# Generalized integration operators on Hardy spaces

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## Populärvetenskapling sammanfattning

Inom komplex analys och operatorteori studerar man vanligtvis begränsade linjära operatorer mellan Banachrum bestående av analytiska funktioner. Detta görs för att kunna erhålla information om själva Banachrummets struktur. Ett klassiskt exempel är Cesaros medelvärdes operator på $H^p$, ett Hardyrum bestående av analytiska funktioner. En generalisering av denna operator är det såkallade Cesaros generaliserad operator, $T_g$, som kan spåras tillbaka till arbetet av Ch. Pommerenke, 1970. Operatorns egenskaper har varit ett aktivt forskningsområde i de senaste 20 åren. I detta arbete, som är inspirerad av studien av $T_g$, försöker vi ge svar till några frågor angående operatorns variation.

## Acknowledgments

## Abstract

Inspired by the study of generalized Cesáro operator $T_g$ (see [1]) we study a variation of this operator,namely $P_{g,a}$, depending on an analytic symbol $g$ and an $n$-tuple of complex numbers, $a$. Regarding the boundedness properties of this operator we prove that $P_{g,a}$ is a bounded linear operator from $H^p$ to itself if and only if $g$ is an analytic function of bounded mean oscillation and compact if and only if $g$ is of vanishing mean oscillation. Furthermore in the special case $n = 2, a = 0$ we completely characterized the functions $g$ for which $P_{g,a}$ is bounded from $H^p$ to $H^q, 0 < p, q < \infty$. As an application of our theorem we prove a factorization theorem for any derivative of an $H^p$ function, and also a theorem about solutions of complex linear differential equations.

# Contents

# 1    Spaces of analytic functions

In this section we introduce the spaces of analytic functions with which we will be concerned in the sequel. We are mostly interested in spaces defined in the unit disc $\mathbb{D}$ of the complex plane. We denote by $\mathbb{T}$ the boundary of $\mathbb{D}$. Let also $dm$ be the normalized Lebesgue measure on $\mathbb{T}$ and $dA$ the normalized area measure on $\mathbb{D}$. With $H(\mathbb{D})$ we denote the space of analytic function in the unit disc.

For $0 < p < \infty$ the usual Hardy space $H^p$ consists of the analytic functions $f$ in $\mathbb{D}$ such that

$$\|f\|_p = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}} < \infty.$$

It is a standard application of Hölder's inequality to prove that $\|\cdot\|_p$ defines a norm for $1 \le p$ such that $H^p$ is a Banach space. For $0 < p < 1$, we can turn $H^p$ into a metric space by defining $d_p(f,g) = \|f - g\|_p$. In any case the set of polynomials is a dense subset of $H^p$.

Furthermore, if $f \in H^p$, its growth near the boundary subjects to a certain restriction. Namely,

$$|f(z)| \le \left(\frac{1+|z|}{1-|z|}\right)^{1/p} \|f\|_p, z \in \mathbb{D}.$$

If $f$ is in $H^2$ then we can express its norm only in terms of its derivative by the classical Littlewood-Paley formula:

$$\|f\|_2^2 = |f(0)|^2 + 2 \int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|z|} dA(z). \tag{1}$$

The space of bounded analytic functions is denoted by $H^\infty$, and is a Banach space with the supremum norm.

Also if $f \in H(\mathbb{D})$ and $a \in \mathbb{D}$ we define $f_a(z) = f((z+a)/(1+\bar{a}z)) - f(a)$ to be the hyperbolic translate of $f$. The space $BMOA$ of function of bounded mean oscillation consists of the functions $f \in H^2$ such that

$$\|f\|_* = |f(0)| + \sup\{\|f_a\|_2 : a \in \mathbb{D}\} < \infty.$$

We say that $f$ is of vanishing mean oscillation or VMOA if

$$\lim_{|a| \longrightarrow 1^-} \|f_a\|_2 = 0.$$

VMOA coincides with the closure of polynomials in the $BMOA$ norm. Another equivalent characterization of $BMOA$ and $VMOA$ functions is in terms of Carleson measures. A Carleson measure is a Borel measure $\nu$ on the unit disc such that there exists a positive constant $C > 0$ which satisfies

$$\nu(S) \leq Ch,$$

for every set $S$ of the form

$$S = \{re^{i\theta} : 1 - h \leq r < 1, \theta_0 < \theta < \theta_0 + h\}.$$

Which is further equivalent to $H^p \subset L^p(\mathbb{D}, \nu)$, for all $0 < p < \infty$. A vanishing Carleson measure, is a Carleson measure such that

$$\frac{\nu(S)}{h} \to 0,$$

as $h \to 0$.

The characterization of $BMOA$ functions in terms of Carleson measures is the followng. $f \in BMOA$ if and only if $\mu_f(z) = |f'(z)|^2 \log \frac{1}{|z|} dA(z)$ is a Cerleson measure and $f \in VMOA$ if and only if $\mu_f$ is a vanishing Carleson measure.

Another space of analytic functions which will turn out to be useful for our purposes is the Bloch space $\mathscr{B}$, consisting of analytic functions $f$ such that

$$\|f\|_{\mathscr{B}} = |f(0)| + \sup_{z \in \mathbb{D}}(1 - |z|^2)||f'(z)| < \infty. \tag{2}$$

There exists an equivalent formulation of condition (2) in terms of the $n-th$ derivative of $f$. Specifically, if $f$ is any holomorphic function in $\mathbb{D}$, and $n \in \mathbb{Z}_+$, there exist $C_1(n), C_2(n) > 0$ such that

$$C_1(n)\|f\|_{\mathscr{B}} \leq \sup_{z \in \mathbb{D}}(1 - |z|^2)^n |f^{(n)}(z)| \leq C_2(n)\|f\|_{\mathscr{B}}.$$

The closure of the polynomials in $\| \cdot \|_{\mathscr{B}}$ is called little Bloch space, and denoted by $\mathscr{B}_0$.

We will also make use of the Lipschitz spaces of analytic functions $\Lambda_\alpha, 0 < \alpha \leq 1$ which consist of analytic functions in $\mathbb{D}$ such that there exists a constant $C$ which satisfies

$$|f(z) - f(w)| \leq C|z - w|^\alpha, \text{ for all } z, w \in \mathbb{D},$$

or equivalent the space of holomorphic functions in the unit disc such that

$$|f'(z)| = \mathscr{O}((1 - |z|)^{\alpha-1}), \text{ when} |z| \to 1.$$

The corresponding "little oh" space $\lambda_\alpha$ is the space of holomorphic functions such that
$$|f'(z)| = o((1 - |z|)^{\alpha-1}), \text{ when } |z| \to 1.$$

There are also various inclusions between these spaces of analytic functions. We simply list the ones that are of some importance for our subsequent work.

$$H^p \subset H^q \text{ if } p > q$$
$$\Lambda_\alpha \subset BMOA \subset H^p$$
$$H^\infty \subset BMOA \subset \mathscr{B}$$
$$VMOA \subset \mathscr{B}_0.$$

Basic facts about the aforementioned spaces can be found for example in [2].

# 2    The operator $T_g$

Let now $g$ be a fixed analytic function in $\mathbb{D}$ and consider the linear operator

$$T_g f(z) = \int_0^z f(t)g'(t)dt, f \in H(\mathbb{D}).$$

The prototype for the operators of this kind is the Cesáro operator

$$Cf(z) = \int_0^z \frac{f(t)}{1-t}dt,$$

which coincides with $T_g$ for $g(z) = \log(1/(1-z))$. It was probably already known to Hardy that C acts as a bounded operator on $H^2$ [3]. But it was not until Pommerenke [4] gave the following elegant proof of the analytic John-Nirenberg inequality that the operators $T_g$ have been an object of intensive study.

