Counterparty Credit Exposures for Interest Rate Derivatives using Stochastic Grid Bundling Method and Change of Measure

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Abstract

The notional amounts outstanding of over-the-counter (OTC) derivatives had grown exponentially for almost two decades and its rapid growth were mainly due the increase in OTC interest rate derivatives. As of december 2014, the total notional amounts outstanding in the global OTC market was 630 trillions USD and the OTC interest rate derivatives represents about 80% of the market.
Trading with OTC derivatives can lead to significant risks. Especially counterparty credit risk has gained particular emphasis due to the credit crisis in 2007. The aim of this thesis is to determine if it is possible to get realistic estimations of counterparty credit risk measures as Expected Exposure (EE) and Potential Future Exposure (PFE) that reflects the "real world" market. In order to do simulations under the historical $\mathbb{P}$-measure, attempts are made to approximate the market price of risk and then calculate exposure profiles for the interest rate derivative Bermudan swaption. The Hull and White one-factor short rate model is used and all calculations are done using the Stochastic Grid Bundling Method(SGBM).

Keywords: OTC, Counterparty credit risk, HW1F, Market price of risk, CVA, Potential Future Exposure, Expected Exposure, Bermudan swaption, Stochastic Grid Bundling Method, SGBM.

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Contents

Abbreviations

BIS Bank of International Settlements
CCR Counterparty Credit Risk
CDS Credit Default Swap
CVA Credit Value Adjustment
EE Expected Exposure
GSR Gaussian Short Rate
HW1F Hull-White 1-Factor model
IRS Interest Rate Swap
ISDA International Swap and Derivatives Association
LSM Least Squares Method
MPoR Market Price of Risk
OTC Over The Counter
PD Probability of Default
PFE Potential Future Exposure
SGBM Stochastic Grid Bundling Method
SGM Stochastic Grid Method
SMM Stochastic Mesh Method

Chapter 1

Introduction

1.1 OTC Contracts

An Over-the-counter (OTC) contract is a bilateral contract that specifies the conditions on how a partucular future trade or agreement is settled. Usually one of the two parts in the contract is an investment bank. This type of contract is negotiated between the counterparties and the trade occurs with commodities, financial instuments (including stocks) and derivatives. OTC derivatives are often refered to as swaps beacause many OTC deals involves cash flows, or obligations, that are swapped or exchanged between the counterparts at defined intervals[5].

The most common reason for engaging an OTC contract instead of an exchange traded contract is that it can be tailored to ones need in contrary to the exchange traded contracts that is more of an one-size-fits-all type of contract. Because of that one can tailor this contracts after the special needs for the trade or agreement it often gives better conditions for a perfect hedge, to trade larger contracts more efficient and accessing liquidity.

For the derivatives the agreements of the contract is often regulated by the International Swaps and Derivatives Association, ISDA, which is a trade organization of participants in the market for OTC derivatives[5].

1.2 OTC Market

The notional amounts outstanding of over-the-counter (OTC) derivatives had grown exponentially for almost two decades when it stabilized back in 2008. This rapid growth were mainly due the increase in OTC interest rate derivatives which is the largest asset class of the OTC market. As of december 2014, the total notional amounts outstanding in the global OTC market was 630 trillions USD and the OTC interest rate derivatives represents about 80% of the market i.e. 505 trillions USD [12]. In Figure 1.1, the growth of the OTC market is illustrated.

Trading with OTC derivatives can lead to significant risks. Especially counterparty risk has gained particular emphasis due to the credit crisis in 2007. Counterparty risk is the risk that the counterparty at a future date may default and cannot fullfill the payment obligations that is required by the contract. Therefore the bank needs to estimate the total risk towards a particular counterparty and use this estimation when creating a capital buffer i.e., the capital requirement, to cover for losses due to a default. There are other ways to limit the risk towards the counterparty for example diversification, netting, collateralisation and hedging. All of these methods are focusing on controlling credit exposure[4].

Even though the term counterparty risk always has been connected to OTC contracts, there has been a tendency of underestimating this term before the financial crisis in 2007. This was due to a general market view which was that large companies were *"too-big-to-fail"*. The years following the financial crisis showed that this general view of the market did not reflect the reality because a couple of those *"too-big-to-fail"* companies went bankrupt[4].

A couple of years before the crisis the second Basel Accord, Basel II, was published. This is standards and recommendations on banking laws and regulations issued by the Basel Committee on Banking Supervision. It is intended to be used as international standards that control how much capital banks need to hold to be safe against the financial and operational risks banks face. Although it was established in 2004, it was not implemented in the major economies until the years around the financial crisis. Due to the events following the crisis the Basel Committee made an extension of the Basel II accords and published new standards that contains stricter regulations, the Basel III accord. These standards are supposed to lead to better quality of capital, increased coverage of risk for capital market activities and

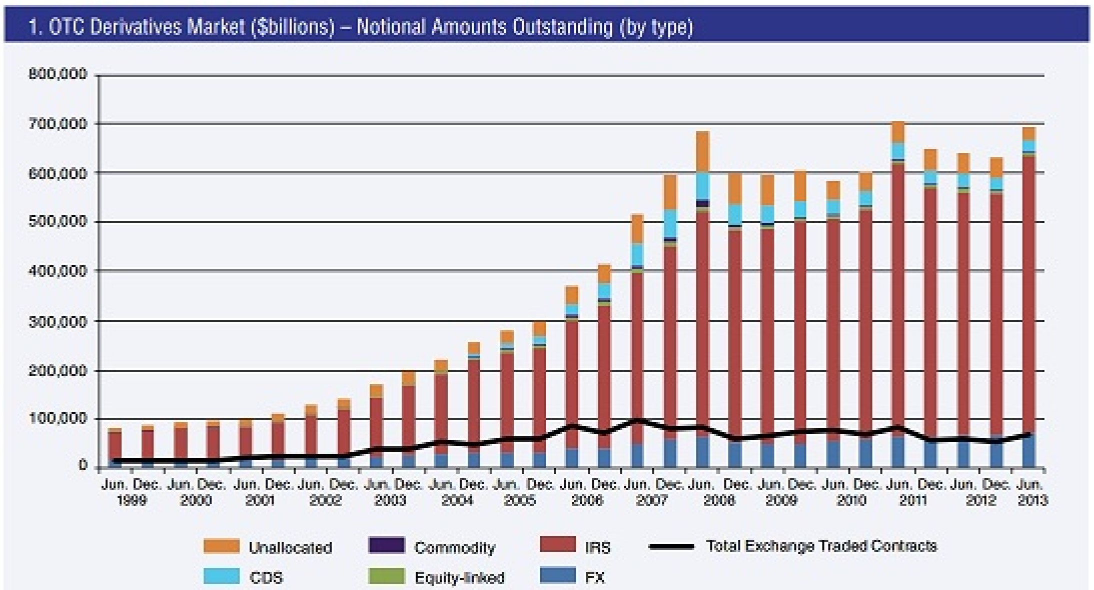


Figure 1.1: To show how fast the growth of the different derivatives on the OTC market has been during the last two decades. Based on statistical data from BIS.

better liquidity standards. The financial crisis has changed the market view and increased the concern of counterparty risk and the need for better risk management when engaging OTC contracts. In the Basel accords, Basel II and III, there are regulatory standards for risk management, setting up capital requirements to cover for eventual losses in case of a counterparty default[1].

In the Basel II accord the requirements consist of computing what is generally referred to as counterparty credit exposure i.e. the amount of money that can be lost if default occurs. Examples of these measures are the Expected Exposures (EE) and the Potential Future Exposure (PFE). In the Basel III accord the standards require the estimation of Credit Value Adjustment (CVA) charges. CVA is an adjustment to the price of the derivative to compensate for possible counterparty default. So the value of an OTC contract is the sum of the risk-free price and the CVA charge[4].

1.3 Description

Aim and framing of questions

From the introduction, it becomes clear why counterparty credit risk is an interesting matter. This thesis is based on the paper "Counterparty Credit Exposures for Interest Rate Derivatives using the Stochastic Grid Bundling Method" written by P. Karlsson, S. Jain and C. W. Oosterlee[9]. Therefore the methodology and models used in this paper will be similar or in some parts identical to the ones used in the underlying paper.
Today, most of the computations regarding exposure profiles are done under the risk-neutral probability measure along with the pricing of the contracts although it should be done under the impact of the real world market. Under real world measure the volatility has a much bigger effect than the drift on short-term risk exposures[7].
Since risk-related measures often are concerned with relatively long-term exposures it becomes more important to obtain a reasonable estimate of the real world drift. When moving from the risk-neutral world to the real world, or vice versa, Girsanov's theorem shows that the drift of variables changes but the volatilities remains the same. So in order to change measure we need estimate theese changes in drift[7].
The aim of this thesis is then to see if it is possible to get realistic estimations of the exposure profiles (EE and PFE) that reflects the "real world" market, i.e. simulate and estimate the mentioned exposure measures under the historical $\mathbb{P}$-measure also refered to as the real-world measure. This will be done for the interest rate derivative Bermudan swaption.
To reach any conclusion regarding the aim of this thesis, the following framing of questions are to be analyzed:

- How do we know that the models and methods that is used produces reasonable results during simulations?
- How is the market price of risk affected when the term structure is fitted to different yield curves?
- Is there a significant difference in results calculated under the risk neutral measure compared to the ones under real world measure?
- What happens during real world simulations for both short-term and long-term contracts?

Model Assumptions

To simplify computations the following assumptions have been made throughout the thesis:

- Only future payments are included in the estimations of the exposures.
- Assumes fixing dates and payment dates to be the same in all tenors.
- Assumes that the short rate has relatively short "memory" and by this assumption make it reasonable to use less amount of historical data in calculations.
- Assumes that there is no wrong-way risk.[1]
- For simplicity the future rates, that should be used to calculate the historical market price of risk, is instead approximated by the 20 year instantaneous forward rate.

Outline

Chapter 2 will define the one-factor Hull-White interest rate model, in different measures and other concepts that will be used in this model. Chapter 3 describes swaptions as a contract and the pricing of Bermudan swaptions. Chapter 4 will present the counterparty credit risk concepts that are relevant to the paper. Chapter 5 explains the Monte Carlo algorithm used for computations, the Stochastic Grid Bundling Method, SGBM. In chapter 6 the setup and numerical results obtained during simulations will be presented. In chapter 7 there is a discussion about the results and arguments leading up to the conclusions in chapter 8.

[1]More about the assumption on no wrong-way risk in the section about CVA.

Chapter 2

Short Rate Model

A short rate model is a model that describes the future evolution of interest rate by modelling the short rate, also known as the instantaneous short rate. Some of the first instantaneous short rate models were time-homogeneous and one-factor models. This means that the short rate dynamics only depends on constant variables and a Brownian motion. Early models, like Vasicek (1977) and that of Cox, Ingersoll and Ross (1985), were considered successful mainly due to their possibility of pricing analytically bonds and bond options. However since these models only have a finite number of free parameters it is not possible to calibrate these models to observed market prices[3].
Ho and Lee (1986) were the first to solve this problem by allowing a parameter to vary deterministically with time. This led to further development of more advanced models which includes both single and multi-factor[1] models that has one or more time-dependent variables that can be calibrated to market data.[3]
Before the short rate model that is used in this thesis, Hull and White (1990), is explained the reader will be introduced to some definitions and concepts that are of importance to interest rate simulations.

2.1 Interest Rate Notations

Let the price at time $t \in [0, T]$ of a zero-coupon bond with maturity T be denoted as $P(t, T)$. As defined in [2, Björk, p.304]:

[1]The main difference between a one and multi-factor model is that the diffusion part of the short rate dynamics is dependent on one respective several Brownian motions.

Definition 2.1. Let $0 \leq t \leq S \leq T$.

