Infinite-Dimensional Linear Stochastic Systems with Random Coefficients and Local Interactions

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Abstract

In this thesis some results for a model that is a merge between the random coefficient autoregressive model of order one and a stochastic process with local interaction in time and space are presented. The first part of the thesis deals with some result from the random coefficient model. The second part of this thesis deals with the random coefficient model with local interaction. Results such as expectation, variance bounds and some covariance results are presented.

Keyword: Stochastic processes, Local interaction, autoregressive random coefficients, probability theory.

Contents

1 Introduction

This thesis will cover two main topics, the first is the random coefficient autoregressive process which is a generalisation of the simpler autoregressive process that is studied in most introductory courses in stochastic process and time series analysis. The second topic is an extension for this model to both time and space and allowing local spatial interaction. This interaction takes the form of first scaling the process neighbour to the right by some small constant ϵ and then adding it. This extension moves the problem out of classical work done on scalar or vector valued random coefficient autoregressive processes to a discrete time, discrete space interacting particle system with a markovian property in time.

One of the reasons the autoregressive class of models has seen so much focus is its wide range of application, many things are modelled using some form of autoregressive structure. The random coefficient class of models has seen uses in problems in widely diffrent fields such as economics as mentioned by Nicholls and Quinn [1982] and in hydrology, biology and meteorology as noted by Andel [1976]. This usefulness has lead to significant attention from academics. Several key results such as stationarity [Andel, 1976], stability (regarding similar type of regression models) [Conlisk, 1974], general moments [Feigin and Tweedie, 1985] and several results regarding inference [Nicholls and Quinn, 1982] have been worked out further aiding the usefulness of this class of models. This usefulness comes from that the rigid coefficient regime is lifted and the very nature of the dependency is allowed to change over time which might be a more natural and intuitive way to study the complex systems generating what is possible to observe.

This allowing of interacting in space moves this construction into the realm of interacting particle systems. Interacting particle system models have found use in both in both social science, for example the voter model, and the natural sciences in statistical mechanics as mentioned by Liggett [1985]. In social sciences if two processes can observe the action of each other it is a rather unnatural assumption that they will not be affected by each other. If more processes are added the local interaction will spread trough out the space with sufficient time. This implies that a different type of interactions need to be studied compared to the limited amount of interaction a vector type model can provide. Thus it is rather interesting to join these notion of random coefficient and spatial dependency and study such a model. This is the primary reason for the construction and process in definition two, even if it is done under the harshest of assumptions.

The thesis is composed of two parts. The first part presents a simpler case without dependency and some results regarding its expectation, covariance and some question common for stochastic processes. This is a special case of $\epsilon = 0$ leading of the second case. The second case examins the object with right side dependency in one spatial dimension. This part will contain expectation, variances bounds, and some covariance results under restrictions. The thesis ends with a short conclusion and open problems that have been identified during the work. To keep the thesis concise a section containing all the proofs is provided at the end of the thesis. In general the reader is expected to have a solid grasp of basic probability theory and stochastic processes. For the interested reader Nicholls and Quinn [1982] monograph gives a oversight over the random coefficient autoregressive process. A corresponding overview for interacting particle systems is provided by Liggett [1985].

2 Random coefficient model

2.1 Introduction

Here we consider the random coefficient model of order 1. This model is mostly serves as primer for the results presented in part 2, since most of the results are previously known. The statements in this section is done under no assumption of distribution, but only on assumptions regarding moments. Since the problem of distribution is hard due to the heavy use of summation and multiplication of dependent random variables this will not be discussed or treated in detail. A formal definition is now in order.

Definition 1 *Let $X(t)$ be a discrete time stochastic process defined by the following equation*

$$X(t) = \eta(t)X(t-1) + \xi(t)$$

where $t \in \mathbb{N}$. $\eta(t)$, $\xi(t)$ and $X(0)$ are independent random variables for all t.
The following assumptions are made as a part of the definition.

1. *$E\eta(t) < \infty$, $E\xi(t) < \infty$ and $EX(0) < \infty$ $\forall t \in \mathbb{N}$*
2. *$\eta(t)$ are independent and identically distributed $\forall t \in \mathbb{N}$.*
3. *$\xi(t)$ are independent and identically distributed $\forall t \in \mathbb{N}$.*

This construction is very similar to the construction of the $AR(1)$ process, with the main difference that η is a random variable compared to a constant in the $AR(1)$ case. The stochastic nature of this process can be seen as if a recursive operator is applied to a driving sequence of $(\eta(t), \xi(t))$ and some starting value $X(0)$. Observe that according to construction the process $X(t)$ has a Markovian property since all information about the process history will be contained in $X(t-1)$.

Some motivation of the assumption is clearly needed due to the quite hash restriction they impose. The first assumption is essentially made to avoid uninstresting cases. The identical distribution on $\xi(t)$ and $\eta(t)$ might not be necessary but without them asymptotic invariance for shifts in time might not be obtainable. The independence assumption in time for the coefficient η is done to reduce the complexity of the problem. Some work by Tjøstheim [1986] have been done under but have been proved to be difficult under anything other than the most basic of dependency structure in time imposed on the coefficients. The independence between the noise random variable ξ and parameter random variable η is a natural one and is commonly made.

We study fundamental properties of $X(t)$ such as $E(X(t))$, $V(X(t))$ and $Cov(X(t), X(s))$. Exact formula for $EX(t)$ is derived and in particular exponential decay of $Cov(X(t), X(s))$ is shown when $|t-s| \to \infty$. This part will also contain some statements regarding common questions such as stationarity, martingale properties and convergence. These results are chosen since they will give the reader a basic understanding of the process. Since many of the results are rather elementary they will not be commented in detail. Some concluding comments and comparison with the classical autoregressive process of order will however be made at the end of this first part.

2.2 Results

In this section the results are presented. Beginning with elementary result such as expectation, variance and covariance. Then limiting results are presented and the section ends with a convergence result for $X(t)$.

Theorem 1 *(Full and partial recurrence in time)*
Let $X(t)$ be a stochastic process defined in Definition 1. Then $X(t)$ can be expressed in the following way:

$$X(t) = \Big(\prod_{j=1}^{t} \eta(j)\Big) X(0) + \xi(t) + \sum_{i=1}^{t-1} \Big(\prod_{j=i+1}^{t} \eta(j)\Big)\xi(i). \tag{1}$$

Furthermore, for any s $0 \leqslant s < t$, the following partial recursion in time for $X(t)$ takes place:

$$X(t) = \Big(\prod_{j=s+1}^{t} \eta(j)\Big) X(s) + \xi(t) + \sum_{i=s+1}^{t-1} \Big(\prod_{j=i+1}^{t} \eta(j)\Big)\xi(i). \tag{2}$$

Theorem 2 *(Expectation of $X(t)$)*
Let $X(t)$ be a stochastic process defined by Definition 1. Assume that $E\eta \neq 1$. Then the expectation of the process $X(t)$ is given by:

$$EX(t) = \frac{1-(E\eta)^t}{1-E\eta} E\xi + (E\eta)^t EX(0). \tag{3}$$

Remark 1 *Under the assumptions $E\eta = 1$ and $E\xi = 0$ the expectation of $X(t)$ follows straight from taking the expectation of equation (2) in Theorem 1 and is:*

$$EX(t) = EX(0), \qquad \forall t \geqslant 0$$

Theorem 3 *(Second moment of $X(t)$)*
Let $X(t)$ be a stochastic process defined by Definition 1. Assume $E\eta^2 < \infty$, $E\xi^2 < \infty$ and $E\eta \neq 1$. Then:

$$EX(t)^2 = (E\eta^2)^t EX(0)^2 + 2\sum_{i=1}^{t} (E\eta^2)^{i-1} E\eta E\xi EX(t-i) + E(\xi^2)\frac{1-(E\eta^2)^t}{1-E\eta^2}$$

Then where $EX(t)$ is defined by equation (3).

