On the use of integer and fractional flexible Fourier form Dickey-Fuller unit root tests

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Abstract

In this paper we propose the use of a new set of conservative critical values for the flexible Fourier form Dickey-Fuller unit root test when the Fourier frequency is estimated. We consider both the integer frequency and the fractional frequency version of the test and investigate their size and power properties. We find that the integer frequency test sometimes has zero power when the deterministic component of the data generating process is characterized by a fractional frequency. Furthermore, when the originally proposed critical values are applied both versions of the test are oversized when the frequency is estimated. However, whereas the integer frequency test is only moderately oversized the fractional frequency test is significantly oversized in many cases. To remedy the size problems we simulate new critical values for the case where the frequency is estimated. The critical values are conservative, and hence yields an undersized test in some cases. Nevertheless, the resulting fractional frequency test with the new conservative critical values applied to it has good power properties.

Keywords: Fractional frequency flexible Fourier form, Unit root test, Structural break, Smooth break, Nonlinear trend

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1. Introduction

Since Perron (1989) showed that unit root tests can suffer from zero power when structural breaks are unaccounted for much of the unit root literature has been concerned with the modelling of breaks. Perron (1989) presents the idea that many economic time series occasionally are subject to changes in level and/or trend as a result of structural change. This can cause unit root tests to be unable to reject the null hypothesis, sometimes even asymptotically. To solve the problem he suggests modelling structural change by introducing a dummy variable and/or a change in trend at the time of the change. Whereas the model proposed in Perron (1989) assumes the change to have occurred at a known point in time the research that followed has mainly focused on the unknown case. Some examples of unit root tests with sharp changes in level or trend at an unknown time are Carrion-i-Silvestre et al. (2009), Harvey et al. (2013), Lee and Strazicich (2003), Perron and Rodríguez (2003), Vogelsang and Perron (1998) and Zivot and Andrews (1992).

Although instant changes in the deterministic component of macroeconomic aggregates are quite reasonable in some cases, in general they are not. This is a result of the very nature of the aggregate. For an aggregate to change instantaneously it requires that all aggregated agents act simultaneously. At least when the time series is an aggregate of a very large number of agents, as in the case of many macroeconomic time series, it seems too restrictive to assume that all agents react to new conditions simultaneously. In this case we would rather expect the changes to occur gradually. For example, Leybourne et al. (1998) emphasize the plausible smooth transitions of economic aggregates and develop a unit root test that accounts for a smooth gradual change. The test models the break as a logistic smooth transition between two regimes. Consequently, it is restricted to one functional form.

Typically the researcher has a wide option of econometric tools to model nonlinearity. However, in general the true functional form is unknown and has to be determined. Unfortunately, determining the functional form of deterministic trends in time series is difficult. Furthermore, misspecification of the functional form may cause as much problems to the test as ignoring the trend altogether (Enders and Jones, 2014). Therefore, because the functional form is crucial to the test it is desirable to find a modelling strategy that is flexible enough to capture a large set of unknown functional forms.

As a way of modelling unknown functional forms in time series a class of flexible Fourier form models was developed (see, for example, Bierens, 1997; Davies, 1987; Gallant, 1981; Gallant and Souza, 1991)). The bottom line of this modelling strategy is that Fourier expansions can approximate a very broad set of nonlinear functional forms. In particular this type of modelling is suitable for smooth changes. More recently, a large body of time series literature that uses Fourier approximation of unknown functional forms has emerged. For example, Becker et al. (2004), Harvey et al. (2010) and Perron et al. (2017) develop tests for the presence of nonlinearity in deterministic components. Moreover, Becker et al. (2006) develop a test for stationary and show that Fourier approximation is sufficient to approximate a wide range of functional forms in this context.

The flexible Fourier form approximation of nonlinear deterministic components has also proved useful in unit root testing. Enders and Lee (2012a) adopt the Lagrange multiplier methodology by Schmidt and Phillips (1992) and develop a unit root test using Fourier form approximation. Similarly, Enders and Lee (2012b) develop a Dickey-Fuller type version of the flexible Fourier form unit root test and Rodrigues and Robert Taylor (2012) further develop the generalized least squares unit root test by Elliott et al. (1996) to allow for nonlinear deterministic components. The authors derive the corresponding unit root statistics and show that the tests depend on the nuisance parameter characterizing the Fourier form, namely, the number of frequencies and their values.

All three of the unit root tests are based on the use of a single or multiple number of integer Fourier frequencies. By contrast, Omay (2015) proposes the use of a single fractional frequency. The author shows that when only integer frequencies are allowed for in the regression the Dickey-Fuller type test is sometimes unable to reject the null when the data generating process (DGP) contains a fractional frequency component. Hence, the integer frequency test is more restrictive in terms of which functional forms they can approximate.

In empirical work the Fourier frequencies have to be estimated. So far, in the case where only one frequency is assumed in the model, the main strategy has been to estimate it by minimizing the sum of squared residuals (SSR) over a prespecified grid. Following this methodology a grid of different frequencies is considered where the estimated frequency is the one that minimizes the SSR of the test regression. Note that this is equivalent of choosing the frequency that maximizes the F-statistic that tests the significance of the Fourier component. The performance of this estimation strategy in the prevailing context has, up until now, been overlooked in the literature. However, Enders and Lee (2012a,b) derive the distribution of the F-statistic and show that it depends on the frequency that is used in the test regression. This implies that also the distribution of the SSR depends on the frequency. Because of the dependence using the SSR to estimate the frequency could be problematic in the sense that minimizing the SSR (or maximizing the F-statistics) does not necessarily result in the most significant parameter estimates of the Fourier component. Because the SSR is not equally distributed for different frequencies it is reasonable to suspect that some frequencies will minimize the SSR more often than others. Hence, the procedure may not yield an unbiased estimate of the of the deterministic component[1]. Consequently, it is important to investigate how the method performs in estimating the frequency and how the final unit root test is affected.

So far Monte Carlo studies that investigate how these tests perform when the frequency is estimated are scarce. Enders and Lee (2012a,b) estimate the size and power of their tests for the cases where the underlying DGP is linear and when it contains an integer frequency. There are no studies that investigate the size and power of the test that allows the Fourier frequency to take fractional values when the frequency is estimated. There are also no studies that investigate the properties of the frequency estimation method in Fourier form unit root tests.

In this paper we consider the Enders and Lee (2012b) Dickey-Fuller type test and investigate its size and power when the frequency is estimated. Both the case where only integer frequencies are allowed for, as originally suggested, and the case where fractional frequencies are allowed for, as suggested by Omay (2015), are considered. We show that

[1] There is most likely dependence between the SSR at different grid points since they are estimated on the same data. Therefore, even if the SSR were equally distributed over the frequency grid this would not be a sufficient condition to ensure that that the procedure would be unbiased.

when the frequency is estimated the test is significantly oversized if the underlying DGP is linear or if it contains a Fourier component with a frequency that is small. Furthermore, we show that when only an integer frequency is allowed for the test sometimes has zero power when a fractional frequency component is present in the DGP. To avoid the power problems we advocate the use of the fractional frequency Fourier form unit root test. To remedy the problem that test is oversized we simulate new conservative critical values. We show that although this approach renders an undersized test for large frequencies its power is maintained at a reasonable level.

The paper is organized as follows. In Section 2 the unit root test and the frequency estimation method is presented. Section 3 provides a detailed description of the critical values. The Monte Carlo study and the critical values simulations are setup in Section 4. In Section 5 the results are presented and discussed. Finally, concluding remarks and suggestions for further research are found in Section 6.

2. The flexible Fourier form unit root test

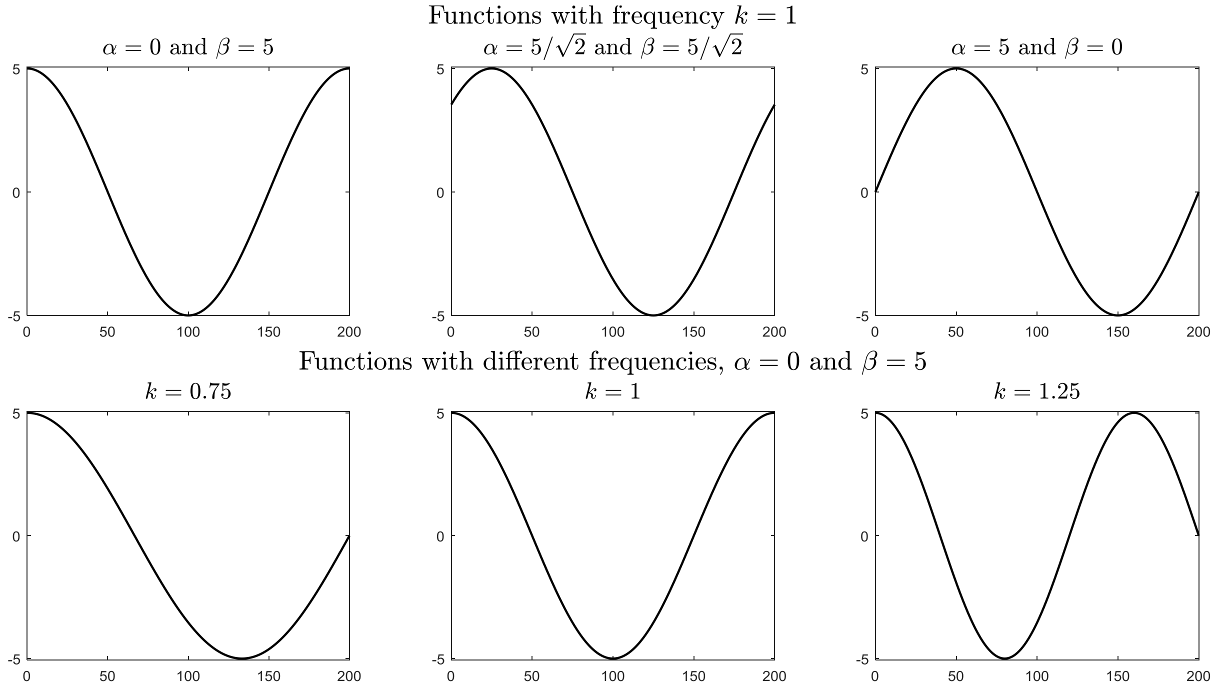
2.1. The Enders and Lee (2012b) Dickey-Fuller type test

This procedure is a version of the augmented Dickey-Fuller test by Said and Dickey (1984) with Fourier terms in the deterministic component. Because the innovation lies in the treatment of the deterministic component we start the discussion by considering the following Fourier function

$$d(t) = c_0 + c_1 t + \sum_{i=1}^{n} \alpha_i \sin(2\pi k_i t/T) + \beta_i \cos(2\pi k_i t/T) \tag{1}$$

where $n \leq \lfloor T/2 \rfloor$, where $\lfloor \cdot \rfloor$ takes the integer part of the element, is the number of frequencies, $t = 1, \dots, T$, T is the number of observations and $k = (k_1, k_2, \dots, k_n)$ with $k_i \in (1,2, \dots, \lfloor T/2 \rfloor)$ represents the Fourier frequencies. The term c_0 is the intercept and $c_1 t$ is a linear trend. Figure 1 shows some single frequency functions that take different shapes depending on the values of α_1, β_1, and k_1 with $c_0 = 0$, $c_1 = 0$ and $T = 200$. Because we only consider a single frequency we skip the subscripts, and thus we have $\alpha = \alpha_1$, $\beta = \beta_1$, and $k = k_1$. The magnitudes of α and β determine the amplitude of the function and the relative magnitudes of α and β in proportion to each other alters where the function takes its maximum and minimum. The frequency determines the number of cycles of the function. Note that integer values of k implies full cycles whereas fractional values of k implies fractional cycles. For example, $k = 1$ implies one full cycle and $k = 1.5$ implies one and a half cycle. Further note that, the function starts and ends at the same value whenever the frequency is equal to an integer. A way to allow the integer frequency function to take different values at the beginning and end of the time span is to include a linear trend.

