June 12, 2017

Master thesis

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Non-linear electronic entanglement witness based on current cross correlations

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Abstract

Demonstration of entanglements of mobile electrons in solid state conductors using minimal entanglement linear witness from electrical current correlations has been given. However, detection of maximally entangled states and some mixed states is lacking if the witness is based on two cross correlation measurements only. This thesis is trying to detect them with the same number of measurements by investigating nonlinear witnesses which, in principle, should be able to detect more states than a linear witness. This is done theoretically with a generic entangler-detector setup. We found that all entangled states are detectable when considering pure states only but when considering mixed states, there is no improvement compared to a linear witness. This was done using the symmetries between the two operators corresponding to each measurement.

Contents

Abbreviations

C-class Concurrence class. 25–27, 31, 32, 37, 40

CP-class Concurrence Purity class. 26, 27, 45

PPT Positive Partial Transpose. 1, 2, 14, 15, 19, 43, 53, 55

PSDH Positive semi-definite hermitian. 54, 55

Acknowledgements

I would like to thank my supervisor Peter Samuelsson for his help, advice, and the freedom he gave for my project. I would like to thank my co-supervisor Fredrik Brange for all the time he spent helping me and answering my questions and for all the interesting discussions we had. I would like to thank all the persons I shared my office with, especially Jim for all the fun we had during the breaks from our work. I would like to thank Lund University and Sweden for paying and allowing me to do my master degree in a different country which was a once in a lifetime experience. I would like to thank my family for their support, they will probably never read this thesis but I'm sure they are proud of it.

Chapter 1

Introduction

Information is one of the main issues of our modern world: how to manipulate, transmit, process, store it with a increasingly high demand in speed, reliability and security. Quantum physics and more particularly quantum information have allowed us to create completely new ways to process information, that cannot be accomplished by means of classical resources. Some examples are quantum computers, quantum teleportation or quantum cryptography. Quantum information is currently receiving enormous investments by private companies as well as public funding. The European Commission will launch in 2018 a 1 billion euros funding for research over the next 20 years in quantum technologies [1]. In the private industry, the biggest tech companies in the world are also investing considerable amount of money into quantum technologies. Google with its Quantum Artificial Intelligence Lab, IBM with IBM Q, Microsoft with Station Q are all spending lots of resources in this field, aware of the fact that a new revolution of the industry is coming [2].

Quantum entanglement is a concept at the root of many of these quantum technologies. It is considered as being the most non-classical manifestation of quantum mechanics and was originally introduced by Einstein, Podolsky and Rosen (EPR) [3] and also by Schrodinger [4] and Bohr [5] at the beginning of the 1930's. We are reaching a point where it will be used in the industry 100 years after its discovery. This emphasizes the importance of past and future research that has been done about it. But it is also an excellent example of the importance of academic research, which provides investments for research of very abstract subjects for the industry that will have practical applications in the long term.

In practice, entanglement is very fragile and it can be degraded by decoherence that results from interaction of the state with its environment. Knowing whether a state is entangled or not is of crucial importance. Different methods have been proposed to detect the presence of entanglement for different types of entangled states, but this problem is still not solved for all kinds of entangled states. The method of entanglement witness is of particular interest as it is a simple method that only requires a few local measurements. Unlike quantum tomography, which provides a full reconstruction of the quantum states, witnesses allow to detect if a state is entangled using a minimal number of measurements.

Witnesses are particularly interesting for qubits travelling in solid state conductors where detection of entanglement is not yet perfectly understood. The background of this

thesis will be mainly based on the article [6] which consider minimal linear entanglement witnesses from electrical current correlations. These minimal entanglement witnesses can detect with only two currents correlation all entangled pure states, with the exception of maximally entangled states, and some mixed states. In this thesis, we will consider non-linear entanglement witnesses which, in principle, should allow us to detect more entangled states and hopefully detect the maximally entangled states.

Chapter 2

Theory

In this chapter, we will describe two different types of quantum states: the separable states and the entangled states. The separability problem consists of making the distinction between those two kinds of states, and we will go through different ways that have been found to partially solve such problem. We will focus particularly on the entanglement witnesses, which is the subject of this thesis. We will also introduce the mixed states as a way to describe statistical ensembles of quantum states, and which requires the formalism of density matrices to describe them.

2.1 Quantum system

We will work with qubits travelling in solid state conductors, the expression flying electron is usually used in that case. Those electrons can be described as a quantum system. We will see how we can describe them mathematically and how we can classify them based on their physical property. All the material in this section can be found in standard text books about quantum information, see e.g. Ref. [7].

2.1.1 Description of a quantum system

We can make the distinction between two types of quantum states: pure states and mixed states.

Pure states

A quantum system can be represented by an Hilbert space $\mathcal{H}$. A *pure state* of such a system is represented by a ray in that space:

$$|\psi\rangle = \sum_i c_i |\phi_i\rangle \tag{2.1}$$

where $\{|\phi_i\rangle\}$ is an orthonormal basis of $\mathcal{H}$, the $c_i = \langle\phi_i|\,\psi\rangle$ are the components of $|\psi\rangle$ in that basis and a ray is a set of vectors which only differ by a phase.

Mixed states

In real conditions, quantum states are interacting with their environment and decoherence can occur: some information about the state is lost into the environment. All we know is

the probability that the system will be in a certain state, i.e., our system is a statistical ensemble of pure states. Such a state is called a *mixed state*, we can see it as a result of some lack of knowledge about a quantum system. A mixed state cannot be written as a vector in $\mathcal{H}$ and, to describe mathematically all the information we have about it, we can use the formalism of density matrices.

Density matrix

Any quantum state, pure or mixed, can be represented by a matrix ρ called *density matrix* which satisfies the following properties:

1. $\rho = \rho^\dagger$
2. $\operatorname{tr} \rho = 1$
3. $\rho \geq 0 \Leftrightarrow$ eigenvalues are non-negative

Any density matrix can be decomposed as follow:

$$\rho = \sum_i p_i \left|\phi_i\right\rangle \left\langle\phi_i\right| \tag{2.2}$$

with $\sum_i p_i = 1$ and where $\left|\phi_i\right\rangle \left\langle\phi_i\right|$ is the density matrix of a pure state. Physically we can see ρ as being a statistical ensemble of pure states $\left|\phi_i\right\rangle$ associated to the probability p_i.

2.1.2 Composite system

Entanglement can occur between two or more quantum systems. In this thesis we will look at the entanglement of pairs of electrons which can be represented by two quantum systems and we would like to describe that system as a whole. Such system is called a *composite system*. We will only focus here on *bipartite* quantum systems, i.e. a quantum system that can always be decomposed into two subsystems. In particular, we only consider subsystems of two dimensions that are represented by 2×2 density matrices. In quantum information such systems are called two qubits, where a *qubit* is a two-state quantum system.

Let's represent those two quantum systems by the two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. Let $\left|a_i\right\rangle$ be a basis of $\mathcal{H}_A$ and $\left|b_i\right\rangle$ a basis of $\mathcal{H}_B$. The composite system of those two subsystems is represented by the space of the tensor product $\mathcal{H}^{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

A pure state of the composite system is described by a vector in $\mathcal{H}^{AB}$ which can be written as:

$$\left|\psi\right\rangle = \sum_{ij} c_{ij} \left|a_i\right\rangle \otimes \left|b_j\right\rangle \tag{2.3}$$

where $\{\left|a_i\right\rangle \otimes \left|b_j\right\rangle\}$ is a basis of $\mathcal{H}^{AB}$.

A mixed state of the composite system is represented by a density matrix ρ_{AB} and can always be decomposed in a chosen product basis as:

$$\rho_{AB} = \sum_{ijkl} P_{kl}^{ij} \left|a_i\right\rangle \left\langle a_j\right| \otimes \left|b_k\right\rangle \left\langle b_l\right| \tag{2.4}$$

Those definitions can be generalized to an arbitrary number of subsystems but those will not be further considered here.

In composite systems further differences appear between classical and quantum objects which make them interesting object to study. Classically, if we take all possible pure states of the A system and all possible pure states of the B system then all the possible combination of states is forming the space of all the bipartite states. In the quantum mechanics world this is not the case anymore. The space of the bipartite states is bigger, we have a new kind of states: the entangled states. Let's define more precisely the different kinds of bipartite states we can encounter.

2.1.3 Type of quantum states

Product states

A state is called *product state* if its density matrix can be written as:

$$\rho = \rho^A \otimes \rho^B \tag{2.5}$$

Physically, we say that the product states are uncorrelated which mean that if we perform a measurement on one of the subsystems, the results will not be correlated with the outcome of measurements on the other subsystems. Those states can be created locally which means that they can be produced independently from each other.

Separable states

A state is called *separable state* if its density matrix can be written as:

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B \tag{2.6}$$

where p_i is the probability associated to each product states $\rho_i^A \otimes \rho_i^B$. So, we have $\sum_i p_i = 1$ and $p_i \geq 0 \ \forall i$.

In the bracket notation, a pure state $\left|\psi\right\rangle \in \mathcal{H}^{AB}$ is separable if it can be written as:

$$\left|\psi\right\rangle = \left|\phi_A\right\rangle \otimes \left|\phi_B\right\rangle \tag{2.7}$$

where $\left|\phi_A\right\rangle \in \mathcal{H}_A$ and $\left|\phi_B\right\rangle \in \mathcal{H}_B$.

Physically, the results of measurements on the subsystem of a separable state can be correlated between the two subsystems but the correlations are coming from the distribution of probability of the product states which can be generated classically and are described by the mixed states. So, the origin of the correlation is not purely quantum mechanical. Product states are a subclass of separable states which do not have any classical correlation.

It can be easily shown that the set of separable states is convex which means that any convex linear combination of separable states is a separable state:

$$\forall \rho_1 \text{ and } \rho_2 \text{ separable}, \exists \lambda > 0 \text{ such that } \quad \rho = \lambda\rho_1 + (1-\lambda)\,\rho_2 \text{ is a separable state} \tag{2.8}$$

From a geometrically point of view, we can visualize this as any pair of separable states can be joined by a straight line which is always in the set of separable states.

Entangled state

A state which is not a separable state is called an entangled state. We cannot write the density matrix in terms of a sum of products of the density matrices of each subsystem. The system cannot be described as two subsystems separately.

In that case, the correlation between the two subsystems is coming from quantum mechanics only. A measurement performed on one of the subsystems will influence the state of the other subsystem thus there is a correlation between local measurements on each subsystem.

An example of well-known entangled states are the Bell states. We are going to refer to them very frequently in this thesis, they can be represented by the following matrices in the standard basis $\{|\uparrow\uparrow\rangle\,|\uparrow\downarrow\rangle\,|\downarrow\uparrow\rangle\,|\downarrow\downarrow\rangle\}$:

$$\rho_{B1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \quad \rho_{B2} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{2.9}$$

$$\rho_{B3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \rho_{B4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Geometrical visualization of the different types of quantum states

The set of all quantum states is also convex. The set of separable states is then a convex subset of the set of all quantum states and the set of entangled states a non-convex subset. Furthermore, the extremal points of the convex set of separable states are product states as any separable states can be written as a convex combination of product states. So, we can visualize those sets as shown in Fig. 2.1.

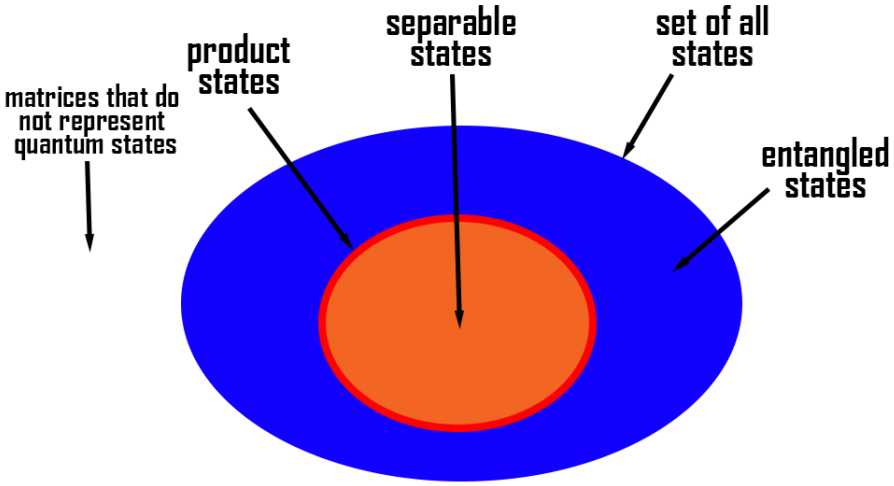

Figure 2.1: Geometrical representation of the different types of quantum states.

This geometrical representation can be sometimes tricky as we are representing a high dimensional convex set into a two-dimensional picture. Therefore, we must be careful in the interpretations we are basing on it. We will come back to the geometrical interpretation a couple of times throughout this thesis as it is a good way to visualize the situation.

2.1.4 Summary

Table 2.1: Summary of the different type of states

	pure states	mixed states	product states	separable states	entangled states
Hilbert space	$\|\psi\rangle = \sum_i c_i \|\phi_i\rangle$	-	$\|\psi\rangle = \|\phi_A\rangle \otimes \|\phi_B\rangle$	-	-
Density matrix	$\rho = \|\psi\rangle\langle\psi\|$	$\rho = \sum_i p_i \|\phi_i\rangle\langle\phi_i\|$	$\rho = \rho^A \otimes \rho^B$	$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B$	-

2.2 Tools for density matrix and multipartite system

Density matrices constitute a different language from the more familiar Dirac notation to describe quantum systems. To compute information about the system and subsystems in this formalism we define here different tools in algebra.

2.2.1 Properties of quantum system

Expectation value

The expectation value of an observable A of the state represented by the density matrix ρ is given by:

$$\langle A \rangle_\rho = \mathrm{tr}\,(\rho A) \tag{2.10}$$

Purity

Given a density matrix ρ the purity P of the state ρ is given by

$$P(\rho) = \mathrm{tr}\left(\rho^2\right) \tag{2.11}$$

where $P(\rho) = 1$ for pure states and $P(\rho) < 1$ for mixed states. The purity is a way to quantify how much a state is mixed. The maximally mixed states are the states where the p_i's of Eq. (2.2) form a uniform probability distribution, all the states occur with the same probability. It can be easily shown that a pure state has only one non-zero eigenvalue equal to 1.

Concurrence

The *concurrence* is a quantity which allows to quantify the amount of entanglement of a given state.

A general pure state of a two-qubit system can always be expressed in the standard basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ as:

$$|\psi\rangle = a\,|\uparrow\uparrow\rangle + b\,|\uparrow\downarrow\rangle + c\,|\downarrow\uparrow\rangle + d\,|\downarrow\downarrow\rangle \tag{2.12}$$

where $a, b, c, d \in \mathbb{C}$ such that $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$
The concurrence C of $|\psi\rangle$ is given by [8] :

$$C(|\psi\rangle) = 2|ad - bc| \tag{2.13}$$

It has also been shown that it is equivalently given by:

$$C(|\psi\rangle) = \sqrt{2\left[1 - \mathrm{tr}\left(\rho_A^2\right)\right]} \tag{2.14}$$

where ρ_A corresponds to the reduced state of $|\psi\rangle$ on the A side which is introduced in Sec. 2.2.2. The concurrence ranges from 0 for separable states to 1 for maximally entangled states such as the Bell states.

For a bipartite mixed state ρ, it has been shown that the concurrence is given by [9]:

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \tag{2.15}$$

where the λ_i are the eigenvalues, in decreasing order, of the hermitian matrix $\sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ with $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\,\rho^*\,(\sigma_y \otimes \sigma_y)$ and ρ^* is the complex conjugate of ρ.