Corollary 1 *(Variance of $X(t)$)*
Under stated assumptions of Definition 2, Theorem 2 and Theorem 3 we have:

$$V(X(t)) = (E\eta^2)^t EX(0)^2 + 2\sum_{i=1}^{t} (E\eta^2)^{i-1} E\eta E\xi EX(t-i) + E(\xi^2)\frac{1-(E\eta^2)^t}{1-E\eta^2} - \Big(\frac{1-(E\eta)^{t-1}}{1-E\eta} E\xi + (E\eta)^t EX(0)\Big)^2,$$

Where $EX(t)$ is defined by equation (3).

Remark 2
Usually it is assumed that $E\xi = 0$. If $E\xi \neq 0$, the variable ξ can be substituted by $\xi := \xi' + E\xi$ where $E\xi' = 0$. If $E\xi = 0$ the following simplifications are obtained from the statement of Corollary 1.

$$VX(t) = (E\eta^2)^t EX(0)^2 + \sigma_\xi^2 \frac{1-(E\eta^2)^{t-1}}{1-E\eta^2} - (E\eta)^{2t}(EX(0))^2.$$

Theorem 4 *(Covariance of $X(t)$)*
Let $X(t)$ be a stochastic process defined by Definition 1. Assume that, $E\eta^2 < 1$. Then the covariance of $X(t)$ and $X(s)$, $\forall t, s \in \mathbb{N}$ is given by:

$$Cov(X(t), X(s)) = (E\eta)^{|t-s|} V(X(s)).$$

Corollary 2 *(Decay of covariance)*
Under assumptions of Theorem 4*,* $X(t)$ *has exponentially fast decay of covariance as* $|t-s| \to \infty$*.*

Corollary 3 *(Asymptotic expectation, variance and covariance)*
Let $X(t)$ *be a stochastic process defined by Definition 1 and assume that* $E\xi = 0$*.*

1. *If* $|E\eta| < 1$ *then* $EX(t) < \infty$ *and* $\lim_{t\to\infty} EX(t) = 0$*.*
2. *If* $E\eta^2 < 1$ *then* $VX(t) < \infty$ *and* $\lim_{t\to\infty} VX(t) = \frac{\sigma_\xi^2}{1-E\eta^2}$*.*
3. *If* $E\eta^2 < 1$ *then* $\lim_{t\to\infty} Cov(X(t+l), X(t+s)) = (E\eta)^{|l-s|}\frac{\sigma_\xi^2}{1-E\eta^2}$*,* $\forall l, s \in \mathbb{N}$*.*
4. *If* $E\eta = 1$ *and* $E\eta^2 > 0$ *then* $\lim_{t\to\infty} VX(t) = \infty$*.*

Theorem 5 *(Conditions for asymptotic stationarity)*
Let $X(t)$ *be a stochastic process defined by Definition 1. Assume further that* $E\eta^2 < 1$*, then:*
$X(t)$ *is asymptotically stationary in a wide sense,*
i.e, $\lim_{t\to\infty} Cov(X(t+l), X(t+s))$ *and* $\lim_{t\to\infty} EX(t)$ *is not dependent on* t*.*

Remark 3 *(Martingale property)*
Let $X(t)$ *be a stochastic process defined by Definition 1, assume* $E\xi(t) = 0$*,* $E\eta = 1\ \forall t \in \mathbb{N}$*. Then:*
$X(t)$ *is a discrete time martingale.*

Theorem 6 *(On convergence of* $X(t)$*)*
Let $X(t)$ *be a stochastic process defined by Definition 1.*
Further let $E\eta^2 < 1$*. Then:*

$$\left| X(t) - \big(\xi(t) + \sum_{i=1}^{t-1}\big(\prod_{j=i+1}^{t} \eta(j)\big)\xi(i)\big) \right| \to 0 \qquad \text{a.s}$$

when $t \to \infty$*.*

Remark 4
Theorem 6 *implies in particular that the limiting distribution of* $X(t)$ *do not depend on* $X(0)$ *if* $E\eta^2 < 1$

2.3 Closing remarks

Many of the presented results in the previous section are similar the known and well understood results from the autoregressive model. The expectation, variance and covariance are the most obvious. The results can loosely be described by substituting the constant for the random variables expectation and second moment, in the case for the variance expression. The properties that follows also follows intuitively by substituting the constant with the expectation. A bold statement regarding the relationship between the autoregressive process and the random coefficient model is that the first is simply a special case of the latter with a one-point distribution. Since the variance is zero for the one-point distribution the n moment is the same as the first moment to the power n. With this stated comments the results should appearer more intuitive.

A comment should be done regarding the difference in distribution between the two processes. For most applications it is assumed that $\xi \in N(0, \sigma_\xi^2)$ for the autoregressive process, which gives the result that the process is in itself normal. Given the result in $Theorem$ 1 and $Theorem$ 6 the process can not be normal, even if both $\eta, \xi \in N(\mu, \sigma^2)$, since the normal distribution only is invariant under addition but not multiplication. The problem of deriving explicit formulas for product distribution is in general hard. Which is one of the main reason the distributional aspects of the process following Definition 1 is not very well understood or atleast difficult hard to obtain. With these priming results the main results of this thesis will now follow in the next section, when neighbourhood dependency is introduced.

3 Markov process with local interaction

3.1 Introduction

The random coefficient model has seen prevalent use in social and natural sciences, and thus have been extensively studied, with generalisation to vector random coefficient models [Nicholls and Quinn, 1982]. The main differences between the vector valued random coefficient model and the object presented below in Definition 2 is that the dependency in space will grow as time increases, making this problem infinite dimensional in both time and space when $t \to \infty$. This heightens the complexity of the problem. We study one sided spatial interaction in one dimension. General case of a fixed interacting neighbourhood and dimension can be treated in a similar way.

Definition 2 *Let $X(t,k)$ be a discrete time, discrete space stochastic process defined by the following equation*

$$X(t,k) = \eta(t,k)X(t-1,k) + \epsilon X(t-1,k+1) + \xi(t,k) \tag{4}$$

where $\eta(t,k)$, $\xi(t,k)$ and $X(0,k)$ are random variables for all $t \in \mathbb{N}$, $k \in \mathbb{Z}$.
The following assumptions are made as a part of the definition.

1. $\left|E\eta(t,k)\right| < \infty$, $\left|EX(0,k)\right| < \infty$ *and* $E\xi(t,k) = 0$ $\forall t \in \mathbb{N}$ $\forall k \in \mathbb{Z}$.
2. $\eta(t,k)$, $\xi(t,k)$ *and* $X(0,k)$ *are independent of each other* $\forall t \in \mathbb{N}$, $\forall k \in \mathbb{Z}$.
3. $\eta(t,k)$ *are identically distributed random variables and independent in both time and space.* $\forall t \in \mathbb{N}$, $\forall k \in \mathbb{Z}$
4. $\xi(t,k)$ *are identically distributed random variables and independent in both time and space.* $\forall t \in \mathbb{N}$, $\forall k \in \mathbb{Z}$
5. $X(0,k)$ *are independent random variable* $\forall k \in \mathbb{Z}$

Many of the assumptions follows from the same reasoning as for the non dependency model. If also $X(0,k)$, $\forall k \in \mathbb{Z}$ are identically distributed. $X(t,k)$ will be invariant under shifts in space. It should be noted that if the process is invariant to shifts in space properties such as expectation, variance and covariance for process following Definition 2 will be the same for all k at a given t. Even if $X(0,k)$ are not identically distributed, this process will also has a Markovian property since all the information in $X(t,k)$ will be contained in the previous time step, if the $X(t-1,k)$ is considered for all k.

