On the variational characterization of quasi-periodic standing waves of the nonlinear Schrödinger equation

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Abstract

We consider quasi-periodic standing wave solutions $U(t,x) = e^{i(\omega t - px)}\Psi(x)$ to the one-dimensional defocusing cubic nonlinear Schrödinger equation, where we assume that $\Psi : \mathbb{R} \mapsto \mathbb{C}$ is $2\pi-$periodic. We study a constrained minimization problem associated with these solutions, and we show that solutions with minimal period of $\Psi(x)$ strictly less than 2π cannot be minimizers, whereas locally the minimum is obtained among those solutions with minimal period 2π.

Populärvetenskaplig sammanfattning

Vi studerar en särskild typ av lösningar till den endimensionella kubiska icke-linjära Schrödingerekvationen (NLS). Denna ekvation dyker upp inom fysiken, t.ex. då man vill modellera Bose-Einstein kondensat. Detta är ett aggregationstillstånd som en gas bestående av bosoner med låg densitet kan övergå till vid nedkylning till temperaturer nära den absoluta nollpunkten, varvid bosonerna delar samma kvantmekaniska grundtillstånd. Fysikerna Bose och Einstein förutspådde existensen av Bose-Einstein kondensat 1924-1925, och detta kunde bevisas experimentellt år 1995. Tillhörande NLS ekvationen finns tre stycken konserveringslagar som vi kallar energin, massan och rörelsemängdsmomentet. Vi hittar villkor på lösningarna till NLS ekvationen för att energin skall kunna vara så liten som möjligt under fixerad massa och rörelsemängdsmoment.

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Contents

1 Introduction

We shall consider the one-dimensional cubic nonlinear Schrödinger equation (NLS)

$$iU_t(t,x) + U_{xx}(t,x) = 2\pi\gamma|U(t,x)|^2 U(t,x) \tag{1.1}$$

with $U : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ a complex-valued function and $\gamma \neq 0$ a constant. The NLS equation is said to be *focusing* if $\gamma < 0$ and *defocusing* if $\gamma > 0$. This equation has applications in physics and it can for example be used under certain conditions when one would like to model a so-called Bose-Einstein condensate [7, 13, 15].

We shall consider a special class of solutions to the NLS equation (1.1), namely the so-called quasi-periodic standing waves, which are solutions of the form

$$U(t,x) = e^{i(\omega t - px)}\Psi(x)$$

in which $\Psi : \mathbb{R} \to \mathbb{C}$ is a complex-valued periodic function and $\omega, p \in \mathbb{R}$. We will consider functions Ψ that are 2π-periodic.

The NLS equation has associated with it three important conservation laws, which in our case take the form

$$E(U) := \int_0^{2\pi} (|U_x(t,x)|^2 + \pi\gamma|U(t,x)|^4)dx, \tag{1.2}$$

$$N(U) := \int_0^{2\pi} |U(t,x)|^2 dx, \tag{1.3}$$

$$M(U) := -i\int_0^{2\pi} \overline{U}(t,x)U_x(t,x)dx, \tag{1.4}$$

and which we shall refer to as the *energy*, *mass* and *momentum*, respectively. That they are conserved means here that the values of the functionals are independent of the time t. We will consider these functionals for the periodic functions $\Psi(x)$. The main motivation for doing so comes from the papers [7,13,14] and in particular we would like to consider a problem discussed in [14], which is that of minimizing the energy $E(\Psi)$ under fixed unit mass $N(\Psi) = 1$ and fixed momentum $M(\Psi) = \ell_0 \in \mathbb{R}$ among the class of 2π-periodic functions $\Psi(x)$. It is known that minimizers do exist, see for example the discussion in ([14], pp.2). Solutions giving the lowest energy are sometimes called *ground state* solutions. In [14] the idea was to consider the minimal period of the function $\Psi(x)$ which is of the form $\frac{2\pi}{n}$ for some integer $n \geq 1$ and numerical computations performed suggested that the lowest energy for each fixed ℓ_0 is given by those solutions with minimal period 2π, that is when $n = 1$. Our aim here is to verify that this is indeed the case and this question is considered in section 4. To motivate the work in section 4 we briefly consider some of the results obtained in [3], [4] and [14] in section 2, as well as state some theoretical arguments in section 3 that will be present in the subsequent discussion.

In the paper [1] the authors studies the stability of standing wave solutions of the defocusing NLS equation (1.1) using integrable systems methods. In particular, they use the squared-eigenfunction connection to study the spectrum of the linearization. It should be possible to use the arguments that they present, especially in section 7, to conclude the result that solutions with minimal period less than 2π cannot be minimizers.

2 Previous results

We start here with discussing the work done in [14]. In there two different minimization problems regarding the energy is studied, one in which one considers the energy with the constraint of unit mass only and the other with the constraints of unit mass and fixed momentum. We start by looking at the singly constrained problem. The Euler-Lagrange equation for this problem is given by

$$-\Psi''(x) + 2\pi\gamma|\Psi(x)|^2\Psi(x) = \mu\Psi(x) \tag{2.1}$$

for a Lagrange multiplier $\mu \in \mathbb{R}$. Here one looks for solutions in the Sobolev space $H^1(\mathbb{T}) = H^1_{per}([0, 2\pi], \mathbb{C})$ (appendix A.2).

The equation (2.1) can be seen as a dynamical system in the variables Ψ, Ψ' and we have the two invariants (alternatively, conserved quantities)

$$H := \frac{1}{2}|\Psi'|^2 - \frac{1}{2}\gamma\pi|\Psi|^4 + \frac{1}{2}|\Psi|^2$$

and

$$J := \mathrm{Im}(\overline{\Psi}\Psi').$$

The bounded solutions depend on the values of these invariants, and depending on the signs of γ and μ, we have different domains $\tilde{D} := \{(J, E)\} \subset \mathbb{R}^2$ for which the bounded solutions exists. Note that if $J \neq 0$, then $\Psi(x) \neq 0$ for all x so one can express Ψ in terms of polar coordinates $\Psi(x) = r(x)e^{i\varphi(x)}$. If $J = 0$ it is fine to use polar coordinates as long as $\Psi(x) \neq 0$ and one can show in this case that $\Psi(x)$ is real-valued up to multiplication by a phase factor $e^{i\theta}$.

We shall consider the second minimization problem for $\gamma > 0$ only, so we briefly mention what type of solutions we have for this case. They are explicitly given by the Jacobi elliptic function $\mathrm{sn}(u; k)$, which is discussed in the appendix. The following table is obtained:

$\gamma > 0$	$J = 0$, $\Psi(x)$	$J \neq 0$, $\|\Psi(x)\|$
$\mu > 0$	$\pm\sqrt{\frac{\mu}{2\pi\gamma}}$, $\frac{\sqrt{\mu}k}{\sqrt{\pi\gamma(1+k^2)}}\mathrm{sn}\big(\sqrt{\frac{\mu}{1+k^2}}x; k\big)$	$\sqrt{s_1 + (s_2 - s_1)\mathrm{sn}^2(\sqrt{\pi\gamma(s_3 - s_1)}x; k)}$
$\mu < 0$	No bounded solutions	No bounded solutions

Here $k \in [0, 1)$ and s_1, s_2, s_3 are roots of a certain polynomial (see section 2.1). We shall go through the solution obtained for $J \neq 0$, since this is of importance for the subsequent discussions. For details, we refer to [14].

2.1 Solutions of the Euler-Lagrange equation with $J \neq 0$

Since $J \neq 0$ we represent $\Psi(x)$ via polar coordinates as $\Psi(x) = r(x)e^{i\varphi(x)}$, where $r(x), \varphi(x)$ are real-valued functions. One notes that the invariants H and J then take the form

$$H = \frac{1}{2}(r')^2 + \frac{J^2}{2r^2} - \frac{1}{2}\pi\gamma r^4 + \frac{1}{2}\mu r^2,$$

$$J = r^2\varphi'.$$

One then introduces the *effective potential*

$$V_J(r) := \frac{J^2}{2r^2} - \frac{1}{2}\pi\gamma r^4 + \frac{1}{2}\mu r^2$$

with $H = \frac{(r')^2}{2} + V_J(r)$. Differentiating $V_J(r)$ and looking for extreme points, one finds that for $J^2 > \frac{\mu^3}{27\pi^2\gamma^2}$ we have $V_J'(r) < 0$ for all $r > 0$, and then we cannot have any bounded solutions since $(r')^2$ would be strictly increasing. For $J^2 < \frac{\mu^3}{27\pi^2\gamma^2}$ one finds that $V_J'(r)$ has two zeros $0 < r_1 < r_2$, and so $V_J(r)$ has local minima and maxima. For $H = V_J(r_1)$ or $H = V_J(r_2)$ we have constant solutions as well as the solution with modulus being homoclinic to r_2 as $x \to \pm\infty$ (see [3]). The latter are not of interest since it is not periodic. In the interior

$$\tilde{D} := \{(J, H) \in \mathbb{R}^2 | V_J(r_1) < H < V_J(r_2)\} \tag{2.2}$$

we have bounded solutions.

