SMALL TOEPLITZ OPERATORS

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Introduction

Toeplitz operators are among the most well studied examples of concrete operators, and entire books have been written about them; in this work the main focus will be on a particular question, a so called cut-off property: if the operator is small enough, does it have to be zero? We will specify the question, suitably modify it to allow for more general situations, and give an answer to it in a number of different Hilbert spaces. In one case the answer is simple and classical and we will just summarize the results. In another important case the question is more complicated and a solution lacked for a while, until it was given quite recently by Daniel Luecking, and then generalized by several mathematicians in the past few years; all of the generalizations use Luecking's theorem as a base case in a way or another. In this thesis we propose a proof of a fairly general theorem that encompasses most of the known theorems in a unified fashion, and does not assume Luecking's theorem; in fact, our method furnishes a new proof as well as a generalization of that theorem.

The plan of the paper is to go through the basics of the spaces and the tools we will consider in Chapter 1, to introduce Toeplitz operators and explore some properties that pertain to our interest in Chapter 2, to state and prove our theorem in Chapter 3, and finally to discuss an interesting application of this type of results to a mathematical physics problem in Chapter 4.

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Contents

Chapter 1

Preliminaries

1.1 Reproducing Kernel Hilbert Spaces

Throughout this thesis we will work with operators that act on some Hilbert spaces of analytic functions. We assume the reader to be familiar with the general theory of Hilbert spaces, and here we will recall the definitions and main properties of the particular spaces we will use. Almost everything will be stated without proofs, which can be found in any text on the subject such as [11], [2], [12]. Throughout this section and in fact throughout this entire work the angular brackets will denote inner product in the space under consideration; since we will be dealing with one space at a time, this should cause no confusion.

A common feature of the spaces we will be considering is that they are *Reproducing Kernel Hilbert Spaces* (RKHS). Here comes the definition:

Definition 1.1.1. Let $\mathcal{H}$ be a Hilbert space of $\mathbb{C}$-valued functions on a set Ω. $\mathcal{H}$ is a RKHS if point evaluation functionals δ_z are bounded on $\mathcal{H}$, that is if for every $z \in \Omega$ and every $f \in \mathcal{H}$ there exists a constant $C_z > 0$ such that

$$|\delta_z(f)| = |f(z)| \leq C_z \|f\|_{\mathcal{H}}. \tag{1.1}$$

The terminology is explained by the Riesz Representation Theorem: since δ_z is a bounded linear functional, its action on $\mathcal{H}$ can be represented by taking inner products with a fixed function; that is, for every $z \in \Omega$ there exists a function $K_z \in \mathcal{H}$ such that for every $f \in \mathcal{H}$

$$f(z) = \delta_z(f) = \langle f, K_z \rangle.$$

The function K_z thus reproduces every function in the space at the point z and it is then called the reproducing kernel (at z). Usually in the literature the reproducing kernels at different points are grouped together in a function $K : \Omega \times \Omega \longrightarrow \mathbb{C}$ defined as

$$K(w, z) = K_z(w),$$

and K is called the reproducing kernel. It is easy to check that $K(w, z) = \overline{K(z, w)}$. Moreover, by definition, $K(\cdot , z)$ belongs to $\mathcal{H}$ for fixed z and $f(z) = \langle f, K(\cdot , z) \rangle$ for any $f \in \mathcal{H}$

and $z \in \Omega$. This last two properties actually characterize the reproducing kernel, namely if a function $H : \Omega \times \Omega \longrightarrow \mathbb{C}$ satisfies them then it coincides with K. Note that

$$\|K_z\|^2 = \langle K_z, K_z \rangle = K_z(z) = K(z,z), \tag{1.2}$$

so that $K(z,z) \geq 0$.

The reproducing kernel yields valuable information about the space $\mathcal{H}$; a way to compute it is if we know an orthonormal basis $\{e_n\}$ of $\mathcal{H}$, for then

$$K_z = \sum \langle K_z, e_n \rangle e_n = \sum \overline{e_n(z)} e_n.$$

If point evaluations also happen to be uniformly bounded on compact subsets of Ω (and such will be the case in every space we will consider), that is for every compact $E \subset \Omega$ there exists a constant C_E such that

$$\sup_{z \in E} |f(z)| \leq C_E \|f\|,$$

then it can be shown that the series $\sum \overline{e_n(z)} e_n(w)$ converges uniformly on compact sets of $\Omega \times \Omega$ to $K(w,z)$.

It is now time to see some examples of RHKS, consisting of functions that are also holomorphic in their domain.

1.1.1 Hardy Space of the Unit Disk

We let $\mathbb{D}$ be the unit disk in the complex plane. The vector space of functions holomorphic on $\mathbb{D}$ will be denoted by $\mathcal{H}(\mathbb{D})$. For such a function and a number $0 < r < 1$, we set

$$\mathcal{M}_2(f,r) := \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\vartheta})|^2 \, d\vartheta \right)^{\frac{1}{2}}$$

its square integral mean on the circle of radius r. The Hardy space of the unit disk $H^2 = H^2(\mathbb{D})$ is then defined as,

$$H^2 = \left\{ f \in \mathcal{H}(\mathbb{D}) : \sup_{0<r<1} \mathcal{M}_2(f,r) < \infty \right\} \tag{1.3}$$

with norm $\|f\|_{H^2} = \sup_{0<r<1} \mathcal{M}_2(f,r)$. It is possible to show that $\mathcal{M}_2(f,r)$ is increasing in r, so that the sup in the definition is actually a limit as r approaches 1^-.

Although we will only deal with the Hilbert space case, we feel compelled to mention that an analogous definition could be given for any $0 < p < +\infty$ in lieu of 2. All the resulting spaces H^p are called Hardy spaces in the literature. The space H^∞ consists of all bounded analytic functions in the disk with the sup norm. Clearly, $H^\infty \subset H^2$.

The peculiar feature about the exponent 2 is that H^2 turns out to be an Hilbert space, as anticipated. Consider the power series expansion of a function in $\mathcal{H}(\mathbb{D})$, $f(z) = \sum_{n=0}^{+\infty} a_n z^n$;

this series converges uniformly on compact subsets of the unit disk, so we can evaluate

$$\mathcal{M}_2(f,r)^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\vartheta})|^2 \, d\vartheta = \frac{1}{2\pi}\int_0^{2\pi} \sum_{n,m=0}^{+\infty} a_n\overline{a_m} r^{n+m} e^{i\vartheta(n-m)} \, d\vartheta$$
$$= \sum_{n=0}^{+\infty} r^{2n}|a_n|^2,$$

since the terms with $n \neq m$ vanish in the integration. We then have

$$\sup_{0<r<1} \mathcal{M}_2(f,r)^2 = \sum_{n=0}^{+\infty} |a_n|^2,$$

so that f is in H^2 if and only if its Taylor coefficients are square summable, with $\|f\|_{H^2} = \left(\sum_{n=0}^{+\infty} |a_n|^2\right)^{\frac{1}{2}}$. This also shows that H^2 is a Hilbert space, with inner product

$$\langle f, g\rangle = \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\vartheta})\overline{g(re^{i\vartheta})} \, d\vartheta.$$

If $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are the sequences of Taylor coefficients of f and g respectively, then the calculation above shows that

$$\langle f, g\rangle = \sum_{n=0}^{+\infty} a_n\overline{b_n} = \langle \{a_n\}, \{b_n\}\rangle_{l^2(\mathbb{N})}. \tag{1.4}$$

It also follows that if $f \in H^2$ with power series expansion as above, then

$$\widetilde{f}(e^{i\vartheta}) := \sum_{n=0}^{+\infty} a_n e^{in\vartheta}$$

is well defined as an element of $L^2(\partial\mathbb{D})$; its Fourier coefficients (which are zero for negative indices) are the Taylor coefficients of f. This is one of the first manifestations of the deep connection between complex and harmonic analysis, and indicates why the Hardy space plays an important role in Fourier analysis. We call $\widetilde{f}$ the boundary value of f; note that $\langle f, g\rangle = \langle \widetilde{f}, \widetilde{g}\rangle_{L^2(\partial\mathbb{D})}$. It can be shown that f can be reconstructed from its boundary value by the Poisson or Cauchy integral; moreover, the Poisson integral of $h \in L^2(\partial\mathbb{D})$ is analytic in $\mathbb{D}$ if and only if the Fourier coefficients of h vanish for negative indices.

All of this can be summarized by saying that there is an isometric isomorphism between H^2 and the subspace $H^2(\partial\mathbb{D})$ of $L^2(\partial\mathbb{D})$ consisting of elements whose Fourier coefficients vanish for negative indices. The operator P defined on $L^2(\partial\mathbb{D})$ by

$$P\left(\sum_{n=-\infty}^{+\infty} a_n e^{in\vartheta}\right) = \sum_{n=0}^{+\infty} a_n e^{in\vartheta}$$

is the orthogonal projection with range $H^2(\partial\mathbb{D})$; it is called the Cauchy–Szegö projection. It is not difficult to show that P is a limit of (Cauchy) integral operators, precisely:

$$P(g)(e^{i\eta}) = \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} g(e^{i(\eta-\vartheta)})\frac{1}{1-re^{i\vartheta}} \, d\vartheta.$$

Using the Cauchy formula in a circle of radius $r < 1$ it is easy to show that point evaluations are bounded linear functionals on H^2, so that H^2 has a reproducing kernel; this has the form

$$K(w,z) = \frac{1}{1 - w\overline{z}}.$$

In fact, $K(\,\cdot\,,z)$ is in H^2 for every fixed $z \in \mathbb{D}$ and the reproducing identity $\langle f, K(\,\cdot\,,z)\rangle = f(z)$ is easily seen using the Cauchy formula.

As a last property in our basic survey, we remark that using the density of trigonometric polynomials in $L^2(\partial\mathbb{D})$ one can obtain the density of analytic polynomials in H^2.

1.1.2 Bergman-type Spaces

Let $\Omega \subset \mathbb{C}$ be a domain (a connected open subset). The Bergman space $L^2_a(\Omega)$ (sometimes also called $\mathcal{A}^2(\Omega)$) is defined as the space of holomorphic function on Ω such that

$$\int_\Omega |f(z)|^2 \, dA(z) < \infty,$$

where dA is the Lebesgue measure on Ω. As norm on $L^2_a(\Omega)$, we set $\|f\| = \left(\int_\Omega |f(z)|^2 \, dA(z)\right)^{\frac{1}{2}}$ and consider $L^2_a(\Omega)$ as a subspace of $L^2(\Omega)$; we also define an inner product on $L^2_a(\Omega)$ which is the restriction of the L^2 inner product, $\langle f, g\rangle = \int_\Omega f(z)\overline{g(z)} \, dA(z)$. Note that the space so defined could well be trivial; for example, Liouville's theorem implies that there are no non–zero entire functions which are square integrable on the whole of $\mathbb{C}$, and thus $L^2_a(\mathbb{C}) = \{0\}$. But if for example $\Omega \neq \mathbb{C}$ is simply connected, it can be shown that $L^2_a(\Omega)$ is infinite dimensional; more simply, if Ω is bounded then the space of analytic polynomials is contained in $L^2_a(\Omega)$.

It is not difficult to see that point evaluation functionals are bounded on $L^2_a(\Omega)$, and actually that the aforementioned stronger property

$$\sup_{z \in E} |f(z)| \leq C_E \|f\| \tag{1.5}$$

holds for any compact set $E \subset \Omega$, for any $f \in L^2_a(\Omega)$ and a constant $C_E > 0$. It follows from (1.5) that convergence in the L^2-norm implies uniform convergence on compacts for functions in $L^2_a(\Omega)$, and therefore that $L^2_a(\Omega)$ is a closed subspace of the Hilbert space $L^2(\Omega)$, thus Hilbert itself.

As anticipated then $L^2_a(\Omega)$ is a RKHS with reproducing kernel $K = K_\Omega$; note that since $K(\,\cdot\,,z) \in L^2_a(\Omega)$ and $K(w,z) = \overline{K(z,w)}$, K is analytic in the first variable and anti-analytic in the second variable. The reproducing kernel provides us with an explicit formula for the orthogonal projection $P : L^2(\Omega) \longrightarrow L^2_a(\Omega)$, called the Bergman projection; in fact, since $Pf \in L^2_a(\Omega)$ for $f \in L^2(\Omega)$, by the reproducing property

$$\begin{aligned} Pf(z) = \langle Pf, K_z\rangle = \langle f, PK_z\rangle = \langle f, K_z\rangle &= \int_\Omega f(w)\overline{K_z(w)} \, dA(w) \\ &= \int_\Omega f(w) K(z,w) \, dA(w), \end{aligned} \tag{1.6}$$

where the second equality holds because P is selfadjoint and the third because $K_z \in L^2_a(\Omega)$. The closed span of the set of finite linear combination with coefficients in $\mathbb{Q}$ of reproducing kernels at points z_n having an accumulation point in Ω is easily seen to be dense in $L^2_a(\Omega)$, because if f is orthogonal to such space than $f(z_n) = 0$ for every n, which by the identity principle for holomorphic functions implies that f is identically zero; so $L^2_a(\Omega)$ is always separable.