Figure 1: Single frequency Fourier functions



Notes: Figure 1 pictures plots of various single frequency Fourier functions of equal amplitude.

The test is identical to the augmented Dickey-Fuller test apart from the deterministic component, which contains trigonometric terms. The test regression is defined as follows

$$\Delta y_t = \gamma y_{t-1} + d(t) + \sum_{j=1}^{p} \vartheta_j \Delta y_{t-j} + \varepsilon_t \qquad (2)$$

where $d(t)$ is the deterministic component defined in equation (1), ε_t is a stationary error term with finite variance σ_ε^2. The unit root hypothesis is setup by estimating equation (2) with OLS and testing the null $H_0: \gamma = 0$ against the alternative $H_1: \gamma < 0$ using the usual t-statistic, τ. The authors show that τ is free of all nuisance parameters but k and T and provide critical values for integer frequencies. Typically the frequencies have to be estimated.

Omay (2015) studies the single frequency case in the context of Enders and Lee (2012b) and emphasise the importance of the frequencies. In order to allow for a wider range of functional forms Omay (2015) suggests letting the frequency take fractional values. For example, in this case the Fourier component itself can attain different values at the beginning and end of the time period. Hence, it is often possible to mimic permanent structural change without the linear trend component, and thus this method sometimes enables the use of a more parsimonious model.

2.2. Frequency estimation

Enders and Lee (2012a,b) propose two strategies for estimating the frequencies. One assumes a single frequency, that is, $n = 1$. In this case, k is estimated by minimizing the SSR of the test regression. The other strategy allows for multiple frequencies. In this case n is allowed to be greater than one and integer frequencies from 1 to n are cumulated. The model is pared down using some information criterion such as the Akaike information criterion or the Bayesian information criterion to estimate the number of cumulated frequencies. After the frequency is estimated an F-test is performed to test for the presence of Fourier components. If the F-test is rejected the data is assumed to contain a nonlinear deterministic component. In this case the unit root test is performed using the estimated frequency. If the F-test is not rejected the nonlinear component is assumed to be absent and the standard linear unit root test is performed.

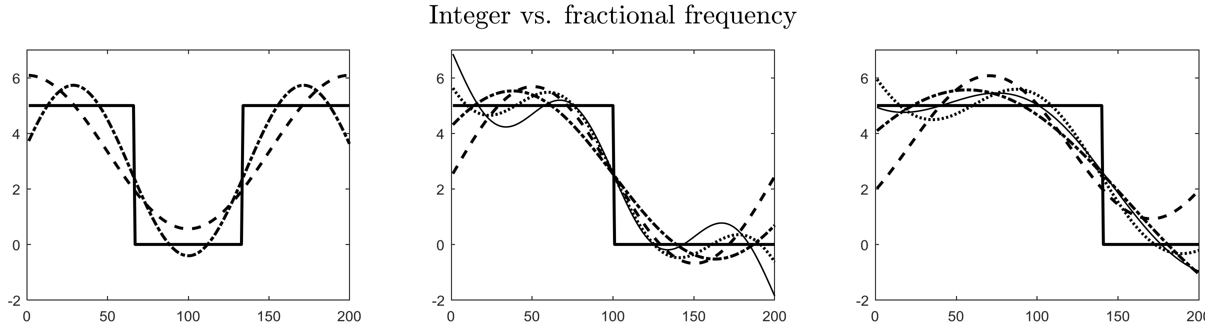
We follow Omay (2015) and consider only the single frequency case and note that Enders and Lee (2012a) and Becker et al. (2006) suggest that a single frequency often provides a good approximation for a wide range of functional forms[2]. Furthermore, note that adding frequencies quickly results in a model with a large number of parameters. In this case an overfitting problem occurs and the power diminishes quite rapidly. For this reason, Enders and Lee (2012b) suggest using a rather small number of frequencies. Restricting the model to one single frequency is a way to circumvent this problem. However, it comes at the cost of a poorer fit. Because, the purpose of this paper is to investigate the frequency estimation

[2] We also considered two frequencies in which case the implication were the same as for the for the single frequency case.

and the importance of the grid we do not perform the F-test in the Monte Carlo simulations in that are presented in Section 5.[3]

In the single frequency case the frequency is estimated by a grid search method. Using this method the test regression is run and the SSR is estimated for each frequency in a prespecified grid. Let $k_{Grid} \in Grid$ be an element in the grid, then the estimated frequency is defined as $\hat{k} = \arg\min SSR$, that is, the argument that minimizes the SSR. As pointed out in the introduction this is equivalent of choosing the frequency that yields the largest F-statistic that tests the joint significance of the parameter estimates of α and β. Furthermore, note that if the fractional grid also contains integer values, of course this is recommended, it will always find a fit that is at least as good as the integer grid in terms of the SSR.

Figure 2: Single frequency Fourier functions fitted to sharp breaks



Notes: Figure 2 shows a linear function with sharp breaks in level and various single frequency Fourier functions fitted to it. Sharp break: ▬, integer frequency: — —, fractional frequency: — · —, integer frequency with trend: —, fractional frequency with trend: ⋯.

Figure 2 shows some Fourier functions that are defined as in equation (1) and fitted to a line with a sharp break in level by minimizing the SSR. In general, Fourier functions will work better in approximating smooth function. However, we use the sharp breaks in this figure to create examples that are visually more illustrative. We compare the integer grid to a fractional grid with 0.1 increments both with and without a linear trend. The first plot in Figure 2 pictures a sharp reverted break in level. In this case the functions with and without a trend are identical, therefore we excluded the functions with a trend form the figure. The dash dot line represents the fractional frequency function and contains a Fourier component with a little more than one cycle, and hence the estimated frequency is large than 1. The

[3] In a preliminary analysis we performed the tests and applied the F-test for nonlinearity. The results had the same implications as presented in this paper. That is with exempt of the case where k is equal to zero or near zero in which case the properties of the test test was improved.

integer frequency is represented by the dashed line and the frequency is estimated equal to 1. The second plot presents one centred break in level with the fitted Fourier functions, now both with and without trends. Consider the two functions that do not contain a linear trend. Although they contain the same number of parameters they differ quite a lot. The integer frequency function is restricted to full cycles, and thus it starts and ends at the same value. In this case, where we have one permanent break in level, this function does not provide a very good approximate for the break. The fractional frequency, on the other hand, is more flexible and in this case the approximation is substantially better compared to its integer equivalent. The solid thin line represents the integer frequency with a linear trend. We see that the fit is improved considerably when a linear trend is included in the regression. Also the fit of the fractional frequency function is improved when we add a linear trend. However, because the fit of the function that does not include a linear trend is already quite good there is not as much room for improvement. The third plot in Figure 2 pictures one break of the same magnitude as before that is shifted towards the end of the sample. The overall conclusion from this picture is the same as for the previous one but it differs in the sense that the integer frequency function performs better than before.

3. Critical values

As we already established, in empirical research the frequency in the unit root test has to be estimated. The standard way to estimate the frequency is to follow the procedure described in the previous section. A further complicating factor in the analysis is that the critical value depends on the frequency. Therefore, when we apply the test we have to choose the critical value according to the frequency in the test regression. Hence, the estimated frequency determines this choice. Consequently, the estimated frequency plays a different role in the test compared to the other estimated parameters.

Both Enders and Lee (2012b) and Omay (2015) provide critical values for the Dickey-Fuller type test. Following standard conventions the critical values are generated under a linear DGP according to

$$u_t = u_{t-1} + v_t \tag{3}$$

where the v_t's are standard normal identically independently distributed (IID) random variables. Note that the DGP does not contain any deterministic component, which would be redundant because the test is invariant in this regard. In order to generate critical values for different frequencies the authors exogenously appoint values to the frequency in the test regression and run the test on the DGP for each desired frequency. For this procedure, we call the frequency that is used in the model the exogenous frequency denoted k_{exo} and denote the corresponding t-statistic $\tau(\hat{k} = k_{exo})$. In this way the resulting critical values become a function of the exogenous frequency $CV(\hat{k} = k_{exo})$, and thus we call them exogenous frequency critical values. Enders and Lee (2012b) provide critical values for the set of integers $k_{exo} \in (1, 2, \dots, 5)$. To obtain the fractional frequency critical values of Omay (2015) the same procedure is followed but here also fractional values of k_{exo} are considered. More specifically, Omay (2015) provides critical values for fractional frequencies with 0.1 increments. Note that, the exogenous integer frequency critical values of Enders and Lee (2012b) are a subset of the fractional frequency critical values.

Enders and Lee (2012b) and Omay (2015) report that the test holds its nominal size and is indifferent of all estimated parameters when a Fourier deterministic component with frequency k is present in the DGP and $k_{exo} = k$. However, even if the true deterministic component were a Fourier function, and thus a true frequency existed, whenever we estimate the frequency it will sometimes differ from the true. Because the critical value is adjusted according to the frequency any inaccuracy in the estimated frequency will also affect the choice of critical value. Enders and Lee (2012b) report the size and power of their test when it is applied to different DGPs. They consider the case where the deterministic component is linear and the case where it contains a Fourier function. In both cases the presence of the Fourier component and its frequency is treated as unknown. They apply an F-test to determine whether any nonlinear trend is present and estimate the frequency by minimizing the SSR. The results shows that the test is somewhat oversized when the function is linear or has a frequency equal to 1. However, they only consider the linear case and the case where the Fourier functions are characterized by integer frequencies. Hence, it is unclear how their test performs under fractional frequencies in the DGP when the frequency is estimated. Omay (2015) only considers the case where the frequency is known.

Now consider the case where the frequency is estimated. To account for the fact that we use $\hat{k} = \arg\min SSR$ in the test regression we consider two alternative ways to generate new conservative critical values. In both cases the frequency is endogenously estimated within the procedure as in Section 2.2. Therefore, we inherit terminology from the sharp break endogenous unit root literature and call the critical values endogenous frequency critical values. In both cases the critical values are generated under the same DGP as in Enders and Lee (2012b) and Omay (2015), that is, the process defined in equation (3).