2.2.2 Properties of quantum subsystems

Partial trace

The partial trace is used to describe parts of a larger quantum system. If we have two physical systems A and B which are described by the density matrix ρ_{AB}, the reduced density matrix of system A is given by $\rho_A = \mathrm{tr}_B(\rho_{AB})$ where tr_B is called the partial trace over B and is defined as follow:

$$\sum_{ijkl} P_{kl}^{ij} \left|a_i\right\rangle \left\langle a_j\right| \otimes \left|b_k\right\rangle \left\langle b_l\right| \;\; \mapsto \;\; \sum_{ijkl} P_{kl}^{ij} \left|a_i\right\rangle \left\langle a_j\right| \otimes \mathrm{tr}\left(\left|b_k\right\rangle \left\langle b_l\right|\right)$$

We can look at some examples to understand physically what happens:

1. If we take the partial trace of a product state we get $\mathrm{tr}_B\left(\rho_A \otimes \rho_B\right) = \rho_A \,\mathrm{tr}_B\left(\rho_B\right) = \rho_A$. No information about state A is lost.

2. Let's consider one of the Bell state ρ_{B1} which is a pure entangled state. The partial trace with respect to the B system only picks up the matrix elements on the diagonal and all the information that are not on the diagonal are lost: $\mathrm{tr}_B\left(\rho_{B1}\right) = \frac{1}{2} I_{2\times 2}$. If we try to describe A as a system by itself, we get a maximally mixed state i.e. we have the minimum amount of information about the state. So, we know everything that can be known about the combination of system A and B but we know nothing about system A and system B.

Expectation values

We can use the partial trace to compute the expectation value for an observable of one of the subsystem. The expectation value of an observable $\mathcal{O}_A$ of the subsystem A is given by:

$$\left\langle O_A\right\rangle = \mathrm{tr}\left(\left(\mathcal{O}_A \otimes I_B\right)\rho_{AB}\right) = \mathrm{tr}_A\left(\mathcal{O}_A \,\mathrm{tr}_B\left(\rho_{AB}\right)\right) \tag{2.16}$$

Partial transposition

The partial transposition is a local operator Γ that acts on the density matrix of composite systems and will be used in the next section. The partial transpose of a state ρ, that can be written as $\rho = \sum_{ijkl} P_{kl}^{ij} \left|i\right\rangle \left\langle j\right| \otimes \left|k\right\rangle \left\langle l\right|$, with respect to the A and B side is defined by:

$$\rho^{\Gamma_A} = \sum_{ijkl} P_{kl}^{ij} \left(\left|i\right\rangle \left\langle j\right|\right)^T \otimes \left|k\right\rangle \left\langle l\right| = \sum_{ijkl} P_{kl}^{ij} \left|j\right\rangle \left\langle i\right| \otimes \left|k\right\rangle \left\langle l\right| = \sum_{ijkl} P_{kl}^{ji} \left|i\right\rangle \left\langle j\right| \otimes \left|k\right\rangle \left\langle l\right| \tag{2.17}$$

$$\rho^{\Gamma_B} = \sum_{ijkl} P_{kl}^{ij} \left|i\right\rangle \left\langle j\right| \otimes \left(\left|k\right\rangle \left\langle l\right|\right)^T = \sum_{ijkl} P_{kl}^{ij} \left|i\right\rangle \left\langle j\right| \otimes \left|l\right\rangle \left\langle k\right| = \sum_{ijkl} P_{lk}^{ij} \left|i\right\rangle \left\langle j\right| \otimes \left|k\right\rangle \left\langle l\right| \tag{2.18}$$

2.3 The separability problem

Given a state ρ, the separability problem is the problem of determining whether ρ is separable or entangled.

There are several criteria to determine if a state is separable or entangled but, up to now, there is no general solution for the separability problem. For a 2×2 or 3×2 systems, there is however a necessary and sufficient condition, as we will see.

2.3.1 Schmidt decomposition

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be two Hilbert spaces, and let $|a_i\rangle$ and $|b_i\rangle$ be two arbitrary basis for $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. Then we know from Eq. (2.3) that any pure state in $\mathcal{H}_A \otimes \mathcal{H}_B$ can be written as:

$$|\psi\rangle = \sum_{i,j} D_{ij} |a_i\rangle |b_j\rangle \tag{2.19}$$

We could choose any basis to write the state $|\psi\rangle$ and the matrix D will change accordingly. Algebra tells us [10] that for every complex matrix D, there is always a unitary transformation U and V such that the matrix UDV is diagonal. This is called the singular value decomposition of D. The direct application of this in quantum information is called the Schmidt decomposition:

Let $|\psi\rangle$ be a bipartite pure state where $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$; then it is always possible to write it as follow:

$$|\psi\rangle = \sum_{i=0}^{d-1} \lambda_i |i\rangle_A |i\rangle_B \tag{2.20}$$

This is called the Schmidt decomposition where we have that:

- The λ_i are real, strictly positive and normalized so that $\sum_{i=0}^{d-1} \lambda_i = 1$.
- The Schmidt coefficients λ_i are uniquely defined.
- $\{|i\rangle_A\}$ and $\{|i\rangle_B\}$ form an orthonormal basis of $\mathcal{H}_A$ and $\mathcal{H}_B$.
- d is called the Schmidt rank and is limited by the dimension of the smaller subsystem. It satisfies $d \leq \min\ \{\dim(\mathcal{H}_A), \dim(\mathcal{H}_B)\}$. So, in our case we have $d \leq 2$.

The Schmidt decomposition allows us to have a deeper insight for the distinction between separable and entangled states. Indeed, as the Schmidt basis elements are separable states, all the information of the entanglement is encoded in the Schmidt coefficients.

If there is only one non-vanishing Schmidt coefficient, then $|\psi\rangle$ is separable. Otherwise, when at least two Schmidt coefficients are different from zero, it is not possible to express $|\psi\rangle$ as a product state. Consequently, we can conclude that a pure state is separable if and only if it has only one non-vanishing Schmidt coefficient.

The Schmidt decomposition provides us with a necessary and sufficient criterion to determine if a pure state is entangled. Unfortunately, for mixed states such a decomposition does not exist [11].

2.3.2 PPT criterion

The PPT criterion, originally introduced in [12], is a simple condition to determine if a state is entangled or not given its density matrix.
We say that a matrix is PPT (positive partial transpose) if its partial transposition defined by Eq. (2.17) and (2.18) is positive semidefinite, i.e. has non-negative eigenvalues.

The PPT criterion tells us that if ρ is a bipartite separable state then ρ is PPT.

Let us note that this is not a necessary and sufficient criterion. In other words, if the state is not PPT then it is not a separable state and it is an entangled state. But if the state is PPT then it could be separable or entangled. From a geometrically point of view, the convex set of the positive partial transpose is larger than the convex set of separable states. So, we can have positive partial state which are not separable states.

We will not give a rigorous proof of the PPT criterion which can be found at [13], but explain the idea:

If we take the transpose of a density matrix the spectrum is preserved [10] hence its trace, hermiticity and positivity is also conserved. So, the transpose of a density matrix is still a density matrix. But the positivity of the density matrix is not necessarily conserved by the partial transpose operator. It is however always conserved for separable states as they can always be written as Eq. (2.6). Indeed, we have

$$(\rho)^{\Gamma_A} = \sum_i p_i \left(\rho_i^A\right)^T \otimes \rho_i^B \tag{2.21}$$

where T is the transpose of ρ_i^A which is a density matrix hence the positivity is preserved so ρ is still a state and a positive matrix. But for entangled states, it is not possible to write the density matrix as in Eq. (2.6) since the positivity of the partial tranpose is not always guaranteed.

2.3.3 Horodecki criterion

For a certain class of states the PPT criterion becomes a necessary and sufficient condition known as the Horodecki theorem:

If the state is 2×2 or 2×3 then the density matrix is PPT if and only if it the state is separable. But in other dimensions, this is not the case. A proof of this theorem can be found at [14].

2.4 Witness

All the previous criteria we saw were requiring to know the exact form of the density matrix of the state. An *entanglement witness* is trying to solve the separability problem by performing only a limited number of measurements on the state. Based on the outcomes of those measurements it is possible to determine if a state is entangled. In this thesis, our goal is to determine if two qubits in a semiconductor are entangled or not by only performing two measurements based on current cross correlations. This section will be a general introduction to witness while chapter 3 will introduce the specific type of witness we are using in this thesis.

2.4.1 Geometrical interpretation

We will start by giving a geometrical intuition of what a witness is and then give a more formal definition.

A corollary of the Hahn–Banach theorem [15] tells us that given a convex set and a point outside of it, there exists a (hyper)plane such that the point is on one side of it and

the convex set is on the other side. This is obvious in two or three dimensions when we look at Fig. 2.2, but this holds for any dimensions.

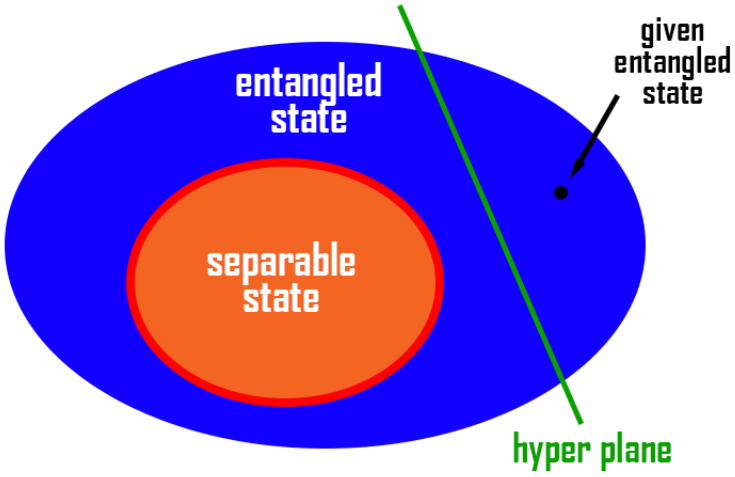

Figure 2.2: Geometrical representation of a linear witness

We can apply this corollary with the convex set of separable states introduced in Sec. 2.1.3. The witness will define the plane which separates the entangled states from the convex set of separable states.

For any points outside the convex set of separable states, corresponding to an entangled state, we can define a plane which separates it from the convex set. Thus, for any entangled state, it is always possible to define a witness to detect it, a formal proof can be found in [13]. All the difficulty is to find that witness. Let's now express those intuitive ideas more mathematically.

2.4.2 Definition

Definition 1

The discussion which will follow is inspired by [16]. Let's consider a vector space V, we can define a plane in V by all the vectors $|v\rangle$ satisfying the inner product $\langle w|v\rangle = C$ where w is a unit vector perpendicular to the plane and C is a constant.

Let's consider the vector space of hermitian operators and let's define the following inner product between two elements of that space:

$$\langle H_1|H_2\rangle = \mathrm{tr}(H_1^\dagger H_2) = \mathrm{tr}(H_1 H_2) \tag{2.22}$$

It can be easily shown that it is indeed an inner product. We can use it to define a plane W in the vector space of the hermitian operators:

$$\mathrm{tr}(W\rho) = C \tag{2.23}$$

where W and ρ are hermitian operators.

Given an entangled state ρ_{ent}, we want to find W such that $\mathrm{tr}(W\rho) > C$ ($\mathrm{tr}(W\rho) < C$) for ρ_{ent} and $\mathrm{tr}(W\rho) < C$ ($\mathrm{tr}(W\rho) > C$) for ρ_{sep}.It corresponds to the picture "being on

each side of the plane" given in the geometrical interpretation where we can choose either side of the plane for the set of separable states. We say that W is a witness if it can detect at least one entangled state. The main difficulty in using witnesses is to find them, we know they exist but we do not know how they look.

The way we define a witness is only a matter of conventions. In most papers, the scientific community usually chooses 0 for the constant C and chooses $\mathrm{tr}(\rho W) > 0$ for the side of the plane with the convex set of separable state and $\mathrm{tr}(\rho W) \leq 0$ for the side of the plane with the entangled state we want to detect. So, the typical formal definition of a witness is given by:

A witness W is an hermitian operator which is positive on all the separable states and negative for at least one entangled states which is equivalent to:

$$\begin{aligned} \mathrm{tr}\,(\rho W) &\geq 0 \quad \text{for all separable states} \\ \mathrm{tr}\,(\rho W) &< 0 \quad \text{for at least one entangled state} \end{aligned} \tag{2.24}$$

So if the expectation value of the witness is positive the state can be either entangled or separable and if the expectation value is negative then the state is entangled.

Looking at the definition 2.10 of the expectation value of a state described by a density matrix, we now understand why we used the inner product defined by 2.22. Indeed, the definition of the witness is expressed in terms of the expectation value of W for the state ρ. This achieves the connection between pure math with the Banach theorem and physics with measurement than can be performed in a lab which makes the witness a very powerful tool.

A witness W is said to be optimal if there is no other (linear) witness which can detect more states than all the entangled states W can already detect. This implies that for any positive operator P, the observable $X = W - P$ is not a witness anymore.Indeed, P will shift some states from the positive side to the negative and it stays a witness if it only shift entangled states. We can see an optimal witness geometrically as the plane which is tangent to the set of separable states. However, this is only a necessary condition, more details can be found in [13]. Our goal is to have a witness which is as optimal as possible but that use observables that are easily accessible in the lab.

Definition 2

The definition 2.24 of a witness is more appropriate for a mathematical treatment of a witness. In this section, we will give an equivalent definition which is more appropriate for experimental applications of a witness and this is the one we will use in this thesis. It was introduced in [6] where a witness is defined as an observable quantity for which at least one entangled state takes on a value outside the range accessible for separable states. So, the definition of a witness becomes:

W is an entanglement witness if there is at least one entangled state ρ such that of one the two following conditions is fulfilled:

$$\begin{aligned} \Delta W_\rho^+ = \mathrm{tr}\,(\rho W) - \max_{\rho_s} \mathrm{tr}\,(\rho_s W) &> 0 \quad \text{for at least one entangled state } \rho \\ \Delta W_\rho^- = \min_{\rho_s} \mathrm{tr}\,(\rho_s W) - \mathrm{tr}\,(\rho W) &> 0 \quad \text{for at least one entangled state } \rho \end{aligned} \tag{2.25}$$

where ρ_s are the density matrices of the separable states and $\Delta W_\rho^\pm$ are called the *detection margins*. In this thesis, we will refer to $\mathrm{tr}\,(\rho W)$ as *witness* and to W as the *witness operator*.

This definition is equivalent to the previous definition 2.24 of a witness. We just use a different constant than 0 and we use two inequalities instead of one:

$$\begin{aligned} \mathrm{tr}\,(\rho W) &> C = \max_{\rho_s} \mathrm{tr}\,(\rho_s W) \quad \text{for at least one entangled state} \\ \mathrm{tr}\,(\rho W) &< D = \min_{\rho_s} \mathrm{tr}\,(\rho_s W) \quad \text{for at least one entangled state} \\ D \leq \mathrm{tr}\,(\rho W) &\leq C \quad \text{for all separable states} \end{aligned} \tag{2.26}$$

Geometrically, we can see that as using two parallel planes to define our witness. It divides the space into three parts where the middle part contains the convex set of separable states and the two other parts contain entangled states only.

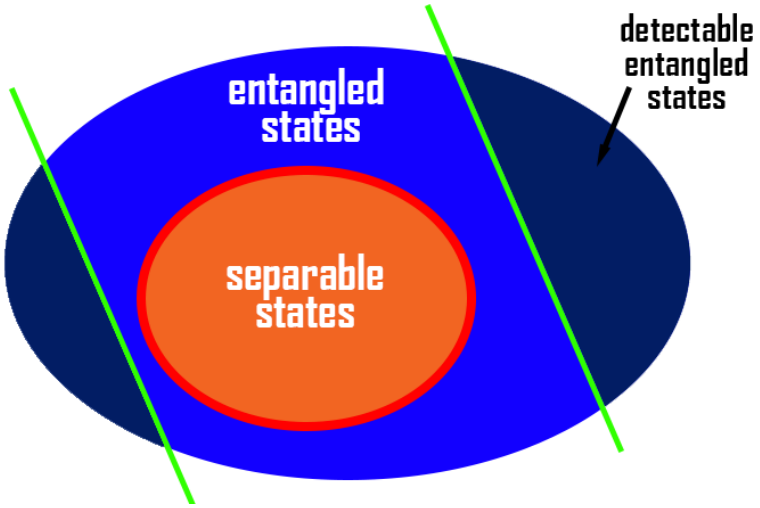

Figure 2.3: Geometrical representation of a linear witness for the definition 2.25

Line representation of a witness

A witness is an operator which associates a state which is a multi-dimensional object to a combination of expectation values which is a real number. We use those values to distinguish the separable and entangled states. We can represent a given witness by a one-dimensional axis which corresponds to the combination of the expectation values. Let's call this the line representation of a witness.