**Proposition 1.** *Let $g \in BMOA$ with $g(0) = 0$ and $\|g\|_* \leq 1$ then there exist absolute constants $A, B > 0$ such that*

$$\| \exp(Ag)\|_2 \leq B.$$

*Proof.* If $f$ is a function in $H^2$ from the Paley-Littlewood formula we get

$$\|T_g f\|_2^2 = 2 \int_\mathbb{D} |(T_g f)'(z)|^2 \log \frac{1}{|z|} dA(z)$$

$$= 2 \int_\mathbb{D} |f(z)|^2 |g'(z)|^2 \log \frac{1}{|z|} dA(z)$$

$$\leq C \|f\|_2^2 \|g\|_*^2.$$

Since such functions $f$ are dense in $H^2$, $T_g$ is bounded. But the spectrum of $T_g$, $\sigma(T_g|H^2)$, is contained in $\bar{D}(0, C\|g\|_*)$. Hence if $|\lambda|^{-1} > C\|g\|_*$ there exists a unique $f \in H^2$ such that $f - \lambda T_g f = 1$. Solving this equation explicitly we get $f = e^{\lambda g} = (I - \lambda T_g)^{-1}1 \in H^2$ and also

$$\|e^{\lambda g}\|_2 \leq \frac{1}{1 - |\lambda|C\|g\|_*} \leq \frac{1}{1 - |\lambda|C},$$

by the power series expansion of $(I - \lambda T_g)^{-1}$.                    □

Another source of motivation for studying the operator $T_g$ is a theorem of Hardy and Littlewood which can be found in [2, Theorem 5.12].

**Theorem 2.** *If $f' \in H^p, 0 < p < 1$ then $f \in H^{p/(1-p)}$.*

In the language of operators this means that $T_z : H^p \to H^{p/(1-p)}$ is a bounded linear operator for $0 < p < 1$.

For a large class of spaces of analytic functions the symbols $g$ for which $T_g$ is a bounded (or compact) linear operator have completely characterized. Of special interest for us is when $T_g$ is bounded between Hardy spaces. Aleman and Cima in [1] gave the characterization in this case. Although the original proof was simplified, the result reads as follows [1, Theorem 1].

**Theorem 3.** *Let $p, q > 0$. Then,*

(i) *For $p > q$ $T_g$ maps $H^p$ into $H^q$ if and only if $g \in H^s$, where $\frac{1}{q} - \frac{1}{p} = \frac{1}{s}$,*

(ii) *$T_g$ maps $H^p$ into itself if and only if $g \in BMOA$,*

(iii) *For $p < q$ and $\frac{1}{p} - \frac{1}{q} \leq 1$, $T_g$ maps $H^p$ into $H^q$ if and only if $g \in \Lambda_{\frac{1}{p} - \frac{1}{q}}$,*

(iv) *If $\frac{1}{p} - \frac{1}{q} > 1$, and $T_g$ maps $H^p$ into $H^q$ then $g$ is constant.*

We are not going to give a proof of this theorem here, which can be found for example in [1] or [5], but we will say a few words about the methods used in the proof. The proof that we have in mind here is not the original one from [1] but a simplified version of it which can be found in [5]. Regarding sufficiency, part (i) follows immediately by the theorem of Fefferman and Stein for the square function (see the discussion in the beginning of section 3). Where that of part (ii) can be settled using again the square function and a clever trick due to Aleksandrov and Peller [6]. Part (iii), uses Carleson type measures and specifically a theorem of Duren [2, Theorem 9.4]. The key in all cases is to find a way to estimate the norm of a function in the target space in terms of its derivative.

The proof of necessity in part (iii) is based on the following proposition [1, Theorem 3].

**Proposition 4.** *Let $p > 0$ and $g \in H^p$. For $a \in \mathbb{D}$, let $\phi_a(z) = (z + a)/(1 + \bar{a}z)$, $k_a(z) = (1 - |a|^2)^{1/p}/(1 - \bar{a}z)^{2/p}$. Then for $0 < t < p/2$, there exists a constant $A_{p,t} > 0$ (depending only on $p$ and $t$) such that $\|g \circ \phi_a - g(a)\|_t^t \leq A_{p,t} \|T_g k_a\|_p^t$.*

Necessity of (i) follows by another interesting lemma [1, Section 4: Proposition].

**Lemma 5.** *Let $g$ analytic in $\mathbb{D}$ and $p > q > 0$ be such that $T_g H^p \subset H^q$. Then $T_g H^{p'} \subset T_g H^{q'}$ whenever $p > p' > q' > 0$ and $1/q - 1/p = 1/q' - 1/p'$.*

Finally necessity of (iii) and (iv) follows immediately by the standard growth estimates for $H^p$ functions.

To demonstrate the power of Theorem 3 we note that it yields a substantial improvement of the Hardy-Littlewood Theorem (Theorem 2).

**Theorem 6.** *[1, Section 1: Theorem] Let $0 < p < 1$ and $p_1 > 0, p_2 > 1$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. If $f'$ is analytic in the unit disc with a factorization of the form $f' = f_1 f_2$ where $f_1 \in H^{p_1}$ and $|f_2(z)| = \mathscr{O}((1 - |z|^{-1/p_2})), |z| \to 1$, then $f \in H^{p/(1-p)}$.*

*Proof.* If $f$ is as in the statement, then

$$f(z) = f(0) + T_{T_z f_2} f_1.$$

By our assumptions, $T_z f_2 \in \Lambda_{1 - \frac{1}{p_2}}$. Therefore, by applying part (iii) of Theorem 3 to the operator $T_{T_z f_2} : H^{p_1} \to H^{p/(1-p)}$, we conclude that $f \in H^{p/(1-p)}$. $\square$

To understand why Theorem 6 is an improvement of the Hardy-Littlewood theorem, note that if $f' \in H^p$ then it can be factorized as $f' = f_1 f_2$ where $f_i \in H^{p_i}, i = 1, 2$ and $p_i, i = 1, 2$ as in the statement of Theorem 6. But, the condition $|f_2(z)| = \mathscr{O}((1 - |z|)^{-\frac{1}{p_2}})$ is much weaker than assuming that $f_2 \in H^{p_2}$. (see for example [2, Theorem 5.10]).

The compactness of the operator $T_g$ is characterized by the corresponding "little-oh" conditions [1, Corollary 1].

**Theorem 7.** *Let $p, q > 0$. Then,*

   *(i) For $p > q$ and $g \in H^s$, $\frac{1}{s} = \frac{1}{q} - \frac{1}{p}$, $T_g : H^p \to H^q$ is compact,*

   *(ii) $T_g : H^p \to H^p$ is compact if and only if $g \in VMOA$,*

   *(iii) For $p < q$ and $\frac{1}{p} - \frac{1}{q} < 1$, $T_g : H^p \to H^q$ is compact if and only if $g \in \lambda_{\frac{1}{p} - \frac{1}{q}}$.*

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# 3 Preliminaries

Let us now collect a few results that will be helpful in the sequel.