A good visualisation of the dependency induced by adding the right neighbour is to visualize a triangle. Since $X(t,k)$ will be dependent on $X(t-1,k+1)$ and $X(t-1,k+1)$ will then intern be dependent on $X(t-2,k+2)$, and so forth. A dependency triangle is formed with its corners at (t,k), $(0,k)$ and $(0,k+t)$. This is more clearly shown in the figure 1, below.

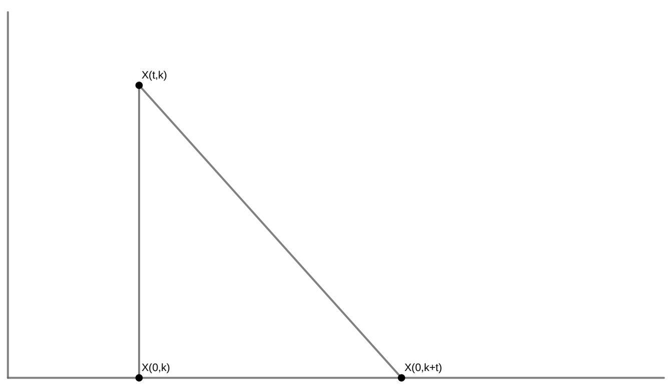

Figure 1.

If there is no overlap of between dependency triangles for two points, then they are independent by assumptions in Definition 2. If a overlap is present, as shown in the figure 2, the points will be dependent due to this overlap.

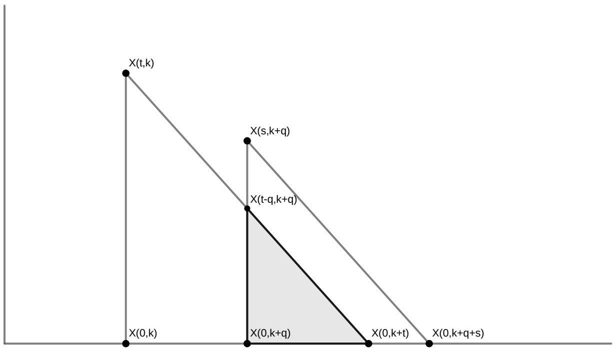

Figure 2.

When t becomes large this triangle will become larger and the number of points that a given point (t, k) is dependent on will increase. It should also be mentioned the covariance function for two points will change depending if one of the points is on, in or outside the other points dependency triangle. Even if $X(t, k)$ is assumed to be invariant under shifts in time. This leads to an added level of complexity when a recursion is to be obtained for the covariance function.

3.2 Results

In this section the results are presented. Beginning with result such as expectation, bounds for the variance and covariance. Then some more precise results under some restrictions are presented.

Theorem 7 *(Full and partial recursions for a given k)*
Let $X(t, k)$ be a stochastic process defined by Definition 2. Then the following recurrence takes place:

$$X(t,k) = \prod_{i=1}^{t} \eta(i,k)X(0,k) + \xi(t,k) + \sum_{i=1}^{t-1} \prod_{j=i+1}^{t} \eta(j,k)\xi(i,k)$$
$$+ \epsilon X(t-1,k+1) + \epsilon \sum_{i=1}^{t-1} \prod_{j=i+1}^{t} \eta(j,k)X(i-1,k+1)$$

Furthermore for any $0 \leqslant s < t$, we have the following partial recursion in time:

$$X(t,k) = \prod_{i=s+1}^{t} \eta(i,k)X(s,k) + \xi(t,k) + \sum_{i=s+1}^{t-1} \prod_{j=i+1}^{t} \eta(j,k)\xi(i,k) +$$
$$\epsilon X(t-1,k+1) + \epsilon \sum_{i=s+1}^{t-1} \prod_{j=i+1}^{t} \eta(j,k)X(i-1,k+1)$$

Theorem 8 *(Expectation of $X(t, k)$)*
Let $X(t, k)$ be a stochastic process defined by Definition 2. Then for all t, k the expectation is:

$$EX(t,k) = \sum_{j=0}^{t} \binom{t}{j} (E\eta)^{t-j} \epsilon^j \mu(0,k+j)$$

Remark 5
If $X(0, k)$ are identically distributed the following follows from Theorem 8, denote $EX(0, k) = \mu_X$:

$$EX(t,k) = \mu_X(\eta + \epsilon)^t$$

Remark 6
If $EX(0,k)$ can be written on the form $c\alpha^k$ where c, α are constants and $|\alpha| < 1$ then:

$$EX(t,k) = c\alpha^k (E\eta + \epsilon\alpha)^t$$

Theorem 9 *(A bound for the variance of X(t,k))*
Let $X(t,k)$ be a stochastic process defined by definition 2. Assume $E(\eta+\epsilon)^2 < 1$, $E(\eta-\epsilon)^2 < 1$, $VX(0,k) < \infty$, $\sigma_\xi^2 < \infty$, and $X(0,k)$ are identically distributed. Then the following bounds holds for all t,k

1. *Case, $\epsilon E\eta > 0$:*

$$\begin{aligned}
(E(\eta-\epsilon)^2)^t V(X(0,k)) + \sum_{i=0}^{t-1} E((\eta-\epsilon)^2)^i \sigma_\xi^2 + \sum_{i=0}^{t-1} E((\eta-\epsilon)^2)^{t-1-i} (EX(0,k))^2 (E\eta+\epsilon)^{2i}\sigma_\eta^2 \\
\leqslant V(t,k) \leqslant \\
(E(\eta+\epsilon)^2)^t V(X(0,k)) + \sum_{i=0}^{t-1} E((\eta+\epsilon)^2)^i \sigma_\xi^2 + \sum_{i=0}^{t-1} E((\eta+\epsilon)^2)^{t-1-i} (EX(0,k))^2 (E\eta+\epsilon)^{2i}\sigma_\eta^2
\end{aligned}$$

2. *Case, $\epsilon E\eta < 0$:*

$$\begin{aligned}
(E(\eta+\epsilon)^2)^t V(X(0,k)) + \sum_{i=0}^{t-1} E((\eta+\epsilon)^2)^i \sigma_\xi^2 + \sum_{i=0}^{t-1} E((\eta+\epsilon)^2)^{t-1-i} (EX(0,k))^2 (E\eta+\epsilon)^{2i}\sigma_\eta^2 \\
\leqslant V(t,k) \leqslant \\
(E(\eta-\epsilon)^2)^t V(X(0,k)) + \sum_{i=0}^{t-1} E((\eta-\epsilon)^2)^i \sigma_\xi^2 + \sum_{i=0}^{t-1} E((\eta-\epsilon)^2)^{t-1-i} (EX(0,k))^2 (E\eta+\epsilon)^{2i}\sigma_\eta^2
\end{aligned}$$

Corollary 4 *(Bound of Variance)*
Under the assumptions of Theorem 9 the following variance bounds holds when $t \to \infty$:

1. *Case $\epsilon E\eta > 0$:*

$$\frac{\sigma_\xi^2}{1-E(\eta-\epsilon)^2} \leqslant \liminf_{t\to\infty} VX(t,k) \leqslant \limsup_{t\to\infty} VX(t,k) \leqslant \frac{\sigma_\xi^2}{1-E(\eta+\epsilon)^2}$$

2. *Case $\epsilon E\eta < 0$:*

$$\frac{\sigma_\xi^2}{1-E(\eta+\epsilon)^2} \leqslant \liminf_{t\to\infty} VX(t,k) \leqslant \limsup_{t\to\infty} VX(t,k) \leqslant \frac{\sigma_\xi^2}{1-E(\eta-\epsilon)^2}$$

A uniform upper bound of $VX(t,k)$ can also be obtained. Let $\epsilon E\eta > 0$, denote $VX(0,k) = \sigma^2_{X(0)}$ and $EX(0,k) = \mu_{X(0)}$ for all k. Then for all k,t:

$$0 \leqslant VX(t,k) \leqslant \sigma^2_{X(0)} + \frac{\sigma_\xi^2 + \mu^2_{X(0)}\sigma_\eta^2}{1-E(\eta+\epsilon)^2}$$

If $\epsilon E\eta < 0$, $E(\eta+\epsilon)^2$ is replaced by $E(\eta-\epsilon)^2$ in the expression above.