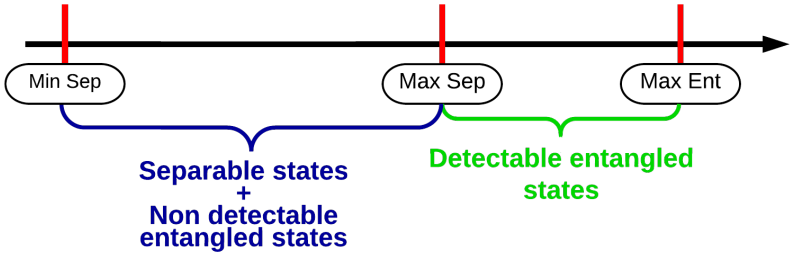

Figure 2.4: Line representation of a witness where the horizontal axis corresponds to the expectation value of the witness operator. The value for the separable state that maximize and minimize the witness and the entangled state that maximize the witness are represented.

Example

We can give a simple example of construction of a witness operator. For any entangled state, we can construct a witness detecting it based on the PPT criterion [17]. Let ρ_{ent} be the density matrix of an entangled state, we know from the PPT criterion that its partial transpose ρ_{ent}^{Γ} has negative eigenvalue λ_{-}. Let $|\psi\rangle$ be the eigenstate associated to this negative eigenvalue then $W = |\psi\rangle\langle\psi|^{\Gamma}$ is a witness detecting ρ according to definition 2.24. Indeed:

$$\mathrm{tr}(\rho_{ent} W) = \mathrm{tr}(\rho_{ent} |\psi\rangle\langle\psi|^{\Gamma}) = \mathrm{tr}(\rho_{ent}^{\Gamma} |\psi\rangle\langle\psi|) = \langle\psi| \rho_{ent}^{\Gamma} |\psi\rangle = \lambda_{-} < 0$$

$$\mathrm{tr}(\rho_{sep} W) = \langle\psi| \rho_{sep}^{\Gamma} |\psi\rangle = \lambda_{+} \geq 0$$

The problem might seem to be solved, we can detect any entangled state we want. But this witness has some drawbacks. W cannot necessarily be expressed in terms of simple measurable observables which makes it much harder to use it practically. Another problem is that the witness is constructed based on the density matrix of the entangled states we want to detect while all the point of a witness is that it does not require to know the explicit form of the density matrix but only requires to know some expectation values of the state. Last but not least, this only works for pure states.

2.4.3 Linear vs non-linear witness

We can make the distinction between two kinds of witnesses: linear witnesses and non-linear witnesses.

A *linear* entanglement witness is defined as being a linear functional $F_L(\rho)$ of expectation values of the density matrix.

A *non-linear* entanglement witness $F_{NL}(\rho)$ is defined as being a non-linear functional of expectation values of the density matrix.

In this thesis, we will focus about non-linear witnesses. Let's briefly summarize first what have been done so far in the scientific literature. The definition 2.24 of a witness is being used in this section.

Example of construction of non-linear witness

It has been shown in [18] that it is always possible for any bipartite linear witness F_L to be improved by non-linear terms $X(\rho)$ which are a non-linear combinations of expectation values. So, the non-linear witness is written as:

$$F_{NL}(\rho) = \mathrm{tr}(W\rho) - X(\rho) \tag{2.27}$$

Improving means here that F_{NL} is still a witness, i.e., is positive on all separable states and F_{NL} can detect all states F_L can detect plus some new states in addition.

The non-linear term is a quadratic function of the expectation values and thus is always positive. By having the line representation of a witness in mind we can see the action of the non-linear term as shifting the states left. But as the F_{NL} is still a witness, all the separable states are still yielding positive values while some entangled states might be shifted enough such that they yield a negative expectation value and thus become detectable hence F_{NL} can detect more states than F_L.

In [19], that method was iterated to generate a sequence of non-linear witness which become better at each iteration.

The main problems with all the methods that have been studied so far is that the non-linear witness cannot necessarily be expressed in terms of very simple observables easily accessible in a lab. Or if we build a non-linear witness starting from a linear witness, the non-linear terms require some additional measurements. We will investigate here only non-linear witnesses which use the same expectation values as the linear ones.

Geometric interpretation

We can visualize a non-linear witness as follow:

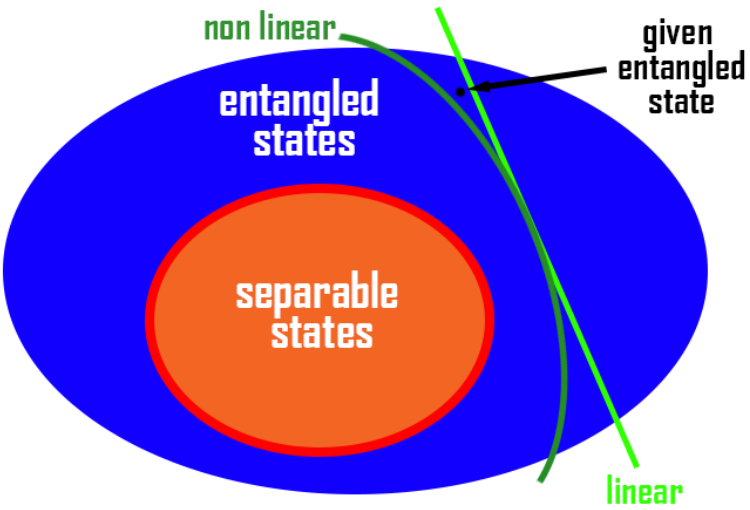

Figure 2.5: Example of a geometrical representation of a non-linear witness constructed from a linear witness where we can see that the non-linear witness can detect more entangled states than another linear witness.

Chapter 3

Witnesses based on cross correlation measurements

This thesis is a follow-up of the paper [6] "Minimal Entanglement Witness From Electrical Current Correlations". In this paper, the authors are constructing and investigating linear witnesses to detect entanglement between two mobile electrons in solid state conductors using only two current cross correlation measurements. The goal of this thesis is to investigate what is possible to do by considering non-linear witnesses based on the same measurements. We will summarize this paper in this section and start by describing a generic entangler-detector set-up from which we will construct the witness.

3.1 Set-up

An entangler emits identical split pairs of spin or orbitally entangled electrons at a rate Γ. The electrons are then separated and go through a pair of detector system A and B.

First, the electrons are going through a polarizer which projects their spin along the direction of the polarizing vectors **a** and **b**. Then, the electrons go through a beam splitter which separates the electrons whether the spin is up or down along the direction of the polarizer. And finally, the electrons reach the detector terminals $A+$, $A-$ or $B+$,$B-$ depending on the orientation of their spin. We suppose all the electrons that arrive at the detector terminals contribute to an electric current which means that the detectors have an ideal efficiency.

We also introduce the beam splitter efficiencies $0 \leq \zeta_A, \zeta_B \leq 1$ to take into account undesirable different physical effects such as dephasing or non-perfect polarization of ferromagnetic detector terminals. It means that an electron at A will be projected along the direction **a** with a probability ζ_A and be transmitted to $A+$ or $A-$ accordingly. And there is a probability $1 - \zeta_A$ that it is not projected along the direction of **a** and has an equal probability to end up on the $A+$ or $A-$ detector. The set-up is schematically represented in Fig. 3.1.

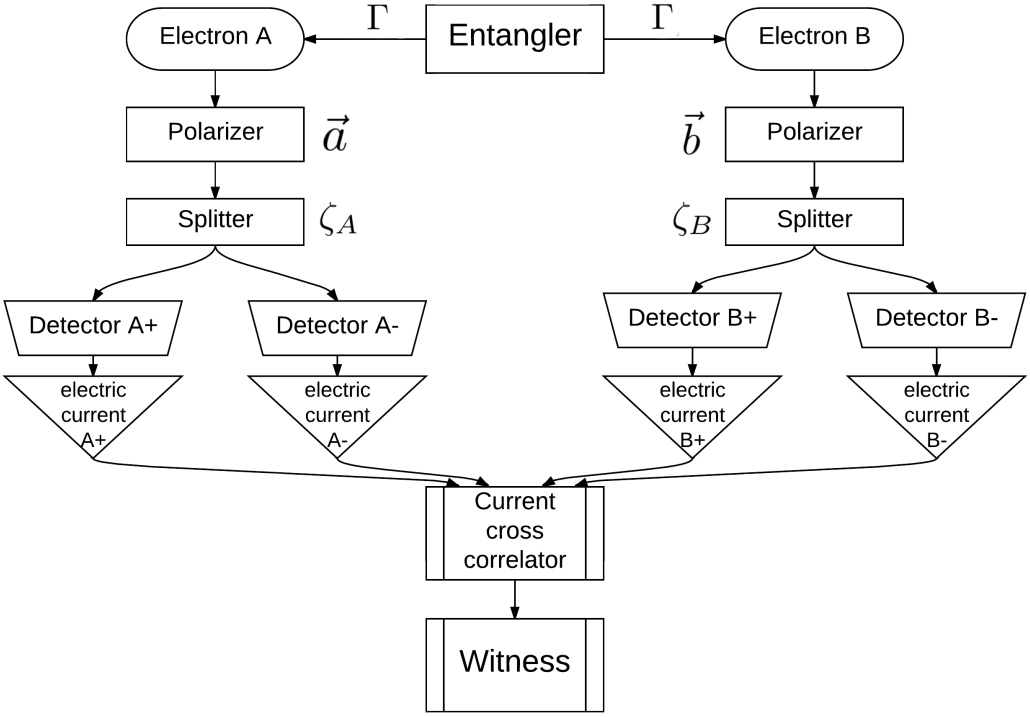

Figure 3.1: Schematic representation of the entangler-detector setup from which the witness is constructed.

3.1.1 Current cross correlation and witness

It is currently very challenging to perform real-time detection of quantum properties of flying electrons. Instead we can use the current generated by all the electrons reaching the detector terminals $A\pm$,$B\pm$ and look at cross correlations of those currents. In the aim of detecting entanglement we can write a current cross correlator as [20]:

$$S_{AB}^{\pm\pm}(\mathbf{a},\mathbf{b}) = e^2\Gamma \,\mathrm{tr}\left[(\mathbb{1} \pm \zeta_A \mathbf{a}\cdot\boldsymbol{\sigma}) \otimes (\mathbb{1} \pm \zeta_B \mathbf{b}\cdot\boldsymbol{\sigma})\,\rho\right] \tag{3.1}$$

where Γ is the rate at which the entangler emits the pair electrons, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is a vector of Pauli matrices and ρ is the density matrix of the two-particle spin. It has been shown in [21] that correlations between the currents is equivalent to a pair measurement so that only correlations between the electrons from the same entangled pair contribute. This justifies why we can use cross correlation measurements to detect entanglement.

We would like to construct an entanglement witness based on such current cross correlations. But to be experimentally viable, we want to minimize the number of correlations measurements and the number of detector terminal pairs $A\pm$,$B\pm$. Based on those requirements, the authors build the following operator based on the measurement of N current cross correlations at two detectors terminals $A+$ $B+$:

$$W^{(N)} = \sum_{i=1}^{N} (\mathbb{1} + \zeta_A \mathbf{a}_i\cdot\boldsymbol{\sigma}) \otimes (\mathbb{1} + \zeta_B \mathbf{b}_i\cdot\boldsymbol{\sigma}) \tag{3.2}$$

The authors of the paper [6] found that by considering Eq. (3.2) where $N = 2$,

$$\begin{aligned} W^{(2)} &= (\mathbb{1} + \zeta_A \mathbf{a}_1 \cdot \boldsymbol{\sigma}) \otimes (\mathbb{1} + \zeta_B \mathbf{b}_1 \cdot \boldsymbol{\sigma}) + (\mathbb{1} + \zeta_A \mathbf{a}_2 \cdot \boldsymbol{\sigma}) \otimes (\mathbb{1} + \zeta_B \mathbf{b}_2 \cdot \boldsymbol{\sigma}) \\ &\equiv W_{(1)} + W_{(2)} \end{aligned} \tag{3.3}$$

they can detect all pure entangled states except the maximally entangled ones and some mixed states. This makes the operator of Eq. (3.3) a witness operator. The witness associated to such witness operator is given by:

$$F_L(\rho) = \mathrm{tr}(\rho W^{(2)}) = \mathrm{tr}(\rho W_{(1)}) + \mathrm{tr}(\rho W_{(2)}) \tag{3.4}$$

This quantity is directly related to the cross correlation measurements by:

$$\mathrm{tr}(\rho W_{(1)}) + \mathrm{tr}(\rho W_{(2)}) = \frac{1}{e^2 \Gamma} \left(S_{AB}^{++}(\mathbf{a}_1, \mathbf{b}_1) + S_{AB}^{++}(\mathbf{a}_2, \mathbf{b}_2) \right) \tag{3.5}$$

3.1.2 Goal

Our starting point are the results given in the last section. We would like to answer the following question: is it possible to obtain a better witness than Eq. (3.4) by using non-linear combination of the same expectation values $\mathrm{tr}(\rho W^{(1)})$ and $\mathrm{tr}(\rho W^{(2)})$? Better means here an improvement of the detection margin ΔW so that we are able to detect more states for a given settings or even detect states that cannot be detected at all in the linear case. Our main goal will be to find a non-linear witness which is able to detect maximally entangled states as they cannot be detected in the linear case.

3.2 Settings and rotation

Another important point that needs to be discussed and which is not fully explained in the paper [6] is the way we define our witness. As we will see in the next chapter, this will be a key point to investigate the non-linear witness.

3.2.1 Fixing the state vs. fixing the witness

To define our witness we use what we call *settings*: the splitters efficiencies ζ_A, ζ_B which are scalars and the polarization vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1, \mathbf{b}_2$ which lie in a three dimensional space. The polarization vectors have a fix length so we need 4×2 parameters to describe them. To define the polarization vectors we have to choose two local basis on the A and B side.

We can choose those local basis such that a given configuration of the polarization vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ ($\mathbf{b}_1$ and $\mathbf{b}_2$) lie in the XZ plane, symmetrically around the Z axis. We will call that the *standard configuration*. The relative angle between $\mathbf{a}_1$ and $\mathbf{a}_2$ ($\mathbf{b}_1$ and $\mathbf{b}_2$) is called θ_A (θ_B). This is represented in Fig. 3.2.

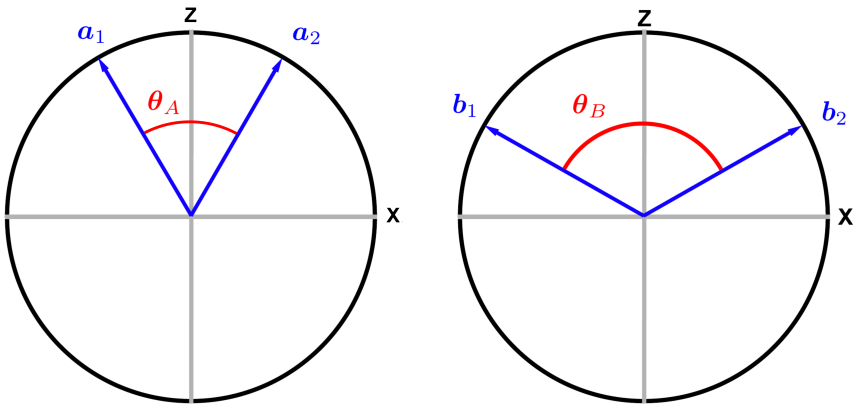

Figure 3.2: Standard configuration of the polarization vectors where they lie in the XZ plane symmetrically around the Z axis.