The first method provides one single unadjusted critical value for all values of $\hat{k}$. In this case we run the test on the data and estimate the frequency by minimizing the SSR over the prespecified search grid to generate the t-statistic $\tau\left(\hat{k} = \arg\min SSR\right)$. The t-statistics is used to calculate the critical value in the usual way, we denote this critical value $CV\left(\hat{k} \in Grid\right)$. Note that, although we do not distinguish between the various frequency estimates the critical values depend on the grid. Therefore, for example, the unadjusted endogenous integer frequency critical value differs from its fractional equivalent. Furthermore, note that in the extreme case where the grid is defined as a single point this method is equivalent to the exogenous frequency case in Enders and Lee (2012b) and Omay (2015).

Finally, note that this method is analogue to what is commonly practiced to generate critical values in the sharp break endogenous unit root literature. For this class of tests the test statistics generally depend on the presence and location of the break. Because the timing and presence of the break is usually unknown the break has to be estimated. Therefore, this class of tests are subject to the same problem as considered here. A common procedure is to assume a fixed number of breaks and to use the same critical value for the entire set of break locations. In this case the critical values are generated without any breaks in the DGP but the possible breaks are still estimated endogenously in the test (see, for example, Perron (2006) for an overview of the literature). Because the critical values are generated under a linear DGP they are constructed such that they hold the nominal size when no change is present in the data. Since it is often unclear whether a changing trend is prevalent in empirical research it is reasonable to construct a test that has this property. Furthermore, a nonstationary process does not revert to a any type of trend. Therefore, it appears particularly difficult to determine the presence and the functional form of a deterministic component in this case. Hence, it is desirable to construct unit root tests that treats any possible change in the deterministic component as unknown.

In the second alternative method we adjust the critical values according to the estimated frequency as recommended in Enders and Lee (2012b) and Omay (2015). However, unlike the previous authors, instead of exogenously appoint a value to the frequency in the test regression we estimate the frequency. So far this procedure is exactly equivalent of the previous alternative method where the frequency is estimated endogenously. The methods differ only in the treatment of the estimated t-statistics. Hence, we generate $\tau(\hat{k} = \arg\min SSR)$ under the DGP defined by (3) just like before. However, to adjust the critical values according to the frequency we sort the resulting statistics according to $\hat{k}$. In this manner we create a sample of t-statistics for each point in the grid based on the $\hat{k}$. That is, the sample of t-statistics for each grid point is selected conditioned on the fact that the frequency was estimated to belong to this point under the linear DGP defined by (3). This yields our endogenous frequency critical values that are adjusted according to the estimated frequency. Hence, the critical value becomes a function of the estimated frequency and is denoted $CV(\hat{k} = \arg\min SSR)$. Note that the critical values again are equal to the exogenous frequency critical values in the extreme case where the grid only contains one single point. Further note that, the critical values depend on both the estimated frequency and the grid. Therefore, the integer critical values are no longer a subset of the fractional frequency critical values as they were when the frequency was exogenously given.

4. Monte Carlo setup

In this section the Monte Carlo experiments that are presented in Section 5 are setup. All critical values are simulated following the procedures described in the previous section and we follow Enders and Lee (2012b) and consider frequencies up to 5. Furthermore, we consider the sample sizes $T \in (100, 200, 500)$. The critical values corresponding to Enders and Lee (2012b) and Omay (2015) are generated using 50,000 repetitions for each

$k_{exo} \in (0.1, 0.2, \dots, 5)$[4]. Because the critical values of Enders and Lee (2012b) are a subset of the critical values of Omay (2015) they are generated simultaneously. The endogenous critical values that correspond to the integer grid are generated using a total sample of 1,000,000 repetitions for $k_{Grid} \in (1, 2, \dots, 5)$. This sample is used to generate both types of endogenous integer frequency critical values. For the unadjusted endogenous critical values the entire sample is used. For the adjusted endogenous critical values the sample is divided according to the estimated frequencies as described above. This yields a little over 94,000 observations for the smallest samples in which case $\hat{k} = 5$ for all T. The endogenous fractional frequency critical values are generate using a total sample of 5,000,000 repetitions for $k_{Grid} \in (0.1, 0.2, \dots, 5)$. This yields a little over 26,000 observations for the smallest samples in which case $\hat{k} = 4.9$ for all T. The critical values are tabulated in the Table 1 to 3 in the Appendix.

In all simulations we use the following DGP

$$d_t = \alpha \sin(2\pi kt/T) + \beta \cos(2\pi kt/T) \quad (4)$$

$$u_t = \rho u_{t-1} + v_t \quad (5)$$

$$y_t = d_t + u_t \quad (6)$$

where d_t is the deterministic component and v_t is standard normal IID and ρ is the autoregressive coefficient. The deterministic component consists of a single Fourier function with frequency equal to k and we set $\alpha = 0$ and $\beta = 5$[5][6]. Finally, we have that $t = 1, \dots, T$. The size and power of the test are calculated using 5% critical values. For all series 50 observations are simulated for the time before the final time series starts. These observations were removed to construct the final time series. In all simulations we generate samples of 5,000 Monte Carlo repetitions. That is, of course, except for the simulations of

[4] Omay (2015) only considers frequencies up to 2. Because it requires little extra work we generate new critical values for the entire grid.

[5] In a preliminary analysis we also considered the cases where $\alpha = 3$ and $\beta = 5$ as well as $\alpha = 3$ and $\beta = 0$ as in Enders and Lee (2012a,b). However, we also considered normalizing the functions to have the same amplitude. In this case the results were almost equivalent for all of the combinations of α and β.

[6] We also applied the exogenous frequency critical values to the test under some multiple frequency DGPs. In general the implications were the same as those presented in this paper, however, in this case the size problems were more pronounced.

the critical values. All codes are written in MATLAB by the author of this paper and are available on request.

Because the purpose of this paper is to point out the importance of the critical values and the frequency search grid we assume the presence of a single Fourier frequency. Consequently, we neither pretest for nonlinearity using the F-test nor do we pare down the model to estimate number of frequencies[7]. Furthermore, we only consider the test that has an intercept and Fourier function in the deterministic component. In a preliminary analysis we also considered the test with a linear trend as well as the Lagrange multiplier test, which also contains a linear trend. However, the implications of these simulations were similar to those presented in this paper. Therefore, we chose to restrict the analysis to the Dickey-Fuller type test with an intercept and a single frequency Fourier component.

In Figure 3 to 5 we consider the size and power of the test with respect to the frequency in the Fourier function. For these simulations the frequency is defined as $k \in (0, 0.25, 0.5, \dots, 5)$ under the null and the alternative hypothesis defined by $\rho \in (1, 0.9, 0.8)$ with sample sizes equal to $T \in (100, 200, 500)$. Note that, because $\beta = 5$ in equation (4) the peak-to-peak amplitude of the deterministic component is $2 \times \beta = 10$. Further note that, for this set of α and β when $k \geq 0.5$ the deterministic component has both a global minimum and a maximum. Hence, magnitude of the change of the function has reached its maximum, which is equal to 10. Conversely, when $k < 0.5$ the trigonometric function has not yet reached its minimum. Hence, the magnitude of the change in the deterministic component is smaller than 10. Moreover, when $k = 0$ the deterministic component is constant, and thus no change is present. Therefore, when $k < 0.5$ the results do not only depend on the frequency but are also driven by the implied magnitude, which is smaller the closer k is to 0. Finally, note that if α and β would approach zero the Fourier function converge to a constant for any k. Hence, the $k = 0$ case is equivalent of the case where $\alpha = \beta = 0$ for any k.

[7] We did similar simulations and considered both assuming a single frequency at most and applying the F-pretest as well as multiple cumulated frequencies and pared down the model using AIC and BIC all with similar implications as reported in this paper.

In Figure 6 to 9 we investigate the distribution of the estimated frequency and the average SSR for each grid point $k_{Grid} \in Grid$. In this case we analyse the data from the simulations above and consider both the null and the alternative defined by $\rho = 1$ and $\rho = 0.8$ respectively. In the present case the deterministic component is characterized by the frequency $k \in (0, 1, 2)$ and the sample size is $T = 100$.

5. Results

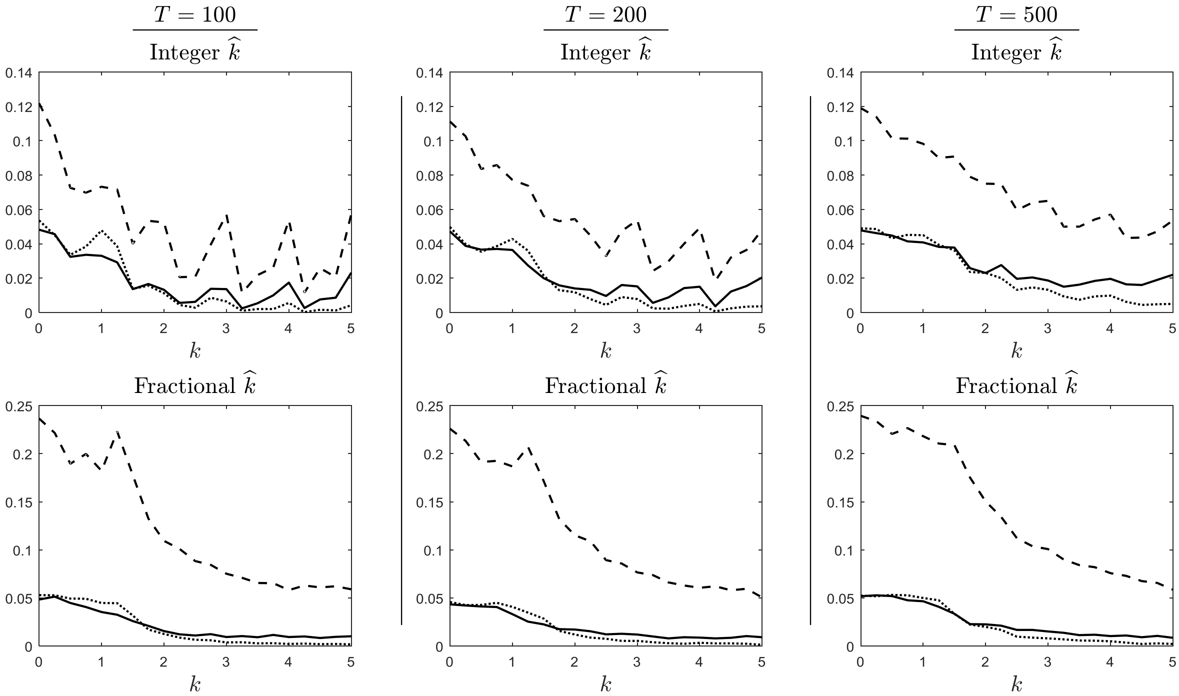
5.1. Size and power when the frequency is estimated

In this section we investigate the size and power of the test when it is applied to different DGPs with various deterministic Fourier components. We consider both the originally proposed exogenous frequency critical values and the two kinds of endogenous frequency critical values that are proposed in this paper.