Put $s = r^2$. One obtains $(s')^2 = -4J^2 + 8Hs - 4\mu s^2 + 4\pi\gamma s^3 = a(s-s_1)(s-s_2)(s-s_3) =: F(s)$, where s_1, s_2, s_3 are the roots of the polynomial $F(s)$ (i.e. the roots of $H - V_J(r)$), with the following relations:

$$\begin{cases} a = 4\pi\gamma \\ 4\mu = a(s_1 + s_2 + s_3) \\ 8H = a(s_1 s_2 + s_1 s_3 + s_2 s_3) \\ 4J^2 = a s_1 s_2 s_3. \end{cases}$$

For $H \in \tilde{D}$ one finds that $F(s)$ has three positive roots $0 < s_1 < s_2 < s_3$. Fixing $(J, H) \in \tilde{D}$, one then obtains the solution

$$r(x) = \sqrt{s_1 + (s_2 - s_1)\mathrm{sn}^2(\sqrt{\pi\gamma(s_3 - s_1)}x; k)}$$

with minimal period

$$T(J, H) = \frac{2K}{\sqrt{\pi\gamma(s_3 - s_1)}}$$

where $k^2 = \frac{s_2 - s_1}{s_3 - s_1} \in (0, 1)$ and $K = K(k)$ is the complete elliptic integral of the first kind ([14, pp.11-12]).

2.2 Minimizing the energy with two constraints

In the problem of minimizing $E(\Psi)$ subject to fixed unit mass $N(\Psi) = 1$ and fixed momentum $M(\Psi) = \ell_0$, the corresponding Euler-Lagrange equation takes the form

$$-\Psi''(x) + i\Omega\Psi'(x) + 2\pi\gamma|\Psi(x)|^2\Psi(x) = \eta\Psi(x) \tag{2.3}$$

for some Lagrange multipliers $\Omega, \eta \in \mathbb{R}$.

In [14], the equation (2.3) is transformed into equation (2.1) via a series of changes of variables, and possible minimizers then corresponds to the solution

$$r(x) = \sqrt{s_1 + (s_2 - s_1)\mathrm{sn}^2(\sqrt{\pi\gamma(s_3 - s_1)}x; k)}.$$

The following expression for the momentum is then obtained

$$\ell_0 = \Big(2\pi - \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{r(x)^2}dx\Big)J \geq 0$$

with $J = -\sqrt{\pi\gamma s_1 s_2 s_3} \leq 0$. One can express s_1, s_2, s_3 in terms of $E(k)$ and $K(k)$ and show that there exists a $k^* > 0$ where s_1 changes sign. It is then conjectured in [14] that ℓ_0 is a strictly increasing function of $k \in [0, k^*)$. We were not able to verify this here.

2.3 A normalized Euler-Lagrange equation

In the paper [3] bounded solutions to the following equation are studied

$$W_{xx}(x) + W(x) - |W(x)|^2 W(x) = 0 \tag{2.4}$$

with $W : \mathbb{R} \to \mathbb{C}$. We will refer to equation (2.4) as the *normalized* Euler-Lagrange equation. We will in section 4 rescale our solution $\Psi(x)$ to a solution of this equation, and use a lot of the results obtained in [3] and [4] regarding these solutions. The bounded solutions to equation (2.4) are found in an analogous region $D = \{(J, H)\} \subset \mathbb{R}^2$ to the region (2.2), and in this context it takes the form

$$D = \{(J, H) \subset \mathbb{R}^2 | J^2 < \frac{4}{27}, V_J(r_Q) < H < V_J(r_q)\} \tag{2.5}$$

where r_Q, r_q are analogous to r_1, r_2 above. The essential part that we want to use is that one is able to scale the function $W(x)$ in such a way that one obtains a part that is 2π-periodic. We briefly describe how this is done. For more details we refer to [3].

If one lets $T(J, H)$ denote the minimal period of the modulus $|W(x)|$ of the solution $W(x) = r(x)e^{i\varphi(x)}$ to (2.4) for $(J, H) \in D$, and $\Phi(J, H)$ denote the increment in the phase $\varphi(x)$ over a period of the modulus, one can show that as $J \to 0^{\pm}$, $\Phi(J, H) \to \pm\pi$. In [3] they then introduce the *renormalized phase* $\Psi(J, H)$ by the expression

$$\Psi(J, H) := \begin{cases} \Phi(J, H) - \pi \text{ sign}(J) \text{ if } J \neq 0 \\ 0 \text{ if } J = 0 \end{cases}$$

and show that this is a smooth function of $(J, H) \in D$ ([3,lemma 2.3]).

Fix $(J, H) \in D$. The solution to (2.4) is unique up to translation and a phase factor ([3]). If one then lets $\lambda := \frac{\pi}{T(J,H)}$ and $\ell := \frac{\Psi(J,H)}{T(J,H)}$ one can make the change of variables $W(x) = e^{i\ell x}P(\lambda x)$. Since the modulus $|W(x)|$ has minimal period $T(J, H)$ we see that $|P(y)|$ is periodic with minimal period π. Moreover, the phase increment over a period shows us that since $W(x+T) = e^{i\Phi}W(x)$, we also get $P(y+\pi) = -P(y)$ such that P is a 2π-periodic function of $y \in \mathbb{R}$. One can also write it in a more convenient form as

$$W(x) = e^{i(\ell+\lambda)x}Q(2\lambda x) = e^{ipx}Q(2\lambda x)$$

for $Q(z) = e^{-iz/2}P(z/2)$ and $p := \ell + \lambda$. This is particularly beneficial since this construction guarantees that $Q(z)$ and $|Q(z)|$ have the same minimal period 2π, and we shall use this in section 4.

3 Theory

In this section we wish to record and review some of the theoretical tools used in the subsequent section. We will however not provide full details for most arguments but instead refer the reader to the relevant literature.

3.1 Some operator theory

Let $\mathcal{H}$ be a (complex) Hilbert space and $B(\mathcal{H})$ the Banach space of all continuous linear operators on $\mathcal{H}$. Let $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ be a densely defined linear operator. Recall that T is said to be *closed* if for every sequence (x_n) in $D(T)$ such that $x_n \to x$ and $Tx_n \to y$, then $x \in D(T)$ and $y = Tx$. Equivalently, the graph

$$G(T) := \{(x, Tx) \mid x \in D(T)\} \subset \mathcal{H} \times \mathcal{H}$$

is closed in the Hilbert space $\mathcal{H} \times \mathcal{H}$. T is *compact* if the image of every bounded set under T is precompact. T is *symmetric* if $\langle Tx, y\rangle = \langle x, Ty\rangle$ for every $x, y \in D(T)$.

The *adjoint domain* $D(T^*)$ is defined as

$$D(T^*) := \{y \in \mathcal{H} \mid \exists \tilde{y} \in \mathcal{H} \text{ such that } \langle Tx, y\rangle = \langle x, \tilde{y}\rangle \ \forall x \in D(T)\}.$$

The *adjoint operator* T^* of T is defined by $T^*y = \tilde{y}$. By denseness of the domain $D(T)$ there can only be one such T^*y for every y, so it is well-defined and one notes that it is linear. T is said to be *self-adjoint* if $T = T^*$ in the sense that T is symmetric and $D(T) = D(T^*)$.

The adjoint of a densely defined linear operator on a Hilbert space is always closed, since if (y_n) is a sequence in $D(T^*)$ such that $y_n \to y$ and $T^*y_n \to z$, then by continuity of the inner product

$$\langle Tx, y\rangle = \lim_{n\to\infty} \langle Tx, y_n\rangle = \lim_{n\to\infty} \langle x, T^*y_n\rangle = \langle x, z\rangle$$

holds for every $x \in D(T)$ which means $y \in D(T^*)$ and $T^*y = z$. Hence we see that every self-adjoint operator is closed.

Let now T be closed. We have the following definitions. The *resolvent set*, $\rho(T)$, of T is defined as the set

$$\rho(T) := \{z \in \mathbb{C} \mid (T - z)^{-1} \text{ exists and is bounded}\}.$$

The operator-valued function $R_T(z) : \mathcal{H} \mapsto D(T)$ with $R_T : z \mapsto (T - z)^{-1}$ for $z \in \rho(T)$ is called the *resolvent*. The *spectrum* of T is the complement of the resolvent set; $\sigma(T) = \mathbb{C} \setminus \rho(T)$. $z \in \sigma(T)$ is an eigenvalue of T if there exists a nonzero vector $x \in \mathrm{Ker}(T - z)$, which is then called an eigenvector. The multiplicity of an eigenvalue z is the dimension of the eigenspace $\mathrm{Ker}(T - z)$. Denote the set of eigenvalues of T as $\sigma_p(T)$. It is clear that a symmetric operator only has real eigenvalues and eigenvectors corresponding to different eigenvalues are orthogonal. We end with a theorem.