(i) The *continuously compounded forward rate* for $[S, T]$ as seen at time t is defined as

$$R(t; S, T) = -\frac{\log P(t, T) - \log P(t, S)}{T - S}.$$

(ii) The *instantaneous forward rate* with maturity T at time t is defined as

$$f(t, T) = -\frac{\partial \log P(t, T)}{\partial T}. \tag{2.1}$$

(iii) The *instantaneous short rate* at time t is defined as

$$r(t) = f(t, t).$$

2.2 The Market-Price-of-Risk

From [3, Brigo & Mercurio, p. 52] we get that the construction of a suitable locally-riskless portfolio, as in Black and Scholes (1973), leads to the existance of a stochastic process. This process depends on the current time and instantaneous spot rate and not from the maturities of the claims constituting the portfolio. Such process, which is commonly referred to as *market-price-of-risk*, defines a Girsanov change of measure from the real-world measure $\mathbb{P}$ to the risk-neutral measure $\mathbb{Q}$. Let us assume that the instantaneous spot rate evolves under the real-world measure $\mathbb{P}$ according to

$$dr(t) = \mu^P(t, r(t))dt + \sigma(t, r(t))dW^P, \tag{2.2}$$

where μ and σ are well-behaved functions and W^P is a $\mathbb{P}$-Brownian motion. It is possible to show the existence of a stochastic process λ so that the following relation hold for the drift part between $\mathbb{P}$- and $\mathbb{Q}$-measure and is given by

$$\mu^P(t, r(t)) = \mu^Q(t, r(t)) + \lambda(t, r(t))\sigma \tag{2.3}$$

which is dependent on $r(t)$ but not on the maturities T.

2.2.1 The historical Market Price of Risk

One estimate of λ is the historical market price of risk[7]. This is a calibration were the price of risk remains constant through time. This expression for λ is given by

$$\lambda_T(t) = \frac{r_0 - F(T)}{\sigma_T}, \tag{2.4}$$

where T indices the rate maturity that the estimate is based on and σ is based on the standard deviation of the daily rate changes in an appropriate period of time. As mentioned in the part about made assumptions, for simplicity reasons $F(T)$ is the 20-year instantaneous forward rate[2] and r_0 is the continuously compounded long run average short-term interest rate.
When there is mean reversion two things tend to happen. First the convexity adjustment used to convert forward prices to future prices is lower so that the estimate of futures rate, $F(T)$ is lower. This lowers the market price of risk. Since forward rates are used instead, the market price of risk will generally be a higher beacuase of that the convexity adjustment lower future rates. Second, for a particular value of $F(T)$ a higher market price of risk is necessary to increase the expected rate by $F(T) - r_0$. This increases the estimation of market price of risk.

2.2.2 Time dependent Market Price of Risk

Another estimate of λ is as a function of time where we can interpret $\lambda(t)$ as local prices of risk at different time steps, t. This estimate of the market price of risk is obtained from [7] but with a modified estimation of the volatility. To simplify estimations of the volatility, σ will be based on the standard deviation of monthly returns in an appropriate period of time.
If constructing a tree describing the average historical behavior of the term structure one can evaluate the market price of risk at each node in each time step as the difference between the expected short rate and the historical average short rate. To estimate these local prices of risk, we assume that the average drift of the short rate at each time is equal to zero in the real world. This means that, when moving from risk-neutral world to the real world, the

[2] $F(T)$ should have been the long run average instantaneous futures rate for maturity T. The long run average instantaneous futures rate is calculated from the long run average forward rate by making a convexity adjustment.

mean value of the short rate at all times should be equal to the historical average short rate.[7]
Since the market price of risk, $\lambda(t)$, will be different here at each time step they are better refered to as *local price of risk*.

2.3 Girsanov Theorem

We quote the following theorem, from [13, Åberg, p. 181]

Girsanov Theorem *Suppose that the stochastic process* $g(t, W_t^P)$ *is bounded and adapted on* $[0, T]$*,* W_t^P *is a standard* $\mathbb{P}$*-BM and the process* W_t^Q *is given by*

$$W_t^Q = W_t^P + \int_0^t g(s, W_s^P) ds. \tag{2.5}$$

$g(t, W_t^P)$ *is called the Girsanov kernel. The process*

$$L_t = \exp\left(-\int_0^t g(s, W_s^P) dW_s^P - \frac{1}{2} \int_0^t g^2(s, W_s^P) ds \right),$$

is then a $\mathbb{P}$*-MG an so is the product* $W_t^Q L_t$*. Finally, define the measure* $\mathbb{Q}$ *by*

$$E^Q[A] = E^P[A L_t] \tag{2.6}$$

Then W_t^Q *is a standard* $\mathbb{Q}$*-BM on* $[0, T]$*.*

2.4 The Hull-White One-Factor Model

The Hull-White one-factor model (HW1F), is a model of future interest rates. It is an extension from the Vasicek model mostly because of the poor fitting of the initial term structure of interest rates. This problem was adressed by John C. Hull and Alan White in their 1990. There hade been others trying to extend the Vasicek model's term-structure earlier but were not regarded as proper extensions due to lack of mean reversion in the short-rate dynamics. The need for an exact fit to the currently-observed yield curve, led Hull and White to the introduction of a time varying parameter in the Vasicek model. They also considered further extending the model by adding another timevarying parameter that would fit a given term structure of volatilities. Such a model, however, may be somehow dangerous when applied to concrete market situations. That and because of it's simplicity and tractability is why the model with only one chosen parameter to be a deterministic function of time, $\theta(t)$, is used amongst many trading desks today[3].
The HW1F implies a normal distribution for the short-rate process at each time. The Gaussian distribution of continously-compounded rates then allows for the derivation of analytical formulas and the construction of efficient numerical procedures for pricing a large variety of derivatives securities, such as the bermudan swaption. Critisism of the model is that it allows for negative short-rates and that it is limited in flexibility for modelling yield curve moments since all points on the yield curve are perfectly correlated[3].
In the underlying article, [9, Karlsson, Jain, Oosterlee], the short-rate model is defined as a general GSR one-factor model[3] but then simplified to a HW1F by making κ a constant. However in this paper, we will skip the general GSR definition and directly define the HW1F model that is later used in the computation part.

[3]The difference in notation between the definition of the general one-factor GSR (by P. Karlsson, S. Jain and C. W. Oosterlee) and the definition of HW1F in this paper, will be that $\kappa = a$ and $\theta(t) = \frac{\Theta(t)}{a}$.

2.4.1 Short Rate dynamics under risk-neutral $\mathbb{Q}$-Measure

In HW1F, the short rate $r(t)$ follows a mean-reverting[4] process of form

$$dr(t) = (\Theta(t) - ar(t))dt + \sigma dW(t), \tag{2.7}$$

where a, σ are constants and W a standard Brownian motion. The parameters a and σ are usually obtaind by calibrating the model to plain vanilla[5] option prices. From [3, Brigo & Mercurio] we get that the deterministic drift function $\Theta(t)$ is chosen so as to exactly fit the current term structure and has the form

$$\Theta(t) = \frac{\partial f(0,t)}{\partial T} + af(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}), \tag{2.8}$$

where f is the instantaneous forward rate defined in equation (2.1). Applying Ito's lemma and substituting Θ with equaqtion (2.8) gives us the following expression for the short-rate

$$r(t) = r(s)e^{-a(t-s)} + \int_s^t e^{-a(t-u)} u^Q(u, r(u))du + \sigma \int_s^t e^{-a(t-u)} dW(u) \tag{2.9}$$

Knowing from [3, Brigo & Mercurio, p. 75] that the HW1F model can be written in affine term structure and thereby the following relation holds

$$P(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du} \middle| \mathcal{F}_t \right] = A(t,T)e^{-B(t,T)r(t)}, \tag{2.10}$$

where $A(t,T)$ and $B(t,T)$ are deterministic functions and $\mathcal{F}_t$ is the filtration at time t generated by W. The deterministic functions, A and B, that fulfills the term structure are

$B(t,T) = \frac{1-e^{-a(T-t)}}{a}$,

$A(t,T) = \frac{P^*(0,T)}{P^*(0,t)} e^{(B(t,T)f^*(0,t) - \frac{\sigma^2}{4a}(1-e^{-2at})B(t,T)^2)}$.

[4]In finance, mean-reversion, is the assumption that the value will tend to move to the average value over time.

[5]The most basic or standardized financial instruments on market are reffered to as plain vanilla.

Then from [2, Björk, p.337] we get the closed form of the bond price, $P(t,T)$, given as

$$P(t,T) = \frac{P^*(0,T)}{P^*(0,t)} e^{(B(t,T)f^*(0,t) - \frac{\sigma^2}{4a}B^2(t,T)(1-e^{-2at}) - B(t,T)r(t))}.$$

One of the great advantages of having the expression in closed form is the possibility to evaluate the short rate at arbitrary time points instead of iterating using small time steps.

2.4.2 Short Rate dynamics under historical $\mathbb{P}$-Measure

The dynamic of $r(t)$ under the risk neutral $\mathbb{Q}$-measure is of the form

$$dr(t) = (\Theta(t) - ar(t))dt + \sigma dW^Q(t), \tag{2.11}$$

where a and σ are constants and W^Q a standard Brownian motion under $\mathbb{Q}$. $\Theta(t)$ is given by equation (2.8).
To obtain the dynamics of $r(t)$ under $\mathbb{P}$-measure we use the Girsanov theorem. First we define our Girsanov kernel $g(t, W^P)$ as the market-price-of-risk, $\lambda(t, r(t))$ from equation(2.3). Before using λ as the girsanov kernel we need to show that λ satisfies the *Sufficient Novikov Condition*, which states that a sufficient condition for the Girsanov Theorem is that $g(t,W)$ satisfies

$$E\left[\exp\left\{\frac{1}{2}\int_0^T g^2(t,W)dt\right\}\right] < \infty.$$

With the Girsanov kernels[6], λ, and equation (2.5) we are ready to change the dynamics of $r(t)$. To obtain the $\mathbb{P}$-dynamics, we just add and subtract the integral of $\lambda(t, r(t))$ which in this differential settings, equation (2.11), takes the form

$$\begin{aligned} dr(t) &= (\Theta(t) - ar(t))dt + \sigma(dW^Q(t) + \lambda(t,r(t))dt - \lambda(t,r(t))dt) \\ &= ((\Theta(t) - ar(t)) + \sigma\lambda(t,r(t)))dt + \sigma \underbrace{(dW^Q(t) - \lambda(t,r(t))dt)}_{=dW^P(t),\ eq.(2.5)}. \end{aligned}$$

[6]The Girsanov kernels are assumed to be bounded since they have been obtained from the article by Hull, Sokol and White [7].

Rewriting this we get the $\mathbb{P}$-dynamics of $r(t)$ as

$$dr(t) = \underbrace{((\Theta(t) - ar(t)) + \sigma\lambda(t, r(t)))}_{u^P(t,r(t)),\ eq.(2.3)} dt + \sigma dW^P(t) \tag{2.12}$$

where W^P is a standard Brownian motion under $\mathbb{P}$. From here on we will use u^P which is defined in equation (2.3).
By applying Itô's lemma to equation (2.12) we will obtain an expression of the short-rate

$$r(t) = r(s)e^{-a(t-s)} + \int_s^t e^{-a(t-u)} u^P(u, r(u))du + \sigma \int_s^t e^{-a(t-u)} dW(u) \tag{2.13}$$

where $u^P(t, r(t))$ is the function defined in equation (2.3).
The bond price, $P(t, T)$, for a zero-coupon bond that matures at time $T > t$ then must satisfy the PDE of form

$$\frac{\partial P}{\partial t} + u^Q(t, r(t))\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial r^2} + r(t)P = 0, \tag{2.14}$$

with boundary condition

$$P(T, T) = 1.$$

According to [2, Björk, p. 70, Proposition 5.6], Feynman-Kač the solution to the PDE will be given by

$$P(t, T) = \mathbb{E}_t^{\mathbb{Q}} \left[1 \cdot e^{-\int_t^T r(u)du} \middle| \mathcal{F}_t^Q \right]. \tag{2.15}$$

where $\mathcal{F}_t^Q$ is the filtration at time t generated by W^Q.