There are some conditions on the domain Ω which ensure that analytic polynomials are dense in $L^2_a(\Omega)$; for example, if Ω is bounded and the boundary is a Jordan curve. However, if Ω is the unit disk (or any disk), then it is easy to show it directly, and every function in $L^2_a(\mathbb{D})$ is approximated in the L^2-norm by its Taylor polynomials; a calculation then shows that the set $\{\varphi_k\}_{k=0}^{+\infty}$ with $\varphi_k = \sqrt{\frac{k+1}{\pi}} z^k$ is an orthonormal set in $L^2_a(\mathbb{D})$, complete by the density of analytic polynomials. But then, as mentioned in the section about general RKHS, the series $\sum_{k=0}^{+\infty} \varphi_k(z)\overline{\varphi_k(w)}$ converges uniformly on compacts to the reproducing kernel of $L^2_a(\mathbb{D})$; the sum of the series can be calculated explicitly, and yields

$$K(z, w) = \frac{1}{\pi} \frac{1}{(1 - \overline{w}z)^2}.$$

Some noteworthy properties of this kernel are that it is never zero, it satisfies $K(z, z) > 0$ for every $z \in \mathbb{D}$, and $K(\,\cdot\,, w)$ is a holomorphic function in a disk larger than $\mathbb{D}$ for fixed w.

Note that functions in the Hardy space H^2 also belong to $L^2_a(\mathbb{D})$, but the converse is not true and the Hardy space norm is strictly stronger than the L^2-norm.

We said that the space $L^2_a(\mathbb{C})$ is trivial and thus not very interesting, but one would still like to work with entire functions in a similar context. One possibility is to consider the Gaussian measure on $\mathbb{C}$,

$$d\nu(z) = \omega(z)dA(z), \quad \text{where } \omega(z) = \frac{1}{\pi} e^{-|z|^2},$$

and to define the Fock space (sometimes also called Segal-Bargmann space) $\mathcal{F}^2(\mathbb{C})$ as the space of all entire functions that belong to $L^2(\mathbb{C}, d\nu)$; then this space is clearly non trivial since for examples analytic polynomials are contained in it. For a reference see [23], where the definition of the Fock space depends on a parameter α incorporated into the measure.

It holds that $\mathcal{F}^2(\mathbb{C})$ is a closed subspace of $L^2(\mathbb{C}, d\nu)$, and thus a Hilbert space with inner product

$$\langle f, g\rangle = \int_{\mathbb{C}} f(z)\overline{g(z)}\; d\nu(z)$$

which gives the norm $\|f\| = \left(\int_{\mathbb{C}} |f(z)|^2\; d\nu(z)\right)^{\frac{1}{2}}$. Also in this space point evaluations are bounded linear functionals, and convergence in $\mathcal{F}^2(\mathbb{C})$ implies uniform convergence on compact subsets of $\mathbb{C}$; then $\mathcal{F}^2(\mathbb{C})$ has a reproducing kernel, which has the form

$$K(z, w) = e^{z\overline{w}}.$$

Analogously to the Bergman space case, then the orthogonal projection $P : L^2(\mathbb{C}, d\nu) \longrightarrow \mathcal{F}^2(\mathbb{C})$ is an integral operator:

$$Pf(z) = \int_{\mathbb{C}} f(w)K(z, w)\; d\nu(w),$$

for all $f \in L^2(\mathbb{C}, d\nu)$. Analytic polynomials turn out to be dense in $\mathcal{F}^2(\mathbb{C})$, and actually it is not difficult to show that the set $\{\psi_n\}_{n=0}^{+\infty}$ with

$$\psi_n(z) = \sqrt{\frac{1}{n!}} z^n$$

is an orthonormal basis for $\mathcal{F}^2(\mathbb{C})$. In summary, using the Gaussian measure we obtain a space whose basic properties are very similar to those of the Bergman space of the unit disk; the differences emerge further in the study, and one of the most interesting aspects of the Fock space is its connection with quantum mechanics. One difference that is already noticeable at this basic stage is that there are no non–constant bounded functions in the Fock space, by Liouville's theorem.

1.2 Compactly Supported Distributions

We assume the reader to know the definitions of distribution, tempered distribution and compactly supported distribution and we will not repeat them. Since we will use the Fourier transform on the space of compactly supported distributions in our proof of the finite rank theorem in Chapter 3, we will recall here the special properties it enjoys; here we are following closely Hörmander [15].

Recall that the Fourier transform is an isomorphism on the space of tempered distribution on $\mathbb{R}^n$, $\mathcal{S}'(\mathbb{R}^n)$; the Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$ is defined by duality with the Schwartz space. However, if a distribution u is compactly supported on $\mathbb{R}^n$, that is u is a continuous linear functional on $\mathcal{C}^\infty(\mathbb{R}^n)$ (the set of such functionals is denoted by $\mathcal{E}'(\mathbb{R}^n)$), then the expression

$$\widehat{u}(\boldsymbol{\xi}) = u(e^{-i\mathbf{x}\cdot\boldsymbol{\xi}}) \tag{1.7}$$

is well defined for $\boldsymbol{\xi}$ and $\mathbf{x}$ in $\mathbb{R}^n$, where u is acting on $\mathbf{x}$ and the dot denotes the $\mathbb{R}^n$ scalar product. The notation is not an abuse, since the expression on the right hand side of (1.7) actually coincides with the Fourier transform of u defined by duality, as shown in [15], chapter 7. Then $\widehat{u}$ thus defined is a $\mathcal{C}^\infty$ function of $\boldsymbol{\xi}$, and the right hand side makes sense also for a complex vector $\boldsymbol{\xi} \in \mathbb{C}^n$; allowing this extension, $\widehat{u}(\boldsymbol{\xi})$ is actually an entire analytic function on $\mathbb{C}^n$. Theorem 7.3.1 in [15] additionally shows that the entire function $\widehat{u}$ grows at most like a polynomial of degree N on $\mathbb{R}^n$, where N is the order of the distribution u; this fact will be frequently used in Chapter 3.

The interesting thing (which will not be used in this work) is that a converse holds: if an entire function on $\mathbb{C}^n$ satisfies some estimate (which implies polynomial growth on $\mathbb{R}^n$), then it is the Fourier transform of a compactly supported distribution. This is usually called the Paley–Wiener–Schwartz theorem.

Chapter 2

Toeplitz operators in spaces of analytic functions

Consider a set $\Omega \subset \mathbb{C}$, and let $\mathcal{A}$ be a closed subspace of $L^2(\Omega)$. Denote by $\mathbf{P}$ the orthogonal projection on $L^2(\Omega)$ with range $\mathcal{A}$, and take a to be a function on Ω; the Toeplitz operator with symbol a is the generally unbounded operator defined as

$$T_a f = \mathbf{P}(af)$$

on its natural domain $\mathcal{D}(T_a) = \{f \in \mathcal{A} : \mathbf{P}(af) \in \mathcal{A}\}$. In order to identify some interesting properties of these operators, it is necessary to be more specific about the space $\mathcal{A}$; in this chapter we will study Toeplitz operators defined on some Hilbert spaces of analytic functions, exploring boundedness and compactness properties. It is the latter that we will be especially interested in, together with some other notions of smallness. In addition, the particular structure of the Hilbert spaces under consideration will allow us to extend the notion of a Toeplitz operator to symbols that are not necessarily functions.

2.1 Hardy Space

The Hardy space on the unit disk was the first in which Toeplitz operators were considered, and the seminal paper [8] by Brown and Halmos proved the basic results and formulated several open problems. The material of this section is largely taken from that paper.

Recall that $H^2 = H^2(\mathbb{D})$ is isometrically isomorphic to the closure of the linear span of $\{e^{in\vartheta} : n \geq 0\}$ in $L^2(\partial\mathbb{D}, \sigma)$, where σ is the normalized arc length. We will use this isomorphism throughout this section and by a small abuse of notation still denote by H^2 the above subspace of $L^2(\partial\mathbb{D}, \sigma)$. We denote by P the projection from $L^2(\partial\mathbb{D})$ onto its closed subspace H^2, and we set $e_n(\vartheta) = e^{in\vartheta}$ for every integer n.

Definition 2.1.1. Let $\varphi \in L^\infty(\partial\mathbb{D})$. The Toeplitz operator with symbol φ is the operator defined as:

$$T_\varphi f = P(\varphi f) \tag{2.1}$$

for every $f \in H^2$, i.e. the compression of the operator of multiplication by φ to H^2.

This choice for the symbol class gives us a bounded operator: It is easy to see that $\|T_\varphi\| \leq \|\varphi\|_\infty$. In [8] it is shown that equality holds.

Remark 2.1.2. If φ happens to be analytic (by this we mean that its Fourier coefficients for negative indices vanish, or equivalently that its harmonic extension in the disk is analytic), then T_φ coincides with the operator of multiplication by φ.

Some immediate consequences of the definition are that $T_{\alpha\varphi+\beta\psi} = \alpha T_\varphi + \beta T_\psi$, and the following:

Lemma 2.1.3. *For f and g in H^2, $\langle T_\varphi f, g\rangle = \langle \varphi f, g\rangle$.*

Proof. From the properties of the projection P:

$$\langle T_\varphi f, g\rangle = \langle P(\varphi f), g\rangle = \langle P(\varphi f), Pg\rangle = \langle \varphi f, P^2 g\rangle = \langle \varphi f, g\rangle.$$

□

Lemma 2.1.4. *The adjoint of the operator T_φ is $T_{\overline{\varphi}}$.*

Proof. For any g in H^2,

$$\langle T_\varphi^* f, g\rangle = \langle f, T_\varphi g\rangle = \overline{\langle T_\varphi g, f\rangle} = \overline{\langle \varphi g, f\rangle} = \langle f, \varphi g\rangle = \langle \overline{\varphi} f, g\rangle = \langle T_{\overline{\varphi}} f, g\rangle$$

where the third equality comes from the previous lemma. It follows that $T_\varphi^* f = T_{\overline{\varphi}} f$ for any f in H^2. □

As a consequence, we see that T_φ is self-adjoint if and only if φ is real valued. The following result is easy but conceptually important for the next chapter.

Proposition 2.1.5. *T_φ is the zero operator if and only if $\varphi = 0$.*

Proof. It would follow directly from $\|T_\varphi\| = \|\varphi\|_\infty$, but since that is a more difficult result we prefer to present a simple direct proof. One direction is trivial. Now suppose that $T_\varphi = 0$; then $P(\varphi e_n) = 0$ for every $n \geq 0$. Consider the Fourier expansion of φ, $\varphi = \sum_{k=-\infty}^{\infty} \widehat{\varphi}(k) e_k$. Then $\varphi e_n = \sum_{k=-\infty}^{\infty} \widehat{\varphi}(k) e_{k+n}$, and $P(\varphi e_n) = 0$ implies that $\widehat{\varphi}(k) = 0$ for every k such that $k + n \geq 0$. By choosing n larger and larger, we obtain $\widehat{\varphi}(k) = 0$ for every k, and thus $\varphi = 0$. □

The infinite matrix associated to a Toeplitz operator has a rather special form: Indeed, let $i, j \geq 0$; by Lemma 2.1.3,

$$\langle T_\varphi e_i, e_j\rangle = \langle \varphi e_i, e_j\rangle = \langle \varphi e_{i+1}, e_{j+1}\rangle = \langle T_\varphi e_{i+1}, e_{j+1}\rangle. \tag{2.2}$$

Thus the matrix $(a_{ij}) = \langle T_\varphi e_i, e_j\rangle$ is constant on the diagonals, and

$$a_{ij} = \widehat{\varphi}(j - i). \tag{2.3}$$

The operator T_{e_1} has an important history in the theory of Hardy spaces; it is called the *unilateral shift* in virtue of the following fact:

$$T_{e_1} e_n = e_{n+1}.$$

This means that in the isomorphism between H^2 and $l^2(\mathbb{N})$ given by the basis $\{e_n : n \geq 0\}$ T_{e_1} acts by shifting the coefficients one position to the right. The *unilateral shift* shows that the set of Toeplitz operators is neither commutative nor closed under multiplication. In fact, $T^*_{e_1}T_{e_1}$ is the identity on H^2 (thus a Toeplitz operator in particular), while $T_{e_1}T^*_{e_1}$ violates condition (2.2) (for example $\langle T_{e_1}T^*_{e_1}e_1, e_1\rangle = 0$ while $\langle T_{e_1}T^*_{e_1}e_2, e_2\rangle = 1$) and therefore cannot be a Toeplitz operator.