Theorem 3.1 ([8, ch.3 §6.8]). Let T be a closed operator in a (complex) Banach space X such that the resolvent exists and is compact for some $z_0 \in \mathbb{C}$. Then $\sigma(T)$ consists entirely of isolated eigenvalues with finite multiplicities.

3.2 Perturbation theory for linear operators

In the papers [3] and [4] a perturbation argument regarding the spectrum of linear operators on a Hilbert space is used to locate the eigenvalues of a certain operator of interest. As we shall use the results obtained via this argument in later sections, we wish to briefly discuss the theory behind it. The book [8] contains a vast amount of information on this subject and we refer the reader to it for a more detailed exposition.

Let X be a Banach space and suppose M, N are two closed subspaces with $X = M \oplus N$. A linear operator T is said to be decomposed according to $X = M \oplus N$ if $PD(T) \subset D(T)$, $TM \subset M$ and $TN \subset N$, with P the projection onto M. In this case the part of T in M, T_M, can be defined as the operator with domain $D(T_M) := D(T) \cap M$ and $T_M x = Tx$ for $x \in D(T_M)$. If T is closed so is T_M since the graph $G(T_M)$ of T_M is just the intersection of the closed set $G(T)$ with the closed set $M \times M$ in the product space $X \times X$. The part T_N is defined similiarly.

Suppose then that we have a closed operator T whose spectrum $\sigma(T)$ can be decomposed into two parts $\sigma_1(T)$ and $\sigma_2(T)$ in the sense that the part $\sigma_1(T)$ consists of a finite number of isolated eigenvalues and is bounded and separated from $\sigma_2(T)$ by a simple closed rectifiable curve Γ. In this case we have the following result.

Theorem 3.2 ([8 ch. 3 §6.4 theorem 6.17]). *T can be decomposed according to $X = M_1 \oplus M_2$ where the spectra of the parts T_{M_1}, T_{M_2} coincide with $\sigma_1(T)$ and $\sigma_2(T)$, respectively.*

One might now be interested in understanding how this finite part $\sigma_1(T)$ of the spectrum of T changes under small perturbations, i.e. when looking at $T + S$ for some operator S close to 0. To get such a result we first need some preliminary discussions. We start with an analyticity result regarding the resolvent of a closed operator.

Lemma 3.3 ([8, ch.3 §6.1 theorem 6.7]) *Let T be a closed operator on a Banach space X and $R_T(z) = R(z)$ be the resolvent. Then the resolvent set $\rho(T)$ is open in $\mathbb{C}$ and $R(z)$ is an analytic function of $z \in \rho(T)$.*

Suppose X is a Banach space and T and A are operators $X \to X$ such that $D(T) \subset D(A)$. Then A is said to be *T-bounded* if there exist constants $a, b > 0$ such that

$$||Ax|| \leq a||x|| + b||Tx|| \tag{3.3}$$

for all $x \in D(T)$. The infimum over all such constants b is called the *T-bound* on A.

Lemma 3.4 ([8, ch.4 §3.6 theorem 3.17]). *Suppose T and A are operators on a Banach space X and T is closed. Suppose A is T-bounded with constants $a, b > 0$ with $b < 1$. Let $S := T + A$ be a perturbed operator. Then S is closed. If there is a point $z \in \rho(T)$ such that*

$$a||R_T(z)|| + b||TR_T(z)|| < 1$$

then $z \in \rho(S)$. If T has compact resolvent then so does S.

We can now discuss the following result.

Theorem 3.5 ([8, ch.4 §3.6 theorem 3.18]). *Suppose T, A and S are as in lemma 3.4, and the spectrum $\sigma(T)$ of T is separated into two parts by a simple, closed and rectifiable curve Γ as in the discussion preceding theorem 3.2, where $\sigma_1(T)$ consists of a finite number of eigenvalues.*

Suppose further that

$$\sup_{z\in\Gamma}(a||R_T(z)|| + b||TR_T(z)||) < 1.$$

Then also $\sigma(S)$ is separated by Γ into two parts $\sigma_1(S)$ and $\sigma_2(S)$ where the spectrum of S restricted to $M_i(S)$ coincide with $\sigma_i(S)$, and $M_i(S)$ is isomorphic to $M_i(T)$, where the M_i are as in theorem 3.2. In particular $\dim M_i(S) = \dim M_i(T)$.

To prove this one notes that from lemma 3.4 we have that S is closed and that every point z on the curve Γ is in the resolvent set $\rho(S)$. Hence $\sigma(S)$ is separated into two parts by Γ. Using some complex analysis and the fact the resolvent is an analytic function one can show that the integrals

$$P(T) := \frac{-1}{2\pi i}\oint_\Gamma R_T(z)dz$$

$$P(S) := \frac{-1}{2\pi i}\oint_\Gamma R_S(z)dz$$

are projections onto $M_1(T)$ and $M_1(S) := P(S)X$ respectively. By theorem 3.2 the spectrum of $T\big|_{M_1(T)}$ coincides with $\sigma_1(T)$ and $P(T)$ is in fact a projection onto the union of the eigenspaces of the finite number of eigenvalues of T contained in $\sigma_1(T)$. One then has to show that the ranges of $P(T)$ and $P(S)$ are isomorphic. The arguments are a bit technical and long so we have to refer the reader to [8].

3.3 Counting negative eigenvalues of constrained self-adjoint operators

Let T be a densely defined, bounded below self-adjoint operator $D(T) \subset X \to X$ on some Hilbert space $(X, \langle\cdot,\cdot\rangle)$. Suppose further that there exists another Hilbert space $Y \subset X \subset Y^*$ dense in X such that $D(T) \subset Y$ and we have the following continuous embeddings $Y \subset X \subset Y^*$. An example would be $H^1(\mathbb{T}) \subset L^2(\mathbb{T}) \subset H^{-1}(\mathbb{T})$. Assume that $T \in B(Y, Y^*)$. We can define a bilinear form $b[u, v] := \langle Tu, v\rangle$ which is then continuous on Y. Suppose that we have a finite co-dimensional subspace $\mathcal{A} \subset Y$ such that $Y = \mathcal{A} \oplus_X \mathcal{A}^\perp$. We call $\mathcal{A}$ the *admissible space* and $\mathcal{A}^\perp$ the *constraint space*. Let $n(T)$ denote the dimension of the largest subspace of Y on which the bilinear form b is negative, that is $b[u, u] < 0, u \neq 0$ and put $z(T) := \dim(\mathrm{Ker}(T))$. One might be interested in finding out how the amount of negative eigenvalues of T changes when one restricts T to act on the admissable space $\mathcal{A}$. Using a min-max principle for self-adjoint operators ([17]) one can show that $n(T)$ is exactly the number of negative eigenvalues (counted with multiplicity) of the operator $T : D(T) \to X$. We call $n(T)$ the negative eigenvalue count of the operator T. If one can then obtain a formula relating $n(T)$ with the count of the restricted operator $n(T_\mathcal{A})$ one gets a bound on the amount of negative eigenvalues one can allow for the restricted operator to be positive. We consider this problem now.

We assume that $n(T) + z(T) < \infty$ and that $\mathcal{A}^\perp \perp \mathrm{Ker}(T)$ in the X inner product. Consider then an orthogonal projection $\Pi : X \to \mathrm{Ran}(\Pi) \subset X$ such that $\Pi(Y) = \mathcal{A}$. Restricting $b[u, v]$ to act on $\mathcal{A}$ induces the operator $T_\mathcal{A} := \Pi T : D(T) \cap \mathcal{A} \subset \Pi(X) \to \Pi(X)$. We look at $\mathrm{Ker}(T_\mathcal{A})$. T has trivial kernel when restricted to $\mathrm{Ker}(T)^\perp$. Denote by $T^{-1} : \mathrm{Ker}(T)^\perp \to \mathrm{Ker}(T)^\perp$ its inverse on its range. Let now $S := \mathcal{A}^\perp = \mathrm{span}\{s_1, s_2, ..., s_m\}$. We want to know when an element $s^\perp \in \mathcal{A}$ is in $\mathrm{Ker}(T_\mathcal{A})$. Either $s^\perp$ is in $\mathrm{Ker}(T)$ or not. If $s^\perp \in \mathrm{Ker}(T)$, then since $\mathrm{Ker}(T) \subset \mathcal{A}$ when acting on Y by assumption on $\mathcal{A}^\perp$ it follows by definition of $T_\mathcal{A}$ that $\mathrm{Ker}(T) \subset \mathrm{Ker}(T_\mathcal{A})$ such that $s^\perp \in \mathrm{Ker}(T_\mathcal{A})$. Suppose then $s^\perp \notin \mathrm{Ker}(T)$ but $s^\perp \in \mathrm{Ker}(T_\mathcal{A})$. Then $0 = T_\mathcal{A}s^\perp = \Pi T s^\perp$

which implies $Ts^{\perp} \in S$ since $\Pi Y = \mathcal{A}$.