We can fix a state and go through all the settings of the witness to see which one can detect it. However, the standard configuration covers only a fraction of all the settings that can parametrize the witness. To parametrize all of them, we can consider all the rotations of those polarization vectors in the standard configuration to cover all the sphere. So, the witness is defined by 2×5 parameters: the relative angle θ_A and θ_B between the polarization vectors, the splitter efficiencies ζ_A, ζ_B and the 2×3 rotation angles R_{1a}, R_{2a}, R_{3a}, R_{1b}, R_{2b}, R_{3b}. The goal is to find for which of those parameters the state is detectable. We will call $(\theta_A, \theta_B, \zeta_A, \zeta_B)$ the *basic settings* and $(\theta_A, \theta_B, \zeta_A, \zeta_B, R_{1a}, R_{2a}, R_{3a}, R_{1b}, R_{2b}, R_{3b})$ the *settings* of the witness, where: $0 < R_{1a}, R_{3a}, R_{1b}, R_{3b} < 2\pi$,$0 < R_{2a}, R_{2b} < \pi$, $0 < \zeta_A, \zeta_B < 1$, $0 < \theta_A, \theta_B < \pi$.

The other option is to choose to fix the basic settings in the standard configuration and look through all the entangled states to see which one yields a positive detection margin and hence is detectable. But this requires to parametrize all the entangled states which is not easy in the mixed case.

Ideally, we could solve the problem by going through all the states and all the settings at the same time but this would require to maximize a function of 25 parameters in the case of the mixed states which is quite complicated. However, we will see in the next section that this will not be necessary when we look at the symmetries of the witness which will allow us to construct classes of states so that we can restrict ourselves to one of the two options described above.

3.2.2 Implementing the rotation

As described in the last section, the polarization vector defining the witness are defined by three dimensional vectors but the witness itself is defined by a 4×4 hermitian matrix so they lie in a different space and thus rotations act differently on them. The rotations in 3D space on the polarization vectors on one side are implemented by elements of the group $SO(3)$ which are 3×3 matrices. But the representation of those rotations on the witness are given by the group $SU(2)$ whose elements are 2×2 matrices. Indeed, there exists a two-to-one surjective homorphism of the group SU(2) onto the group SO(3) which means representation of SO(3) are also representation of SU(2). The Euler parameterization of

an arbitrary matrix $SU(2)$ is given by:

$$e^{iR_1\sigma_z}e^{iR_2\sigma_x}e^{iR_3\sigma_z} \tag{3.6}$$

where $0 \leq R_2 \leq \pi$, $0 \leq R_1 \leq 2\pi$, $0 \leq R_3 \leq 4\pi$. If we want to restrict ourselves to the parametric space of $SO(3)$ we can fix those angles as follow: $0 \leq R_3 \leq 2\pi$ and make the identification $R_3 \sim R_3 + 2\pi$

As the witness is defined by the tensor product of the two operators then the rotations $U \in SU(2) \otimes SU(2)$ of the witness can be written as:

$$U = U_A \otimes U_B = e^{iR_{1a}\sigma_z}e^{iR_{2a}\sigma_x}e^{iR_{3a}\sigma_z} \otimes e^{iR_{1b}\sigma_z}e^{iR_{2b}\sigma_x}e^{iR_{3b}\sigma_z} \tag{3.7}$$

3.2.3 Equivalence classes

Doing an unitary operation on the witness operator is mathematically equivalent in our case to doing an unitary operation on the state. It gives the same result as the trace is cyclic:

$$\mathrm{tr}(\rho UWU^\dagger) = \mathrm{tr}(U^\dagger \rho UW) \tag{3.8}$$

What this implies is that if we can find a state ρ^* which can be detected by a given setting, then we can easily determine the settings to detect all the states generated by all the unitary operations of ρ^* and the settings of the witness that detect it. This will be explained in details in the next section first we need to answer the following question: given a two-qubit state, if we apply all unitary operations possible, what kind of states are we producing?

It can be easily shown that rotation operators preserve the trace, positivity and hermicity and it can also be shown that they preserve the concurrence. So, we know that by considering all the rotations of a given state, we are generating states with the same concurrence.

Any pure two-qubit system which can always be written as $|\psi\rangle = a\,|\uparrow\uparrow\rangle + b\,|\uparrow\downarrow\rangle + c\,|\downarrow\uparrow\rangle + d\,|\downarrow\downarrow\rangle$ with $a^2 + b^2 + c^2 + d^2 = 1$ can always be transformed into $|\Phi\rangle = a'\,|\uparrow\uparrow\rangle + d'\,|\downarrow\downarrow\rangle$ where $a'^2 + d'^2 = 1$ with local unitary transformation $U = U_A \otimes U_B$ acting on $|\psi\rangle$ where $|\psi\rangle$ and $|\Phi\rangle$ have the same concurrence.

Indeed, we can always write the general bipartite state $|\psi\rangle$ as $|\psi\rangle = \sum a_{ij}\,|i\rangle\,|j\rangle$ where the coefficients a_{ij} can be seen as a matrix A characterizing the state. Performing local unitary operation $U \otimes V$ on $|\psi\rangle$ is then equivalent to the transformation of the matrix $A \longrightarrow UAV^T$. Using the singular value decomposition introduced in Sec. 2.3.1, we can diagonalised any matrix A using the proper choice of U and V and thus transform a state of the form $|\psi\rangle$ to a state of the form $|\Phi\rangle$.

We say that two density matrix ρ and ρ' are equivalent under local unitary transformation when there exists an operator $U_A \otimes U_B$ such that

$$\rho' = (U_A \otimes U_B)\,\rho\,(U_A \otimes U_B)^\dagger \tag{3.9}$$

All the density matrices ρ and ρ' which are equivalent under local unitary transformation form what we will call the Concurrence class (C-class) which class the states based on their concurrence.

Equivalently we can introduce Concurrence Purity class (CP-class) for the mixed states where each class is defined by the concurrence and the purity of the mixed state. It can be easily shown that the purity is also conserved for local unitary transformation.

3.2.4 Rotation and equivalence classes

One might see a problem here: all the witnesses are generated by elements of $SU(2) \otimes SU(2)$ while all the states of a given class are generated by elements of $U(2) \otimes U(2)$ which are two different groups. However, $U(2)$ and $U(1) \times SU(2)$ are isomorphic. As $U(1)$ correspond to a phase, any transformation of the form $\rho \to U\rho U^\dagger$, where $U \in U(1)$, leaves ρ invariant: $SU(2)$ and $U(2)$ have the same representation on hermitian matrices. So, when we consider local unitary transformation of states we can restrict ourselves to elements of $SU(2) \otimes SU(2)$. Any transformation between two states of a class correspond to a rotation of the witness. Therefore we will use the term rotation when we apply $SU(2) \otimes SU(2)$, parametrized by Eq. (3.7), on the state or the witness.

3.2.5 Using the equivalence classes

Maximization

If we are using the definition 2.25 of a witness, we need to perform the maximization $\max_{\rho_s} \operatorname{tr}(\rho_s W)$ where ρ_s are all the density matrix of the separable states. Using rotation, it is easy to show that the value of the maximum will not depend on the rotation angles we choose to define the settings of the witness. Let's suppose that ρ_s^* is the separable state which maximizes the witness with the value M:

$$\max_{\rho_s} \operatorname{tr}(\rho_s W) = M = \operatorname{tr}(\rho_s^* W) \tag{3.10}$$

$\forall W'$ which can be written as $UWU^\dagger$ we have:

$$\begin{aligned} \max_{\rho_s} \operatorname{tr}(\rho_s W') &= \max_{\rho_s} \operatorname{tr}\left(\rho_s U W U^\dagger\right) \\ &= \max_{\rho_s} \operatorname{tr}\left(U^\dagger \rho_s U W\right) \\ &= \max_{\rho_s'} \operatorname{tr}(\rho_s' W) \\ &= M \end{aligned}$$

So, the maximum only depends on the basic settings and not on the rotation angles. We can for example choose the calculate the maximum value in the standard configuration and as we will see in the next chapter, this will greatly simplify the problem.

Detection of states

If we want to show that we can detect all the states of a given C-class or CP-class, we just have to show that we can detect one state of that given class. We know that we can also detect all the other states of the same class with the same detection margin with settings that can be found very easily.

If ρ_{ent} is detectable by the witness operator W with a detection margin ΔW by the definition 2.25 of a witness we have:

$$\Delta W = \mathrm{tr}\left(\rho_{ent} W\right) - \max_{\rho_s} \mathrm{tr}\left(\rho_s W\right) > 0 \tag{3.11}$$

then any state ρ_{ent} which is in the same C-class or CP-class as ρ_{ent}, i.e. can be written as $\rho'_{ent} = U\rho_{ent}U^\dagger$, is detectable by the witness operator $W' = UWU^\dagger$ with the same detection margin:

$$\begin{aligned}
\mathrm{tr}\left(\rho'_{ent} W'\right) - \max_{\rho_s} \mathrm{tr}\left(\rho_s W'\right) &= \mathrm{tr}\left(U\rho_{ent}U^\dagger UWU^\dagger\right) - \max_{\rho_s} \mathrm{tr}\left(\rho_s UWU^\dagger\right) \\
&= \mathrm{tr}\left(\rho_{ent} W\right) - \max_{\rho_s} \mathrm{tr}\left(\rho_s W\right) \\
&= \Delta W
\end{aligned}$$

Conclusion

The goal is to find a pair state-setting such that the state is detectable and in the best case such that the detection margin is maximized. Then we know how to detect all the other states in the same C-class or CP-class. There are two options to do that:

We can choose a state in a given class and find the settings which can detect it and ideally find the setting which give the biggest detection margin ΔW. But this requires to go through all the parameters $(\theta_A, \theta_B, \zeta_A, \zeta_B, R_{1a}, R_{2a}, R_{3a}, R_{1b}, R_{2b}, R_{3b})$.

The other option is to fix the rotation angles and find a state in a C-class or CP-class which is detectable. This requires to parametrize the separable states to perform the maximization and to parametrize the entangled state to find a state we can detect. The maximization can be performed in any configuration and as we will see in the next chapter performing it in the standard configuration will greatly simplify the problem.

Chapter 4

Result and analysis of a non-linear witness

In this section, we will analyze a non-linear witness which uses the same expectation values as the linear witness introduced in [6]. First, we will investigate the case of pure states and compare it with the results obtained with the linear witness. To have a deeper insight on these differences, we will use the standard configuration, where we will see that the witness has an interesting symmetry. Then we will do the same in the more general case of mixed states, which is much bigger than the set of pure states and requires much more parameters to describe it. But once again, looking at the problem in the standard configuration will simplify the problem.

4.1 Definition of the non-linear witness

The first non-linear combination of expectation values of $W^{(1)}$ and $W^{(2)}$ that comes to mind is simply the multiplication of those two expectation values. So, we consider the following functional:

$$F_{NL} = \mathrm{tr}(\rho W^{(1)})\,\mathrm{tr}(\rho W^{(2)}) \tag{4.1}$$

We want to verify if F_{NL} is a witness and can detect at least one entangled state. By using definition 2.25 of a witness this requires to find settings for $W^{(1)}$ and $W^{(2)}$ such that for at least one entangled state, we have:

$$\mathrm{tr}(\rho W^{(1)})\,\mathrm{tr}(\rho W^{(2)}) - \max_{\rho_s}\left[\mathrm{tr}(\rho_s W^{(1)})\,\mathrm{tr}(\rho_s W^{(2)})\right] > 0 \tag{4.2}$$

The first step is to perform the maximization over the separable states. This requires to parametrize all the separable states which is not difficult in the pure case but becomes much more complicated in the mixed case.

4.2 Detection of pure states

In this section, we only restrict ourselves to the pure states meaning that in Eq. (4.2), ρ and ρ_s are pure states only.

4.2.1 Parametrization of the pure separable states

We parametrize the density matrices ρ_{sep} of the two qubits pure separable states $|\psi_S\rangle$ as follow:

By using the Eq. (2.2), we can write the density matrix of any pure state $|\psi_S\rangle$ as

$$\rho_{sep} = |\psi_S\rangle \langle\psi_S| \tag{4.3}$$

By using the Eq. (2.7), any bipartite pure separable state can be written as :

$$|\psi_S\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \tag{4.4}$$

In the case when $|\psi_A\rangle$ and $|\psi_B\rangle$ are qubits, we can parametrize them as:

$$|\psi_A\rangle = \cos\phi_A |\uparrow\rangle + e^{i\Psi_A} \sin\phi_A |\downarrow\rangle \tag{4.5}$$

$$|\psi_B\rangle = \cos\phi_B |\uparrow\rangle + e^{i\Psi_B} \sin\phi_B |\downarrow\rangle \tag{4.6}$$

where $0 < \phi_A, \phi_B < \pi$ and $0 < \Psi_A, \Psi_B < 2\pi$.

By putting together Eq. (4.3), Eq. (4.4), Eq. (4.5) and Eq. (4.6) we can parametrize all the density matrices of the two-qubits pure separable states in terms of the four parameters $\{\phi_A, \phi_B, \Psi_A, \Psi_B\}$.

Using that parametrization, we can compute the analytical expression of F_{NL} in terms of those parameters. We choose the standard configuration of our witness and thus we can express it only in terms of the basic settings $\{\xi_A, \xi_B, \alpha, \beta\}$. Using Mathematica to compute F_{NL} with the parametrized pure states, we get the following expression:

$$\begin{aligned}\mathrm{tr}(W^{(1)}\rho_{sep})\,\mathrm{tr}(W^{(2)}\rho_{sep}) = \\ \left[\left(1+\zeta_A\cos\left(\frac{\alpha}{2}\right)\cos(2\phi_A)\right)^2 - \left(\zeta_A\cos(\Psi_A)\sin\left(\frac{\alpha}{2}\right)\sin(2\phi_A)\right)^2\right]\times \\ \left[\left(1+\zeta_B\cos\left(\frac{\beta}{2}\right)\cos(2\phi_B)\right)^2 - \left(\zeta_B\cos\left(\Psi_B\right)\sin\left(\frac{\beta}{2}\right)\sin\left(2\phi_B\right)\right)^2\right]\end{aligned} \tag{4.7}$$

4.2.2 Maximization over the pure states

We want to find the maximum value of the expression given in Eq. (4.7) which requires to find at least on separable state which maximizes it. So, we need to maximize Eq. (4.7) over the parameters $\{\phi_A, \phi_B, \Psi_A, \Psi_B\}$ while keeping the basic settings general. The details can be found in the appendix A. The final result is given by:

$$\max_{\rho_{sep}} \mathrm{tr}(W^{(1)}\rho_{sep})\,\mathrm{tr}(W^{(2)}\rho_{sep}) = \left(1+\zeta_A\cos(\frac{\alpha}{2})\right)^2\left(1+\zeta_B\cos(\frac{\beta}{2})\right)^2 \tag{4.8}$$

This expression of the maximum is valid for any set of basic settings. Moreover, we know from Sec. 3.2.5 that the value of the maximum does not depend on the rotation, so all the results we found for the standard configuration can be used in any configuration.

4.2.3 Detection of the entangled states with F_{NL}

Now that we have an analytical expression for the value of the maximum of F_{NL} for separable states, let's see which entangled states we can detect.

Bell states and maximally entangled states

Let's directly go to what the linear witness could not do: see if we can detect maximally entangled states. We choose to fix a maximally entangled state and perform a maximization of the detection margin ΔW over the parameters $(\theta_A, \theta_B, \zeta_A, \zeta_B, R_{1a}, R_{2a}, R_{3a}, R_{1b}, R_{2b}, R_{3b})$ using Matlab. We performed that maximization with the singlet state ρ_{B1} and found that a maximum was reached with the following settings:

$$\theta_A = \pi, \theta_B = \pi, \zeta_A = 1, \zeta_B = 1 \tag{4.9}$$

$$R_{1a} = 0, R_{2a} = 0, R_{3a} = 0, R_{1b} = 0, R_{2b} = 0, R_{3b} = 0 \tag{4.10}$$

with a detection margin of 3. It turns out that with the same settings we can also detect the Bell state ρ_{B3} with the same detection margin $\Delta W = 3$.