First we will look at some variations of the Hardy-Stein identity (1). To do so we shall introduce the Stolz angle $\Gamma_\sigma(e^{i\theta})$ with vertex at $e^{i\theta}$ and aperture $\sigma$. That is, the interior of the convex hull of the point $e^{i\theta}$ and the disc $D(0, \sigma)$. Then we can define the so called square function or Lusin area function $S_f$.

$$S_f(\zeta) = \Big( \int_{\Gamma_\sigma(\zeta)} |f'|^2 dA \Big)^{1/2}, \zeta \in \mathbb{T}.$$

A well known result of Fefferman and Stein (see [7]) states that if $0 < p < \infty$ there exist constants $C_1, C_2 > 0$, depending only on $\sigma$ and $p$, such that

$$C_1 \|f\|_p^p \le |f(0)|^p + \int_{\mathbb{T}} S_f^p dm \le C_2 \|f\|_p^p,$$

for any $f$ analytic in $\mathbb{D}$. Another related function is the Paley-Littlewood $G-$function, which is defined by

$$G(f)(t) = \Big( \int_0^1 |f'(re^{it})|^2 (1-r) dr \Big)^{1/2}, \ \ t \in \mathbb{R}.$$

The Paley-Littlewood $G-$ function enjoys the same property as the Lusin area function, i.e. there exist $C_1, C_2 > 0$, depending only on $\sigma$ and $p$, such that

$$C_1 \|f\|_p^p \le |f(0)|^p + \int_{\mathbb{T}} G^p(f) dm \le C_2 \|f\|_p^p, \tag{3}$$

for any $f$ analytic in $\mathbb{D}$. For more information about these functions the reader is referred to [7] and [8].

For our purposes, we need a version of the Paley-Littlewood $G-$function, involving only the $n-th$ derivative of $f$. We start with a lemma.

**Lemma 8.** *Let $f$ be an analytic function in the unit disc, then for $z \in \mathbb{D}$*

$$|f^{(n)}(z)|^2 \le \frac{n!(n-1)!2^{2n}}{(1-|z|)^{2n}} \int_{D(z, \frac{1-|z|}{2})} |f'(\zeta)|^2 dA(\zeta).$$

*Proof.* It is a standard exercise to prove that

$$\int_{\mathbb{D}} |f'(\zeta)|^2 dA(\zeta) = \sum_{k \geq 0} k|a_k|^2.$$

Then,

$$|f^{(n)}(0)|^2 = (n!)^2 |a_n|^2 \leq n!(n-1)! \sum_{k \geq 0} k|a_k|^2 = n!(n-1)! \int_{\mathbb{D}} |f'(\zeta)|^2 dA(\zeta).$$

$$(4)$$

Now let $z \in \mathbb{D}$ fixed and set $r = \frac{1-|z|}{2}$. Applying (4) to the function $f_z(\zeta) = f(z + r\zeta), \zeta \in \mathbb{D}$, and using a change of variables:

$$r^{2n}|f^{(n)}(z)|^2 \leq n!(n-1)! \int_{\mathbb{D}} r^2 |f'(z+r\zeta)|^2 dA(\zeta)$$

$$= n!(n-1)! \int_{D(z,\frac{1-|z|}{2})} |f'(\zeta)|^2 dA(\zeta),$$

which gives the desired inequality. $\qquad\square$

Now, let $f$ analytic in $\mathbb{D}$. We define the Paley-Littlewood $G_k-$function of order $k$ to be

$$G_k(f)(t) = \left( \int_0^1 |f^{(k)}(re^{it})|^2 (1-r)^{2k-1} dr \right)^{1/2}. \qquad (5)$$

**Proposition 9.** *Let $p > 0$ and $f$ analytic in $\mathbb{D}$ with $f^{(i)}(0) = 0, 0 \leq i < k$ then, for any $k \in \mathbb{Z}_+$, there exists constants $C_1, C_2$ depending only on $p$ and $k$ such that*

$$C_1 \|f\|_{H^p} \leq \|G_k(f)\|_{L^p} \leq C_2 \|f\|_{H^p}.$$

*Proof.* We prove the left inequality for $k = 2$, the general case follows by induction. First, let $f$ be analytic in an open set containing the closure of $\mathbb{D}$. Then for $t \in \mathbb{R}$ fixed

$$G_1^2(f)(t) = \int_0^1 |f'(re^{it})|^2 (1-r) dr = \frac{1}{2} \int_0^1 \frac{\partial |f'(re^{it})|^2}{\partial r} (1-r)^2 dr$$

$$= \int_0^1 Re(e^{it} f''(re^{it}) \bar{f}'(re^{it}))(1-r)^2 dr$$

$$\leq \int_0^1 |f''(re^{it})||f'(re^{it})|(1-r)^2 dr$$

$$\leq G_1(f)(t)G_2(f)(t),$$

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by Cauchy-Schwarz. Dividing through by $G_1(f)(t)$ we have the result for $f$ analytic in a larger disc. Now if $f$ is an arbitrary analytic function, fix $0 < \rho < 1$ and consider the dilations $f_\rho(z) = f(\rho z)$. Then,

$$G_1(f_\rho)(t) \leq \rho^4 \int_0^1 |f''(r\rho e^{it})|^2 (1-r)^3 dr$$
$$\leq \rho \int_0^\rho |f''(u e^{it})|^2 (1-u)^3 du.$$

Then by taking liminf in both sides as $\rho \longrightarrow 1^-$ and applying Fatou's lemma on the left and monotone convergence on the right we conclude that $G_1(f) \leq G_2(f)$. And the result follows by (3).

To prove the right inequality, we will use Lemma 8. First note that for any $z = re^{i\theta} \in \mathbb{D}, D(re^{i\theta}, \frac{1-r}{2}) \subset \Gamma_{\frac{1}{2}}(e^{i\theta})$, which, together with Lemma 8, justifies the following calculation

$$G_k^2(f)(\theta) = \int_0^1 |f^{(k)}(re^{i\theta})|^2 (1-r)^{2k-1} dr$$
$$\leq C_k \int_0^1 \int_{D(re^{i\theta}, \frac{1-r}{2})} |f'(\zeta)|^2 dA(\zeta)(1-r)^{-1} dr$$
$$= C_k \int_{\Gamma_{\frac{1}{2}}(e^{i\theta})} |f'(\zeta)|^2 \int_0^1 \chi_{D(r, \frac{1-r}{2})}(\zeta)(1-r)^{-1} dr dA(\zeta).$$

It is routine to check that if $|\zeta| < \frac{3r-1}{2}$ or $|\zeta| > \frac{1+r}{2}$ then $\zeta \notin D(r, \frac{1-r}{2})$. Hence

$$\int_0^1 \chi_{D(r, \frac{1-r}{2})}(\zeta)(1-r)^{-1} dr \leq \int_{2|\zeta|-1}^{\frac{2|\zeta|+1}{3}} (1-r)^{-1} dr = \log 3$$

Therefore we have proven that $G_k^2(f)(\theta) \leq S_f^2(\theta)$ and the estimate follows by Fefferman-Stein's theorem. $\qquad\square$

The following result was proved by Aleman and Cima in [1]. We include a proof here because it contains an interesting technique.

**Proposition 10.** *For every $F \in H^p, p > 0$ there exists $G \in BMOA$ and $f \in H^p$ such that $F' = G'f$.*

*Proof.* Let

$$F(z) = \Big( \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} |f(\zeta)|^{p/2} dm(\zeta) \Big)^{2/p}.$$

Then by M. Riesz's theorem [2, Theorem 4.1], $F \in H^p$, and since $F$ has positive real part $\log F \in BMOA$. Also $|F^{p/2}(\zeta)| \geq Re(F^{p/2}(\zeta)) = |f(\zeta)|^{p/2}$. Let $h = f/F \in H^\infty$. Then $f = hF$, hence

$$f' = h'F + hF' = F(h' + h\frac{F'}{F}) = F(h' + h(\log F)') = Fg',$$

where

$$g(z) = \int_0^z h'(t) + h(t)(\log F)'(t)dt.$$

Then by the necessity part of Theorem 3(ii) it follows that $g \in BMOA$. $\square$


We conclude this section with a lemma of technical nature, which will be used in the sequel. Its proof essentially involves only linear algebra. For $\gamma > 0$ we use the notation $(\gamma)_0 = 1, (\gamma)_k = \gamma(\gamma + 1) \cdots (\gamma + k - 1)$ for $k \geq 1$.

