek (1987) and Merton (1973). In the Merton model an obligor will default if at maturity the value of the obligor's assets is smaller than the payment due. If the asset value process is given by $A(t)$ at time t and the debt is B, the obligor will default at maturity T if

$$A(T) < B. \tag{2.4}$$

The model proposed by Vasicek (1987) states that the asset value of obligor i can be described by a geometric Brownian motion with drift μ_i and volatility σ_i according to

$$\mathrm{d}A_i = \mu_i A_i \mathrm{d}t + \sigma_i A_i \mathrm{d}Y_i, \tag{2.5}$$

where Y_i is the Brownian motion driving the asset return of obligor i. This differential equation gives that

$$\ln A_i(T) = \ln A_i(0) + (\mu_i - \frac{1}{2}\sigma_i^2)T + \sigma_i\sqrt{T}\Phi_i, \tag{2.6}$$

where $\Phi_i \in N(0,1)$. Using equation 2.6 and the assumption that default occurs if the asset value is below B_i, the PD for obligor i is found as

$$\begin{aligned} PD_i &= P(A_i(T) < B_i) \\ &= P(\ln A_i(0) + \mu_i T - \frac{1}{2}\sigma_i^2 T + \sigma_i\sqrt{T}Y_i < \ln B_i) \\ &= P\left(Y_i < \frac{\ln B_i - \ln A_i(0) - \mu_i T + \frac{1}{2}\sigma_i^2 T}{\sigma_i\sqrt{T}}\right) \\ &= N(c_i), \end{aligned} \tag{2.7}$$

where $N(\cdot)$ is the cumulative normal distribution function and $c_i = \frac{\ln B_i - \ln A_i(0) - \mu_i T + \frac{1}{2}\sigma_i^2 T}{\sigma_i\sqrt{T}}$.

The ASRF approach builds on this definition of PD_i and the asset value process together with two assumptions. The first assumption states that the portfolio is infinitely granular, meaning that the number of obligors is large enough so that the idiosyncratic risk is diversified away. The second assumption is that there is a single common systematic risk factor that influences the creditworthiness of all obligors (Hibbeln, 2010).

The standardized asset return using the ASRF approach can be stated as

$$Y_i = w_i X + \sqrt{1 - w_i^2}\xi_i, \tag{2.8}$$

where X is the single systematic risk factor, ξ_i is the idiosyncratic risk factor and w_i is the factor loadings of obligor i. The factor loadings are calculated as

$$w_i = \sqrt{\rho_i}, \tag{2.9}$$

where ρ_i is the asset correlation with the systematic risk factor for obligor i. The systematic and the idiosyncratic risk factors are both standard normal distributed but they are not correlated. Furthermore the idiosyncratic factors are uncorrelated for different obligors, that is $E[\xi_i\xi_j] = 0$ for $i \neq j$ (Grippa and Gornicka, 2016).

As stated in equation 2.7 default will occur if the asset return is smaller than c_i. The probability of default conditional on the systematic risk factor can now be expressed as

$$\begin{aligned} PD(X = x) &= P(Y_i < c_i | X = x) \\ &= P(w_i X + \sqrt{1 - w_i^2}\xi_i < c_i | X = x) \\ &= P(w_i X + \sqrt{1 - w_i^2}\xi_i < N^{-1}(PD_i) | X = x) \\ &= P(w_i x + \sqrt{1 - w_i^2}\xi_i < N^{-1}(PD_i)) \\ &= P(\xi_i < \frac{N^{-1}(PD_i) - w_i x}{\sqrt{1 - w_i^2}}) \\ &= N\left(\frac{N^{-1}(PD_i) - w_i x}{\sqrt{1 - w_i^2}}\right). \end{aligned} \tag{2.10}$$

The obligors in the portfolio are mutually independent and the ASRF assumes an infinitely granular portfolio, which makes it possible to estimate the VaR by using equation 2.10 to write the conditional PD_i. The VaR can therefore be calculated as,

$$VaR_i = PD_i(X = x) \cdot LGD_i \cdot EAD_i = N\left(\frac{N^{-1}(PD_i) - w_i x}{\sqrt{1 - w_i^2}}\right) \cdot LGD_i \cdot EAD_i. \quad (2.11)$$

The capital requirement for the entire portfolio can now be calculated as the difference between the sum over all N obligors of the expected loss for obligors in the portfolio and the unconditional expected loss $EL_i = PD_i \cdot LGD_i \cdot EAD_i$. As the attentive reader will notice, this is the UL defined previously in this section. The UL can thus be calculated as (Grippa and Gornicka, 2016),

$$UL_\alpha = \sum_{i=1}^{N} (VaR_\alpha^i - EL_i). \quad (2.12)$$

2.4 Internal Ratings-Based Approach

The Basel II regulations state that required credit risk capital can be calculated using two methods. The first, called the standardised approach, will use external approximations of the needed risk measures while the second approach, the Internal Ratings Based (IRB), allows the bank to use internal approximations of the risk measures (BCBS, 2004b). However, before adapting the IRB approach, a bank needs clearance from financial supervisors. If a bank is cleared to use the IRB approach the risk measures to be supplied are PD, LGD, EAD and the effective maturity M (BCBS, 2004b, par. 211). The IRB approach is based on finding the VaR for the 99.9% confidence level and using this to find the UL. To calculate the VaR the ASRF approach is used, and the assumptions made on this model are assumed to be fulfilled. The UL can be calculated using equation 2.12 (Hull, 2012).

For some cases Basel II assumes that there is a relationship between the probability of default and the correlation ρ. These cases are exposures of the type corporate, sovereign and bank. The correlation for a corporate with an annual turnover above 50 million euro is calculated as

$$\rho_i = 0.12 \cdot \frac{1 - \exp(-50 \cdot PD_i)}{1 - \exp(-50)} + 0.24 \cdot \left(1 - \frac{1 - \exp(-50 \cdot PD_i)}{1 - \exp(-50)}\right). \quad (2.13)$$

The factor loadings of obligor i can then be calculated using equation 2.9.

A bank also needs to make a maturity adjustment to account for events that might occur for maturities longer than one year, for example a decline in credit rating. This adjustment for maturity M is calculated as,

$$MA = \frac{1 + (M - 2.5)b}{1 - 1.5b}. \quad (2.14)$$

The constant $b = \big(0.11852 - 0.05478 \cdot \ln(PD)\big)^2$ and for $M = 1$ the maturity adjustment equals 1 as well.

When a bank has chosen, and been given clearance, to use the IRB-approach there are two possible methods. One is the Foundation IRB where banks are expected to provide their internal estimations of the probability of default for their borrowing grades. All other relevant risk measures comes from measures estimated by the supervisor (BCBS, 2004b). The other method is the Advanced IRB. For this method the bank should provide their own estimations or calculations for all necessary risk measures (BCBS, 2004b).

2.5 Credit concentration risk

The ASRF model in Pillar 1 gives the capital requirement for credit risk but it does not state what happens if the assumptions of the model are not met. A portfolio that does not fulfill the assumptions of the ASRF model is said to contain credit risk concentration, which is not considered in the credit risk capital requirements of Pillar 1 (Grippa and Gornicka, 2016). However the Basel accords state in Pillar 2 that concentration risk needs to be considered when calculating the capital requirement of credit risk (BCBS, 2004b). Below the two subgroups of credit concentration risk considered in this thesis are presented.

2.5.1 Name concentration risk

Name concentration risk arises when the size of the portfolio is small or the exposure to a few individual borrowers is large compared to the total exposure of the portfolio. This violates one of the assumptions of the ASRF model, which states that the portfolio should be infinitely granular. A bank can have portfolios large enough to assume that the risk is diversified away in accordance with the assumptions made in the ASRF, but in many cases name concentration is present in a portfolio. If this concentration risk is not considered it can lead to an understatement of the capital requirement (Grippa and Gornicka, 2016).

2.5.2 Sector concentration risk

The second subgroup of concentration risk is sector risk. This type of risk stems from portfolio groups that have a common underlying risk factor (Hibbeln, 2010). This leads to imperfectly correlated groups, or sectors, and it violates the second assumption of the ASRF model, which states that the model assumes a single systematic risk factor (Grippa and Gornicka, 2016). Industry concentration relates to sectors that can be defined by a specific industry while geographical concentration can contain both region and country specific sectors (Hibbeln, 2010). The capital needed for sector concentration risk will depend on the correlation between the sectors' underlying risk factors. A high correlation between two risk factors will lead to a higher capital requirement while a low correlation can be seen as a hedge and will therefore lead to a lower capital requirement.

Chapter 3

Mathematical Theory

This chapter will present the reader with the main theory needed for understanding the methods used later in this thesis. While the previous chapter focused on the theory and workings of credit risk, this chapter includes the additional mathematical theory necessary for this thesis. The theory is presented within the thesis limitations and the interested reader is referred to the sources for a more extensive presentation.

3.1 Monte Carlo

The main idea behind Monte Carlo (MC) simulation is to give an estimate of the expected value of a certain function of a random variable, that is, for the r.v. $\mathbf{X}$, estimate the value $E[h(\mathbf{X})]$. Given that $\mathbf{X}$ has the density function f, the expected value can be calculated as,

$$E[h(\mathbf{X})] = \int h(x)f(x)dx \tag{3.1}$$

Before introducing the MC estimator the law of large numbers is presented as found in Ross (2014),

Theorem 3.1.1 (Law of large numbers)**.** *Let $X_1, X_2, ...$ be a sequence of independent random variables having a common disitribution, and let $E[X_i] = \mu$. Then with probability 1,*

$$\frac{X_1 + X_2 + ... + X_n}{n} \to \mu, \text{ as } n \to \infty.$$

The Monte Carlo estimator of the expected value in equation (3.1) is calculated as

$$\mu_{MC} = \frac{1}{N}\sum_{i=1}^{N} h(\mathbf{X}), \tag{3.2}$$

and by the law of large numbers, $\mu_{MC} \to \mu$ as $N \to \infty$. Using this, fairly complicated values can be estimated as long as they can be rewritten as the expectation in equation (3.1). The variance of the Monte Carlo estimator becomes

$$\sigma^2_{MC} = \frac{1}{N-1}\sum_{i=1}^{N}\Big(h(\mathbf{X}) - \mu_{MC}\Big)^2. \tag{3.3}$$

The variance will decrease as N increases. This implies that for some cases a large number of steps are required before the Monte Carlo expectation converges towards the real value

and the computational burden might prove very heavy (Givens and Hoeting, 2013). Monte Carlo simulation performed without any variance reducing technique is often referred to as Crude Monte Carlo (CMC).

3.1.1 Monte Carlo simulation for estimating risk quantiles

When using CMC to estimate a quantile of a distribution the approach is slightly different than the one described above. Looking at the definition of VaR in equation 2.2, the CMC approach to estimate this quantile is to replace the unknown distribution function $F_{Loss}(l)$ with an empirical distribution $\hat{F}_{Loss}(l)$. The empirical function is found as the step function

$$\hat{F}_{Loss}(l) = \frac{1}{S_{MC}} \sum_{i=1}^{S_{MC}} \mathbb{1}_{\{L_i \leq l\}}, \tag{3.4}$$

where S_{MC} is the number of MC steps and $\mathbb{1}_{\{L_i \leq l\}}$ is the indicator function taking value 1 if $L_i \leq l$ and 0 otherwise (Brereton, Chan, and Kroese, 2013). In Brereton, Chan, and Kroese (2013) the following description of the algorithm implemented to estimate the VaR at level α is presented:

Algorithm to estimate VaR

1. Generate an iid sample $L_1, ..., L_N$
2. Order the sample from smallest to largest as $L_{(1)} \leq ... \leq L_{(N)}$
3. Return the quantile estimator $\hat{q}_\alpha = L_{(\lceil \alpha \cdot S_{MC} \rceil)}$

3.2 Importance Sampling

As seen in equation (3.3) the variance of the MC estimator decreases as the number of steps increase. However using a large number of steps can be computationally burdensome and in the case of estimating a quantile a large number of steps will be necessary. As the quantile represents an extreme event, it naturally does not occur often and a large number of simulations will be needed to find a converging estimation of the quantile. If the number of steps needs to be reduced or a rare event is to be estimated, variance reducing techniques can be implemented. These are important tools to reduce the number of steps without loosing measurement accuracy. One such technique is Importance Sampling (IS). IS uses an importance sampling distribution so that samples that have a low probability in the target distribution function now become sampled more frequently. To compensate for shifting the distribution, importance weights are introduced. The use of IS can be helpful when the sought for value is a rare event in the distribution function (Givens and Hoeting, 2013).

The idea behind IS is to rewrite the expectation in (3.1) as

$$E[h(\mathbf{X})] = \int h(x)f(x)dx = \int h(x)\frac{f(x)}{g(x)}g(x)dx, \tag{3.5}$$

where $g(x)$ is the IS distribution. By letting $\omega(x) = \frac{f(x)}{g(x)}$ one sees that equation (3.5) becomes the expectation of $h(\mathbf{X})$ if $\mathbf{X}_{1:S_{MC}}$ are iid observations generated from the distribution function $g(x)$. The estimated expected value now becomes

$$\hat{\mu}_{IS} = \frac{1}{S_{MC}} \sum_{i=1}^{S_{MC}} h(\mathbf{X}_i)\omega(\mathbf{X}_i). \tag{3.6}$$

The factor $\omega(\mathbf{X})$ is the previously mentioned importance weight that compensates for drawing $\mathbf{X}$ from the IS distribution. Just as for CMC, the IS estimator will converge towards the target expected value as the number of simulation steps grows large. However, some caution should be taken when choosing $g(x)$. By choosing $f(\mathbf{X})/g(\mathbf{X})$ in such a way that the fraction is large only when $h(\mathbf{X})$ is small, a relatively small variance is ensured (Givens and Hoeting, 2013).

3.2.1 Importance sampling for estimating risk quantiles

The theory devoted to efficient Monte Carlo methods for rare events has mainly been developed in the context of estimating rare-event probabilities of the form $l = \mathbb{P}(S(\boldsymbol{X}) > \gamma)$ for some real-valued function S, threshold γ and random vector $\boldsymbol{X}$. Glynn (1996) suggested an importance sampling approach to quantile estimation. The CMC estimator of the empirical distribution function is replaced with the IS estimator

$$\hat{F}^{IS}_{Loss}(l) = 1 - \frac{1}{N} \sum_{i=1}^{N} W(L_i)\mathbb{1}_{\{L_i > l\}}, \tag{3.7}$$

where the $\{L_i\}$ are drawn from the IS density g and $W(l) = f_{Loss}(l)/g_{Loss}(l)$ is the likelihood ratio. The estimator suggested by Glynn (1996) focuses on the right tail of the distribution, making it suitable in the case of VaR estimations. The reader is referred to the article for a motivation of the focus of the estimator. The IS VaR estimator then becomes

$$VaR^{IS}_{\alpha} = \inf\{l : \hat{F}^{IS}_{Loss}(l) \geq \alpha\}. \tag{3.8}$$

If g is chosen such that draws from the right tail of L happen more frequently, the estimator could provide considerably better performance than the CMC estimator. Brereton, Chan, and Kroese (2013) suggest that a good choice of g is the density g that minimizes the variance of

$$\hat{l}^{IS} = \frac{1}{N} \sum_{i=1}^{N} W(L_i)\mathbb{1}_{\{L_i > VaR_{\alpha}\}}, \tag{3.9}$$

where the $\{L_i\}$ are drawn from g. This is the standard IS estimator for

$$l = \mathbb{P}(L > VaR_{\alpha}). \tag{3.10}$$

A disadvantage to this method is that the computation of $\hat{l}^{IS}$ involves VaR_{α} which is the unknown quantity we seek to estimate. However the IS estimator for VaR is able to provide large efficiency gains even when the initial estimate of VaR_{α} is quite inaccurate. A rough estimate of VaR_{α} is easy obtained by an initial simulation using CMC. Another problem of this method is that importance sampling cannot be applied directly to $\{L_i\}$ as the density f_{Loss} is usually unknown. Instead we seek to represent L as a function of a random vector $\boldsymbol{X}$ with known density f_x to which we can apply importance sampling. So, the main idea is to first calculate an initial estimate of VaR_{α}, denoted $\widehat{VaR}_{\alpha}$, and then to find an appropriate importance sampling density for estimating $\mathbb{P}(L > \widehat{VaR}_{\alpha})$.