So, we can detect ρ_{B1} (and ρ_{B3}) and from Sec. 3.2.5 it implies we can also detect any maximally entangled states with the same detection margin using the same basic settings but by considering rotations. Let's do that explicitly and use the rotations to generate the three witnesses which can detect the other Bell states. The details can be found in the appendix B and the final results are given in the table 4.1:

Table 4.1: Settings that maximize the detection margin for the Bell states obtain by using the rotations on the basic settings 4.9 as seen in appendix B.

Witness	ζ_A	ζ_B	α	β	R_{1a}	R_{2a}	R_{3a}	R_{1b}	R_{2b}	R_{3b}	Bell states detectable	ΔW
W1	1	1	π	π	0	0	0	0	0	0	ρ_{B1}, ρ_{B3}	3
W2	1	1	π	π	0	0	π	π	0	π	ρ_{B2}, ρ_{B4}	3
W3	1	1	π	π	π	π	$\pi/2$	π	π	$\pi/2$	ρ_{B3}, ρ_{B2}	3
W4	1	1	π	π	π	π	$\pi/2$	π	0	$\pi/2$	ρ_{B4}, ρ_{B1}	3

Another thing we could do is to look at what range of settings are able to detect the singlet state ρ_{B1} and look what happens around the optimal point $(\pi, \pi, 1, 1)$. We are limited to represent two dimensions of settings only and thus will look at symmetric settings on system A and B. In Fig. 4.1, we represent the detection margin for the singlet state using a color bar with respect to symmetric settings $\theta = \theta_A = \theta_B$ and $\zeta = \zeta_A = \zeta_B$:

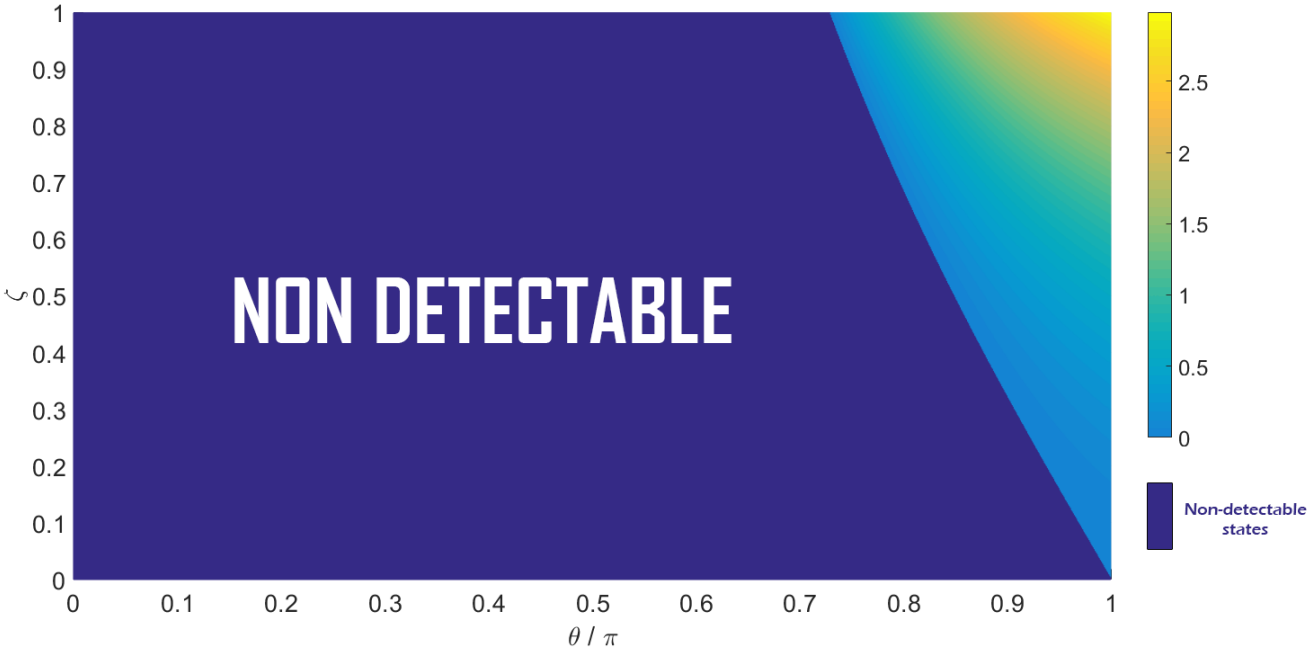

Figure 4.1: Detection margin ΔW for the singlet state ρ_{B1} in terms of symmetric settings $\theta = \theta_A = \theta_B$ and $\zeta = \zeta_A = \zeta_B$ in the standard configuration. The area in dark blue represents the settings for which ΔW is negative, i.e., the singlet state is not detectable but the color intensity in this area is not proportional to ΔW. The lighter area which corresponds to the color bar represents settings for which the singlet state is detectable.

We can see that all the settings represented by the area around the point $(\pi, \pi, 1, 1)$ are able to detect the singlet state. For any values of ζ, we can find orientation of the polarizing vectors such that the singlet state is detectable. This means that even in the presence of undesirable physical effects on the beam splitter, no matter how important they are, we can still detect the singlet state and in consequence all the maximally entangled states.

In the four dimensional space of the four basic settings, we can visualize the zone where we can detect the singlet state as a hypervolume around the point $\theta_A = \pi, \theta_B = \pi, \zeta_A = 1, \zeta_B = 1$.

All pure entangled states

We showed in the last paragraph that the non-linear witness with appropriate settings could detect all maximally entangled states. Another thing we could do is to determine if the non-linear witness is able to detect all the other pure entangled states that the linear witness could detect. To do that we parametrize one entangled state for each C-class and maximize the detection margin over all the settings of the witness.

We know from Sec. 3.2.3 that we can parametrize an element of every C-class of two-qubits pure entangled states as follow:

$$|\psi(\omega)\rangle = \cos\omega\,|\uparrow\uparrow\rangle + \sin\omega\,|\downarrow\downarrow\rangle \tag{4.11}$$

where $\omega \in \left]0, \frac{\pi}{4}\right]$ and the associated density matrix is given by $\rho = |\psi(\omega)\rangle\langle\psi(\omega)|$. If we can show we can detect $|\psi(\omega)\rangle$ for all ω then it means we can detect all pure entangled states with the same detection margin using different settings. Using matlab we performed

a maximization of ΔW over all the settings of the witness. We did it for 100 values of ω and plot the resulting ΔW with respect to the concurrence introduced in Sec. 2.2.1:

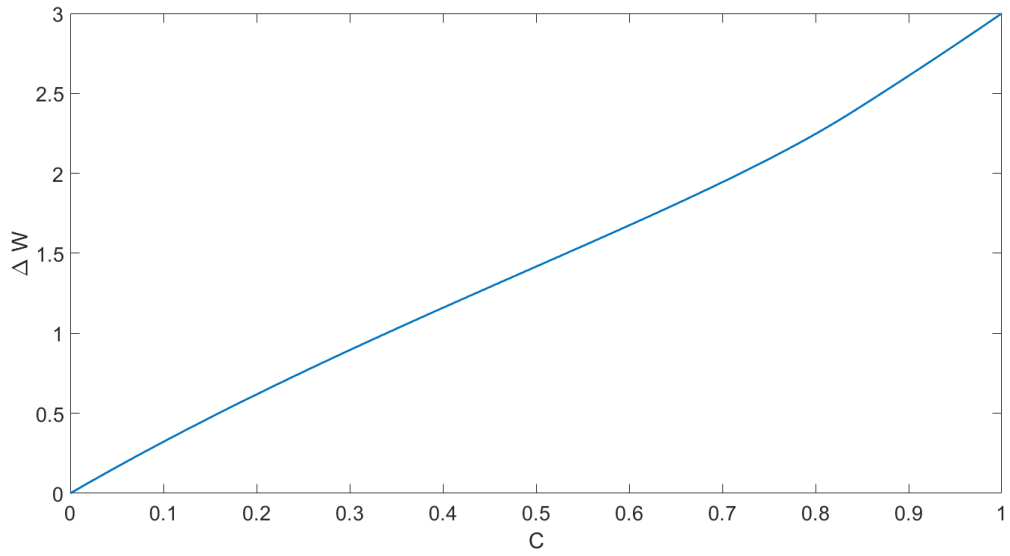

Figure 4.2: Maximization of ΔW over all the settings with respect to the concurrence of the entangled states in the pure case.

We can see that for any value of ω, we have a positive detection margin which means that we can detect all pure entangled state with the witness F_{NL}. As expected, for a concurrence of 0, which corresponds to a separable state, the detection margin equals to 0. Another interesting thing is that the detection margin is monotonically increasing with respect to the concurrence. So, the value of the detection margin could be a new quantity to quantify the degree of entanglement of a state which would be directly measurable by the witness. So, the witness would not only be able to determine if the state is entangled or not but in addition it gives information about the degree of entanglement of the state.

Investigation of the settings $\theta_A = \pi, \theta_B = \pi, \zeta_A = 1, \zeta_B = 1$

In the last section, we numerically calculated the settings that were maximizing ΔW for a pure state of every C-class. The next step would be to do that analytically and have an expression of the best set of settings in terms of ω which would allow us to easily determine the best witness to detect any given entangled state. However, this would require long calculation and will not provide us with a very useful result. So, instead let's try to find one basic setting that can detect all entangled states and which curve is as close as possible as the one for the optimal case in Fig. 4.2.

The following settings $\theta_A = \pi, \theta_B = \pi, \zeta_A = 1, \zeta_B = 1$ is a good option for that. Let's compute the corresponding ΔW with respect to ω and let's compare it with the optimal result:

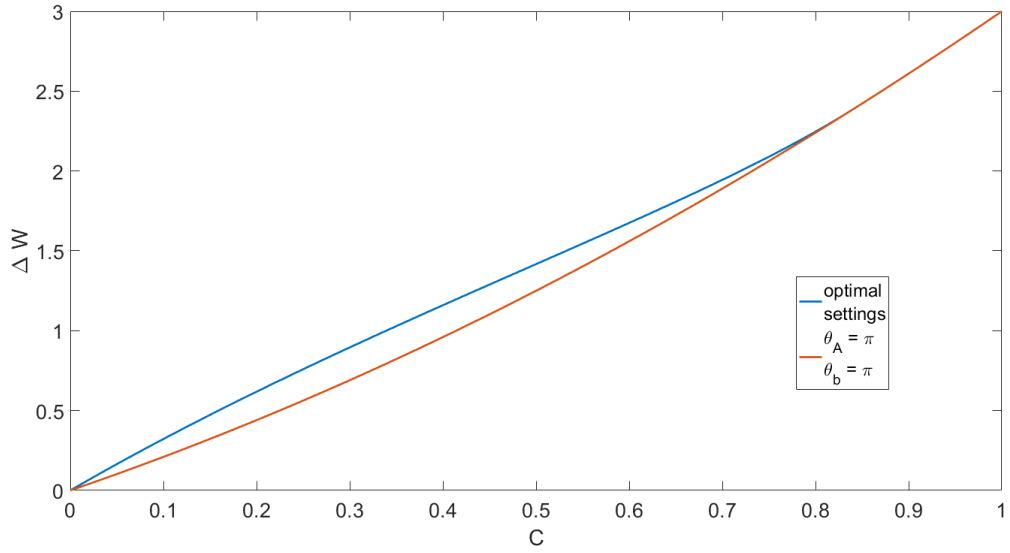

Figure 4.3: In blue is the same curve as in Fig. 4.2 where the detection margin is maximized over all settings and in orange it is the curve when the settings are fixed to $\theta_A = \pi, \theta_B = \pi, \zeta_A = 1, \zeta_B = 1$.

As expected, the curves are the same for the maximally entangled states but also for states with an high entanglement. Then the orange curve gets slightly lower than the optimal curve and then meet again, as expected, at the point (0,0).

So, the best setting to detect highly entangled states of the form 4.11 is $\theta_A = \pi, \theta_B = \pi, \zeta_A = 1, \zeta_B = 1$ while the state that are non highly entangled correspond to a different regime of optimal settings. We could construct all the witness starting from that configuration, it will not maximize the detection margin in every case but will still provide a good enough solution to detect all the entangled states.

4.2.4 Comparison of the linear witness and non-linear witness using the symmetries in the standard configuration

We showed in the last section that we can detect the maximally entangled states using the non-linear witness defined by Eq. (4.1) which was not possible with linear witnesses used in [6] as explained in Sec. 3.1.1. Let's try to understand more in details where that improvement is coming from by using the symmetries of the problem and let's illustrate that by comparing numerical results for the linear and non-linear witness. All the results in this section will be reused to study the case of mixed states.

Decomposition of the witness operator

Let's put ourselves into the standard configuration where the polarization vectors lie in the XZ plane. In this case the witness operators $W^{(1)}$ and $W^{(2)}$ have the following form:

$$W^{(1)} = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}, W^{(2)} = \begin{pmatrix} a & -b & -c & d \\ -e & f & g & -h \\ -i & j & k & -l \\ m & -n & -o & p \end{pmatrix} \tag{4.12}$$

which allows us to rewrite them as:

$$W^{(1)} = \begin{pmatrix} a & 0 & 0 & d \\ 0 & f & g & 0 \\ 0 & j & k & 0 \\ m & 0 & 0 & p \end{pmatrix} + \begin{pmatrix} 0 & b & c & 0 \\ e & 0 & 0 & h \\ i & 0 & 0 & l \\ 0 & n & o & 0 \end{pmatrix} \equiv W_1 + W_2 \tag{4.13}$$

$$W^{(2)} = \begin{pmatrix} a & 0 & 0 & d \\ 0 & f & g & 0 \\ 0 & j & k & 0 \\ m & 0 & 0 & p \end{pmatrix} + \begin{pmatrix} 0 & -b & -c & 0 \\ -e & 0 & 0 & -h \\ -i & 0 & 0 & -l \\ 0 & -n & -o & 0 \end{pmatrix} \equiv W_1 - W_2 \tag{4.14}$$

So, the two witness operators $W^{(1)}$ and $W^{(2)}$ can be expressed in terms of the two operators W_1 and W_2 which have a particular form. Let's call matrix of the form W_1 the *X-form* and matrix of the form W_2 the *O-form*. All the matrix which do not have an X form are called non-X and same thing for non-O. Let's see what the linear and non-linear witness become under that decomposition.

Expressing the witness in terms of new operators

The linear witness becomes:

$$\begin{aligned} \operatorname{tr}\left(\rho W^{(1)}\right) + \operatorname{tr}\left(\rho W^{(2)}\right) &= \operatorname{tr}\left(\rho\left(W_1 + W_2\right)\right) + \operatorname{tr}\left(\rho\left(W_1 - W_2\right)\right) \\ &= 2\operatorname{tr}\left(\rho W_1\right) \end{aligned} \tag{4.15}$$

The witness only depends on the X elements of the density matrix as W_1 has the X-form. Thus, changing the O-elements of the density matrix don't change the value of the witness.

The non-linear witness becomes:

$$\begin{aligned} \operatorname{tr}\left(\rho W^{(1)}\right) \operatorname{tr}\left(\rho W^{(2)}\right) &= \operatorname{tr}\left(\rho\left(W_1 + W_2\right)\right) \operatorname{tr}\left(\rho\left(W_1 - W_2\right)\right) \\ &= \left(\operatorname{tr}\left(\rho W_1\right)\right)^2 - \left(\operatorname{tr}\left(\rho W_2\right)\right)^2 \end{aligned} \tag{4.16}$$

The witness now depends on a difference of two positive terms where the first one depends only on X-elements of the density matrix while the second one depends on O-elements of the density matrix only. The second term is the main difference with the linear witness so let's investigate its effect in more details.