Proposition 1 *(Covariance in time for a given position k)*
Let $X(t,k)$ be a stochastic process defined by Definition 2. Further let $E(\eta+\epsilon)^2 < 1$. Then for a given pair of points (t,k) and (s,k), $0 < t < s$ the covariance can be expressed as follows:

$$Cov(X(t,k),X(s,k)) = (E\eta)^{t-s} VX(s,k) + \epsilon Cov(X(t-1,k+1),X(s,k)) + \epsilon \sum_{i=s+1}^{t-1} (E\eta)^{t-i} Cov(X(i-1,k+1),X(s,k))$$

Conjecture 1 *(Decay of Covariance in time for a given position k)*
A more precise expression of the statement of proposition 1 can be obtained by further expanding the covariances, with this more coeficents will emerge. This leads to the following Conjecture:
$Co(X(t,k), X(s,k))$ *has exponential decay when* $|t-s| \to \infty$

Proposition 2 *(Explicit expression of variance when $E\eta = 0$)*
Let $X(t,k)$ be a stochastic process defined by Definition 2, further let $E\eta = 0$, $E(\eta+\epsilon)^2 < 1$ and $X(0,k)$ be i.i.d. Then the following result is obtained:

$$VX(t,k) = (E(\eta+\epsilon)^2)^t VX(0,k) + \sigma_\xi^2 \frac{1-(E(\eta+\epsilon)^2)^t}{1-E(\eta+\epsilon)^2} + (EX(0,k))^2 \sum_{i=0}^{t-1} \epsilon^{2(t-1-i)}(E(\eta+\epsilon)^2)^i$$

Proposition 3 *(Covariance in time, when $E\eta = 0$)*
Let $X(t,k)$ be a stochastic process defined by Definition 2 but without the independence assumption placed on $X(0,k)$. Further let $E\eta = 0$ and $s < t$. Then this result is obtained:

$$Cov(X(t,k), X(s,k)) = \epsilon^{t+s} Cov(X(0,k+t), X(0,k+s)).$$

Proposition 4 *(Covariance in space, when $E\eta = 0$)*
Let $X(t,k)$ be a stochastic process defined by Definition 2 but without the independence assumption placed on $X(0,k)$. Further let $E\eta = 0$ for all, t,k. Then following expression can be derived:

$$Cov(X(t,k), X(t,k+q)) = \epsilon^{2t} Cov(X(0,k+t), X(0,k+q+t))).$$

Conjecture 2 *(Covariance without parameter restriction)*
A bound for $Cov(X(t,k), X(t,k+q))$ even for the case $E\eta \neq 0$ will again be in the form

$$\left|Cov(X(t,k), X(t,k+q))\right| \leqslant VX(t-q,k)(E\eta+\epsilon)^{2q} + C(t,q),$$

where C is some constant dependent on (t,q). This can be obtained with the help of Fourier transform. However, this will be detailed in future work.

Proposition 5 *(Diagonal covariance when $E\eta = 0$)*
Let $X(t,k)$ be a stochastic process defined by Definition 2, $E\eta = 0$ and $VX(t,k)$ be well defined and finite. Then the covariance along the right diagonal can be expressed as the following:

$$Cov(X(t,k), X(t-s,k+s)) = \epsilon^s VX(s,k+s)$$

3.3 Closing remarks

Some results have been presented regarding expectation, variance and covariance under some restrictions. The results are in some ways similar to the results presented in the first part of the thesis. This local interaction increases leads to more interesting results even under some restrictions. Even if the interaction is local in the sense that the right neighbour at the previous time step is added. If the process, for a given space point, increases considerably due to the random nature of ξ this shock will propagate throughout the space to the left of the point as the process moves forwards in time. This shock will decrease when time moves on since ϵ is some small constant and the more distant a point is form the shock the more times it will be weighted by ϵ. Therefore the choice of both ϵ and the statistical properties of η are crucial for the overall stability of the system. This is an interesting insight into this infinite dimensional linear system that is studied in this thesis.

This problem will also have some interesting combinatorial properties. The initial values of the process will be weighted when time moves on either by a random η or some fixed ϵ. Since the neighbouring point is added at every point in time and for all points. Some initial points will be added more times while some less. This is perhaps more clearly seen when figure 3 is examined in detail, and how many paths for example $X(0,2)$ can take and still being a part of $X(3,0)$.

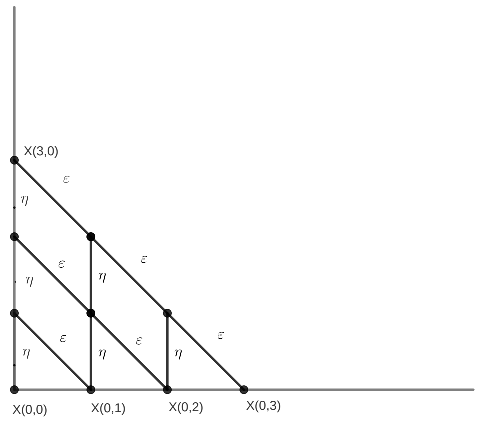

Figure 3.

The figure is a schematic over these paths the initial values can take for the point $X(3,0)$ and what they will be weighted with. If this complexity can be shown for $t = 3$, it should be noted that the number of paths will grow very rapidly when time increases. The more precise nature of the combinatorial side of this problem will however not be explored deeper in this thesis.

4 Conclusion

4.1 Conclusion

The problem of interacting particles and random coefficient models have generated a wide interest from a select group of mathematicians and statistician since it first inception in the seventies. This attention has lead to significant progress during the past decades. However the study of stochastic models that might capture the slowly changing dynamics of dependency structure is still an ongoing area of research. The results form this thesis are with most certainty not trivial, but limited in scope. The stated object in Definition 2 is simple in its construction but leads to many interesting structures and results even under the harshest assumptions of independence and identical distribution, as presented in the second part.

Further generalisations and reduction of assumption would be interesting to pursue since a lot of the underlying characteristics of this problem is yet to be solved. Due to the limited scoop of a master thesis a larger or deeper collection of results will not be presented. The results presented are quite fundamental to its nature, and such only a few comments will be needed for the results. They are provided in the closing remarks section of each part. In general most of the results are to be expected. In a wider logical sense and a lot of previous results regarding general autoregressive processes seems to carry over to the objects studied in this thesis. One of the primary example of this is for a result not presented in thesis but is a part of the work done, and that is the conditions on ϵ and $E\eta$ under symmetric dependency in d dimensions. Then the expectation exist when $|E\eta + 2d\epsilon| < 1$, this is intuitive since an increasing the number of spatial dimensions, under symmetric dependency, will lead to more neighbours being interacted with. This implies that either $E\eta$ or ϵ needs to become smaller. This is both an inserting and intuitive result.

In general this type on simple construction that gives rise to an interesting structure is a multifaceted problem in stochastic processes. Only elementary results are presented here. It should be noted that, as with many open questions in academia the process of answering one have created many more. This is certainly the case in this thesis too.