Consider now the space $T^{-1}(S) := \text{span}\{T^{-1}s_1, ..., T^{-1}s_m\}$. Under the assumption that $T^{-1}(S) \cap \mathcal{A} = \{0\}$ we have that the following mxm matrix

$$D_{ij} := \langle s_i, T^{-1}s_j \rangle$$

is non-singular. It is actually an "if and only if" statement. If it were singular then there would be at least one non-zero element of the kernel of D_{ij}, and using this one can construct a nonzero element $s \in S$ such that $T^{-1}s \in \mathcal{A}$, i.e. the intersection is not trivial. The m×m Hermitian matrix $D := D_{ij}$ is called the *constraint matrix*. Under the assumptions above, we have the following result.

Proposition 3.6 ([9, chapter 5.3, theorem 5.3.1]). Suppose $S \subset \text{Ker}(T)^{\perp}$ is an m-dimensional subspace and that the Hermitian constraint matrix $D \in \mathbb{C}^{m \times m}$ is non-singular. The difference in the negative eigenvalue count of T on X and T restricted to the admissable space $\mathcal{A}$, $T_{\mathcal{A}}$, is given by the negative eigenvalue count of D, that is we have the relation

$$n(T_{\mathcal{A}}) = n(T) - n(D).$$

4 The minimization problem

4.1 Preliminaries

We turn now to the question of characterizing those 2π-periodic solutions $\Psi(x)$ to the defocusing NLS equation ($\gamma > 0$) that give the lowest energy subject to fixed unit mass and fixed momentum, the so-called ground state solutions. The functions $\Psi(x)$ lie in the real Hilbert space $H^1(\mathbb{T})$ with the inner product

$$(f, g) := \mathrm{Re} \int_0^{2\pi} (f(x)\overline{g}(x) + f'(x)\overline{g}'(x))dx.$$

We identify the dual space $(H^1(\mathbb{T}))^*$ of H^1 with $H^{-1}(\mathbb{T})$ via the dual pairing

$$\langle f, g\rangle := \mathrm{Re} \int_0^{2\pi} f(x)\overline{g(x)}dx$$

with $f \in H^{-1}$, $g \in H^1$ (see appendix).

We start off by computing the first and second order derivatives of the functionals E, M and N. These play a fundamental role in this study. The first order derivatives will be bounded linear operators from H^1 into $\mathbb{R}$, which we identify with elements of H^{-1}. These are found to be

$$E'(\Psi) = -2\Psi'' + 4\pi\gamma|\Psi|^2\Psi,$$

$$N'(\Psi) = 2\Psi,$$

$$M'(\Psi) = -2i\Psi'.$$

The second order derivatives are bounded linear operators from H^1 into H^{-1} and are found to be

$$E''(\Psi) = -2\partial_{xx} + 4\pi\gamma\Psi^2 K + 8\pi\gamma|\Psi|^2,$$

$$N''(\Psi) = 2I,$$

$$M''(\Psi) = -2i\partial_x.$$

where K is the complex conjugate operator $Kz := \overline{z}$ and I the identity. The Euler-Lagrange equation (2.3) arises as the first derivative of the *modified energy*

$$\hat{E}(\Psi) := E(\Psi) - \eta N(\Psi) - \Omega M(\Psi)$$

where $\eta, \Omega \in \mathbb{R}$ are some Lagrange multipliers. That is, a solution to (2.3) is a critical point to $\hat{E}$. To further study these solutions, we will have to consider the second derivative of $\hat{E}$. This is given by

$$\hat{E}''(\Psi) = E''(\Psi) - \eta N''(\Psi) - \Omega M''(\Psi) = -2\partial_{xx} + 2\Omega i\partial_x + 4\pi\gamma\Psi^2 K + 8\pi\gamma|\Psi|^2 - 2\eta.$$

We put $\mathcal{L} := \hat{E}''(\Psi)$ and consider this as a linear operator $L^2(\mathbb{T}) \to L^2(\mathbb{T})$ with domain $H^2(\mathbb{T})$. This is a second order differential operator. It is clear that $\mathcal{L}$ is symmetric in L^2, and one can even show that it is self-adjoint, and that it has compact resolvent ([3,4]). By the discussion in section 3, the spectrum of $\mathcal{L}$ consists entirely of a countable number of isolated real eigenvalues.

The idea is now to consider these eigenvalues and see how they depend on the minimal period of the 2π-periodic function $\Psi(x)$, which is of the form $\frac{2\pi}{n}$ for $n \in \mathbb{Z}^+$. A necessary condition for a function $\psi(x)$ in some suitable function space to be a minimum of an integral functional $J(\varphi)$ is that the first derivative vanishes and the second derivative is positive in the sense that if $\langle\cdot,\cdot\rangle$ denotes a dual pairing, then $\langle J'(\psi), \varphi\rangle = 0$ and $\langle J''(\psi)\varphi, \varphi\rangle \geq 0$ for all φ in the set of possible candidate functions. A sufficient condition is that the second derivative is strongly positive, i.e. $\langle J''(\psi)\varphi, \varphi\rangle \geq C||\varphi||^2$ for all $\varphi \neq 0$ and some constant $C > 0$ ([5]). In our case this is later shown to hold for $n = 1$ in the minimal period of Ψ but for $n \geq 2$ the second derivative $\mathcal{L}$ has negative eigenvalues among the class of candidate functions φ such that it clearly cannot even be positive. This means that the only possible minimizers in our case are the class of 2π-periodic solutions $\Psi(x)$ with minimal period 2π, which was conjectured in [14]. A very similar problem has been studied in the papers [3] and [4] although with a different NLS equation. They study standing wave solutions $U(x,t) = e^{-i\omega t}W(x)$ to the following defocusing NLS equation

$$iU_t(x,t) + U_{xx}(x,t) - |U(x,t)|^2U(x,t) = 0 \tag{4.1}$$

where they take $\omega = +1$ such that $W(x)$ satisfies the normalized Euler-Lagrange equation (2.4). As discussed in section 2.3, one can scale the solution $W(x)$ as $W(x) = e^{ipx}Q(2\lambda x)$ where $Q(z)$ and $|Q(z)|$ have minimal period 2π. By construction $Q(z)$ satisfies the equation

$$4\lambda^2 Q_{zz} + 4ip\lambda Q_z + (1-p^2)Q - |Q|^2Q = 0 \tag{4.2}$$

which is very similar to the Euler-Lagrange equation (2.3). In fact, it is the Euler-Lagrange equation to an analogous problem to our minimization problem but with different functionals where the Lagrange multipliers have been fixed. Our first step is then to transform our solution $\Psi(x)$ in such a way that we can use the analysis of the 2π-periodic functions $Q(z)$ considered in [3] and [4]. We start by writing it in terms of a function W satisfying the normalized Euler-Lagrange equation (2.4). By using the change of variables

$$\Psi(x) = e^{i\frac{\Omega}{2}x}\psi(x)$$

we get that $\psi(x)$ then is a solution to the equation (2.1), $-\psi'' + 2\pi\gamma|\psi|^2\psi = \mu\psi$ with $\mu := \eta + \frac{\Omega^2}{4} > 0$. If we now put

$$\psi(x) = \sqrt{\frac{\mu}{2\pi\gamma}}W(\sqrt{\mu}x)$$

we see that $W(y)$ satisfies the equation (2.4).

To be able to use the scaling of W we first need to check that we end up in the correct domain $(J,H) \subset D \subset \mathbb{R}^2$ (see section 2.3). To do this it suffices to check that the invariant $J_W = \mathrm{Im}(\overline{W}W')$ corresponding to the solution W satisfies $J_W^2 < \frac{4}{27}$. Since $J = \mathrm{Im}(\overline{\psi}\psi')$ satisfies $J^2 < \frac{\mu^3}{27\pi^2\gamma^2}$ and $W(y) = W(\sqrt{\mu}x) = \sqrt{\frac{2\pi\gamma}{\mu}}\psi(x)$ we find that $J_W = \frac{2\pi\gamma}{\sqrt{\mu^3}}J$ and so $J_W^2 = \frac{4\pi^2\gamma^2}{\mu^3}J^2 < \frac{4}{27}$. We can then scale our $W(y)$ to obtain a function $Q(2\lambda y) = Q(2\lambda\sqrt{\mu}x)$. Thus we can write $\psi(x) = \sqrt{\frac{\mu}{2\pi\gamma}}e^{ip\sqrt{\mu}x}Q(2\lambda\sqrt{\mu}x)$ and so

$$\Psi(x) = \sqrt{\frac{\mu}{2\pi\gamma}}e^{i(\frac{\Omega}{2}+p\sqrt{\mu})x}Q(2\lambda\sqrt{\mu}x)$$

which we for simplicity write as $\Psi(x) = ce^{iax}Q(bx)$ for $a, b, c \in \mathbb{R}$, $c > 0$.