Effect of the second term on the separable states that maximize the witness

The linear witness cannot detect any maximally entangled states. Which means that:

$$\forall W_1, \forall \rho_{ent}^{max}, \exists \rho_{sep} \text{ such that } \operatorname{tr}(\rho_{sep} W_1) \geq \operatorname{tr}(\rho_{ent}^{max} W_1) \tag{4.17}$$

In particular, it was found that for any given witness W_1, for the maximally entangled states $\tilde{\rho}_{ent}^{max}$ that maximize the witness it is always possible to find a separable state which yield the same value. Or alternatively for any given maximally entangled states, for the witness operator ${}^*W_1$ which settings maximize the witness, it is always possible to find a separable state ρ_s which yield the same value:

$$\forall W_1 \text{such that } \operatorname{tr}(\tilde{\rho}_{ent}^{max} W_1) = \max_{\rho_{ent}^{max}} \operatorname{tr}(\rho_{ent}^{max} W_1) \Rightarrow \operatorname{tr}(\tilde{\rho}_{ent}^{max} W_1) = \max_{\rho_s} \operatorname{tr}(\rho_s W_1) \tag{4.18}$$

$$\forall \rho_{ent}^{max} \text{such that } \operatorname{tr}(\rho_{ent}^{max} W_1^*) = \max_{W_1} \operatorname{tr}(\rho_{ent}^{max} W_1) \Rightarrow \operatorname{tr}(\rho_{ent}^{max} W_1^*) = \max_{\rho_{sep}} \operatorname{tr}(\rho_{sep} W_1^*) \quad (4.19)$$

So, if we find a way to lower $\max_{\rho_{sep}} \operatorname{tr}(\rho_{sep} W_1^*)$ more than $\operatorname{tr}(\rho_{ent}^{max} W_1^*)$ or $\max_{\rho_{sep}} \operatorname{tr}(\rho_{sep} W_1)$ more than $\operatorname{tr}(\tilde{\rho}_{ent}^{max} W_1)$ then we can detect any maximally entangled states. And this is exactly what the second term in the non-linear witness will do. As this second term only depends the O-elements of the density matrix, this will put more constraint on the state maximizing the witness and allow us to break that symmetry. Let's first show that the maximum is indeed shifted.

As the non-linear witness can be expressed in terms of a difference of two squared term, if we find a state which maximizes the first term and minimizes the second term separately, then this state would maximize the non-linear witness but also the linear witness which only depends on the first term and both witnesses would be equivalent.

However, it can be shown that for some settings, this is not possible if we consider only pure states. We can see that if $\Omega_1 = \{$separable states which maximize the firts term of 4.16$\}$ and $\Omega_2 = \{$separable states which maximize the second term of 4.16$\}$ then $\Omega_1 \cap \Omega_2 = \emptyset$ for some settings. Indeed, Ω_1 is the same set of states that maximize the linear witness and by looking at the expression of the states of Ω_2 we see that they have no elements in common with Ω_1 for the settings for which the singlet state is detectable.

Effect of the second term on the X states in the standard configuration

In the second term of 4.16, W_2 has the O-form so there is only a dependence on the O-elements of ρ. Thus, this term will be 0 for all the X-states and will not make any difference compared to the linear witness. This explains why we can now detect the Bell states: the value of the maximum is shifted left while the value of the Bell states stay the same as they have the X-form. So, all the Bell states that in the standard configuration were yielding the same value as a separable state are now detectable. It can be visualized in Fig. 4.4 with the line representation of the witness:

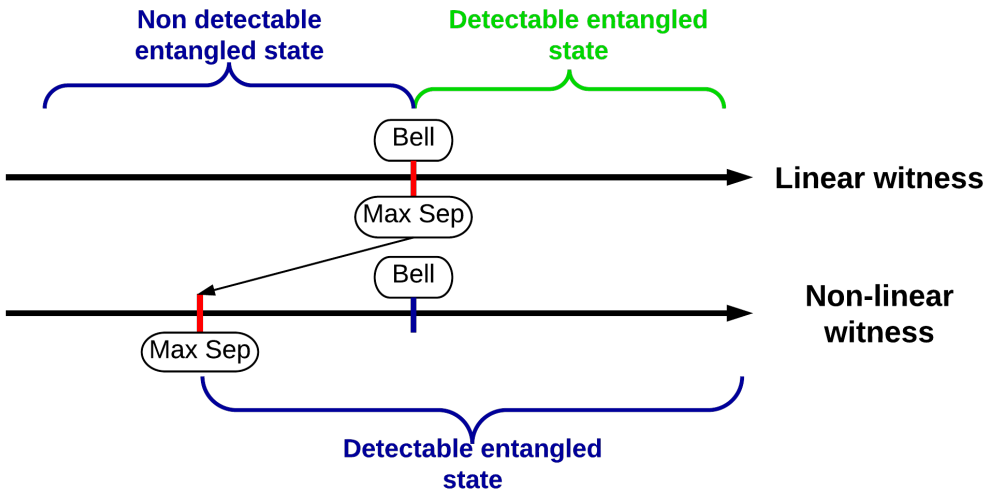

Figure 4.4: Line representation of a witness that illustrates why the singlet state becomes detectable. The values of the detection margin are different here for the linear witness and the non-linear witness and cannot be compared but what is important is the relative position of the states.

This can be illustrated numerically by doing the same thing we did in Fig. 4.2 but for the linear witness and then compare both together:

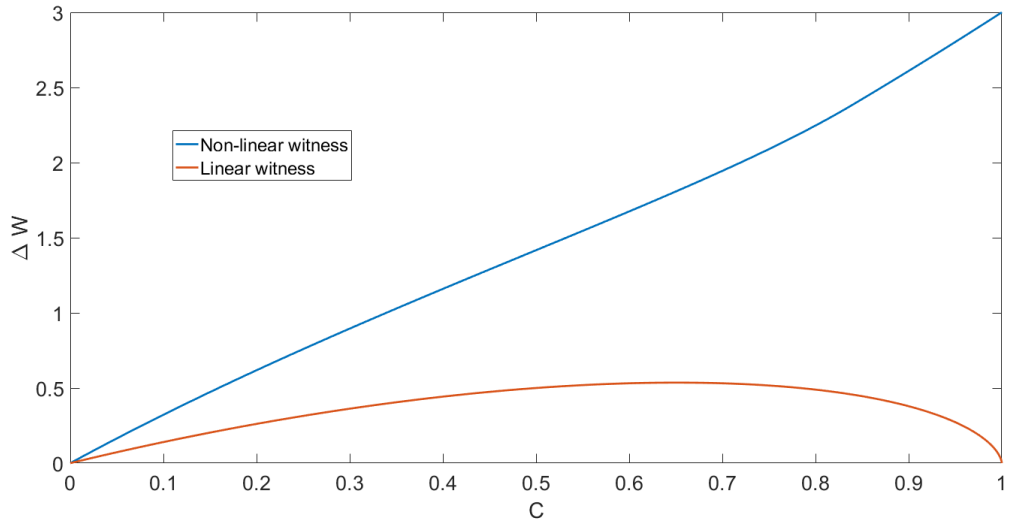

Figure 4.5: Maximization of ΔW over all the settings for the linear witness and the non-linear witness

Let's first emphasis here that we cannot compare the numerical values of both detection margins as they are coming from different witnesses. We can only make observations on the sign of the detection margin and on the shape of the curve.

We see that the detection margin of maximally entangled states is 0 for the linear witness hence making them undetectable. While the detection margin is positive for the non-linear witness. We also see that the detection margin is maximized around a concurrence of 0.7 and then is decreasing to 0 for a concurrence of 1. While for the non-linear witness, the detection margin is monotonically increasing with the concurrence. So, a shift of the maximum allows a better detection of all the high entangled states in general and not only the maximally entangled ones.

Effect of the second term on the non-X states in the standard configuration

What will happen with all the non-X states? As the second term is always positive, it will shift them towards lower values, which will reduce the detection margin and might even make it negative for some states that were detectable. So, the outcome will depend on the clash between those two effects: the shifting of the maximum separable state left which increase the detection margin and the shifting of the non-X states left which decrease the detection margin. Here is a visual example where the shift would allow us to detect an additional entangled state:

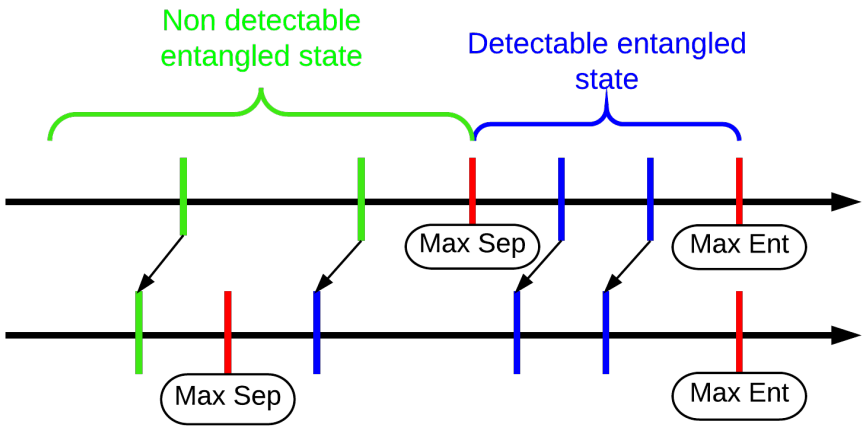

Figure 4.6: Line representation of a witness to illustrate a case where the shift of the maximum is bigger than the shift of the entangled states which allows to detect more entangled states.

Rotation

All the analysis above was done in the standard configuration which fixes the rotation angles. We learned that it was shifting the maximum left, not changing the position of the X states but was shifting all the non-X states left. But if we consider other settings, we no longer have the symmetries of the standard configuration and we cannot talk about X-states anymore. So, what happens for settings in the non-standard configuration?

To consider settings in the non-standard configuration, we can use the rotations introduced in Sec. 3.2.2. As explained in Sec. 3.2.3 those rotations leave the detection margin invariant and transform one state into another state of the same C-class.

The standard configuration will be the best configuration to detect X states as the second term is then minimized. The maximization performed to produce Fig. 4.5 was performed over all settings parameters. But as the entangled states are parametrized here by the X-state, we know the best settings will be in the standard configuration. So, performing the maximization in the standard configuration only, which reduce the number of parameters over which we maximize by 6, would yield the exact same result. Let's call the set of X states Ω_X and let's call the set of witnesses in the standard configuration W_X.

If we use a rotation R to go into another configuration $R^\dagger W_X R$ and transform Ω_X accordingly, the best configuration to detect $R^\dagger \Omega_X R$ will be $R^\dagger W_X R$ as the detection margin is always constant under any rotation.

The decomposition of the witness in terms of the difference of the square of the expectations values of two other observables will also still be valid. In the standard configuration, those new observables take the X-form and O-form. In other configurations, they will have different form but the second term will still be canceled by the new set $R^\dagger \Omega_X R$.

So, using the standard configuration allows us to greatly simplify the problem and deduce what will happen in other configurations. This will be even more useful when we consider mixed states where the problem becomes much more complicated.

4.3 Detection of mixed states

In the last section, we performed a maximization over the pure separable states which were easily parametrizable. By only considering pure states, everything looks great: we can detect the maximally entangled states with the same measurements we use for the linear witness, our goal is completed. Let's now see what happens if we consider the mixed states.

4.3.1 Why do we need to consider the mixed states?

In practice, it is already hard to create entangled states but what is even harder is to maintain and manipulate them under laboratory conditions. A quantum system could interact with its environment which could result in entanglement between our system and the environment; this is called decoherence. As we saw in Sec. 2.2.2, when we take the partial trace of a Bell state to describe a subsystem we get a mixed state: we cannot describe the part of an entangled state by a pure state. So, if our system interacts with the environment and becomes entangled with it, we cannot describe it by a pure state. The pure state is the theoretical ideal solution while the mixed states are representing states under laboratory conditions.

4.3.2 Maximization over the mixed states

Linear vs non-linear witness

In the case of the linear witness, the problem was easier, as we know we only have to consider pure states in the maximization. Indeed, we know from Eq. (2.2) that any mixed states can be written as the convex combination of pure states. The linear witness can then be written as:

$$\operatorname{tr}\left(\rho^{mix}\left(W^{(1)}+W^{(2)}\right)\right)=\sum_i \lambda_i \operatorname{tr}\left(\rho_i^{pure}\left(W^{(1)}+W^{(2)}\right)\right) \tag{4.20}$$

then it is easy to see that we have:

$$\sum_i \lambda_i \operatorname{tr}\left(\rho_i^{pure}\left(W^{(1)}+W^{(2)}\right)\right) \leq \max_{\rho_i^{pure}} \operatorname{tr}\left(\rho_i^{pure}\left(W^{(1)}+W^{(2)}\right)\right). \tag{4.21}$$

This means that the linear witness can always be maximized by pure states only and we don't need to perform the maximization over mixed states.

In the case of non-linear witness, the argument no longer holds because we now have cross terms and thus the witness can be maximized by mixed states only:

$$\sum_{ij} \lambda_i \lambda_j \operatorname{tr}\left(\rho_i^{pure} W^{(1)}\right) \operatorname{tr}\left(\rho_j^{pure} W^{(2)}\right) \geq \max_{\rho_{pure}} \operatorname{tr}\left(\rho_{pure} W^{(1)}\right) \operatorname{tr}\left(\rho_{pure} W^{(2)}\right) \tag{4.22}$$

This means that we must perform the maximization of the non-linear witness over all the mixed states.

Strategy to follow

The problem to solve is given by $\max_{\rho_{sep}} \operatorname{tr}\left(\rho W^{(1)}\right) \operatorname{tr}\left(\rho W^{(2)}\right)$, where the maximization is performed over all the pure and mixed states. Maximizing over all the mixed states would require to parametrize all the mixed separable state which is a difficult task given the number of parameters. Furthermore, maximizing over all those parameters becomes a very difficult problem.

We first tried to solve that problem by using ingenious brute forcing. By doing this, we discovered the existence of the X-states introduce in Sec. 4.2.4 and they will allow us to greatly simplify the problem.

Solving the problem

Once again, let's put ourselves in the standard configuration where we can write our witness as:

$$\begin{aligned}\operatorname{tr}\left(\rho W^{(1)}\right) \operatorname{tr}\left(\rho W^{(2)}\right) &= \operatorname{tr}\left(\rho\left(W_1 + W_2\right)\right) \operatorname{tr}\left(\rho\left(W_1 - W_2\right)\right) \\ &= \operatorname{tr}^2\left(\rho W_1\right) - \operatorname{tr}^2\left(\rho W_2\right)\end{aligned}$$

In the pure case, it was not possible to find a state which could maximize the first term and minimize the last term. For the mixed case, this is however the case. Let's see how we can construct this state:

1. We find a separable state which maximizes the first term. This state will also maximize the linear witness.

2. We replace all the non-X elements of the density matrix by 0 to get a density matrix of the X-form, let's call this operation the *X-map*. It can be shown that applying the X-map on a separable state will always yield another separable state. However, this doesn't work in the pure case as this operation always transform a pure state into a mixed state. The proof can be found in appendix C.

W_1 has the X-form so the first term only depends on the X-elements of ρ_{sep}. As the X-map only modifies the X-elements, the X-states will still maximize the first term. W_2 has the O-form so the second term only depends on the O-elements of ρ_{sep}. All the O-elements of the X-state are 0 so the second term will be 0 and thus be minimized.

Thus we have found a state which maximizes $\operatorname{tr}\left(\rho W^{(1)}\right) \operatorname{tr}\left(\rho W^{(2)}\right)$ and found a way to construct it for any settings of the witness in the standard configuration. As usual, the equivalent state that maximizes the witness in other configurations always exists and can be easily found.

Let's note that it could be possible to find a state that maximizes the first term and minimize the second term separately but which does not have the X-form. But this state would yield the same value as an X-state and in our case, we are not particularly interested in finding all the states which maximize the witness but are more interested in the value of that maximum.