The above also disproves the tempting guess $T_{\varphi\psi} = T_\varphi T_\psi$, which is in fact true only for a rather small subset of the symbol class; in [8], using the matrix characterization, it is proved that a product $T_\varphi T_\psi$ is a Toeplitz operator if and only if ψ is analytic or $\overline{\varphi}$ is analytic and in this case the multiplication identity holds (the sufficiency is clear since if ψ is analytic then T_ψ is multiplication by ψ, and the case φ co-analytic is handled by passing to the adjoints). From the above and using the fact that zero sets of holomorphic functions have zero measure it follows that a product of two Toeplitz operator is zero if and only if at least one of the two factors is zero, a fact which can be expressed by saying that there are no zero-divisors in the set of Toeplitz operators on the Hardy space. The corresponding result for an arbitrary finite product of Toeplitz operators was an open problem for long and has been settled only fairly recently in [1].

An important, albeit easy, consequence of the matrix structure, which in turn follows from the fact that we know explicitly an orthonormal basis of $L^2(\partial\mathbb{D})$ which extends that of H^2, is that Toeplitz operators on the Hardy space cannot be too small; namely, the following holds:

Theorem 2.1.6. *The only compact Toeplitz operator on H^2 is the zero operator.*

Proof. The sequence $\{e_n\}_{n\geq 0}$ converges weakly to 0, therefore if T_φ is compact $\|T_\varphi e_n\| \longrightarrow 0$. For every integer k, take $n \geq 0$ such that $n + k \geq 0$; by (2.2) we have:

$$\langle \varphi, e_k\rangle = \langle T_\varphi e_n, e_{n+k}\rangle \leq \|T_\varphi e_n\|$$

Sending n to ∞ we see that $\widehat{\varphi}(k) = 0$ for every integer k, which gives the result. □

2.2 Bergman Space

For Toeplitz operators on the classical Bergman space of the unit disk $\mathbb{D}$ or on a general bounded domain $\Omega \subset \mathbb{C}$ (Bergman–Toeplitz operators from now on), several questions have been asked and answered since the 1970's. As we will shortly see, Bergman–Toeplitz operators are in some sense much less rigid than Hardy–Toeplitz operators. Extensive discussions for the unit disk case can be found in the references [2], [23]; in this section we have drawn freely from these, making changes whenever it was necessary to our needs.

Let now P be the projection from $L^2(\Omega, dA)$ (where dA is the two-dimensional Lebesgue measure on Ω, normalized so that Ω has measure 1) onto its closed subspace $L^2_a(\Omega)$; for $\varphi \in L^\infty(\Omega)$, the Toeplitz operator acting on $L^2_a(\Omega)$ can be defined in much the same way as in (2.1): $T_\varphi f := P(\varphi f)$ for any f in L^2_a. Using the integral representation of the Bergman projection,

$$T_\varphi f(z) = \int_\Omega \varphi(w) f(w) \overline{K_z(w)}\, dA(w).$$

With this choice of symbols, since the Bergman projection has norm 1, it obviously holds that $\|T_\varphi\| \leq \|\varphi\|_\infty$, so T_φ is a bounded operator. However, unlike in the Hardy space case, equality does not hold in general and this bound is far from optimal: there are even unbounded symbols that give rise to bounded Toeplitz operators; in [10] (rather intricate) explicit examples are constructed.

If φ is analytic, so that $\varphi \in H^\infty(\Omega)$, T_φ coincides with multiplication by φ on L^2_a. The easy algebraic properties hold as well (α and β are complex numbers):

$$T_{\alpha\varphi+\beta\psi} = \alpha T_\varphi + \beta T_\psi, \tag{2.4}$$

$$T^*_\varphi = T_{\overline{\varphi}}. \tag{2.5}$$

The equivalent of Lemma 2.1.3 is also true with the same proof and f and g in L^2_a.

We should mention that the problem of when $T_\varphi T_\psi$ is a Toeplitz operator is harder than its Hardy space counterpart, and it is still not known whether the product of two Bergman-Toeplitz operator being zero implies that one of the factors is zero.

The fact that in general for $L^2(\Omega)$ an explicit description of an orthonormal basis is lacking means that the matrix form of a Toeplitz operator does not possess the easy structure of (2.3) even when the orthonormal basis of the Bergman space is known explicitly, such as for $L^2_a(\mathbb{D})$.

2.2.1 Measures as Symbols

Now we see how Bergman-Toeplitz operators can be defined for more general classes of symbols than bounded functions; we start with the case of a finite complex Borel measure μ on the unit disk $\mathbb{D}$.

Definition 2.2.1. Let the sesquilinear form $\mathfrak{t}_\mu$ be defined on analytic polynomials by:

$$\mathfrak{t}_\mu(f,g) = \int_{\mathbb{D}} f(z)\overline{g(z)}\, d\mu(z). \tag{2.6}$$

If this form is bounded, i.e. if there exists a constant $C > 0$ such that $|\mathfrak{t}_\mu(f,g)| \leq C\|f\|\|g\|$, where the norms are in $L^2_a(\mathbb{D})$, then since analytic polynomials are dense in $L^2_a(\mathbb{D})$ $\mathfrak{t}_\mu$ can be extended to $L^2_a(\mathbb{D}) \times L^2_a(\mathbb{D})$ and it follows from the Riesz representation theorem that there exists a linear bounded operator $T_\mu : L^2_a(\mathbb{D}) \longrightarrow L^2_a(\mathbb{D})$ such that

$$\langle T_\mu f, g\rangle = \mathfrak{t}_\mu(f,g) \tag{2.7}$$

for every f and g in $L^2_a(\mathbb{D})$. We call T_μ the Toeplitz operator with symbol μ; its norm does not exceed C.

Remark 2.2.2. The unit disk setting is not absolutely necessary and the same definition could be given for a bounded domain Ω, but we would need to identify a dense subset of $L^2_a(\Omega)$ where the form $\mathfrak{t}_\mu(f,g)$ is well defined, since analytic polynomials might not be dense in $L^2_a(\Omega)$.

We will now discuss an alternative approach to the definition of Toeplitz operators with finite measures as symbols.

Let K_z be the reproducing kernel of $L^2_a(\mathbb{D})$ at the point z. Mimicking the standard definition as compression of the multiplication operator, the Toeplitz operator with symbol μ could be formally defined as:

$$T_\mu f(z) = \int_{\mathbb{D}} f(w)\overline{K_z(w)}\, d\mu(w) \tag{2.8}$$

for $f \in L^2_a(\mathbb{D})$. Without further conditions on the measure, this operator may be unbounded. It is densely defined, since the integral on the right hand side of (2.8) converges for f being an analytic polynomial or even $f \in H^\infty(\mathbb{D})$ (because K_z is a bounded function for a fixed z in the disk). Even when the integral converges, $T_\mu f$ defined in this way may not be square integrable, although it is always an analytic function.

Note that formally, for $g \in L^2_a(\mathbb{D})$, by the reproducing property of the kernel,

$$\begin{aligned}
\langle T_\mu f, g\rangle &= \int_{\mathbb{D}} \left(\int_{\mathbb{D}} f(w)\overline{K_z(w)}\, d\mu(w)\right) \overline{g(z)}\, dA(z) \\
&= \int_{\mathbb{D}} f(w) \left(\int_{\mathbb{D}} K_w(z)\overline{g(z)}\, dA(z)\right)\, d\mu(w) \\
&= \int_{\mathbb{D}} f(w)\overline{g(w)}\, d\mu(w) \\
&= \mathfrak{t}_\mu(f,g).
\end{aligned}$$

We see then that if the exchange of integrals above can be justified, the two definitions given by (2.7) and (2.8) coincide.

Perhaps more transparently, note that if the form $\mathfrak{t}_\mu$ satisfies the boundedness condition in Definition 2.2.1 we can evaluate

$$T_\mu f(z) = \langle T_\mu f, K_z\rangle = \mathfrak{t}_\mu(f, K_z) = \int_{\mathbb{D}} f(w)\overline{K_z(w)}\, d\mu(w).$$

Therefore at least when $|\mathfrak{t}_\mu(f,g)| \leq C\|f\|\|g\|$ the expressions (2.7) and (2.8) define the same (bounded) operator.

Let us note that if $d\mu(w) = \varphi(w)\, dA(w)$ for some φ bounded in the disk, then the form $\mathfrak{t}_\mu$ is bounded because of Cauchy-Schwarz, (2.8) defines a bounded operator and $T_\mu f = P(\varphi f) = T_\varphi f$ for every $f \in L^2_a(\mathbb{D})$.

The approach through (2.8) is for instance the one chosen by [23]; we have chosen the definition through a sesquilinear form because that is the expression we will need when proving the *Finite Rank Theorem* in the next chapter, and ultimately in the literature related to such problems it is the formulation that turns out to be most useful.

The question of boundedness and compactness of T_μ (that is, of $\mathfrak{t}_\mu$) is related to a very important topic in the theory of Banach spaces of holomorphic functions, that of Carleson measures. We will now define such objects and prove some sufficient conditions for the Toeplitz operator to be bounded or compact.

Definition 2.2.3. A finite positive Borel measure μ on $\mathbb{D}$ is called a Carleson measure for L^2_a if the inclusion map

$$\imath : L^2_a(\mathbb{D}) \longrightarrow L^2(\mathbb{D}, d\mu)$$

is bounded, i.e. if there exists $C > 0$ such that

$$\int_{\mathbb{D}} |f(z)|^2 \, d\mu(z) \leq C \int_{\mathbb{D}} |f(z)|^2 \, dA(z) \tag{2.9}$$

for every $f \in L^2_a(\mathbb{D})$. If the inclusion map is required to be compact, then μ is called a *vanishing Carleson measure.*

Remark 2.2.4. This definition could also be given for $L^p_a(\mathbb{D})$, for any $p \geq 1$; in [23] (which on its turn draws on a number of older sources) equivalent conditions for μ to be either a Carleson measure or a vanishing Carleson measure are proved, from which it is apparent that the definition is actually independent of $p \geq 1$.

Proposition 2.2.5. *Let μ be a finite complex Borel measure on $\mathbb{D}$. If its total variation $|\mu|$ is a Carleson measure, then the sesquilinear form $\mathfrak{t}_\mu$ is bounded on $L^2_a(\mathbb{D})$.*

Proof. By Remark 2.2.4, if $|\mu|$ is a Carleson measure it holds that

$$\int_{\mathbb{D}} |h(z)| \, d|\mu|(z) \leq C \int_{\mathbb{D}} |h(z)| \, dA(z)$$

for any h in $L^1_a(\mathbb{D})$. Noticing that for f and g in $L^2_a(\mathbb{D})$ the product fg is in $L^1_a(\mathbb{D})$ we can now estimate:

$$\begin{aligned} |\mathfrak{t}_\mu(f,g)| = \left| \int_{\mathbb{D}} f(z)\overline{g(z)} \, d\mu(z) \right| &\leq \int_{\mathbb{D}} |f(z)\overline{g(z)}| \, d|\mu|(z) \\ &\leq C \int_{\mathbb{D}} |f(z)g(z)| \, dA(z) \\ &\leq C\|f\|\|g\|. \end{aligned}$$

□

Theorem 2.2.6. *Let μ be a finite complex measure such that its total variation $|\mu|$ is vanishing Carleson; then the operator*

$$T_\mu : L^2_a(\mathbb{D}) \longrightarrow L^2_a(\mathbb{D})$$

is compact on $L^2_a(\mathbb{D})$.

Proof. By definition for f in L^2_a we have

$$\|T_\mu f\| = \sup\{|\langle T_\mu f, g\rangle| : g \in L^2_a, \quad \|g\| = 1\} = \sup\left\{\left|\int_{\mathbb{D}} f(z)\overline{g(z)}\ d\mu(z)\right| : g \in L^2_a, \quad \|g\| = 1\right\}.$$

Using the Cauchy-Schwarz inequality on the space $L^2(d|\mu|)$, we have:

$$\left|\int_{\mathbb{D}} f(z)\overline{g(z)}\ d\mu(z)\right| \leq \int_{\mathbb{D}} |f(z)g(z)|\ d|\mu|(z) \leq \|f\|_{L^2(d|\mu|)}\|g\|_{L^2(d|\mu|)}.$$

Since $|\mu|$ is Carleson, there exists a constant $C > 0$ such that $\|g\|_{L^2(d|\mu|)} \leq C\|g\|$ for any g in L^2_a. Substituting the inequalities in the expression above for the norm, we get

$$\|T_\mu f\| \leq C\|f\|_{L^2(d|\mu|)}.$$

Now take a sequence $\{f_n\}$ in L^2_a converging weakly to zero. By definition, $|\mu|$ being vanishing Carleson then implies that $\|f_n\|_{L^2(d|\mu|)} \longrightarrow 0$ and thus $\|T_\mu f_n\|$ converges to zero, which shows that T_μ is compact. □

Remark 2.2.7. If the measure μ is positive, the conditions expressed in Proposition 2.2.5 and Theorem 2.2.6 are also necessary for respectively boundedness and compactness of the Toeplitz operator with symbol μ; see [23], where also the sufficient conditions are only stated for a positive measure.