Glasserman and Li (2005) suggest an approach to find an appropriate importance sampling density and the following method is based on the their article. They propose two methods to reduce the variance when estimating a tail event. The first is exponential twisting of

the conditional default probabilities and the second is exponential twisting followed by applying IS to the systematic risk factor. The aim of the exponential twisting is to twist the conditional probability of default $PD_i(x)$ towards larger values in order to achieve a low variance estimator of a tail probability (Glasserman and Li, 2005). The tail probability is written as $P(L > L_\alpha)$, where L as usual is the loss and L_α is a large loss in the tail of the loss distribution. The exponential twisting assumes that all obligors are independent and that there exists a default indicator D_i, which will take value 1 if there is default and 0 otherwise. By replacing the PD_i with new probabilities Q_i the likelihood ratio relating the new distribution of the default indicator to the old one becomes

$$LR = \prod_{i=1}^{N} \left(\frac{PD_i}{Q_i} \right)^{D_i} \left(\frac{1 - PD_i}{1 - Q_i} \right)^{1-D_i}. \tag{3.11}$$

The tail probability can now be written as an expectation using the new probabilities Q_i according to

$$P(L > L_\alpha) = E_Q[\mathbb{1}_{\{L>L_\alpha\}} LR]. \tag{3.12}$$

This means that the estimator of $P(L > L_\alpha)$ is unbiased if the default indicators are sampled using the new default probabilities. Given that $c_i = LGD_i \cdot EAD_i$, the suggested exponential twisting is written as

$$Q_i = \frac{PD_i e^{\theta c_i}}{1 + PD_i(e^{\theta c_i} - 1)}. \tag{3.13}$$

As can be seen in equation 3.13 these new probabilities will only be larger than the previous PD_i if $\theta > 0$. Using the new probabilities the LR in equation 3.11 simplifies to

$$LR = \exp(-\theta L + \Psi(\theta)), \tag{3.14}$$

where $\Psi(\theta) = \log(E[e^{\theta L}]) = \sum_{i=1}^{N} \log(1 + PD_i(e^{\theta c_i} - 1))$. The unbiased estimator of $P(L > L_\alpha)$ can thus be found as $E[\mathbb{1}_{\{L>L_\alpha\}} e^{\theta L + \Psi(\theta)}]$. However, θ still needs to be decided. This is done be recalling why the IS is used, to reduce the variance. Reducing the variance is equivalent to minimizing the second moment of the estimator, which is written as

$$E_\theta[\mathbb{1}_{\{L>L_\alpha\}} e^{-2\theta L + 2\Psi(\theta)}], \tag{3.15}$$

where $E_\theta[\cdot]$ indicates that the expected value is found using θ to calculate the twisted probabilities. The second order moment is difficult to minimize and Glasserman and Li therefore suggest to minimize the upper bound, which is $e^{-2\theta L_\alpha + 2\Psi(\theta)}$. To minimize the upper bound, $\theta L_\alpha - \Psi(\theta)$ is maximized for $\theta \geq 0$. As $\Psi(\theta)$ is strictly convex the solution becomes

$$\theta = \begin{cases} \text{Unique solution to } \Psi'(\theta) = L_\alpha, & L_\alpha > \Psi'(0) \\ 0, & x \leq \Psi'(0). \end{cases} \tag{3.16}$$

When θ has been found the twisting can be done and the new probabilities of default are found.

The second method presented here to reduce the variance suggests to combine the exponential twisting explained above with applying IS to the systematic risk factors. As the theory of the exponential twisting has already been explained the following section will focus on the IS for the systematic factors. To motivate their approach, Glasserman and Li conclude that the variance of the estimator $\hat{p}$ of the tail probability $P(L > L_\alpha)$ can be divided according to,

$$V[\hat{p}] = E[V[\hat{p}|\boldsymbol{X}]] + V[E[\hat{p}|\boldsymbol{X}]]. \tag{3.17}$$

The vector $\boldsymbol{X}$ contains S systematic risk factors, L is the loss and L_α signifies a tail event, that is a large loss. By applying the exponential twisting the conditional variance $V[\hat{p}|\boldsymbol{X}]$ in equation 3.17 becomes small, which indicates that the focus in order to reduce the variance of the estimator should be on the term $V[E[\hat{p}|\boldsymbol{X}]]$, when applying the IS to the systematic risk factors. As $E[\hat{p}|\boldsymbol{X}] = P(L > L_\alpha|\boldsymbol{X})$ the sought for IS distribution should be chosen such that it reduces the variances when estimating the integral of $P(L > L_\alpha|\boldsymbol{X})$. In Glasserman and Li the suggested distribution is a function proportional to

$$X \mapsto P(L > L_\alpha|\boldsymbol{X} = \boldsymbol{x})e^{-\boldsymbol{x}^T\boldsymbol{x}/2}. \tag{3.18}$$

However, in order to normalize this function we need to divide by the probability $P(L > L_\alpha)$. As this is the sought for probability this function is not suitable to use as IS distribution. Instead a normal distribution with the same mode as the optimal density above is suggested. The mode μ can be found as the maximization of 3.18 according to

$$\boldsymbol{\mu} = \max_{\boldsymbol{x}} P(L > L_\alpha|\boldsymbol{X} = \boldsymbol{x})e^{-\boldsymbol{x}^T\boldsymbol{x}/2}. \tag{3.19}$$

By finding $\boldsymbol{x}$ that maximizes the expression above, the means of the normal distribution proposed as the IS density to be applied to the systematic risk factors are found. However, it is not always straight forward to find the exact solution to 3.19 and Glasserman and Li propose several ways of simplifying the problem through further approximation. This thesis utilizes the *constant approximation* that suggests to replace the loss L by $E[L|\boldsymbol{X} = \boldsymbol{x}]$ and the tail probability $P(L > L_\alpha|\boldsymbol{X} = \boldsymbol{x})$ with the indicator function $\mathbb{1}_{\{E[L|\boldsymbol{X}=\boldsymbol{x}]>L_\alpha\}}$. The problem in 3.19 can thus be rewritten as

$$\boldsymbol{\mu} = \min_{\boldsymbol{x}} \boldsymbol{x}^T\boldsymbol{x} \tag{3.20}$$

$$\text{s.t} \quad E[L|\boldsymbol{X} = \boldsymbol{x}] > L_\alpha. \tag{3.21}$$

Note that $E[L|\boldsymbol{X} = \boldsymbol{x}] = \sum_i PD_i(\boldsymbol{x}) \cdot EAD_i \cdot LGD_i$, which is inserted in the condition of the minimization problem. To restate the problem, $\boldsymbol{\mu}$ will be found as the minimization w.r.t. $\boldsymbol{x}$ of $\boldsymbol{x}^T\boldsymbol{x}$, conditional on $\sum_i PD_i(\boldsymbol{x}) \cdot EAD_i \cdot LGD_i > L_\alpha$. When this problem has been solved and $\boldsymbol{\mu}$ attained, the systematic risk factors $\boldsymbol{X}$ can now be drawn from $N(\boldsymbol{\mu}, I)$. To compensate for the shifted distribution, the estimator of the tail probability becomes

$$\mathbb{1}_{\{L>L_\alpha\}}e^{-\boldsymbol{\mu}^T\boldsymbol{X}+\boldsymbol{\mu}^T\boldsymbol{\mu}/2}, \tag{3.22}$$

where the added term $e^{-\mu^T\boldsymbol{X}+\mu^T\mu/2}$ is the likelihood ratio relating the density of the shifted distribution to the original standard normal distribution (Glasserman and Li, 2005).

3.3 Cholesky decomposition

Cholesky decomposition is a method for writing a matrix as a product of another matrix and its transpose. The definitions needed are stated below before stating the theorem for the Cholesky decomposition.

Definition 3.3.1. A matrix $A = [a_{ij}] \in M_n$ is *Hermitian* if $A = A^*$.

Definition 3.3.2. A Hermitian matrix $A \in M_n$ is *positive definite* if $x^*Ax > 0$ for all non-zero $x \in C^n$. It is *positive semidefinite* if $x^*Ax \geq 0$ for all nonzero $x \in C^n$.

Theorem 3.3.1 (Cholesky decomposition)**.** *Let $A \in M_n$ be Hermitian. Then A is positive semidefinite (respectively, positive definite) if and only if there is a lower triangular matrix $L \in M_n$ with non-negative (respectively, positive) diagonal entries such that $A = LL^*$. If A is positive definite, L is unique.*

For the proof of the theorem and more theory on Cholesky decomposition we refer to Horn and Johnson (2012).

The following theorem builds upon the Cholesky decomposition and will be needed later in this thesis when the models for sector concentration are presented.

Theorem 3.3.2. *Let ϵ be a random vector with identically independent normally distributed elements and let L be the matrix in the Cholesky decomposition of a hermitian positive semidefinite matrix A. Then $L\epsilon$ will be random variables with zero mean and unit variance but where the internal correlations of the random variables will be equal to the matrix A.*

The proof can be seen in equation 3.23 below,

$$\begin{aligned} var[L\epsilon] &= E[(L\epsilon)(L\epsilon)^*] - E[L\epsilon]E[L\epsilon]^* \\ &= E[(L\epsilon)(L\epsilon)^*] - LE[\epsilon]E[\epsilon^*]L^* \\ &= E[L\epsilon\epsilon^*L^*] \\ &= LE[\epsilon\epsilon^*]L^* \\ &= LL^* \\ &= A. \end{aligned} \tag{3.23}$$

3.4 A series expansion for the bivariate normal integral

The bivariate cumulative normal distribution function is defined as

$$N_2(x, y, \rho) = \int_{-\infty}^{x}\int_{-\infty}^{y} n_2(u, v, \rho)dudv, \tag{3.24}$$

where the bivariate normal density is given by

$$n_2(u, v, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} exp\bigg(-\frac{1}{2}\frac{u^2 - 2\rho uv + v^2}{1-\rho^2}\bigg). \tag{3.25}$$

The tetrachoric series is a standard procedure for calculating the bivariate normal distribution function (Vasicek, 1998)

$$N_2(x, y, \rho) = N(x)N(y) + n(x)n(y)\sum_{k=0}^{\infty}\frac{1}{(k+1)!}He_k(x)He_k(y)\rho^{k+1}, \tag{3.26}$$

where $N(\cdot)$ is the cumulative normal distribution function, $n(\cdot)$ is the normal density function and

$$He_k(x) = \sum_{i=o}^{[k/2]}\frac{k!}{i!(k-2i)!}(-1)^i 2^{-i}x^{k-2i}, \tag{3.27}$$

are the Hermite polynomials.

It should be noted that the tetrachoric series in equation 3.26 converges only slighter faster than a geometric series with quotient ρ and is therefore not very practical to use when ρ is large in absolute value. An alternative series that converges approximately as a geometric series with quotient $(1-\rho^2)$ is given in Vasicek (1998).

Chapter 4

Credit concentration risk under FI

As previously stated, the capital requirement for credit risk calculated under Pillar 1 does not take credit concentration risk into account. Instead concentration risk is considered when calculating the capital requirement in Pillar 2. One of many tasks of the Swedish FI is to control that banks consider the credit concentration risk and as a part of this FI has proposed their own models to be used for calculating the capital requirement for concentration risk (Finansinspektionen, 2014).

4.1 Name concentration risk under FI

FI proposes two methods to calculate the capital requirement for name concentration risk; both will be presented below. The first method is for banks using the standardized approach to calculate the capital requirement for credit risk while the second method is for banks using the IRB approach. The second method is more finely calibrated than the first method but requires more extensive data material, which firms using the Standardized approach cannot be assumed to be able to provide.

The first method, the standardized approach, uses the Herfindahl index of the 30 largest exposures (Finansinspektionen, 2014). Given that EAD_i is the i:th largest exposure, the Herfindahl index can be calculated as

$$HI_{name} = \sum_{i=1}^{30} \sigma_i^2, \tag{4.1}$$

where σ_i is the fraction of exposure i and the total exposure for the 30 largest exposures

$$\sigma_i = \frac{EAD_i}{\sum_{j=1}^{30} EAD_j}. \tag{4.2}$$

In order to take into account how large the proportion of the 30 largest exposures are of the entire portfolio the Adjusted Herfindahl (AHI) index is used by FI. For a portfolio of totally N exposures this measure is found according to

$$AHI = HI_{name} \cdot \frac{\sum_{i=1}^{30} EAD_i}{\sum_{i=1}^{N} EAD_i}. \tag{4.3}$$

Finally, FI uses the AHI to calculate the captial requirment p_{name}^{SA} as a percentage of the credit risk capital requirement of pillar 1 and the formula is given as (Finansinspektionen, 2014),

$$p_{name}^{SA} = 9(1 - \exp(-18 \cdot AHI)). \tag{4.4}$$

The second approach to calculate the capital requirement for name concentration risk is based on the results from Gordy and Lütkebohmert (2013). The model proposed is formulated in the CreditRisk$^+$ framework, a credit risk model that assumes a gamma distribution for the systematic risk factor (Gordy and Lütkebohmert, 2013). FI does not use the full expression found in Gordy and Lütkebohmert but rather a simplified version. The capital requirement is calculated using the following equation,

$$p_{name}^{GL} = \frac{100}{2K^2} \sum_{i=1}^{N} s_i^2 (0.25 + 0.75 LGD_i) \cdot (4,83(K_i + R_i) - K_i). \tag{4.5}$$

The total number of exposures are N and LGD_i is the loss given default of the i:th exposure. The rest of the parameters in equation (4.5) are presented below. The exposure at default for the i:th exposure is EAD_i, EL_i is the expected loss and UL_i is the unexpected loss for the i:th exposure (Finansinspektionen, 2014).

$$\begin{aligned} R_i &= \frac{EL_i}{EAD_i} \\ K_i &= \frac{UL_i}{EAD_i} \\ K &= \frac{\sum_{i=1}^{N} UL_i}{\sum_{i=1}^{N} EAD_i} \\ s_i &= \frac{EAD_i}{\sum_{i=1}^{N} EAD_i}. \end{aligned}$$

The capital requirement for name concentration p_{name}^{GL} is again given as a percentage of the capital requirements for credit risk under Pillar 1.