**Lemma 11.** *Suppose that $f_0, f_2, \ldots f_{n-1}$ are complex valued functions on the unit disc (not necessarily analytic), such that for any $\gamma \in \mathbb{R}$ sufficiently large there exists $C_\gamma > 0$ such that*

$$\Big| \sum_{k=0}^{n-1} f_k(z)(\gamma)_k \Big| \leq C_\gamma, z \in \mathbb{D}. \tag{6}$$

*Then all $f_k$ are bounded.*


*Proof.* Choose distinct $\gamma_0, \gamma_2, \ldots \gamma_{n-1}$ sufficiently large. It is a tedious but standard calculation that

$$\det \begin{bmatrix} (\gamma_0)_1 & (\gamma_0)_2 & \cdots & (\gamma_0)_{n-1} \\ (\gamma_1)_1 & (\gamma_1)_2 & \cdots & (\gamma_1)_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ (\gamma_{n-1})_1 & (\gamma_{n-1})_2 & \cdots & (\gamma_{n-1})_{n-1} \end{bmatrix} = \prod_{0 \leq i < j < n} (\gamma_j - \gamma_i) \neq 0.$$

In other words, the vectors $\Gamma_k = ((\gamma_k)_0, \ldots, (\gamma_k)_{n-1}), k = 0, 1, \ldots n - 1$ form a basis of $\mathbb{R}^n$. Therefore for a fixed $k, 0 \leq k < n$, there exist $r_0, \ldots, r_{n-1} \in \mathbb{R}$, such that

$$(0, \ldots, 0, 1, 0, \ldots, 0) = \sum_{i=0}^{n-1} r_i \Gamma_i,$$

where the vector on the left of the above equation has all components, except for the $k-$th, equal to zero. Therefore,

$$f_k(z) = \sum_{j=0}^{n-1} f_j(z) \Big( \sum_{i=0}^{n-1} r_i(\gamma_i)_j \Big)$$

$$= \sum_{i=0}^{n-1} r_i \sum_{j=0}^{n-1} f_j(z)(\gamma_i)_j.$$

Hence, by the our assumptions,

$$|f_k(z)| \leq \sum_{i=0}^{n-1} |r_i| C_{\gamma_i}.$$

$\square$

# 4  Main results

Another way to think of the operator $T_g$ is the following. Let $f, g$ be analytic functions in the unit disc. Then, the usual rule of differentiation gives $(fg)' = f'g + fg'$. Hence $T_g f$ can be thought of as a primitive of the first term of the Leibniz's rule. In this light a natural way to generalize $T_g$ is to consider operators of the form

$$T_{g,a}f(z) = \int_0^z \int_0^\zeta f(t)g''(t) + af'(t)g'(t)dtd\zeta, a \in \mathbb{C}, \tag{7}$$

or even more general

$$P_{g,a}f = I^n(fg^{(n)} + a_1 f'g^{(n-1)} + \cdots + a_{n-1}f^{(n-1)}g'),$$

where $g$ is an analytic symbol , $a = (a_1, a_2, \ldots, a_{n-1})$ is a $(n-1) - tuple$ of complex numbers, and $I$ is the integration operator defined by

$$I(f)(z) = \int_0^z f(t)dt.$$

Occasionally we will use the symbol $\int$ for the operator $I$. When $n = 1$ we adopt the convention $P_{g,-} = T_g$.

The purpose of this work is to investigate boundedness and compactness criteria for the operators $P_{g,a}$. Our main results in this direction are the following.

**Theorem 12.** *Let* $0 < p < \infty$ *and* $a \in \mathbb{C}^{n-1}$. *Then,* $P_{g,a}$ *is bounded from* $H^p$ *to itself if and only if* $g \in BMOA$. $P_{g,a}$ *is compact if and only if* $g \in VMOA$.

**Theorem 13.** *Let* $0 < p < q < \infty$ *and* $a \in \mathbb{C}^{n-1}$. *Define also* $\alpha = \frac{1}{p} - \frac{1}{q} > 0$ *and* $k = \max\{l : a_l \neq 0\}$ *or* $k = 0$ *if* $a = 0$. *Then if* $l < \alpha \leq l + 1 \leq n - k$ *for some* $l \in \mathbb{N}$, $P_{g,a}$ *is bounded from* $H^p$ *to* $H^q$ *if and only if* $g^{(l)} \in \Lambda_{a-l}$. *If* $\alpha > n - k$ *and* $P_{g,a}$ *is bounded from* $H^p$ *to* $H^q$ *then* $P_{g,a}$ *is the zero operator.*

**Theorem 14.** *Let* $0 < q < p < \infty$ *and* $a \in \mathbb{C}^{n-1}$. *If* $g \in H^s$, *where* $\frac{1}{s} = \frac{1}{q} - \frac{1}{p}$ *then* $P_{g,a}$ *is bounded from* $H^p$ *to* $H^q$. *In the special case that* $n = 2$ *and* $a = 0$, *if* $P_{g,a} : H^p \longrightarrow H^q$ *is bounded, then* $g \in H^s$.

In Theorem 14 the following question remains unanswered.

**Problem.** *Is it true that if $P_{g,a} : H^p \longrightarrow H^q, 0 < q < p < \infty$ is bounded then $g$ must be in $H^s, \frac{1}{s} = \frac{1}{q} - \frac{1}{p}$?*

Theorems 12, 13 and 14 show that the behavior, regarding boundedness, of the operator $P_{g,a}$ is essentially the same with this of $T_g$ when $p \geq q$. In the case that $p < q$ the operator exhibits a different behavior because its boundedness depends on its last non zero term. What is interesting about these operators is that when they are bounded, every term comprising the operator is forced to be bounded. In other words there is no cancellation between the terms which make up the operator $P_{g,a}$. Even if this observation is not yet verified when $p = q$ there is strong evidence that this should be the case (see Lemma 20).

A motivation for the study of the operator $P_{g,a}$ comes from Pommerenke's proof of the John-Nirenberg inequality in [4]. The proof of Theorem 12 combined with Pommerenke's argument imply the following proposition. The details of the proof are given in section 4.3.

**Proposition 15.** *Let $n \in \mathbb{N}, 0 < p < \infty, f_0 \in H^p, G \in BMOA$ and $g_i \in \mathcal{B}, 1 \leq i < n$. There exists a constant $A > 0$ depending on $p$ such that if $\|G\|_*, \|g_i\|_{\mathcal{B}} < A$, every solution of the non homogeneous linear differential equation*

$$G^{(n)}f + g_1^{(n-1)}f' + \cdots + g_{n-1}'f^{(n-1)} + f^{(n)} = f_0^{(n)}$$

*is in $H^p$. If $G \in VMOA$ and $g_i \in \mathcal{B}_0$, the same result holds without the restriction in the norm of $g_i$ and $G$.*

The proofs of the sufficiency parts of theorems 12 13 and 14, except for compactness, are given in section 4.1 and of the necessity parts in 4.2. Compactness in Theorem 12 is treated in section 4.3.