We consider the case where the peak-to-peak amplitude is fixed and we let the frequency vary. The first three plots of Figure 3 pictures the size of the integer frequency type test in various sample sizes. The dashed line pictures the size of the test when the exogenous frequency critical values are applied as recommended in Enders and Lee (2012b). In the first plot, where $T = 100$, we can see that using exogenous frequency critical values the test is moderately oversized at 12% when $k = 0$, that is, when the trend is constant. When the frequency is equal to 1 the test is still somewhat oversized and when larger integer frequencies are present the size is close to the nominal 5%. However, the size does not converge monotonically in k. When the frequency takes fractional values the size is substantially lower compared to the when it is equal to an integer. Although, when we consider the larger sample sizes in the second and third plot of the figure we see that the difference in size between the fractional and integer cases is smaller when the sample size is increased.

The solid line in the figure represents the test with the adjusted endogenous frequency critical values and the dotted line corresponds to the unadjusted endogenous frequency critical value. First notice that, for both types of critical values the test holds the nominal size of 5% when $k = 0$. This result is of course by construction but is still worth noting as it justifies the use of these critical values. As the frequency increases the size diminishes also in the case where the endogenous frequency critical values are applied. However, because the size diminishes from the nominal level of 5% the test is undersized when large Fourier frequencies are in the DGP. Hence, both types of the endogenous frequency critical values are conservative. Interestingly, when the unadjusted critical values are applied the size is closer to the nominal level for frequencies up to about 1.5. Thereafter the size diminishes very quickly and approaches zero when the frequency increases. When the adjusted critical values are applied the size diminishes more gradually and settles at about 2% for large integer frequencies. The size is smaller when the Fourier component is characterized by fractional frequencies similar to when the exogenous frequency critical values are applied. This result is expected since the tests only differ in the critical values and not in the regression model. Hence, any problem inherent to the model, such as the grid, remains and thereby the problem persists.

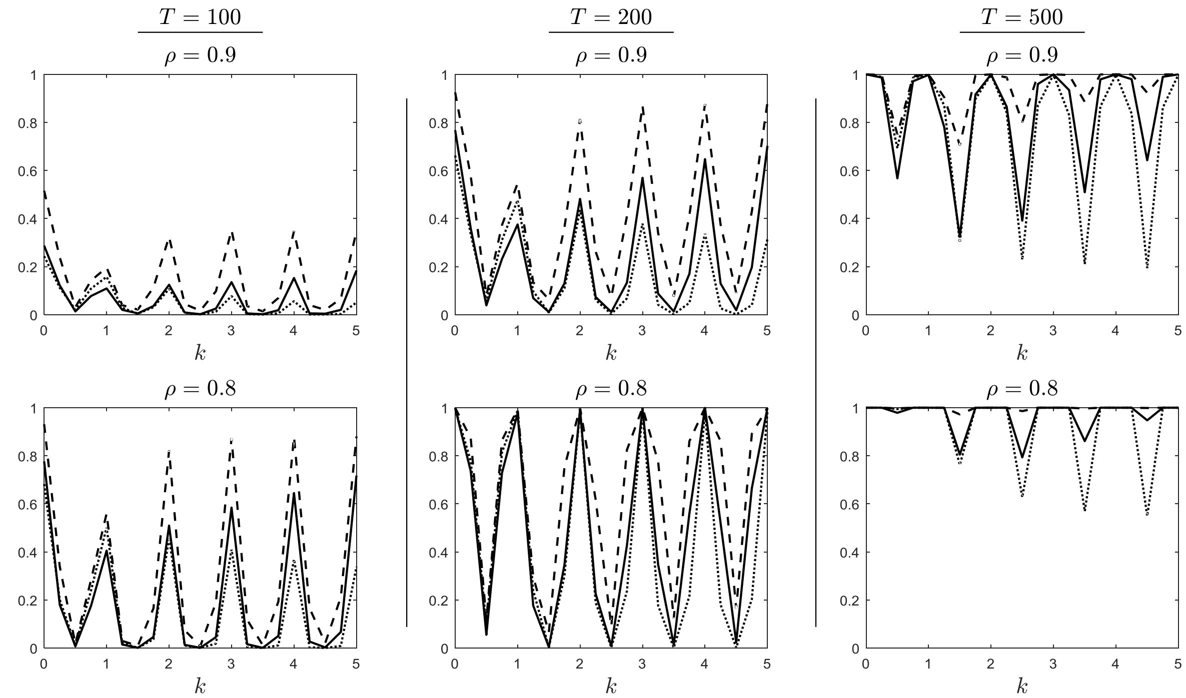
Figure 3: Size with respect to k



Notes: Figure 3 shows the size of the integer and fractional frequency unit root tests calculated using 5% critical values. The data is based on Monte Carlo simulations with 5,000 repetition where the stochastic component of the DGP has standard normal IID errors. Exogenous frequency critical values: — —, adjusted endogenous frequency critical values: —, unadjusted endogenous frequency critical values: ⋯.

The bottom three plots in Figure 3 shows the size of the test that uses the fractional frequency grid. The most striking observation here is that the test is considerably oversized when the exogenous frequency critical values are applied. When the $k = 0$ the size is well above 20% in all sample sizes. As the frequency increases the rejection rate is quite constant until the frequency is a little larger than 1. Thereafter it diminishes quite rapidly. For frequencies equal to 2 the size lies at around 10% and it finally approaches the nominal 5% for large frequencies.

Figure 4: Power of the integer frequency test with respect to k.



Notes: Figure 4 shows the power of the integer frequency unit root tests calculated using 5% critical values. The data is based on Monte Carlo simulations with 5,000 repetition where the stochastic component of the DGP has standard normal IID errors. Exogenous frequency critical values: — —, adjusted endogenous frequency critical values: —, unadjusted endogenous frequency critical values: ⋯.

Also when we use the fractional grid the test holds the nominal size when we apply the endogenous frequency critical values. The main difference from the integer frequency test is that the size does not differ depending on whether the DGP contains a fractional or integer frequency. Apart from that the tendency is similar to the integer case where the size is equal to the nominal 5% by construction at $k = 0$ and diminishes as the frequency increases. However, for the adjusted endogenous frequency critical values the size settles at about 1% for large frequencies instead of the 2% level in the integer grid case.

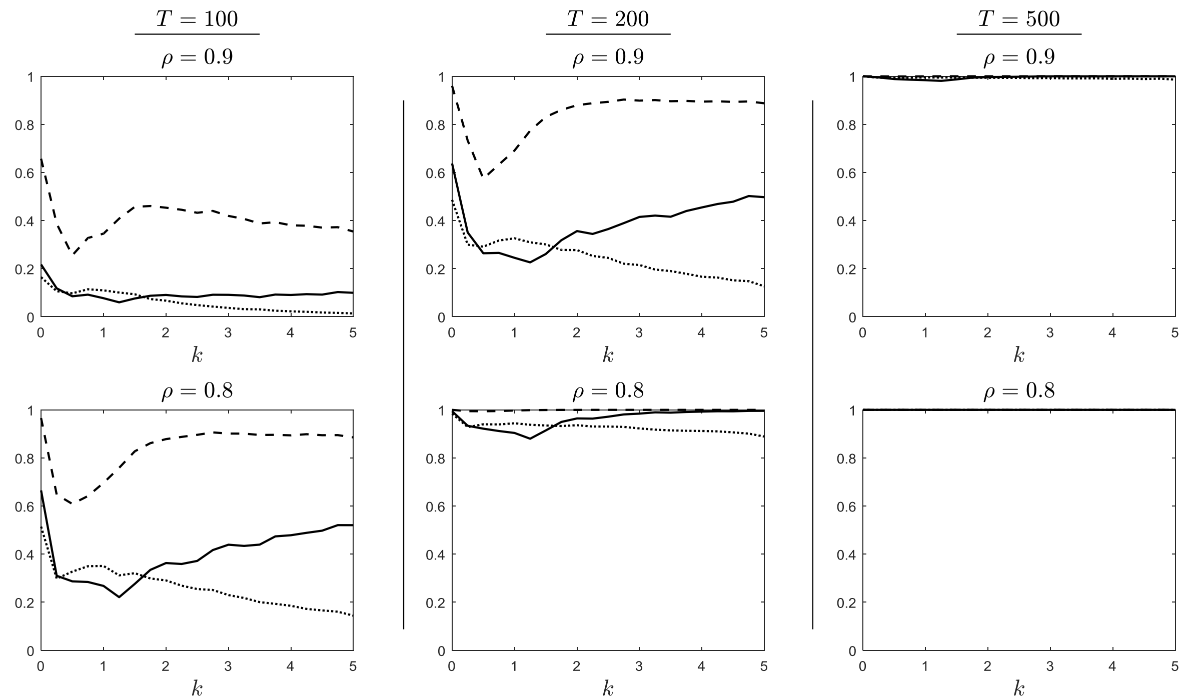
The test was reported to hold its nominal size when the frequency was exogenously appointed in Enders and Lee (2012b). Consequently, we conclude that the size problems must be a consequence of the frequency estimation. An interesting but undesirable detail is that the size problems are almost unchanged when the sample is increased. Most often we would expect the properties of the test to improve when the sample size is increased. Therefore, other things equal, the size should approach the nominal as the sample is increased. However, the problems associated with the error in the frequency estimate may also increase with the sample size. In this case the final performance of the test does not necessarily need to improve unless the frequency estimate would be correct each time. From Figure 3 we see that the size is almost unchanged when the sample is increased. Hence, the problem that the test is oversized for small frequencies is not only restricted to small samples. Note, however, that the dips in size when the integer grid is used and the DGP contains a fractional frequency are substantially smaller in the largest sample.

Figure 4 pictures the power of the integer frequency test with respect to the frequency in the Fourier component. In the three upper plots the case with the autoregression coefficient $\rho = 0.9$ is presented. In this setting the autoregression coefficient is relatively close to 1 and in the smallest sample the power is very low. However, it is a well known fact that unit root tests have low power under near unit roots in small samples. Hence, this result is in line with the unit root literature. First consider the integer values of k, in this case power of the test is the largest when the deterministic component is constant. For all three types of critical values the relatively high power at $k = 0$ falls substantially when k increases. Both when we apply the exogenous frequency critical values as well as the adjusted endogenous frequency critical values the power at the integer frequencies reaches their lowest levels at $k = 1$. Thereafter the power increases slowly with the frequency. The power of the test when we apply the unadjusted endogenous frequency critical values, on the other hand, diminishes monotonically with respect to the integer frequencies. This result is quite expected since t-statistic still varies with the frequency. When the corresponding critical value is not adjusted the power of the test deteriorates as the frequency increases.

When the deterministic component in the DGP contains a fractional frequency the power is very low. For the samples $T = 100$ and $T = 200$ the power is almost zero in the area between the integer frequencies. In the larges sample, although the power has converged to 1 at integer frequencies, the test is substantially less powerful at fractional frequencies.