2.5 Simulation scheme of the calibrated HW1F model

This scheme for simulating the short rate has been provided by my mentor, Magnus Wiktorsson. In appendix A, one can follow the derivation of the short rate dynamics that is used to construct this scheme for simulating the short rate under the Hull-White one-factor model on an equidistant time grid.
To simulate a trajectory of r on an equidistant time grid with spacing Δ, the easiest way is then to define

$$y_k = r(k\Delta) - f^*(0, k\Delta) - \frac{\sigma^2}{2a^2}\left(1 - e^{-ak\Delta}\right)^2.$$

Using Eq. (A.2) we obtain the following recursive relation for y (standard AR(1) process)

$$\begin{aligned} y_0 &= 0 \\ y_k &= \alpha(\Delta) y_{k-1} + e_k, \; k = 1, \cdots \end{aligned}$$

where

$$\begin{aligned} \alpha(\Delta) &= e^{-a\Delta} \\ e_k &\in \mathrm{N}\left(0, \frac{\sigma^2}{2a}(1 - e^{-2a\Delta})\right), k = 1, \cdots, \text{ and they are all iid.} \end{aligned}$$

Having generated the sequence

$$\{y_k\}_{k=1}^{n}$$

we obtain r as

$$r(k\Delta) = y_k + f^*(0, k\Delta) + \frac{\sigma^2}{2a^2}\left(1 - e^{-ak\Delta}\right)^2, \; k = 1, \cdots, n.$$

This has the advantage that we can change the initial forward curve without having to re-simulate y.

Chapter 3

Interest Rate Swaps and Swaptions

3.1 Interest Rate Swap (IRS)

An Interest Rate Swap, IRS, is a derivative instrument in which two parties agree to exchange interest rate cash flows, based on a specified notional amount from a fixed rate to a floating rate or from one floating rate to another. There are many different kinds of IRS and as a part of the OTC contracts they can be tailored after specific needs of the parties involved. Most commonly is to use the IRS for hedging or speculating purposes. As stated in the introduction part of this paper, the total amount of traded interest rate contracts in the OTC-market was about 505 trillions USD and of those the swaps stands for a total traded amount of about 381 trillions USD[12] [8].

In a fix for floating swap, a payment stream at a fixed rate of interest, known as the swap rate, is exchanged for a payment stream at a floating rate. The floating rate are often indexed to a reference rate like the LIBOR rate. These payment streams are called the *legs* of the swap, fixed leg and floating leg. Since the agreement has two counterparties there are two positions of the contract, payer or receiver. The party that pays at a fixed rate and receives at a floating is the *payer*, and the party that pays at a floating rate in exchange for fixed rate is the *recevier*.

3.2 Swaptions

A swaption is an option granting the owner the right but not the obligation to enter into an underlaying swap. Although options can be traded on a variety of swaps, the term *swaption* typically refers to an IRS as underlaying swap. There are two types of swaption contracts, a payer swaption or a receiver swaption. These are defined the same way as for the swaps i.e, the one with the right to enter into a swap where they either pay or receives payments at the swap rate is the owner of the payer- or receiver swaption[8].
There are three main categories of swaptions, although these contracts may be derived after customized demands. These standard varieties of swaptions are:

- European, in which the owner has the right to enter the swap only on expiration date.
- Bermudan, in which the owner has the right to enter the swap on multiple prespecified dates.
- American, in which the owner the right to enter the swap on any day that falls within a range of two predetermined dates.

The participants of these contracts are predominantly large corporations, banks, financial institutions and hedge funds.

3.2.1 Valuation of Bermudan Swaption

The valuation of the Bermudan Swaption will be defined as in [9, Karlsson, Jain, Oosterlee].
A vanilla interest rate swap is a contract that allows one to change payments between two different legs, often a floating leg against a fixed leg. The value of the forward swap rate $S_{n,m}(t)$ and swap annuity $A_{n,m}(t)$ at time t with payments $T_{n+1}, ..., T_m$ are given by

$$S_{n,m}(t) = \frac{P(t, T_n) - P(t, T_m)}{A_{n,m}}, \quad A_{n,m}(t) = \sum_{i=n}^{m-1} P(t, T_{i+1})\tau_i,$$

where $\tau_i = T_{i+1} - T_i$. Given a lockout i.e., a no-call period up to time T_1, the Bermudan swaption gives the holder the right, at a set of fixing dates T_n, for

$T_n \in \mathcal{T} = \{T_1, T_2, ..., T_{m-1}\}$ to enter into a fixed for floating swap $S_{n,m}$ with fixing date T_n and last payment date T_m. The Bermudan swaption with the fixed coupon k, exercised at time T_n corresponds to the payoff given by

$$U_n = \phi \mathcal{N} A_{n,m}(T_n)(S_{n,m}(T_n) - k),$$

where $\mathcal{N}$ denotes the notional, and $\phi \in \{-1, +1\}$ the payer or receiver factor (+1 for a payer and -1 for a receiver swaption). The holder of a payer Bermudan will pay the fixed swap leg and receive the floating swap leg. The present value V_0 of a Bermudan swaption is the supremum taken over all discrete stopping times of all conditional expected discounted payoffs, that is

$$V_0 = B(T_0) \sup_{T_n \in \mathcal{T}} \mathbb{E}_{T_0} \left[\frac{U_n}{B(T_n)} \right]. \tag{3.1}$$

For practical reasons we now choose to work under the spot measure, $\mathbb{Q}^B$. Where we have, $B(t)$, as the numeraire. $B(t)$ is the discrete version of the continuously compounded money market account with rolling certificate of deposit given from

$$B(t) = P(t, T_{i+1}) \prod_{n=0}^{i} P(T_n, T_{n+1})^{-1}, t \in (T_i, T_{i+1}],$$

with corresponding fixed discrete tenor structure $0 = T_0 < T_1 < \cdots < T_N$. One of the beneficial reasons to work with the spot measure is that the numeraire asset $B(t)$ is alive throughout the tenor. This allows for simulating paths irrespective of tenor which i practical for Bermudan swaptions since the contract can mature randomly at any of the preset dates in the discrete tenor structure.

Let $\mathbb{E}_t^B = \mathbb{E}$ denote the conditional expectation with respect to the measure induced by $B(t)$. The value at time t of a contract paying $V(T)$ at time T is then given by

$$V(t) = B(t) \mathbb{E}_t \left[\frac{V(T)}{B(T)} \right].$$

The option value at an arbitrary time T_n is the maximum of the intrinsic value U_n and the conditional continuation value H_n given as

$$V_n = \max(U_n, H_n), \tag{3.2}$$

where $H_m = 0$ at maturity T_m. Further, the continuation value H_n is the conditional expected option value at a future time T_{n+1} and given by

$$H_n = B(T_n)\mathbb{E}_{T_n}\left[\frac{V_{n+1}}{B(T_{n+1})}\right]. \tag{3.3}$$

The valuation problem is solved via backward induction starting at the time of maturity, T_m, and solved by recursively evaluate (3.2) and (3.3) at each step until we reach time T_0. Then the value of V_0 will be the value of the Bermudan swaption contract.

Chapter 4

Counterparty Credit Risk

In this chapter some of the concepts that is of importance to estimate the counterparty credit risk measure (EE, CVA and PFE) in this paper will be defined and described.

4.1 Credit Default Swaps

A credit default swap (CDS) is a financial agreement that the seller of the CDS will compansate the buyer in the event of a loan default or other credit event. It is often seen as an insurance against the risk of default of some company, called the *reference entity* which is not a party in the agreement.[8]
The buyer of a CDS makes a series of payment to the seller and in return the buyer recieves a payoff from the seller in case of default of the reference entity. The value of the CDS varies with the credit quality of the reference entity, which means that it should not only be seen an insurance against default but as an object for speculation on the credit quality of the reference entity. This is why the CDS often is used when calibrating the probability of default.

4.2 Probability of Default

The probability of default, PD, is a financial term describing the probability of a counterparty defaulting in a particular time horizon. These probabilities

are either obtained as real, $\mathbb{P}$, or risk-neutral, $\mathbb{Q}$, default probabilities. These different kinds of default probabilities are used in different types of analyzes. Often when pricing or calculating the change of price of a financial contract due to default probabilities, the risk-neutral method is used to account for the market price of risk. When calculating for potential future losses from default, real-world default probabilities are used. [4]
The default probabilities for a given counterparty is usually bootstrapped from a quoted CDS. Basel III states that "Whenever a CDS spread is not available, the bank must use a proxy spread that is appropriately based on rating, industry and region of a counterparty". Some calibration methods that are used to estimate PD are from CDS spreads or bond spreads (if traded and quoted in the market), from a rating transition matrix (normally given by credit rating agencies or institutions) and from proxies such as stock price or reported fundamental data. In this paper we will define $PD(t)$ under the risk-neutral $\mathbb{Q}$-measure since we will use $PD(t)$ to price the CVA. Let τ_C be the counterparty's default time and the probability that the counterparty C defaults before time t can then be expressed as $PD(t) = \mathbb{Q}(\tau_C < t)$. We define the default probabilities as

$$PD(t) = 1 - e^{\left(-\int_0^t \gamma(t)dt\right)}, \tag{4.1}$$

where the probability factor $\gamma(t)$ is called the hazard rate or instantaneous credit spread. The hazard rate can be interpreted as the probability that the counterparty defaults in dt years given that it has not defaulted so far. [9]

4.3 Expected Exposure

Let $V^i(t)$ be the default-free value of a contract i at time t. Then the exposure of this one contract is given by

$$E(t) = max(V^i, 0).$$

The value of a portfolio consisting of N contracts towards a specific counterparty C at time t can be written as

$$V_C(t) = \sum_{i=1}^{N} V^i(t).$$

The portfolio exposure $E(t)$ against a counterparty C is then given by the portfolio value

$$E(t) = max(V_C, 0).$$

Since one never gain on a counterparty default, the exposure is always positive. Let τ_C denote the counterparty's default time. The expected exposure $EE(t)$ can now be defined as

$$EE(t) = \mathbb{E}[E(t)|t = \tau_C].$$

And the risk-neutral discounted expected exposure, $EE^*(t)$ is then expressed as

$$EE^*(t) = B(0) \cdot \mathbb{E}^{\mathbb{Q}^{\mathbb{B}}} \left[\frac{E(t)}{B(t)} \middle| t = \tau_C \right], \tag{4.2}$$

where $\mathbb{E}$ is the expectation with respect to the spot measure $\mathbb{Q}^{\mathbb{B}}$ and the numeraire is the discrete version of the continous compounded money market account with rolling certificate of deposit, $B(t)$.

4.4 CVA

Credit Value Adjustment can be seen as the price of or as a measure of the market value of counterparty credit risk towards a specific counterparty. The CVA is a measure that adjusts the risk-free price of the derivatives to account for the counterparty credit risk. CVA is defined as

$$CVA = P_{risk-free} - P_{risky}, \tag{4.3}$$

where CVA is the adjustment between the risk-free price, $P_{risk-free}$ and the risky (true) price, P_{risky}, of the derivative. CVA has become one of the main focuses in the Basel III accord and trading desks today are required to estimate CVA charges towards each of their counterparties. The CVA might differ between different counterparties due to different conditions in terms of risk, this means that there will be different prices of a derivative for each counterparty.