The condition of being a vanishing Carleson measure is rather abstract at this point; the following lemma provides us with many concrete examples of them, and therefore, in stark contrast with the Hardy space case, with many compact Bergman-Toeplitz operators.

Lemma 2.2.8. *If μ is a finite complex measure compactly supported in $\mathbb{D}$, $|\mu|$ is a vanishing Carleson measure.*

Proof. The supports of μ and $|\mu|$ coincide, so $|\mu|$ is compactly supported as well. Consider a bounded sequence $\{f_n\}$ in $L^2_a(\mathbb{D})$, i.e. such that $\sup_n \|f_n\| \leq C$. Then, since pointwise evaluations are locally uniformly bounded in the Bergman space, $\{f_n\}$ is an equibounded family of holomorphic functions and by Montel's theorem there is a subsequence $\{f_{n_k}\}$ that converges uniformly on compacts to an holomorphic function f; in particular the convergence is uniform on a compact set K containing the support of $|\mu|$. We can estimate

$$\int_{\mathbb{D}} |f_{n_k} - f|^2\ d|\mu| \leq \|f_{n_k} - f\|^2_{\infty,K}|\mu|(\mathbb{D})$$

and the right hand side tends to zero as we let k to infinity. We have shown that the family $\{f_n\}$ is relatively compact in $L^2(d|\mu|)$ and thus that the inclusion

$$\imath : L^2_a(\mathbb{D}) \longrightarrow L^2(d|\mu|)$$

is compact. □

Remark 2.2.9. Compactness of the Toeplitz operator with a compactly supported measure as symbol can be proved directly without going through the machinery of Carleson measures; as in Lemma 2.2.8 the proof is an easy consequence of the local uniform boundedness of point evaluations in the Bergman space. For these measures, the unit disk setting is really unnecessary and we can do everything on a bounded domain Ω; the form in (2.7) (with integration over Ω) is bounded, again by local uniform boundedness of point evaluations.

We can get even more:

Lemma 2.2.10. *If μ is a measure compactly supported in Ω, the Toeplitz operator*

$$T_\mu : L^2_a(\Omega) \longrightarrow L^2_a(\Omega)$$

belongs to the trace class.

Proof. Let $\{e_n\}_{n\geq 1}$ be an orthonormal basis of $L^2_a(\Omega)$; from Chapter 1 we know that the reproducing kernel of $L^2_a(\Omega)$ has the expression:

$$K(z,w) = \sum_{n=1}^{+\infty} e_n(z)\overline{e_n(w)}.$$

By definition of the operator T_μ we have

$$\langle T_\mu e_n, e_n\rangle = \int_\Omega |e_n(z)|^2 \; d\mu(z).$$

We can then estimate:

$$\begin{aligned}\sum_{n=1}^{+\infty} |\langle T_\mu e_n, e_n\rangle| &\leq \int_\Omega \sum_{n=1}^{+\infty} |e_n(z)|^2 \; d|\mu|(z)\\ &= \int_\Omega K(z,z) \; d|\mu|(z)\\ &\leq \sup_{z\in F} |K(z,z)||\mu|(\Omega)\end{aligned}$$

where F is any compact set in Ω containing the support of μ, and the last quantity is thus finite. Therefore the series defining the trace is absolutely convergent, and the lemma is proved. $\square$

From the proof of the lemma we see that, when T_μ is in the trace class or is positive, the trace has the expression:

$$tr(T_\mu) = \int_\Omega K(z,z) \; d\mu(z).$$

To conclude this survey of results about smallness of Toeplitz operator with symbol measure, let us mention that Daniel Luecking in [16] gave sufficient conditions (which become also necessary in the case of a positive measure) on a finite complex measure on the unit disk for the corresponding Toeplitz operators to belong to a Schatten class; the conditions on the measure are of geometric Carleson-type, and the results are proved for a rather wide class of Hilbert spaces of analytic functions which include standard weighted Bergman spaces.

2.2.2 A Necessary Condition for Compactness

An easy consequence of the above sufficient conditions for compactness is the following observation: if φ is a continuous function on $\overline{\Omega}$ that vanishes on the boundary of Ω, the operator T_φ is compact on $L^2_a(\Omega)$. This is because we can approximate φ in the sup norm by compactly supported functions φ_n, and then

$$\|T_\varphi - T_{\varphi_n}\| \leq \|\varphi - \varphi_n\|_\infty \longrightarrow 0.$$

Since T_{φ_n} is compact and the compact operators form a closed ideal, T_φ is compact.

For symbols φ in $\mathcal{C}(\overline{\Omega})$ the condition is not exactly necessary for general domains; it turns out that the vanishing of φ is only required on a portion of the boundary, see [4] for the statement and proof. We can however prove that the condition is necessary for the unit disk; this would follow from the results of [4], but in the unit disk case a direct proof is available using a tool called the Berezin transform. Since the latter object has proven to be very useful in the study of Toeplitz operators and has connections to different areas of mathematics, we have chosen to present the proof.

Let k_z be the normalized reproducing kernel at z of $L^2_a(\mathbb{D})$,

$$k_z(w) = \frac{1-|z|^2}{(1-w\overline{z})^2}.$$

We will need the following:

Lemma 2.2.11. *k_z converges weakly to 0 in $L^2_a(\mathbb{D})$ when $|z| \longrightarrow 1$.*

Proof. By the reproducing property, for any f in $L^2_a(\mathbb{D})$,

$$\langle f, k_z\rangle = (1-|z|^2)f(z).$$

We then have that $\langle f, k_z\rangle \longrightarrow 0$ as $|z| \longrightarrow 1$ for bounded analytic functions, and since those are dense in $L^2_a(\mathbb{D})$ the lemma follows. $\square$

The Berezin symbol of a linear operator on $L^2_a(\mathbb{D})$ (bounded for simplicity, even though it is not necessary) is the function on the unit disk defined as

$$\widetilde{A}(z) = \langle Ak_z, k_z\rangle.$$

The Berezin transform of a bounded function φ is defined as the Berezin symbol of the associated Toeplitz operator T_φ, $\widetilde{\varphi} := \widetilde{T_\varphi}$. By definition of the Toeplitz operator we have the following explicit form for the Berezin transform:

$$\widetilde{\varphi}(z) = \int_{\mathbb{D}} \varphi(w)|k_z(w)|^2\, dA(w). \tag{2.10}$$

Recall that the function

$$\varphi_z(w) = \frac{w-z}{1-w\overline{z}}$$

is an idempotent automorphism of the disk for $z \in \mathbb{D}$ fixed, and note that $\varphi'_z(w) = k_z(w)$. By changing the variable w for $\varphi_z(w)$ in the integral (2.10), since the Jacobian of the transformation is $|\varphi'_z(w)|^2 = |k_z(w)|^2$, we obtain an alternative description of the Berezin transform:

$$\widetilde{\varphi}(z) = \int_{\mathbb{D}} \varphi(\varphi_z(w))\ dA(w).$$

Note that $\widetilde{\varphi}(z)$ is a continuous function of z in the disk; if φ is a continuous function on the closed disk, then the same is true for $\widetilde{\varphi}(z)$ and their values coincide on $\partial\mathbb{D}$. In fact, if $z_0 \in \partial\mathbb{D}$ and $z \longrightarrow z_0$, $\varphi_z(w) \longrightarrow z_0$ for any $w \in \mathbb{D}$. Then

$$\widetilde{\varphi}(z_0) = \lim_{z \to z_0} \int_{\mathbb{D}} \varphi(\varphi_z(w))\ dA(w) = \varphi(z_0),$$

where the second equality holds by dominated convergence. We are now ready to prove

Proposition 2.2.12. *If φ is continuous in the closed disk, then T_φ is compact if and only if φ vanishes on the boundary.*

Proof. The sufficiency was proved in an observation above. Assume now that T_φ is compact; by Lemma 2.2.11, $T_\varphi k_z$ converges to 0 in $L^2_a(\mathbb{D})$ as $|z|$ approaches 1. Then $\widetilde{\varphi}(z) = \langle T_\varphi k_z, k_z\rangle \longrightarrow 0$ as $|z| \longrightarrow 1$ and by the paragraph above this means that also $\varphi = 0$ on $\partial\mathbb{D}$. □

For general bounded symbols, the vanishing of φ in an almost-everywhere sense is far from being necessary: in [2], an example of a compact Toeplitz operator whose symbol does not have a limit on the boundary is given. What we proved in the above proposition is actually that the vanishing of the *Berezin transform of* φ is necessary for arbitrary bounded symbols (continuity was only needed to assert the equality of boundary values); Axler and Zheng in [5] proved that it is also sufficient.

2.2.3 Distributions as Symbols

The sesquilinear form through which we defined Toeplitz operator does not necessarily have the form (2.6), and could be associated to a more general analytic object. In the following we will need to use a compactly supported distribution as a symbol.

Compactly supported distributions on a domain Ω coincide with continuous linear functionals on $\mathcal{C}^\infty(\Omega)$, so the number

$$\mathfrak{t}_\Theta(f, g) := \Theta(f\overline{g}) \tag{2.11}$$

is well defined for f and g in $L^2_a(\Omega)$ and Θ a compactly supported distribution.

Since Θ as every compactly supported distribution has finite order, that is it can be extended to a continuous linear functional on $\mathcal{C}^k(\Omega)$ for a certain positive integer k, the right hand side of (2.11) is less than a constant times the product of the $\mathcal{C}^k$-norm of f and g in a compact set containing the support of Θ. But, by the Cauchy formula for derivatives of analytic functions, the $\mathcal{C}^k$-norm of a function in $L^2_a(\Omega)$ is controlled by its $L^2_a(\Omega)$-norm on compact sets just like the $\mathcal{C}^0$-norm. We then see that there exists a constant $C > 0$ such that

$$\mathfrak{t}_\Theta(f,g) \leq C\|f\|\|g\|$$

for all f and g in $L^2_a(\Omega)$, and so the sesquilinear form $\mathfrak{t}_\Theta$ analogously to the case of a measure gives rise to a bounded operator T_Θ on $L^2_a(\Omega)$ such that

$$\langle T_\Theta f, g\rangle = \mathfrak{t}_\Theta(f,g).$$

As in the case of a measure, it is possible to get a pointwise expression for $T_\Theta f$ using the reproducing kernel K_z for $L^2_a(\Omega)$ as follows:

$$T_\Theta f(z) = \langle T_\Theta f, K_z\rangle = \Theta(f\overline{K_z}). \tag{2.12}$$

Remark 2.2.13. Compactly supported distributions not only give rise to bounded Toeplitz operators, but indeed to compact ones: much in the same way as for a measure, consider a sequence $\{f_n\}$ weakly converging to 0 in L^2_a; by boundedness of pointwise evaluation functionals, it converges to 0 pointwise and uniformly on compacts. Now taking a compact K containing the support of Θ we evaluate

$$\begin{aligned}\|T_\Theta f_n\| = \sup_{\|g\|\leq 1} |\langle T_\Theta f_n, g\rangle| &= \sup_{\|g\|\leq 1} |\Theta(f_n\overline{g})| \\ &\leq \sup_{\|g\|\leq 1} C\|f_n\|_{\mathcal{C}^k(K)}\|g\|_{\mathcal{C}^k(K)}\end{aligned}$$

for some positive integer k. The $\mathcal{C}^k$-norm of g is controlled by its L^2_a-norm, so that for $\|g\| \leq 1$ it is controlled by a constant, and $\|f_n\|_{\mathcal{C}^k(K)}$ converges to 0 by uniform convergence of derivatives on compacts. This implies then that $T_\Theta f_n$ converges to 0 in norm.

We mention that it is not immediate to extend the definition of Toeplitz operator to distributions that are not compactly supported.

2.3 Fock Space

Much of what was said for Bergman spaces of a bounded domain can be repeated for the Fock space; in this short section we will try to highlight some differences and mention the techniques used to prove results equivalent to that of the previous section, without reporting the proofs. The interested reader may consult the recent text [24], where previous results are collected and systematized.