4.2 Sector concentration risk under FI

The second concentration risk considered is sector concentration which FI divides into industry and geographical concentration. For industry concentration the systematic risk factor is industry specific, meaning that obligors belonging to the same industry sector will share a common systematic risk factor. The same is true for geographical sector concentration but the risk factors will now be common for the obligors in the same geographical sectors.

To find the capital requirement for industry concentration risk, FI uses the HI calculated for the 12 industry sectors considered by FI. These sectors are: credit institutions; housing loans; other lending to households; real estate activities; commerce; hotels and restaurants; construction; manufacturing; transportation; forestry and agriculture; other service activities; and other corporate lending. The method to find the HI is much the

same as described above in the case of name concentration. However there are some important differences, the summarization is in this case done over the twelve industry sectors for a slightly different σ_i. The HI for industries is thus written as,

$$HI_{ind} = \sum_{i=1}^{12} \sigma_i^2, \tag{4.6}$$

where σ_i this time is found as the total exposure to sector i divided by the total exposure of the entire portfolio. This means that σ_i indicates the share sector i has in the portfolio. Given the HI_{ind}, the capital requirement is then calculated as

$$p_{ind} = 8(1 - \exp(-5 \cdot HI_{ind}^{1.5})). \tag{4.7}$$

To find the capital requirement for geographical sector concentration risk the same kind of calculation as for industry concentration is applied by FI. The Herfindahl index is calculated as in equation 4.6 but the sectors are now decided by the geographical regions defined by FI. The capital requirement is found as

$$p_{geo} = 8(1 - \exp(-2HI_{geo}^{1.7})). \tag{4.8}$$

The 16 geographical sectors considered by FI are Sweden, Norway, Denmark, Finland, Estonia, Latvia, Lithuania, Germany, Poland, Great Brittain, rest of Europe, Russia, Japan, North America and other countries.

Chapter 5

Methods for measuring Name concentration

While FI uses the methods presented in Chapter 4, a bank may choose internal methods to calculate the required capital for concentration risk. By allowing internal methods it is possible for a bank to customize the calculations to the bank's specific risks and conditions. However, FI states that internal models are often complicated and can give rise to an increased model risk and that the result from an internal method should not differ too much from the result of FI's method (Finansinspektionen, 2014). In this chapter two suggestions on alternative methods to calculate name concentration risk capital are presented.

5.1 Full Monte Carlo

The first method chosen is the Full Monte Carlo (FMC) building on the theory presented in section 3.1 and 3.2. The value to be estimated is the VaR at a 99.9% level and following the theory of using MC to estimate a quantile, the loss is to be represented by an empirical loss distribution. To reduce the variance and find a converging estimation of the VaR the IS-approach suggested by Glasserman and Li (2005) is implemented. The theory of the IS is presented in a vectorized form but in the case of risk estimation for name concentration there is only one systematic risk factor. The steps presented below are followed when implementing the FMC method:

- Implement the IS according to section 3 and find a suitable μ. The IS is applied only to the systematic risk factor and the exponential twisting is therefore not implemented.
- Draw the idiosyncratic risk factors ξ_i from a standard normal distribution and the systematic risk factor X from a normal distribution with mean μ and unit variance.
- The asset return Y_i is calculated for each obligor according to equation 2.8.
- The calculated asset return is compared to the PD threshold specified for each obligor. Default occurs if $Y_i < c_i$ is true and as can be seen in equation 2.7, $c_i = N^{-1}(PD_i)$. The loss for the defaulting obligors is calculated as $LGD_i \cdot EAD_i$.

The total loss is calculated as the sum of all individual losses, that is

$$Loss = \sum_{i=1}^{M} LGD_i \cdot EAD_i \cdot \mathbb{1}_{\{Y_i < N^{-1}(PD_i)\}}. \tag{5.1}$$

- To account for shifting in the distribution of the systematic risk factor, the VaR is found in equation 3.8 using $\alpha = 99.9\%$. The UL is then given from equation 2.12.

5.2 Partial Portfolio Approach

This section describes the Partial Portfolio Approach (PPA) suggested by Grippa and Gornicka (2016). It is very similar to the method suggested by Higo (2006) and in this thesis features of both models will be implemented. The main idea of the PPA is to give an estimation of the VaR that is not to computationally heavy but that at the same time addresses the fact that real life portfolios are not infinitely granular. This is done by splitting the portfolio into two separate sub-portfolios. The first portfolio, henceforth referred to as portfolio A, is based on the M largest exposures and will require a granularity adjustment. The second sub-portfolio, henceforth referred to as portfolio B, is the rest of the exposures in the portfolio and is assumed to be diversified. The authors justify this division by arguing that the granularity adjustment is the most important for the largest exposures. By dividing the portfolio in this way the simulations that are extra computationally heavy will be performed only for the M largest exposures belonging to portfolio A instead of the entire portfolio.

The steps to implement the PPA are described below but before starting, the number of exposures in each portfolio needs to be decided. It needs to be taken into account that there are enough obligors in portfolio A to ensure that the calculated VaR is close to the VaR for the full portfolio. At the same time, the number of obligors in portfolio A can not become too large as this will reduce the computational gain. The literature proposes two different ways of defining the sub-portfolios. Grippa and Gornicka classify as large, thus belonging to portfolio A, those obligors whose exposure share in the total portfolio is above a certain threshold. The threshold is chosen between 0 and 100 percent. A threshold at 0% will put all obligors in the non-granular sub-portfolio A while a threshold at 100% means that all belong to portfolio B. Thus, the higher the threshold the fewer exposures will be classified as large. Grippa and Gornicka do not make any general recommendations on how to choose the threshold. A second suggestion comes from Higo (2006), who proposes a way of defining the sub-portfolios where the sum of exposure weights in the granular sub-portfolio B is studied to find a sufficiently fine grained sub-portfolio B,

$$\frac{\sum_{i=M+1}^{N} EAD_i^2}{\left(\sum_{i=1}^{N} EAD_i\right)^2} = \sum_{i=M+1}^{N} \sigma_i^2 \approx 0. \tag{5.2}$$

It is not known how small the sum of exposure weights squared in sub-portfolio B should be to have an accurate approximation of the VaR of the entire portfolio. The two different ways of choosing the threshold will be analyzed. The goal is to find a measure that gives good result for portfolios with different levels of sector concentration. Good results are defined as results close to the results of a full Monte Carlo simulation but where the number of exposures in portfolio A is relatively small compared to the number of exposures in portfolio B.

Assuming that there are a total of N obligors and M of these are in portfolio A, the PPA can be implemented in much the same way as for the FMC but with adjustments described in the implementation steps below:

- As for the FMC, implement the IS according to section 3 and find a suitable μ. The IS is applied only to the systematic risk factor and the exponential twisting is not implemented.
- Draw the systematic risk factor X from the distribution $N(\mu, 1)$.
- For each exposure in portfolio A draw the idiosyncratic risk factor ξ_i from a standard normal distribution.
- The asset return Y_i is calculated for each obligor in portfolio A as in equation (2.8).
- The calculated asset return is compared to the PD threshold specified for each obligor. If $Y_i < c_i$ is true, meaning that obligor i defaults, the loss for the defaulting obligors in portfolio A is calculated as $LGD_i \cdot EAD_i$. The total loss in portfolio A is calculated as the sum of all individual losses, that is

$$Loss^A = \sum_{i=1}^{M} LGD_i \cdot EAD_i \cdot \mathbb{1}_{\{Y_i < N^{-1}(PD_i)\}}. \tag{5.3}$$

- As sub-portfolio B is assumed to be diversified the loss in this portfolio is calculated as the sum of the conditional expected losses according to

$$Loss^B = \sum_{i=1}^{N-M} LGD_i \cdot N\left[\frac{N^{-1}(PD_i) - w_i \cdot X}{\sqrt{1 - w_i^2}}\right] \cdot EAD_i. \tag{5.4}$$

- The total loss is calculated as

$$Loss = Loss^A + Loss^B. \tag{5.5}$$

- The VaR is found as in equation 3.8 using $\alpha = 99.9\%$. The UL is then found as in equation 2.12.

To further increase the speed of the method, Higo (2006) proposes to segment the obligors in subportfolio B into $K \times H$ homogeneous groups of PD_i and w_i. Equation 5.4 can then be rewritten as

$$Loss^G = \sum_{k=1}^{K} \sum_{h=1}^{H} \left(N\left[\frac{N^{-1}(PD_k^{group}) - w_h^{group} \cdot X}{\sqrt{1 - (w_h^{group})^2}}\right] \sum_{i \in \{k,h\}} EAD_i LGD_i \right), \tag{5.6}$$

where PD_k^{group} and w_h^{group} denote the default probability of group $k (k = 1, 2, ..., K)$ and the correlation of group $h (h = 1, 2, ..., H)$.

Chapter 6

Methods for measuring Sector concentration

In this chapter the models chosen for sector concentration risk are presented. As sector concentration in a portfolio implies that there are several underlying systematic risk factors, the multi-factor Merton model is used to implement the chosen methods.

6.1 Multi-factor Merton Model

Consider a portfolio of loans to M distinct borrowers. As before, PD_i is the probability of default for borrower i and default happens when a variable Y_i describing the financial well-being of borrower i falls below a threshold. Assuming that variables $\{Y_i\}$ are standard normally distributed the default threshold for an individual borrower is given by $N^{-1}(PD_i)$. Borrower i's standardized asset return depends linearly on S normally distributed systematic risk factors, K_s $(s = 1, ..., S)$ with a full-rank correlation. Each borrower has its own specific combination of these factors X_i (known as a composite factor)

$$Y_i = w_i X_i + \sqrt{1 - w_i^2}\xi_i \tag{6.1}$$

where ξ_i is the idiosyncratic risk which also has a standard normal distribution.

In the single factor Merton model the correlation structure of each firm is completely described by the correlation ρ, while in the multi-factor model we distinguish between inter-sector correlation ρ_{inter} and intra-sector correlation ρ_{intra}. The correlation between the systemic risk factors are described by the inter-sector correlation and the intra-sector correlation characterizes the sensitivity of the asset return to the composite factor.

Assume that the S original correlated systematic factors are decomposed into S independent normal systematic factors Z_k $(k = 1, ..., S)$. The relation between $\{Z_k\}$ and the composite factor is given by

$$X_i = \sum_{k=1}^{S} \alpha_{ik} Z_k. \tag{6.2}$$

To ensure that X_i has unit variance it must hold that $\sum_{k=1}^{S} \alpha_{ik}^2 = 1$. The factor weights α_{ik} may be calculated via Cholesky decomposition of the inter-sector correlation matrix,

as stated in theorem 3.3.2.

If borrower i defaults, the amount of loss is determined by its loss-given default stochastic variable $L\tilde{G}D_i$ with mean μ_{LGD_i} and standard deviation σ_{LGD_i}. It is assumed that $\{L\tilde{G}D_i\}$ are independent between themselves as well from all other variables in the model. Portfolio loss L can be written as a sum of individual portfolio losses L_i

$$L = \sum_{i=1}^{N} L_i = \sum_{i=1}^{N} EAD_i \mathbb{1}_{\{Y_i \leq N^{-1}(PD_i)\}} L\tilde{G}D_i \tag{6.3}$$

where $\mathbb{1}_{\{.\}}$ is the indicator function. If the portfolio is large enough to be considered fine-grained the portfolio losses are driven primarily by the systematic factors as the idiosyncratic risk is diversified away and equation 6.2 can be replaced by the limiting loss distribution of an infinitely fine-grained portfolio

$$L^{\infty} = E[L|\{Z_k\}] = \sum_{i=1}^{N} EAD_i \mu_{LGD_i} N\Big[\frac{N^{-1}(PD_i) - w_i \sum_{k=1}^{S} \alpha_{ik} Z_k}{\sqrt{1 - w_i^2}}\Big]. \tag{6.4}$$

Equation 6.4 is much simpler than equation 6.2 but Monte Carlo simulation of the systematic factors is still required when the number of factors is greater than one.

6.2 Implementation of a multi-factor Merton Model

We will implement a version of the multi-factor Merton model where it is assumed that the asset return of obligor i in sector s can be represented by

$$Y_{s,i} = w_i K_s + \sqrt{1 - w_i^2}\xi_i \tag{6.5}$$

where K_s is the sector risk factor (with $s = 1, ..., S$). So the difference from 6.1 is that instead of having its own specific combination of the systematic risk factors the asset return of each borrower only depends on one sector factor. In the same way as in 6.2 we may present the correlated sector risk factors K_s as a combination of independently and standard normally distributed factors $Z_k (k = 1, ..., S)$,

$$K_s = \sum_{k=1}^{S} \alpha_{s,k} Z_k \tag{6.6}$$

in which the factor weights $\alpha_{s,k}$ are calculated via a Cholesky decomposition of the inter-sector correlation matrix. This means that the inter-sector correlation is given as

$$\rho_{s,t}^{inter} = Corr(K_s, K_t) = \sum_{k=1}^{S} \alpha_{s,k} \cdot \alpha_{t,k} \tag{6.7}$$

From 6.1 and 6.2 the asset correlation between i in sector s and obligor j in sector t is given by

$$Corr(Y_{s,i}, Y_{t,j}) = \begin{cases} 1 & \text{if } s = t \text{ and } i = j \\ w_i \cdot w_j & \text{if } s = t \text{ and } i \neq j \\ w_i \cdot w_j \cdot \sum_{k=1}^{K} \alpha_{s,k} \cdot \alpha_{t,k} & \text{if } s \neq t \text{ and } i \neq j \end{cases} \tag{6.8}$$

From our definitions we divided the concentration risk into two parts, name and sector concentration. Name concentration arises in non-granular portfolios where the idiosyncratic risk cannot be ignored whereas sector concentration arises in portfolios that are not well diversified across sectors. In this thesis we have divided our methods into models for calculating an add-on for name concentration and sector concentration respectively, following the method of FI. Hence for our multi-factor model we are only interested in the risk coming from the sector distribution and will assume that the portfolio is fine grained so that the loss can be calculated from 6.4. In each Monte Carlo step then only the sector factors need to be simulated. If the combined concentration risk arising from both sector and name concentration are of interest the simulations are easily extended to incorporate name concentration as well.

As in the case of sub-portfolio B in the PPA method it is possible to rewrite equation 6.4 as follows when the obligors are segmented into $S \times K \times H$ homogeneous groups with respect to s, PD_i and w_i.

$$L^{\infty} = \sum_{s=1}^{S} \sum_{k=1}^{K} \sum_{h=1}^{H} \left(N\left[\frac{N^{-1}(PD_k^{group}) - w_h^{group} \cdot X_s}{\sqrt{1-(w_h^{group})^2}} \right] \sum_{i \in \{s,k,h\}} EAD_i LGD_i \right) \tag{6.9}$$

where again PD_k^{group} and w_h^{group} denote the default probability of group $k (k = 1, 2, ..., K)$ and the correlation of group $h (h = 1, 2, ..., H)$.