When estimating CVA there are mainly two directions, unilateral and bilateral CVA. Unilateral CVA only considers the counterparties risk while bilateral CVA accounts for the credit quality of both parties. Both these

charges are calculated with Monte Carlo methods. Unilateral CVA requires 2 factors, the market and the counterparty credit, and an assumption on a single correlation between the two factors. When calculating the bilateral CVA one need to account for the own credit risk as well, which means that there are 3 factors and a $3x3$ correlation matrix between the factors. In this paper, as mentioned before, we will only focus on the unilateral CVA but the methods presented can be extended towards bilateral CVA.
CVA is computed as the integral over all points in time of the discounted expected exposure given that the counterparty defaults at that time, multiplied with the default probability and the Loss Given Default, i.e. one minus the recovery rate R_C. Mathematically, the unilateral CVA is given by

$$CVA = (1 - R_C) \cdot B(0) \int_0^T \mathbb{E}\left[\frac{E(t)}{B(t)}\delta(t - \tau_C)\right] dt, \tag{4.4}$$

where δ is the Dirac delta function, $\mathbb{E}$ is the expectation with respect to the spot measure $\mathbb{Q}^B$ and T is the maturity of the instrument within the portfolio with the longest maturity.
We will further assume that there i no wrong-way risk. ISDA defines the wrong-way risk as "the risk that occurs when exposure to a counterparty is adversely correlated with the credit quality of that counterparty" . In short it arises when default risk and credit exposure increases together, this codependence will increase the CVA on the forward contract and will make the CVA higher than when the effects were independent. Making this assumption will simplify the calculations for CVA and it means that we assume that exposure and default probability is independent. With this CVA now can be expressed as

$$CVA = (1 - R_C) \cdot B(0) \int_0^T \mathbb{E}\left[\frac{E(t)}{B(t)}\middle| t = \tau_C\right] \mathbb{E}[\delta(t - \tau_C)]dt. \tag{4.5}$$

The second expectation in the integrand above is recognized as the default probability function $PD(t)$. So we rewrite the CVA as

$$CVA = (1 - R_C) \cdot B(0) \int_0^T \mathbb{E}\left[\frac{E(t)}{B(t)}\middle| t = \tau_C\right] dPD(t). \tag{4.6}$$

If we define a discrete grid $0 = T_0 < T_1 < \cdots < T_m = T$ of observation dates and recognize that the integrand expectation part actually is the risk-neutral discounted expected exposure $EE^*(t)$. Equation (4.6) can now be approximated by

$$CVA \approx (1 - R_C) \sum_{n=1}^{N-1} EE^*(T_n) \cdot (PD(T_{n+1}) - PD(T_n)). \tag{4.7}$$

Equation (4.7) can be seen as a weighted average of the risk-neutral discounted expected exposure with the weights given by the default probabilities. The most complex part of the CVA estimation is the evaluation of the exposure. There are guidelines for how to simulate market scenarios and how many simulations required to get a satiesfied value of CVA in the sense of convergence. With American Monte Carlo methods a large number of market scenarios of factors such as yield and inflation curves, FX rates, equity and commodity prices, credit spreads etc. are simulated.

4.5 Potential Future Exposure

Potential Future Exposure (PFE) is a measure of counterparty risk or credit risk and it is often refered to as sensitivity of risk with respect to market prices. It should be seen as an upper bound on a confidence interval for future credit exposure.
For a given date t and some confidence level α, PFE_α is the maximum exposure of a contract or portfolio with a high degree of statistical confidence α defined as

$$PFE_\alpha(t) = \inf\{x : \mathbb{P}(EE(t) \leq x) \geq \alpha\}, \tag{4.8}$$

where $\mathbb{P}$ is the real-world probability measure.

Chapter 5

Stochastic Grid Bundling Method

5.1 Background of the SGBM

The pricing of American-style contracts can be complicated i.e., contracts in which the holder can choose the time of exercise. The traditional valuation methods, such as lattice- or tree based techniques are often impractical due to the curse of dimensionality, and are therefore mainly used on low-dimensional cases. More attractive methods are simulation methods. They are based on stochastic sampling of paths of the underlaying state vector and converges in proportion to the square root of the number of paths that are generated, independent of the dimension of the problem. [11]
In the case of American-style contracts an optimal exercise policy has to be determined via a dynamic programming approach. The difficulty then arises in combining the forward evolution of simulated paths with the backward induction of dynamic programing. [11]
Several simulation-based methods which combines Monte Carlo path generation and dynamic programming techniques have been developed over the years. Amongst these methods, there are a class of *regression-based methods*. In 2012 Jain and Oosterlee derived the regression-based method called *Stochastic Grid Method*, SGM. This method consists of elements from the

Least Squares Method (LSM) [1], the *Stochastic Mesh Method* (SMM)[2] and *stratified state aggregation along the pay-off* method [3]. [10]
SGM follows the dynamic programming style of SMM by recursively computing the option price backwards in time. The functional approximation of the option price at a given time step is used to compute the price at the previous time step which is obtained using regression. An advantage of this method is that the dimensionality of the problem is reduced using the pay-off as a mapping function. In comparance with LSM, for which the basis functions used for regression grows fast with the dimensions of the problem, the number of basis functions in SGM is independent of the dimensions of the problem. [10]
Another approach is based on approximating the transition probabilities using either *bundling*, *partitioning* or *quantization* of the state space. or computing weights to approximate these conditional probabilities, as in the stochastic mesh method. The bundling approach was introduced by James A. Tilley.[11]
In 2013, Jain and Oosterlee further developed the stochastic grid method into a method that is a hybrid of *regression*- and *bundling*- based approaches, the *Stochastic Grid Bundling Method*, SGBM. It uses regressed value functions together with bundling of the state space to approximate continuation values at different time steps. To make the algorithm less computationally expensive, the bundling in SGBM is used to cluster grid points.[11]

5.2 General Summary of the SGBM

The steps involved in the SGBM algorithm will be stated and described as in Jain and Oosterlee (2013). [11]

[1] by Francis A. Longstaff and Eduardo S. Schwartz can be found in *Valuing American options by simulation: A simple least-squares approach*, The Review of Financial Studies Vol. 14, Iss. 1 (2001), pp. 113-147

[2] by Mark Broadie and Paul Glasserman can be found in *A stochastic mesh method for pricing high-dimensional American option*, Journal of Computational Finance 7 (July, 2004), pp. 35-72

[3] by Jérôme Barraquand and Didier Martineau can be found in *Numerical valuation of high-dimensional multivariate American securities*, Journal of Financial and Quantitative Analysis. Vol. 30, Iss. 3 (Sept, 1995), pp. 383-405

Step I: Generating grid points

The grid points are generated by simulating N independent copies of sample paths with all of them starting from the same initial state R_{t_0}, $\{R_{t_0}(n), ..., R_{t_M}(n)\}$, $n = 1, ..., N$, of the underlying process R_t. The n-th grid point at time step t_m is then $R_{t_m}(n)$ with $n = 1, ..., N$. If the underlying process is in closed form it can be simulated directly otherwise an appropriate discretization scheme should be applied to generate the sample paths.

Step II: Option value at terminal time

The option value at terminal time, t_M, is given by

$$V_{t_M}(R_{t_M}) = \max(U(R_{t_M}), 0)$$

where $U(R_{t_M})$ is the intrinsic value. This relation is used to compute the option value for all grid points at the final time step.

The next three steps are subsequently performed for each time step, $t_m, m \leq M$ recursively, moving backwards in time starting from t_M.

Step III: Bundling

The grid points at t_{m-1} are bundled into $\mathcal{B}_{t_{m-1}}(1), ..., \mathcal{B}_{t_{m-1}}(v)$ non overlapping sets or partitions. We will use the *Recursive bifurcation algorithm*[4] for partitioning but there are other approaches that can be used for partitioning.

Step IV: Mapping high-dimensional state space to a low-dimensional space

Corresponding to each bundle $\mathcal{B}_{t_m}(\beta), \beta = 1, ..., v$, a parametrizied value function $Z : \mathbb{R}^d \, x \, \mathbb{R}^K \mapsto \mathbb{R}$. This function assigns computed values $Z(R_{t_m}, \alpha^{\beta}_{t_m})$ to state R_{t_m}. Here $\alpha^{\beta}_{t_m} \in \mathbb{R}^K$ is a vector of free parameters. The objective then is to choose, for each t_m and β, a parameter vector $\alpha^{\beta}_{t_m}$ so that

$$Z(R_{t_m}, \alpha^{\beta}_{t_m}) \approx V_{t_m}(R_{t_m}).$$

[4]For a description of the Recursive bifurcation algorithm see Appendix B

This can be done by representing $Z(R_{t_m}, \alpha_{t_m}^{\beta})$ as linear combination of basis functions using regression. Doing this will give us the following expression

$$Z(R_{t_m}, \alpha_{t_m}^{\beta}) \approx V_{t_m}(R_{t_m}) \approx \sum_{i=0}^{q} \alpha_{i,n}\zeta_i(t_m) \tag{5.1}$$

where ζ_i is a set of q basis functions, $\zeta_i : \mathbb{R}^d \ x \ \mathbb{R}^K \mapsto \mathbb{R}$ and constants $\alpha_{i,n}$.

Step V: Computing the continuation and option values at t_{m-1}

The continuation values for

$$R_{t_{m-1}}(n) \in \mathcal{B}_{t_{m-1}}(\beta), n = 1, ..., N, \beta = 1, ..., v,$$

are approximated by

$$\hat{H}_{t_{m-1}}(R_{t_{m-1}}(n)) = \mathbb{E}\left[Z(R_{t_m}, \alpha_{t_m}^{\beta}) \middle| \, R_{t_{m-1}}(n) \right].$$

The approximated option value, $\hat{V}_{t_{m-1}}$, are then given by

$$\hat{V}_{t_{m-1}} = \max(U(R_{t_{m-1}}), \hat{H}_{t_{m-1}}(R_{t_{m-1}}(n))).$$

5.3 SGBM to calculate EE, CVA and PFE

Once the approximated values, V_n, have been obtianed through the regression it is time to compute the exposure. This is done as described in the underlying article by Karlsson, Jain and Oosterlee. [9]
The exposure, $E_n(t_n)$ at time t_n, is calculated using the law of iterated expectations, given as

$$\begin{aligned} E_n(t_n) &= B(t_n)\mathbb{E}\left[\frac{V_{n+1}(t_{n+1})}{B(t_{n+1})} \middle| \, X(t_n) \right] \\ &= B(t_n)\mathbb{E}\left[\mathbb{E}\left[\frac{V_{n+1}(t_{n+1})}{B(t_{n+1})} \middle| \, \zeta(t_{n+1}), X(t_n) \right] \middle| \, X(t_n) \right] \end{aligned} \tag{5.2}$$

where $X(t_n)$ is grid points. To simplify, we rewrite this expression and decompose it into two parts that needs to be computed by the SGBM. First we will compute the expected option value, refered to as inner expectation, given by

$$Z(t_n) = \mathbb{E}\left[\frac{V_{n+1}(t_{n+1})}{B(t_{n+1})} \,\middle|\, \zeta(t_{n+1}), X(t_n)\right] \tag{5.3}$$

In the second step we will compute the exposure, refered to as the outer expectation, given by

$$E_n(t_n) = B(t_n)\mathbb{E}\left[Z(t_n) \,\middle|\, X(t_n)\right]. \tag{5.4}$$