We will denote by ν the gaussian measure on the complex plane, i.e.

$$d\nu(z) = \omega(z)dV(z), \quad \text{where } \omega(z) = \frac{1}{\pi}e^{-|z|^2}.$$

Here dV denotes the unnormalized Lebesgue measure on the plane.

Recall that the Fock space $\mathcal{F}^2(\mathbb{C})$ is the closed subspace of $L^2(\mathbb{C}, d\nu)$ consisting of entire functions. Using the orthogonal projection $P : L^2(d\nu) \longrightarrow \mathcal{F}^2$ we can define the Toeplitz operator with symbol $\varphi \in L^\infty(\mathbb{C})$ in the usual way,

$$T_\varphi f = P(\varphi f).$$

Toeplitz operators acting on the Fock space are usually called Berezin–Toeplitz or Bargmann–Toeplitz operators. One difference we note with the Bergman space setting is that, since there are no bounded nonconstant entire functions by Liouville's theorem, in the Fock space there are no Toeplitz operators with analytic symbols that are not constant multiples of the identity, at least with this classical definition.

As in the other spaces, T_φ is bounded with $\|T_\varphi\| \leq \|\varphi\|_\infty$. As in the Bergman space case and in contrast to the Hardy space, this bound is far from optimal and we can considerably enlarge the class of symbols.

Pointwise T_φ is described by the formula

$$T_\varphi f(z) = \frac{1}{\pi} \int_{\mathbb{C}} K(z,w) f(w) \varphi(w) e^{-|w|^2} \, dV(w),$$

where $K(w,z) = \overline{K(z,w)} = K_z(w) = e^{\overline{z}w}$ is the reproducing kernel of $\mathcal{F}^2$, and we have

$$\langle T_\varphi f, g \rangle = \frac{1}{\pi} \int_{\mathbb{C}} f(w) \overline{g(w)} \varphi(w) e^{-|w|^2} \, dV(w).$$

As we did in the previous section, this suggests to define for a σ-finite complex Borel measure on $\mathbb{C}$ the following sesquilinear form:

$$\mathfrak{t}_\mu(f,g) = \frac{1}{\pi} \int_{\mathbb{C}} f(w) \overline{g(w)} e^{-|w|^2} \, d\mu(w), \tag{2.13}$$

for f and g analytic polynomials (recall those are dense in the Fock space). If this form satifies a bound $|\mathfrak{t}_\mu(f,g)| \leq C\|f\|\|g\|$, then by Riesz representation there exists a unique bounded linear operator T_μ such that $\langle T_\mu f, g \rangle = \mathfrak{t}_\mu(f,g)$ which coincides with the standard definition through the projection when $d\mu(z) = \varphi(z) \, dV(z)$ for some bounded φ.

A positive Borel measure μ on $\mathbb{C}$ is called a Fock–Carleson measure for $\mathcal{F}^2(\mathbb{C})$ if there exists a positive constant C such that

$$\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} d\mu(z) \leq \int_{\mathbb{C}} |f(z)|^2 d\nu(z)$$

for any f in $\mathcal{F}^2(\mathbb{C})$.

Analogously to the Bergman space case, using Cauchy–Schwarz inequality we can show that if $|\mu|$ is a Fock-Carleson measure the form $\mathfrak{t}_\mu$ is bounded. It is also true that if μ is positive, the boundedness of T_μ implies that μ is Fock-Carleson. The proof of this uses the Berezin transform, which in the Fock space setting has a particularly interesting form. We will briefly illustrate it.

The normalized reproducing kernel in the Fock space has the form

$$k_z(w) = K_z(w)/\sqrt{K(z,z)} = e^{\overline{z}w - \frac{1}{2}|z|^2}.$$

The Berezin symbol of an operator A on the Fock space is defined as in the Bergman space as:

$$\widetilde{A}(z) = \langle Ak_z, k_z \rangle.$$

The Berezin transform of a measure μ is the Berezin symbol of the associated Toeplitz operator (we will not mention the precise conditions under which this makes sense and we refer to [24]):

$$\begin{aligned} \widetilde{\mu}(z) = \widetilde{T_\mu}(z) &= \frac{1}{\pi} \int_{\mathbb{C}} |k_z(w)|^2 e^{-|w|^2} \, d\mu(w) \\ &= \frac{1}{\pi} \int_{\mathbb{C}} e^{-|z-w|^2} \, d\mu(w). \end{aligned}$$

In particular we see that when $d\mu(z) = \varphi(z) \, dV(z)$ a rescaling of the Berezin transform yields the heat transform, the operator that assigns to a function φ the value of the solution of the heat equation at a time instant with initial datum φ. Much of the interest in the Berezin transform on the Fock space is related to this fact.

A positive Borel measure μ is called a vanishing Fock–Carleson measure if

$$\lim_{n\to\infty} \int_{\mathbb{C}} |f_n(z)|^2 e^{-|z|^2} \, d\mu(z) = 0$$

for every sequence $\{f_n\} \subset \mathcal{F}^2$ converging weakly to 0 in $\mathcal{F}^2$. Repeating almost *verbatim* the proof of Theorem 2.2.7 we have that if $|\mu|$ is vanishing Fock–Carleson than the operator T_μ is compact. Again the converse holds for positive measures and it is proved through the Berezin transform. Schatten ideals membership conditions for $p \geq 1$ also follow very closely the ones for the Bergman space given first in [16].

Analogously to what was done for Bergman spaces, Toeplitz operators can also be defined for compactly supported distributions Θ through the sesquilinear form

$$\widetilde{\mathfrak{t}_\Theta}(f,g) = \Theta(\frac{1}{\pi} e^{-|\cdot|^2} f\overline{g}).$$

Note however that the weight inside the parenthesis is actually irrelevant, and at the cost of substituting Θ with $\frac{1}{\pi}e^{-|\cdot|^2}\Theta$ (which is still a distribution with the same support as that of Θ), we can (and will in the following chapter) use the sesquilinear form which has the same appearance as in the Bergman space:

$$\mathfrak{t}_\Theta(f,g) = \Theta(f\overline{g}). \tag{2.14}$$

This form satisfies the bound $|\mathfrak{t}_\Theta(f,g)| \leq C\|f\|\|g\|$; then there exists a bounded operator T_Θ such that $\langle T_\Theta f, g\rangle = \mathfrak{t}_\Theta(f,g)$, and in the same way as in Remark 2.2.13 this operator is compact.

Berezin–Toeplitz operators are more related to quantum mechanics and mathematical physics than Toeplitz operators on other spaces; there is a strong relation between them and certain Weyl pseudo-differential operators on $L^2(\mathbb{R}, dx)$ ([24]), and much effort has been dedicated to identifying a class of symbols adequate for a good symbol calculus.

The difficulties of such problems are well exemplified by the following fact, which is very relevant for the next chapter and shows that without further assumptions the quantization map $\varphi \longrightarrow T_\varphi$ is not one-to-one:

Remark 2.3.1. There exists a nonzero (unbounded) radial function such that the corresponding Toeplitz operator is the zero operator on $\mathcal{F}^2$. The example was first constructed in [13].

Chapter 3

Finite rank Toeplitz Operators

We have seen in the previous chapter that both on the Fock space and in Bergman spaces of various domains compact Toeplitz operators are in abundance, even for classical symbols in L^∞; we mentioned that there are also Schatten ideals membership conditions, which essentially involve the way in which the function converges to the boundary (for a bounded domain). In particular if the symbol is compactly supported the corresponding Toeplitz operator belongs to every Schatten class. At this point it is natural to ask whether a cutoff property, that is the deduction that if a Toeplitz operator is small enough then the symbol must be zero, exists on the finite rank level. Due to Remark 2.3.1, we will consider only compactly supported analytic objects as symbols.

Note that on the Hardy space of the unit disk the cutoff happens at the compactness level. However, if we deal with particular subspaces of the Hardy space, the situation may change drastically; we will start our investigation of finite rank Toeplitz operators with a short overview of this case, which has attracted much attention recently.

3.1 Truncated Toeplitz Operators

An *inner* function is a bounded holomorphic function on $\mathbb{D}$ such that $|u(z)| \leq 1$ on $\mathbb{D}$ and $|\widetilde{u}(e^{i\vartheta})| = 1$ almost everywhere in $\partial\mathbb{D}$, where $\widetilde{u}$ denotes the boundary value function of g as defined in Chapter 1. The classical Beurling theorem in $H^2(\mathbb{D})$ characterizes the unilateral shift nonzero closed invariant subspaces of H^2 as those of the form uH^2 for an inner function u. Then the subspaces invariant for the adjoint of the shift, the *backward shift*, are of the form

$$\mathcal{K}_u = H^2 \ominus uH^2.$$

The notation means that $\mathcal{K}_u$ is the orthogonal complement of uH^2. These are called model spaces, due to the fact that the compression of the shift operator to them serves as a model for a large class of contractions on Hilbert spaces. An introduction to model spaces is provided for instance by the survey [14].

There are several more concrete function theoretic descriptions of the spaces $\mathcal{K}_u$, but we will not mention them. Each $\mathcal{K}_u$ is a reproducing-kernel Hilbert space, with the reproducing kernel at the point $\lambda \in \mathbb{D}$ being

$$k_\lambda(z) = \frac{1 - \overline{u(\lambda)}u(z)}{1 - \overline{\lambda}z}.$$

It is necessary to say that each model space has an isometric conjugate-linear involution (a conjugation) $C : \mathcal{K}_u \longrightarrow \mathcal{K}_u$ defined on boundary functions as

$$Cf(\zeta) = \overline{f(\zeta)\zeta}u(\zeta).$$

The so-called conjugate reproducing kernel of $\mathcal{K}_u$ is defined as

$$\widetilde{k_\lambda}(z) = Ck_\lambda(z) = \frac{u(z) - u(\lambda)}{z - \lambda}.$$

If we let $P_u : L^2(\partial\mathbb{D}) \longrightarrow \mathcal{K}_u$ be the projection onto the model space, the *truncated Toeplitz operator* with symbol $\varphi \in L^2$ is defined as

$$A^u_\varphi(f) = P_u(\varphi f)$$

with natural domain

$$\mathcal{D}(A^u_\varphi) = \{f \in \mathcal{K}_u : P_u(\varphi f) \in \mathcal{K}_u\}.$$

Here we are looking at P_u as an integral operator from $L^1(\partial\mathbb{D})$ to the space of holomorphic functions on $\mathbb{D}$. The operator A^u_φ is thus densely defined, since $\mathcal{K}_u \cap H^\infty$ is included in $\mathcal{D}(A^u_\varphi)$ and is dense in $\mathcal{K}_u$ (the set of kernels $\{k_\lambda\}$ is included in $\mathcal{K}_u \cap H^\infty$ and is dense in $\mathcal{K}_u$ for λ varying in a uniqueness set for $\mathcal{K}_u$, for example a set having an accumulation point on the disk).

If φ is in L^∞ then A^u_φ is everywhere defined and bounded with $\|A^u_\varphi\| \leq \|\varphi\|_\infty$. Note that the aforementioned very important compressed shift is a special case of the above definition with $\varphi(z) = z$.

Note that for $\varphi \in uH^2$, φf is in uH^2 for every $f \in \mathcal{K}_u$ and then $P_u(\varphi f) = 0$, so A^u_φ is the zero operator. The same happens if $\overline{\varphi}$ is in uH^2, since $\langle \varphi f, h\rangle = \langle f, \overline{\varphi}h\rangle = 0$ for every f and h in $\mathcal{K}_u$ which means that φf is L^2-orthogonal to $\mathcal{K}_u$. Uniqueness of the symbol for truncated Toeplitz operators is therefore out of question; Sarason in [22] proved that the two above cases are the only way in which uniqueness can fail, namely he showed that A^u_φ is the zero operator if and only if $\varphi = \psi + \overline{\chi}$ for some ψ and χ in uH^2.

Since truncated Toeplitz operators seem to be very different from their full Hardy space counterpart, we can expect that there may be nonzero compact Toeplitz operators; in fact, there are even nonzero *finite rank* truncated Toeplitz operators operators. Consider the rank one operator $Ck_\lambda \otimes k_\lambda$ (where $f \otimes g(h) = \langle h, g\rangle f$); its action on $f \in \mathcal{K}_u$ is

$$[Ck_\lambda \otimes k_\lambda](f)(w) = f(\lambda)\frac{u(w) - u(\lambda)}{w - \lambda}.$$

Let us now check how the truncated Toeplitz operator with symbol $\varphi(z) = \frac{u(z)}{z-\lambda}$ acts on f:

$$\begin{aligned}
A^u_{\frac{u}{z-\lambda}} f(w) = P_u\left(\frac{u}{z-\lambda}f\right)(w) &= \left\langle \frac{u}{z-\lambda}f, k_w \right\rangle \\
&= f(\lambda)\left\langle \frac{u-u(\lambda)}{z-\lambda}, k_w \right\rangle \\
&+ \left\langle \frac{u}{z-\lambda}(f-f(\lambda)), k_w \right\rangle \\
&+ f(\lambda)u(\lambda)\left\langle \frac{1}{z-\lambda}, k_w \right\rangle \\
&= f(\lambda)\frac{u(w)-u(\lambda)}{w-\lambda},
\end{aligned}$$

since the term in the third line of the equation vanishes because $u\frac{f-f(\lambda)}{z-\lambda} \in uH^2$ and the term in the fourth line vanishes because $\frac{1}{z-\lambda}$ has only negative Fourier coefficients on the boundary.