Suppose now that $|\Psi(x)|$ has minimal period $\frac{2\pi}{n}$. Then since $|Q(bx + \frac{b2\pi}{n})| = |\Psi(x + \frac{2\pi}{n})| = |\Psi(x)| = |Q(bx)|$ and $|Q(z)|$ has minimal period 2π we get $b = n$. Similiarly, $\Psi(x) = \Psi(x+2\pi) = e^{ia2\pi}\Psi(x)$ gives us that $a \in \mathbb{Z}$, say $a = k$. The potential eigenfunctions of the operator $\mathcal{L}$ then takes the form

$$u(x) = e^{ikx}v(nx)$$

where $n \in \mathbb{Z}^+$ is determined by the minimal period of $|u(x)|$, $k \in \mathbb{Z}$ and $v(y)$ is a 2π-periodic function. However, the class of 2π-periodic functions $v(y)$ are not enough for us to get all 2π-periodic functions $u(x)$ so we will need a larger class of such functions. If $u(x)$ is 2π-periodic, then $v(y) = e^{-iky/n}u(y/n)$ is $2\pi n$-periodic, and conversely if $v(y)$ is $2\pi n$-periodic then u(x) is 2π-periodic so to get all possible eigenfunctions we have to consider the class of $2\pi n$-periodic functions $v(y)$.

The mass, momentum and energy relations between Ψ and Q are

$$N(\Psi) = 2c^2\tilde{N}(Q),$$

$$M(\Psi) = 2c^2(k\tilde{N}(Q) - n\tilde{M}(Q)),$$

$$E(\Psi) = 2c^2(k^2\tilde{N}(Q) - 2kn\tilde{M}(Q) + \mu\tilde{E}(Q)),$$

where

$$\tilde{N}(Q(z)) := \frac{1}{2}\int_0^{2\pi} |Q(z)|^2 dz,$$

$$\tilde{M}(Q) := \frac{i}{2}\int_0^{2\pi} \overline{Q}(z)Q'(z)dz,$$

$$\tilde{E}(Q) := \int_0^{2\pi} (2\lambda^2|Q_z(z)|^2 + \frac{1}{4}|Q(z)|^4)dz,$$

which are exactly the functionals studied in [3] and [4]. Hence to minimize $E(\Psi)$ it suffices to minimize $\tilde{E}(Q)$ subject to fixed mass and momentum. Here $Q(z)$ is a critical point to the following modified energy

$$\mathcal{E}(Q) := \tilde{E}(Q) - (1 - p^2)\tilde{N}(Q) - 4p\lambda\tilde{M}(Q)$$

where the corresponding second derivative is the linear operator $L^2(\mathbb{T}) \to L^2(\mathbb{T})$

$$T := -4\lambda^2\partial_{zz} - 4ip\lambda\partial_z - (1 - p^2) + Q^2K + 2|Q|^2$$

with domain $H^2(\mathbb{T})$. To actually see that we have a correspondence between the eigenfunctions of T and $\mathcal{L}$ we need to check that a function $v(z)$ above that is an eigenfunction to T with eigenvalue ζ, say, produces an eigenfunction $u(x)$ to $\mathcal{L}$. A straightforward calculations shows that indeed $u(x)$ becomes an eigenfunction for $\mathcal{L}$ with the dilated eigenvalue $2\mu\zeta$ such that the amount of negative eigenvalues is preserved. We can thus turn to the problem of minimizing $Q(z)$ among the class of $2\pi n$ periodic functions $v(z)$.

Following [3] and [4] we show that Q is a local minimum among the class of 2π-periodic functions in section 4.2. We then consider the class of $2\pi n$ periodic functions in section 4.3 and finally apply all these results to our original problem in section 4.4.

4.2 The case of 2π-periodic functions

In this section we follow the work done in [3] and [4]. Consider quasi-periodic wave solutions to the NLS (4.1) above of the form $U(t,y) = e^{-it}W(y) = e^{i(py-t)}Q(2\lambda y)$, where $W(y)$ solves the equation (2.4) and $Q(z)$, $|Q(z)|$ are 2π-periodic. For fixed $(J,H) \in D$ (where D is the domain from section 2.3) the solution $W(y)$ is a unique solution to (2.4) up to translation and a phase factor ([3]). This is emphasized by writing $W(y) = W_{J,H}(y)$ and $Q(2\lambda y) = Q_{J,H}(2\lambda y)$. Here our functionals are $\tilde{E}, \tilde{N}$ and $\tilde{M}$ where we have not put any specific constraints on $\tilde{M}$ and $\tilde{N}$ yet.

The modified energy is of the form $\mathcal{E}_{J,H}(Q) = \tilde{E}(Q) - (1-p^2)\tilde{N}(Q) - 4p\lambda\tilde{M}(Q)$ with the wave profile $Q_{J,H}$ being a critical point. The operator $T_{J,H} = \mathcal{E}''_{J,H}(Q_{J,H}) = -4\lambda^2\partial_{zz} - 4ip\lambda\partial_z - (1-p^2) + Q^2_{J,H}K + 2|Q_{J,H}|^2$ is considered as a linear operator $L^2(\mathbb{T}) \to L^2(\mathbb{T})$ with domain $H^2(\mathbb{T})$, and one can even show that it is self-adjoint in L^2 with compact resolvent ([3]). Of main interest is the kernel of $T_{J,H}$, which since $T_{J,H}$ is a linear second order ordinary differential operator will be a four-dimensional (real) vector subspace of $C^2(\mathbb{R},\mathbb{C})$. However, we are only interested in functions in $H^2(\mathbb{T})$. One can show that we can always find at least two linearly independent functions in $\mathrm{Ker}(T_{J,H}) \cap H^2$. Indeed, noting that equation (4.2) is invariant under phase rotation and space translation and the fact that the functions Q are 2π-periodic allows us to conclude that the equation is invariant under the action of the two-torus $G := \mathbb{T}^2$ acting on $H^1(\mathbb{T})$ through the unitary representation

$$(\mathcal{R}_{(\varphi,\xi)}f)(z) = e^{-i\varphi}f(z+\xi)$$

for $(\varphi,\xi) \in G$. In particular, $\mathcal{E}'(\mathcal{R}_{(\varphi,\xi)}Q_{J,H}) = 0$ and differentiating this with respect to φ and ξ at $(\varphi,\xi) = (0,0)$ we obtain the $T(Q'_{J,H}) = 0 = T(-iQ_{J,H})$. The functions $Q'_{J,H}$ and $-iQ_{J,H}$ are seen to be linearly independent in the domain of T, since otherwise $|Q_{J,H}| = |W_{J,H}|$ would be constant which would contradict the assumptions that (J,H) lies in D (see [3]). This shows that the multiplicity of zero as an eigenvalue of $T_{J,H}$ in L^2 is always at least 2. One can produce two other linearly independent eigenfunctions for zero and show that they cannot have periodicity 2π. This is done in [3] by parametrizing the solutions $W(x) = e^{ipx}Q(2\lambda x)$ to (2.4) in terms of p and λ, and then introducing the functions

$$R_1(2\lambda x) = e^{-ipx}\frac{\partial W_{J,H}}{\partial_\lambda}(x) = \frac{\partial Q_{J,H}}{\partial_\lambda}(2\lambda x) + 2xQ'_{J,H}(2\lambda x),$$

$$R_2(2\lambda x) = e^{-ipx}\frac{\partial W_{J,H}}{\partial_p}(x) = \frac{\partial Q_{J,H}}{\partial_p}(2\lambda x) + ixQ_{J,H}(2\lambda x).$$

Using this one can show that $T_{J,H}R_1 = 0 = T_{J,H}R_2$ so both R_1 and R_2 are in the kernel of $T_{J,H}$. It is then shown in [3] that the four functions $Q'_{J,H}, -iQ_{J,H}, R_1, R_2$ are linearly independent over $\mathbb{R}$ but no non-trivial linear combination of R_1 and R_2 is 2π-periodic, so we must have $\mathrm{Ker}(T_{J,H}) \cap H^2 = \mathrm{span}\{Q'_{J,H}, -iQ_{J,H}\}$. Hence we can conclude.