So to compute equation (5.3) and (5.4) we need to numericly approximate $Z_n(t_n)$. One way to do this, is by condition $V_{n+1}(t_{n+1})$ on $X(t_n)$ and then use bundling[5]. This is a method to partition the state space into non-overlapping regions, so that each point in space can be identified to lie in exactly one of the bundled regions. The idea behind bundling is that for a large set of paths use the fact that neighbouring paths will have a similar continuation values. One can therefore perform local-averaging over nearby paths.
The goal is to construct bundles by generating K replications of the underlaying asset path and their grid points, $X(t_n, w_k)$. Here w_k represents the k-th path, $n = 1, ..., N$ and $k = 1, ..., K$. Then one need to bundle them at each time point, t_n, into $a_n(K)$ non-overlapping sets, $\mathcal{B}^s(t_n) = (\mathcal{B}^1(t_n), ..., \mathcal{B}^a(t_n))$, with a threshold, $b_n(K)$, on the number of path points in each set or bundle. Defining the s-th bundle at time t_n as

$$\mathcal{B}^s(t_n) = \left\{X(t_n, w_k) : \|X(t_n, w_k) - \mu_n^s\|_2 \leq \|X(t_n, w_k) - \mu_n^\ell\|_2 \ \forall 1 \leq \ell \leq a_n(K)\right\}, \tag{5.5}$$

for $k = 1, ..., K$, where μ_n^s is the mean of the points in $\mathcal{B}^s(t_n)$. For a deeper knowledge and description of the the recusive bifurcation algorithm that is used for bundling turn to Appendix B.
So to compute the inner expectation, equation (5.3), we approximate Z_n onto a polynomial subspace where the value are linear combinations of the basis functions. This is done by regression on the option values at t_n for those paths that originate from the bundle containing $X(t_n, w_k)$. So we express Z_n as in equation (5.1), that is

$$\hat{Z}_n(t_n, w_k) = \sum_{i=0}^{q} a_{i,n}\zeta_i(t_n, w_k), \tag{5.6}$$

[5]Bundling was introduced by Tilley and then extended to higher dimensions in State Space Partitioning Method (SSPM) by Jin, Tana and Sun.

for $k \in \mathcal{B}(t_n, w_k)$ and the following residual is to be minimized

$$\min_{\alpha} \sum_{l \in \mathcal{B}(t_n, w_k)} \left(\hat{Z}_n(t_n, w_l) - V_n(t_n, w_l) \right)^2 .$$

To compute the outer expectation i.e. the approximate of the exposure, $\hat{E}_n(t_n, w_k)$, at time t_n for the path w_k is then given by

$$\hat{E}_n(t_n, w_k) = B(t_n, w_k) \mathbb{E} \left[\frac{\hat{V}_{n+1}(t_{n+1}, w_k)}{B(t_{n+1}, w_k)} \middle| \mathcal{B}(t_n, w_k) \right] \tag{5.7}$$

where bundle $\mathcal{B}(t_n, w_k)$ is the set of path-indices of path that lies in the bundle containing $X(t_n, w_k)$. If combining the relation in equation (5.3) and (5.7) we can rewrite the expression for the approximated exposure as

$$\hat{E}_n(t_n, w_k) = B(t_n, w_k) \sum_{i=0}^{q} a_{i,n} \mathbb{E} \left[\frac{\zeta_i(t_{n+1}, w_k)}{B t_{n+1}, w_k} \middle| \mathcal{B}(t_n, w_k) \right] . \tag{5.8}$$

In the computation of the exposure we have decomposed the problem into two parts, the inner- and the outer expectation. We use T-forward measure, with corresponding expectation $\mathbb{E}^T$ and T-maturity zero coupon bond $P(t, T)$ as numeraire, to express the outer expectation in closed form and the with the advantage that it decouples the payoff $V(T)$ from the numeraire and takes the discount factor out of the expectation i.e.

$$\begin{aligned} V(t) &= B(t) \mathbb{E}_t \left[\frac{V(T)}{B(T)} \right] \\ &= P(t, T) \mathbb{E}_t^T \left[V(T) \right] . \end{aligned}$$

As mentioned earlier one benefit with the spot measure compared to the T-forward measure is that the numeraire asset $B(t)$ is alive throughout the tenor and therefore allows for simulating paths irrespective of tenor. This is one reason for computing the inner expactation under spot measure with regression.

Since both the spot- and T_{n+1}-forward coincides over the intervall $[t_n, t_{n+1}]$ we can use *hybrid measure* when calculating the exposure. We can once again rewrite the expression for the approximated exposure under these hybrid measure as

$$\begin{aligned}
\hat{E}_n(t_n, w_k) &= B(t_n, w_k)\mathbb{E}\left[\frac{\sum_{i=0}^{q} a_{i,n}\zeta_i(t_{n+1}, w_k)}{B(t_{n+1}, w_k)}\middle|\mathcal{B}(t_n, w_k)\right] \\
&= P(t_n, t_{n+1}, w_k)\mathbb{E}^{T_{t+1}}\left[\sum_{i=0}^{q} a_{i,n}\zeta_i(t_{n+1}, w_k)\middle|\mathcal{B}(t_n, w_k)\right] \\
&= P(t_n, t_{n+1}, w_k)\sum_{i=0}^{q} a_{i,n}\mathbb{E}^{T_{t+1}}\left[\zeta_i(t_{n+1}, w_k)\middle|\mathcal{B}(t_n, w_k)\right].
\end{aligned}$$

Once we have calculated the approximate exposures at each time t_n we can approximate the expected exporure at each time step as the mean of exposures. Given as

$$\hat{EE}(t_n) \approx \frac{1}{K}\sum_{k=1}^{K} \hat{E}_n(t_n, w_k),$$

and then further we get the approximation of the discounted expected exposure as

$$\hat{EE}^*(t_n) \approx \frac{1}{K}\sum_{k=1}^{K} B(t_0, w_k)\frac{\hat{E}_n(t_n, w_k)}{B(t_n, w_k)},$$

where $k = 1, ..., K$ are the number of generated paths. We are then ready to compute the CVA and PFE according to equation (4.7) and (4.8).

5.4 Summary of the SGBM

For clarity, a summarize of the steps of SGBM algorithm that will be used for the computation of both the inner- and outer expectation, which is needed in order to further compute the counterparty credit risk measures.

First step: Approximation of the Mapping Functions, Z_n.

1. Generate K_1 paths, $w_1, ..., w_{K_1}$, from the underlying dynamics.
2. For each time point, t_n, and path, w_k, for $n = 1, ..., N-1$ and $k = 1, ..., K_1$ compute the state variable, $X(t_n, w_k)$ and the terminal time option values $V_n(t_n, w_k)$.

3. Moving backwards in time, for each $n = N-1, ..., 1$, perform the following scheme.
    (a) Bundle the grid points at t_{n-1} into a distinct bundles, except for t_0 which is the starting point of alla paths, using the recursive bifurcation algorithm.
    (b) Using regression to compute the mapping function, $Z_n^s, s = 1, ..., a$, given by equation (5.6). Using the option values $V_n(t_n)$ at all the time points t_n with paths that originates from the s-th bundle, $\mathcal{B}^s(t_{n-1})$ at time t_{n-1}.

Second Step: Estimate the Counterparty Credit Risk Measures.

1. Generate a new set of K_2 paths, $w_1^{'}, ..., w_{K_2}^{'}$, from the underlying dynamics in order to compute new state variables $X(t_n, w_k^{'})$ and the terminal time option values $V_n(t_n, w_k^{'})$.
2. Moving backwards in time, for each $n = N-1, ..., 1$, perform the following scheme.
    (a) Bundle the grid points at t_{n-1} into a distinct bundles, except for t_0 which is the starting point of alla paths, using the recursive bifurcation algorithm.
    (b) Compute the approximate exposures, $\hat{E}_n(t_n, w_k^{'})$ for the grid points that lies in bundle s, at time t_n, using equation (5.8) for those paths for which $X(t_{n+1}, w_k)$ belongs to the bundle $\mathcal{B}^s(t_n)$, for $s = 1, ..., a$. Use the regressed functions, $Z_{n+1}(t_{n+1})$, obtained from the first step of simulation.
    (c) Compute the $EE_n(t_n)$, $EE_n^*(t_n)$ and $PFE_\alpha(t_n)$ for the grid points in bundle s at time t_n for those paths for which $X(t_{n+1}, w_k^{'})$ belongs to the bundle $\mathcal{B}^s(t_n)$, for $s = 1, ..., a$.
3. The CVA charge is then approximated as in equation (4.7). (Only for the R-dynamics generated under the risk-neutral measure, $\mathbb{Q}$, since it is a matter of pricing.)

5.5 Remarks on the SGBM

- The CVA, EE and PFE could all be evaluated directly from the first step of the algorithm if needed, for reasons like computer power. By decomposing the simulation in two steps, where we first obtain the mapping function Z_n and then on a fresh set of generated paths approximate the risk measures, we make sure that we get unbaised values when using our mapping functions to a new set of paths. [9]

- Some of the difficulties in American Monte Carlo methods lies in the choice of the regression variables, so it is crucial to choose a suitable set of explanatory variables and parametric functions. In this framework this can be considered to be a combination of both art and science. Usually the basis functions are selected based on the problem and relies on human experience. They are often chosen such that they represent the most salient properties of a given state. [11]

Chapter 6

Results

This chapter will start of with a description of the setup in the simulations and then later on present some results of the simulations.

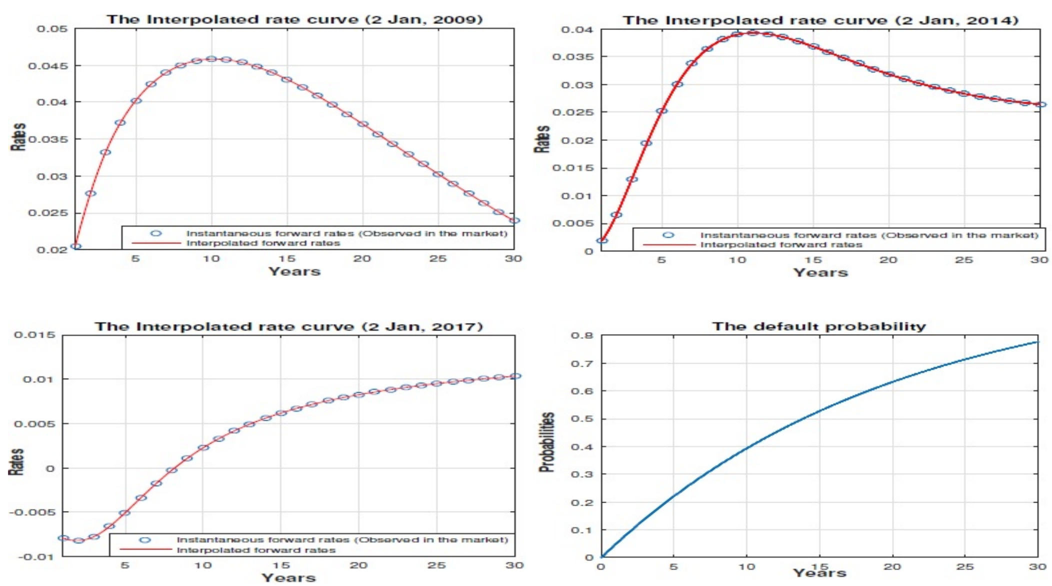


Figure 6.1: Upper left: The interpolated instantaneous forward rates observed in the market from AAA-rated euro area central government bonds on January 2, 2009. Upper right: observed on January 2, 2014. Lower left: observed on January 2, 2017. Lower right: The default probability function, $PD(t) = 1 - e^{(-\int_0^t \gamma(t)dt)}$ with hazard rate $\gamma(t) = 0.05$.