6.3 Intra-correlations in the multi factor model

Special attention needs to be paid to the choice of the intra-correlations in the multi factor model. In the single factor model a suggestion for the correlation is proposed by the Basel committee, see equation 2.13. The Basel correlation formula is calibrated on well-diversified portfolios (BCBS, 2006). This implies that the correlation formula is chosen in such way that the single-risk factor model leads to a good approximation of the risk based on the full correlation structure in a multi factor model.

Using the Basel correlation formula for the intra-sector correlations in a multi-factor model is equivalent to saying that the regulatory capital calculated in Pillar 1 is an upper barrier of the true risk. The Pillar 1 capital could only be an upper barrier if only one sector exists or if all sectors are perfectly correlated. In all other cases using the Basel correlations in a multi factor model will lead to an effect of sector diversification, which leads to a lower capital requirement compared to under Pillar 1. Cespedes et al. (2006) did the above and used the Basel correlation in their analysis. Afterwards they recognized the criticism and mentioned that it should be possible using some scaling for the intra-sector correlations. A scaling up of the intra-sector correlations is suggested by Hibbeln (2010). Hibbeln found the following intra-sector correlation to be a good match for portfolios with different quality distributions:

$$\rho_i = 0.185 \cdot \frac{1-\exp(-50 \cdot PD_i)}{1-\exp(-50)} + 0.34 \cdot \left(1 - \frac{1-\exp(-50 \cdot PD_i)}{1-\exp(-50)}\right). \tag{6.10}$$

These new intra-sector correlations will be used for all multi-factor models in this thesis. For more information of the derivation of this formula we refer to Hibbeln (2010).

6.4 Pykhtin's method

In his paper "Multi-factor adjustment" published in 2004, Michael Pykhtin presents an analytical method for calculating portfolio VaR in the multi-factor Merton framework. The target quantity is the quantile at a confidence level q of the portfolio loss L, $t_q(L)$. However, there is no straight forward analytical expression of this quantile in the multi-factor model and Pykhtin therefore assumes that a random variable $\bar{L}$ can be constructed. The variable is constructed in such way that its quantile $t_q(\bar{L})$ can be calculated analytically and is close enough to $t_q(L)$. The loss can then be seen as $L = \bar{L} + U$ where U is the perturbation. To describe the scale of the perturbation Pykhtin also introduce a perturbed variable $L_\epsilon = \bar{L} + \epsilon U$. For high enough confidence levels Martin & Wilde (2002) showed that $t_q(L)$ can be calculated via the Taylor expansion in powers of the perturbation variable ϵ around $t_q(\bar{L})$. By keeping terms up to quadratic the quantile can thus be expressed as,

$$t_q(L) = t_q(\bar{L}) + \frac{dt_q(L_\epsilon)}{d\epsilon}\Big|_{\epsilon=0} + \frac{1}{2}\frac{d^2 t_q(L_\epsilon)}{d\epsilon^2}\Big|_{\epsilon=0} \tag{6.11}$$

The first two derivatives of VaR were derived in Gourieroux, Laurent, and Scaillet (2000). The first derivative is given by the expectation of the perturbation conditional on $\bar{L} = t_q(\bar{L})$:

$$\frac{dt_q(L_\epsilon)}{d\epsilon}\Big|_{\epsilon=0} = E\Big[U|\bar{L} = t_q(\bar{L})\Big] \tag{6.12}$$

while the second derivative is

$$\frac{d^2 t_q(L_\epsilon)}{d\epsilon^2}\Big|_{\epsilon=0} = -\frac{1}{f_{\bar{L}}(l)}\frac{d}{dl}(f_{\bar{L}}(l)V[U|\bar{L} = l])\Big|_{l=t_q(\bar{L})} \tag{6.13}$$

where $f_{\bar{L}}(\cdot)$ is the probability density function for $\bar{L}$ and $\text{var}[U|\bar{L} = l]$ is the variance of U conditional on $\bar{L} = l$. For a derivation of the derivative expressions the reader is referred to Gourieroux, Laurent, and Scaillet (2000).

As equation 6.11 gives an expression to calculate $t_q(L)$ the main problem is now to find a suitable $\bar{L}$. As the target is to find an analytical expression of $t_q(\bar{L})$, the variable $\bar{L}$ can not be expressed in the multi-factor model and is therefore defined in the one-factor framework, via the limiting loss distribution, as

$$\bar{L} = l(\bar{X}) = \sum_{i=1}^{N} EAD_i \mu_{LGD_i} PD_i(\bar{X}) \tag{6.14}$$

where $PD_i(\bar{X})$ is the conditional default probability given by

$$PD_i(x) = N\left[\frac{N^{-1}(PD_i) - a_i x}{\sqrt{1 - a_i^2}}\right] \tag{6.15}$$

and a_i is the effective factor loading for borrower i, $\bar{X}$ is the single systematic risk factor and PD_i is the unconditional PD for borrower i. The systematic risk factor has a standard normal distribution. As we are now in the one factor framework there will be only one systematic risk factor compared to the multiple risk factors in the multi factor framework. Pykhtin further states that since $\bar{L}$ is a deterministic monotonically decreasing function of $\bar{X}$ it is possible to calculate the quantile of $\bar{L}$ at level q as the function value at $\bar{X} = N^{-1}(1-q)$:

$$t_q(\bar{L}) = l(N^{-1}(1-q)) = \sum_{i=1}^{N} EAD_i \mu_{LGD_i} PD_i(N^{-1}(1-q)). \tag{6.16}$$

Using that the derivatives in equations 6.12 and 6.13 are given by expressions conditional on $\bar{L} = t_q(\bar{L})$ and that this is equivalent to conditioning on $\bar{X} = N^{-1}(1-q)$, the derivatives in equation 6.11 can now be expressed as

$$\frac{dt_q(L_\epsilon)}{d\epsilon}\Big|_{\epsilon=0} = E\Big[U\Big|\bar{X} = N^{-1}(1-q)\Big] \tag{6.17}$$

and

$$\frac{d^2t_q(L_\epsilon)}{d\epsilon^2}\Big|_{\epsilon=0} = -\frac{1}{n(x)}\frac{d}{dx}n(x)\frac{v(x)}{l'(x)}\Big|_{x=N^{-1}(1-q)} \tag{6.18}$$

where $v(\cdot)$ is the conditional variance of U defined as $v(y) = \text{var}[U|\bar{X} = x]$, $l'(\cdot)$ is the first derivative of $l(\cdot)$ and $n(\cdot)$ is the standard normal density, which is the probability density of $\bar{X}$.

As $\bar{L}$ is defined in the one factor model, $\bar{L}$ needs to be related to the portfolio loss L defined in the multi factor framework. This is done by relating the single systematic factor $\bar{X}$ to the original systematic factors $\{Z_k\}$. Pykhtin assumes there is a linear relationship given by

$$\bar{X} = \sum_{k=1}^{S} b_k Z_k \tag{6.19}$$

where the coefficients must satisfy $\sum_{k=1}^{S} b_k^2 = 1$ to preserve unit variance of $\bar{X}$. To complete the specification of $\bar{L}$ we need to specify the effective factor loadings $\{a_i\}$ and the systematic factor coefficients $\{b_k\}$. In order to specify these coefficients Pykhtin starts by requiring that $\bar{L} = E[L|\bar{X}]$. As Pykhtin states, this condition is intuitively appealing and it will also guarantee that the first-order term in the Taylor series, given in equation 6.17, vanishes for any confidence level q. To calculate the expected conditional loss, the composite risk factor for borrower i is represented as

$$X_i = \bar{\rho}_i\bar{X} + \sqrt{1-\bar{\rho}_i^2}\eta_i, \tag{6.20}$$

where η_i is a standard normal variable independent of the systematic risk factor $\bar{X}$ and $\bar{\rho}_i$ is the correlation between X_i and $\bar{X}$ given by

$$\bar{\rho}_i \equiv \text{cor}(X_i, \bar{X}) = \sum_{k=1}^{S} \alpha_{ik} b_k, \tag{6.21}$$

where α is the same as in equation 6.2. With these notations the asset return given by equation 6.1 can be rewritten as

$$Y_i = w_i\bar{\rho}_i\bar{X} + \sqrt{1-(w_i\bar{\rho}_i)^2}\xi, \tag{6.22}$$

where ξ is a standard normal variable independent of $\bar{X}$. It then follows that the conditional expectation of L is

$$E[L|\bar{X}] = \sum EAD_i \cdot \mu_{LGD_i} \cdot N\Big[\frac{N^{-1}(PD_i) - w_i\bar{\rho}_i\bar{X}}{\sqrt{1-(w_i\bar{\rho}_i)^2}}\Big]. \tag{6.23}$$

By comparing equation 6.14 with equation 6.23 we see that $\bar{L} = E[L|\bar{X}]$ if and only if

$$a_i = w_i\bar{\rho}_i = w_i\sum_{k=1}^{N}\alpha_{ik}b_k. \tag{6.24}$$

Let us now assume that the effective factor loadings $\{a_i\}$ are given by equation 6.24 and that the first derivative of VaR is zero so that the correction to $t_q(\bar{L})$ is only given by the second derivative of VaR given in equation 6.18. Unlike the choice of the effective factor loadings $\{a_i\}$ the choice of $\{b_k\}$ is not critical to the presented method. As $\{b_k\}$ specifies the zeroth-order term $t_q(\bar{L})$ the method will work with many alternative specifications as long as they yield $t_q(\bar{L})$ close enough to the unknown target function $t_q(L)$. To do this the ideal solution would be to find a set $\{b_k\}$ that minimizes the difference between $t_q(\bar{L})$ and $t_q(L)$. This is however complicated and Pykhtin gives an easier specification of $\{b_k\}$. Pykhtin's solution builds on that intuitively one would expect the optimal single risk factor $\bar{X}$ to have as much correlation as possible with the composite risk factors $\{X_i\}$. Mathematically this translates to the following maximization problem

$$\max_{\{b_k\}} \left(\sum_{i=1}^{N} c_i \text{cor}(\bar{X}, X_i) \right) \quad \text{such that} \quad \sum_{K=1}^{S} b_k^2 = 1 \tag{6.25}$$

where the condition comes from equation 6.19 to assure unit variance of $\bar{X}$. By using equation 6.21 we find that the solution to this maximization problem is given by

$$b_k = \sum_{i=1}^{N} (c_i/\lambda)\alpha_{ik} \tag{6.26}$$

where the positive constant λ is the Lagrange multiplier chosen so that $\{b_k\}$ satisfy the constraint. There is no clear way of choosing the coefficients $\{c_i\}$ and Pykhtin tries several different specifications but finds the following to be one of the best-performing choices:

$$c_i = EAD_i \cdot \mu_{LGD_i} \cdot N\left[\frac{N^{-1}(PD_i) + w_i N^{-1}(\alpha)}{\sqrt{1 - w_i^2}} \right]. \tag{6.27}$$

To summarize, the loss quantile $t_q(L)$ can be expressed analytically, by introducing the r.v. $\bar{L}$, as in equation 6.11. However this expression can be simplified as described above by requiring that $\bar{L} = E[L|\bar{X}]$, which yields that the quantile can now be expressed as,

$$t_q(L) = t_q(\bar{L}) + \frac{1}{2} \frac{d^2 t_q(L_\epsilon)}{d\epsilon^2}\Big|_{\epsilon=0}. \tag{6.28}$$

6.4.1 Multi-factor adjustment

The multi-factor adjustment is defined, using equation 6.28, as

$$\Delta t_q \equiv t_q(L) - t_q(\bar{L}) = \frac{1}{2} \frac{d^2 t_q(L_\epsilon)}{d\epsilon^2}\Big|_{\epsilon=0}. \tag{6.29}$$

By calculating the derivative in equation 6.18 and using the relation $n'(x) = -xn(x)$, equation 6.29 can be rewritten as

$$\Delta t_q = -\frac{1}{2l'(x)} \left[v'(x) - v(x) \left(\frac{l''(x)}{l'(x)} + x \right) \right] \Bigg|_{x=N^{-1}(1-q)}. \tag{6.30}$$

The first and second derivatives of the function $l(x)$ are obtained by differentiation of equation 6.14,

$$l'(x) = \sum_{i=1}^{N} EAD_i \mu_{LGD_i} PD_i'(x) \tag{6.31}$$

and

$$l'(x) = \sum_{i=1}^{N} EAD_i \mu_{LGD_i} PD_i''(x) \tag{6.32}$$

where $PD_i'(x)$ and $PD_i''(x)$ are the first and second order derivatives of the conditional probability of default. The conditional probability of default is given in equation 6.15 and the differentiations yield

$$PD_i'(x) = -\frac{a_i}{\sqrt{1-a_i^2}} n\left[\frac{N^{-1}(PD_i) - a_i x}{\sqrt{1-a_i^2}}\right] \tag{6.33}$$

and

$$PD_i''(x) = -\frac{a_i^2}{1-a_i^2} \frac{N^{-1}(PD_i) - a_i x}{\sqrt{1-a_i^2}} \left[\frac{N^{-1}(PD_i) - a_i x}{\sqrt{1-a_i^2}}\right] \tag{6.34}$$

The derivative of the conditional variance $v(x)$ will only be presented in this report and the reader is referred to the full article of Pykhtin (2004) for the derivation. Pykhtin divides $v(x)$ into two parts, one part can be seen as quantifying the difference between the multi- and single-factor limiting loss distribution and is given by

$$v_\infty(x) = \sum_{i=1}^{N}\sum_{j=1}^{N} EAD_i EAD_j \mu_{LGD_i}\mu_{LGD_j} \left[N_2\Big(N^{-1}[PD_i(x)], N^{-1}[PD_j(x)], \rho_{ij}^X\Big) - PD_i(x)PD_j(x)\right] \tag{6.35}$$

where $N_2(\cdot,\cdot,\cdot)$ is the bivariate normal cumulative distribution function and ρ_{ij} is the element of a correlation matrix used in Pykhtin. The element in row i, column j of the correlation matrix ρ^X is given by

$$\rho_{ij}^X = \frac{r_i r_j \sum_{k=1}^{S} \alpha_{ik}\alpha_{jk} - a_i a_j}{\sqrt{(1-a_i^2)(1-a_j^2)}}. \tag{6.36}$$

The second term of $v(x)$ can be seen as a term describing the effect that a finite number of obligors will have on the portfolio and is expressed as,

$$v_{GA}(x) = \sum_{i=1}^{N} EAD_i^2 \Big(\mu_{LGD_i}^2\Big[PD_i(x) - N_2\Big(N^{-1}\big[PD_i(x)\big], N^{-1}\big[PD_i(x)\big], \rho_{ii}^X\Big)\Big] + \sigma_{LGD_i}^2 PD_i(x)\Big). \tag{6.37}$$

The derivatives of equations 6.35 and 6.37 are found as

$$v_\infty'(x) = 2\sum_{i=1}^{N}\sum_{j=1}^{N} EAD_i EAD_j \mu_{LGD_i}\mu_{LGD_j} PD_i'(x) \left[N\left(\frac{N^{-1}[PD_j(x)] - \rho_{ij}^X N^{-1}[PD_i(x)]}{\sqrt{1-\left(\rho_{ij}^X\right)^2}}\right) - PD_j(x)\right] \tag{6.38}$$

and

$$v'_{GA}(x) = \sum_{i=1}^{N} EAD_i^2 PD'_i(x) \left(\mu^2_{LGD_i} \left[1 - 2N\left(\frac{N^{-1}[PD_i(x)] - \rho^X_{ii} N^{-1}[PD_i(x)]}{\sqrt{1 - \left(\rho^X_{ii}\right)^2}} \right) \right] + \sigma^2_{LGD_i} \right). \quad (6.39)$$

As $v_{GA}(x)$ describes the effect of the finite number of loan in the portfolio, v_{GA} vanishes as the number of counterparties $N \to \infty$.