The bottom three plots in Figure 4 pictures the case where the stochastic process is characterized by the autoregression coefficient $\rho = 0.8$. In this case the power is improved at the integer frequencies for all sample sizes. When k equals an integer plus 0.5, on the other hand, the power is still more or less equal to 0 in sample sizes $T = 100$ and $T = 200$. Only when $T = 500$ the power has reached high levels for all critical values.

Figure 5: Power of the fractional frequency test with respect to k.



Notes: Figure 5 shows the power of the fractional frequency unit root tests calculated using 5% critical values. The data is based on Monte Carlo simulations with 5,000 repetition where the stochastic component of the DGP has standard normal IID errors. Exogenous frequency critical values: — —, adjusted endogenous frequency critical values: —, unadjusted endogenous frequency critical values: ⋯.

The power of the fractional frequency test is investigated in Figure 5. The difference between power of the integer frequency test and the fractional frequency test is similar to that of the size. For example, the fractional frequency test does not suffer from the power problems that the integer frequency test has when a fractional frequency is in the deterministic component. In the smallest sample with $T = 100$ and $\rho = 0.9$ the power is very low when the endogenous frequency critical values are applied. The unadjusted endogenous critical value renders a test with power between 16% and 10% for frequencies up to 1.5. Thereafter we see a monotonic decline of the power as the frequency grows larger and it finally reaches its minimum at just over 1% at $k = 5$. The use of the adjusted endogenous frequency critical values results in 22% power when $k = 0$. The power falls instantly as the frequency increases and takes its minimum at 6% when $k = 1.25$. After the power has reached its minimum it slowly increases to about 10% for large frequencies. The power of the test when the exogenous frequency critical values are applied is much larger compared to the when the endogenous critical values are used. However, this should be seen in the light of the fact that the test is significantly oversized when these critical values are applied. Therefore, although the nominal power is higher when the exogenous frequency critical values are applied it is not comparable to the other cases, and hence we cannot say that the test is more powerful in general.

When the sample is increased to $T = 200$ the power curves are shifted upwards. The adjusted endogenous and the exogenous frequency critical values tests are improved quite a lot for the cases where the frequency is large or equal to zero. Unfortunately, the power improves more slowly over the span where the frequency is small to moderate. This creates a valley in the power curve where the frequencies are between about 0.25 and 1.5. When $T = 500$ the power has reached almost 100% in all cases. When the autoregression coefficient is changed to $\rho = 0.8$ and $T = 100$ the power curves are shifted in a similar way as they were when the sample size was extended to $T = 200$ and ρ was fixed. When $\rho = 0.8$ and the sample is increased to $T = 200$ the power is high for all sets of critical values. The power of the exogenous frequency critical values test is 100% and both of the endogenous frequency critical values tests have a power of almost 90% or more. When $\rho = 0.8$ and $T = 500$ the power has reached 100% in all cases.

In Figure 3 and 4 we saw that both the size and power of the integer frequency test crucially depend on the functional form of the deterministic component in the DGP. This result indicates that integer frequencies often are insufficient in approximating the shape of fractional frequencies. Consequently, the properties of the test deteriorates the more the frequency in the deterministic component differs from integer values. The issue appears particularly problematic when the power is considered. For example, the power varies between 0% and 100% in the cases where $\rho = 0.8$ and $T = 200$ when the endogenous frequency critical values are applied. Hence, we can easily find a functional form for which the test is unable to reject the null. Moreover, note that in this case the power is even smaller than the size. In empirical research it is not unlikely that the functional form of a trend would be close to some Fourier function with a fractional frequency. For example, this may be the case if the data contains a cyclical component and the data spans half cycles. Another plausible example where the trend could be approximated by a fractional frequency Fourier function is where there is a smooth permanent break in level. Therefore, the flexibility of the fractional frequencies provides an advantage over integer frequencies.

The use of fractional frequencies solves some of the size and power problems of the test that uses integer frequencies. However, when the exogenous frequency critical values are applied to the fractional frequency test it becomes much more oversized, and hence a new problem arises. A researcher could perhaps accept a test that is oversized at between 11% and 12% at most, which is the case of the integer frequency test. However, the fractional frequency test is oversized at a level of more than 20% for frequencies ranging from 0 to 1.5. At this level the size problems are severe and the validity of the test can be seriously questioned. Therefore, when applying the fractional frequency test there is much to gain from using the conservative endogenous frequency critical values.

Both types of endogenous frequency critical values are constructed to hold the nominal size for the case where $k = 0$. As the frequency increases the size of the test diminishes for both sets of critical values, however, at different speed. Figure 3 indicates that the test with the unadjusted critical values weakly dominates the test with the adjusted critical values in terms of the size for frequencies up to about 1.5. With exception for the case where the frequency is equal to or close to zero this also holds for the power. For frequencies that are larger than 1.5, on the other hand, the test that uses the adjusted endogenous critical values dominates its unadjusted equivalent. Hence, neither set of critical values dominates the other for the entire set of frequencies. Therefore, which set of critical values that are to prefer depends on the deterministic component in the DGP. However, in empirical research any deterministic component is in general unknown. Consequently, to say which critical values that are to prefer for given dataset would require assumptions about the functional form of the deterministic component. However, the reason for using a Fourier function to approximate the deterministic trend is to avoid having to define its functional form. Hence, it is inherent to the choice of using this type of modeling strategy that we do not want to make such assumptions.

Because neither method dominates the other in all cases one may consider the extent to which the methods dominates the other in the different cases. When the unadjusted critical values are applied the power of the test never exceeds its adjusted equivalent by more than 10% in the considered cases. The adjusted critical values, on the other hand, dominates the unadjusted by more than 10% quite soon after it has exceeded it. Furthermore, for large frequencies the difference in power is almost 40% in many cases. This result is somewhat in favor of the adjusted critical values. However, because the use of the adjusted critical values is not superior in all cases the argument for these critical values is not unambiguous.

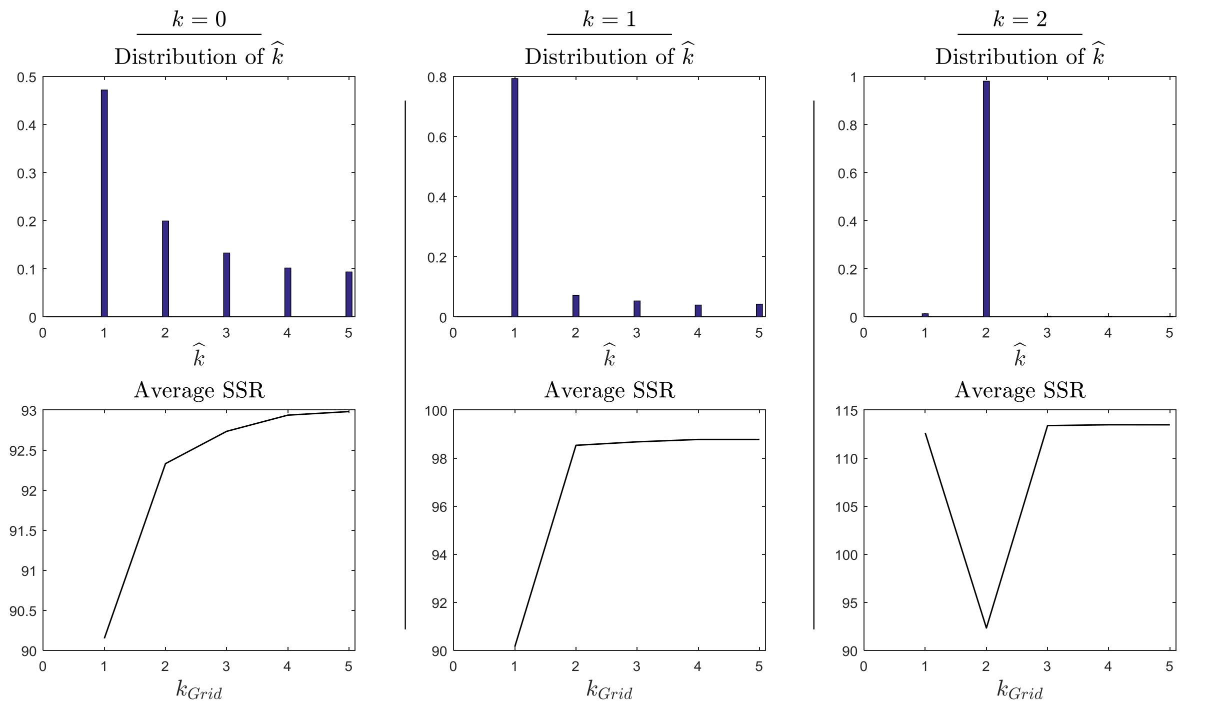
In this section we considered the cases where the change in the deterministic component of the DGP is either absent or of magnitude 10. It is important to point out that if we had considered a smaller magnitudes then the size and power curves would have been closer to the case where the change is absent. Consequently, even for large k the implications would have been similar to those of the case where $k = 0$ if the magnitude of the change would have been small enough.

5.2 Distribution of the estimated frequency

In this section we investigate the distribution of the estimated frequency and the SSR. As established in Enders and Lee (2012a,b) the SSR depends on the frequency in the test regression. Therefore, it is interesting to have a closer look at the SSR as well as the distribution of the frequency estimate to get a better understanding of the properties of the test. We consider the cases where $T = 100$ for $k \in (0, 1, 2)$ under both the null and the alternative with $\rho = 0.8$. Figure 6 to 9 show normalized histograms of the distribution of $\hat{k}$

and plots of the expected value of the SSR for each frequency in the search grid. Note that, only integer values of the frequency in the deterministic component are considered. Hence, it is always possible for the estimated frequency to equal the true, that is, with exception of when $k = 0$. Note that we only consider the average SSR for each grid point. It would of course have been interesting to look at the entire distribution of the SSR at each point as well as their joint distributions. However, this would create a multi dimensional problem that quickly becomes very complicated with the number of grid points. Hence, for simplicity we restrict the analysis to only cover the average SSR.

Figure 6: Integer frequency estimation under the null



Notes: Figure 6 considers the frequency estimation of the integer Fourier frequency unit root test under the null with $T = 100$. The three upper plots in the figure show normalized histograms of the distribution of the estimated frequency. The three plots at the bottom show the average SSR for each grid point in the frequency estimation grid. The data is based on Monte Carlo simulations with 5,000 repetition where the stochastic component of the DGP has standard normal IID errors.