## 4.1 Sufficiency

In order to understand the behavior of the operator $P_{g,a}$ it will be useful to consider the operators $P(g; n, k)$ defined for an analytic function $g$ and natural numbers $n, k$ such that $0 \leq k < n$, by the formula

$$P(g; n, k)f = I^n(f^{(k)}g^{(n-k)}).$$

The main step to prove the sufficiency part of Theorem 12, is the following proposition.

**Proposition 16.** *Let $n, k \in \mathbb{Z}_+, k < n$, and $g \in \mathscr{B}$. Then the operator $P(g; n, k) : H^p \longrightarrow H^p$ is bounded.*

*Proof.* Let $f$ be an analytic function in an open set containing the closure of $\mathbb{D}$. We can assume without loss of generality that $f^{(i)}(0) = 0, 0 \leq i < k$ because it readily checked that $P(g; n, k)$ maps the set of polynomials in $H^p$. Then we have that

$$G_n(P(g; n, k)f)(t) = \Big( \int_0^1 |f^{(k)}(re^{it})|^2 |g^{(n-k)}(re^{it})|^2 (1-r)^{2n-1} dr \Big)^{1/2}$$

$$\lesssim \|g\|_{\mathscr{B}} \Big( \int_0^1 |f^{(k)}(re^{it})|^2 (1-r)^{2k-1} dr \Big)^{1/2}$$

$$\lesssim \|g\|_{\mathscr{B}} G_k(f).$$

The result follows immediately from Proposition 9. □

To prove sufficiency in Theorem 12, notice that

$$T_g = P(g; n, 0) + \sum_{k=1}^{n-1} \binom{n-1}{k} P(g; n, k). \tag{8}$$

Therefore if $g \in BMOA \subset \mathscr{B}$ by Proposition 16 and Theorem 3, $P(g; n, 0)$ is bounded. Hence $P_{g,a}$ is bounded as well.

The next proposition generalizes a result which is known to hold for one derivative ([5, Prposition 6.1]) and is needed in order to prove the sufficiency part of Theorem 13. The proof we will give here is similar to the on in the case of one derivative [9, Proposition 1], [5, Prposition 6.1], but using the sufficiency part of Theorem 12 instead.

**Proposition 17.** *Let $f \in H^p$, then there exist $F \in H^p$ and $G_n \in BMOA, n \in \mathbb{N}$ such that $f^{(n)} = FG_n^{(n)}$.*

We shall use the following notation

$$A_g f = P(g; n, n-1)f = I^n(f^{(n-1)}g').$$

The next lemma is of some interesting on its own right.

**Lemma 18.** *If $g$ is in BMOA the operator $A_g$ is a bounded linear operator from $\mathscr{B}$ to BMOA.*

*Proof.* Let $f \in \mathscr{B}$ and set $F = A_g f$. Then for an arbitrary $h \in H^2$

$$P_{F,0}h = I^n(f^{(n-1)}g'h) = P(f;n,1)T_g h.$$

But $T_g h \in H^2$ since $g \in BMOA$ and by Proposition 16, $P(f;n,1)$ is bounded on $H^2$. Hence $P_{F,0}$ is bounded on $H^2$ as well. By the necessity part of Theorem 12 we get that $F \in BMOA$. $\qquad\square$

*Proof of Proposition 17.* By examining the proof for the case $n = 1$ we see that $g' = FG'_1$ where $G_1 \in BMOA$ and $F$ is given by

$$F(z) = \left( \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} |f(\zeta)|^{p/2} dm(\zeta) \right)^{2/p}.$$

Since $F$ has positive real part, $\log F \in BMOA$. Now we proceed by induction.

$$f^{(n)} = (FG^{(n-1)}_{n-1})'$$
$$= F((\log F)' G^{(n-1)}_{n-1} + G^{(n)}_{n-1})$$
$$= F(A_{\log F} G_{n-1} + G_{n-1})^{(n)}.$$

But $\log F \in BMOA \subset \mathscr{B}$, hence, by the previous lemma $A_{\log F} G_{n-1} \in BMOA$. $\qquad\square$

We are now ready to proceed to the proof of the sufficiency in Theorem 13. As before we prove a slightly stronger statement.

**Proposition 19.** *Let $0 < p < q < \infty, n \in \mathbb{N}$ and $0 \le k < n$ fixed. Then set $\alpha = \frac{1}{p} - \frac{1}{q}$. If $l < \alpha \le l+1 \le n-k$ for some $l \in \mathbb{N}$ and $g^{(l)} \in \Lambda_{\alpha-l}$, then $P(g;n,k)$ is bounded from $H^p$ to $H^q$.*

*Proof.* For $n = 1$ the proposition reduces to Theorem 3 part (iii). Suppose now that the statement is true for some $n > 1$ and proceed by induction. We distinguish two cases. If $n - k < \alpha \le n + 1 - k$, take an arbitrary $f \in H^p$

and write $f^{(k)} = FG^{(k)}$ for some $F \in H^p$ and $G \in BMOA$. It is easy to check that

$$P(g; n+1, k)f = I^{(n-k)}P(G; k+1, 1)T_{g^{(n-k)}}F.$$

By the assumption that $g^{(n-k)} \in \Lambda_{\alpha-n+k}$ and Theorem 3 we conclude that $T_{g^{(n-k)}}$ is bounded from $H^p$ to $H^{p'}$, where $\frac{1}{p'} = \frac{1}{q} + n - k$. It follows, using Proposition 16 and the boundedness properties of $I$ that in this case $P_{g,a} : H^p \longrightarrow H^q$ is bounded. Suppose now that $\alpha \leq n - k$. Assuming without loss of generality that $g$ has sufficient zero multiplicity at the origin. Integrating by parts we get

$$P(g; n+1, k)f = P(g; n, k)f - P(g; n+1, k+1)f.$$

The first term on the right hand side is bounded by the induction hypothesis. To prove the boundedness of the second term, factorize $f^{(k+1)}$ as before. Then

$$P(g; n+1, k+1)f = P(G; n+1, n-k)P(g; n-k, 0)F.$$

The result follows by the induction hypothesis and Theorem 12 applied to the operator $P(G; n+1, n-k)$. $\qquad\qquad\square$

The proof of sufficiency in Theorem 14 is similar but easier. Again we will prove by induction on $n$ that for $0 \leq k < n$, $P(g; n, k)$ is bounded from $H^p$ to $H^q$. As usual write $f^{(k)} = FG^{(k)}$ and note that the same recursive formula holds as before.

$$P(g; n+1, k)f = P(g; n, k)f - P(G; n+1, n-k)P(g; n-k, 0)F.$$

which by induction proves our claim.

**Remark.** *It is a standard application of the closed graph theorem to prove that whenever $P_{g,a}$ is bounded from $H^p$ to $H^q$ then there exist a constant $C_{p,q}$ such that*

$$\|P_{g,a}\|_{p,q} \leq C_{p,q}\|g\|_{X_{p,q,a}},$$

*where the space $X_{p,q,a}$ is the space which $g$ must be in such that $P_{g,a}$ is bounded.*

## 4.2 Necessity

First we prove necessity in Theorem 12. We suppose that $P_{g,a} : H^p \longrightarrow H^p$ is bounded. We will prove that in that case $g \in \mathscr{B}$ which together with Theorem 3, Proposition 16 and equation (8) imply that $g \in BMOA$.