4.2 Open problems

This problem is in most aspects an interesting one and many more hours is needed to grasp the fundamental nature of the problem, in which space it belongs, convergence results and more. Due to the short time this type of thesis is supposed to be written many problems are left open or solved with rough approximation that might be removed if more time is spent on this problem. Since continued work will be done by the author on this project not many open problems will be posed. There is manly three open problems that might be stated:

1. What is the distribution of both the random coefficient model and the Markov process with local interaction, for a given distribution of η, ξ and $X(0,k)$?

2. Is there a distribution that is invariant or asymptomatic invariant under the operation of multiplication and addition?

3. What are corresponding results when the type of interaction is changed, for example if the neighbours at time t are added instead of $t-1$ or if the two nearest neighbours are added to the left and right?

These are interesting questions that will require dedicated work and deep insight of the field of mathematics and probability theory. They are however left to the reader.

5 Proofs

In each proof section, the proofs are in the same order as presented in the corresponding result sections.

5.1 Proofs for part 1

Proof of Theorem 1 (Solution of difference equation)
Let $X(t)$ be a stochastic process defined by Definition 1. Then the proof is done using induction. Let P_n denote the proposition that $X(n)$ can be written in the form stated in the theorem.
Proof of P_1

$$X(1) = \eta(1)X(0) + \xi(1) = \prod_{j=1}^{1} \eta(j)X(0) + \xi(1) + \sum_{i=1}^{1-1} \prod_{j=i+1}^{1} \eta(j)\xi(i).$$

Where the summation is an empty sum and is thus defined to 0, whereby P_1 is proven.
Proof of $P_n \Rightarrow P_{n+1}$, suppose that P_n is true.

$$\begin{aligned} X(n+1) &= \eta(n+1)[X(n)] + \xi(n+1) \\ &= \eta(n+1)\Big[\prod_{j=1}^{n} \eta(j)X(0) + \xi(n) + \sum_{i=1}^{n-1} \prod_{j=i+1}^{n} \eta(j)\xi(i)\Big] + \xi(n+1) \\ &= \prod_{j=1}^{n+1} \eta(j)X(0) + \eta(n+1)\xi(n) + \sum_{i=1}^{n-1} \eta(n+1) \prod_{j=i+1}^{n} \eta(j)\xi(i) + \xi(n+1). \end{aligned}$$

Note that the added terms now comprise the $(n+1)$:st term in the sum and the following expression is obtained.

$$X(t) = \prod_{j=1}^{n+1} \eta(j)X(0) + \xi(n+1) + \sum_{i=1}^{n+1-1} \prod_{j=i+1}^{n+1} \eta(j)\xi(i)$$

Thus it is shown $P_n \Rightarrow P_{n+1}$ and by the principle of mathematical induction the proposition P_n is true $\forall n \in \mathbb{N}$.

Furthermore the result of the partial recurrence is obtained by writing out the process in the following way.:

$$X(t) = \eta(t)\big[X(t-1)\big] + \xi(t) = \eta(t)\eta(t-1)\big[X(t-2)\big] + \eta(t)\xi(t-1) + \xi(t)$$

A clear structure appears and this can be done recursively to $X(s)$ whereby a formal induction proof as the one above can be obtained for arbitrary t and s. A formal proof will not be provide due to its similarity to the one above.

Q.E.D.

Proof of Theorem 2 (Expectation of $X(t)$)
Let $X(t)$ be a stochastic process defined by Definition 1, under the assumptions of *Theorem* 2. Using the results form equation (2) of *Theorem* 1 we derive:

$$EX(t) = E\sum_{i=1}^{t-1}\Big(\prod_{j=i+1}^{t}\eta(j)\Big)\xi(i) + E\xi(t) + E\Big((\prod_{j=1}^{t}\eta(j))X(0)\Big)$$

Independence assumption is used to split $E(\xi\eta) = (E\xi)(E\eta)$ and constant expectation for all t.

$$\begin{aligned}EX(t) &= \sum_{i=1}^{t-1}\Big(\prod_{j=i+1}^{t}E\eta(j)\Big)E\xi(i) + E\xi(t) + \Big(\prod_{j=1}^{t}E\eta(j)\Big)EX(0)\\ &= \sum_{i=1}^{t-1}(E\eta)^{t-i}E\xi + E\xi + (E\eta)^{t}EX(0)\\ &= \sum_{i=0}^{t-1}(E\eta)^{i}E\xi + (E\eta)^{t}EX(0)\\ &= \frac{1-E\eta^{t}}{1-E\eta}E\xi + E\eta^{t}EX(0).\end{aligned}$$

Which corresponds to the statement in the theorem.

Q.E.D.

Proof of Theorem 3 (Second moment of $X(t)$)
The theorem is proved by proving that for all t that $EX(t)^2$ can be written in the form presented in the theorem, this is done by induction.
Let $X(t)$ be a stochastic process defined by Definition 1 and further let the assumptions of the theorem be fulfilled. Further let P_n be the proposition that $EX(n)^2$ can be written in the form presented in the theorem.
Proof of P_1

$$\begin{aligned}EX(1)^2 &= E(\eta(1)X(0)+\xi(1))^2 = E\eta^2 EX(0)^2 + 2E\eta E\xi EX(0) + E\xi^2\\ &= (E\eta^2)^1 EX(0)^2 + 2\sum_{i=1}^{1}(E\eta^2)^{i-1}E\eta E\xi EX(1-i) + \sum_{i=1}^{1}(E\eta^2)^{i-1}E(\xi^2).\end{aligned}$$

P_1 is thereby shown.
Assume that P_n holds, now $P_n \Rightarrow P_{n+1}$ will be shown.

$$\begin{aligned}EX(n+1)^2 &= E(\eta(n+1)X(n)+\xi(n+1))^2 = E\eta^2 EX(n)^2 + 2E\eta E\xi EX(n) + E\xi^2\\ &= E\eta^2\Big[(E\eta^2)^n EX(0)^2 + 2\sum_{i=1}^{n}(E\eta^2)^{i-1}E\eta E\xi EX(n-i) + \sum_{i=1}^{n}(E\eta^2)^{i-1}E\xi^2\Big]\\ &+ 2E\eta E\xi EX((n+1)-1) + E\xi^2\\ &= (E\eta^2)^{n+1}EX(0)^2 + 2E\eta E\xi EX((n+1)-1) + 2E\eta^2\sum_{i=1}^{n}(E\eta^2)^{i-1}E\eta E\xi EX(n-i)\\ &+ E\eta^2\sum_{i=1}^{n}(E\eta^2)^{i-1}E(\xi^2) + E\xi^2.\end{aligned}$$

The second terms in the last equality now corresponds to $i = 1$ in the $n+1$ sum and can be moved in to the summation. The same argument is done to the last sum in the expression but for $i = 1$.

$$EX(n+1)^2 = (E\eta^2)^{n+1}EX(0)^2 + 2\sum_{i=1}^{n+1}(E\eta^2)^{i-1}E\eta E\xi EX(n+1-i) + \sum_{i=1}^{n+1}(E\eta^2)^{i-1}E(\xi^2).$$

Then it is shown that $P_n \Rightarrow P_{n+1}$ and by the principle of induction that P_n is true $\forall n \in \mathbb{N}$.

Q.E.D

Proof of Corollary 1 (Variance of $X(t)$) This result follows clearly from the result of *Theorem* 2 and *Theorem* 3, and $VX(t) = EX(t)^2 - (EX(t))^2$.