Similarly for any $\lambda \in \mathbb{D}$ we have that

$$k_\lambda \otimes Ck_\lambda = A^u_{\frac{\overline{u}}{\overline{z}-\overline{\lambda}}}.$$

We have thus found nonzero rank one truncated Toeplitz operators, a stark contrast to the Hardy space case. In model spaces it is possible to define reproducing kernels at points in a subset of $\partial\mathbb{D}$ that satisfies a certain property that we will leave unspecified; tensoring those boundary kernels also gives rank one operators. Donald Sarason in [22] proved that if a truncated Toeplitz operator has rank one then it is a scalar multiple of one of the operators described above.

Now consider the map: $\mathbf{\Phi} : \mathbb{D} \longrightarrow \mathcal{B}(\mathcal{K}_u)$ such that $\mathbf{\Phi}(\lambda) = Ck_\lambda \otimes k_\lambda$; if we denote its n-th derivative with respect to λ by $D^n[Ck_\lambda \otimes k_\lambda]$, by Leibniz formula we get an operator of rank n; this operator can also be defined at those special points on the boundary mentioned above. Similarly the operator $\overline{D}^n[k_\lambda \otimes Ck_\lambda]$ can be defined as the derivative with respect to $\overline{\lambda}$ of $k_\lambda \otimes Ck_\lambda$ with the same consideration for the boundary points. As in the rank one case, it turns out that

$$\begin{aligned}
D^n[Ck_\lambda \otimes k_\lambda] &= A^u_\varphi, \\
\overline{D}^n[k_\lambda \otimes Ck_\lambda] &= A^u_{\overline{\varphi}}
\end{aligned}$$

where $\varphi(z) = \frac{n!\cdot u(z)}{(z-\lambda)^{n+1}}$ and equality also holds in the admissible boundary points in the sense of boundary values.

Sarason asked whether every finite rank truncated Toeplitz operator was a finite linear combination of the operators above; recently Bessonov in [6] answered this question in the affirmative. In conclusion, we see that on model spaces no cutoff points exists and the small Toeplitz operators are essentially related to the reproducing kernels.

3.2 The finite rank theorem

If we want to prove a cutoff theorem on Bergman and Fock spaces that holds for objects as general as compactly supported distributions, the statement cannot be that the symbol is zero if the operator has finite rank. Indeed, consider a measure of the form $\mu = \sum_{j=1}^{N} c_j \delta_{z_j}$, where $z_j \in \Omega$ for the Bergman space of a bounded domain or $z_j \in \mathbb{C}$ for the Fock space; then by definition $T_\mu f(z) = \sum_{j=1}^{N} c_j f(z_j) K(z, z_j)$, where $K(z, w)$ is the reproducing kernel at w of the appropriate space under consideration. This means that the range of T_μ is contained in the linear span of $\{K(z, z_j)\}_{j=1}^{N}$, i.e. it is finite dimensional. If we consider a distribution of the form

$$\Theta = \sum_{j=1}^{N} L_j \delta_{z_j}, \tag{3.1}$$

where the L_j are differential operators, then the range will be contained in the span of reproducing kernels and their derivatives (up to the order of L_j) at the points z_j.

Our task will be to prove that if the Toeplitz operator T_Θ has finite rank for a compactly supported distribution Θ, then Θ must be of the form (3.1). In case Θ coincides with a compactly supported function, this implies that it must be zero. Therefore for symbols-function the cutoff really happens at the finite rank level.

3.2.1 Problem setting and previous results

After having described conditions for Schatten class membership, naturally Luecking in [16] deals with the finite rank hypothesis for symbol-measures. The proof proposed there was later discovered to be seriously flawed, and the finite rank hypothesis remained unsettled for more than twenty years.

Finally in 2008 in the paper [17] Luecking himself produced a correct proof of the hypothesis. Luecking's elegant proof is very algebraic, and he actually proves a theorem about operators from the space of analytic polynomials to the space of linear functionals on conjugate analytic polynomials. To be precise, he proves the following:

Theorem (D. Luecking, [17]). *Let μ be a measure compactly supported in $\mathbb{C}$, and T_μ the operator defined by $T_\mu f(g) = \int f\overline{g}\, d\mu$ for f and g analytic polynomials. Then T_μ has finite rank if and only if μ has finite support.*

The proof ultimately involves using the Stone–Weierstrass theorem on the space of symmetric continuous function, and this is the reason why the symbol of the operator is required to be a measure and to have compact support.

In the years following the publication of Luecking's paper several generalizations of the theorem appeared, the most notable ones being to severable complex variables and to a symbol that is a compactly supported distribution. A direct extension of Luecking's argument in order to cover compactly supported distributions does not seem to work because of the aforementioned reliance on some form of the Stone–Weierstrass theorem, and the existing proofs of the finite rank theorem for distributions use induction on the (finite) order of the distribution, with the base case being Luecking's theorem, solving iteratively the $\overline{\partial}$-equation to produce a less singular distribution at every step.

The survey [21] contains the details, as well as references to other papers that proved it and a discussion of the multidimensional case.

After reading the paper [7] we became aware of another approach to the proof of the finite rank theorem, which essentially uses the Fourier transform. The idea is quite far from the algebraic methods of [17] and actually originated in the failed proof of [16]; the authors of [7] use it to prove that if a Toeplitz operator on the Fock space having as symbol a function with certain growth restrictions at infinity has finite rank, then the symbol must be zero.

We found out that this approach with some modifications worked in the Bergman space setting as well; since the method uses the decay at infinity of the Fourier transform of an integrable function (which in [7] is the symbol times the weight), there was some skepticism that it could be extended to symbols that are measures or distributions. Surprisingly, with some effort this turned out to be possible in the case of compactly supported distribution. The advantage of our approach is that it furnishes a unified treatment of the Fock and Bergman space case, and does not require to distinguish between symbol functions or more general objects. In fact, the theorem being proved directly for compactly supported distributions without relying on induction, it gives an alternative proof (and a generalization) of the original theorem of Luecking.

In order to streamline the presentation of the proof and treat together the Bergman and Fock spaces, we will modify some notation. We will let Ω stand either for an arbitrary bounded domain in $\mathbb{C}$ or for the whole of $\mathbb{C}$, and we will denote by $L^2_a(\Omega)$ respectively the Bergman space or the Fock space. We let Θ be a distribution with compact support in Ω; the sesquilinear form defining the Toeplitz operator has the form

$$\mathfrak{t}_\Theta(f,g) = \Theta(f\overline{g})$$

whatever Ω is. The Toeplitz operator with symbol Θ is then defined as usual by $\langle T_\Theta f, g\rangle = \mathfrak{t}_\Theta(f,g)$. We are now ready to state:

Theorem 3.2.1. *The operator $T_\Theta : L^2_a(\Omega) \longrightarrow L^2_a(\Omega)$ has finite rank, say equal to N, if and only the support of Θ consists of N points.*

3.2.2 Proof of the theorem

The easy implication, that if Θ has finite support the operator has finite rank, was proved at the beginning of the chapter. In order to prove the converse we will start by setting some lemmas. First notice that having rank equal to N for T_Θ means that there exist orthogonal sets $\{f_j\}_{j=1}^N$ and $\{g_j\}_{j=1}^N$ such that $T_\Theta f = \sum_{j=1}^N \langle f, f_j\rangle g_j$, where the scalar product is in $L^2_a(\Omega)$; in terms of the sesquilinear form:

$$\mathfrak{t}_\Theta(f,g) = \sum_{j=1}^N \langle f, f_j\rangle \langle g_j, g\rangle. \tag{3.2}$$

We now show how the problem can be reduced to a Bergman space in which analytic polynomials are dense, and actually to the Bergman space of a disk. We will momentarily denote the Toeplitz operator with symbol Θ acting on $L^2_a(\Omega)$ by T^{Ω}_{Θ}.

Lemma 3.2.2. *If the operator T^{Ω}_{Θ} has rank less than or equal to N, then the infinite matrix A_Θ with entry $\mathfrak{t}_\Theta(z^n, z^m)$ at the index (n, m) (n and m range from 0 to infinity) has rank less than or equal to N.*

Proof. It follows from the assumption that there exists a linear relation

$$\sum_{n=0}^{N} c_n T^{\Omega}_{\Theta} z^n = 0$$

for some c_n which are not all zero. Taking the scalar product of this relation with z^m we obtain a linear combination with nontrivial coefficients of the first $N+1$ rows of the matrix A_Θ, which means that the rank of this matrix cannot exceed N. □

Note now that the number $\mathfrak{t}_\Theta(z^n, z^m) = \Theta(z^n \overline{z^m})$ (analytic polynomials are contained in every space we are considering), and consequently the matrix A_Θ, does not depend on the domain Ω as long as the latter contains the support of Θ. We take a disk D that contains such support, and we think of Θ as acting on $\mathcal{C}^\infty(D)$. We remark that Θ is still the same object: a distribution compactly supported in a set $U \subset \mathbb{C}$, which is a continuous linear functional on $\mathcal{C}^\infty(U)$, is identified with a continuous linear functional on $\mathcal{C}^\infty(\mathbb{C})$ which has support contained in U. See Hörmander's book [15], Theorem 2.3.1, for a proof of this seemingly obvious fact.

Lemma 3.2.3. *The rank of the matrix A_Θ is equal to the rank of T^D_Θ.*

Proof. We have just shown that the rank of the matrix cannot exceed the rank of T^D_Θ. For the reverse inequality, let the rank of A_Θ be M and suppose that the rank of T^D_Θ is greater than M. Then there exist $M+1$ functions $u_1, ..., u_{M+1}$ in $L^2_a(D)$ such that the set $\{T^D_\Theta u_j\}_{j=1}^{M+1}$ is linearly independent. The matrix with entries $\{\langle T^D_\Theta u_j, T^D_\Theta u_k\rangle\}$ for $j, k = 1, ..., M+1$ is thus non-singular. Take analytic polynomials p_j that approximate u_j and analytic polynomals q_k that approximate $T^D_\Theta u_k$ in the $L^2_a(D)$ norm (this is possible because of the density of polynomials in $L^2_a(D)$); the matrix with elements $\{\langle T^D_\Theta p_j, q_k\rangle\}$ for $j, k = 1, ..., M+1$ is a submatrix of a matrix obtained from A_Θ by linear manipulation of rows and columns, thus its determinant must be zero since rank $A_\Theta \leq M$. But it should approximate a matrix with nonzero determinant, which gives the desired contradiction. □

Combining together the two lemmas, we obtain the desired reduction: if T^{Ω}_{Θ}, for Ω a bounded domain or $\mathbb{C}$, has finite rank, then the matrix A_Θ has finite rank which implies that the operator T^D_Θ has finite rank, say N. If we prove Theorem 3.2.1 for T^D_Θ, then the form of Θ we will obtain implies that also the rank of T^{Ω}_{Θ} is N. Incidentally, this proves that Lemma 3.2.3 holds for the Bergman space of a domain in which analytic polynomials are not dense; the rank of the operator is solely determined by its rank on analytic polynomials, even if those are not dense. From now on we will suppress the superscript D and we will think of the operator T_Θ as acting on the Bergman space of a disk that contains the support of Θ.

We will now prove the rank zero case, that is the uniqueness of the symbol. This follows from the general case below, but since it serves as intuition for the subsequent steps we thought it would be better to isolate it.