For $\lambda \in \mathbb{D}, \gamma > 1/p$ set

$$f_{\lambda,\gamma}(z) = \frac{(1 - |\lambda|^2)^{\gamma - 1/p}}{(1 - \bar{\lambda}z)^{\gamma}}.$$

A simple calculation shows that

$$f_{\lambda,\gamma}^{(k)}(\lambda) = \frac{(\gamma)_k \bar{\lambda}^k}{(1 - |\lambda|^2)^{k+1/p}}, \quad k \in \mathbb{N}, \tag{9}$$

where $(\gamma)_0 = 1, (\gamma)_k = \gamma(\gamma + 1) \cdots (\gamma + k - 1)$. Also there exists a positive constant $C_\gamma$ such that $\|f_{\lambda,\gamma}\|_p \leq C_\gamma, \lambda \in \mathbb{D}$. Then the standard estimates for $H^p$ functions give

$$\frac{C_\gamma}{(1 - |\lambda|^2)^{n+1/p}} \geq |(P_{g,a}f_{\lambda,\gamma})^{(n)}(\lambda)|$$

$$= \Big| \sum_{k=0}^{n-1} \frac{a_k \bar{\lambda}^k (\gamma)_k}{(1 - |\lambda|^2)^{k+1/p}} g^{(n-k)}(\lambda) \Big|.$$

Hence

$$\Big| \sum_{k=0}^{n-1} g^{(n-k)}(\lambda)(1 - |\lambda|^2)^{(n-k)} a_k \bar{\lambda}^k (\gamma)_k \Big| \leq C_\gamma. \tag{10}$$

By applying Lemma 11 we can infer that $\sup_{\lambda \in \mathbb{D}} |g^{(n)}(\lambda)|(1 - |\lambda|^2)^n < \infty$, i.e. $g \in \mathscr{B}$.

The proof of necessity in Theorem 14 is similar. Consider again the family of test functions $f_{\lambda,\gamma}$. As before

$$\Big| \sum_{k=0}^{n-1} \frac{a_k \bar{\lambda}^k (\gamma)_k}{(1 - |\lambda|^2)^k} g^{(n-k)}(\lambda) \Big| \leq \frac{C_\gamma}{(1 - |\lambda|^2)^{\frac{1}{q} - \frac{1}{p} + n}},$$

by applying again Lemma 11 we can separate the previous condition to the following.

$$|g^{(n-k)}(\lambda)| \lesssim \frac{1}{(1 - |\lambda|^2)^{\frac{1}{q} - \frac{1}{p} + n - k}}, \quad \lambda \in \mathbb{D}, \tag{11}$$

where $k = \max\{l : a_l \neq 0\}$. Since $a_k \neq 0$ it follows immediately that $g^{(l)} \in \Lambda_{\alpha-l}$ if $\alpha \leq n - k$ and $g^{(n-k)} = 0$ if $\alpha > n - k$.

The proof of necessity in Theorem 14 is more involved and is based on the following lemma which can be proved for arbitrary $a \in \mathbb{C}$.

**Lemma 20.** *For every $q > 0$ there exists a positive constant $C = C(q) > q$ such that if $p > C(q)$ and $P_{g,a} : H^p \longrightarrow H^q$ then $g \in H^s, \frac{1}{s} = \frac{1}{q} - \frac{1}{p}$.*

*Proof.* Assume that $g(0) = g'(0) = 0$, and multiple integration by parts we can rewrite the formula expressing $P_{g,a}$ as

$$P_{g,a}f = fg + (1-a)\int\int f''g + (a-2)\int f'g \tag{12}$$

$$= fg + (1-a)\int\int\left(\frac{f''}{f} - \frac{f'^2}{f^2}\right)gf + (1-a)\int\int\frac{f'^2}{f^2}gf + (a-2)\int\frac{f'}{f}gf, \tag{13}$$

for any $f \in H^p$, where we used the convention

$$\int f = \int_0^z f(t)dt.$$

Now let $\epsilon > 0$ and $G = \mathscr{H}\left(\frac{|g|^\alpha}{(1+\epsilon|g|)^\alpha}\right)^\beta$, where $\alpha$ and $\beta$ are positive constants to be specified later and $\mathscr{H}$ denotes the Herglotz transform, i.e.

$$G(z) = \left(\int_{\mathbb{T}}\frac{|g(\zeta)|^\alpha}{(1+\epsilon|g(\zeta)|)^\alpha}\frac{\zeta+z}{\zeta-z}dm(\zeta)\right)^\beta. \tag{14}$$

$G$ has positive real part , hence $\|\log G\|_* \lesssim \beta$ and also $G \in H^\infty$. If we assume that $\beta p > 1$ then by M. Riesz's theorem [2, Theorem 4.1]

$$\|G\|_p^p = \int_{\mathbb{T}}\mathscr{H}\left(\frac{|g|^\alpha}{(1+\epsilon|g|)^\alpha}\right)^{\beta p}dm \tag{15}$$

$$\lesssim \int_{\mathbb{T}}\left(\frac{|g|}{1+\epsilon|g|}\right)^{\alpha\beta p}dm. \tag{16}$$

The last estimate we need for $G$ is that $|G| \geq (ReG^{1/\beta})^\beta = \left(\frac{|g|}{1+\epsilon|g|}\right)^{\alpha\beta}$. Which gives

$$\|Gg\|_q^q = \int_{\mathbb{T}}|Gg|^q dm \geq \int_{\mathbb{T}}\left(\frac{|g|}{1+\epsilon|g|}\right)^{(\alpha\beta+1)q}. \tag{17}$$

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With these preliminaries we are going to estimate $\|P_{g,a}G\|_q$. First we estimate separately the last three terms in the right hand side of (13).

$$\left\| \int \int (\log G)'' Gg \right\|_q \le C_1(q)\|\log G\|_*\|Gg\|_q \tag{18}$$

$$\le C_2(q)\beta\|Gg\|_q, \tag{19}$$

by the sufficiency part of Theorem 12.

$$\left\| \int \int \frac{G'^2}{G^2} gG \right\|_q = \left\| \int \int (\log G)' \left( \int (\log G)' gG \right)' \right\|_q$$
$$\le C_3(q)\beta^2\|gG\|_q.$$

$$\left\| \int \frac{G'}{G} gG \right\|_q \le C_4(q)\beta\|gG\|_q. \tag{20}$$

Then (13) and the boundedness of $P_{g,a}$ give

$$\|Gg\|_q(1 - C_5(q)\beta - C_6(q)\beta^2) \lesssim \|P_{g,a}\|_{p,q}\|G\|_p. \tag{21}$$

Furthermore, if $\beta C_5(q) + \beta^2 C_6(q) < 1$ (or equivalently $\beta < C(q)$ where $C(q)$ is a continuous function of $C_5, C_6$) we arrive at

$$\|Gg\|_q \le C_{q,\beta}\|P_{g,a}\|_{p,q}\|G\|_p, \tag{22}$$

which together with estimates (16) and (17) give

$$\left( \int_{\mathbb{T}} \left( \frac{|g|}{1 + \epsilon|g|} \right)^{(\alpha\beta+1)q} \right)^{1/q} \le \tilde{C}_{q,\beta}\|P_{g,a}\|_{p,q} \left( \int_{\mathbb{T}} \left( \frac{|g|}{1 + \epsilon|g|} \right)^{\alpha\beta p} \right)^{1/p}. \tag{23}$$

Now, choose $\alpha$ such that $(\alpha\beta + 1)q = \alpha\beta p$ and note that in this case $(\alpha\beta + 1)q = \frac{1}{s}$. Hence

$$\left( \int_{\mathbb{T}} \left( \frac{|g|}{1 + \epsilon|g|} \right)^s dm \right)^{1/s} \le \tilde{C}_{q,\beta}\|P_{g,a}\|_{p,q}.$$