Proof of Theorem 4 (Covariance of $X(t)$, $X(s)$)
Let $X(t)$ be a stochastic process defined by Definition 1, and let the assumptions of the theorem be fulfilled. Consider the equation (2) form *Theorem* 1. Suppose that $s \leqslant t$. We have:

$$E(X(t)X(s)) = E\Big(\prod_{j=s+1}^{t} \eta(j)X(s)^2\Big) + E\Big(X(s)\sum_{i=s+1}^{t-1}\prod_{j=i+1}^{t}\eta(j)\xi(i)\Big) + E\xi(t)X(s).$$

and

$$(EX(t))(EX(s)) = E\prod_{j=s+1}^{t}\eta(j)(EX(s))^2 + EX(s)E\sum_{i=s+1}^{t-1}\prod_{j=i+1}^{t}\eta(j)\xi(i) + E\xi(t)EX(s).$$

Then the following is derived.

$$\begin{aligned} Cov(X(t),X(s)) &= E\prod_{j=s+1}^{t}\eta(j)X(s)^2 - E\prod_{j=s+1}^{t}\eta(j)(EX(s))^2 \\ &= (E\eta)^{t-s}(EX(s)^2 - (EX(s))^2) \\ &= (E\eta)^{t-s}VX(s). \end{aligned}$$

Q.E.D

Proof of Corollary 2 (Decay of covariance)
Under the assumption from Theorem 4, mainly that $E\eta^2 < 1$ it follows that $|E\eta| < 1$. With this remark the proof is trivial.

Proof of Corollary 3 (Asymptotic results)
The proof follows simply from convergence criterion of a geometric sum. The variance statements follows from a corollary of Jensen's inequality; that for a arbitrary random variable with well defined moments $V\chi < \infty \Leftrightarrow E\chi^2 < \infty$. With these two comments the formal proof will not be provided.

Proof of Theorem 5 (Conditions for asymptotic stationarity of $X(t)$)
For a stochastic process to be stationary in wide sense its sufficient that the expectation and covariance are time invariant. From 1 and 3 in Lemma 2 it follows clearly that asymptotically $EX(t)$ and $Cov(X(t), X(s))$ are invariant under shifts in time.

Proof of Theorem 6 (On convergence of $X(t)$)
We will show that $\prod_{j=1}^{t}\eta(j)X(0) \to 0$ $(a.s)$ when $t \to \infty$. A sufficient criterion for almost sure convergence [Shiryaev, 1996] is

$$\sum_{k=0}^{\infty} P\Big(\big(\prod_{j=1}^{k}|\eta(j)|\big)|X(0)| > \delta\Big) < \infty$$

For any $\delta > 0$. Using Markov's inequality $P(|\chi| > \epsilon) < \frac{E|\chi|}{\epsilon}$ a bound of the sum is obtained as follows:

$$\begin{aligned} \sum_{k=1}^{\infty} P\big(\prod_{j=1}^{k}|\eta(j)X(0)| > \delta\big) &\leqslant \sum_{k=1}^{\infty}\frac{E\big(\prod_{j=1}^{k}|\eta(j)|\big)E|X(0)|}{\delta} \\ &= \frac{1}{\delta}\sum_{k=1}^{\infty}\big(E|\eta|\big)^k E|X(0)| \\ &= \frac{E|X(0)|}{\delta}\sum_{k=1}^{\infty}(E|\eta|)^k < \infty \end{aligned}$$

By assumption, $E\eta^2 < 1$ hence the last sum converges, for all $\delta > 0$ which implies the statement of the Theorem.

Q.E.D

5.2 Proofs for part 2

Proof of Theorem 7 (Solution of difference equation) Let $X(t,k)$ be a stochastic process defined by Definition 2. Then the proof is done using induction. Let P_n denote the proposition that $X(n)$ can be written in the form stated in the theorem.
Proof of P_1

$$X(1,k) = \eta(1,k)X(0,k) + \epsilon X(0,k+1) + \xi(1)$$
$$X(1,k) = \prod_{i=1}^{1} \eta(i,k)X(0,k) + \xi(1,k) + \sum_{i=1}^{1-1} \prod_{j=i+1}^{1} \eta(j,k)\xi(i,k)$$
$$+ \epsilon X(1-1,k+1) + \epsilon \sum_{i=1}^{1-1} \prod_{j=i+1}^{1} \eta(j,k)X(i-1,k).$$

One should note that the summations are empty sums and is thus 0, whereby P_1 is proven.
Proof of $P_n \Rightarrow P_{n+1}$, suppose that P_n is true.

$$\begin{aligned} X(n+1,k) &= \eta(n+1,k)[X(n,k)] + \epsilon X(n,k+1) + \xi(n+1,k) \\ &= \eta(n+1,k)\Big[\prod_{i=1}^{n} \eta(i,k)X(0,k) + \xi(n,k) + \sum_{i=1}^{n-1} \prod_{j=i+1}^{n} \eta(j,k)\xi(i,k) \\ &+ \epsilon X(n-1,k+1) + \epsilon \sum_{i=1}^{n-1} \prod_{j=i+1}^{n} \eta(j,k)X(i-1,k)\Big] + \epsilon X(n,k+1) + \xi(n+1,k) \\ &= \eta(n+1,k)\prod_{i=1}^{n} \eta(i,k)X(0,k) + \eta(n+1,k)\xi(n,k) + \eta(n+1,k)\sum_{i=1}^{n-1} \prod_{j=i+1}^{n} \eta(j,k)\xi(i,k) \\ &+ \eta(n+1,k)\epsilon X(n-1,k+1) + \eta(n+1,k)\epsilon \sum_{i=1}^{n-1} \prod_{j=i+1}^{n} \eta(j,k)X(i-1,k) + \epsilon X(n,k+1) + \xi(n+1,k). \end{aligned}$$

Note that the added terms now comprise the n+1 term in the sum and the following expression is obtained:

$$X(n+1,k) = \prod_{i=1}^{n+1} \eta(i,k)X(0,k) + \xi(n+1,k) + \sum_{i=1}^{n+1-1} \prod_{j=i+1}^{n+1} \eta(j,k)\xi(i,k) + \epsilon X((n+1)-1,k+1)$$
$$+ \epsilon \sum_{i=1}^{n+1-1} \prod_{j=i+1}^{n+1} \eta(j,k)X(i-1,k).$$

Thus it is shown $P_n \Rightarrow P_{n+1}$ and by the principal of mathematical induction the proposition P_k is true $\forall k \in \mathbb{N}$.

This partial recurrence result is obtained by writing out:

$$\begin{aligned} X(t,k) &= \eta(t,k)\big[X(t-1,k)] + \epsilon X(t-1,k+1) + \xi(t,k) \\ &= \eta(t,k)[\eta(t-1,k)X(t-2,k) + \epsilon X(t-2,k+1) + \xi(t-1,k)] + \epsilon X(t-1,k+1) + \xi(t,k) \\ &= \eta(t,k)\eta(t-1,k)X(t-2,k) + \eta(t,k)\epsilon X(t-2,k+1) + \eta(t,k)\xi(t-1,k) + \epsilon X(t-1,k+1) + \xi(t,k). \end{aligned}$$

A clear structure appears and this can be done recursively to $X(s,k)$ whereby a formal induction proof as the one above can be obtained for arbitrary t and s. As with the non interacting result, a formal proof will not be provide due to its similarity to the one above.