Lemma 4.1 For any $(J,H) \in D$, zero is an eigenvalue of $T_{J,H}$ with multiplicity 2.

We can now study the spectrum of $T_{J,H}$. What we are interested in is the amount of negative and positive eigenvalues. The solutions $Q_{J,H}(z)$ depends continuously on $(J,H) \in D$ so the coefficients of the matrix operator are continuos functions of (J,H). Hence the eigenvalues depends continuously on $(J,H) \in D$ and in particular they cannot jump when moving in D. Since zero has constant multiplicity 2 in D it is sufficient to locate the eigenvalues of $T_{J,H}$ for *one* value of $(J,H) \in D$. We shall consider the eigenvalues for some (J,H) in a neighborhood of $(0,0)$.

We now discuss what happens when $(J,H) \to (0,0)$. When (J,H) is small one can show that the solution $W_{J,H}(y) = e^{i\ell y} P_{J,H}(\lambda y) = e^{ipy} Q_{J,H}(2\lambda y)$ tends to zero uniformly, and in particular the period $T(J,H) \approx \pi$ and the renormalized phase $\Psi(J,H) \approx 0$ ([3]). Thus $\lambda \approx 1$ and $\ell \approx 0$. The domain D is not smooth near the origin ([3], see fig. 1 section 2), so one would like to use some different parameters than (J,H) for small solutions. Let $W(y) = e^{\ell y} P(\lambda y)$ be a bounded solution to (2.4). The choice in [4] are the first order Fourier coefficients of the function $P(\lambda y) = P(w)$; $a := \int_0^{2\pi} P(w) e^{iw} dy$ and $b := \int_0^{2\pi} P(w) e^{-iw}$. Replacing $P(w)$ with $e^{-i\varphi} P(w+\xi)$ one can assume that both a and b are real. Using a Lyapunov-Schmidt reduction argument (see for example [10]) one obtains in a neighborhood of $(a,b) = (0,0)$ the expansions

$$\lambda_{(a,b)} = 1 - \frac{3}{4}(a^2+b^2) + \mathcal{O}(a^4+b^4),$$

$$\ell_{(a,b)} = \frac{1}{4}b^2 - a^2 + \mathcal{O}(a^4+b^4),$$

$$P_{(a,b)}(w) = ae^{-iw} + be^{iw} - \frac{a^2 b}{8} e^{-3iw} - \frac{ab^2}{8} e^{3iw} + \mathcal{O}(|ab|(|a|^3+|b|^3)),$$

and

$$Q_{(a,b)}(z) = ae^{-iz} + b - \frac{a^2 b}{8} e^{-2iz} - \frac{ab^2}{u} e^{iz} + \mathcal{O}(|ab|(|a|^3+|b|^3)).$$

Now for (a,b) close to zero the operator $T_{J,H}$ has coefficients depending on (a,b) rather then (J,H) which we emphasize by writing $T_{(a,b)}$. As $(a,b) \to (0,0)$, $T_{(a,b)}$ converges to the constant coefficient operator $T_0 := -4\partial_{zz} - 4i\partial_z$. We have the following result regarding the spectrum of this operator in the complexification of the real space L^2.

Proposition 4.2 The spectrum of T_0 in $L^2(\mathbb{T})$ is given by $\sigma(T_0) = \{4k(k+1) \mid k \in \mathbb{Z}\}$.

Proof. One can show that T_0 is self-adjoint with compact resolvent in $L^2(\mathbb{T})$ ([4]). Hence its spectrum is countable and purely a point spectrum. Each $f \in L^2(\mathbb{T})$ can be expanded in a Fourier series $f(z) = \sum_{k\in\mathbb{Z}} \hat{f}(k) e^{ikz}$ with $\hat{f}(k) = \frac{1}{\sqrt{2}} \langle f, \frac{1}{\sqrt{2}} e^{ikz} \rangle_{L^2}$. If ζ is an eigenvalue of T_0 then working instead with Fourier series we have the equation

$$\sum_{k\in\mathbb{Z}} (4k^2 + 4k - \zeta) \hat{f}(k) e^{ikz} = 0.$$

The exponential monomials are linearly independent so the only possible eigenfunctions are the monomials with eigenvalues $\zeta = 4k(k+1)$. For $k \in \mathbb{Z}$ we see that the functions $e^{ikz}, ie^{ikz}, e^{-i(1+k)z}$ and $ie^{-i(1+k)z}$ are all linearly independent over $\mathbb{R}$ and satisfy the equation $T_0 f = 4k(k+1)f$. Hence they generate the eigenspace and each eigenvalue has multiplicity four. □

Note that this in particular shows that 0 is a quadruple eigenvalue of T_0, while the rest of the spectrum is positive and greater or equal to 8. Although the spectrum is real due to self-adjointness we can still picture it as lying in the complex plane. Picking som curve, say a circle of radius 6 centred at the origin we have a finite system of eigenvalues in the interior of the circle separated from the rest of the spectrum. To use the perturbation argument from section 3.2 with T_0 as our "base" operator and $T_{(a,b)} = T_0 + (T_{(a,b)} - T_0) := T_0 + A_{(a,b)}$ we have to establish that $A_{(a,b)}$ is T_0-bounded with T_0-bound less than 1. Now for functions $u \in H^2(\mathbb{T})$ one can establish the interpolation inequality

$$||u'||_{L^2} \leq \epsilon ||u''||_{L^2} + C_\epsilon ||u||_{L^2}$$

where $\epsilon > 0$ is arbitrary and C_ϵ is a constant depending on ϵ. For a second order differential operator T this can be used to establish a $T-$bound of a small perturbation of its coefficients, where the bounds in (3.3) can be made arbitrarily small as long as the perturbation is kept small. For an example of this see ([8, ch.4, §1.2 example 1.10]). In particular, on the circle of radius 6 around the origin our operator is invertible and the quantities $||R_{T_0}(z)||$ and $||T_0 R_{T_0}(z)||$ for z on the circle are fixed. Using the definition of $T_{(a,b)}$ one can further show that the expression $\langle T_{(a,b)}u, u\rangle$ is bounded below by $-C||u||^2_{L^2}$ for some constant $C > 0$ sufficiently large. Then, keeping the perturbation $T_{a,b}$ small, the difference $A_{(a,b)}$ satisfies theorem 3.5 and we can then conclude that the spectrum of $T_{(a,b)}$ decomposes as

$$\sigma(T_{(a,b)}) = \{\zeta^{(0)}_{a,b}, \zeta^{(1)}_{a,b}, \zeta^{(2)}_{a,b}, \zeta^{(3)}_{a,b}\} \cup \sigma_1(T_{(a,b)})$$

with $\sigma_1(T_{(a,b)}) \subset [6, +\infty)$ for sufficiently small (a, b). The four eigenvalues $\zeta^{(i)}_{a,b}$ are the continuation of the quadruple eigenvalue of T_0 at the origin. One can then show that two of these are zero, as expected, and that one is negative and one is positive. For this we refer to ([4, appendix]).

Consider now for any $(J, H) \in D$ the critical point $Q_{J,H}(z)$ for the modified energy $\mathcal{E}_{J,H}$ above. The above argument shows that we cannot conclude that $Q_{J,H}$ is a minima among the class of functions $Q \in H^1(\mathbb{T})$ since the operator $T_{J,H}$ always has a negative eigenvalue in L^2. However, we are interested in the case of constrained functionals, i.e. we consider the subclass of functions $Q \in H^1(\mathbb{T})$ such that $\tilde{N}(Q) = \tilde{N}(Q_{J,H})$ and $\tilde{M}(Q) = \tilde{M}(Q_{J,H})$. In particular we restrict ourselves to the set $\Sigma_{J,H}$ of $H^1(\mathbb{T})$ given by

$$\Sigma_{J,H} := \big\{Q \in H^1(\mathbb{T}) \mid \tilde{N}(Q) = \tilde{N}(Q_{J,H}),\ \tilde{M}(Q) = \tilde{M}(Q_{J,H})\big\}.$$

With the global chart (H^1, I), with I the identity operator, $H^1(\mathbb{T})$ has the structure of an infinite-dimensional (Hilbert) manifold, i.e. a manifold modelled on an infinite-dimensional Hilbert space. A subset $N \subset M$ of a Hilbert manifold M modelled on a Hilbert space V is called a submanifold if there is a vector subspace $W \subset V$ such that for every point $x \in N$ there is a chart (U, ϕ) on M with $x \in U$ satisfying

$$\phi(U \cap N) = \phi(U) \cap W$$

see ([6]). For $\Sigma_{J,H}$ we have the following result.

Lemma 4.3
$\Sigma_{J,H}$ is a codimension 2 submanifold of H^1.