By dividing the conditional variance $v(x)$ into the two parts stated above, Pykhtin calculates the the multi-factor adjustment as $\Delta t_q = \Delta t_q^{\infty} + \Delta t_q^{GA}$. Both parts are calculated using equation 6.30 but the first uses only v_{∞} and the second part only v_{GA}. As v_{GA} can be seen as the granularity adjustment this term does not need to be included if the portfolio is assumed to be well diversified. Following the discussion in section 6.2 we are only interested in the add-on corresponding to the sector concentration. Hence we will calculate the sought for VaR as the quantile $t_q(L) = t_q(\bar{L}) + \Delta t_q^{\infty}$ using the coefficients and derivatives found in this section.

Chapter 7

Data

To evaluate our proposed models we will use simulated data to construct plausible loan portfolios. The simulated portfolios will contain different levels of concentration risk and have some features similar to those found in real banking portfolios. Different portfolios will be simulated for testing the models for name and sector concentration. The portfolios for testing the level of name concentration risk will differ in the distribution of EAD and the portfolios used to test the models for sector concentration will differ in the distribution between sectors. All portfolios will have common features and these, together with their individual characteristics, will be presented in this chapter.

7.1 General characteristics of simulated portfolios

As all portfolios, regardless if they are used to test the models for name or sector concentration, have some common characteristics. These will be presented before separating the portfolios into name and sector portfolios. The first assumption made is that each portfolio contains 10 000 counterparties and that each counterparty has only one loan. The size of the portfolio is selected to operate in the same order of magnitude as any standard bank. It is reasonable to assume counterparties with a single loan since if a portfolio consists of counterparties with multiple loans, the loans may be grouped into one loan per counterparty. For simplicity all counterparties are assumed to be big corporates and all loans are assumed to have a maturity of one year. This means that the maturity adjustment in equation 2.14 becomes 1 and will thus not affect the calculation of the IRB capital requirement.

Each counterparty is given a rating from 1 to 7 where 1 is the best, corresponding to a low probability of default. All counterparties with the same rating are assigned the same PD, where the PDs reach from 0.03% to 15%. The choice of 0.03% as the lowest value of PD is based on this being stated as the lowest possible PD in BCBS (2004b). The distribution between rating classes can be found in figure 7.1. The rating distribution is chosen in such way that there is a large number of counterparties with a good rating but there is still enough counterparties in the higher rating classes to make the distribution somewhat realistic.

There should be a dependence between EAD and PD in a portfolio as it is unlikely that a counterparty with a high PD is granted a large loan. Instead the counterparties with a low PD should have a better chance of obtaining a bigger loan. So in the creation of our portfolios we start by assigning a rating according to the distribution in figure 7.1. The

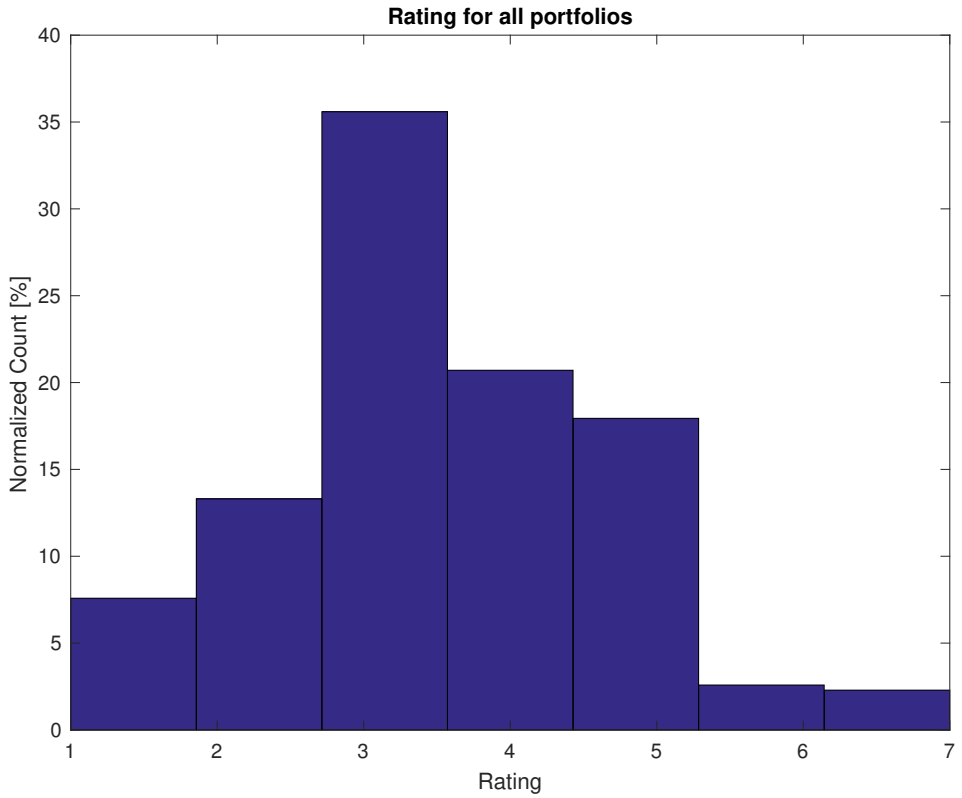

FIGURE 7.1: Rating distribution of simulated portfolios.

next step is then to assign an EAD given the rating and the overall distribution of EAD for the portfolio that is wished. The EAD distribution of each portfolio will differ and will be explained more closely in coming sections.

All exposures are given a LGD of 45%, which is based on BCBS (2004b) choosing this level under the foundation approach for senior claims on corporates. We therefore find this value for LGD suitable to use for all portfolios.

7.2 Portfolios containing name concentration

To test our models for calculating the name concentration risk we wish to create portfolios with a varying degree of name concentration. This is done by simulating portfolios with an increasing number of counterparties with large exposures. To measure the degree of name concentration the Herfindahl Index is used. Portfolio 1 will have the largest concentration, meaning that a few counterparties will have large exposures compared to the rest of the portfolio exposures. The number of large exposures is then increased, meaning that the name concentration will decrease, and Portfolio 5 will have the lowest degree of concentration. So the five final portfolios will go from a high concentration to a lower concentration. Table 7.1 shows the HI for each portfolio and in figure 7.2 the sorted EAD values for each portfolio are plotted. As can be seen, the EAD is more evenly distributed for an increasing portfolio number. Apart from the different distribution of EAD all five portfolios follow the general setup established for the creation of portfolios in this thesis.

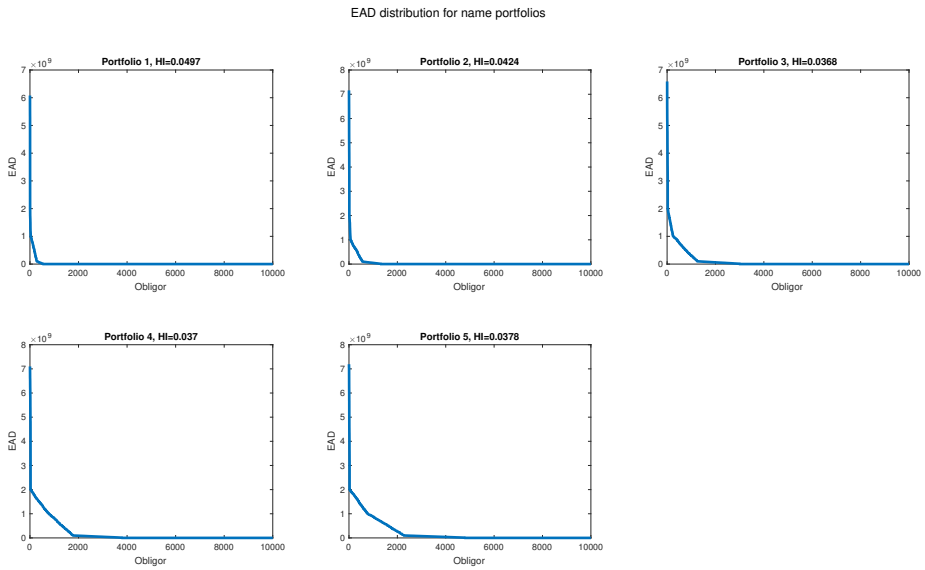

FIGURE 7.2: EAD distribution of simulated name portfolios.

	HI
Name Portfolio 1	0.0497
Name Portfolio 2	0.0424
Name Portfolio 3	0.0368
Name Portfolio 4	0.0370
Name Portfolio 5	0.0378

TABLE 7.1: HI of the five simulated name portfolios. The HI is calculated as defined by FI, that is HI is based on the 30 largest exposures of the portfolio.

7.3 Portfolios containing sector concentration

7.3.1 Correlation matrices

Before describing the characteristics of the portfolios used to test the models for measuring sector concentration risk, the correlation matrices for the correlation between sectors are presented. In chapter 4 the sectors used by FI to calculate measures of industry and sector concentration risk are presented and ideally these sectors would be used when simulating portfolios. However, the correlation matrices available are not based on the same sector division as the one used by FI. The matrices will be referred to as matrix A and matrix B. Matrix A is defined as a mixture between geographical and industry sectors, for example Europe Materials, and it is not possible to map the sectors at hand to those sectors used by FI. As the model used by FI to measure sector concentration risk is based on the sectors defined by FI it is not possible to apply FI's models to portfolios simulated with sectors from the correlation matrix A.

In an attempt to apply the models of FI on the sector portfolios correlation matrix B is used. The sector division of matrix B does not correspond perfectly to the one defined by FI for geographical sectors. To compensate for this the simulated portfolios will only contain those sectors that are shared with the geographical sectors of FI. These are Sweden,

Norway, Finland, Denmark, UK and North America.

The correlation matrices are calculated from publicly available industry/country indices with data from 1991-01-31 to 2015-11-30.

7.3.2 Characteristics of portfolios

To test the models for measuring the amount of sector concentration we wish to create portfolios with a varying degree of sector concentration. In contrast to the portfolios used for name concentration, the EAD distribution will remain the same for all portfolios. The used EAD distribution will be the same as seen in Portfolio 3 for name concentration. This choice is based on portfolio 3 not having any extreme EAD distribution. Again HI is used to measure the degree of concentration in each portfolio but it is now calculated on a sector level. The portfolios are divided into two subgroups, the first group will be simulated using the inter-sector correlations of matrix A and the second group will be simulated using the inter-sector correlations of matrix B, meaning that FI's methods can be applied only to the portfolios in the second group.

In the first group four portfolios will be simulated, all with varying sector distribution. The sector distribution of the first three portfolios is loosely based on the distribution presented on p. 189 in Hibbeln (2010). The sectors presented there do not perfectly match the ones in matrix A but can still be used to get a sense of a plausible sector distribution. Portfolio 1 is the least concentrated and the level of concentration increases up until Portfolio 4, which has the highest concentration. The counterparties in Portfolio 4 are distributed only among two sectors. These sectors are the real estate sectors of Sweden and Europe and this portfolio can therefore be seen as an representation of a real life real estate portfolio. The sector distribution of the simulated portfolios in the first group can be seen in figure 7.3 and the HI for each portfolio is found in table 7.2.

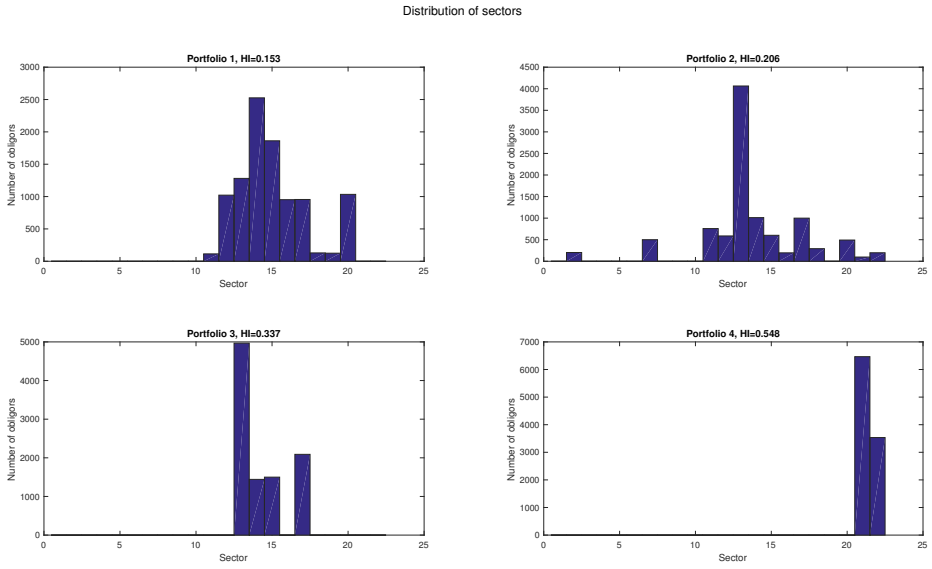

FIGURE 7.3: Sector distribution of simulated sector portfolios.

	HI
Sector Portfolio 1	0.153
Sector Portfolio 2	0.206
Sector Portfolio 3	0.337
Sector Portfolio 4	0.548

TABLE 7.2: HI of the four simulated sector portfolios

In the second group all sectors are only geographically defined and this group will therefore be referred to as the geographical sector portfolios. For the geographical sectors, three portfolios will be simulated. As mentioned above, the sectors of the simulated geographical portfolios must conform with those chosen by FI and not all geographical sectors can therefore be included. The first of the geographical portfolios is the most diversified among sectors and has a larger share in the Nordic countries as we considered this a more plausible distribution for a Nordic loan portfolio. The other two geographical portfolios are divided on only two sectors and will therefore have the same HI. However, the difference between these portfolios is that Portfolio 2 is divided among two sectors with a relatively low inter-sector correlation while the sectors of Portfolio 3 have a high inter-sector correlation. The sector distributions of the geographical portfolios are found in figure 7.4 and the HI is presented in table 7.3.