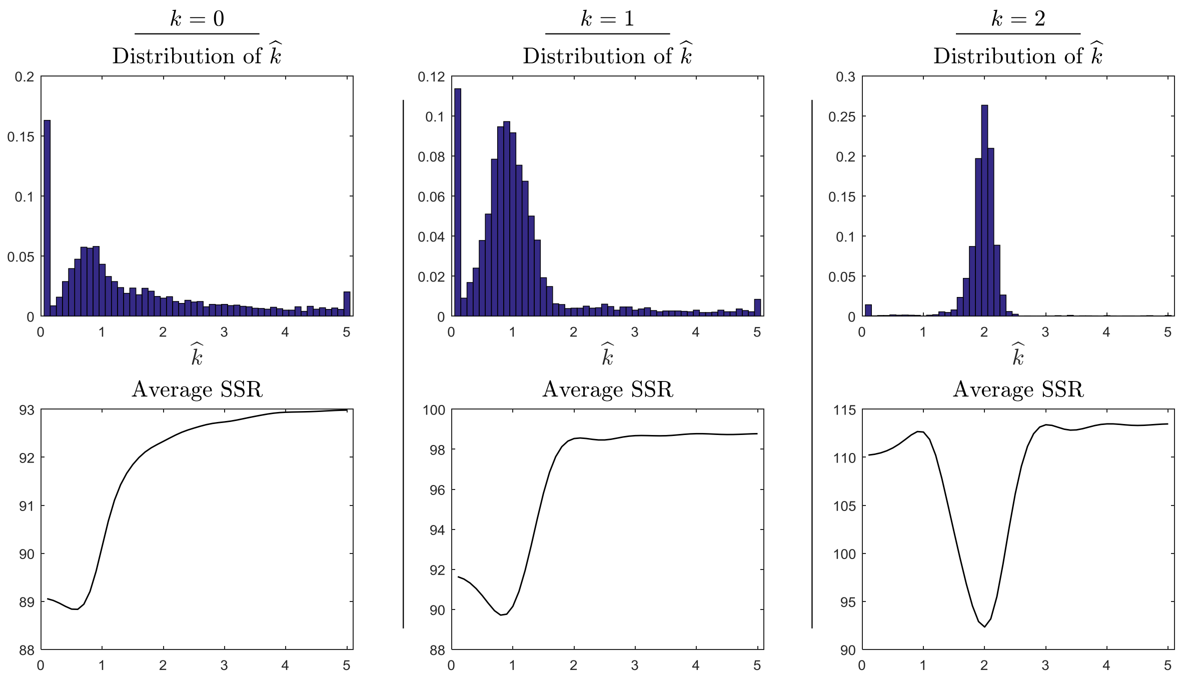
Figure 6 pictures the integer frequency case under the null. The first histogram shows the distribution of the estimated frequency when $k = 0$. In this case, because the deterministic component is in fact linear neither value of $\hat{k}$ is correct. However, although all frequencies in the grid differ from the true the distribution is concentrated at $\hat{k} = 1$, where we find almost 50% of its mass. Hence, the frequency estimate is skewed towards the smallest point in the grid if we apply the test on a process with a constant deterministic trend. Note that also the expected value of the SSR is minimized at $k_{Grid} = 1$. When $k = 1$ the distribution is shifted even more to the left and a greater part of the mass is situated at $\hat{k} = 1$. In a similar way the expected value of the SSR at this point differs a bit more from the others grid points compared to what it did in the previous case. When $k = 2$ almost all of the mass is situated at $\hat{k} = 2$. In this case the average SSR at this point differs more distinctively from the other grid points.

The overall ability to estimate the frequency in this case seems quite good and the estimate equals the true frequency in 80% of the cases when $k = 1$ and in nearly 100% of the cases when $k = 2$. However, it is important to point out that when we only consider integer frequencies we restrict the analysis to functions that completes full cycles. In this case, where the frequency in the DGP also appears in the search grid, there is in a sense a lot of information in the grid. In this case, we are effectively are choosing between the number of cycle we wish to model the data with. Conversely, if k would have been a fractional number the true frequency would not have appeared in the grid. In this case the estimated frequency would never be correct. Whether this is would be a problem or not depend on how the test performs in terms of size and power. As we saw in the previous section in the prevailing situation the power is equal to zero when k is equal to an integer plus 0.5.

Figure 7 illustrates the case where the fractional frequency test is considered under the null. Similar to the integer frequency case, when $k = 0$ there is a high concentration of the mass of the distribution at the smallest point of the grid, that is, when $\hat{k} = 0.1$. Interestingly, whereas $\hat{k} = 0.2$ is estimated rarely with probability close to 0.01 the following frequencies are estimated much more frequently. This creates a bump in the distribution that is centered around 0.8. In the corresponding plot for the average SSR wee see that this is also where the SSR has its minimum. When $k = 1$ the distribution is shifted to the right as expected. However, the most frequently estimated value is still 0.1. Furthermore, exempt from the first grid point $\hat{k} = 0.9$ is the most frequently estimated followed by $\hat{k} = 0.8$, which in turn is followed by $\hat{k} = 1$. Moreover, about 35% of the estimated frequencies are off the true frequency by 0.5 or more, that is, by half a cycle or more. Hence, estimating the frequency by minimizing the SSR does not seem to perform very well for frequencies of this magnitude.

In the corresponding plot of the average SSR we can see that the SSR takes its minimum at 0.8. Consequently, the SSR attain its minimum at the same point as it did when $k = 0$. This illustrates the problem of minimizing the SSR to estimate the frequency when SSR depends on k_{Grid}. Because the average SSR is minimized at a different value than k it is perhaps not surprising that the most frequently estimated $\hat{k}$ also differs from k. Of course the entire distribution of SSR and the dependence between the grid points also determines $\hat{k}$. However, everything else held constant, a smaller average SSR for some grid point k_{Grid} would imply that it minimizes the SSR more often than the others. When $k = 2$ the distribution of $\hat{k}$ takes its maximum value at the correct frequency and at about 85% of the times the estimate is within 0.2 frequencies from the true. The plot of the corresponding average SSR now shows that it is distinctively lower at the true frequency.

Figure 7: Fractional frequency estimation under the null

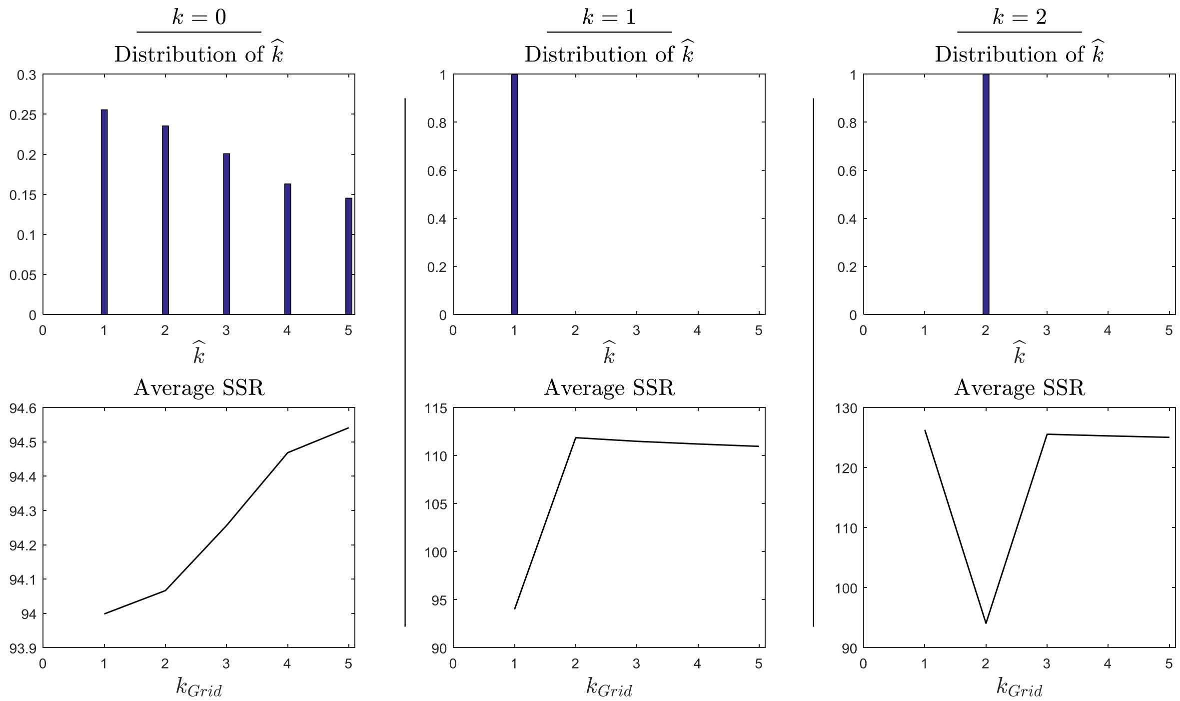


Notes: Figure 7 considers the frequency estimation of the fractional Fourier frequency unit root test under the null with $T = 100$. The three upper plots in the figure show normalized histograms of the distribution of the estimated frequency. The three plots at the bottom show the average SSR for each grid point in the frequency estimation grid. The data is based on Monte Carlo simulations with 5,000 repetition where the stochastic component of the DGP has standard normal IID errors.

Overall, the difference in both the distribution of $\hat{k}$ and the average SSR is relatively small between the cases where $k = 0$ and $k = 1$. When the Fourier frequency is changed to $k =$

2, on the other hand, the distribution of $\hat{k}$ and the average SSR clearly indicates where the true frequency lies.

Figure 8: Integer frequency estimation under the alternative $\rho = 0.8$



Notes: Figure 8 considers the frequency estimation of the integer Fourier frequency unit root test under the alternative with autoregression coefficient $\rho = 0.8$ and $T = 100$. The three upper plots in the figure show normalized histograms of the distribution of the estimated frequency . The three plots at the bottom shows the average SSR for each grid point in the frequency estimation grid. The data is based on Monte Carlo simulations with 5,000 repetition where the stochastic component of the DGP has standard normal IID errors.