Conclusion

If we compare the expression of the linear and the non-linear witness, we now see that any state which maximizes the non-linear witness also maximize the linear witness.

$$\mathrm{tr}\left(\rho W^{(1)}\right)\mathrm{tr}\left(\rho W^{(2)}\right) = \mathrm{tr}^2\left(\rho W_1\right) - \mathrm{tr}^2\left(\rho W_2\right) \tag{4.23}$$

$$\mathrm{tr}\left(\rho W^{(1)}\right) + \mathrm{tr}\left(\rho W^{(2)}\right) = 2\,\mathrm{tr}\left(\rho W_1\right) \tag{4.24}$$

As in the standard configuration, it is always possible to find a state which cancels the second term and maximizes the first one, then the same thing will be possible in any configurations and the maximum for the linear and non-linear witness will always be given by the same state for any settings. Does it mean that the linear witness and non-linear witness are exactly the same? The value of the maximum will obviously change but the relative position will stay the same for both witnesses. It means that all the entangled states that were smaller or bigger than the maximum will stay the same way thus will not allow to detect more states. One last thing needs to be investigated, the second term which might change the relative position of some states.

4.3.3 Effect of the second term in equation 4.23

In the case of mixed cases, the second term will have the same effect than it had in the pure cases: shifting all the state that do not cancel the second term left. But this time it will not shift the maximum left. So, it is making the situation worse than the linear witness. Let's see what happens with pure states.

In Fig. 4.7, we performed the same maximization we did in Fig. 4.2 but this time using the new maximum for mixed states. As we are parametrizing the pure states by X states, the second term will always be equal to zero so we expect to get a curve identical to the linear witness just with a different scale.

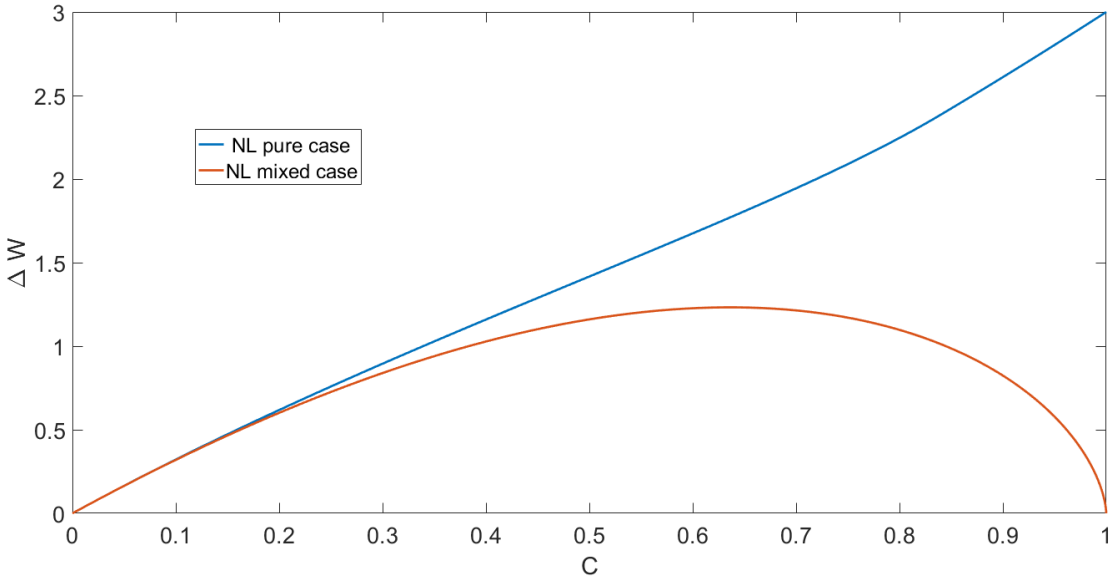

Figure 4.7: Comparison of the non-linear witness to detect pure entangled states in the pure and mixed case. In the mixed case, we are back to a curve similar to the linear witness and thus do not have any improvements.

It is indeed what we get. So, we are still able to find settings to detect all pure entangled states except the maximally pure entangled ones. But in a given C-class, the

non-linear witness will be worse than the linear witness to detect states that do not cancel the second term.

4.3.4 Conclusion

When considering all the mixed states, the non-linear witness given by Eq. (4.1) is equal or worse than the linear witness used in [6]. Meaning that we can detect the same or less entangled states, pure or mixed, than the linear witness. In other words, the non-linear witness is useless in the mixed case.

Chapter 5

Outlook

5.1 Summary

We looked at two different types of states: the pure states and the mixed states and considered the non-linear witness defined by Eq. (4.1) to detect entanglements.

We found that by considering only the pure states, we can detect, with appropriate settings, all the pure entangled states, which is an improvement compared to the linear witness which couldn't detect the maximally entangled states.

When considering mixed states, a linear witness can always be maximized by a pure state while non-linear witness can be maximized by mixed states only. This require to perform the maximization over all the mixed states which is a much more complicated problem. However, we found that it was always possible to construct a state which is maximizing the linear and the non-linear witness at the same time. We found that it implies that the non-linear witness is equal or worse than the linear witness.

However, going from pure states to mixed states is a big step and can be seen as two extreme cases. It would be for example possible to limit the purity of the set of mixed states to a certain value.

Furthermore, we only looked at a very specific non-linear witness. But one could try to look at different non-linear witnesses and try to build one which is not maximized by the same states as the linear witness. In our case, we were lucky because the symmetries allowed us to use the X-states which simplified the problem however this cannot be used for any non-linear combination of the expectation values and the problem of maximizing over all the mixed states might not be avoided.

Another path to follow would be to generalize the definition of a witness and see if with only two measurements it is possible to detect or not the maximally entangled states no matter what the witness is. But again, avoiding to look at the parametrization of all mixed states might be difficult.

All those ideas are discussed with more details in the next sections.

5.2 Purity

The process explained in Sec. 4.3.2 to construct a state that maximize the non-linear witness is starting with a state that maximize the first term of Eq. (4.23) or equivalently the linear witness and then consist of setting the non-X elements of the matrix to 0. This

operation always reduces the purity of a non-X state. Indeed, the purity of a general 4×4 hermitian is given by:

$$\text{tr}\left(\rho_{\text{sep}}^2\right) = \rho_{11}^2 + \rho_{22}^2 + \rho_{33}^2 + \rho_{44}^2 + 2|\rho_{12}|^2 + 2|\rho_{13}|^2 + 2|\rho_{23}|^2 + 2|\rho_{14}|^2 + 2|\rho_{24}|^2 + 2|\rho_{34}|^2 \quad (5.1)$$

Setting the non-X elements of the density matrix to 0 sets some terms of Eq. (5.1) to 0 and as it is a sum of positive terms, this always reduce the purity.

The question is what is the minimum amount by which the purity is reduced? Knowing that, we could impose a limit on the purity such that the process can no longer work. Answering that question is not easy but a motivation to investigate it can be given by Fig. 5.1. To make this graph, we generated five million random density matrix, we used the PPT criterion to make the distinction between entangled and separable states and then we plotted the expectation value $\text{tr}\left(\rho W^{(1)}\right) \text{tr}\left(\rho W^{(2)}\right)$ with respect to the purity of the state.

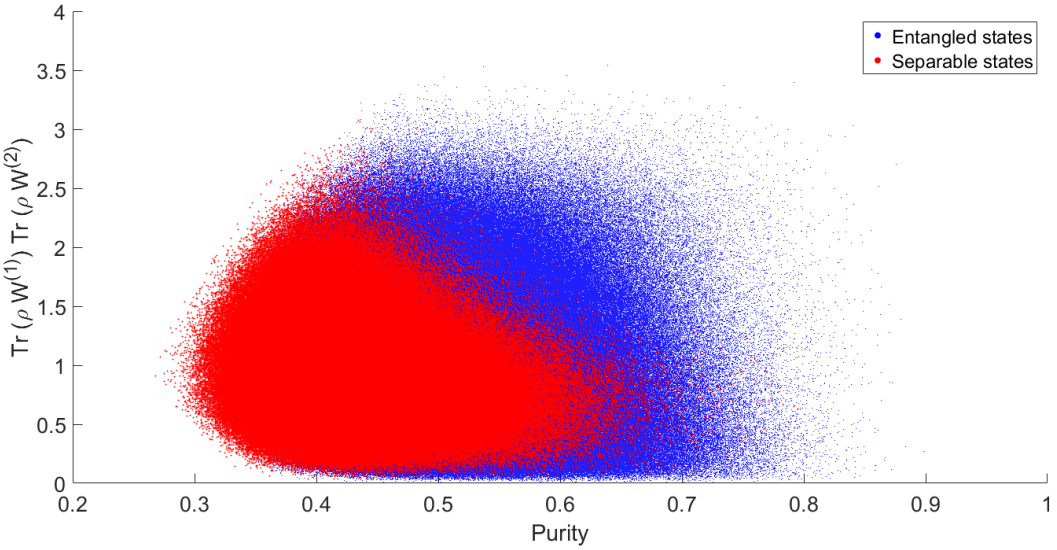

Figure 5.1: $\text{tr}\left(\rho W^{(1)}\right) \text{tr}\left(\rho W^{(2)}\right)$ (non-linear witness) in terms of the purity of the states for 5 million density matrices randomly generated.

We can see that the higher the purity is the more entangled states yields a bigger value than the separable state and hence are detectable.

If we take the example of the Bell states which would be represented on this graph at the point $(1, 4)$ then we know that we have a mixed state at the point $(0.5, 4)$ which yields the same expectation value making it undetectable. The question is what is the minimum purity required such that no separable states can give a value of 4 for the witness. Based on this graph we can see that the expectation value of separable state decrease with the purity so trying to do that would make sense. However, five millions is not a lot for an object that has 15 degrees of freedom and we have to be careful with the conclusions we can draw with such a plot.

If we can solve that problem and can modelize the interaction of the qubit with the environment we know what is the minimum purity we could expect for the state we want to detect after decoherence and thus determine if it would be possible to practically detect those states in the lab.

Another interesting thing to look at is the same type of plot than Fig. 5.1 but for the linear witness and compare the situation between the two. This plot can be seen on Fig. 5.2.

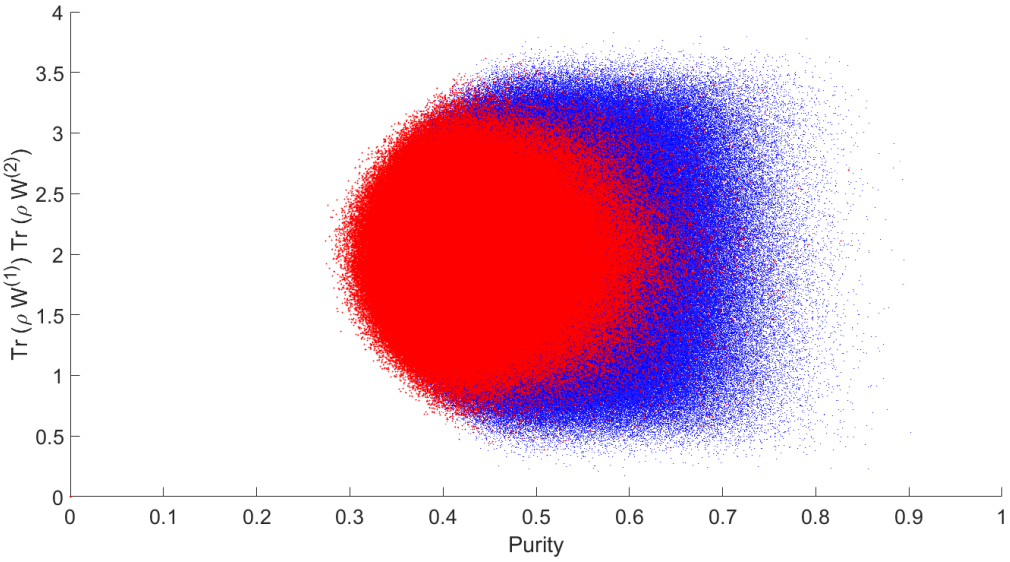

Figure 5.2: $\mathrm{tr}\left(\rho W^{(1)}\right) + \mathrm{tr}\left(\rho W^{(2)}\right)$ (linear witness) in terms of the purity of the states for 5 million density matrices randomly generated.

We can see a clear difference between the linear witness and the non linear witness for states having an high purity. Indeed, the states with high purity tend to yield higher values for the linear witness than the non-linear one. Thus, if we limit the purity there are still a difference between the linear and non-linear witness where the non-linear one should perform better.

5.3 Generalization of the problem

The building blocks of our cross correlation witness are the two expectation values of the observables $W^{(1)}$ and $W^{(2)}$. A functional f of those observables allows us to distinguish between separable states and entangled states when the value of f for an entangled state yields a value outside of the range of the separable states and then f is called a witness. We can express in terms of a simple idea in which case it is possible to make of f a witness. Indeed, every time the two expectation values of an entangled state ρ_{ent} cannot be reproduced by a separable state, we can build a function which is maximized for ρ_{ent} and thus satisfies the definition 2.25 of a witness. This more general definition can be expressed as follow:

We can build a witness of the form $f(\mathrm{tr}\left(\rho W^{(1)}\right), \mathrm{tr}\left(\rho W^{(2)}\right))$ if $\exists \rho_{ent}$ such that

$$\mathrm{tr}\left(\rho_{ent} W^{(1)}\right) \neq \mathrm{tr}\left(\rho_{sep} W^{(1)}\right) \text{ and/or } \mathrm{tr}\left(\rho_{ent} W^{(2)}\right) \neq \mathrm{tr}\left(\rho_{sep} W^{(2)}\right) \quad \forall \rho_{sep} \tag{5.2}$$

We can also say that an entangled state ρ_{ent} will not be detectable by any witness of the form $f(\mathrm{tr}\left(\rho W^{(1)}\right), \mathrm{tr}\left(\rho W^{(2)}\right))$ if $\exists \rho^*_{sep}$ such that:

$$\mathrm{tr}\left(\rho_{ent}W^{(1)}\right) = \mathrm{tr}\left(\rho^*_{sep}W^{(1)}\right) \text{ and } \mathrm{tr}\left(\rho_{ent}W^{(2)}\right) = \mathrm{tr}\left(\rho^*_{sep}W^{(2)}\right) \tag{5.3}$$

We could for example use definition 5.2 to determine if we could detect one of the Bell states. This would require to pick a set of settings for the witness and go through all the mixed separable state or to parametrize one state in every CP-class and go through all the settings using the rotation of the witness. Using definition 5.3 to show that it is not possible to detect a given entangled states might be slightly easier to use as it doesn't necessarily require to go through all the mixed separable state but just to find on.

I personally think that the solution resides in the use of the symmetries in the structure of the sets of separable states, entangled states and witnesses. This would require to use group theory and for example study in more details the structure of the X states and symmetries in general. I think some ideas could be inspired by the use of group theory physicists do in theoretical particle physics.

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Appendix A

Maximization of the pure states

Let's maximize the following expression over the parameters $\phi_A, \Psi_A, \phi_B, \Psi_B$ while keeping the basic settings $\alpha, \beta, \zeta_A, \zeta_B$ general. The result might then depend on the values of the basic settings but their values are restricted on $0 < \alpha, \beta < \pi$ and $0 < \zeta_A, \zeta_B < 1$.

$$\begin{aligned}\operatorname{tr}(W^{(1)}\rho_{sep})\operatorname{tr}(W^{(2)}\rho_{sep}) &= \left[\left(1+\zeta_A\cos(\frac{\alpha}{2})\cos(2\phi_A)\right)^2 - \left(\zeta_A\cos(\Psi_A)\sin(\frac{\alpha}{2})\sin(2\phi_A)\right)^2\right]\times \\ &\quad \left[\left(1+\zeta_B\cos(\frac{\beta}{2})\cos(2\phi_B)\right)^2 - \left(\zeta_B\cos(\Psi_B)\sin(\frac{\beta}{2})\sin(2\phi_B)\right)^2\right]\end{aligned}$$

We see that the expression is a function of the form $f(A)f(B)$ where $A = \{\phi_A, \Psi_A, \xi_A, \alpha\}$ and $B = \{\phi_B, \Psi_B, \xi_B, \beta\}$ so the two terms are symmetric and independent from each other. It can also be easily shown that $f(A) \geqslant 0 \quad \forall A$.