Fatou's lemma then gives $g \in H^s$. $\qquad\square$

We can now prove the necessity part in Theorem 14. Let $P_{g,0} : H^p \longrightarrow H^q, p > q$ be bounded. And set $\frac{1}{s} = \frac{1}{q} - \frac{1}{p}$. Note that by using an interpolation argument (see [10, Remark 2.2.5]) we can choose the constant $C = C(q)$

in the previous lemma to stay bounded if $0 < \epsilon < q < 1/\epsilon$ for some $\epsilon > 0$. Therefore

$$C_0 = \sup_{\frac{1}{s} \leq q \leq \frac{1}{s} + \frac{1}{p}} C(q) < \infty.$$

Pick a natural number $n$ such that $np > C_0$ and define $p' = np$. Then if $f_i \in H^{p'}$

$$P_{g,0}(f_1 f_2 \cdots f_n) = P_{P_{g,0}(f_2 f_3 \cdots f_n),0}(f_1).$$

Keeping $f_2, f_3, ..., f_n$ fixed and applying the previous lemma to the operator $P_{P_{g,0}(f_2 f_3 \cdots f_n),0}$ we have that $P_{g,0}(f_2 f_3 \cdots f_n) \in H^{q_1}, \frac{1}{q_1} = \frac{1}{q} - \frac{1}{p'}$. Continuing inductively we arrive at $g \in H^{q_n}, \frac{1}{q_n} = \frac{1}{q} - \frac{n}{p'} = \frac{1}{s}$.

## 4.3   Compactness

Now, we are going to prove the compactness part of Theorem 12. The proof is similar to the one for boundedness, therefore, first we prove the result corresponding to Proposition 16 for the operators $P(g; n, k)$ when $g$ is in the little Bloch space.

**Proposition 21.** *Let* $0 < p < \infty$ *and* $g \in \mathscr{B}_0$. *Then for* $n > 1, 1 < k < n$ *the operator* $P(g; n, k)$ *is a compact operator from* $H^p$ *to itself. If* $g \in VMOA, P(g; n, 0)$ *is compact from* $H^p$ *to itself.*

*Proof.* By integration by parts one can write $P(g; n, k)$ in the following form

$$P(g; n, k)f = c_1 I^{(k)} M_{g^{(k)}} f + c_2 I^{(k+1)} M_{g^{(k+1)}} f + \cdots + c_{n-k} I^n M_{g^{(n)}} f.$$

Where $M_\phi$ is the multiplication operator defined by

$$M_\phi f(z) = f(z)\phi(z).$$

If $g$ is polynomial, $M_{g^{(i)}}$, for $k \leq i \leq n$ is bounded on $H^p$. It is a well known fact (see [1]) that $I$ is compact on $H^p$, hence if $g$ is polynomial $P(g; n, k)$ is a compact operator. Suppose now that $g \in \mathscr{B}_0$. Then there exists a sequence of polynomials $g_n$ converging in the Bloch norm to $g$. Therefore by the remark in the end of section 4.1

$$\|P(g; n, k) - P(g_m; n, k)\|_p \leq C_p \|g_m - g\|_{\mathscr{B}},$$

i.e. $P(g; n, k)$ is compact as the norm limit of compact operators.

The proof of the second part is identical, since $VMOA$ is the closure of the set of polynomials in $BMOA$. □

By Proposition 21 and the fact that $VMOA \subset \mathscr{B}_0$, it follows immediately that $P_{g,a}$ is compact if $g \in VMOA$.

To show necessity it suffices to show that $g \in \mathscr{B}_0$, because then by Proposition 21 and formula (8) it follows that $T_g$ is compact and therefore $g \in VMOA$. Let $f_{\lambda,\gamma}$ be defined by (9). For fixed $\gamma$, $f_{\lambda,\gamma}$ converges to zero on compact sets as $|\lambda| \longrightarrow 1^-$. The compactness of $P_{g,a}$ then implies that $P_{g,a} f_{\lambda,\gamma}$ converges strongly to 0.

Using the same estimates as in the proof of necessity of Theorem 12, we get the following inequality.

$$\left| \sum_{k=0}^{n-1} g^{(n-k)}(\lambda)(1 - |\lambda|^2)^{(n-k)} a_k \bar{\lambda}^k(\gamma)_k \right| \le \|P_{g,a} f_{\lambda,\gamma}\|_p.$$

Again, by applying Lemma 11 we can separate the above estimate.

$$|g^{(n)}(\lambda)|(1 - |\lambda|^2)^n \lesssim \|P_{g,a} f_{\lambda,\gamma}\|_p,$$

which gives the desired result, since the right part converges to zero as $|\lambda| \longrightarrow 1^-$.

We are now in position to prove Proposition 15.

*Proof of Proposition 15.* Proposition 16 and 12 imply that the operator defined by

$$Af = I^n(fG^{(n)} + f'g_1^{(n-1)} + \cdots + f^{(n-1)}g'_{n-1})$$

is bounded on $H^p$. By an application of the closed graph theorem, there exists a constant $C$ depending only on $p$ such that

$$\|A\|_p \le C(\|G\|_* + \|g_1\|_{\mathscr{B}} + \cdots + \|g_{n-1}\|_{\mathscr{B}}).$$

If $\|G\|_*, \|g_i\|_{\mathscr{B}} < 1/(nC)$, $-1$ is in the resolvent set of the operator $A$ on $H^p$. Let now $f$ any solution of the above differential equation. There exists $F \in H^p$ such that $F - f_0$ is a polynomial of degree less than $n$ and $F^{(i)}(0) = f^{(i)}(0), 0 \le i < n$. Then for some $\tilde{f} \in H^p, A\tilde{f} + \tilde{f} = F$. But $\tilde{f}$ and $f$

satisfy the same differential equation with the same initial conditions, hence $f = \tilde{f} \in H^p$. If $G \in VMOA$ and $g_i \in \mathscr{B}_0$, $A$ is compact. It is easy to check that it has no eigenvalues, therefore its spectrum is the singleton $\{0\}$, thus the result follows. $\qquad\square$

# 5    The spectrum of the operator $T_g$

In this chapter we will mention some known results about the spectrum of the operator $T_g$ in the Hardy space and we will connect this to a problem about integration operators on the Hardy spaces of simply connected domains. The spectrum of the operator $T_g$ in $H^p$ was investigated in [11] by Aleman and Pelaez and it relates naturally with norm inequalities for the weighted square function.

The resolvent set $\rho(T_g|H^p)$ of $T_g$, consists of the complex numbers $\lambda \neq 0$ such that for any $h \in H^p$ the differential equation

$$\lambda f'(z) - f(z)g'(z) = h'(z), \quad f(0) = \frac{1}{\lambda}h(0)$$

has a unique solution in $H^p$. It is easy to solve this equation explicitly and get a formula for the resolvent of $T_g$.

$$R_{\lambda,g}h(z) = \frac{1}{\lambda}h(0)e^{g(z)/\lambda} + \frac{1}{\lambda}e^{g(z)/\lambda} \int_0^z h'(t)e^{-g(t)/\lambda}dt.$$

By plugin $h(z) \equiv 1$ in the above equation we have that $\lambda \in \rho(T_g|H^p)$ is equivalent to the conditions

$$e^{g/\lambda} \in H^p \text{ and } S_{g,\lambda}h(z) = e^{g(z)/\lambda} \int_0^z h'(t)e^{-g(t)/\lambda}dt$$

is a bounded linear operator on $H^p$.