6.1 Setup for simulation

In all the simulations, the market information are represented as a 1-dimensional state variable $X(t)$, and the short rate $r(t)$ has been simulated under the dynamics of the HW1F model, described by equation (2.7). For simplicity the parameters a and σ were chosen to be constants while the model parameters otherwise has been calibrated to the initial zero coupon bond prices observed in the market Januari 2, 2009, 2014 or 2017. The probability of default function, $PD(t)$, has been calculated from equation (4.1) with $\gamma(t) = 0.05$. See Figure 6.1 for an illustration of the observed yield curves and $PD(t)$.
For the regression part, in the SGBM algorithm, a third order polynomial with the short rate as basis functions is choosen. This means that equation(5.1) is approximated by the following equation

$$Z_n(t_n, \alpha_{t_n}^{\beta}) \approx V_n(R_{t_n}) \approx a_0 + a_1 r(t_n, \alpha_{t_n}^{\beta}) + a_2 r(t_n, \alpha_{t_n}^{\beta})^2 + a_3 r(t_n, \alpha_{t_n}^{\beta})^3.$$

The computation of the continuation value in equation (5.8) requires the computation of

$$\mathbb{E}\left[r(t_n, w_k)^q | r(t_{n-1}, w_k)\right],$$

for $q = 1, 2, 3$. In order to compute the expected value of the basis functions in the equation above one need to use the moments for the short rates, more about these in appendix C.
In all simulations the Bermudan swaption is considered exercisable once a year. Simulations has been done for the two different values of the constant parameters $a = 0.01, 0.02$ and $\sigma = 0.01, 0.02$. The strike was in the underlaying article set to respresent different states of Moneyness, i.e. the spot vs. strike ratio of 80%, 100% and 120%. The same states is used in this thesis but with a minor adjustment on the strike that represents the states of moneyness 80% and 120%.
The short rates are simulated with monthly time steps. In the first step of SGBM we simulate $K_1 = 4096$ different paths for the different years to maturity, $T = 5, 10, 15, 20$. These short rates are simulated using the Mersenne twister pseudo random number generator in matlab. These generated paths are then used to estimate the the regressed functions. In the second step of the SGBM we generate $K_2 = 8192$ paths using quasi Monte Carlo Sobol paths. These paths then uses the regressed functions to estimate the swaption value, EE, PFE and CVA charges.

The values are then compared with the ones obtained in the underlaying paper and used in some sense to validate if the simulations computes reasonable values. These values are presented in appendix D.
In both the first and second step of the SGBM, the bundling at each time step is done by the recursive bifurcation algorithm into 8 bundles, this is further described in appendix B. The reported values are always obtained from the second step of the SGBM algorithm. The prices are reported in basis points and the notional amount used is $\mathcal{N} = 10.000$.

6.2 Valuation and CVA

The values and CVAs for the bermudan swaption with maturities $T = 5, 10$ years that has been computed during simulations, with the setup presented in previous section, can be found in Table E.1 in appendix D. In this table the computed values are compared with the ones presented in the underlaying paper, and used in some sense to validate or verify if the simulation provides reasonable values.
In Figure E.1, appendix D, the computations for the exposure profile is presented together with approximate values from the underlying article.
Further on, in Table E.2 the deviation between each simulated value and the ones from the underlying article is presented, both as basis points and as a percentage deviation. Table 6.1 provides a summary on the results presented in appendix D.

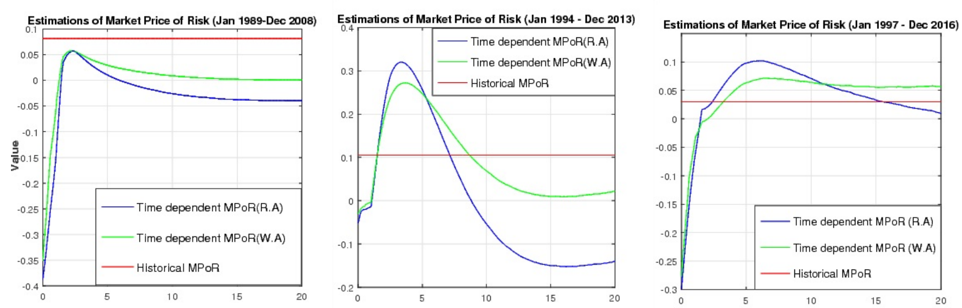
Deviation from value in underlying article				
Maturity	Price		CVA	
5 Years	3.71	(0.58%)	0.34	(0.47%)
10 Years	38.87	(2.42%)	1.90	(0.82%)

Table 6.1: The table shows the mean deviation from the values in the underlying article in both basis points and percentage as a summary of the subfocus. For a full view of the deviation go to appendix D.

Summary of Confidence intervals		
Maturity	Price	CVA
5 Years	(100%)	(75%)
10 Years	(58%)	(50%)

Table 6.2: The table shows the percentage of simulated values that includes the underlying article values in its confidence interval at the significance level $\alpha = 0.05$. The confidence intervals are not presented in the report but could easily be calculated with the values presented in the tables in appendix D.

6.3 Market Price of Risk



(a) MPoR, 2 jan 2009 (b) MPoR, 2 jan 2014 (c) MPoR, 2 jan 2017

Figure 6.2: Each graph shows the three different estimations of the market price of risk at three different dates. These estimates has been evaluated from simulating 500.000 short rates and historical data going back 20 years.

For the simulations done under the historical or real-world measure, $\mathbb{P}$, one need to try to estimate the market price of risk, λ. As described earlier this will be done in two ways, with the historical- and the time dependent market price of risk.
In the process of estimating the historical market price of risk, the assumption that a reasonable used amount of years with historical data would be 20 years.[1] This makes the historical market price of risk constant even though

[1]All data on historical short rates are collected as the mean historical short rate in the Euro area, which includes 19 contries. Although with the exception of the data from before 1994 which is the short rate of France. This one was choosen since it was the closest

the maturities changes since the used amount of historical data is the same. For the time dependent or local market price of risk there is less amount of historical data[2], only going back 18 months. This amount of historical data is used to compute the average shortrate for the last 18 months and then constantly moving from historical towards simulated data with each timestep. This estimation of the "moving" average is done in two different ways, as a regular average and as a weighted average. Calculated according to these formulas

$$\begin{aligned} Regular\ average &= \frac{1}{18}\sum_{i=0}^{17} r(t_{18-i}), \\ Weighted\ average &= \frac{1}{18}(w_0(\sum_{i=0}^{5} r(t_{18-i})) + w_1(\sum_{6}^{11} r(t_{18-i})) + w_2(\sum_{12}^{17} r(t_{18-i}))). \end{aligned}$$

Where the weights $w_k = \frac{2^k}{(2^n-1)}$ and with $k = 0, 1, 2, .., n-1$ periods of 6-months intervals. The monthly historical data is denoted as $r(t_{18-i})$ where $i = 1, ..., 18$ is the index of the months. This is the two different time dependent market prices of risk. The weighted average method is constructed with the assumption that the most recent monthly values of the short rate should impact the average short rate the most.

6.4 Exposure Profiles

The main focus of this thesis is to analyse the exposure profiles computed under risk neutral measure agianst the ones computed under the real world impact. To make the result easier to oversight a little more narrow approach has been chosen, i.e. only for the swaptions with maturities $T = 5, 20$ years and at each maturity with parameters $a = 0.01$ and $\sigma = 0.01$.
Figures (6.3)-(6.8) shows all the computed values through time for the exposure measures $PFE_{5\%}$, EE and $PFE_{95\%}$. In all figures, the exposures under risk neutral $\mathbb{Q}$ measure is denoted with "+" and compared with the three models simulated under the historical or real-world $\mathbb{P}$-measure, denoted with "o". All simulations are done with observed yield curves from the 2 january 2009, 2014 and 2017.

to the Euro data.

[2]This is because of the assumption mentioned in the begining about the short rate having short memory.

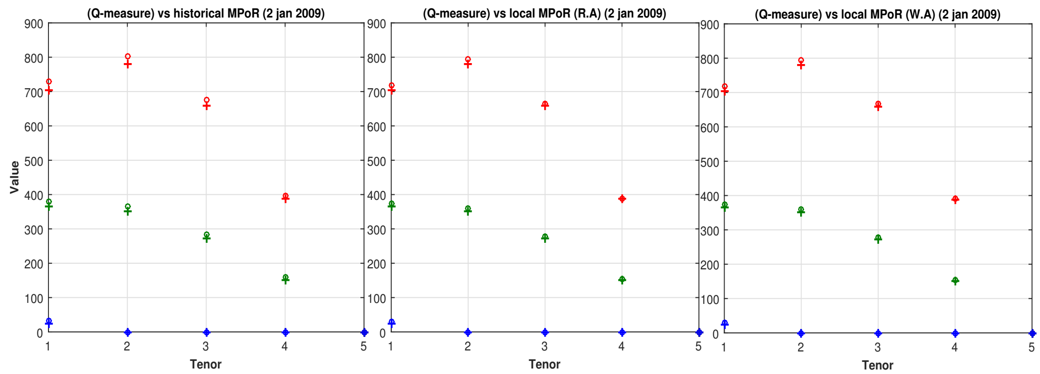


Figure 6.3: The three graphs shows exposure profiles for the three different market prices of risk at 2 january 2009. From left: historical MPoR, local MPoR with regular average and the local MPoR with weighted average.

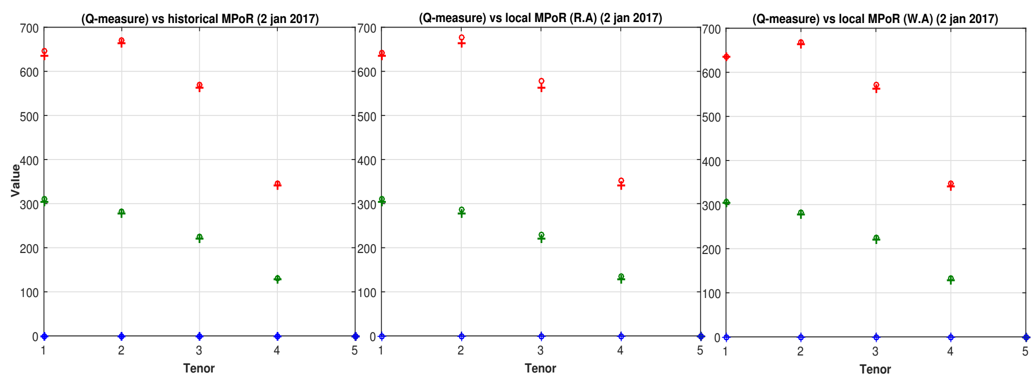


Figure 6.5: The three graphs shows exposure profiles for the three different market prices of risk at 2 january 2017. From the left: historical MPoR, local MPoR with regular average and the local MPoR with weighted average.

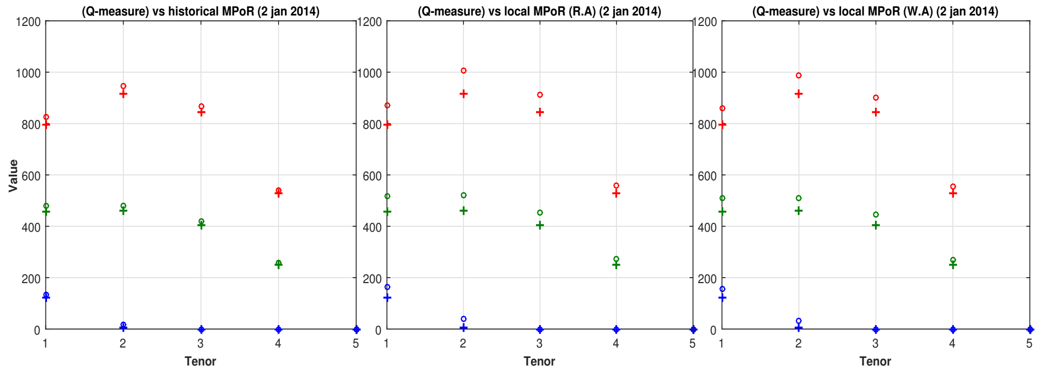


Figure 6.4: The three graphs shows exposure profiles at 2 january 2014. From the left: historical MPoR, local MPoR with regular average and the local MPoR with weighted average.