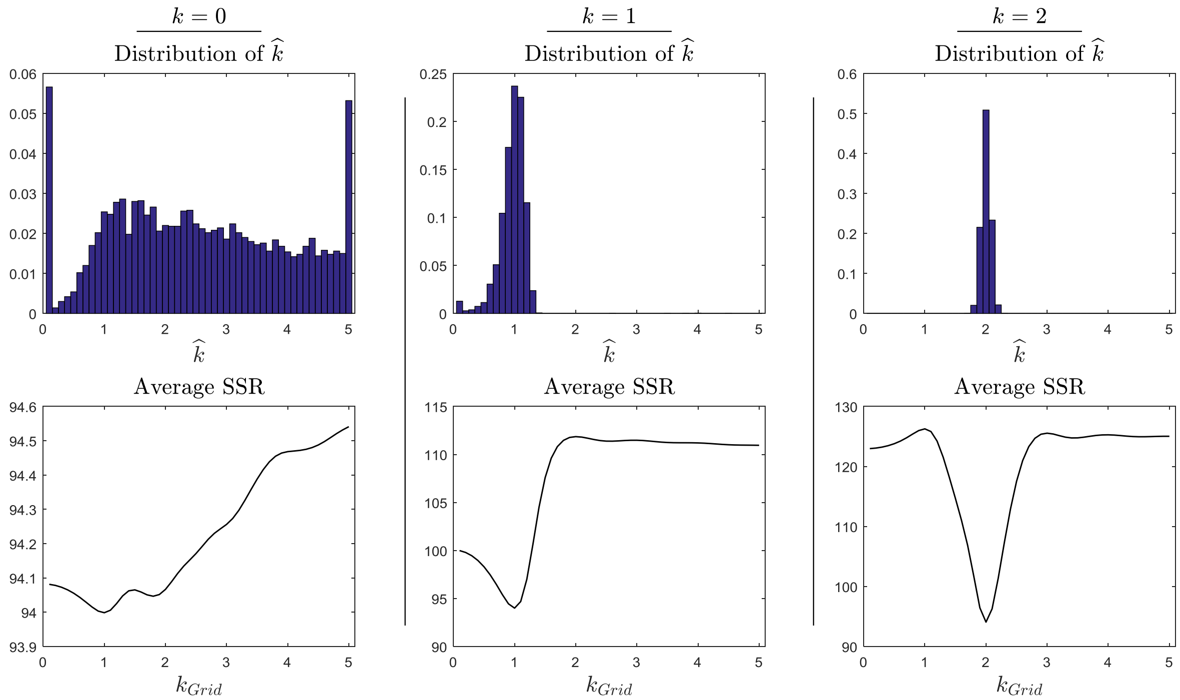
In Figure 8 we consider the integer frequency test under the alternative with $\rho = 0.8$. The first plot pictures the distribution of $\hat{k}$ for the case where $k = 0$, now the distribution is quite close to uniform. Hence, a lot of the bias towards the left endpoint has disappeared. Turning our attention to the first plot at the bottom we can see that the average SSR takes a quite different shape from how it looked like under the null. It still takes its minimum at the first grid point, however, the difference between the smallest and the largest value is much smaller compared to Figure 6. The difference is now only 0.5 in contrast to around 3 under the null. For the cases when $k = 1$ and $k = 2$ the estimate equals the true 100% of the times. This is also reflected in the average SSR plots where there is large difference between the cases where the k_{Grid} is equal to k and where k_{Grid} differs from k.

The distribution of the estimated fractional frequency under the alternative is illustrated in Figure 9. Again the first plot shows the distribution of the estimated frequency when $k = 0$. The mass at the left endpoint is greatly reduced and the bump that characterized the distribution under the null has almost disappeared. We conclude that the distribution is closer to uniform in this case as well. However, the smallest values of the estimate just next to the leftmost endpoint of the grid are still much less frequent than the others. Note that the mass at the rightmost endpoint has increased substantially. The corresponding average SSR looks similar to how it looked like for the integer grid where the difference between the smallest and largest value is quite small. When $k = 1$ the precision of the estimate is improved and the distribution of the $\hat{k}$ now has its peak at k. In the case where $k = 2$ the estimate is correct in about 50% of the time. Moreover, the estimate is only off by 0.2 or more in approximately 4% of the cases. The improvements in the precision of the estimated frequency are also reflected in the average SSR plots, in which case the minimums are found at the true frequencies. However, note that, although the frequency estimate when $k = 1$ is improved quite a lot when we go from the null to the alternative it is still much less accurate compared to the $k = 2$ case.

The consequence of the varying precision of $\hat{k}$ at different values of k is also reflected in the size and power plots in the previous section. As reported in Enders and Lee (2012b) the test holds its size when the frequency is exogenously given. Consequently, that the test is oversized when we estimate the frequency must be a result of the estimation. The results in this section show that with the method at hand the frequency is much more difficult to estimate when $k = 1$ compared to when $k = 2$[8]. If we reconsider Figure 3 we can see that there is a considerable difference of the extent to which the test is oversized for $k = 1$ and $k = 2$. Similarly, in Figure 4 and 5 the region that ranges from values of k close to zero to about 1.5 is characterized by lower power compared to the other values of k. This is approximately the region that covers the bump in the first plot in Figure 7. Note that $k = 1$, which is difficult to estimate, lies in this region whereas $k = 2$, which is much easier to estimate, does not.

[8] In a preliminary analysis we considered other values of k with the result that showed that the same argument can be made small and large values of k in general rather than just for $k = 1$ and $k = 2$.

Figure 9: Fractional frequency estimation under the alternative $\rho = 0.8$



Notes: Figure 9 considers the frequency estimation of the fractional Fourier frequency unit root test under the alternative with autoregression coefficient $\rho = 0.8$ and $T = 100$. The three upper plots in the figure show normalized histograms of the distribution of the estimated frequency . The three plots at the bottom shows the average SSR for each grid point in the frequency estimation grid. The data is based on Monte Carlo simulations with 5,000 repetition where the stochastic component of the DGP has standard normal IID errors.

6. Conclusions

In this paper we have investigated the size and power properties of the flexible Fourier form Dickey-Fuller unit root test proposed by Enders and Lee (2012b). We considered the cases where the Fourier frequency is estimated using an integer search grid as originally suggested as well as where a fractional search grid is used as proposed by Omay (2015). By Monte Carlo simulations we show that when the integer grid is used the test sometimes has zero power in small samples when the DGP is characterized by a fractional frequency. When we allow the estimated frequency to take fractional values the power problem at the fractional frequencies is solved. However, in this case the test is considerably oversized when the deterministic component in the DGP is either linear or contains a Fourier function with a frequency that is small.

To get a better understanding of the properties of the test we investigated the distribution of the estimated frequency. We found that the frequency estimate is biased towards the smallest grid point under the null when the deterministic component is linear or contains a frequency that is small. When we consider larger frequencies and when we go from the null to the alternative hypothesis the frequency estimate is improved. However, also under the alternative it still difficult to estimate the frequency of the Fourier deterministic component when it is small.

To solve the size problems of the fractional frequency test we simulate two types of new critical values. The critical values are conservative and are constructed to take into account that the frequency is endogenously estimated in the test procedure. The first type provides a single unadjusted critical value for the entire grid of frequencies. The second type provides a set of critical values in which each critical value is adjusted to the estimated frequency. Neither of the new critical values dominates the other in all cases. However, the adjusted critical values yields a test such that when it is dominated by the other it is only a little less powerful. Conversely, the unadjusted critical value renders a test that is substantially less powerful than the other in many cases. Therefore, the results are somewhat in favor of the adjusted endogenous frequency critical values. We find that although the test is sometimes undersized when using these critical values its power is still at a reasonable level.

Finally, in this paper we only considered Fourier frequency deterministic components in the DGP of the Monte Carlo study. However, the reason for using Fourier frequencies to model time series is that they are suppose to be able to approximate a large variety of other functional forms. Therefore, an area of further research is to investigate how the test performs when the deterministic component takes other functional forms. Furthermore, it would also be interesting to compare it to other unit root tests that models the deterministic component differently.

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Appendix

Table 1: Endogenous integer frequency critical values

Adjusted critical values for τ , $CV\left(\hat{k} = \arg\min SSR\right)$									
	$T = 100$			$T = 200$			$T = 500$		
$\hat{\boldsymbol{k}}$	1%	5%	10%	1%	5%	10%	1%	5%	10%
1	-4.66	-4.08	-3.79	-4.59	-4.04	-3.76	-4.55	-4.01	-3.74
2	-4.48	-3.84	-3.50	-4.41	-3.81	-3.48	-4.38	-3.79	-3.48
3	-4.30	-3.62	-3.26	-4.26	-3.60	-3.24	-4.22	-3.59	-3.25
4	-4.13	-3.44	-3.06	-4.08	-3.41	-3.06	-4.09	-3.42	-3.06
5	-4.01	-3.28	-2.91	-3.97	-3.29	-2.93	-3.96	-3.29	-2.93
	Unadjusted critical values for τ , $CV\left(\hat{k} \in Grid\right)$								
	$T = 100$			$T = 200$			$T = 500$		
	1%	5%	10%	1%	5%	10%	1%	5%	10%
	-4.53	-3.92	-3.60	-4.46	-3.88	-3.57	-4.43	-3.86	-3.56
	Critical values for the F-statistic								
	$T = 100$			$T = 200$			$T = 500$		
	1%	5%	10%	1%	5%	10%	1%	5%	10%
	10.44	7.61	6.35	10.05	7.42	6.23	9.89	7.33	6.17