Proof. Define the mapping $F : H^1 \to \mathbb{R}^2$ by $F(Q) := (\tilde{N}(Q), \tilde{M}(Q))$. The functions $\tilde{N}'(Q_{J,H}) = Q_{J,H}$ and $\tilde{M}'(Q_{J,H}) = -iQ'_{J,H}$ are linearly independent and therefore the differential $DF(Q_{J,H})$ is surjective. Using theorem D from [6] we find that the preimage $F^{-1}(\tilde{N}(Q_{J,H}), \tilde{M}(Q_{J,H}))$ is a submanifold of H^1 of codimension equal to the dimension of $\mathbb{R}^2$ with tangent space $\mathcal{T}_{J,H} = \mathrm{Ker}(DF(Q_{J,H})) = \big\{Q \in H^1(\mathbb{T}) \mid \langle Q_{J,H}, Q\rangle = \langle -iQ'_{J,H}, Q\rangle = 0\big\}$ at the point $Q_{J,H}$. □

To show that this restriction gets rid of the negative eigenvalue of $T_{J,H}$ we can us the result discussed in section 3.3. In our case the admissible space $\mathcal{A}$ is the tangent space $\mathcal{T}_{J,H}$ and $Y = H^1(\mathbb{T})$. We need to find the complementary space S, L^2-orthogonal to $\mathcal{T}_{J,H}$ such that $H^1 = \mathcal{T}_{J,H} \oplus_{L^2} S$. One can show that the function $Q_{J,H}(z)$ is a member of a family of so-called traveling and rotating waves of the form $e^{-i\omega t}Q^{\omega,c}_{J,H}(z + ct)$ for (ω, c) lying in a neighborhood of $(0, 0)$ in $\mathbb{R}^2$ with $e^{-i\omega t}Q^{\omega,c}_{J,H}$ being a smooth function of (ω, c) with $e^{-i\omega t}Q^{0,0}_{J,H} = Q_{J,H}$. For a proof see ([3, pp.846]). Using this one gets by construction that the function $Q^{\omega,c}_{J,H}$ is a critical point of $\mathcal{E}^{\omega,c}_{J,H}(Q) := \mathcal{E}_{J,H}(Q) - \omega\tilde{N}(Q) - c\tilde{M}(Q)$.

Now put

$$\partial_\omega Q_{J,H} := \frac{\partial}{\partial_\omega} Q^{\omega,c}_{J,H}\Big|_{(\omega,c)=(0,0)},$$

$$\partial_c Q_{J,H} := \frac{\partial}{\partial_c} Q^{\omega,c}_{J,H}\Big|_{(\omega,c)=(0,0)}.$$

The space $\mathcal{N}_{J,H} := \text{span}\{\partial_\omega Q_{J,H}, \partial_c Q_{J,H}\}$ is called the normal space. Since $Q^{\omega,c}_{J,H}$ is a critical point of $\mathcal{E}^{\omega,c}_{J,H}$ we must have that $\mathcal{E}'_{J,H}(Q^{\omega,c}_{J,H}) = \omega\tilde{N}'(Q^{\omega,c}_{J,H}) + c\tilde{M}'(Q^{\omega,c}_{J,H})$. Then differentiating this relation with respect to ω and c at $(\omega, c) = (0, 0)$ one obtains

$$T_{J,H}(\partial_\omega Q_{J,H}) = \tilde{N}'(Q_{J,H}) = Q_{J,H},$$

$$T_{J,H}(\partial_c Q_{J,H}) = \tilde{M}'(Q_{J,H}) = -iQ'_{J,H}.$$

and so $\langle T_{J,H}Q_1, Q_2\rangle = 0$ for every $Q_1 \in \mathcal{N}_{J,H}$, $Q_2 \in \mathcal{T}_{J,H}$, i.e. we have that S is L^2-orthogonal to $\mathcal{T}_{J,H}$ with $S := \text{span}\{T_{J,H}Q_1 \mid Q_1 \in \mathcal{N}_{J,H}\}$. The corresponding constraint matrix $D_{i,j} := \langle s_i, T^{-1}_{J,H}s_j\rangle$ can be computed. Indeed, if one defines the function $d_{J,H}(\omega, c) := \mathcal{E}^{\omega,c}_{J,H}(Q^{\omega,c}_{J,H})$ for (ω, c) sufficiently close to $(0, 0)$, the Hessian of $d_{J,H}$ is given by

$$\mathcal{H}_{J,H} := \begin{pmatrix} \frac{\partial^2 d_{J,H}}{\partial\omega^2} & \frac{\partial^2 d_{J,H}}{\partial\omega\partial c} \\ \frac{\partial^2 d_{J,H}}{\partial c\partial\omega} & \frac{\partial^2 d_{J,H}}{\partial c^2} \end{pmatrix}\Bigg|_{(\omega,c)=(0,0)}.$$

One can see that $\frac{\partial}{\partial\omega} d_{J,H}(\omega, c) = -\tilde{N}(Q^{\omega,c}_{J,H})$ and $\frac{\partial}{\partial c} d_{J,H}(\omega, c) = -\tilde{M}(Q^{\omega,c}_{J,H})$, which means that the Hessian takes the form

$$\mathcal{H}_{J,H} := -\begin{pmatrix} \langle T_{J,H}(\partial_\omega Q_{J,H}), \partial_\omega Q_{J,H}\rangle & \langle T_{J,H}(\partial_c Q_{J,H}), \partial_\omega Q_{J,H}\rangle \\ \langle T_{J,H}(\partial_\omega Q_{J,H}), \partial_c Q_{J,H}\rangle & \langle T_{J,H}(\partial_c Q_{J,H}), \partial_c Q_{J,H}\rangle \end{pmatrix}.$$

But from this we see $D_{i,j} = -\mathcal{H}_{J,H}$. One can show that the Hessian is non-degenerate and that $\det(\mathcal{H}_{J,H}) < 0$ for every $(J, H) \in D$ (see [3 prop. 3.5]). Note that $\mathcal{H}_{J,H}$ is a symmetric matrix and so it has exactly one negative eigenvalue. Also, since it is non-degenerate we automatically get, by the discussion in section 3.3, that $\mathcal{T}_{J,H} \cap T^{-1}_{J,H}S = \mathcal{T}_{J,H} \cap \mathcal{N}_{J,H} = \{0\}$. Since $T_{J,H}$ has exactly one negative eigenvalue in L^2, we get via proposition 3.6, that the restriction of $T_{J,H}$ to the admissible space $\mathcal{T}_{J,H}$ has exactly 0 negative eigenvalues and the bilinear form $\langle T_{J,H}Q, Q\rangle$ is positive. This means that $Q_{J,H}$ is a candidate for being a minimizer. One can further show that the bilinear form is strongly positive on $\mathcal{T}_{J,H}$ with respect to $||\cdot||_{H^1}$ such that $Q_{J,H}$ is a local minimum for $\mathcal{E}_{J,H}$ restricted to the manifold $\Sigma_{J,H}$. See [3, pp.849] for details.

4.3 The case of $2\pi n$-periodic functions

We consider now the case of $2\pi n$-periodic functions $Q(z)$ for $n \in \mathbb{Z}, n \geq 2$. These functions then lie in the Sobolev space $H^1_{per}([0, 2\pi n], \mathbb{C})$. We can use the same results and arguments as in the 2π-periodic case, with a difference in the negative eigenvalues of the operator $T_{J,H}$. In this case the constant coefficient operator T_0 instead has the following spectrum in the complexification of L^2.

Proposition 4.4. The spectrum of $T_0 = -4\partial_{zz} - 4i\partial_z$ in $L^2[0, 2\pi n]$ is $\sigma_n(T_0) = \{4(\frac{k}{n})((\frac{k}{n}) + 1) \mid k \in \mathbb{Z}\}$.

Proof. In this case each function $f \in L^2[0, 2\pi n]$ has a Fourier series expansion taking the form $\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{i(\frac{k}{n})z}$. By the same argument as in proposition 4.2 the result follows. $\square$

We thus still have that zero is a quadruple eigenvalue and under small perturbations, i.e. for $(J, H) \in D$ close to $(0, 0)$ one of these goes out to the negative x-axis and one to the positive x-axis as discussed in the case of 2π-periodic perturbations. However, now we have much more negative eigenvalues, in fact since $(\frac{k}{n})^2 + (\frac{k}{n}) < 0$ for $-n < k < 0$ and $(\frac{k}{n})^2 - (\frac{k}{n}) < 0$ for $0 < k < n$ we get $2n - 1$ negative eigenvalues of the operator $T_{J,H}$ in $L^2[0, 2\pi n]$ and the Hessian $\mathcal{H}_{J,H}$ still has exactly one negative eigenvalue meaning that, when restricted to the admissible space $\mathcal{T}_{J,H}$, we get $2n - 2$ negative eigenvalues of $T_{J,H}$. Hence according to proposition 3.6 it is not positive for $n \geq 2$ as an operator $H^1 \to H^{-1}$, leaving us only with case of the 2π-periodic functions $Q_{J,H}$ being minimizers.