Lemma 3.2.4. *If T_Θ is the zero operator, then Θ is identically zero.*

Proof. Note that the function $e_z(w) = e^{i\bar{z}w}$ is in $L^2_a(D)$ for every $z \in \mathbb{C}$. Since $T_\Theta = 0$, the expression $\mathfrak{t}_\Theta(e_{-z}, e_z)$ is equal to 0 for every $z \in \mathbb{C}$. Think of z as $z = x + iy$, and Θ acting on $w = u + iv$. Then

$$\begin{aligned} 0 = \mathfrak{t}_\Theta(e_{-z}, e_z) &= \Theta(e^{-i\bar{z}w - iz\overline{w}}) \\ &= \Theta(e^{-2i \operatorname{Re} z\overline{w}}) = \Theta(e^{-2i\mathbf{x}\cdot\mathbf{u}}) \\ &= \mathcal{F}(\Theta)(2\mathbf{x}), \end{aligned}$$

where $\mathbf{x} = (x, y) \in \mathbb{R}^2$, $\mathbf{u} = (u, v) \in \mathbb{R}^2$, the scalar product in $\mathbb{R}^2$ is indicated with a dot and $\mathcal{F}$ is the Fourier transform in $\mathbb{R}^2$. Note that the Fourier transform is well defined since Θ is compactly supported, and it being zero for every $\mathbf{x}$ implies that $\Theta = 0$. $\square$

We are going to build up on the idea of using the Fourier transform. Let T_Θ have rank N, so that $\mathfrak{t}_\Theta$ is in the form (3.2). We have, in the same notation of the above lemma and for every f and g in $L^2_a(D)$,

$$\mathfrak{t}_\Theta(e_{-z}f, e_z g) = \sum_{j=1}^{N} \langle e^{-i\bar{z}(\cdot)}f, f_j\rangle\langle g_j, e^{i\bar{z}(\cdot)}f\rangle = \sum_{j=1}^{N} \overline{\varphi_j(z)}\psi_j(z), \tag{3.3}$$

where both φ_j and ψ_j are entire functions of z. In the same way as in the previous lemma we have that the first term in the above equalities is equal to $\mathcal{F}(\Theta f\overline{g})(2\mathbf{x})$; note that the support of the distribution $\Theta f \overline{g}$ is contained in that of Θ. By the Paley–Wiener–Schwartz theorem, the Fourier transform of a compactly supported distribution in $\mathbb{C} \cong \mathbb{R}^2$ can be extended to an entire function on $\mathbb{C}^2$ which grows polinomially on $\mathbb{C}$.

Denoting by ∂ the derivative with respect to z and by $\overline{\partial}$ the derivative with respect to $\overline{z}$, we have

$$\partial^\alpha \overline{\partial^\beta} \mathfrak{t}_\Theta(e_{-z}f, e_z g) = (-i)^{\alpha+\beta} \mathcal{F}(\Theta f \overline{g} w^\beta \overline{w}^\alpha)(2\mathbf{x})$$

for all positive integers α and β. The distribution being Fourier transformed is still compactly supported, which implies that also the mixed derivatives of $\mathfrak{t}_\Theta(e_{-z}f, e_z g)$ grow polinomially on $\mathbb{C}$.

By equation (3.2) then

$$\partial^\alpha \overline{\partial^\beta} \sum_{j=1}^{N} \overline{\varphi_j(z)}\psi_j(z) = \sum_{j=1}^{N} \overline{\varphi_j^\beta(z)}\psi_j^\alpha(z)$$

also grows polynomially and if we let α and β vary between 0 and $N - 1$ we have that the right hand side is the product of the conjugate wronskian matrix of the φ_j with the transpose wronskian matrix of the ψ_j, and this product matrix has entries that grow at most polynomially. But then the modulus of the determinant (determinant denoted by W) grows at most polynomially, and it follows that

$$|W(\varphi_1, ..., \varphi_N)W(\psi_1, ..., \psi_N)|(z) \leq r(z)$$

where r is a polynomial whose degree is bounded by a fixed multiple of the order of the distribution Θ.

The function inside the modulus is entire and bounded by a polynomial, so it has to be a polynomial. Since the two Wronskians are entire functions of exponential type and their product is a polynomial, we can then write:

$$W(\varphi_1, ..., \varphi_N) = e^{\alpha z}p(z) \quad , \quad W(\psi_1, ..., \psi_N) = e^{-\alpha z}q(z) \tag{3.4}$$

where the degree of p and q is bounded by a multiple of the order of Θ. Since by the definition of φ_j and ψ_j the Wronskians $W(\varphi_1, ..., \varphi_N)$ and $W(\psi_1, ..., \psi_N)$ depend respectively only on f and only on g, it follows that α is independent of both f and g.

Consider now the vector subspace V of $\mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^{2N})$ generated by

$$\mathbf{\Phi}_{f,g}(t) = (\langle e^{-it(\cdot)}f, f_1\rangle, ..., \langle e^{-it(\cdot)}f, f_N\rangle, \langle e^{-it(\cdot)}f, g_1\rangle, ..., \langle e^{-it(\cdot)}f, g_N\rangle)^T$$

as f and g vary in $L^2_a(D)$. Note that the first N components are just the φ_j's restricted to the real line, while for j between $N+1$ and $2N$ the j-th component is equal to $\overline{\psi_j(-\cdot)}$ restricted to the real line.

Since differentiating $\mathbf{\Phi}_{f,g}$ with respect to t produces a multiple of the integration variable in the first component of each inner product, the space we constructed is differentiation invariant.

It follows by the special form of the Wronskian of the components of $\mathbf{\Phi}_{f,g}$ that the closure of V in the natural topology of $\mathcal{C}^\infty$ is not the whole of $\mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^{2N})$, and actually has infinite codimension in it. To see this, consider the C^∞ vector

$$\mathbf{e}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = (e^{\lambda_1 t}, ..., e^{\lambda_N t}, e^{\mu_1 t}, ..., e^{\mu_N t})$$

for λ_j's and μ_j's complex numbers, $\lambda_j \neq \lambda_i$ and $\mu_j \neq \mu_i$ for $i \neq j$. The Wronskian of the first N components is equal to $\mathcal{V}(\lambda_1, ..., \lambda_N)e^{(\sum_{k=1}^N \lambda_k)t}$ and the Wronskian of the second N components is equal to $\mathcal{V}(\mu_1, ..., \mu_N)e^{(\sum_{k=1}^N \mu_k)t}$, where $\mathcal{V}$ denotes the Van der Monde determinant. Since by equation (3.4) the Wronskian of both the first N components and the second N components of an element of V is an exponential of fixed order multiplied by a polynomial of bounded degree, if either $\sum_{k=1}^N \lambda_k$ or $\sum_{k=1}^N \mu_k$ grows enough the vectors $\mathbf{e}_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ cannot be approximated by a sequence in V.

Since the space of distributions with compact support is the dual of $\mathcal{C}^\infty$ with the natural topology, the above paragraph implies that there exists a distribution $\mathbf{u} = (u_1, ..., u_{2N})$, compactly supported in $\mathbb{R}$, such that

$$\mathbf{u}(\mathbf{\Phi}_{f,g}) = 0 \quad \forall f, g \in L^2_a(D).$$

By definition $\mathbf{u}(\mathbf{\Phi}_{f,g}) = \sum_{j=1}^{N} u_j(\overline{\varphi_j}) + \sum_{j=1}^{N} u_{N+j}(\overline{\psi_j(-\cdot)})$ (thinking of φ_j and ψ_j as restricted to the real line), where the terms in the sum are the usual action of a distribution on a scalar function. From now on, we will take $f = g = 1$.

Let us examine the term $u_j(\overline{\varphi_j})$; since u_j being compactly supported has finite order, by the structure theorem for distributions it can be written as a *finite* sum $u_j = \sum_k \left(\frac{d}{dt}\right)^k u_{jk}$, where the u_{jk}'s are continuous and bounded functions with compact support on $\mathbb{R}$ and the derivative is intended in the distributional sense. Then we have

$$u_j(\overline{\phi_j}) = \sum_k (-1)^k \int_{\mathbb{R}} \left(\frac{d}{dt}\right)^k \overline{\varphi_j}(t) u_{jk}(t)\ dt = \sum_k (-1)^k \int_{\mathbb{R}} \left(\frac{d}{dt}\right)^k \left(\int_D e^{-itw} \overline{f_j}(w)\ dA(w)\right) u_{jk}(t)\ dt$$

$$= \int_D \left(\sum_k \int e^{-itw} (iw)^k u_{jk}(t) dt\right) \overline{f_j}(w) dA(w) = \int_D \widehat{u_j}(w) \overline{f_j}(w) dA(w),$$

and the last term is the $L^2_a(D)$ scalar product between the Fourier transform of u_j extended to the complex domain (which is an entire function, and thus belongs to $L^2_a(D)$) and f_j. Calculations for the terms $u_{N+j}(\overline{\psi_j(-\cdot)})$ are exactly the same. The condition $\mathbf{u}(\mathbf{\Phi}_{1,1}) = 0$ then translates to

$$\int_D \widehat{\mathbf{u}} \cdot (\overline{\mathbf{f}}, \overline{\mathbf{g}})^T = 0$$

where $\widehat{\mathbf{u}} = (\widehat{u_1}, ..., \widehat{u_{2N}})$, $\mathbf{f} = (f_1, ..., f_N)$, $\mathbf{g} = (g_1, ..., g_N)$ and the dot denotes scalar product in $\mathbb{C}^{2N}$.

Since V is differentiation invariant, we also have for any polynomial P that $\mathbf{u}(P\left(\frac{d}{dt}\right)\mathbf{\Phi}_{1,1}) = 0$; by repeating the calculation above this translates to $\int_D P(w)\widehat{\mathbf{u}}(w) \cdot (\overline{\mathbf{f}}, \overline{\mathbf{g}})^T(w) = 0$. Select now distributions $\mathbf{u}_1, ..., \mathbf{u}_{2N}$ such that

$$\int_D P_i \widehat{\mathbf{u}}_i \cdot (\overline{\mathbf{f}}, \overline{\mathbf{g}})^T = 0 \tag{3.5}$$

for every polynomial P_i, $i = 1, ..., 2N$; this is possible since the closure of V has infinite codimension (in particular it is greater than $2N$ whatever N is) in $\mathcal{C}^\infty$, and the distributions can be taken in such a way that $\widehat{\mathbf{u}_1}, ..., \widehat{\mathbf{u}_{2N}}$ are linearly independent. By density of polynomials in $L^2_a(D)$, we obtain relation (3.5) for P_i being any function in $L^2_a(D)$. Summing on i we have:

$$\int_D \left(\sum_{i=1}^{2N} P_i \widehat{\mathbf{u}}_i\right) \cdot (\overline{\mathbf{f}}, \overline{\mathbf{g}})^T = 0.$$

Denote by $\widehat{\mathbf{U}}$ the $2N \times 2N$ matrix having as columns the functions $\widehat{\mathbf{u}_1}, ..., \widehat{\mathbf{u}_{2N}}$, and by $\mathbf{P}$ the vector whose components are the P_i's. We are then left to check which vectors $\mathbf{H}$ can be written in the form $\left(\sum_{i=1}^{2N} P_i \widehat{\mathbf{u}}_i\right)$,that is we want to know for which vectors $\mathbf{H}$ whose component are entire functions the matrix equation

$$\left[\widehat{\mathbf{U}}\right] \mathbf{P} = \mathbf{H}$$

can be solved for $\mathbf{P}$ whose components are in $L^2_a(D)$. Consider vectors of the form $\mathbf{H} = (0, ..., H_j, ..., 0)^T$, where H_j has the same zeros of det $\left[\widehat{\mathbf{U}}\right]$ in $\overline{D}$ (the determinant is not identically zero, since the columns of the matrix are linearly independent), and note that the latter are in a finite number in $\overline{D}$ since $\det\left[\widehat{\mathbf{U}}\right]$ is an entire function; call the set of these zeros $\mathcal{Z}$. Then $\mathbf{P} = \left[\widehat{\mathbf{U}}\right]^{-1}\mathbf{H}$ has components in $L^2_a(D)$ and the equation can be solved. We thus discovered that

$$\int_D (0, ..., H_j, ..., 0) \cdot (\overline{\mathbf{f}}, \overline{\mathbf{g}})^T = 0$$

for every entire function H_j with zeros in $\mathcal{Z}$. By letting j run from 1 to $2N$, this means that all the f_j's and the g_j's are $L^2_a(D)$-orthogonal to every such entire function. We do not need to worry about the possible zeroes on ∂D; this is because $(z-\lambda)L^2_a(D)$ is dense in $L^2_a(D)$ for λ on the boundary on D, see for instance [3]. This means that being orthogonal to every entire function with zeros in $\mathcal{Z}$ is equivalent to being orthogonal to every entire function with zeros in $\mathcal{Z} \cap D$. By density of polynomials in $L^2_a(D)$, this is equivalent to being orthogonal to every function in $L^2_a(D)$ with zeros in $\mathcal{Z} \cap D$. But functions orthogonal to those are finite linear combinations of reproducing kernels and derivatives of reproducing kernels (derivatives arise for zeros of multiplicity greater than one) at points $z_k \in \mathcal{Z} \cap D$.