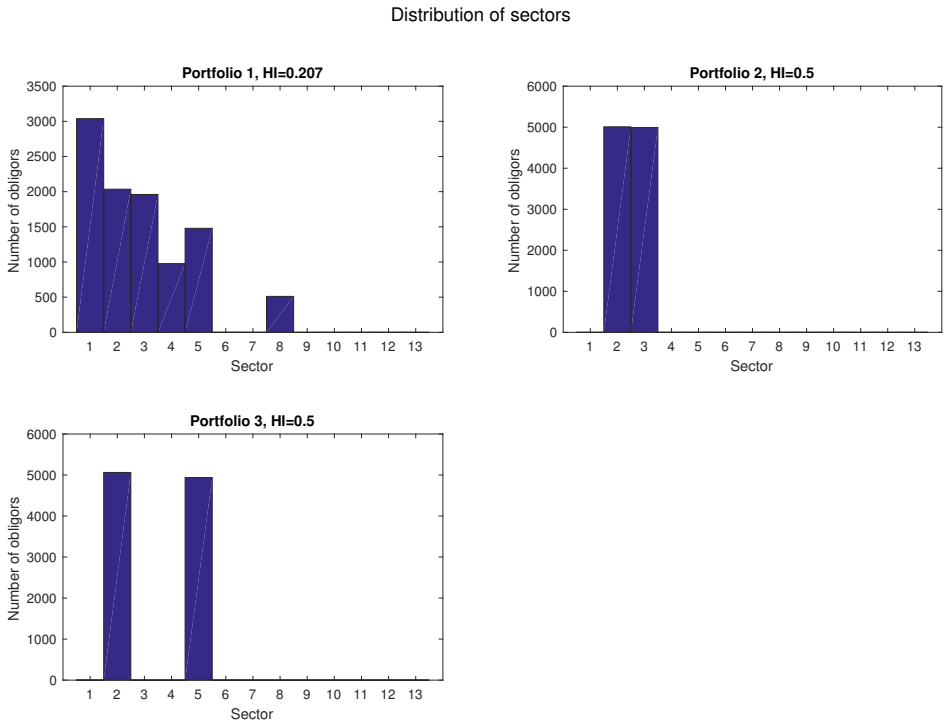

FIGURE 7.4: Sector distribution of simulated geographical portfolios.

	HI
Geographical Portfolio 1	0.207
Geographical Portfolio 2	0.5
Geographical Portfolio 3	0.5

TABLE 7.3: HI of the four simulated geographical portfolios

Chapter 8

Results

This chapter will present the results when applying the methods in previous chapters to the simulated portfolios described in chapter 7. Both the add-on and the computational time of each method is presented. The add-on is given as a percentage add-on to the IRB capital requirement for credit risk under Pillar 1. The results are divided into one section for each type of concentration risk.

Both types of credit concentration risk will be evaluated using full Monte Carlo simulation and the importance sampling presented in section 3.2.1. However the exponential twisting of PD will not be implemented, instead IS is only applied to the systematic risk factors. This is due to the fact that the exponential twisting will require a new θ in each MC-step, which becomes quite computationally burdensome. By implementing IS only on the systematic risk factors we still get a considerable reduction of variance but are only required to solve for μ one time per simulation.

8.1 Results for name concentration

The results below are the results related to the FMC and the PPA described in section 5. The results also include those of FI's method for name concentration, which is presented in section 4. The number of MC steps is set to 10^7 as this gives a percentage standard deviation of $2.26 \cdot 10^{-4}$, which is deemed to be small enough.

8.1.1 Choice of M

Before evaluating the PPA on the simulated name portfolios, the number of counterparties in portfolio A, M, needs to be chosen. Different suggestions, henceforth called patterns, for defining the size of portfolio A for each proposed method are presented in the tables 8.1 and 8.2. The resulting number of counterparties in portfolio A is found in Appendix A. The accuracy as well as the computational time is taken into consideration when making the choice. The method of Higo and the method of Grippa and Gornicka to choose M are tested, both are presented in section 5.2.

	$\sum_{i=M+1}^{N} w_i^2$
Pattern 1	$5 \cdot 10^{-6}$
Pattern 2	10^{-5}
Pattern 3	$5 \cdot 10^{-5}$
Pattern 4	10^{-4}
Pattern 5	$5 \cdot 10^{-4}$
Pattern 6	10^{-3}

TABLE 8.1: **Higo**: Definition of patterns for the Higo method. M is the number of obligors in sub-portfolio A.

	Threshold (%)
Pattern 1	0.005
Pattern 2	0.01
Pattern 3	0.05
Pattern 4	0.1
Pattern 5	0.3
Pattern 6	0.5

TABLE 8.2: **Grippa and Gornicka**: Definition of patterns for the Grippa and Gornicka method.

In tables A.1 and A.2 it can be seen that the number of obligors in sub-portfolio A differs between the two methods. Higo's method puts fewer obligors in sub-portfolio A for the first patterns but more obligors for the last patterns than Grippa and Gornicka's method. The methods have in common that the number of obligors in sub-portfolio A are reduced for a higher pattern and that the later portfolios have more obligors in sub-portfolio A for low patterns than the first portfolios.

The computational time for each pattern and portfolio is presented in Appendix A. The computational time is reduced with M so the computational times will follow the same structure as the numbers of obligors in sub-portfolio A, also presented in Appendix A.

The percentage deviation from the FMC-VaR is presented in tables 8.3 and 8.4. The deviance gets larger for each pattern as more obligors are put in portfolio B. The magnitude of the deviance is of the same order for all portfolios and for almost all patterns for the Higo method. With Grippa and Gornicka's method the size of the deviance varies more across portfolios. The limit 0.1 is set and all deviances with an absolute value smaller than the limit are deemed to be small enough. For Higo's method pattern 1 and 2 give values smaller than 0.1 for all portfolios. In the case of Grippa and Gornicka's method the approved patterns are pattern 1 and 2 for all portfolios. Pattern 3 gives enough accuracy for the first two portfolios and for portfolio 1 even pattern 4 is accurate enough.

	Portfolio 1	Portfolio 2	Portfolio 3	Portfolio 4	Portfolio 5
Pattern 1	**-0.0103**	**-0.0082**	**-0.0161**	**-0.0235**	**-0.0475**
Pattern 2	**-0.0762**	**-0.0196**	**-0.0465**	**-0.0357**	**-0.0254**
Pattern 3	-0.1275	-0.1350	-0.2042	-0.1253	-0.1401
Pattern 4	-0.1997	-0.3354	-0.2646	-0.3328	-0.3337
Pattern 5	-1.0547	-1.2789	-1.2734	-1.2497	-1.0105
Pattern 6	-1.9592	-2.4524	-1.9530	-1.7621	-1.2369

TABLE 8.3: **Higo**: Percentage deviance from the FMC-VaR for all combination of patterns and portfolios. The deviances with an absolute value below 0.1 are marked in bold font.

	Portfolio 1	Portfolio 2	Portfolio 3	Portfolio 4	Portfolio 5
Pattern 1	**-0.0084**	**-0.0616**	**-0.0431**	**0.0097**	**-0.0168**
Pattern 2	**-0.0040**	**-0.0615**	**-0.0534**	**-0.0035**	**0.0077**
Pattern 3	**-0.0181**	**-0.1387**	-0.2131	-0.3206	-0.4159
Pattern 4	**-0.0382**	-0.3055	-1.0095	-1.2452	-1.0914
Pattern 5	-1.0845	-2.3914	-1.7488	-1.4275	-1.2543
Pattern 6	-4.1188	-3.0478	-2.0473	-1.7518	-1.3173

TABLE 8.4: **Grippa and Gornicka**: Percentage deviance from the FMC-VaR for all combination of patterns and portfolios. The deviances with an absolute value below 0.1 are marked in bold font.

The choice of M is discussed further in chapter 9. When testing the PPA on the simulated portfolios M is chosen according to Higo's pattern 2 for all portfolios.

8.1.2 Results for name portfolios

Following the previous results the PPA is evaluated on all portfolios using the choice of M from Higo's Pattern 2. The method of FI, presented in equation 4.5, is also tested on the portfolios as well as the FMC.

	Add-on Lütkebohmert(%)	Time
Portfolio 1	17.7142	< 0.1s
Portfolio 2	9.3920	< 0.1s
Portfolio 3	4.0867	< 0.1s
Portfolio 4	3.2909	< 0.1s
Portfolio 5	2.4853	< 0.1s

TABLE 8.5: Add-on for name concentration risk calculated using FI's method presented together with the calculation time. The add-on is given as a percentage of the capital requirement for credit risk under Pillar 1.

	Add-on FMC(%)	**Time**
Portfolio 1	12.5138	55 min 47 s
Portfolio 2	5.1200	49 min 42 s
Portfolio 3	1.8159	52 min 18 s
Portfolio 4	0.4643	52 min 3 s
Portfolio 5	0.2731	50 min 23 s

TABLE 8.6: Add on for name concentration risk calculated using full Monte Carlos simulations presented together with the calculation time. The add-on is given as a percentage of the capital requirement for credit risk under Pillar 1.

By comparing the results from FI's method (table 8.5) with the result from the FMC (table 8.6) it is clear that the FMC consistently produces lower results than FI. That lower concentration gives lower add-ons is true for both methods as the add-on for the later portfolios is substantially lower than for the first portfolios for both methods.

	Add-on PPA(%)	**Time**
Portfolio 1	12.5043	7 min 16 s
Portfolio 2	5.1190	8 min 41 s
Portfolio 3	1.8151	10 min 18 s
Portfolio 4	0.4641	11 min 37 s
Portfolio 5	0.2730	13 min 1 s

TABLE 8.7: Add-on for name concentration risk calculated using the Partial Portfolio Approach with pattern 2 presented together with the calculation time. The add-on is given as a percentage of the capital requirement for credit risk under Pillar 1.

The goal of PPA is to produce results similar to those of FMC but with a shorter computational time. The results in table 8.7 agree well with the results from FMC in table 8.6 and the computational time is notably reduced.

8.2 Results for sector concentration

Below the results for the chosen methods to calculate the capital requirement due to sector concentration risk are presented. The results are divided according to the sector and the geographical sector portfolios in the same way as in chapter 7. The Pykhtin method is tested for both type of portfolios and three different versions of the Pykhtin method are tested. The first method is the one explained in section 6.4 and the second method is almost the same but utilizes the bivariate approximation in section 3.4. The third version of Pykhtin uses the grouping in section 6.2. As the bivariate normal distribution becomes computationally heavy when implemented in MATLAB the bivariate approximation and the grouping are used to try and reduce the computational time. The approximation will be implemented using both the five first and the ten first Hermitian polynomials. As stated in section 3.4 the bivariate normal approximation is not recommended for large values of the correlation ρ_{ij}^X found in equation 6.36. In order to control that the correlations are in a range suitable for the approximation, the distribution of the correlations is included.

8.2.1 Results for sector portfolios

The four portfolios used here are simulated based on correlation matrix A. As previously discussed, the sector division of this correlation matrix does not agree with the division of FI and the results are therefore not compared to the method used by FI. The methods evaluated for the sector portfolios are therefore only the FMC and Pykhtin.

	Add-on FMC (%)	**Time (w/o grouping)**	**Time (w. grouping)**
Portfolio 1	2.1507	1h 29 min 7s	7 min 55s
Portfolio 2	12.1489	1h 25 min 35s	8 min 27s
Portfolio 3	24.7369	1h 30 min 31s	7 min 46s
Portfolio 4	30.1344	1h 27 min 20s	8 min 25s

TABLE 8.8: Add-on for sector concentration risk calculated using full Monte Carlo simulations presented together with the calculation time for both with and with out grouping. The add-on is given as a percentage of the capital requirement for credit risk under Pillar 1.

The results from FMC in table 8.8 are seen as the true results and will be the benchmark for the other methods. By comparing the time with and without the grouping it is clear that the grouping of exposures is a very effective way to speed up the calculations. This of course given that it is possible to group the obligors into a reasonable number of groups.

	Add-on Pykhtin (%)	**Time**
Portfolio 1	1.9938	2h 35 min
Portfolio 2	12.1269	2h 34 min
Portfolio 3	24.9256	2h 35 min
Portfolio 4	30.0850	2h 34 min

TABLE 8.9: Add-on for sector concentration risk calculated using the Pykhtin method presented together with the calculation time. The add-on is given as a percentage of the capital requirement for credit risk under Pillar 1.

The results in table 8.9 agree well with the result from FMC in table 8.8 but the computational time is longer than for FMC.

	5 step approx.		**10 step approx.**	
	Add-on(%)	**Time**	**Add-on(%)**	**Time**
Portfolio 1	4.3641	1 min 3s	4.3641	2 min 33s
Portfolio 2	12.8680	0 min 59s	12.8680	2 min 30s
Portfolio 3	25.9929	0 min 55s	25.9929	2 min 29s
Portfolio 4	32.3268	0 min 55s	32.3268	2 min 30s

TABLE 8.10: Add-on for sector concentration risk calculated using the Pykhtin method with the 5 step and 10 step bivariate normal approximation presented together with the calculation time. The add-on is given as a percentage of the capital requirement for credit risk under Pillar 1.

The first way to try to speed up Pykhtin is to use the bivariate normal approximation in section 3.4. The results from both the 5 step and 10 step approximation are found in table 8.10. There is no difference in the add-on between the shorter and longer approximation. The only difference is in the computational time, where the shorter takes less than half the

time of the longer. Both methods are much faster than the original Pykhtin. However, comparing the add-on with the add-on from Pykhtin and FMC there is a substantial deviance for most portfolios where Pykhtin with approximation over-estimates the add-on in all cases.

	Add-on Pykhtin w. grouping (%)	**Time**
Portfolio 1	1.9938	0.84 s
Portfolio 2	12.1269	0.75 s
Portfolio 3	24.9256	0.10 s
Portfolio 4	30.0850	0.02 s

TABLE 8.11: Add-on for sector concentration risk calculated using the Pykhtin method with grouping presented together with the calculation time. The add-on is given as a percentage of the capital requirement for credit risk under Pillar 1.

Another way to speed up Pykhtin is to group the obligors. The results in table 8.11 are identical with the results produced by Pykhtin in table 8.9 but with a dramatic decrease in computational time.

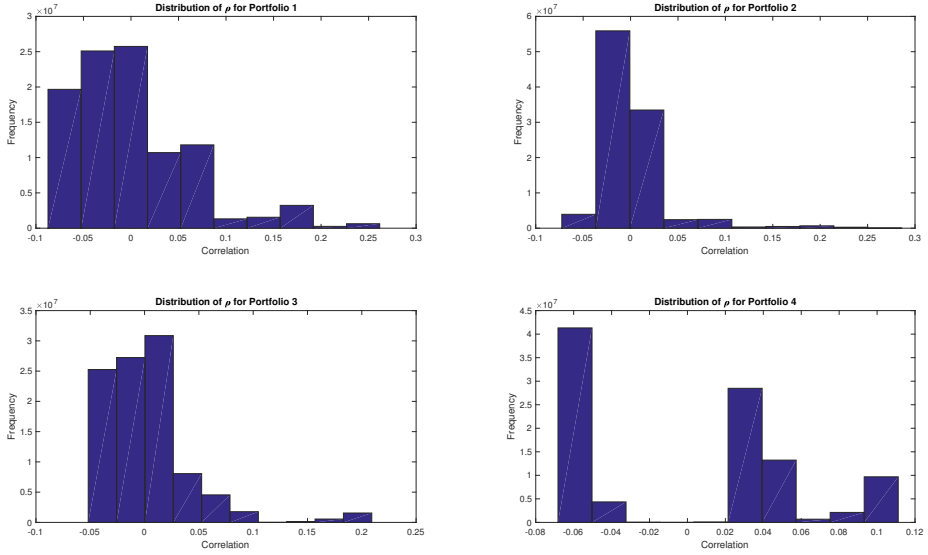

FIGURE 8.1: Distribution of the correlation matrix consisting of the elements ρ_{ij}^{X} given by equation 6.36. The distribution is given for the four sector portfolios.