4.4 Applications

We can now apply the results obtained in section 4.2 and 4.3 to our original problem. Having obtained a solution $\Psi(x)$ to the Euler-Lagrange equation (2.3), we transform it as described above to obtain a function $Q(2\lambda\mu x) = Q(z)$ being 2π-periodic, and we get relations between the functionals as described above. To minimize $E(\Psi)$ then amounts to minimize $\tilde{E}(Q)$, with some constraints on the mass and momentum. The functionals $\tilde{E}(Q), \tilde{N}(Q), \tilde{M}(Q)$ are exactly those studied in section 4.2 and the manifold we wish to minimize over takes to form of $\Sigma_{J,H}$. Thus we see that the functions $\Psi(x)$ with minimal period $\frac{2\pi}{n}$ for $n \geq 2$ are excluded as minimizers, whereas the minimum is obtained among those functions $\Psi(x)$ with minimal period 2π.

A Appendix

Here we collect some notions and results that are used throughout this thesis. The proofs will be left out and for these we will have to refer the reader to the referenced literature. We also take a rather restrictive approach in our notions in the sense that the majority of the results included can be made much more general.

A.1 Elliptic functions

For a reference on elliptic integral and elliptic functions, see for example [11].
Elliptic integrals
The *incomplete* elliptic integrals of the first and second kind are defined respectively as

$$K(\varphi, k) := \int_0^{\varphi} \frac{1}{\sqrt{1-k^2\sin^2(\theta)}} d\theta = \int_0^{x} \frac{1}{(1-t^2)(1-k^2t^2)} dt,$$

$$E(\varphi, k) := \int_0^{\varphi} \sqrt{1-k^2\sin^2(\theta)} d\theta = \int_0^{x} \sqrt{\frac{1-k^2t^2}{1-t^2}} dt,$$

where $x = \sin(\varphi), t = \sin(\theta), \varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $k \in (0,1)$. The *complete* elliptic integrals of the first and second kind are defined respectively as

$$K = K(k) := K(\pi/2, k) = \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2(\theta)}} d\theta = \int_0^{1} \frac{1}{(1-t^2)(1-k^2t^2)} dt,$$

$$E = E(k) := E(\pi/2, K) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2(\theta)} d\theta = \int_0^{1} \sqrt{\frac{1-k^2t^2}{1-t^2}} dt,$$

If we now write $u(\varphi, k) := K(\varphi, k) = K(\arcsin(x), k)$ for $-1 \le x \le 1$ we see that u is a strictly increasing function of x. The inverse to u will be denoted as $x = \mathrm{sn}(u; k)$ for $-K(k) \le u \le K(k)$. By definition of u and x, we have $\varphi = K^{-1}(u, k) := \mathrm{am}(u)$ and so

$$\mathrm{sn}(u; k) = \sin\, \mathrm{am}(u).$$

The inverse to u, $\mathrm{am}(u)$, is called the amplitude of u and $\mathrm{sn}(u; k)$ the elliptic sine function. This is one of the so-called Jacobi elliptic functions. Its minimal period is given by $4K$, $\mathrm{sn}(u; k) = \mathrm{sn}(u + 4K; k)$.

A.2 Sobolev spaces

See [12] and [16] for more details on Sobolev spaces and distributions.

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} = [0, 2\pi]$ be the unit circle. Consider the space $C^\infty(\mathbb{T})$ consisting of smooth 2π-periodic complex-valued functions. Giving it the topology induced by the semi-norms

$$||\varphi||_n := \sup_{0 \le x \le 2\pi} |\varphi^{(n)}(x)|$$

we obtain the space $\mathcal{D}(\mathbb{T})$ of *periodic test functions*. The topological dual $\mathcal{D}'(\mathbb{T})$ with the weak-$*$ topology is called the space of *periodic distributions*. Each $\varphi \in \mathcal{D}(\mathbb{T})$ is a periodic distribution via the pairing

$$\langle \varphi, \psi \rangle := \int_{\mathbb{T}} \varphi \psi dx.$$

Also $L^2(\mathbb{T}) \subset \mathcal{D}'(\mathbb{T})$ via this pairing.

The Fourier transform on $L^2(\mathbb{T})$ can be defined as the function $\hat{f} : \mathbb{Z} \mapsto \mathbb{C}$ with

$$\hat{f}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-inx} dx.$$

The Schwartz space $\mathcal{S}(\mathbb{Z})$ consists of all rapidly decaying sequences, i.e. all functions $f : \mathbb{Z} \mapsto \mathbb{C}$ such that for any $M < \infty$ there exists a constant $C_{f,M}$ satisfying $|f(n)| \leq C_{f,M}(1+n^2)^{-M}$ for all $n \in \mathbb{Z}$. The topology on $\mathcal{S}(\mathbb{Z})$ is induced by the semi-norms $p_k(f) := \sup_{n\in\mathbb{Z}}(1+n^2)^k |\hat{f}(n)|$. Its topological dual in the weak-$*$ topology is called the space of *tempered distributions* and denoted $\mathcal{S}'(\mathbb{Z})$. Using the definition of the Fourier transform above one can show that it is a continuous linear operator $\mathcal{D}(\mathbb{T}) \mapsto S(\mathbb{Z})$ and it can be extended to the space of periodic distributions $\mathcal{D}'(\mathbb{T})$ as a map $\mathcal{D}'(\mathbb{T}) \mapsto S'(\mathbb{Z})$.

For $s \in \mathbb{R}$ the *Sobolev space* $H^s(\mathbb{T})$ can be defined as the space of $u \in \mathcal{D}'(\mathbb{T})$ such that

$$||u||_s^2 := \sum_{n\in\mathbb{Z}} (1+n^2)^s |\hat{u}(n)|^2 < \infty.$$

These are all Hilbert spaces with the corresponding inner product. For $k \in \mathbb{Z}^+$ one could define $H^k(\mathbb{T})$ as the space of $u \in L^2(\mathbb{T})$ with distributional derivatives up to order k all lying in $L^2(\mathbb{T})$, using the norm

$$||f||_{H^k(\mathbb{T})} = \sqrt{\sum_{j=0}^{k} ||f^{(j)}||^2_{L^2(\mathbb{T})}}.$$

Using Fourier series this norm can be shown to be equivalent to $||\cdot||_{H^k(\mathbb{T})}$ above and the corresponding spaces the same. There is also a notion of duality between $H^s(\mathbb{T})$ and $H^{-s}(\mathbb{T})$ where $H^{-s}(\mathbb{T})$ is identified with the dual of $H^s(\mathbb{T})$ via the pairing

$$\langle u, v \rangle = \int_{\mathbb{T}} u(x) v(x) dx$$

with $u \in H^{-s}(\mathbb{T})$ and $v \in H^s(\mathbb{T})$. For details see [12].

A.3 Calculus in normed spaces

Let X, Y be normed spaces. By $B(X, Y)$ we mean the space of all bounded linear operators $X \to Y$.

Fréchet derivative

Let X, Y be normed spaces, $U \subset X$ be a non-empty open subset and $f : U \to Y$ be a map. We say that f is *Fréchet differentiable* at a point $x_0 \in U$ if there exists a bounded linear operator $L \in B(X, Y)$ satisfying

$$\lim_{h \to 0} \frac{||f(x_0 + h) - f(x_0) - Lh||}{||h||} = 0.$$

The operator L is then called the Fréchet derivative of f at the point x_0, and we denote it as $df[x_0]$. If f is differentiable at each point in U we call the function $df[\cdot] : U \to B(X,Y)$ the (Fréchet) derivative of f. The derivative $df[x_0]$ at a point x_0 is unique.

One can define the *second derivative* of f in a similiar fashion, by saying that the map f is twice (Fréchet) differentiable at a point $x_0 \in U$ if the function $x \to df[x]$ is differentiable at x_0. The derivative of $df[x]$ at the point x_0 will be a bounded linear operator $X \to B(X,Y)$ and we denote it as $d^2f[x_0]$. If f is twice differentiable at each point in U we call $d^2f[\cdot] : U \to B(X, B(X,Y))$ the second derivative of f.

Suppose X is a real normed space. Consider the special case when $Y = \mathbb{R}$. Then the derivative $df[x_0]$ is a bounded linear operator $X \to \mathbb{R}$, i.e. it is an element of the dual space X^*, and the second derivative $d^2f[x_0]$ is a bounded linear operator from X to X^*. If $\langle \cdot, \cdot \rangle$ denotes a dual pairing between X^* and X, then $d^2f[x_0]$ acts as $\langle d^2f[x_0]x, y \rangle$, $x, y \in X$.

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