Let now $\omega$ be a (positive) weight function on $\mathbb{T}$ and denote by $W$ the outer function corresponding to $\omega$ i.e.

$$W(z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \omega(\zeta)dm(\zeta)\right).$$

Throughout what follows we assume that $\omega, \log\omega \in L^1(\mathbb{T})$. Then we define the weighted Hardy space $H^p(\omega), p > 0$ to be $W^{-1/p}H^p$. The norm in this space is given by

$$\|f\|_{\omega,p} = \|W^{1/p}f\|_p,$$

The weighted square function is defined as follows

$$S_{\omega,p,f}(\zeta) = \left(\int_{\Gamma_\sigma(\zeta)} |W|^{2/p}|f'|^2dA\right)^{1/2}, \quad \zeta \in \mathbb{T}.$$

The important observation is that the boundedness of $S_{g,\lambda}$ is equivalent to a weighted norm inequality for the square function. To be more specific let $\omega$ be the weight defined by $\omega(\zeta) = \exp(pg(\zeta)/\lambda), \zeta \in \mathbb{T}$. If $g \in BMOA$, $S_{g,\lambda}$ is always bounded from below, because its inverse is a bounded linear operator, thus if we set $H = e^{-g/\lambda}S_{g,\lambda}h$, so that $h' = e^{g/\lambda}H'$, we can reformulate the boundedness of $S_{g,\lambda}$ as follows

$$\|e^{g/\lambda}H\|_p \sim \int_{\mathbb{T}} \Big( \int_{\Gamma_\sigma(\zeta)} e^{2Re(g/\lambda)}|H'|^2 dA \Big)^{p/2} dm(\zeta),$$

which can be rewritten with the previously introduced notation for the weighted square function as

$$\|H\|_{\omega,p} \sim \int_{\mathbb{T}} S^p_{\omega,p,H} dm. \tag{24}$$

Therefore the problem boils down to the characterization of the weights such that an estimate of the form (24) holds.

We say that a weight $\omega$ satisfies the $\mathscr{A}_\infty$ condition if

$$\frac{1}{m(I)} \int_I \omega dm \lesssim \exp \Big( \frac{1}{m(I)} \int_I \log \omega dm \Big)$$

for all open arcs $I \subset \mathbb{T}$.

With this preparation we can state the theorem which characterizes the spectrum of $T_g$ from [11].

**Theorem 22.** *Assume that $\lambda \in \mathbb{C} - \{0\}, 0 < p < \infty$ and $g \in BMOA$. Then, the following assertions are equivalent:*

*(i)* $\lambda \in \rho(T_g|H^p)$,

*(ii)* $e^{g/\lambda} \in H^p$ and $\omega(e^{i\theta}) = \exp(pRe(g(e^{i\theta})/\lambda))$ satisfies the $\mathscr{A}_\infty$ condition.

Let us now turn to a problem from [9]. If $\Omega$ is a simply connected domain in the complex plain, $0 \in \Omega$, let $H^p(\Omega)$ denote the conformally invariant Hardy space of this domain. That is, the space of analytic functions in $\Omega$ that they have a harmonic majorant.

In [9] Aleman and Siskakis asked whether the operator $J_\Omega$ defined by

$$J_\Omega f(z) = \frac{1}{z} \int_0^z f(t) dt$$

is always a bounded linear operator on $H^p(\Omega)$. In order to use the tools developed in this chapter we introduce the Riemann mapping $h : \mathbb{D} \longrightarrow \Omega$, $h(0) = 0$, $h'(0) > 0$ corresponding to $\Omega$. Let $C_h(f) = f \circ h$ be the standard isometry from $H^p(\Omega)$ to $H^p$. Then it is clear that $J_\Omega$ is bounded if and only if $A_h = C_h J C_h^{-1}$ is bounded on $H^p$. By a change of variables and an integration by parts we get

$$A_h(f)(z) = \frac{1}{h(z)} \int_0^z f(t) h'(t) dt$$

$$= f(z) - \frac{1}{h(z)} \int_0^z f'(t) h(t) dt.$$

Motivated by the similarity between $A_h$ and the resolvent of the operator $T_g$ we prove the following.

**Proposition 23.** *If $\Omega$ is a simply connected domain, $0 \in \Omega$ and $h$ is the corresponding conformal mapping, then the following are equivalent:*

*(i) $J_\Omega$ is bounded from $H^p(\Omega)$ to itself,*

*(ii) The operator $P_h$ defined by*

$$P_h f(z) = \frac{z}{h(z)} \int_0^z f'(t) \frac{h(t)}{t} dt$$

*is bounded on $H^p$.*

*Proof.* Let

$$B_h f(z) = \frac{1}{h(z)} \int_0^z f'(t) h(t) dt = A_h f(z) - f(z) \tag{25}$$

and $M_z$ be the multiplication by the independent variable operator. It is readily checked that

$$P_h M_z = M_z B_h + P_h T_z. \tag{26}$$

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By (25) and (26) it follows immediately that if $P_h$ is bounded then so is $A_h$. To prove the converse again by equations (25) and (26) it suffices to show that $P_h T_z$ is bounded. Let first $f \in H^p, f(0) = 0$ and set $F(z) = f(z)/z$ then,

$$P_h T_z f(z) = \frac{z}{h(z)} \int_0^z f(t) \frac{h(t)}{t} dt$$

$$= \frac{z}{h(z)} \int_0^z F(t) h(t) dt = M_z B_h T_z F,$$

which is in $H^p$ by our assumptions. Since for an arbitrary function $\in H^p$

$$P_h T_z f(z) = P_h T_z (f - f(0))(z) + f(0) P_h T_z 1,$$

it remains to show that $P_h T_z 1$ is an $H^p$ function. To see this note that $h^{-1}(z) \int_0^z h(t) = z - A_h(id)(z)$ is an $H^p$ function and also

$$\frac{1}{h(z)} \int_0^z h(t) dt = \frac{1}{h(z)} \int_0^z t \left( \int_0^t \frac{h(\zeta)}{\zeta} d\zeta \right)' dt \qquad (27)$$

$$= \frac{z}{h(z)} \int_0^z \frac{h(t)}{t} dt - \frac{1}{h(z)} \int_0^z \int_0^t \frac{h(\zeta)}{\zeta} d\zeta dt, \qquad (28)$$

by integration by parts. Combining the fact that $h \in H^q, 0 < q < 1/2$ (see for example [2, Theorem 3.16]) and the Hardy-Littlewood Theorem (Theorem 2) we get that the second term of the right hand side of (28) is in every $H^q$ space $0 < q < \infty$. Hence $P_h T_z 1$ is in $H^p$. $\qquad \square$

**Corollary 24.** *Let $0 < p < \infty$ and $\Omega$ a simply connected domain. Then the following are equivalent.*

(i) *The weight $\omega(\zeta) = |h(\zeta)|^{-p}$ satisfies the $\mathscr{A}_\infty$ condition,*

(ii) *$\frac{1}{p}$ is in the resolvent set of $T_g$ on $H^1$, where $g(z) = -\log(\frac{h(z)}{z})$,*

(iii) *The operator $J_\Omega$ is bounded on $H^p(\Omega)$.*

*Proof.* Let $g(z) = -\log(\frac{h(z)}{z})$. Then $g \in BMOA$ (see [12]). The result follows immediately from Theorem 22 applied to the operator $T_g$ and Proposition 23. $\qquad \square$

From the fact that the spectrum of a bounded operator is compact we can conclude that if $p < C$, for some positive constant $C$ depending only on $\Omega$, then $J_\Omega$ is bounded on $H^p(\Omega)$.

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