But then, substituting in (3.2) the special form we found, we get for the Fourier transform of Θ:

$$\begin{aligned}\mathcal{F}(\Theta)(2\mathbf{x}) = \mathfrak{t}_\Theta(e_{-z}, e_z) &= \sum_{j=1}^N \left\langle e_{-z}, \sum_k P_k(\partial)K_{z_k}\right\rangle \left\langle \sum_l Q_l(\partial)K_{z_l}, e_z\right\rangle \\ &= \sum_{j=1}^N \left(\sum_k P_k(-i\overline{z})e^{-i\overline{z}z_k}\right)\left(\sum_l Q_l(-iz)e^{-iz\overline{z_l}}\right),\end{aligned}$$

where P_k and Q_l are polynomial expressions (the inner sums depend on j, but we do not want to make the notation heavier). But now remember that the Fourier transform of Θ should grow at most polinomially in $\boldsymbol{x}$ (or z, which is the same); this implies that the mixed terms in the sum that contain $e^{-i(\overline{z}z_k - z\overline{z_l})}$ for $k \neq l$ must be zero, because $\overline{z}z_k - z\overline{z_l}$ can be made to have a nonzero and positive imaginary part if $k \neq l$ and then $e^{-i(\overline{z}z_k - z\overline{z_l})}$ will grow more than polinomially. So each inner sum contains only one term, and we can write

$$\begin{aligned}\mathcal{F}(\Theta)(2\mathbf{x}) &= \sum_{j=1}^N P_j(-i\overline{z})e^{-i\overline{z}z_j}Q_j(-iz)e^{-iz\overline{z_j}} \\ &= \sum_{j=1}^N P_j(-i\overline{z})Q_j(-iz)e^{-2i\boldsymbol{x}\cdot\boldsymbol{z}_j}.\end{aligned}$$

But the latter expression is the Fourier transform of a linear combination of delta distributions and derivatives of delta distributions at the points z_j, and by uniqueness of the Fourier transform we then get the desired result, namely

$$\Theta = \sum_{j=1}^N L_j \delta_{z_j}$$

for L_j being differential operators.

Chapter 4

Quantum Mechanical Applications

Luecking in [17] highlights a consequence of the finite rank theorem, concerning finite codimensional subspaces in Bergman spaces. However, what captured our interest was a less direct connection with mathematical physics and spectral theory, identified by G. Rozenblum and others for instance in [18],[20]. We will now attempt to briefly illustrate this connection.

Consider a charged, spinless particle moving in a plane under the action of a constant magnetic field perpendicular to the plane. Let us say that the particle is confined to the xy plane and the magnetic field has the form $\mathbf{B} = (0, 0, b)^T$. The Hamiltonian of this system has the form

$$H_0 = (-i\nabla - \mathbf{A})^2,$$

where $\mathbf{A}(x, y) = (A_1(x, y), A_2(x, y))$ is a magnetic vector potential for the field, that is we have $\partial_x A_2 - \partial_y A_1 = b$. The equations define $\mathbf{A}$ up to a gauge transform; we choose the gauge in which $\mathbf{A}(x, y) = (-\frac{b}{2}y, \frac{b}{2}x)$. With this choice,

$$H_0 = \left(-i\frac{\partial}{\partial x} - \frac{b}{2}y\right)^2 + \left(-i\frac{\partial}{\partial y} + \frac{b}{2}x\right)^2.$$

This operator is defined on $\mathcal{C}_c^\infty(\mathbb{R}^2)$ and essentially self-adjoint on $L^2(\mathbb{R}^2)$, which means that its closure in $L^2(\mathbb{R}^2)$ (which will still be denoted by H_0) is self-adjoint.

The spectrum of H_0 can be determined explicitly and has been known for several decades; one way to find it is by means of the so called creation and annihilation operators, defined respectively as follows:

$$\mathbf{a} = -2i\overline{\partial} + \frac{b}{2}y - i\frac{b}{2}x \quad , \quad \mathbf{a}^* = -2i\partial + \frac{b}{2}y + i\frac{b}{2}x. \tag{4.1}$$

As usual, $\partial = \frac{1}{2}(\partial_x - i\partial_y)$ and $\overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$. In the same way as for H_0, the operators $\mathbf{a}$ and $\mathbf{a}^*$ are defined on $\mathcal{C}_c^\infty(\mathbb{R}^2)$ and then closed in $L^2(\mathbb{R}^2)$. The notation is consistent since $\mathbf{a}^*$ is the adjoint of $\mathbf{a}$. Note that these operators can also be expressed in terms of the function $\Psi(z) = \frac{b}{4}|z|^2$, which is the scalar potential of the magnetic field, as follows:

$$\mathbf{a} = -2ie^{-\Psi}\overline{\partial}e^{\Psi}, \quad \mathbf{a}^* = -2ie^{\Psi}\partial e^{-\Psi}. \tag{4.2}$$

A calculation shows that they satisfy the commutation relation:

$$[\mathbf{a}, \mathbf{a}^*] = 2b\mathbf{I}, \tag{4.3}$$

where $\mathbf{I}$ is the identity operator in $L^2(\mathbb{R}^2)$. We also have $H_0 = \mathbf{a}^*\mathbf{a} + b\mathbf{I}$, which implies that the spectrum of H_0 lies in $[b, +\infty)$. Actually b is an eigenvalue of H_0, with eigenspace $\ker \mathbf{a}^*\mathbf{a} = \ker \mathbf{a}$. The equation $\mathbf{a}u = 0$ for $u \in L^2$ is equivalent to $\overline{\partial}(e^{\Psi}u) = 0$, which means that $f = e^{\Psi}u$ is an entire function such that $e^{-\Psi}f$ is in L^2. Recalling the explicit form of Ψ, f belongs to the space

$$\mathcal{F}_b = \left\{ f \text{ entire} : \int_{\mathbb{C}} |f(z)|^2 e^{-\frac{b}{2}|z|^2} dA(z) < \infty \right\}.$$

This space of holomorphic function is precisely the Fock space for $b = 2$. The eigenspace of H_0 corresponding to the first eigenvalue $\Lambda_0 = b$ is thus the infinite dimensional space $\mathcal{L}_0 = e^{-\Psi}\mathcal{F}_b$.

The relation (4.3) then shows that $\mathcal{L}_0$ is the eigenspace of the operator $\mathbf{a}\mathbf{a}^*$ with respect to its first eigenvalue $2b$ (note that $\mathbf{a}\mathbf{a}^* \geq 2b\mathbf{I}$ in the sense of positive operators). Since the spectra without the zero of $\mathbf{a}\mathbf{a}^*$ and $\mathbf{a}^*\mathbf{a}$ coincide, $2b$ is also an eigenvalue of $\mathbf{a}^*\mathbf{a}$ with relative eigenspace $\mathcal{L}_1 = \mathbf{a}^*\mathcal{L}_0$. The spectrum of $\mathbf{a}^*\mathbf{a}$ between 0 and $2b$ is empty, because otherwise also the spectrum of $\mathbf{a}\mathbf{a}^*$ to the left of $2b$ would be non-empty. $\mathcal{L}_1$ is then the eigenspace of H_0 relative to the second eigenvalue $\Lambda_1 = 3b$. Continuing in this way, we obtain that the spectrum of H_0 consists only of isolated eigenvalues

$$\Lambda_q = b(2q - 1) \quad q = 1, 2, 3, ... \tag{4.4}$$

each with associated eigenspace of infinite dimension

$$\mathcal{L}_q = (\mathbf{a}^*)^q \, \mathcal{L}_0. \tag{4.5}$$

In the literature the Λ_q's are called *Landau Levels.* Note that the spectrum of H_0 is purely essential. The creation and annihilation operators act between Landau eigenspaces in the following way:

$$\mathbf{a}^* : \mathcal{L}_q \longrightarrow \mathcal{L}_{q+1} \ , \ \mathbf{a} : \mathcal{L}_q \longrightarrow \mathcal{L}_{q-1} \ , \ \mathbf{a} : \mathcal{L}_0 \longrightarrow 0.$$

Now the question is what happens to the Landau Levels under a perturbation introduced by the action of an electric field. More precisely, let V be a measurable, essentially bounded and compactly supported real valued function in $\mathbb{R}^2$; we will indicate by the same letter V the operator of multiplication by V in $L^2(\mathbb{R}^2)$. Consider the perturbed operator $H = H_0 + V$. Since the operator V is relatively compact with respect to H_0, Weyl's theorem on the essential spectrum implies that the essential spectrum of H and that of H_0 coincide; therefore the essential spectrum of H consists of the Landau Levels and the spectrum of H consists entirely of eigenvalues. New eigenvalues, necessarily of finite multiplicity, may appear with the perturbation, with only possible accumulation points at the Landau Levels. The authors of the papers cited above are interested in whether the new eigenvalues generated are finite in every spectral gap (interval between two consecutive Landau Levels), and in case they are not to the rate of convergence to the Landau Levels.

Denote by P_q the L^2 orthogonal projection with range $\mathcal{L}_q$. It turns out that the asymptotics of the eigenvalues near Λ_q are determined by the eigenvalue asymptotics of the compact Toeplitz-like operator $P_qV|_{\mathcal{L}_q}$. This was established in precise terms for example in [18] for positive potentials, using a variational characterization of eigenvalues; in particular it shows that if the number of eigenvalues generated by the perturbation is finite in the interval $(\Lambda_q, \Lambda_{q+1})$ then the operator $P_qV|_{\mathcal{L}_q}$ has finite rank. In the papers above the potential was assumed to be positive from the beginning because of lack of knowledge about the spectrum of a Toeplitz operator with symbol of variable sign. The finite rank theorem had not been established yet, so it was not even known whether the Toeplitz operator could have finite rank without the potential being zero. In [19] the relation between eigenvalues of the perturbed Hamiltonian with potential of variable sign around a Landau Level and eigenvalues of the corresponding Toeplitz-like operator is stated in exact terms and proved, but only for the ground level Λ_0. The proof uses some more advanced spectral theory techniques than [18]. We tried for some time to produce a simpler proof of the fact that if the eigenvalues in the spectral gap are finite than the Toeplitz operator has finite rank, but we were unsuccessful. Let us now assume that this is true.

The operator corresponding to the ground state $P_0V|_{\mathcal{L}_0}$ is practically one of the type we studied in the above chapters, as there is an isometry between $\mathcal{F}_b$ and $\mathcal{L}_0$ given by $f \mapsto e^{-\Psi} f$ and thus the quadratic form of $P_0V|_{\mathcal{L}_0}$ is unitarily equivalent to the quadratic form of the Toeplitz operator T_V on the Fock space. Assume that the eigenvalue cluster around Λ_0 consists of a finite number of eigenvalues: then the operator $P_0V|_{\mathcal{L}_0}$ has finite rank, and consequently T_V has finite rank on the Fock space; but by the main result of the previous chapter this implies that V is identically zero.

For higher Landau Levels, this is not as simple since the spaces $\mathcal{L}_q$ do not immediately fit into the framework of Luecking's theorem. However, it is proved in [9], Corollary 9.3, that the operator $P_qVP_q \colon L^2(\mathbb{R}^2) \longrightarrow L^2(\mathbb{R}^2)$ for V bounded and compactly supported is unitarily equivalent to the operator $P_0WP_0 \colon L^2(\mathbb{R}^2) \longrightarrow L^2(\mathbb{R}^2)$, where $W = \mathcal{D}_q(\Delta)V$ with $\mathcal{D}_q$ being a polynomial of degree q with positive coefficients and differentiation is in the sense of distribution if V is not smooth. This implies that if $P_qV|_{\mathcal{L}_q}$ has finite rank, the same is true for $P_0W|_{\mathcal{L}_0}$; but on $\mathcal{L}_0$ we can apply the theorem, so $W = \mathcal{D}_q(\Delta)V$ (not necessarily a function) is a finite linear combination of delta distributions and their derivatives. By Fourier transforming the equation defining W, we obtain that $\mathcal{D}_q(-|\xi|^2)\widehat{V}(\xi)$ is a finite combination of exponentials multiplied by polynomials; then either $\widehat{V}$ has poles (the zeros of $\mathcal{D}_q(-|\xi|^2)$) or $\mathcal{D}_q(-|\xi|^2)$ divides the right hand side, so $\widehat{V}$ is itself a combination of exponentials multiplied by polynomials. The former case cannot happen since $\widehat{V}$ is an entire function on the complex plane, and the latter case cannot happen either unless V is zero, because otherwise V itself would be a combination of delta functions and their derivatives, but V is a function.

We thus obtained that if a perturbation gives rise to a finite number of eigenvalue around one Landau Level, then the perturbation is zero (and consequently the finite number of new eigenvalues is in fact zero around *any* Landau Level); in other words, as soon as the perturbation is nonzero an infinite number of eigenvalues of the perturbed Hamiltonian, necessarily having as only accumulation point the corresponding Landau Level, appear in every spectral gap. Information on the rate of convergence can be found in the papers cited above.

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