Q.E.D

Proof of Theorem 8(Expectation of $X(t,k)$)
Let $X(t,k)$ be a stochastic process defined by definition 2. Then this result can either be obtained straight from taking the expectation of the result from *Theorem* 7 if $X(0,k)$ are identically distributed. However the proof provided below do not rely on using that $X(t,k)$ is invariant under shifts in space. Consider that $n := k+1$ in this transformation to deal with the neighbouring point.

$$EX(t,k) = \mu(t,k) = E\eta(t,k)EX(t-1,k) + \epsilon EX(t-1,k+1) \tag{5}$$

The discrete time Fourier transform is defined by:

$$\tilde{g}(\lambda) = \sum_{n=-\infty}^{\infty} g(n)e^{\lambda in},$$

The inverse discrete time Fourier transform is given by:

$$g(n) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{g}(\lambda)e^{-i\lambda n}d\lambda.$$

Apply the transform, w.r.t k, to the expected value function (5):

$$\begin{aligned}
\tilde{\mu}(t,\lambda) &= \sum_{k=-\infty}^{\infty} E\eta\mu(t-1,k)e^{\lambda ik} + \epsilon \sum_{k=-\infty}^{\infty} E\eta\mu(t-1,k+1)e^{\lambda ik+\lambda i-\lambda i} \\
&= \sum_{k=-\infty}^{\infty} E\eta\mu(t-1,k)e^{\lambda ik} + \epsilon \sum_{n=-\infty}^{\infty} E\eta\mu(t-1,n)e^{\lambda in-\lambda i} \\
&= \tilde{\mu}(t-1,\lambda)E\eta + \tilde{\mu}(t-1,\lambda)\epsilon e^{-\lambda i} \\
&= \tilde{\mu}(t-1,\lambda)(E\eta + \epsilon e^{-\lambda i}) \\
&= [\tilde{\mu}(t-2,\lambda)(E\eta + \epsilon e^{-\lambda i})](E\eta + \epsilon e^{-\lambda i}),
\end{aligned}$$

A recursion can now be observed for the Fourier transformed expected value function.

$$\begin{aligned}
\tilde{\mu}(t,\lambda) &= \tilde{\mu}(0,\lambda)(E\eta + \epsilon e^{-\lambda i})^t \\
&= \tilde{\mu}(0,\lambda)\sum_{j=0}^{t}\binom{t}{j}(E\eta)^{t-j}\epsilon^j e^{-i\lambda j}.
\end{aligned}$$

The inverse Fourier transform is preformed on the recursion. Since $EX(0,k) < \infty$, the integral will not diverge.

$$\begin{aligned}
\mu(t,k) &= \frac{1}{2\pi}\int_0^{2\pi} \tilde{\mu}(0,\lambda)\sum_{j=0}^{t}\binom{t}{j}(E\eta)^{t-j}\epsilon^j e^{-i\lambda j}e^{-\lambda ik}d\lambda \\
&= \sum_{j=0}^{t}\binom{t}{j}(E\eta)^{t-j}\epsilon^j \frac{1}{2\pi}\int_0^{2\pi} \tilde{\mu}(0,\lambda)e^{-i\lambda(k+j)}d\lambda \\
&= \sum_{j=0}^{t}\binom{t}{j}(E\eta)^{t-j}\epsilon^j \mu(0,k+j).
\end{aligned}$$

Which is the expression presented in the theorem.

Q.E.D

Proof of Theorem 9 (A bound of $V(t,k)$)
Here a proof for the bound in Case 1 will be presented, the proof for case 2 follows in a similar way. Consider

$$\begin{aligned}VX(t,k) &= Cov(X(t,k),X(t,k))\\ &= Cov(\eta(t,k)X(t-1,k)+\epsilon X(t-1,k+1)+\xi(t,k), \eta(t,k)X(t-1,k)+\epsilon X(t-1,k+1)+\xi(t,k)\\ &= Cov(\eta(t,k)X(t-1,k),\eta(t,k)X(t-1,k))\\ &+ 2\epsilon E\eta Cov(X(t-1,k),X(t-1,k+1)) + \epsilon^2 Cov(X(t-1,k+1),X(t-1,k+1)) + \sigma_\xi^2\end{aligned}$$

Under independence $V(\zeta_1\zeta_2) = E\zeta_2^2 V\zeta_1 + (E\zeta_1)^2 V\zeta_2$ which leads to:

$$VX(t,k) = E\eta(t,k)^2 VX(t-1,k) + (EX(t-1,k)^2\sigma_\eta^2 + 2\epsilon E\eta Cov(X(t-1,k),X(t-1,k+1)) + \epsilon^2 VX(t-1,k+1) + \sigma_\xi^2.$$

Using that if $X(t,k)$ is invariant under shifts in space, the variance is the same for all k, assuming that $E\eta\epsilon > 0$ (Case 1). However if $E\eta\epsilon < 0$ the inequalities are reversed (Case 2). The following bounding can be made:

$$-2\epsilon E\eta VX(t-1,k) \leqslant 2\epsilon E\eta Cov(X(t-1,k),X(t-1,k+1)) \leqslant 2\epsilon E\eta VX(t-1,k)$$

Using this bound the following is derived:

$$\begin{aligned}&E\eta(t,k)^2 VX(t-1,k) + (EX(t-1,k))^2\sigma_\eta^2 - 2\epsilon E\eta VX(t-1,k) + \epsilon^2 VX(t-1,k) + \sigma_\xi^2\\ &\qquad\leqslant VX(t,k) \leqslant\\ &E\eta(t,k)^2 VX(t-1,k) + (EX(t-1,k))^2\sigma_\eta^2 + 2\epsilon E\eta VX(t-1,k) + \epsilon^2 VX(t-1,k) + \sigma_\xi^2,\end{aligned}$$

Which is equivalent to,

$$\begin{aligned}&E(\eta(t,k)^2 - 2\epsilon\eta(t,k) + \epsilon^2)VX(t-1,k) + (EX(t-1,k))^2\sigma_\eta^2 + \sigma_\xi^2\\ &\qquad\leqslant VX(t,k) \leqslant\\ &E(\eta(t,k)^2 + 2\epsilon\eta(t,k) + \epsilon^2)VX(t-1,k) + (EX(t-1,k))^2\sigma_\eta^2 + \sigma_\xi^2.\end{aligned}$$

Hence

$$\begin{aligned}&E((\eta-\epsilon)^2)VX(t-1,k) + (EX(t-1,k))^2\sigma_\eta^2 + \sigma_\xi^2\\ &\qquad\leqslant VX(t,k) \leqslant\\ &E((\eta+\epsilon)^2)VX(t-1,k) + (EX(t-1,k))^2\sigma_\eta^2 + \sigma_\xi^2.\end{aligned}$$

A recursion in time can now be obtained for the bounds. The upper bound will be shown but the lower follows the same structure.

$$\begin{aligned}&E((\eta+\epsilon)^2)\left[E((\eta+\epsilon)^2)VX(t-2,k) + (EX(t-2,k))^2\sigma_\eta^2 + \sigma_\xi^2\right] + (EX(t-1,k))^2\sigma_\eta^2 + \sigma_\xi^2.\\ &= (E(\eta+\epsilon)^2)^2 VX(t-2,k) + E((\eta+\epsilon)^2)(EX(t-2,k))^2\sigma_\eta^2 + E((\eta+\epsilon)^2)\sigma_\xi^2 + (EX(t-1,k))^2\sigma_\eta^2 + \sigma_\xi^2.\end{aligned}$$

The recursion will become:

$$(E(\eta+\epsilon)^2)^t V(X(0,k)) + \sum_{i=0}^{t-1} E((\eta+\epsilon)^2)^i\sigma_\xi^2 + \sum_{i=0}^{t-1} E((\eta+\epsilon)^2)^{t-1-i}(EX(i,k))^2\sigma_\eta^2$$