Table 2: Endogenous fractional frequency critical values

Adjusted critical values for τ, $CV(\hat{k} = \arg\min SSR)$

	$T = 100$			$T = 200$			$T = 500$				$T = 100$			$T = 200$			$T = 500$		
$\hat{k}$	1%	5%	10%	1%	5%	10%	1%	5%	10%	$\hat{k}$	1%	5%	10%	1%	5%	10%	1%	5%	10%
0.1	-4.84	-4.28	-4.00	-4.75	-4.22	-3.96	-4.70	-4.20	-3.94	**2.6**	-4.79	-4.12	-3.79	-4.70	-4.08	-3.75	-4.65	-4.06	-3.73
0.2	-5.09	-4.52	-4.24	-5.00	-4.45	-4.19	-4.93	-4.42	-4.16	**2.7**	-4.74	-4.09	-3.72	-4.64	-4.04	-3.71	-4.62	-4.01	-3.68
0.3	-5.11	-4.54	-4.25	-5.02	-4.46	-4.19	-4.94	-4.42	-4.16	**2.8**	-4.72	-4.05	-3.70	-4.64	-4.00	-3.65	-4.58	-3.98	-3.66
0.4	-5.11	-4.53	-4.24	-4.99	-4.45	-4.19	-4.94	-4.43	-4.16	**2.9**	-4.73	-4.01	-3.65	-4.61	-3.97	-3.62	-4.58	-3.97	-3.64
0.5	-5.11	-4.53	-4.24	-5.00	-4.46	-4.19	-4.94	-4.43	-4.17	**3.0**	-4.68	-3.99	-3.63	-4.61	-3.96	-3.61	-4.53	-3.94	-3.61
0.6	-5.11	-4.53	-4.25	-5.00	-4.46	-4.19	-4.94	-4.43	-4.17	**3.1**	-4.67	-4.00	-3.63	-4.58	-3.95	-3.60	-4.52	-3.94	-3.60
0.7	-5.10	-4.54	-4.25	-5.00	-4.47	-4.20	-4.94	-4.43	-4.17	**3.2**	-4.63	-3.96	-3.60	-4.57	-3.91	-3.59	-4.56	-3.93	-3.59
0.8	-5.11	-4.54	-4.25	-5.01	-4.47	-4.21	-4.95	-4.44	-4.18	**3.3**	-4.61	-3.95	-3.59	-4.59	-3.90	-3.56	-4.54	-3.90	-3.58
0.9	-5.12	-4.56	-4.27	-5.02	-4.49	-4.22	-4.96	-4.45	-4.19	**3.4**	-4.66	-3.94	-3.57	-4.57	-3.94	-3.57	-4.54	-3.91	-3.57
1.0	-5.13	-4.57	-4.29	-5.04	-4.51	-4.24	-4.98	-4.46	-4.21	**3.5**	-4.59	-3.92	-3.55	-4.53	-3.90	-3.53	-4.51	-3.89	-3.55
1.1	-5.15	-4.60	-4.32	-5.05	-4.53	-4.26	-4.99	-4.49	-4.23	**3.6**	-4.55	-3.89	-3.52	-4.49	-3.85	-3.51	-4.52	-3.85	-3.51
1.2	-5.17	-4.61	-4.33	-5.05	-4.53	-4.27	-5.02	-4.50	-4.24	**3.7**	-4.54	-3.84	-3.49	-4.49	-3.83	-3.48	-4.47	-3.82	-3.48
1.3	-5.18	-4.61	-4.32	-5.06	-4.54	-4.27	-5.00	-4.49	-4.23	**3.8**	-4.53	-3.82	-3.44	-4.46	-3.81	-3.45	-4.39	-3.81	-3.45
1.4	-5.14	-4.57	-4.28	-5.03	-4.50	-4.22	-4.97	-4.45	-4.19	**3.9**	-4.51	-3.80	-3.44	-4.44	-3.78	-3.43	-4.41	-3.78	-3.44
1.5	-5.09	-4.50	-4.21	-4.96	-4.44	-4.15	-4.92	-4.40	-4.13	**4.0**	-4.49	-3.80	-3.43	-4.45	-3.76	-3.41	-4.40	-3.77	-3.43
1.6	-5.03	-4.42	-4.11	-4.90	-4.37	-4.07	-4.86	-4.32	-4.04	**4.1**	-4.46	-3.78	-3.41	-4.44	-3.77	-3.42	-4.38	-3.75	-3.41
1.7	-4.95	-4.35	-4.04	-4.87	-4.30	-4.00	-4.82	-4.27	-3.98	**4.2**	-4.49	-3.75	-3.38	-4.40	-3.74	-3.39	-4.38	-3.74	-3.40
1.8	-4.92	-4.31	-3.99	-4.85	-4.26	-3.95	-4.79	-4.21	-3.91	**4.3**	-4.51	-3.78	-3.40	-4.45	-3.76	-3.41	-4.39	-3.76	-3.41
1.9	-4.90	-4.28	-3.95	-4.81	-4.22	-3.91	-4.77	-4.20	-3.90	**4.4**	-4.43	-3.73	-3.36	-4.41	-3.73	-3.37	-4.38	-3.74	-3.40
2.0	-4.86	-4.26	-3.92	-4.79	-4.21	-3.90	-4.75	-4.19	-3.89	**4.5**	-4.43	-3.73	-3.36	-4.37	-3.70	-3.36	-4.35	-3.73	-3.38
2.1	-4.88	-4.24	-3.91	-4.79	-4.19	-3.88	-4.74	-4.17	-3.86	**4.6**	-4.41	-3.70	-3.33	-4.36	-3.71	-3.35	-4.34	-3.68	-3.36
2.2	-4.84	-4.23	-3.90	-4.77	-4.18	-3.87	-4.74	-4.17	-3.85	**4.7**	-4.35	-3.64	-3.30	-4.36	-3.66	-3.31	-4.30	-3.65	-3.33
2.3	-4.86	-4.22	-3.89	-4.77	-4.17	-3.85	-4.73	-4.15	-3.84	**4.8**	-4.35	-3.68	-3.30	-4.34	-3.67	-3.31	-4.27	-3.63	-3.31
2.4	-4.88	-4.20	-3.87	-4.76	-4.15	-3.83	-4.73	-4.14	-3.83	**4.9**	-4.38	-3.64	-3.28	-4.32	-3.65	-3.29	-4.34	-3.65	-3.29
2.5	-4.81	-4.17	-3.84	-4.74	-4.14	-3.81	-4.66	-4.10	-3.79	**5.0**	-4.33	-3.62	-3.26	-4.29	-3.62	-3.28	-4.25	-3.62	-3.27

Unadjusted critical values for τ, $CV(\hat{k} \in Grid)$

$T = 100$			$T = 200$			$T = 500$		
1%	5%	10%	1%	5%	10%	1%	5%	10%
-4.99	-4.41	-4.11	-4.90	-4.35	-4.06	-4.84	-4.31	-4.04

Critical values for the F-statistic

$T = 100$			$T = 200$			$T = 500$		
1%	5%	10%	1%	5%	10%	1%	5%	10%
13.35	10.26	8.84	12.74	9.91	8.60	12.41	9.73	8.47

Table 3: Exogenous frequency critical values

Critical values for τ, $CV(\hat{k} = k_{exo})$																			
	$T = 100$			$T = 200$			$T = 500$				$T = 100$			$T = 200$			$T = 500$		
$\hat{k}$	1%	5%	10%	1%	5%	10%	1%	5%	10%	$\hat{k}$	1%	5%	10%	1%	5%	10%	1%	5%	10%
0.1	-4.49	-3.89	-3.59	-4.45	-3.87	-3.57	-4.40	-3.84	-3.56	**2.6**	-3.84	-3.13	-2.78	-3.79	-3.13	-2.77	-3.80	-3.13	-2.78
0.2	-4.50	-3.90	-3.59	-4.46	-3.88	-3.58	-4.41	-3.85	-3.56	**2.7**	-3.80	-3.10	-2.76	-3.77	-3.10	-2.76	-3.77	-3.11	-2.76
0.3	-4.51	-3.91	-3.60	-4.46	-3.88	-3.59	-4.41	-3.86	-3.57	**2.8**	-3.78	-3.08	-2.74	-3.75	-3.09	-2.74	-3.75	-3.09	-2.74
0.4	-4.51	-3.92	-3.61	-4.46	-3.89	-3.60	-4.42	-3.87	-3.59	**2.9**	-3.76	-3.06	-2.72	-3.74	-3.07	-2.73	-3.73	-3.07	-2.72
0.5	-4.52	-3.93	-3.62	-4.48	-3.91	-3.61	-4.43	-3.88	-3.60	**3.0**	-3.75	-3.05	-2.71	-3.73	-3.05	-2.71	-3.71	-3.06	-2.71
0.6	-4.53	-3.94	-3.63	-4.49	-3.92	-3.62	-4.41	-3.88	-3.60	**3.1**	-3.74	-3.04	-2.70	-3.71	-3.04	-2.70	-3.69	-3.05	-2.71
0.7	-4.53	-3.93	-3.62	-4.48	-3.91	-3.61	-4.42	-3.87	-3.59	**3.2**	-3.74	-3.04	-2.70	-3.71	-3.04	-2.70	-3.68	-3.04	-2.70
0.8	-4.51	-3.91	-3.61	-4.48	-3.89	-3.58	-4.41	-3.85	-3.57	**3.3**	-3.72	-3.03	-2.69	-3.69	-3.03	-2.69	-3.67	-3.03	-2.69
0.9	-4.48	-3.87	-3.56	-4.45	-3.84	-3.53	-4.39	-3.81	-3.52	**3.4**	-3.71	-3.02	-2.68	-3.68	-3.02	-2.68	-3.66	-3.03	-2.69
1.0	-4.44	-3.81	-3.49	-4.38	-3.78	-3.48	-4.33	-3.76	-3.46	**3.5**	-3.71	-3.01	-2.68	-3.65	-3.02	-2.67	-3.65	-3.01	-2.69
1.1	-4.39	-3.75	-3.41	-4.33	-3.71	-3.40	-4.28	-3.70	-3.38	**3.6**	-3.68	-3.00	-2.67	-3.64	-3.00	-2.67	-3.63	-3.00	-2.68
1.2	-4.33	-3.67	-3.34	-4.26	-3.65	-3.33	-4.24	-3.64	-3.31	**3.7**	-3.67	-2.99	-2.66	-3.63	-2.99	-2.66	-3.64	-3.00	-2.67
1.3	-4.27	-3.61	-3.27	-4.22	-3.59	-3.26	-4.18	-3.57	-3.24	**3.8**	-3.65	-2.98	-2.65	-3.61	-2.98	-2.66	-3.64	-2.99	-2.66
1.4	-4.21	-3.55	-3.20	-4.17	-3.53	-3.19	-4.12	-3.51	-3.18	**3.9**	-3.63	-2.98	-2.64	-3.60	-2.98	-2.65	-3.64	-2.98	-2.65
1.5	-4.15	-3.49	-3.14	-4.10	-3.47	-3.12	-4.08	-3.45	-3.11	**4.0**	-3.63	-2.97	-2.63	-3.58	-2.97	-2.64	-3.64	-2.98	-2.65
1.6	-4.11	-3.44	-3.08	-4.06	-3.41	-3.06	-4.03	-3.39	-3.05	**4.1**	-3.62	-2.96	-2.63	-3.58	-2.97	-2.64	-3.63	-2.97	-2.64
1.7	-4.08	-3.39	-3.02	-4.01	-3.36	-3.00	-4.00	-3.35	-3.00	**4.2**	-3.61	-2.95	-2.63	-3.58	-2.97	-2.63	-3.62	-2.97	-2.64
1.8	-4.03	-3.34	-2.98	-3.95	-3.32	-2.96	-3.97	-3.31	-2.96	**4.3**	-3.61	-2.95	-2.62	-3.58	-2.96	-2.63	-3.61	-2.96	-2.64
1.9	-4.01	-3.31	-2.94	-3.91	-3.29	-2.93	-3.95	-3.29	-2.93	**4.4**	-3.60	-2.94	-2.62	-3.58	-2.96	-2.63	-3.60	-2.96	-2.64
2.0	-3.98	-3.28	-2.91	-3.91	-3.26	-2.89	-3.91	-3.26	-2.90	**4.5**	-3.58	-2.94	-2.62	-3.57	-2.96	-2.63	-3.58	-2.95	-2.63
2.1	-3.95	-3.26	-2.89	-3.92	-3.23	-2.87	-3.90	-3.23	-2.88	**4.6**	-3.58	-2.93	-2.62	-3.56	-2.95	-2.62	-3.58	-2.95	-2.63
2.2	-3.93	-3.23	-2.86	-3.90	-3.20	-2.84	-3.88	-3.21	-2.85	**4.7**	-3.58	-2.93	-2.61	-3.56	-2.94	-2.61	-3.57	-2.94	-2.63
2.3	-3.92	-3.21	-2.84	-3.88	-3.18	-2.82	-3.85	-3.20	-2.83	**4.8**	-3.57	-2.93	-2.61	-3.54	-2.94	-2.61	-3.56	-2.94	-2.62
2.4	-3.89	-3.19	-2.82	-3.85	-3.17	-2.81	-3.84	-3.17	-2.81	**4.9**	-3.57	-2.92	-2.61	-3.54	-2.93	-2.61	-3.56	-2.94	-2.62
2.5	-3.87	-3.16	-2.80	-3.82	-3.15	-2.79	-3.83	-3.15	-2.80	**5.0**	-3.58	-2.92	-2.60	-3.53	-2.93	-2.60	-3.55	-2.94	-2.62