It implies that to maximize the whole expression, we can maximize $f(A)$ and $f(B)$ separately and we just need to maximize one of the them as they are the same with different variables name. This make the problem already much easier. Let's consider the first term:

$$\left[\left(1+\zeta_A\cos(\frac{\alpha}{2})\cos(2\phi_A)\right)^2 - \left(\zeta_A\cos(\Psi_A)\sin(\frac{\alpha}{2})\sin(2\phi_A)\right)^2\right] \tag{A.1}$$

We see that it is a difference of two positive terms. But we cannot maximize the first term and minimize the second term separately as they depend on the same parameter ϕ_A unless the value of ϕ_A which maximizes the first term also minimize the second term. It turns out this is the case as we will see below.

We have to take into account the sign of $\cos(\frac{\alpha}{2})$ but as α is restricted between 0 and π we only have the two following cases to consider: $\cos(\frac{\alpha}{2}) > 0 \Leftrightarrow \alpha \in [0, \pi[$ and $\cos(\frac{\alpha}{2}) = 0 \Leftrightarrow \alpha = \pi$.

A.1 $\cos(\frac{\alpha}{2}) = 0$

In the case where $\cos(\frac{\alpha}{2}) = 0$, the second term of A.1 is 0 so the expression becomes:

$$1 - \left(\zeta_A\cos(\Psi_A)\sin(2\phi_A)\right)^2 \tag{A.2}$$

To maximize this expression the second term has to be equal to 0 which can be done with the following solutions:

$$\Psi_A = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \quad \cup \quad \phi_A \in [0, \pi] \quad \cup \quad \zeta_A \in]0, 1] \tag{A.3}$$

$$\phi_A = 0 \text{ or } \frac{\pi}{2} \text{ or } \pi \quad \cup \quad \Psi_A \in [0, 2\pi] \quad \cup \quad \zeta_A \in]0, 1] \tag{A.4}$$

$$\zeta_A = 0 \quad \cup \quad \Psi_A \in [0, 2\pi] \quad \cup \quad \phi_A \in [0, \pi] \tag{A.5}$$

In those cases the maximum is given by:

$$\max_{\rho_{sep}} \text{tr}(W^{(1)}\rho_{sep}) \, \text{tr}(W^{(2)}\rho_{sep}) = 1 \tag{A.6}$$

A.2 $\cos(\frac{\alpha}{2}) > 0$

The values of Ψ_A and ϕ_A that minimize the second term of A.1 are the same as the one given in the last section. To maximize the first term of A.1 we have $\cos(2\phi_A) = 1 \Leftrightarrow \phi_A = 0$ or π. So, the solution that maximizes the first term and minimize the second term at the same time are given by:

$$\phi_A = 0 \text{ or } \pi \quad \cup \quad \Psi_A \in [0, 2\pi] \quad \cup \quad \zeta_A \in]0, 1] \tag{A.7}$$

$$\zeta_A = 0 \quad \cup \quad \Psi_A \in [0, 2\pi] \quad \cup \quad \phi_A \in [0, \pi] \tag{A.8}$$

A.3 Value of the maximum

The value of the maximization over the pure separable states in terms of the settings is simply given by:

$$\max_{\rho_{sep}} \text{tr}(W^{(1)}\rho_{sep}) \, \text{tr}(W^{(2)}\rho_{sep}) = \left(1 + \zeta_A \cos(\frac{\alpha}{2})\right)^2 \left(1 + \zeta_B \cos(\frac{\beta}{2})\right)^2 \tag{A.9}$$

In the case $\cos(\frac{\alpha}{2}) = 0$, we have much more states which maximize the witness than the case where $\cos(\frac{\alpha}{2}) > 0$. But in the former, the maximum will always be given by 1 while in the latter, the maximum can vary between 1 and 4 depending on the settings.

We also note that only X states are maximizing the witness for the setting $\cos(\frac{\alpha}{2}) > 0$ while X states and non-X states can maximize the witness for the case $\cos(\frac{\alpha}{2}) = 0$.

But let's keep in mind that those are the separable states that maximize the witness in the standard configuration. The states that maximize the witness for other settings takes different form and can be obtain using the rotations.

Appendix B

Investigation of the Bell states

We found that we can detect ρ_{B1} and ρ_{B3} in the standard configuration with the following basic settings: $\theta_A = \pi, \theta_B = \pi, \zeta_A = 1, \zeta_B = 1$. From that information let's build other witnesses which can detect the other Bell states.

The first step is to find the unitary operations which link ρ_{B1} and ρ_{B3} to the other Bell states. Those unitary operations are not unique and we will not go into details to characterize all of them but we will just give here a set of unitary operations that do the job. Those transformations can be parametrized by:

$$U = U_A \otimes U_B = e^{iR_{1a}\sigma_z} e^{iR_{2a}\sigma_x} e^{iR_{3a}\sigma_z} \otimes e^{iR_{1b}\sigma_z} e^{iR_{2b}\sigma_x} e^{iR_{3b}\sigma_z} \tag{B.1}$$

They allow us to link the four Bell states as represented in Fig. B.1 where each color corresponds to a set of parameters, given in table B.1, of Eq. (B.1).

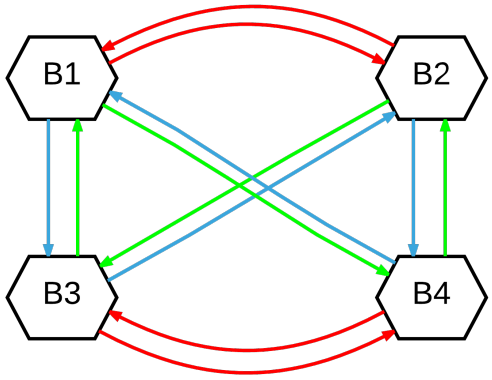

Figure B.1: Unitary operations on the Bell states are represented by arrows with different colors.

Table B.1: Settings associated to each unitary operator represented in Fig. B.1

Color	R_{1a}	R_{2a}	R_{3a}	R_{1b}	R_{2b}	R_{3b}
Red	0	0	π	π	0	π
Blue	π	π	$\pi/2$	π	π	$\pi/2$
Green	π	π	$\pi/2$	π	0	$\pi/2$

By applying the red, green and blue unitary operator to the original witness W1 which can detect ρ_{B1} and ρ_{B2} we can generate 3 new witnesses which will be able to detect the other Bell states:

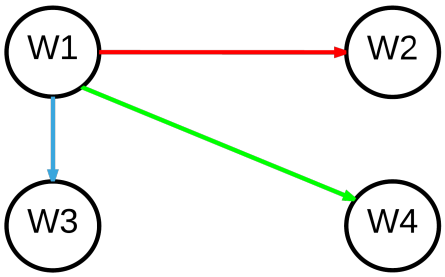

Figure B.2: Unitary operations on the witnesses are represented by arrows with different colors

By using the diagrams B.1 and B.2 we can easily determine which Bell states we will be able to detect with the witnesses constructed in Fig. B.2 as shown in table B.2.

Table B.2: Witness generated by applying the transformations on W1 and the corresponding Bell states we can detect.

Witness	**Transformation**	**Bell states detectable**
W1		ρ_{B1} ρ_{B3}
W2	Red	$\rho_{B1} \rightarrow \rho_{B2}$ $\rho_{B3} \rightarrow \rho_{B4}$
W3	Blue	$\rho_{B1} \rightarrow \rho_{B3}$ $\rho_{B3} \rightarrow \rho_{B2}$
W4	Green	$\rho_{B1} \rightarrow \rho_{B4}$ $\rho_{B3} \rightarrow \rho_{B1}$

We can visualize this with the line representation of a witness. Here is for example what happens if we apply the red transformation on the witness W1 which is equivalent to applying them to the states.

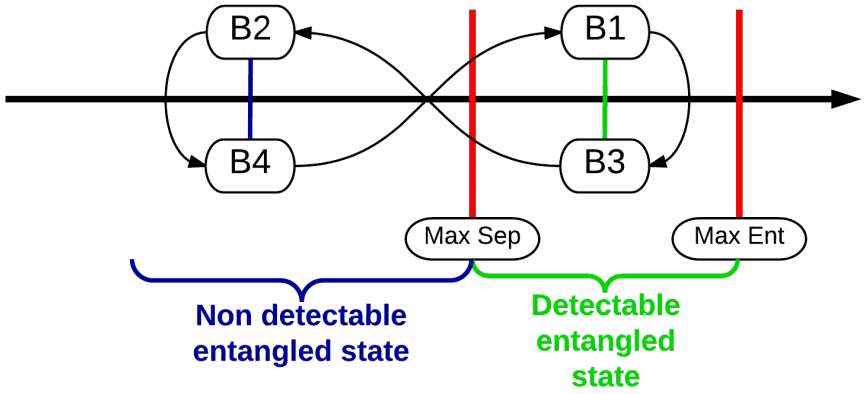

Figure B.3: Line representation of the witness that illustrate what happens when we apply the red transformation on the Bell states and the witness in the standard configuration. As we can see, there is a cyclic permutation of the Bell states

Appendix C

Transformation of separable states into X states

C.1 Map

We want to show that if we are starting from a density matrix of a separable state:

$$\rho_{\text{sep}} = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} \tag{C.1}$$

And replace all the non-X elements by 0,

$$\rho_{\text{X}} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix} \tag{C.2}$$

ρ_{X} still represents a separable state. Let's call that operation the X-map. We have to show that the following properties are conserved under a X-map: trace one, positive semi-definite, hermitian, PPT.

C.2 Hermicity and trace one

The trace is invariant and still equal to one as we are not modifying any diagonal elements. The hermicity is conserved as the 0 elements are put symmetrically around the diagonal.

C.3 Positivity

The X-matrix is still positive semi-definite. To show that, we use the following criteria: a Hermitian matrix is positive semidefinite if and only if all of its principal minors are non-negative. The principal minors are defined as follow:

Let M be a $n \times n$ matrix. The $k \times k$ submatrices of M formed by deleting the $n - k$ rows of M and the same $n - k$ columns of M are called the *principal submatrices* of M. The determinants of the principal submatrices are called the *principal minors* of M.

ρ_{sep} is positive semi-definite, so all of its principal minors are non-negative. For our proof, we will only need second order principal minors where two rows and columns are deleted:

$$\rho_{33}\rho_{44} - |\rho_{34}|^2 \geqslant 0 \tag{C.3}$$
$$\rho_{22}\rho_{44} - |\rho_{24}|^2 \geqslant 0 \tag{C.4}$$
$$\rho_{22}\rho_{33} - |\rho_{23}|^2 \geqslant 0 \tag{C.5}$$
$$\rho_{11}\rho_{44} - |\rho_{14}|^2 \geqslant 0 \tag{C.6}$$
$$\rho_{11}\rho_{33} - |\rho_{13}|^2 \geqslant 0 \tag{C.7}$$
$$\rho_{11}\rho_{22} - |\rho_{12}|^2 \geqslant 0 \tag{C.8}$$

Let's show that this imply that all the principal minors of the X-matrix are non-negative:

1st order The principal minors of 1st order are the diagonal elements $\rho_{11}, \rho_{22}, \rho_{33}, \rho_{44}$ of ρ_{X} which are the same as the diagonal elements ρ_{sep}. And they are all real and non-negative by the definition of a density matrix.

2nd order The principal minors of 2nd order are given by: $\rho_{33}\rho_{44}, \rho_{22}\rho_{44}, \rho_{22}\rho_{33}, \rho_{11}\rho_{44}, \rho_{11}\rho_{33}, \rho_{11}\rho_{22}$ which are product of diagonal elements, hence non-negative.

3rd order The principal minors of 3rd order are given by:

$$\rho_{44}\left(\rho_{22}\rho_{33} - |\rho_{23}|^2\right) \tag{C.9}$$
$$\rho_{33}\left(\rho_{11}\rho_{44} - |\rho_{14}|^2\right) \tag{C.10}$$
$$\rho_{22}\left(\rho_{11}\rho_{44} - |\rho_{14}|^2\right) \tag{C.11}$$
$$\rho_{11}\left(\rho_{22}\rho_{33} - |\rho_{23}|^2\right) \tag{C.12}$$

and are all positive by using the Eq. (C.5) and (C.6) and the fact that diagonals elements are positive.

4th order The principal minors of 4th order is given by the determinant of ρ_{sep} which is given by:

$$\begin{aligned}
\det(\rho_{\text{X}}) &= |\rho_{14}|^2|\rho_{23}|^2 - |\rho_{14}|^2\rho_{22}\rho_{33} - \rho_{11}\rho_{44}|\rho_{23}|^2 + \rho_{11}\rho_{22}\rho_{33}\rho_{44} \\
&= \left(\rho_{22}\rho_{33} - |\rho_{23}|^2\right)\left(\rho_{11}\rho_{44} - |\rho_{14}|^2\right) \\
&\geqslant 0
\end{aligned}$$

So, ρ_{X} is indeed semi positive as all of its principal minors are non-negative. To summarize we showed that the X-map of any Positive semi-definite hermitian (PSDH) matrix is still a PSDH matrix.

C.4 PPT

The last step is to show that the ρ_X is still separable. To do so, we can use the PPT criterion and show that the partial transpose of ρ_X is positive semidefinite.

ρ_{sep} is PPT as it is a separable state so its partial transpose will be a PSDH matrix. By using the result from last section, we know that its partial transpose will also be positive. And we showed in last section that the X-map of any PSDH matrix is still a PSDH matrix. Hence the X-state is PPT and it is then a separable state. The proof can be easily visualize by looking at Fig. C.1.

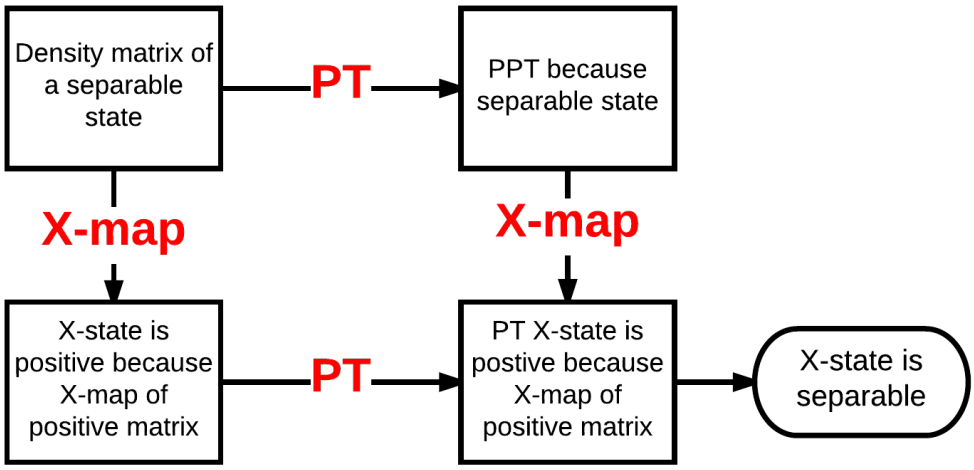

Figure C.1: Diagram that shows why ρ_X is still separable.

C.5 Pure state

The X-map do not preserve the purity of a state. Indeed, the purity of ρ_{sep} is given by $\text{tr}\left(\rho_{sep}^2\right)$ which is equal to: $\rho_{11}^2 + \rho_{22}^2 + \rho_{33}^2 + \rho_{44}^2 + 2|\rho_{12}|^2 + 2|\rho_{13}|^2 + 2|\rho_{23}|^2 + 2|\rho_{14}|^2 + 2|\rho_{24}|^2 + 2|\rho_{34}|^2$. So, setting the non-X elements to 0 will change the value of $\text{tr}\left(\rho_{sep}^2\right)$ and hence might transform a pure state into a mixed state. That is another reason to see why the maximum is shifted in the pure case but not in the mixed case.