8.2.2 Results for geographical sector portfolios

The geographical sector portfolios are simulated using correlation matrix B and as, opposed to previous sector portfolios, these sectors are in agreement with FI's sectors the results of FI's method for geographical sector concentration can now be included. So the methods evaluated on the geographical sector portfolios are FI's method, FMC and Pykhtin.

	Add-on FI (%)	Calculation time
Portfolio 1	1.0248	< 0.001s
Portfolio 2	3.6783	< 0.001s
Portfolio 3	3.6787	< 0.001s

TABLE 8.12: Add-on for sector concentration risk calculated using FI's method presented together with the calculation time. The add-on is given as a percentage of the capital requirement for credit risk under Pillar 1.

The results from FI's method are found in table 8.12. The more concentrated a portfolio is the higher the add-on. Portfolio 2 and 3 have the same add-on as the portfolios are equally concentrated.

	Add-on FMC (%)	Time (w/o grouping)	Time (w. grouping)
Portfolio 1	10.0402	1h 26 min	7 min 15s
Portfolio 2	15.7884	1h 30 min	6 min 47s
Portfolio 3	34.2627	1h 27 min	6 min 45s

TABLE 8.13: Add-on for geographical sector portfolios calculated using full Monte Carlo simulations presented together with the calculation time. The add-on is given as a percentage of the capital requirement for credit risk under Pillar 1.

The add-on from FMC in table 8.13 is much higher than FI's for all portfolios. The add-on increases with increasing sector concentration but here portfolio 3 requires a higher add-on than portfolio 2 even though they are equally concentrated.

	Add-on Pykhtin (%)	Time
Portfolio 1	9.8846	2h 30 min
Portfolio 2	14.5785	2h 32 min
Portfolio 3	33.9028	2h 30 min

TABLE 8.14: Add-on for geographical sector portfolios calculated using the Pykhtin method presented together with the calculation time.

The result from Pykhtin's method in table 8.14 are compared with the FMC result in table 8.13. The results agrees well but again Pykhtin is slower than FMC.

	5 step approx.		10 step approx.	
	Add-on(%)	Time	Add-on(%)	Time
Portfolio 1	12.0460	1 min 3s	12.0460	2 min 38s
Portfolio 2	21.8063	0 min 59s	21.8063	2 min 27s
Portfolio 3	35.6295	0 min 58s	35.6295	2 min 30s

TABLE 8.15: Add-on for geographical sector portfolios calculated using the Pykhtin method with the 5 step and the 10 step bivariate normal approximation presented together with the calculation time. The add-on is given as a percentage of the capital requirement for credit risk under Pillar 1.

The results from the Pykhtin with approximation are found in table 8.15. The add-ons are compared with the add-on calculated using the original Pykhtin method and using the approximation leads to an over-estimation of the add-on. The error is especially large for portfolio 2.

	Add-on Pykhtin w. grouping (%)	**Time**
Geo. Portfolio 1	9.8846	0.28 s
Geo. Portfolio 2	14.5785	0.03 s
Geo. Portfolio 3	33.9028	0.03 s

TABLE 8.16: Add-on for geographical sector portfolios calculated using the Pykhtin method with grouping presented together with the calculation time. The add-on is given as a percentage of the capital requirement for credit risk under Pillar 1.

Pykhtin with grouping again gives the same results, found in table 8.16, as the original method but to a much shorter computational time.

A general observation to be made is that regardless of method, the results in tables 8.13-8.16 do not agree with the results of FI in table 8.12. Possible explanations behind this deviation will be discussed further in the following chapter but it should be noted that the deviation is large, meaning that the methods studied in this thesis for sector concentration do not agree with the method of FI.

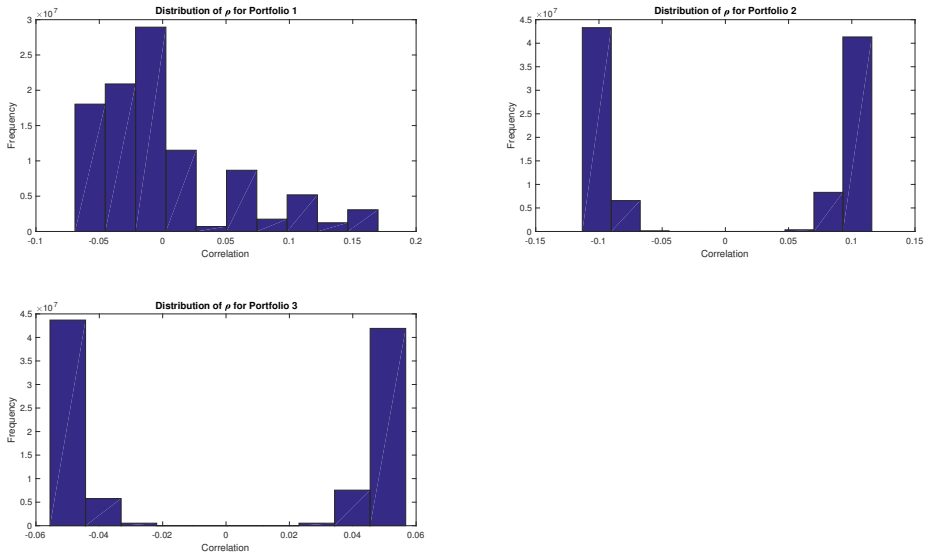

FIGURE 8.2: Distribution of the correlation matrix consisting of the elements ρ_{ij}^{X} given by equation 6.36. The distribution is given for the three geographical sector portfolios.

Chapter 9

Discussion

The results of chapter 8 will be discussed in this chapter. The tested methods for name and sector concentration risk are discussed in two separate sections. A final section gives some suggestions for further research.

9.1 Name concentration

9.1.1 Choice of M

The idea behind PPA is to divide the portfolio into two sub-portfolios where sub-portfolio A is non-granular while sub-portfolio B can be seen as infinitely granular. The two extreme cases are that either all exposures are in sub-portfolio A or that all are found in sub-portfolio B. The case where all exposures are in sub-portfolio A is the same as in the FMC. The other extreme case, where all exposures are placed in sub-portfolio B, is equal to the ASRF assumption that there is no idiosyncratic risk and hence no name concentration in the portfolio. The higher the threshold, the more exposures are put in the sub-portfolio B and the add-on will approach zero as the portfolio with name concentration becomes smaller and smaller. Consideration has to be taken in the choice of the threshold as a too high threshold will result in an underestimation of the true name concentration risk. There is no risk in choosing a low threshold as this only contributes to a more accurate estimation. However, we do not wish to choose the threshold too low as a lower threshold increases the computational time. Thus the choice of cut off point is a trade off between accuracy and computational time. Although computational time is an important factor we choose to prioritize the accuracy. A slow method can not do much harm but a method producing inaccurate capital requirements can be dangerous and should be avoided.

When choosing M, the number of obligors in sub-portfolio A, both the methods described by Grippa and Gornicka (2016) and Higo (2006) are tested. Since a general choice of M will be less time consuming than choosing an individual M for each portfolio we wish to find the M that works for all portfolios. We choose to let all patterns with an absolute value of the deviation ≤ 0.1 be considered good enough. The choice of 0.1 as a threshold is based on the standard deviation of the FMC. In table A.2 and A.1 the number of obligors in sub-portfolio A are found for each portfolio and each patterns. The number varies for the different portfolios and it seems that the less concentrated the portfolio is the more exposures are included in portfolio A. When comparing tables 8.4 and 8.3, Higo's method gives errors of the same magnitude for each pattern while there is a much greater variation in the magnitude of the errors for the patterns chosen according to Grippa and Gornicka's

method. It is possible that by using Higo's method the choice of M can be generalized to a greater extent than by using Grippa and Gornicka's method to choose M. We therefore base our choice on Higo and as can be seen in 8.3, pattern 2 will be used for all portfolios.

9.1.2 Comparison to FI

When comparing the results in table 8.6 and 8.7 to the ones in table 8.5 it is clear that the methods tested in this thesis consequently produce results lower than the results computed using FI's method. A large deviation from the results in table 8.5 would indicate that the method chosen is not suited for the specific type of portfolio. It can be difficult to state the size of deviation in order for it to be considered large but we believe the results produced by the FMC and the PPA are good enough. A more experienced person might have a better insight to how plausible the results are in reference to those of FI but we believe that although there is a deviation, it is small enough to not dismiss the methods based only on this. As FI's method is supposed to suit all Swedish banks it is possible that portfolios characteristics are not accounted for when using FI. The deviation in results when comparing FI to the methods presented in this thesis could be due to the FMC and PPA being able to better capture the individual characteristics of the portfolios. It is possible that FI's method lands on the conservative side since FI is likely to believe that it is more dangerous to underestimate than to overestimate the risk level. By choosing a different method, which to a larger extent can include specific portfolio characteristics, it is possible that the capital requirement becomes lower. This does not necessarily mean that the calculation is wrong but simply that it is more suitable for the specific portfolio and can better estimate the risk level.

Another possible explanation behind the deviation is the model's different assumptions on the distribution of the systematic risk factor. In FI's model the distribution of the systematic risk factor is the gamma distribution and in both the FMC and the PPA the same random variable is assumed to be normally distributed. This model difference can be a possible explanation as to why the results differ. Depending on the shape of the gamma distribution it can behave rather differently than the normal distribution and can lead to different values of the systematic risk factor.

The given reasons for the difference in results are reasons we deem plausible but if none of these reasons hold and the methods underestimate the capital requirement, this can be a big problem. By not holding enough capital the bank will be exposed to a large risk and one could argue that it is preferable to overstate rather than understate the capital requirement. So before choosing to implement either the FMC or the PPA it is wise to investigate further if the lowering of capital requirement compared to FI is due to the methods being able to take the portfolio characteristics into account or if there is an underlying problem leading to an understatement of capital.

9.1.3 Comparison between FMC and PPA

Tables 8.6 and 8.7 show that the two methods produce similar results. This justifies the choice of M for the PPA as we expect that a good choice of M should produce the same result as the FMC. The main goal of the PPA is to reduce the computational time without loosing accuracy and looking at the tables it is clear that the PPA reduces the computational time greatly compared to FMC. The PPA does seem to be the preferred method as it produces accurate results but is faster than the FMC. This indicates that only the

largest exposures have a significant effect on the level of name concentration.

One possible disadvantage of the PPA is that it requires choosing M for each portfolio. The basics of the FMC and the PPA are very similar and the implementation should therefore require about the same time and effort but while the FMC will be ready to use for all portfolios, the PPA will still require an M. The choice of M is very important as this will affect both the computational time and the accuracy of the answer. Higo's method for choosing M seems to give results with errors of around the same magnitude for the different portfolios studied here. Although Higo's method to choose M appears to be the preferable method in this case the basis is to small to give an universal recommendation suitable for all types of portfolios.

9.2 Sector concentration

9.2.1 Comparison between FMC and Pykhtin

The two methods chosen for sector concentration risk are the Pykhtin model and the FMC. To compare these, table 8.8 is compared to tables 8.9-8.11 for the sector portfolios and table 8.13 with tables 8.14-8.16 for the geographical sector portfolios. In general it can be said that Pykhtin has one advantage as Pykhtin is an analytical method. The FMC is not an analytical expression and the FMC estimator will have a variance. This variance can be reduced but at the cost of computational time as it will require an increase in the number of MC steps. The result of Pykhtin can not be improved in any similar manner and the computational time will remain the same, just as the final result. This can be seen as both negative and positive. As mentioned, an analytical result can be more reliable as no variance needs to be considered but the computational time of Pykhtin can be troublesome. The large computational time is due to the fact that Pykhtin's method requires the calculation of a bivariate normal distribution in every combination of obligors, as seen in equation 6.35 and 6.37. In a portfolio of N obligors, this means N^2 calls to the bivariate normal distribution function. This thesis uses MATLAB, a programing language intended for numerical computations with matrices, for all calculations. MATLAB is optimized for working with matrices and is not optimal for cases when regular for-loops are used. This poses a problem in the implementation of Pykthin as the MATLAB-function *mvncdf* used for the calculations of the bivariate normal distribution function can not handle matrix input in the way that would have been needed in this thesis. Looping over all combinations of obligors and calling *mvncdf* takes a long time, much longer than the FMC simulations do. As the idea behind Pykhtin's method is to be a faster alternative to FMC simulations this is a big problem. An alternative could be to change the programming language and try a language more suited for the necessary calculations.

In order to reduce the computational time this thesis tries two methods to work around the problem of MATLAB's function *mvncdf*. The first is to use an approximation for the bivariate normal distribution. The approximation is known to work well as long as the correlations given by equation 6.36 are low. However, looking at the results of Pykhtin when using the bivariate normal approximation, see tables 8.10 and 8.15, the computational time is greatly reduced but the results differ from those of Pykhtin without the approximation. Pykhtin without the approximation is more in agreement with the result from the FMC simulations. It does not appear to be any improvement when using the ten step approximation compared to the five step one. The lack of improvement is probably due to the high number of times, N^2, that the answer from the approximations is used leading to an accumulative effect where the small error from the approximation builds up

to a large error in the final result. The approximation is not faulty but it is not accurate enough to suit the need of Pykhtins method. Since the bivariate normal distribution function is used numerous times the approximation needs to be extremely accurate to not lead to a large error in the final answer. A more accurate approximation comes at the cost of higher computational time and as the main goal of Pykhtin is to reduce computational time, Pykhtin with the approximation does not appear to be the favorable choice.

Another problem when using the approximation for Pykhtin is that the error of the approximation varies from portfolio to portfolio. Looking at Sector portfolio 2 in table 8.10, the approximation seems to provide a reasonably accurate answer. On the other hand, for Geographical sector portfolio 2 in table 8.15, there is a substantial deviation. As mentioned, the approximation works best for a low absolute value of correlation but no definition is given for what can be considered a low correlation. Figures 8.1 and 8.2 show the histograms of the correlation found in equation 6.36. Sector portfolio 2 have a distribution where the correlations are centered around 0, so most are lower than 0.05 and almost no correlations exceed 0.1. Looking at figure 8.2, most of the correlations of geographical portfolio 2 have absolute values around 0.1, with only a few smaller correlations. This is a likely reason why the approximation performs so differently for the different portfolios. If the bivariate normal approximation is to be implemented in the Pykhtin method, close attention needs to be paid to the correlation structure of the portfolio to make sure that the approximation will be accurate enough. Due to the limitations in this thesis the validity of the approximation has not been further investigated and no recommendations can be given on how and when the approximation could be suitable to include.