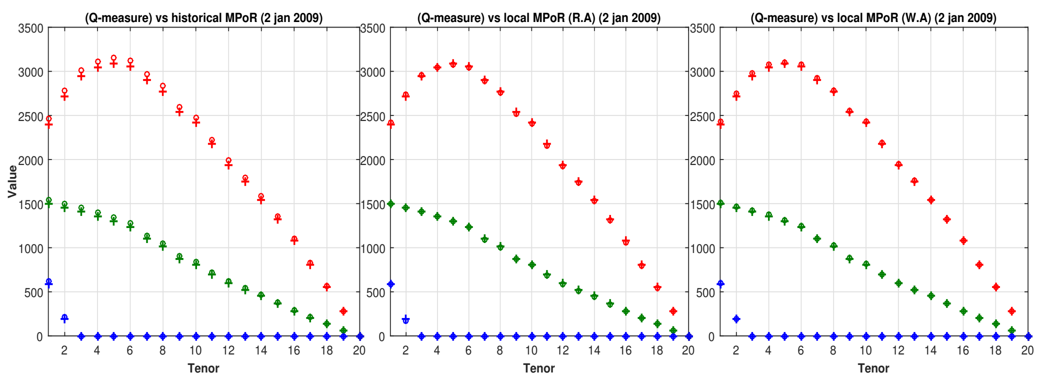


Figure 6.6: The three graphs shows exposure profiles for the three different market prices of risk at 2 january 2009. From the left: historical MPoR, local MPoR with regular average and the local MPoR with weighted average.

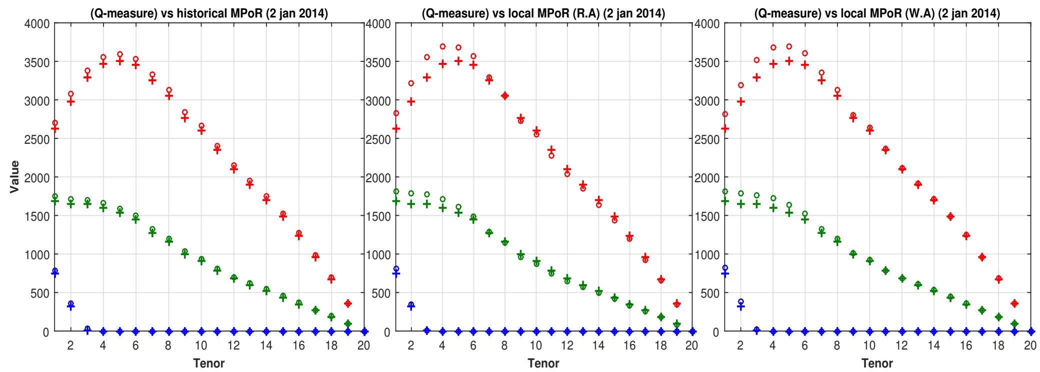


Figure 6.7: The three graphs shows exposure profiles for the three different market prices of risk at 2 january 2014. From the left: historical MPoR, local MPoR with regular average and the local MPoR with weighted average.

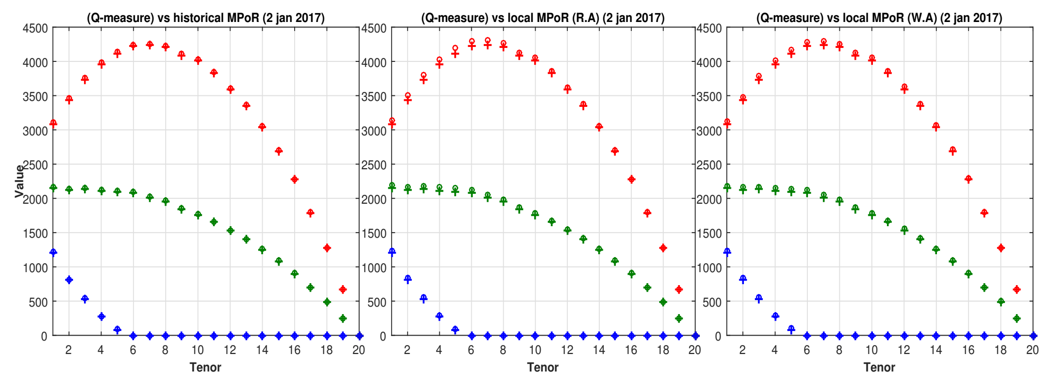


Figure 6.8: The three graphs shows exposure profiles for the three different market prices of risk at 2 january 2017. From the left: historical MPoR, local MPoR with regular average and the local MPoR with weighted average.

Chapter 7

Discussion

Starting with the subfocus of the thesis, which is to validate the algorithm and determine a suitable setup for the parameters of the models. Amongst the values obtained during simulations more than half of the values, 64.5%, were inside a reasonable deviation i.e. less than 1% from the underlying article values. Only 35.5% of the values differ more from the ones in the underlaying paper. A good measure of the deviation would be if more than 50% of the simulated values have a confidence interval includes the comparing values at some signaficance level, i.e. it can not be said with total certainty that the values do differ. The confidence intervals with significance level at $\alpha = 0.05$ states that 34 of the 48 simulated values can not with certainty be said to differ from the ones in the underlying article.
Although there are some values that differ a little to much from the ones compared with and that the standard deviation of the values are a bit to high I have decided to accept the simulated values since more than half of the values can not be said to differ with total certainty. The deviation of the simulated values are most likely a result from a number of uncertainties or unknown factors that follows from the underlying article. Such as setting of the random number generators, the interpolation of the yield curves, the size of the time steps used in the simulations of the short rate, the bundling technique and access to data used for model setup. All theese factors could be a cause for deviation from the values in the underlying article. This deviation will also most likely grow with the increasing number of computations that comes with contracts that are longer. This would explain why the deviations are larger in the 10 year contracts compared to the 5 year contracts.
Figure (6.2) shows the three different attempts of approximating the market

price of risk for the three different yield curves. In all of the three graphs the historical market price of risk is constant. It is affected both by the yield curve value at $T = 20$ and also by the twenty years of historical rates leading up to the time for computations. This means that the size of the historical market price of risk will mostly be decided by the prediction of the twenty year forward rate. The part affected by the long term historical short rate will not differ that much since it is computed from historical long-term average.

Studying the two time dependent approximations of the market prices of risk, which is reffered to as local prices of risk, one can see from Figure (6.2), that the differencies is that the one based on regular average of the last 18-months is less stable than the one based on a wieghted average of the last 18-months. In all of these graphs the local prices of risk are negative near zero and in some sense tend to increase as the shape of the yield curve. If the yield curve is growing fast, so is the value of the local price of risk.

This approximations is done by continously comparing the simulated rates with a historical mean of the short rate. In this thesis I have choose to make the assumption that the short rate doesn't have a memory and therefore only based the historical data on the last eighteen months. This might be the reason why the local prices of risk stabilizes a few years into the simulations. If the historical data were based on a longer periods of time this stabilizing period would in most cases also be longer.

Observing the PFE plots, see Figure (6.3)-(6.8), one can directly see that the differences between simulations done in risk neutral measure and the real world measure follows the graphs of market prices of risk, from Figure (6.2).

From the market prices of risk that are derived from 2009 yield curve, one can see in Figure (6.2a) that most likely only the historical market price of risk will impact the real world simulations. The PFE plots for both the short-term and the long-term contracts, see Figure (6.3) and (6.6), shows that it is only from the historical market price of risk we can observe a visible impact on the real world values.

Moving on to the plots from the 2014 yield curve, Figure (6.4) and (6.7), one can detect a larger real world impact from both the historical- and the two time-dependent market prices of risk. Just as mentioned in previous part about the market price of risk, it shows clearly that in the short-term contracts there are impact all the way throughout the contract but for the long-term contract the impact decreases with time as the two time-dependent

market prices of risk stabilizes around zero somewhere near year 10.
Finaly studying the PFE plots from the 2017 yield curve, see Figure (6.5) and (6.8). In both these sets of PFE plots there is just a very small impact from the time-dependent market price of risk that is approximated with a regular average. This effect is also only visible for a about 10 years, that means througout the short-term contract and for about half way through the long-term contract.
The effects I did notice for the short-term compared to the long-term contracts are that there was more impact on the short-term contracts than the long-term contracts from the two time-dependent market prices of risk. And, the constant historical market price of risk has the impact spread out evenly over time.

Chapter 8

Conclusions

To summarize this thesis and the three attempts of approximating the market price of risk. Going through the results from the simulations it becomes clear that all of the three approximations has drawbacks that makes them no good to use.

If I start with the historical market price of risk, it has a huge drawback as it is constant and that it doesn't adapt to changes in the market through time. Both the two time-dependent approximations of the market price of risk did combine the forward- and historical data better than the historical market price of risk. Then after a few years it did stabilize around zero which only made it active for a part of the contract. It is not very realistic that the market price of risk would not change more for the second part of the contract. That quality, to only impact on the short-term contract, is a drawback since many interest rate derivatives are long-term contracts.

My conclusion is that these approximations of the market price of risk are not complex enough to capture all the market factors that impacts on both short-term and long-term. These appriximations might be seen as a start of something to further develop into a multi-factor model, where one could try to capture and combine the market factors that can impact both on short-term and long-term interest rates. The three approximations that were analyzed in this thesis could be used in combination as an extra safety margin towards exposure and as a way of gaining more insight or a hint of how the impact from the market is changing.

For a very long time, the focus of research have been on the risk-neutral measure because of its role in pricing of derivatives. During the last years, focus have started to change direction towards risk management and that have also led to the realization that the real-world measure should be given more attention. This is why I gained interest in this topic and I think it will be a topic that only will gain more interest over the years to come since risk management is getting more and more important amongst banks and institutions all over the financial world.

Appendix A

Derivation of short rate dynamics

From [2, Björk, p. 335, Eq. 22.36] we have that

$$\mathrm{d}r(t) = (\theta(t) - ar(t))\mathrm{d}t + \sigma \mathrm{d}W(t).$$

Solving this we obtain that

$$r(t) = e^{-a(t-s)}r(s) + \int_s^t e^{-a(t-u)}\theta(u)\mathrm{d}u + \int_s^t \sigma e^{-a(t-u)}\mathrm{d}W(u). \tag{A.1}$$

The problem now is to evaluate the integral

$$\int_s^t e^{-a(t-u)}\theta(u)\mathrm{d}u$$

for the case where θ is calibrated to the initial forward curve $f^*(0, \cdot)$.
From [2, Björk, p. 336, Eq. 22.44] we have that

$$f^*(0, T) = e^{-aT}r(0) + \int_0^T e^{-a(T-u)}\theta(u)\mathrm{d}u - \frac{\sigma^2}{2a^2}\left(1 - e^{-aT}\right)^2.$$

We can then rewrite this relationship to obtain

$$\int_0^T e^{au}\theta(u)\mathrm{d}u = e^{aT}f^*(0, T) + e^{aT}\frac{\sigma^2}{2a^2}\left(1 - e^{-aT}\right)^2 - r(0).$$

Evaluating this for $T = t$ and $T = s$ and taking the difference we obtain

$$\int_s^t e^{au}\theta(u)\mathrm{d}u = e^{at}f^*(0,t)+e^{at}\frac{\sigma^2}{2a^2}\left(1-e^{-at}\right)^2-e^{as}f^*(0,s)-e^{as}\frac{\sigma^2}{2a^2}\left(1-e^{-as}\right)^2.$$

Finally by multiplying both sides with e^{-at} we obtain

$$\int_s^t e^{-a(t-u)}\theta(u)\mathrm{d}u = f^*(0,t)-e^{-a(t-s)}f^*(0,s)+\frac{\sigma^2}{2a^2}\left(\left(1-e^{-at}\right)^2-e^{-a(t-s)}\left(1-e^{-as}\right)^2\right).$$

We then plug this into (A.1) to get that

$$\begin{aligned} r(t) &= f^*(0,t)+\frac{\sigma^2}{2a^2}\left(1-e^{-at}\right)^2+e^{-a(t-s)}\left(r(s)-f^*(0,s)-\frac{\sigma^2}{2a^2}\left(1-e^{-as}\right)^2\right) \\ &\quad+\int_s^t \sigma e^{-a(t-u)}\mathrm{d}W(u). \end{aligned} \tag{A.2}$$

Appendix B

Market-implied default probabilities

This section will be about finding an approximate relation between CDS and the hazard rate in order to approximate market-implied default probabilities. As mentioned, in the section about default probabilities , the hazard rate γ defines the probability of default in an infitely small interval dt, with condition on no prior default before time u, as $\gamma_u dt$. The hazard rate is related to the cumulative default probability from

$$F(u) = 1 - S(u) = 1 - e^{-\gamma u}, \tag{B.1}$$

here we have assumed a constant hazard rate. From [4, Gregory, Appendix 6B] we get that an approximate relation between the hazard rate and the CDS premium is

$$\gamma \approx \frac{X_{CDS}}{(1 - R_C)}, \tag{B.2}$$

where X_{CDS} is the CDS premium(as percentage) and R_C is the assumed recovery rate(also in percentage). This approximation assumes that the CDS curve is flat i.e., the CDS premiums for all maturities are equal.
If we combine equation (B.1) and the approximate relation in equation (B.2) we can derive an approximate relation between the cumulative default probability and the CDS premium at a certain maturity $T(X_T^{CDS})$ as

$$F(T) = 1 - e^{-\frac{X_T^{CDS}}{(1-R_C)}T} \approx -\frac{X_T^{CDS}}{(1 - R_C)}T. \tag{B.3}$$

Equation (B.3) generally works as a good approximation of market-implied default probabilities although to compute the probabilities acurately one need to solve numerically for the correct hazard rate, assuming a certain underlying functional form.