The result presented is derived from this recursion and expectation is substituted with the result from *Theorem* 8 under invariance of shifts in space, leading to:

$$\begin{aligned}&(E(\eta-\epsilon)^2)^t V(X(0,k)) + \sum_{i=0}^{t-1} E((\eta-\epsilon)^2)^i\sigma_\xi^2 + \sum_{i=0}^{t-1} E((\eta-\epsilon)^2)^{t-1-i}(EX(0,k))^2(E\eta+\epsilon)^{2i}\sigma_\eta^2\\ &\qquad\leqslant V(t,k) \leqslant\\ &(E(\eta+\epsilon)^2)^t V(X(0,k)) + \sum_{i=0}^{t-1} E((\eta+\epsilon)^2)^i\sigma_\xi^2 + \sum_{i=0}^{t-1} E((\eta+\epsilon)^2)^{t-1-i}(EX(0,k))^2(E\eta+\epsilon)^{2i}\sigma_\eta^2\end{aligned}$$

Q.E.D

Proof of Corollary 4 (Bound of Variance for large t)
This results follows form the bounds in *Theorem* 9. The first case is shown here but the proof for case 2 follows the same structure. Consider the upper variance bound presented in *Theorem* 9:

$$(E(\eta+\epsilon)^2)^t V(X(0,k)) + \sum_{i=0}^{t-1} E((\eta+\epsilon)^2)^i \sigma_\xi^2 + \sum_{i=0}^{t-1} E((\eta+\epsilon)^2)^{t-1-i} (EX(0,k))^2 (E\eta+\epsilon)^{2i} \sigma_\eta^2$$

Then the following observations can be made when $t \to \infty$

1. If $E(\eta+\epsilon)^2 < 1$ the first term in the upper bound converges to 0.
2. The second term is a geometric sum and converges if $E(\eta+\epsilon)^2 < 1$ to $\frac{\sigma_\xi^2}{1-E(\eta+\epsilon)^2}$
3. Using that $\sum_{i=0}^{n-1} a^i b^{n-i} = \frac{a^n-b^n}{a-b}$, if $a, b > 0$ and $a \neq b$.
   Since $E(\eta+\epsilon)^2 < 1 \Rightarrow (E(\eta+\epsilon))^2 < 1$ both are positive, not equal and less then 1 the last term converges to 0.

With these three observations the upper bound presented is derived. The lower bound is obtained in the same manor since $E(\eta-\epsilon)^2 < 1$ by assumption.
The uniform bound is obtained from the upper bound in *Theorem* 9 for case 1. By bounding the first term with $VX(0,k)$ and the last term with $EX(0,k)$ since $EX(t,k)^2 \leqslant EX(0,k)^2$ if $E\eta+\epsilon < 1$, making the last term a geometric sum. Then the following is obtained:

$$V(X(0,k)) + \sum_{i=0}^{t} E((\eta+\epsilon)^2)^i \sigma_\xi^2 + (EX(0,k))^2 \sum_{i=0}^{t} E((\eta+\epsilon)^2)^i \sigma_\eta^2 \leqslant \sigma_{X(0)}^2 + \frac{\sigma_\xi^2 + \mu_{X(0)}^2 \sigma_\eta^2}{1-E(\eta+\epsilon)^2}$$

This is the result presented.

Proof of Proposition 1 (Covariance of $X(t,k)$ and $X(s,k)$)
Let $X(t,k)$ be a stochastic process defined by Definition 2, by using the partial recursion expression and independence of η and ξ this result is obtained. Suppose further that $0 < s < t$ then we derive the following using the partial recursion form *Theorem* 7:

$$\begin{aligned} Cov(X(t,k),X(s,k)) &= Cov\Big(\big(\prod_{i=s+1}^{t} \eta(i,k)\big)X(s,k) + \xi(t,k) + \sum_{i=s+1}^{t-1}\big(\prod_{j=i+1}^{t}\eta(j,k)\big)\xi(i,k) \\ &\quad + \epsilon X(t-1,k+1) + \epsilon \sum_{i=s+1}^{t-1}\big(\prod_{j=i+1}^{t}\eta(j,k)\big)X(i-1,k+1)\ ,\ X(s,k)\Big) \\ &= (E\eta)^{t-s} VX(s,k) + \epsilon Cov(X(t-1,k+1),X(s,k)) \\ &\quad + \epsilon \sum_{i=s+1}^{t-1} (E\eta)^{t-i} Cov(X(i-1,k+1),X(s,k)). \end{aligned}$$

This is the expression presented in the theorem and thus the proof is complete.

Q.E.D

Proof of Proposition 2 (Explicit expression of variance when $E\eta = 0$)
This proof follows from the following recursion and the results from *Theorem* 8.

$$\begin{aligned} VX(t,k) &= E(\eta+\epsilon)^2 VX(t-1,k) + \sigma_\xi^2 + EX(t-1,k) \\ &= (E(\eta+\epsilon)^2)^2 VX(t-2,k) + E(\eta+\epsilon)^2(\sigma_\xi^2 + EX(t-2,k)) + \sigma_\xi^2 + EX(t-1,k). \end{aligned}$$

By substituting in $VX(t-2,k)$ the recursion now becomes clear and the proof is concluded.

Proof of Proposition 3 (Covariance in time, when $E\eta = 0$)
This result is obtained form the following recursion, suppose that $s < t$ and $E\eta = 0$:

$$\begin{aligned} Cov(X(t,k), X(s,k)) &= Cov(\eta(t,k)X(t-1,k) + \xi(t,k) + \epsilon X(t-1,k+1), X(s,k)) \\ &= \epsilon Cov(X(t-1,k+1), X(s,k)) = \epsilon^2 Cov(X(t-2,k+2), X(s,k)) \end{aligned}$$

The recursion can be preformed until the point $(s, k+s)$. Then the result from *Proposition* 4 can be used to obtain the following:

$$\begin{aligned} Cov(X(t,k), X(s,k)) &= \epsilon^{t-s} Cov(X(t-(t-s), k+(t-s)), X(s,k)) \\ &= \epsilon^{t-s}\epsilon^{2s} Cov(X(t-(t-s)-s, k+(t-s)+s), X(s-s, k+s)) \\ &= \epsilon^{t+s} Cov(X(0,k+t), X(0,k+s)). \end{aligned}$$

This is what is expressed in the proposition and thus the proof is conclude.

Q.E.D

Proof of Proposition 4 (Covariance in space, when $E\eta = 0$)
Suppose the conditions of the proposition is fulfilled, then the following recursion can be made:

$$\begin{aligned} Cov(X(t,k), X(t,k+q)) &= Cov(\eta(t,k)X(t-1,k) + \epsilon X(t-1,k+1) + \xi(t,k) \ ; \\ &\qquad \eta(t,k+q)X(t-1,k+q) + \epsilon X(t-1,k+q+1) + \xi(t,k+q)) \\ &= \epsilon^2 Cov(X(t-1,k+1), X(t-1,k+q+1)) \\ &= \epsilon^2(\epsilon^2 Cov(X(t-2,k+2), X(t-2,k+q+2))) \end{aligned}$$

The recursion can be continued until the initial time. Then the following is derived:

$$Cov(X(t,k), X(t,k+q)) = \epsilon^{2t} Cov(X(0,k+t), X(0,k+q+t))).$$

Which is stated result.

Q.E.D

Proof of Proposition 5 (Diagonal covariance when $E\eta = 0$)
Under the assumptions of the proposition and of Definition 2. This result is obtained by the following recursion:

$$\begin{aligned} Cov(X(t,k), X(t-s,k+s) &= Cov(\eta(t,k)X(t-1,k) + \xi(t,k) + \epsilon X(t-1,k+1), X(t-s,k+s)) \\ &= \epsilon Cov(X(t-1,k+1), X(t-s,k+s)) \\ &= \epsilon\big(\epsilon Cov(X(t-2,k+2), X(t-s,k+s))\big) \\ &= ... = \epsilon^s Cov(X(t-s,k+s), X(t-s,k+s)) \\ &= \epsilon^s VX(t-s,k+s). \end{aligned}$$

Q.E.D

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