The second method tested in order to reduce the computational time is Pykhtin with grouping of exposures. As can be seen in tables 8.11 and 8.16 this method appears to work very well. The results are the same as for the full Pykhtin but the computational time is greatly reduced. It is faster than Pykhtin with the bivariate normal approximation and has a better accuracy. Pykhtin with the grouping of exposures does appear to be the best method in this case. However, it is not always possible to group exposures in the manner done in this thesis. When simulating the portfolios we assume that all obligors in the same rating class will be assigned the same PD. This is not necessarily true in the real world as a bank might have other models to calculate and assign PD and if this is the case, the grouping of the exposures might not be as efficient. The computational time of the grouped methods depends on the numbers of groups needed to correctly group the portfolio, hence on the numbers of sectors, PDs and intra-correlations. As the number of groups grow, the computational time will go towards the computational time for the method without grouping. Grouping can probably reduce the computational time in most cases but maybe not as much as for our simulated portfolios.

Looking beyond the computational time of Pykhtin one positive aspect of the method is that it offers the possibility to have LGD as a distribution and not a set value. This thesis uses $LGD = 0.45$ for all simulated portfolios but it is possible to have LGD given as a variable with a mean and a variance. In Pykhtin this can be handled without significantly changing the computational time. On the other hand there is no restriction on FMC that prevents variables being drawn from distributions. However, by needing to draw variables for each obligor this could add to the computational time of the FMC while for Pykhtin the mean and variance of the LGD are already part of the expression and do not need to be simulated. Another possible advantage Pykhtin has over the FMC is that it is possible to use Pykhtin to calculate both name and sector concentration without significantly increasing the computational time. In section 6.4.1 it is shown that the quantile adjustment

used in Pykhtin can be split into two parts, one part compensating for the shift from the multi-factor model to the single-factor model and the other part compensates for the effect of having a finite number of counterparties in the portfolio. The second part can be seen as the granularity adjustment and by adding this part to the quantile expression the name concentration can be calculated in addition. This would not significantly increase the computational time of Pykhtin and if both the name and sector concentration risk needs to be measured for one portfolio, Pykhtin could be a possible alternative. It should be remembered that grouping is not possible when calculating the granularity adjustment. This should however not lead to a dramatic increase in computational time as the time consuming steps are for measuring the impact of sector concentration, for which the grouping may still be used.

9.2.2 Choice of intra-correlations

As explained in section 6.3 the intra-correlations used in this thesis are based on the suggestions of Hibbeln (2010) and can be found in equation 6.10. The choice of intra-correlations will have a great effect on the resulting capital requirement for sector concentration risk. For example, also presented in section 6.3, the use of the intra-correlations suggested by the Basel regulations would give a negative add-on to the Pillar 1 capital requirement in the case of sector concentration. We have not conducted any trials on our own to determine how to best calculate the intra-correlations and have based our choice solely on the recommendation of Hibbeln (2010). We deem this source to be trustworthy but we do not know how the authors of the book have concluded that this specific formula is the best for the intra-correlations. It is possible that the intra-correlations are not suitable for our specific portfolios or that there are other restrictions that we are not aware of that would eliminate the chosen formula as possible correlations. Another possibility is that even though the chosen correlations are acceptable there are other ways to calculate the correlations that take into account circumstances specific to the bank. Regardless of how the intra-correlations are chosen the choice will greatly affect the final result. In our case this means that our results could be both higher or lower depending on what correlations are used.

9.2.3 Comparison to FI

When comparing the results in table 8.12 with the results presented in tables 8.13-8.16 it is obvious that the results based on the method used by FI do not agree with the results of the methods used in this thesis. It is possible, as previously discussed for name concentration, that the methods of this thesis are better at including portfolio characteristics and will therefore give a different result than that of FI. However in this case the deviation is quite large, which indicates that there might be difficulties when comparing the results of the thesis methods to the result of FI. As previously mentioned the sectors defined by FI are not the same as the sectors used in this thesis, leading to FI's method only being applied to the geographical sectors. As FI's method is based on and defined for FI's 16 specific geographical sectors it can be difficult to use this method for sectors defined differently. So even if the chosen sectors of the Geographical sector portfolios agree with FI's sectors, all the sectors of matrix B do not agree with FI. To be able to apply FI's method, the sectors of matrix B might need to be mapped to match those of FI exactly or the formula in equation 4.8 needs to be altered to take into account the new geographical sectors.

Regardless of the underlying reason it is problematic that the results agree so poorly with the results of FI. A small difference could be expected but such a large difference can be an indication that the methods proposed in this thesis are not appropriate to use when measuring sector concentration risk in a loan portfolio. However, in order to investigate this fully a sector correlation matrix corresponding perfectly with the sectors of FI would be needed. Portfolios based on the sectors of FI could then be simulated and the method proposed by FI could be evaluated without needing to question if the formula is suitable for the used sectors.

A general problem when it comes to sector concentration is that it is not clear what the benchmark portfolio should be. When it comes to name concentration an infinitely granular portfolio is the natural benchmark to measure the name concentration risk relative the asymptotic single risk factor model. However, for sector concentration there exists no common definition of a benchmark. As it is not known on what sector-wise well diversified portfolio the IRB correlations are calibrated on it is difficult to compare the implemented multi-factor model with the ASRF model as well as with any other model which might be another reason to the large difference between our implemented multi-factor models and the model proposed by FI.

All portfolios used in this thesis are corporate portfolios consisting of only big corporates. Typically the total portfolio of a bank contains other kinds of exposure like retail and sovereign with the corporate portfolios being only a fraction of the total loan portfolio. In this theses only the impact of sector concentration on the corporate credit portfolio is studied. Although the presence of sector concentration might have significant impact on the capital requirement for the corporate credit portfolio it might have a smaller impact in terms of a bank's total credit portfolio. In FI's model for geographical sector concentration no regards are taken to the exposure types and the add-on will be the same no matter the obligors. So the fact that only a subset of a real banking portfolio is studied in this thesis might be another reason why there is such a big difference between the multi-factor models and FI's approach.

The method used by FI is easy to implement as the only input needed is the aggregated exposures to each sector to be able to calculate HI. The calculated add-on only depends on how concentrated the portfolio is and does not take the inter-correlation structure in the portfolio into account. This is a big drawback as the concentration risk is strongly dependent on the correlations between sectors. A portfolio concentrated in sectors with high inter-correlations will be riskier than a portfolio concentrated in sectors with lower inter-correlations. Concentration in sectors with a low inter-correlation may even be seen as a form of hedge. By not taking the correlation structure into account the model of FI may both under- and overestimate the sector risk in portfolios. Looking at Geographical portfolio 2 and 3 FI's add-ons are identical for both portfolios while the add-on calculated with the multi factor models is twice as high for Geographical portfolio 3 than for Geographical portfolio 2. Geographical portfolio 3 is in the multi-factor models seen as riskier as it is concentrated in two sectors with a higher inter-correlation.

One of the advantages with Pykhtin and the FMC proposed here is that they are not sector specific. As long as there is a correlation matrix available the methods will work for any type of sectors. FI has very well defined sectors, which makes their methods specific to these sectors, while both Pykhtin and FMC can handle any sector division. If the goal is that the method should be compatible with the method used by FI the sectors will still need to be mapped to the sectors of FI. However this must not always be the case and it

can be an advantage to have methods that are independent of the sector division.

The methods implemented in this thesis are more advanced than the method used by FI. They may capture the risk in a portfolio in a better way but they are more complicated and require assumptions about the correlation structure. As the correlations can be hard to estimate and the results are strongly dependent on the assumed correlation structure there exists a big model risk with the more advanced models. FI's method on the other hand might be too simple but it has advantages when it comes to stability and lower model risk.

9.3 Further research

If going forward with the methods presented in this thesis we suggest that some subjects should be considered first. In the case of Pykhtin, using the grouping works well for the portfolios simulated in this thesis but the grouping of exposures will not work well for all types of portfolios. As mentioned it is possible that a bank assigns individual PDs, meaning that the grouping will not be as efficient. If this is the case full Pykhtin or Pykhtin with approximation are possible alternatives. The main problem with full Pykhtin is that the computational time becomes large when the method is implemented in MATLAB. A possible way forward is to implement Pykhtin using a different programming language to try and speed up the method. If this is not possible or if there is no alternative programming language suitable for the needs of Pykhtin, some time should be spent trying to speed up the method. A possible way to approach this is to find a more suitable approximation to the bivariate normal distribution. Only the approximation presented in section 3.4 was implemented in this thesis and it is possible there is an approximation that is not as dependent on the value of ρ^X and would work better for Pykhtin. We believe both of these approaches are interesting but if there is a possibility to speed up Pykhtin without the approximation we believe this is to be preferred. Using the approximation adds uncertainty to the result and it is therefore better to use Pykhtin without the approximation if possible.

Another area that should be examined further before implementing any of the sector concentration methods is the choice of the intra-sector correlation. As discussed the choice is based on the recommendations of Hibbeln (2010) and due to the limitations of this thesis, we have not examined any alternatives. Further research should be spent trying to find the optimal intra-sector correlations. The choice of correlations highly effect the resulting capital requirement and care should be taken before choosing these. If this thesis had no time constraint we would have liked to spend more time researching the choice of the intra-sector correlations.

In the case of name concentration, the choice of M for the PPA could be optimized. Based upon our simulated portfolios Higo's method for choosing the number of obligors in sub-portfolio A seems to give consistent results for all studied portfolios. It would be interesting to investigate this further to see if the same results would hold for a bigger collection of portfolios.

Chapter 10

Summary and Conclusions

The aim of this thesis has been to evaluate methods for calculating capital requirements for name and sector concentration risk and compare them with the methods chosen by FI for the same calculation. To achieve this; four methods where chosen, two for name and two for sector concentration risk. The chosen methods were evaluated on simulated loan portfolios and the capital requirement for each portfolio was calculated as a percentage of the credit risk capital requirement under Pillar 1.

The models chosen for name concentration were the FMC and the PPA. The PPA is similar to the FMC but the portfolio is split into two sub-portfolios. One sub-portfolio is assumed to be diversified while the other is assumed to contain name concentration and will require a granularity adjustment. By calculating the granularity adjustment only for one of the sub-portfolios the computational time can be reduced. To test the methods five portfolios with a varying degree of name concentration were simulated.

For sector concentration the FMC was used as well as a model suggested by Pykhtin (2004). Pykhtin suggests an analytical model and the aim when implementing this model is to find a faster alternative to the FMC. For the sector concentration two sets of portfolios were simulated. The first set contained four portfolios with a varying degree of sector concentration. The second set contained three portfolios, also with a varying degree of sector concentration but the sectors used in this set are chosen so that they correspond to sectors defined by FI. FI's method could only be applied to the second portfolio set as the first set contained sectors that did not correspond with FI's sectors.

The accuracy of the results for the FMC and the PPA were similar but the PPA had a much shorter computational time once the size of each sub-portfolio had been determined. Both methods produced lower results than the results of FI. This can be due to the FMC and the PPA being able to capture the individual characteristics of the portfolio and therefore lowering the capital requirement or a model mismatch as the methods are based on different models. However, if the low results are an indication that the methods do not produce reliable results this should be seen as a serious error as understating of capital requirements can be a large risk.

For both sets of the sector portfolios Pykhtin's method and FMC were evaluated. Pykhtin's method was implemented in full and as an effort to reduce the computational time Pykhtin's method was also implemented using an approximation and by grouping the exposures. The full Pykhtin's method produced results similar to the FMC but was slower

than FMC. Pykhtin's method with the approximation was faster than FMC but produced results that differed from the FMC. Using the grouping on the full Pykhtin's method lead to results matching those of full Pykhtin's method and reduced the computational time greatly. The results of Pykhtin's method and FMC for the second set of sub-portfolios did not agree with the results of FI. This is not necessarily an indication that the method is faulty since the sector division is slightly different than that of FI. Furthermore the FMC and Pykhtin's method are more complicated methods than FI's method and require assumptions on the underlying model. The deviation in results should be considered as a disadvantage and the reasons behind it considered before moving forward with any of the methods.

The conclusions of this thesis is that the PPA appears to be a good alternative to FMC for measuring name concentration. It is equally accurate but as the computational time is shorter the PPA appears to be the preferable method. Consideration should however be taken as to how the non-granular portfolio is chosen and how this will affect the total time of the PPA. For sector concentration Pykhtin's method with grouped exposures appears to be the best alternative in this case. It performs similar to FMC and is faster. However, the computational time will depend on the numbers of groups needed to correctly classify the portfolio. None of the methods studied agree with the methods of FI and the reasons behind this should be considered before implementing any of the methods.

Appendix A

Tables

	Portfolio 1	**Portfolio 2**	**Portfolio 3**	**Portfolio 4**	**Portfolio 5**
Pattern 1	377	781	1306	1539	1938
Pattern 2	329	623	1083	1406	1748
Pattern 3	238	415	760	1018	1194
Pattern 4	212	352	592	790	858
Pattern 5	139	152	117	92	15
Pattern 6	93	41	8	0	0

TABLE A.1: **Higo**: Number of obligors in sub-portfolio A for each combination of pattern and portfolio.

	Portfolio 1	**Portfolio 2**	**Portfolio 3**	**Portfolio 4**	**Portfolio 5**
Pattern 1	528	1196	2125	1851	2255
Pattern 2	501	1010	1241	1670	2086
Pattern 3	278	484	724	822	683
Pattern 4	251	358	227	80	25
Pattern 5	133	44	24	15	5
Pattern 6	28	15	5	0	0

TABLE A.2: **Grippa and Gornicka**: Number of obligors in sub-portfolio A for each combination of pattern and portfolio.

	Portfolio 1	**Portfolio 2**	**Portfolio 3**	**Portfolio 4**	**Portfolio 5**
Pattern 1	7 min 25 s	9 min 43 s	11 min 17 s	12 min 1 s	13 min 59 s
Pattern 2	7 min 16 s	8 min 41 s	10 min 18 s	11 min 37 s	13 min 1 s
Pattern 3	6 min 47 s	7 min 30 s	8 min 54 s	9 min 53 s	10 min 39 s
Pattern 4	6 min 40 s	7 min 14 s	8 min 12 s	8 min 55 s	9 min 13 s
Pattern 5	6 min 21 s	6 min 22 s	6 min 6 s	6 min 3 s	5 min 38 s
Pattern 6	6 min 8 s	5 min 47 s	5 min 35 s	6 min 8 s	6 min 10 s

TABLE A.3: **Higo**: Computational time for all combinations of pattern and portfolio.

	Portfolio 1	**Portfolio 2**	**Portfolio 3**	**Portfolio 4**	**Portfolio 5**
Pattern 1	8 min 7 s	10 min 56 s	14 min 45 s	13 min 36 s	15 min 14 s
Pattern 2	7 min 58 s	10 min 8 s	11 min 7 s	12 min 50 s	14 min 29 s
Pattern 3	7 min 3 s	7 min 56 s	8 min 56 s	9 min 16 s	8 min 41 s
Pattern 4	7 min 3 s	7 min 24 s	6 min 47 s	6 min 8 s	5 min 46 s
Pattern 5	6 min 27 s	5 min 55 s	5 min 46 s	5 min 45 s	5 min 38 s
Pattern 6	5 min 51 s	5 min 44 s	5 min 36 s	6 min 9 s	6 min 10 s

TABLE A.4: **Grippa and Gornicka**: Computational time for all combinations of pattern and portfolio.

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