Appendix C

Bundling - Recursive Bifurcation Algorithm

If bundling K_s grid points at time t_n, given by $X(t_n, w_k)$, where $k = 1, ..., K_s$ then following steps are performed recursively.

1. Start by computing the mean of the given set of grid points at time t_n,

$$\mu_n^s = \frac{1}{K_s} \sum_{k=1}^{K_s} X(t_n, w_k).$$

2. Bundling the grid points is done by dividing the grid points into two groups, depending on whether the grid points, price of the underlying asset, is greater than or less than the mean of the grid points, mean of the underlying asset pricees, for that given set of grid points:

$$\begin{aligned} \mathcal{B}^1(t_n, w_k) &= 1(X(t_n, w_k) > \mu_n^s), \\ \mathcal{B}^2(t_n, w_k) &= 1(X(t_n, w_k) \leq \mu_n^s), \end{aligned}$$

   for $k = 1, ..., K_s$. $\mathcal{B}^1(t_n, w_k)$ then returns *true* when the state value is greater than the mean, μ_n^s, and belongs to bundle 1. $\mathcal{B}^2(t_n, w_k)$ then returns *true* when the state value is less than or equal to the mean and belongs to bundle 2.

3. Each bundle can be split again by returning to step 1, until they are the right size or until there enough bundles.

Appendix D

Moments for the HW1F

Let $M_k(s,t) = \mathbb{E}[r(t)^k|s]$ be the k-th moment. The three first moments for the HW1F is given by,

$$M_1(t,s) = f^*(0,t) + \frac{\sigma^2}{2a^2}(1-e^{-at\Delta})^2 + e^{-a\Delta}Y(s) \tag{D.1}$$

$$M_2(t,s) = M_1^2(s,t) + \frac{\sigma^2}{2a}(1-e^{-2a\Delta}) \tag{D.2}$$

$$M_3(t,s) = M_1^3(s,t) + 3M_1(s,t)(M_2(s,t) - M_1^2(s,t)). \tag{D.3}$$

The formulas for the moments has been derived in colaboration with my mentor Magnus Wiktorsson.

Appendix E

Validation of simulated data

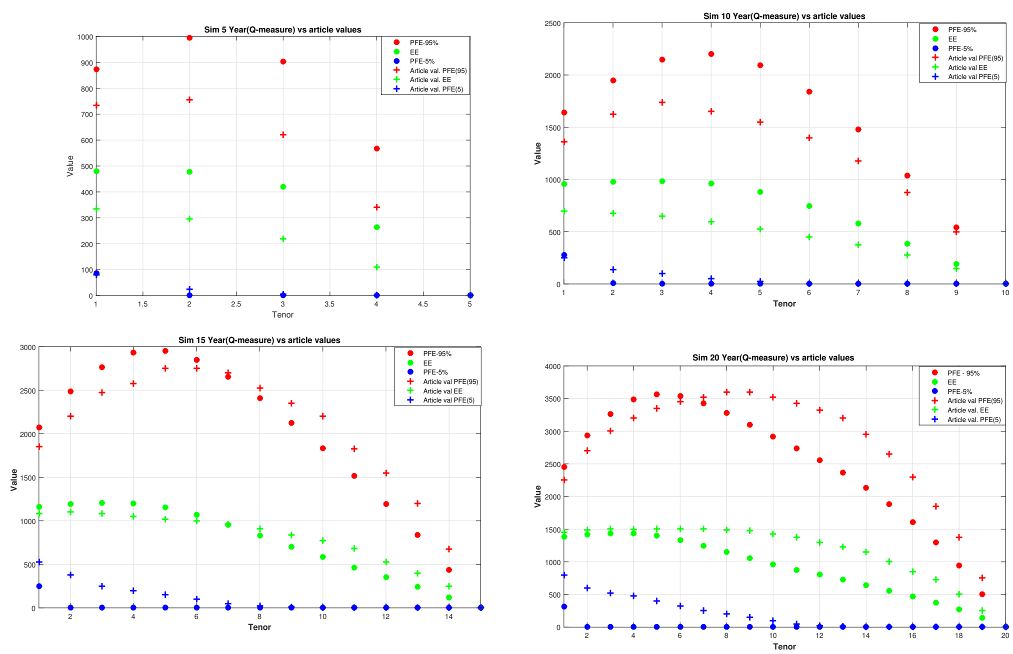


Figure E.1: Above, the simulated value of $PFE_{5\%}$, EE and $PFE_{95\%}$ for different maturities has been plotted, as dots, against the approximate values for the same maturities from the underlying article, as +.[9]

The SGBM-data for the 5 year swaption under ℚ-measure						
Parameters			Results from simulations		Results underlying article	
MN	a	σ	Price	CVA	Price	CVA
0.8	0.01	0.01	471.57 (4.76)	51.60 (0.20)	477.30 (0.1732)	51.70 (0.0144)
1.0	0.01	0.01	542.04 (5.04)	58.42 (0.20)	548.12 (0.1827)	58.58 (0.0140)
1.2	0.01	0.01	595.50 (4.98)	63.37 (0.21)	599.25 (0.1824)	63.41 (0.0141)
0.8	0.01	0.02	733.33 (9.05)	76.61 (0.33)	736.43 (0.4557)	76.31 (0.0347)
1.0	0.01	0.02	801.68 (8.95)	82.53 (0.32)	801.25 (0.4585)	82.29 (0.0327)
1.2	0.01	0.02	848.42 (8.40)	86.58 (0.32)	846.78 (0.4973)	86.45 (0.0337)
0.8	0.02	0.01	469.86 (4.55)	51.35 (0.18)	471.07 (0.1776)	51.06 (0.0149)
1.0	0.02	0.01	538.92 (5.55)	58.13 (0.21)	542.15 (0.1792)	57.99 (0.0140)
1.2	0.02	0.01	591.07 (4.58)	63.15 (0.19)	593.49 (0.1783)	62.86 (0.0132)
0.8	0.02	0.02	726.78 (8.14)	75.87 (0.34)	723.13 (0.4520)	75.03 (0.0347)
1.0	0.02	0.02	793.76 (8.84)	81.75 (0.33)	788.16 (0.4595)	80.96 (0.0343)
1.2	0.02	0.02	841.70 (8.61)	85.86 (0.32)	833.95 (0.4567)	85.13 (0.0324)

The SGBM-data for the 10 year swaption under ℚ-measure						
Parameters			Results from simulations		Results underlying article	
MN	a	σ	Price	CVA	Price	CVA
0.8	0.01	0.01	954.29 (10.32)	177.26 (0.61)	947.3 (0.2326)	175.04 (0.0340)
1.0	0.01	0.01	1167.10 (10.92)	212.32 (0.69)	1187.0 (0.2355)	215.55 (0.0346)
1.2	0.01	0.01	1362.86 (10.99)	244.41 (0.62)	1367.7 (0.2259)	245.44 (0.0327)
0.8	0.01	0.02	1502.30 (17.94)	284.08 (1.12)	1584.1 (0.5393)	283.59 (0.0747)
1.0	0.01	0.02	1736.63 (20.17)	321.33 (1.25)	1805.5 (0.5513)	319.16 (0.0760)
1.2	0.01	0.02	1888.92 (18.63)	344.76 (1.21)	1966.0 (0.5317)	344.73 (0.0691)
0.8	0.02	0.01	939.61 (9.69)	173.81 (0.60)	921.2 (0.2283)	170.12 (0.0337)
1.0	0.02	0.01	1151.08 (9.89)	209.00 (0.68)	1162.7 (0.2170)	211.08 (0.0324)
1.2	0.02	0.01	1348.73 (10.25)	241.42 (0.51)	1345.0 (0.2367)	241.23 (0.0344)
0.8	0.02	0.02	1464.06 (16.41)	275.71 (1.12)	1529.5 (0.5057)	273.61 (0.0717)
1.0	0.02	0.02	1705.70 (15.55)	313.36 (0.95)	1752.3 (0.5428)	309.44 (0.0763)
1.2	0.02	0.02	1853.08 (17.13)	336.81 (1.02)	1914.2 (0.5630)	335.13 (0.0729)

Table E.1: The simulated values and CVAs for the bermudan swaption with 5 and 10 years maturity. The calculations was done with 4096/8192 simulated paths for the first/second pass of the SGBM-algorithm.

Validation of the SGBM-data for 5/ 10 year bermudan swaption under $\mathbb{Q}$-measure						
Parameters			Deviation from value in underlying article			
MN	a	σ	Price 5Y	CVA 5Y	Price 10Y	CVA 10Y
0.8	0.01	0.01	5.73 (1.20%)	0.10 (0.19%)	6.99 (0.74%)	2.22 (1.27%)
1.0	0.01	0.01	6.08 (1.11%)	0.16 (0.27%)	19.90 (1.68%)	3.23 (1.50%)
1.2	0.01	0.01	3.75 (0.63%)	0.04 (0.06%)	4.84 (0.35%)	1.03 (0.42%)
0.8	0.01	0.02	3.06 (0.42%)	0.30 (0.39%)	81.80 (5.16%)	0.49 (0.17%)
1.0	0.01	0.02	0.43 (0.05%)	0.24 (0.29%)	68.87 (3.81%)	2.17 (0.68%)
1.2	0.01	0.02	1.64 (0.19%)	0.13 (0.15%)	77.08 (3.92%)	0.03 (0.008%)
0.8	0.02	0.01	1.21 (0.26%)	0.29 (0.57%)	18.40 (2.00%)	3.69 (2.17%)
1.0	0.02	0.01	3.23 (0.60%)	0.14 (0.24%)	11.62 (1.00%)	2.08 (0.99%)
1.2	0.02	0.01	2.42 (0.41%)	0.29 (0.46%)	3.73 (0.28%)	0.19 (0.08%)
0.8	0.02	0.02	3.65 (0.50%)	0.84 (1.12%)	65.44 (4.28%)	2.10 (0.77%)
1.0	0.02	0.02	5.60 (0.71%)	0.79 (0.98%)	46.60 (2.66%)	3.92 (1.27%)
1.2	0.02	0.02	7.75 (0.93%)	0.73 (0,86%)	61.12 (3.19%)	1.68 (0.50%)
Average deviation			3.71 (0.58%)	0.34 (0.47%)	38.87 (2.42%)	1.90 (0.82%)

Table E.2: In the table all the deviations, in basispoint and percentage, from the values in the underlying article.

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